SSY080

Transformer, Signaler och System

Examiner: Silvia Muceli muceli@chalmers.se

Date: 05/01/21, Time: 4 h (14.00-18.00)

Grading system

10 Quest A	**1 point each**	**10 points in total**	**5/10 necessary to pass**
3 Quest B	**5 points each**	**15 points in total**	**7/15 necessary to pass**

Note: only a **complete answer** results in the **full point** (A) / **points** (B).

Points	**12-15**	**16-20**	**20-25**
Final grade	**3**	**4**	**5**

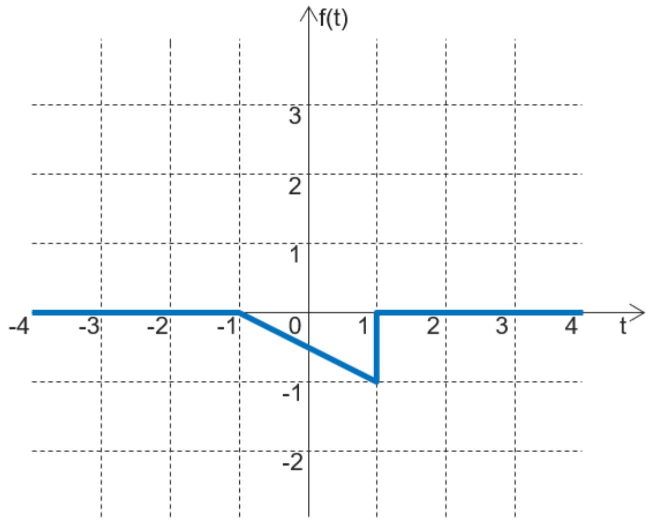
At the top of the first page, report **which questions you have answered** (e.g. A1, A3, A10, B2).

All answers must be written in **English**.

The solutions must be complete and easy to follow.

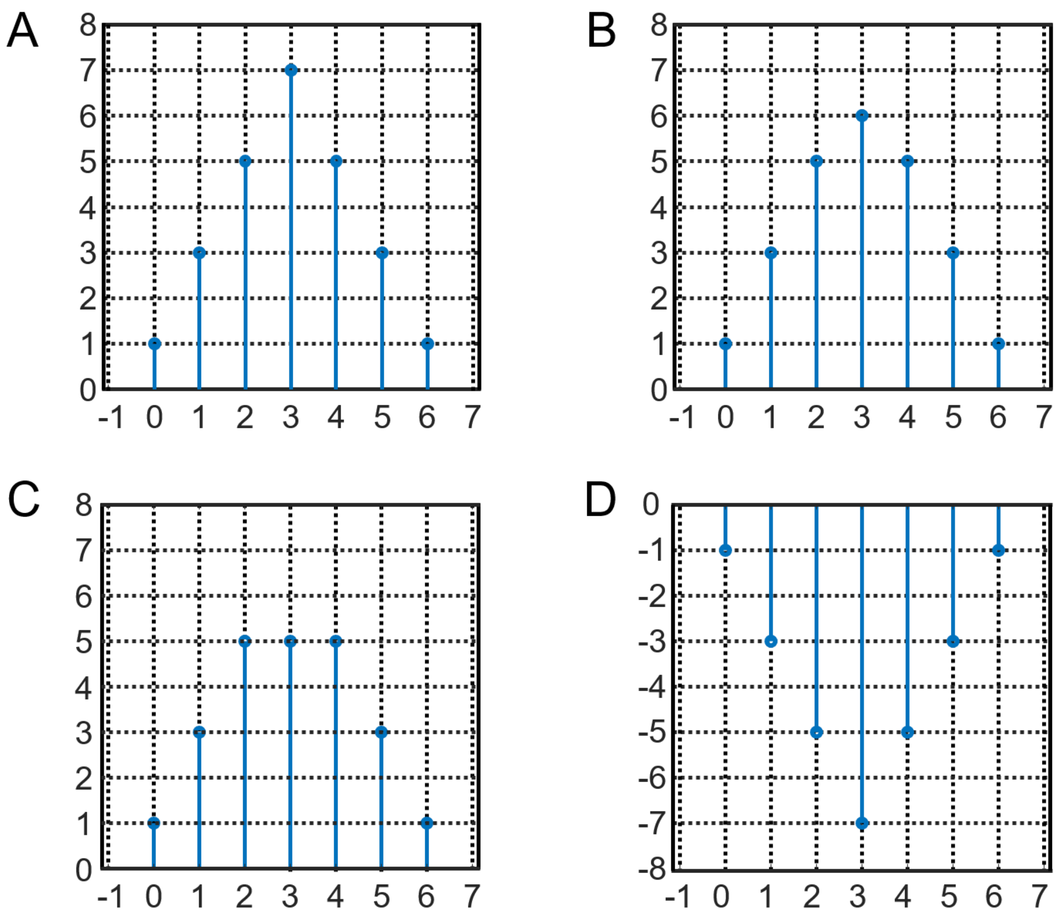
You can either write by hand or on a computer.

A1. Given the signal f(t) in the figure

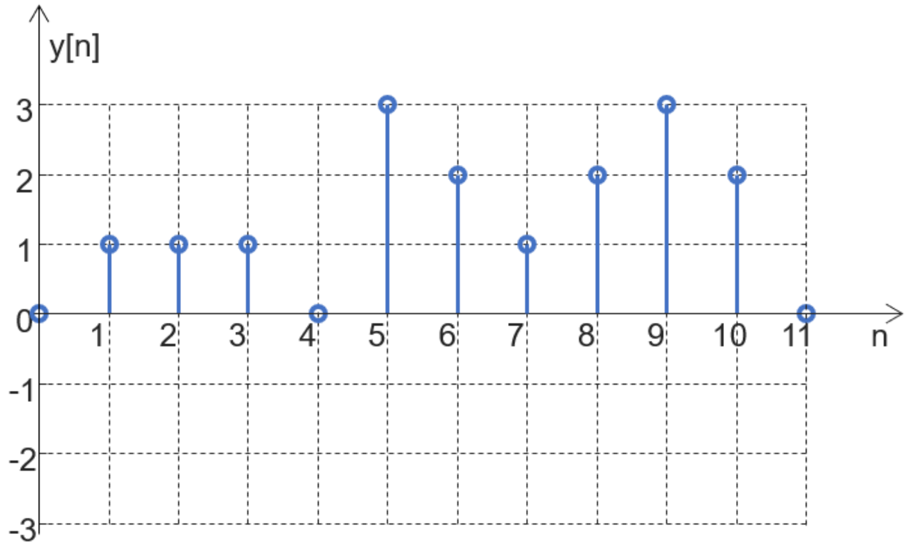


Plot the signal y(t) = 1 - f(t). Motivate your answer.

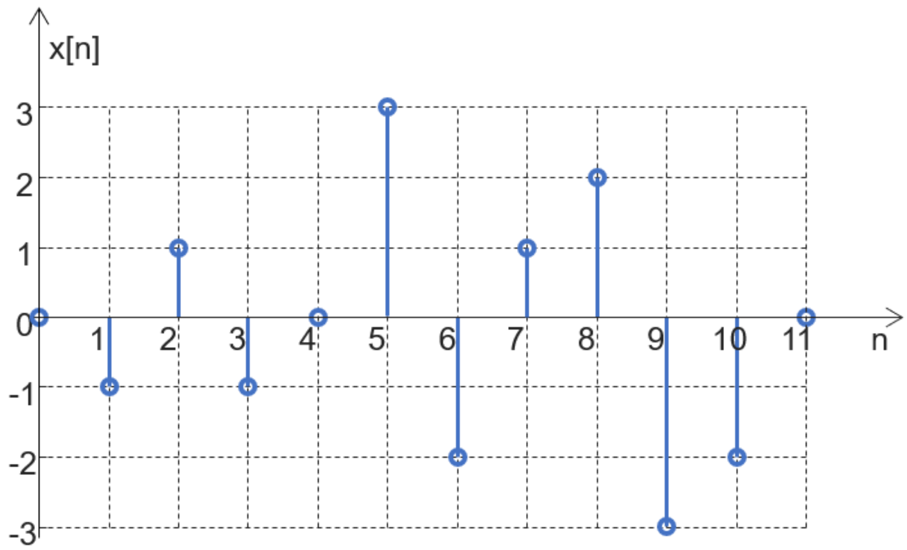
A2. Given the two sequences $x_1[n] = u[n] - u[n-4]$ and $x_2[n] = \delta[n] + 2\,\delta[n-1] + 2\delta[n-2] + \delta[n-3]$, determine the convolution x[n]= $x_1[n]$ * $x_2[n]$. One of the 4 options (A, B, C, D) is correct. Motivate your answer.



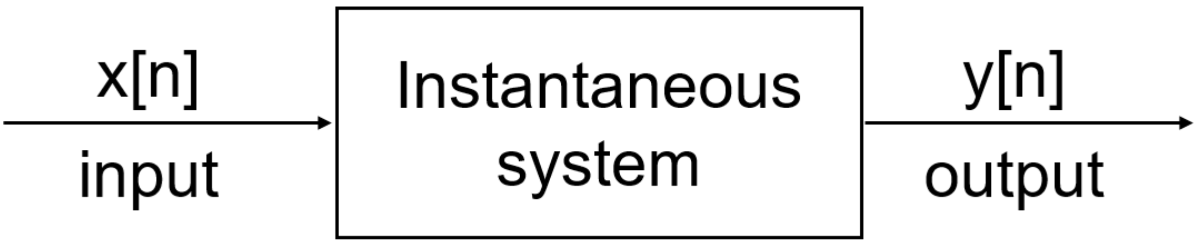
A3. Given an instantaneous system that provides as output the sequence y[n] represented below



when fed with the sequence x[n] as input,



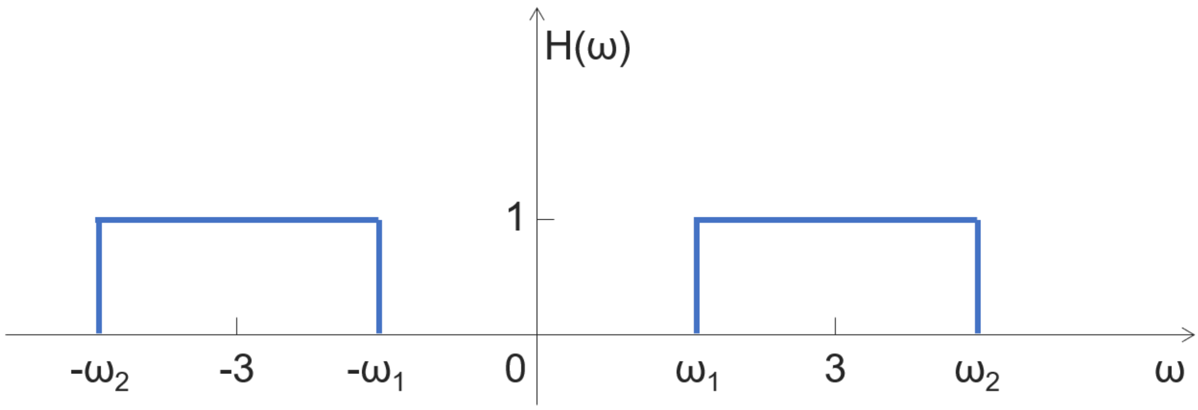
determine the mathematical function that relates the input x[n] to the output y[n] (i.e. y[n] = …). Motivate your answer.



Is the system invertible? Motivate your answer.

A4. Given a LTI system with

- input x(t)=sin(t)+cos(3t)
- and impulse response h(t) with Fourier transform $H(\omega)$ represented in the figure



Determine the values of ω_1 and ω_2 so that the resulting system output y(t) is equal to

$$y(t) = \alpha \cos(3t)$$

with α constant and ≠ 0. Motivate your answer.

A5. Given the periodic signal $x(t) = cos(3t) + 2\sin(9t)$, compute the coefficient c_0 of the complex Fourier series

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}.$$

Motivate your answer.

A6. Determine the inverse Laplace transform $f(t)$ of the function

$$F(s) = \frac{1}{s} - \frac{3}{2}\frac{1}{s+2}$$

Motivate your answer.

A7. Consider the LTI system described by the following differential equation

$$\frac{d^2y(t)}{dt^2} + 10\frac{dy(t)}{dt} + 16\, y(t) = x(t),$$

where x(t) is the system input and y(t) the system output. Determine the system transfer function in the Laplace domain. Motivate your answer.

A8. Given a system with transfer function

$$H(s) = \frac{1}{s^2 + 2s + 5}$$

determine if the system is stable. Motivate your answer.

A9. Compute the following summation using the z-transform $\sum_{n=0}^{+\infty} (0.4)^n$.

A10. Determine the inverse z transform $f[k]$ of the function

$$F[z] = \frac{4}{z-3} + \frac{5}{z-2}$$

Motivate your answer.

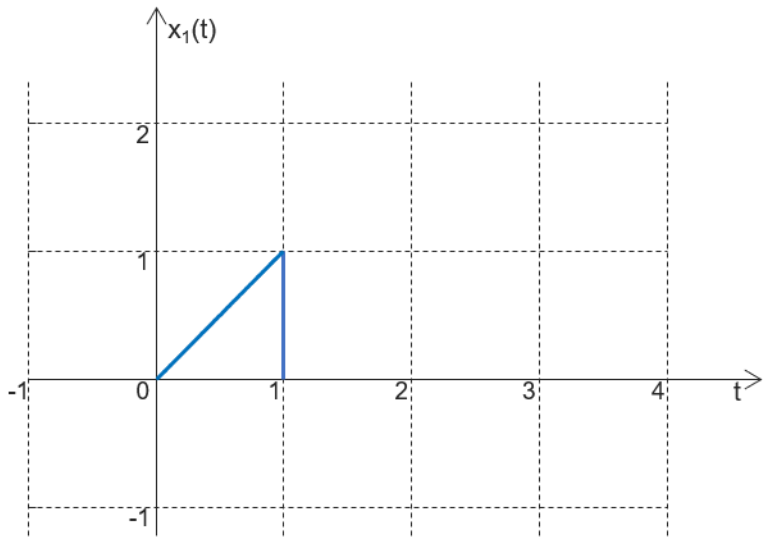
B1. Given the signal $x(t) = 7 + sin(2t) + 5\cos\left(4t + \frac{\pi}{3}\right)$,

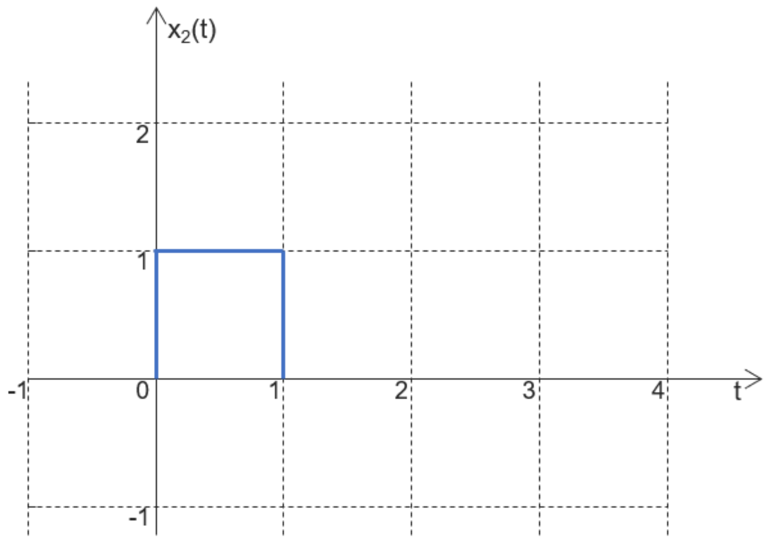
a. Determine the fundamental frequency ω_0

b. Determine the coefficients c_k of the complex Fourier series

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}.$$

B2. a. Compute the convolution between $x_1(t)$ and $x_2(t)$.





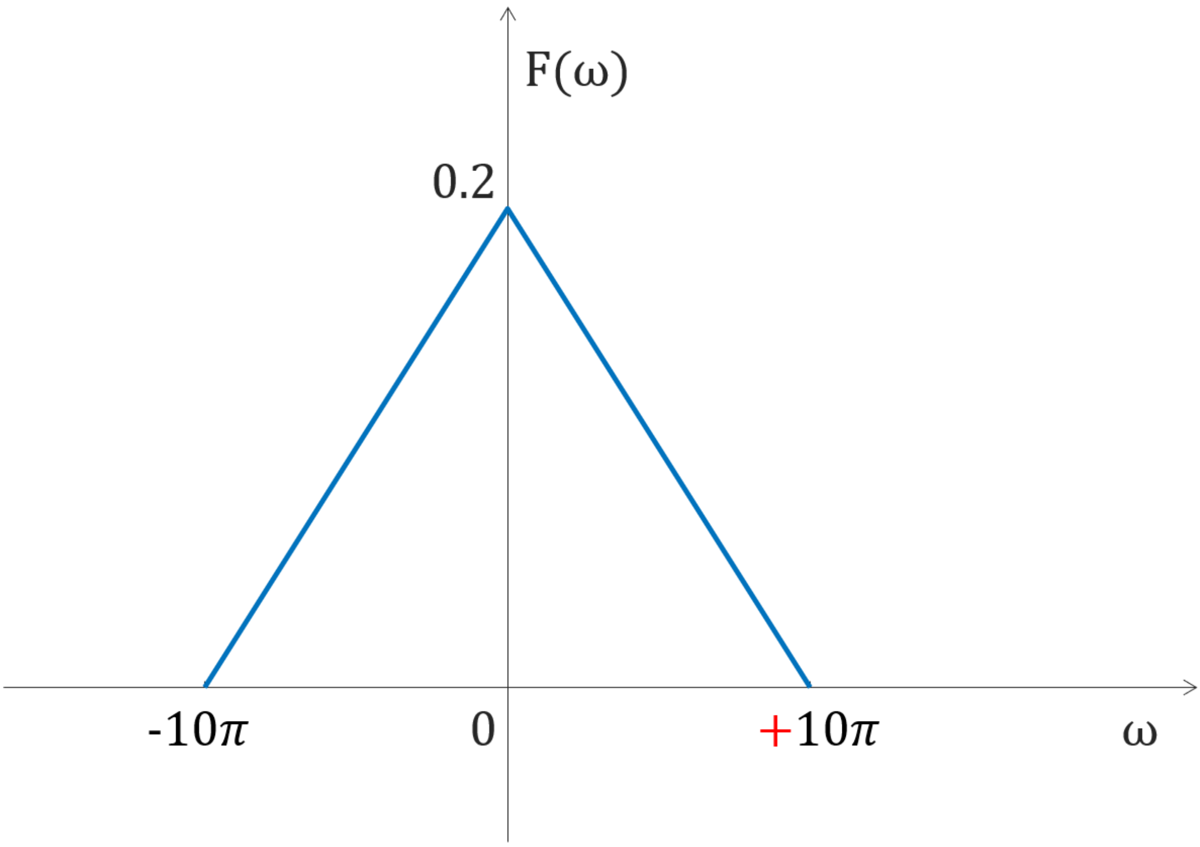
b. Show that the commutative property of the convolution holds true by computing the convolution between $x_2(t)$ and $x_1(t)$.

NOTE: If you use a software to calculate the integrals, you need to indicate which software you used. Even if you solved the integrals with a software, you need to indicate the mathematical expression of the integrals.

I report a random example to clarify what I mean:

- $\int_a^b x(\tau)d\tau = \frac{b^2-a^2}{2}$ will NOT be considered correct, because the mathematical expression of $x(\tau)$ is not specified
- $\int_a^b x(\tau)d\tau = \int_a^b \tau\, d\tau = \left.\frac{\tau^2}{2}\right|_a^b = \frac{b^2-a^2}{2}$ will be considered correct
- In case you use a software to calculate the integrals, $\int_a^b x(\tau)d\tau = \int_a^b \tau\, d\tau = \frac{b^2-a^2}{2}$ will also be considered correct because the expression of $x(\tau)$ has been explicitly reported (provided that you indicate which software you used to calculate the integral).

B3. Given a signal $f(t)$ with Fourier transform $F(\omega)$ sketched in the following figure



with ω expressed in radians / second, plot the Fourier spectrum $\bar{F}(\omega)$ of the signal $\bar{f}(t)$ obtained by sampling the signal $f(t)$ at a rate of $\mathcal{F}_S = \frac{1}{T}$ in the following two cases

a. $T = 0.4\ seconds$
b. $T = 0.05\ seconds$

Explain the procedure you followed.

Based on the two plots (a and b), discuss which of the two sampling intervals T (a or b) is suitable in order to be able to recover $f(t)$ from its samples. Motivate your answer.

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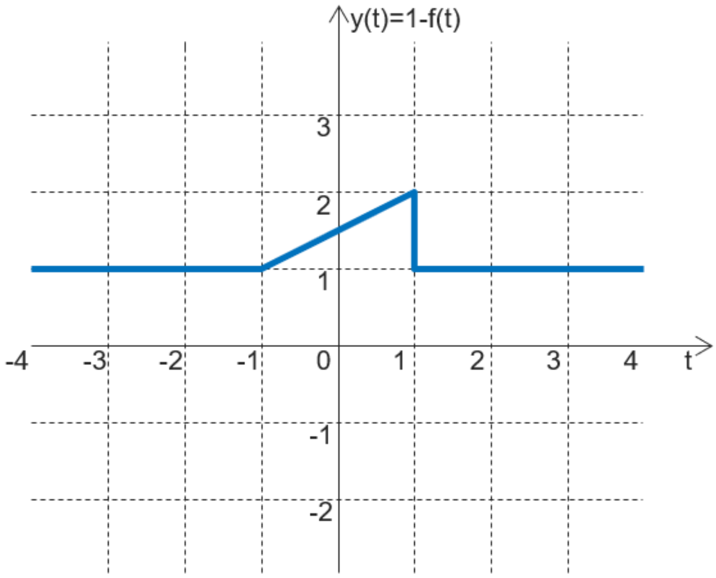
Solution

A1.



- f(t): flip about the t axis

1: DC

A2. B is the correct solution.

D cannot be correct, because the convolution of two positive sequences yields a positive output.

A, B, and C differ for the sample x[3], which is obtained in case of maximal overlap between x1[n] and x2[n] ($\sum_{n=0}^{3} x_1[n]x_2[n] = 6$).

A3. The system is instantaneous. Therefore $y[n]$ depends only on $x[n]$. Comparing input and output, it can be observed that

$$y[n] = \begin{cases} x[n] & if\ x[n] \geq 0 \\ -x[n] & if\ x[n] < 0 \end{cases}$$

Therefore $y[n] = |x[n]|$

The system is not invertible. For instance, given $y[5] = 3$, we cannot conclude if $x[5] = 3$ or $x[5] = -3$.

A4. $Y(\omega) = H(\omega)X(\omega)$

$\sin(t)$ contributes to $X(\omega)$ with terms proportional to δ centered at $\omega = \pm 1$ radians/seconds, and we want to filter them out

$\cos(3t)$ contributes to $X(\omega)$ with terms proportional to δ centered at $\omega = \pm 3$ radians/seconds, and we want to maintain them

Therefore, $0 < \omega_1 < 3$, and $3 < \omega_2 < \infty$

A5. The terms cos and sin are oscillatory and do not contribute to DC. Therefore, $c_0 = 0$.

A6.

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = u(t) - \frac{3}{2}e^{-2t}u(t) = \left(1 - \frac{3}{2}e^{-2t}\right)u(t)$$

A7. Transfer function: ratio of the system output to input in the Laplace domain, assuming zero initial conditions

$$s^2\, Y(s) + 10\, s\, Y(s) + 16\, Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 10\, s + 16}$$

A8. The system has poles in $p = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm j2$, i.e. in the LHP. Therefore, the system is stable.

A9.

$$\sum_{n=0}^{+\infty} 0.4^n = \sum_{n=0}^{+\infty} 0.4^n\, z^{-n}\Big|_{z=1} = \mathcal{Z}\{0.4^n\, u[n]\}\big|_{z=1} = \frac{z}{z-0.4}\bigg|_{z=1} = \frac{1}{0.6} = \frac{10}{6} = \frac{5}{3}$$

A10.

$$f[k] = \mathcal{Z}^{-1}\{F[z]\} = 4\,\mathcal{Z}^{-1}\left\{\frac{1}{z-3}\right\} + 5\,\mathcal{Z}^{-1}\left\{\frac{1}{z-2}\right\} = [4\,(3)^{k-1} + \ 5\,(2)^{k-1}]\, u[k-1]$$

B1. a. Fundamental frequency

$$\omega_1 = 2 \quad \left[\frac{\mathrm{r}}{\mathrm{s}}\right], T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{2} = \pi \quad [\mathrm{s}]$$
$$\omega_2 = 4 \quad \left[\frac{\mathrm{r}}{\mathrm{s}}\right], T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{4} = \frac{\pi}{2} \quad [\mathrm{s}]$$

Common period $T = m_1 T_1 = m_2 T_2$, with $m_1, m_2 \in \mathbb{Z}$

$$T = \pi \ [s], m_1 = 1, m_2 = 2$$
$$\omega_0 = \frac{2\pi}{\pi} = 2 \quad \left[\frac{\mathrm{r}}{\mathrm{s}}\right]$$

b. Coefficients

$$x(t) = 7 + sin(2t) + 5\cos\left(4t + \frac{\pi}{3}\right) = 7 + \frac{e^{j2t} - e^{-j2t}}{2j} + 5\frac{e^{j4t}e^{\frac{j\pi}{3}} + e^{-j4t}e^{-\frac{j\pi}{3}}}{2} =$$

$$= 7e^{jk\omega_0 t}\Big|_{k=0} + \frac{1}{2j}e^{jk\omega_0 t}\Big|_{k=1} - \frac{1}{2j}e^{jk\omega_0 t}\Big|_{k=-1} + \frac{5}{2}e^{\frac{j\pi}{3}}e^{jk\omega_0 t}\Big|_{k=2} + \frac{5}{2}e^{-\frac{j\pi}{3}}e^{jk\omega_0 t}\Big|_{k=-2}$$

$$c_k = \begin{cases} 7 & 0 \\ \pm\frac{1}{2j} & k = \pm 1 \\ \frac{5}{2j}e^{\pm\frac{j\pi}{3}} & k = \pm 2 \\ 0 & otherwise \end{cases}$$

B2. a. $y_a(t) = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\ d\tau$

$t \le 0$

$y_a(t) = 0$

$0 \le t \le 1$

$$y_a(t) = \int_0^t \tau\, d\tau = \frac{1}{2}\tau^2\Big|_0^t = \frac{1}{2}t^2$$

$1 \le t \le 2$

$$y_a(t) = \int_{-1+t}^{1} \tau\, d\tau = \frac{1}{2}\tau^2\Big|_{t-1}^{1} = \frac{1}{2} - \frac{1}{2}(t-1)^2 = t - \frac{1}{2}t^2$$

$t \ge 2$

$y_a(t) = 0$

b.

$$y_b(t) = x_2(t) * x_1(t) = \int_{-\infty}^{+\infty} x_2(\tau)\, x_1(t-\tau)\ d\tau$$

$t \le 0$

$y_b(t) = 0$

$0 \le t \le 1$

$$y_b(t) = \int_0^t (t-\tau)\, d\tau = t\,\tau|_0^t - \frac{1}{2}\tau^2\Big|_0^t = t^2 - \frac{1}{2}t^2 = \frac{1}{2}t^2$$

$1 \le t \le 2$

$$y_b(t) = \int_{-1+t}^{1} (t-\tau)\, d\tau = t\,\tau|_{t-1}^{1} - \frac{1}{2}\tau^2\Big|_{t-1}^{1} = t\,(1-t+1) - \frac{1}{2} + \frac{1}{2}(t-1)^2 = t - \frac{1}{2}t^2$$

$t \ge 2$

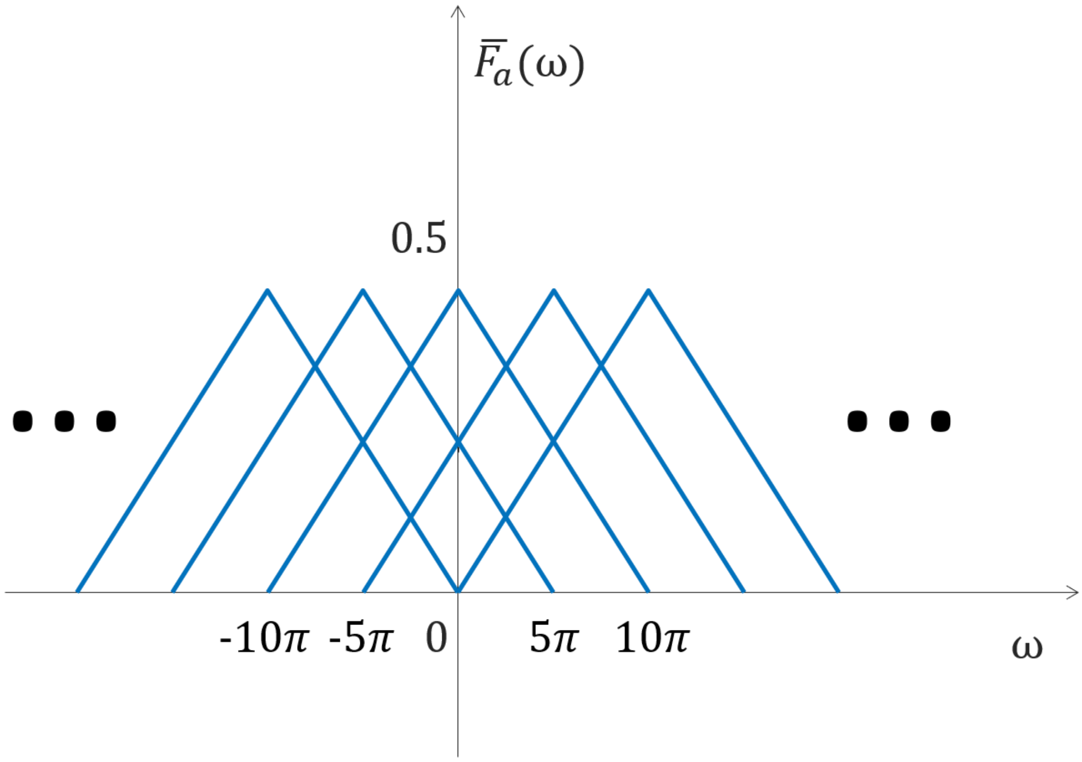
$y_b(t) = 0$

$y_a(t) = y_b(t)$, as expected since the convolution is commutative

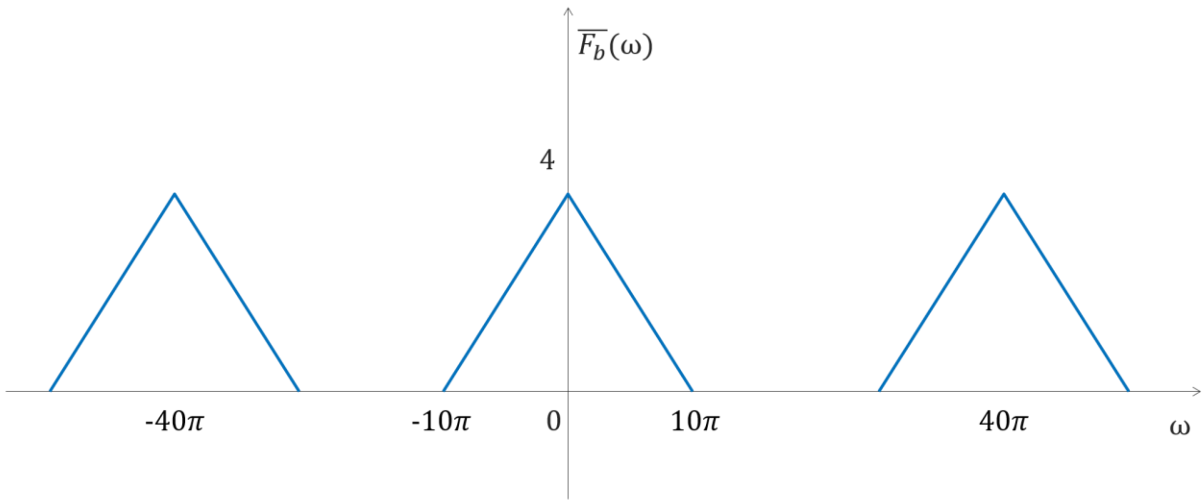
B3.

$$\bar{F}(\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} F(\omega - n\,\omega_s)$$

a. $\omega_{sA} = \frac{2\pi}{T_A} = \frac{2\pi}{0.4} = 5\,\pi \quad [\frac{radian}{second}]$



b. $\omega_{sB} = \frac{2\pi}{T_B} = \frac{2\pi}{0.05} = 40\,\pi \quad [\frac{radian}{second}]$



Note: this figure is not in scale.

Case a: aliasing

Case b: sampling higher than Nyquist rate, we can recover the original signal (even using a practical filter).