TDA206/DIT370: Discrete Optimization Final Exam

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Instructions:

- Write your answers to the point. Credit will only be assigned based on the correctness of what is requested e.g. if it is a LP, credit will be assigned based on the choice of decision variables, objective and constraints only. Anything else you write will not receive any credit. So use extra sheets to work out the answer and then only write the requested answer in your solution.
- You must upload **one pdf file** to FIRE as per the instructions on the course page. The pdf can be generated via LaTeX (preferred option) but it can also be handwritten and then scanned as a pdf (or photo image converted to pdf). Make sure to **join scanned images into one pdf** containing all your pictures/scans. The scanned images must be easily readable so make sure that you try and test out your scanning or imaging beforehand
- You may use the textbook and all other reading material mentioned together with the lectures on the course page.
- If you have questions, send an email at the address given above, they will be promptly answered between 8:30 AM and 12:30 AM.

1. (10 points) A company is planning its production schedule over the next six months (it is currently the end of month 2). The demand (in units) for its product over that timescale is as shown below:

Month	3	4	5	6	7	8
Demand	5000	6000	6500	7000	8000	9500

The company currently has in stock: 1000 units which were produced in month 2; 2000 units which were produced in month 1; 500 units which were produced in month 0.

The company can only produce up to 6000 units per month and the managing director has stated that stocks must be built up to help meet demand in the coming months. Each unit produced costs SEK 15 and the cost of holding stock is estimated to be SEK 0.75 per unit per month (based upon the stock held at the beginning of each month).

The company has a major problem with deterioration of stock in that the stock inspection which takes place at the end of each month regularly identifies ruined stock (costing the company SEK 25 per unit). It is estimated that, on average, the stock inspection at the end of month t will show that 11% of the units in stock which were produced in month t are ruined; 47% of the units in stock which were produced in month $t-1$ are ruined; 100% of the units in stock which were produced in month $t-2$ are ruined. The stock inspection for month 2 is just about to take place.

The company wants a production plan for the next six months that avoids stockouts.

(a) Formulate the problem of minimizing cost to satisfy all production constraints as a linear program. (HINT: Introduce (among others) variables for the number of units in stock at the beginning and end of each month t which were produced in month $i, i = t, t-1, t-2$.)

(b) Because of the stock deterioration problem the managing director is thinking of directing that customers should always be supplied with the oldest stock available. How would this affect your formulation of the problem?

2. (10 points) Consider the LP:

$$\begin{array}{rcl} \max \quad z & = & 3x_1 + 4x_2 + 5x_3 \\ \text{s.t.} & & 4x_1 + x_2 + 4x_3 \leq 4 \\ & & -2x_1 + x_2 + 2x_3 \leq 10 \\ & & x_1, x_2, x_3 \geq 0 \end{array}$$

Solve the LP using the Simplex algorithm. Show the tableau, the BFS and the values of the objective z at each step.

3. (10 points) Consider the problem of computing a *global* mincut in an undirected graph $G = (V, E)$ (with at least two vertices) and a cost function $c_e \geq 0, e \in E$. That is, we want to find a partition $(S, V \setminus S)$ to minimize $\sum_{i \in S, j \notin S} c_{i,j}$. Introduce the decision variables $x_v \in \{0, 1\}, v \in V$ to describe the partition ($x_v = 1$ iff $v \in S$) and $z_{u,v} \in \{0, 1\}$ to denote that the edge crosses the cut i.e. $x_u \neq x_v$.

(a) Give an *exact* formulation of the global mincut via an integer LP using the given decision variables. (HINT: Be careful to ensure that a feasible solution is a non–trivial cut i.e. all vertices in the graph don't get the same label.)

(b) Would it work to just change min to max to get an exact formulation of the maxcut problem? Justify.

(c) Consider the natural LP relaxation of the ILP in (a) and give the optimum value and corresponding solution, justifying briefly why it is feasible and optimal. Comment on whether this LP relaxation is useful to find a mincut.

4. (10 points) Consider the LP:

$$\begin{array}{rcl} \max \quad z & = & c_1x_1 + c_2x_2 \\ \text{s.t.} & & -x_1 + x_2 \leq 1 \\ & & x_1 + 2x_2 \leq 2 \\ & & 2x_1 + x_2 \geq 0 \\ & & 2x_1 - 2x_2 \leq 1 \\ & & x_1, x_2 \geq 0 \end{array}$$

(a) Write the dual LP.

(b) Using complementary slackness, write the primal solution corresponding to the dual solution $v = (0, 1/3, 0, 2/3)$.

(c) For which values of c_1 and c_2 is the primal solution optimal? Justify.

5. (10 points) The graph in Figure 1 has vertices of two types, a set $H = \{1, 2, 3, 4, 5, 6, 7\}$ of *hubs* and a set $C = \{a, b, c, d, e, f\}$ of *connectors*. A subset of hubs $S \subseteq H$ is called *dominant* if every connector in C is connected by an edge to at least one hub in S. For example, $S = \{1, 3, 7\}$ is dominant because a, b, c have edges to 1, d, f have edges to 7 and e has an edge to 3.

(a) Formulate an ILP to find a set of dominant hubs of minimum size in this particular graph.

(b) Pass to the LP relaxation and write its dual.

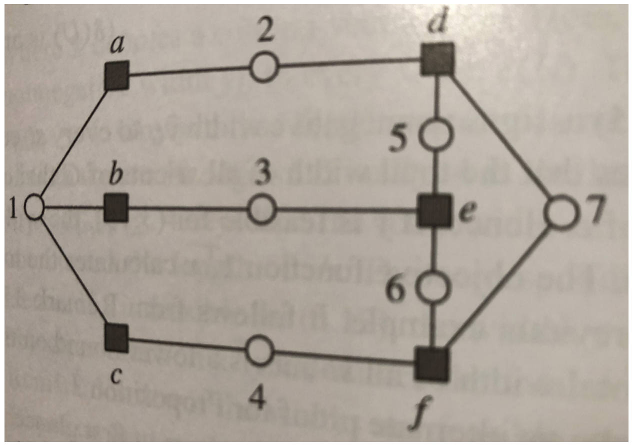

Figure 1: Hubs and connectors

(c) Apply the primal-dual algorithm to compute a pair of approximately optimal solutions. Show the initialization and the primal and dual variables at each step. Give the dominant set of hubs computed by your algorithm.

6. (10 points) As we discussed in class, the MINCUT problem can be solved in polynomial time – provided the costs on the edges are non–negative. What happens otherwise? This problem explores this. Let $G = (V, E)$ be a an undirected graph and let $c_{u,v}, (u, v) \in E$ be arbitrary costs on the edges, *positive or negative*. (For a natural example, consider a grocery shopping site where the vertices are food items and edges denote similarities in tastes: then beef would have an edge to hamburger bread with positive weight but beef would have an edge to a vegan soya product with very large negative weight!) Given an integer $k \geq 2$, we want to partition the vertices V into k disjoint subsets to minimize the resulting cut. That is, let $\ell : V \to \{1, \cdots, k\}$ denote the partition i.e. for a vertex $v \in V$, the subset that it belongs to is $\ell(v)$. Then we want to find a partition to minimize $\sum_{(u,v)\in E, \ell(u)\neq\ell(v)} c_{u,v}$. Call this problem $\pm$MINkCUT.

(a) Show that this problem is NP-hard by giving a reduction from MAXCUT to $\pm$MINkCUT. That is, given an input to the MAXCUT problem, show how to convert it efficiently into an instance of the $\pm$MINkCUT whose optimal solution allows you to easily find the optimum to the original MAXCUT problem.

(b) Consider a vector formulation where instead of the labels $\{1, \cdots, k\}$, we allow each vertex $u \in V$ to be assigned a vector $\mathbf{z}_u \in \{\mathbf{e}_1, \cdots, \mathbf{e}_k\}$ where $\mathbf{e}_i := (0, \cdots, 1, \cdots, 0)$ is the basis vector in $\mathbb{R}^k$ with 1 in coordinate i and 0 in other coordinates. Give an *exact* formulation of the $\pm$MINkCUT problem as a vector program using these vectors. Explain in two lines why your vector program is an exact formulation of $\pm$MINkCUT.

(c) Pass to the SDP relaxation of the vector program in (b) by introducing a matrix whose entries stand for dot products of the vector labels of the vertices in (b).

(d) Give a randomized rounding algorithm to convert the optimal solution to the SDP in (c) back to a partition of the vertices into k disjoint subsets. (Hint: Look at the dot product of the vector assigned to a vertex with the basis vectors $\{\mathbf{e}_1, \cdots, \mathbf{e}_k\}$.)

(e) Give an analysis of the value of the resulting cut.