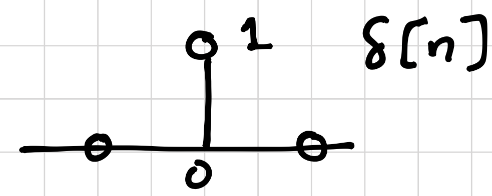
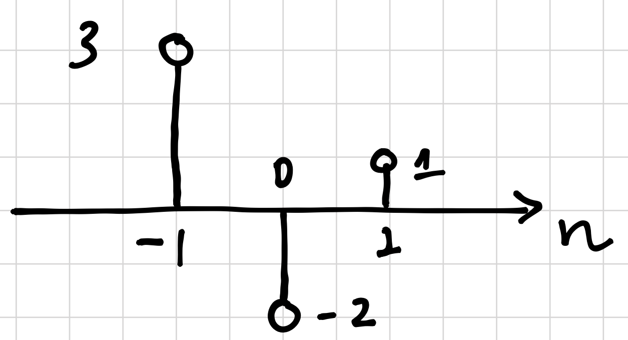
Chapter 2: Linear Time-Invariant (LTI) Systems in Time Domain

The Convolution Sum (DT Convolution)

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{otherwise} \end{cases}$$



Ex



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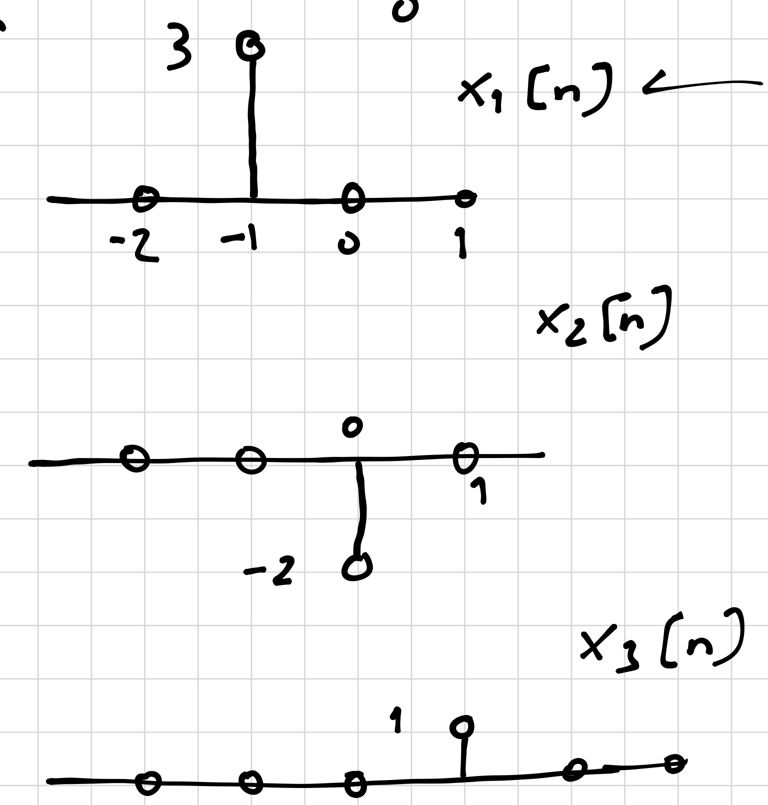


$x_1[n] = x[-1] \cdot \delta[n+1]$

$x_2[n] = x[0]\,\delta[n]$

$x_3(n) = x[1] \cdot \delta[n-1]$

$+$

$x[n] = \ldots + x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1] + \ldots$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

} product of $x[n]$ and time shifted impulse sequence

Let's consider a DT LTI system $\mathcal{H}$

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \mathcal{H}\{x[n]\}$$

$$y[n] = \mathcal{H}\left\{\sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]\right\}$$

Using linearity property /* $x_1 + x_2 \xrightarrow{\mathcal{H}} y_1 + y_2$ */

$$y[n] = \sum_{k=-\infty}^{+\infty} \mathcal{H}\{x[k]\,\delta[n-k]\}$$

(using superposition)

($a_1 x_1 \xrightarrow{\mathcal{H}} a \cdot y_1$)

Homogenity

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\mathcal{H}\{\delta[n-k]\}$$

Using time-invariance

$$h[n-k] \triangleq \mathcal{H}\{\delta[n-k]\} \Rightarrow h[n] = \mathcal{H}\{\delta[n]\}$$

$h[n]$ is called the Impulse Response

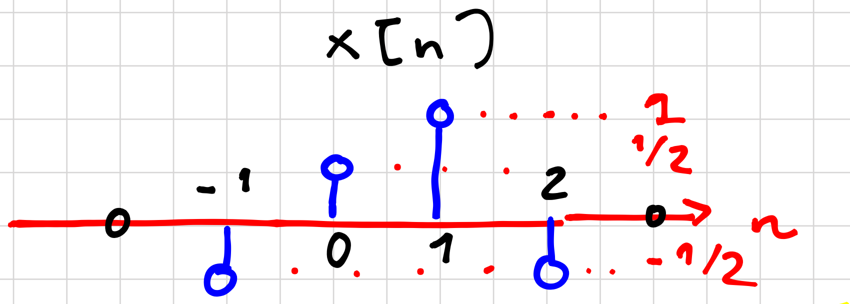
$$\delta[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow h[n]$$

Impulse Response

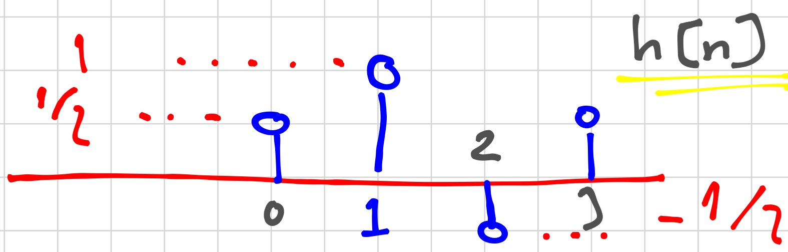
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k] \quad \Big\} \text{ Convolution Sum}$$

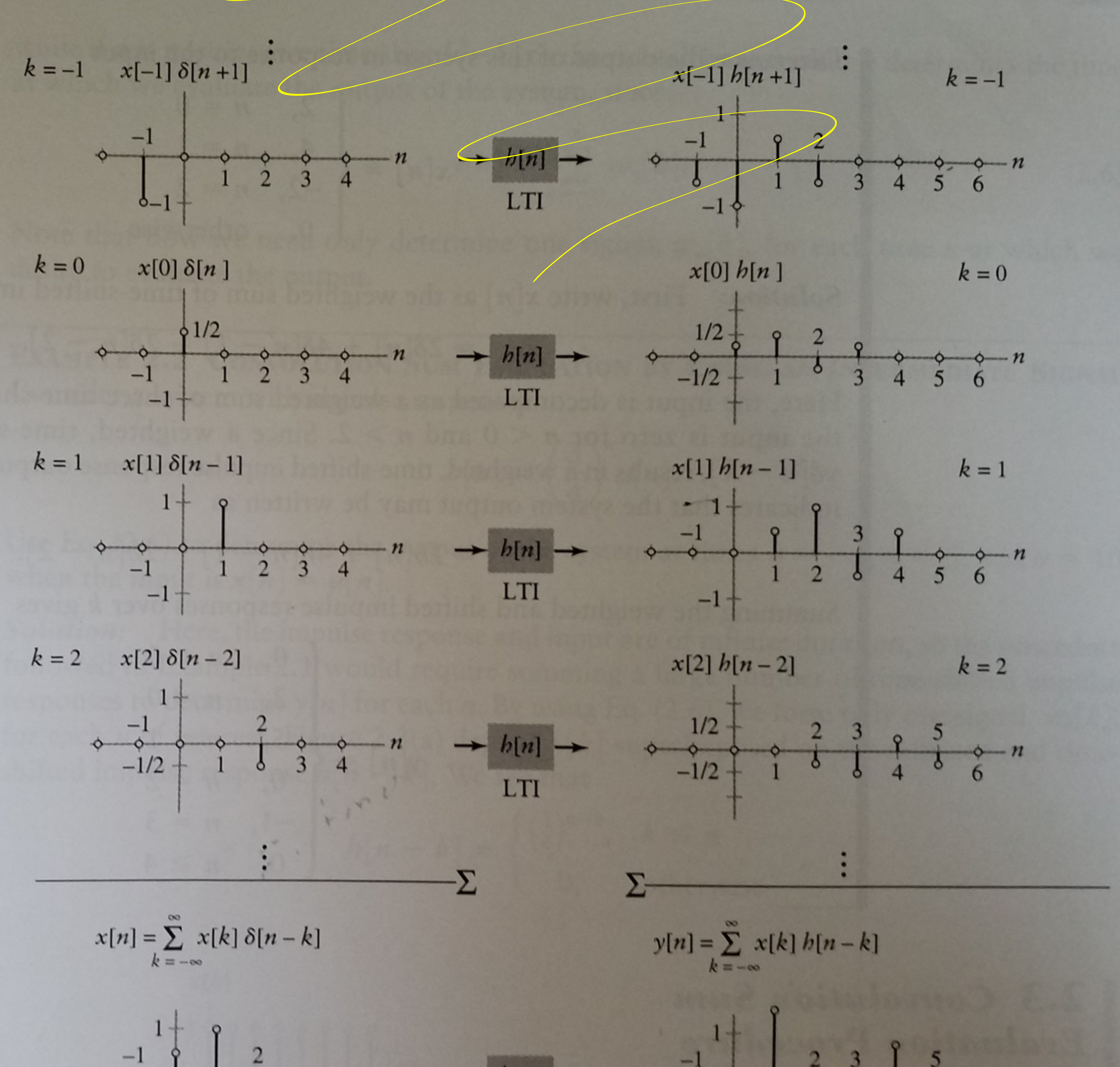
$$y[n] = x[n] * h[n]$$

Ex



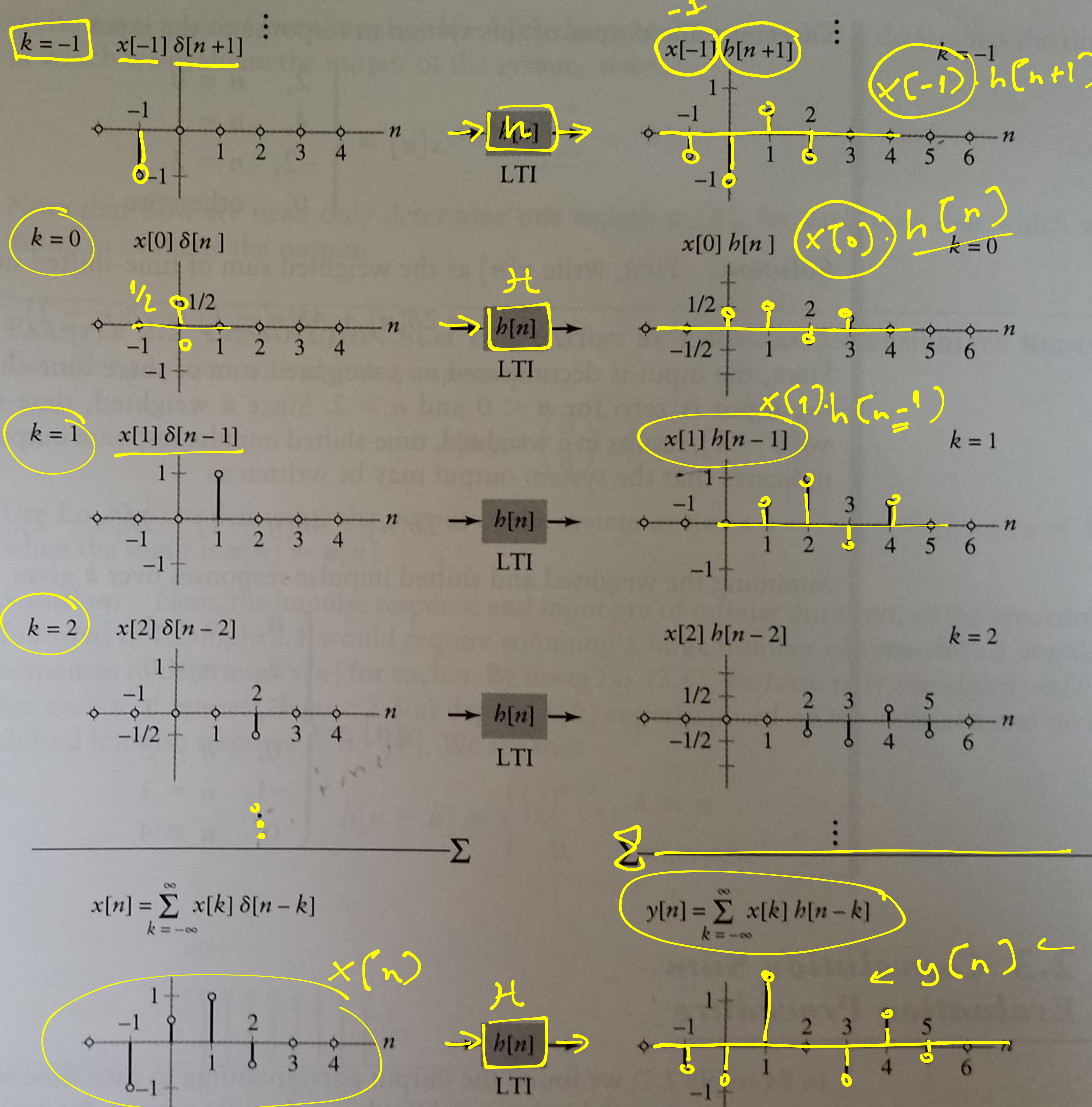
$$x[n] = \begin{cases} -1/2, & x = 2 \\ 1, & x = 1 \\ 1/2, & x = 0 \\ -1/2, & x = -1 \\ 0, & \text{otherwise} \end{cases}$$





$$x(n)=\sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

$$y[n]=\sum_{k=-\infty}^{+\infty} x[k]\cdot h[n-k]$$



Ex

$$y[n] = x[n] + \frac{1}{2}x[n-1]$$

$$x[n] = \begin{cases} 2, & n=0 \\ 4, & n=1 \\ -2, & n=2 \\ 0, & \text{otherwise} \end{cases}$$

Find the output signal when the input signal is $x[n]$.

Sol

$$h[n] = \delta[n] + \frac{1}{2}\delta[n-1]$$

$$\delta[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow h[n]$$

$$x[n] = 2\cdot\delta[n] + 4\cdot\delta[n-1] - 2\,\delta[n-2]$$

$$y[n] = \mathcal{H}\{x[n]\} \qquad \delta[n] \to \boxed{} \to h[n]$$

$$= \mathcal{H}\{2\,\delta[n] + 4\,\delta[n-1] - 2\,\delta[n-2]\}$$

$$= 2 \cdot \mathcal{H}\{\delta[n]\} + 4\,\mathcal{H}\{\delta[n-1]\} - 2\,\mathcal{H}\{\delta[n-2]\}$$

$$= 2 \cdot h[n] + 4 \cdot h[n-1] - 2 \cdot h[n-2]$$

n	-1	0	1	2	3
$h[n]$	0	1	1/2	0	0
$2h[n]$		2	1		
$4h[n-1]$			4	2	
$-2h[n-2]$				-2	-1
+		2	5	0	-1

$$h[n] = \delta[n] + \frac{1}{2}\delta[n-1]$$

$$y[n] = \begin{cases} 2, & n=0 \\ 5, & n=1 \\ -1, & n=3 \\ 0, & \text{otherwise} \end{cases}$$

Convolution Sum Evaluation Procedure

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k]\,h[n-k]}$$

- We define an intermidiate signal

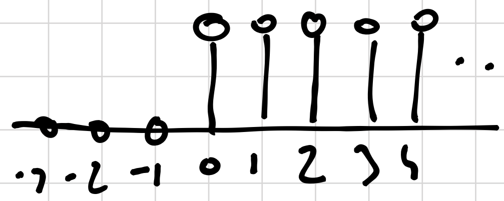
$$w_n[k] = x[k] \cdot h[n-k]$$

k is the independent variable for the intermediate signal

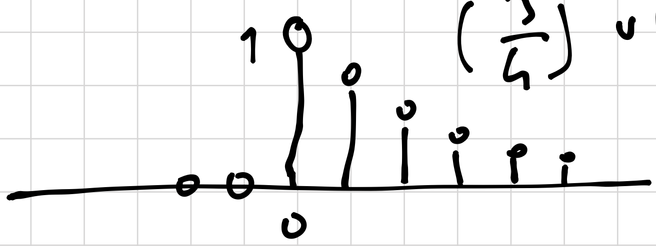
$$y[n] = \sum_{k=-\infty}^{+\infty} w_n[k]$$

Ex

$$\left.\begin{aligned} h[n] &= \left(\tfrac{3}{4}\right)^n \cdot u[n] \\ x[n] &= u[n] \end{aligned}\right\}$$ ~~$y[n]=?$~~



$u[n]$



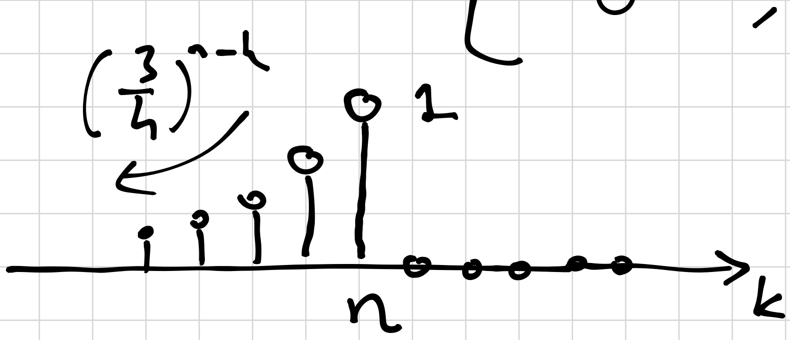
$\left(\tfrac{3}{4}\right)^n u[n]$

Determin $y[-5]$, $y[5]$ and $y[10]$.

$$w_n[k] = x[k] \cdot h[n-k]$$

$$h[n-k] = \left(\tfrac{3}{4}\right)^{n-k} u[n-k]$$

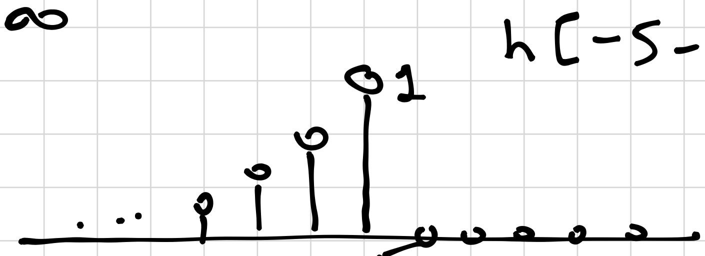
$$= \begin{cases} \left(\tfrac{3}{4}\right)^{n-k} & k \leq n \\ 0, & \text{otherwise} \end{cases}$$



$h[n-k]$

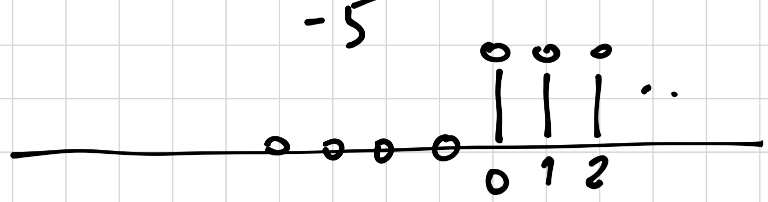
$$y[-5] = \sum_{k=-\infty}^{+\infty} w_5[k]$$

$h[-5-k]$



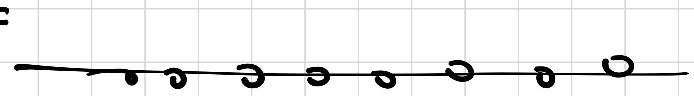
$h[-5-k]$

$x[k]$

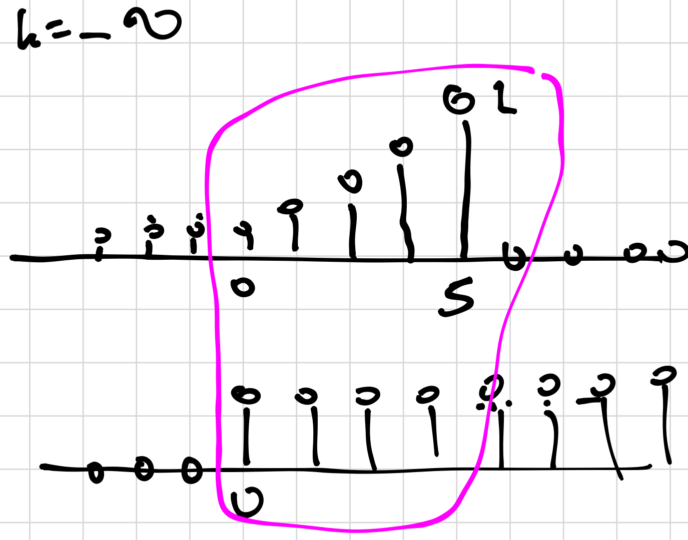


$*$

$w_{-5}[k] =$

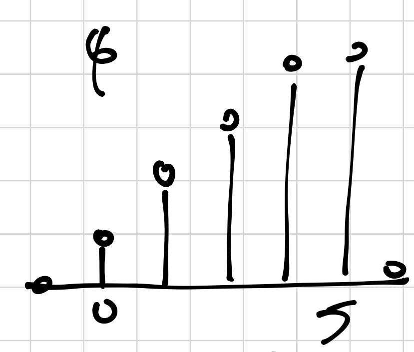


$$y[-5] = \sum_{k=-\infty}^{+\infty} w_{-5}[k] = 0$$



$h[5-k]$

$x[k]$



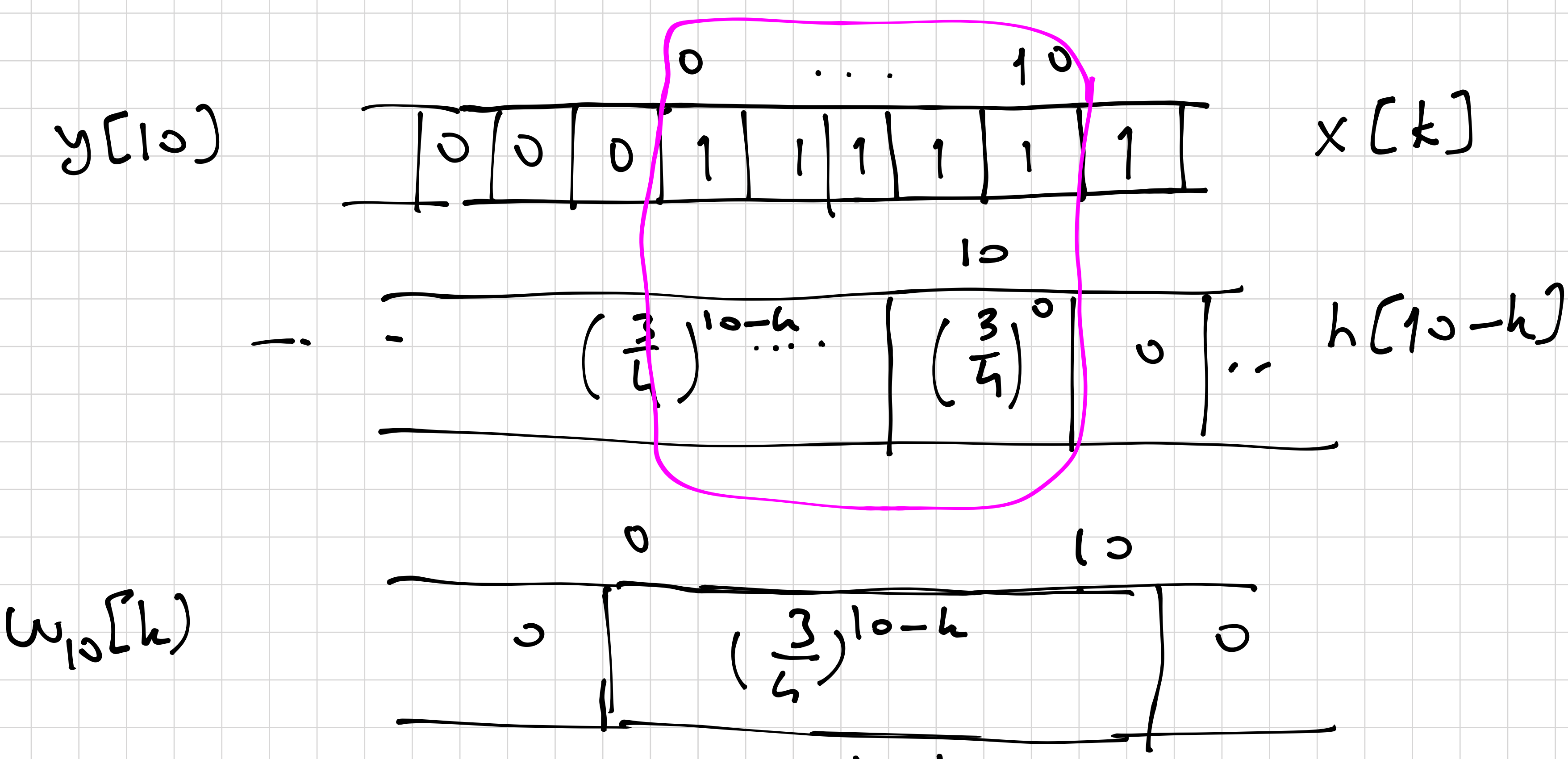
$w_5[k]$

$$y[5] = \sum_{k=-\infty}^{+\infty} w_5[k] = \sum_{k=0}^{5} \left(\frac{3}{4}\right)^{5-k}$$

$$= \left(\frac{3}{4}\right)^5 \sum_{k=0}^{5} \left(\frac{4}{3}\right)^k$$

$$= \left(\frac{3}{4}\right)^5 \frac{1-\left(\frac{4}{3}\right)^6}{1-\left(\frac{4}{3}\right)}$$

$$y[5] = 3.288$$



$$w_{10}[k] = \begin{cases} \left(3/4\right)^{10-k} & , \ 0 \leq k \leq 10 \\ 0 & , \ \text{otherwise} \end{cases}$$

$$y[10] = \sum_{k=-\infty}^{+\infty} w_{10}[k] = \sum_{k=0}^{10} \left(\frac{3}{4}\right)^{10-k}$$

$$y[10] = \quad \ldots \quad = \boxed{3.831}$$

 $h[n-k] = h[-k-(-n)]$

Procedure 2.1: Reflect and Shift Convolution Sum Evaluation

1. Graph both $x[k]$ and $h[n-k]$ as a function of the independent variable k. To determine $h[n-k]$, first reflect $h[k]$ about $k = 0$ to obtain $h[-k]$. Then shift by $-n$.
2. Begin with n large and negative. That is, shift $h[-k]$ to the far left on the time axis.
3. Write the mathematical representation for the intermediate signal $w_n[k]$.
4. Increase the shift n (i.e., move $h[n-k]$ toward the right) until the mathematical representation for $w_n[k]$ changes. The value of n at which the change occurs defines the end of the current interval and the beginning of a new interval.
5. Let n be in the new interval. Repeat steps 3 and 4 until all intervals of time shifts and the corresponding mathematical representations for $w_n[k]$ are identified. This usually implies increasing n to a very large positive number.
6. For each interval of time shifts, sum all the values of the corresponding $w_n[k]$ to obtain $y[n]$ on that interval.

Ex

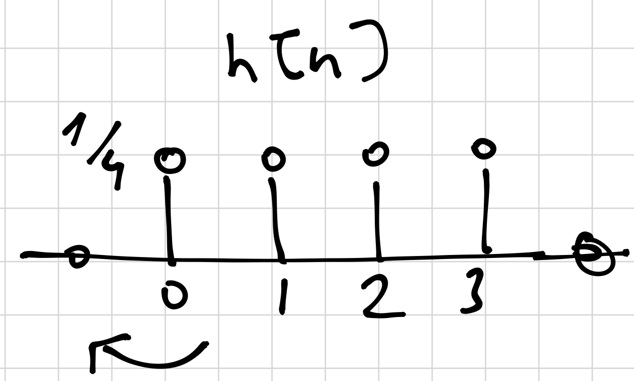
$$y[n] = \frac{1}{4}\sum_{k=0}^{3} x[n-k]$$

$$x[n] = u[n] - u[n-10] \Rightarrow y[n] = ?$$

Sol

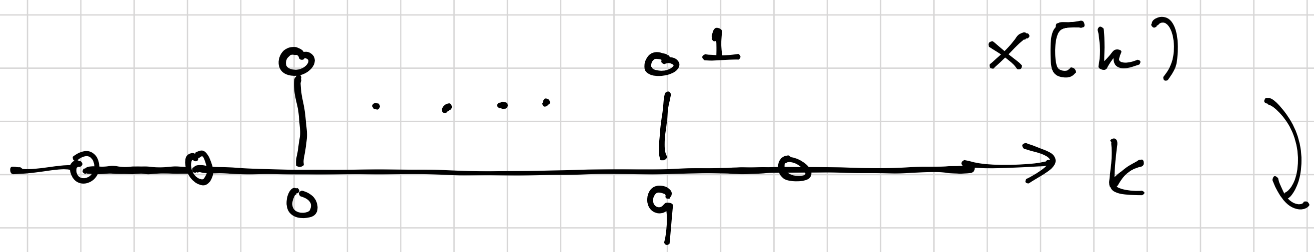
* Find the impulse response

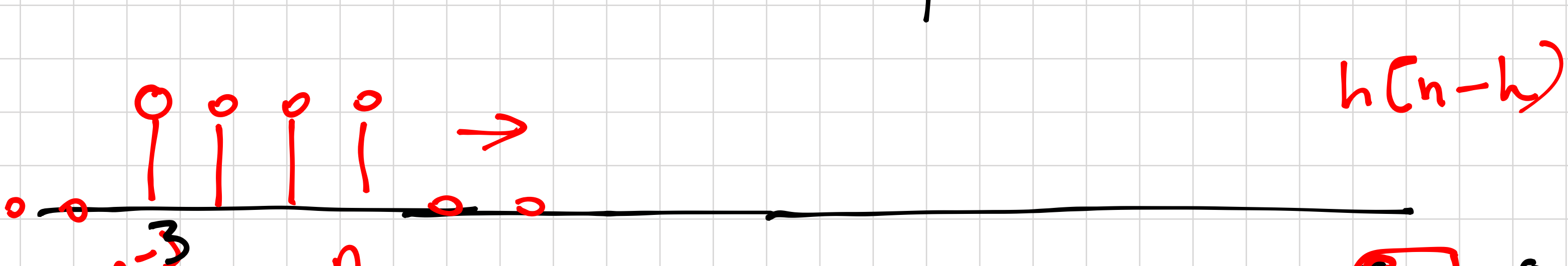
$$\delta[n] \to \boxed{h} \to h[n]$$



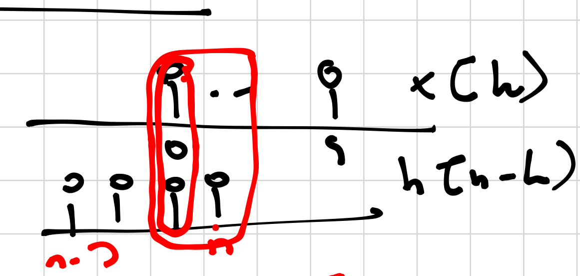
$$h[n] = \frac{1}{4}\sum_{k=0}^{3} \delta[n-k]$$

$$h[n] = \frac{1}{4}\left(u[n] - u[n-4]\right)$$



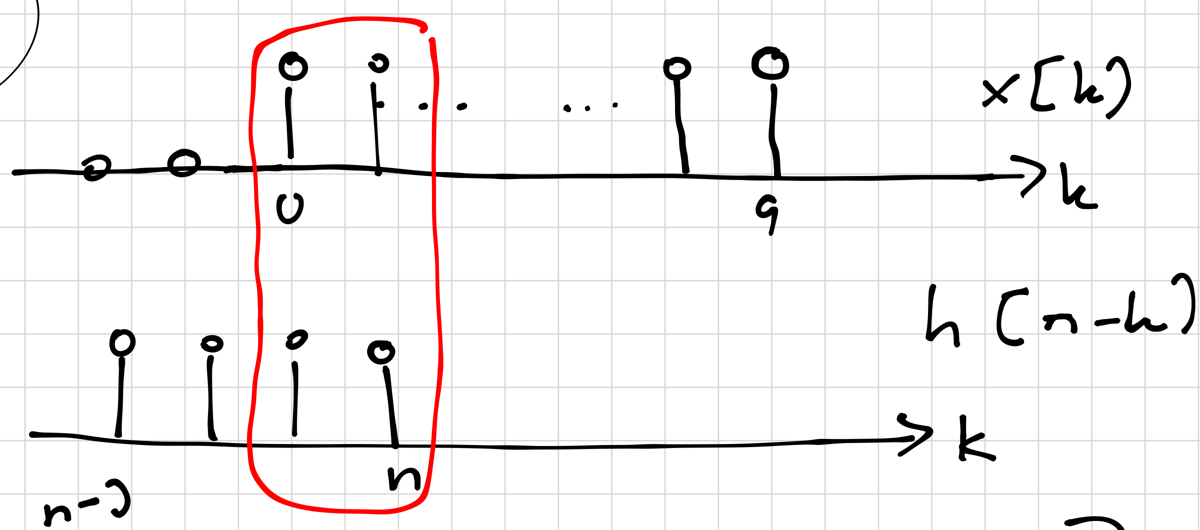


(1) $n < 0 \Rightarrow w_n[k] = 0$



(2) $\left.\begin{array}{l} n-3 < 0 \\ n \geq 0 \end{array}\right\} 0 \leq n < 3$ $\quad w_n[k] = \begin{cases} 1/4, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$

$0 \leq n < 3$

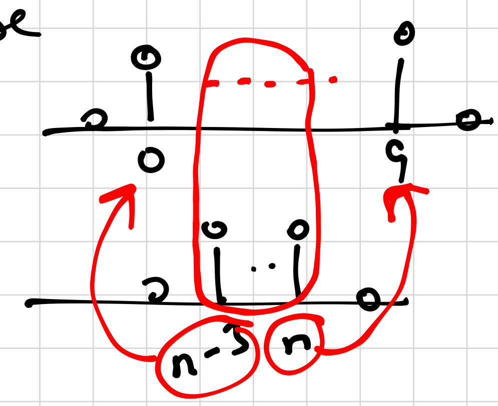


$$w_n[k] = \begin{cases} (1/4), & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

$$y[n] = \sum_{k=-\infty}^{+\infty} w_n[k] = \sum_{k=0}^{n} \frac{1}{4} = \frac{n+1}{4}$$

- $n-3 \geq 0$, $n \geq 3$, $n \leq 9$

$$w_n[k] = \begin{cases} 1/4, & n-3 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$



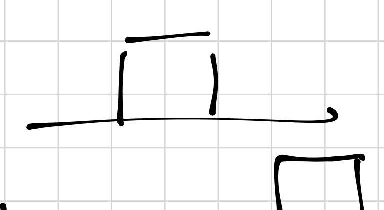
$$y[n] = \sum_{k=n-3}^{n} 1/4 = 1$$

- $n > 9$ & $n-3 \leq 9$

$9 < n \leq 12$

$$w_n[k] = \begin{cases} 1/4 & n-3 \leq k \leq 9 \\ 0, & \text{otherwise} \end{cases}$$

$$y(n) = \sum_{k=n-3}^{9} \frac{1}{4} = \frac{13-n}{4}$$



$$\sum_{k=9}^{b} 1 = b-9+1$$

- $n > 12$ $\quad w_n[k] = 0$

$$y(n) = 0$$

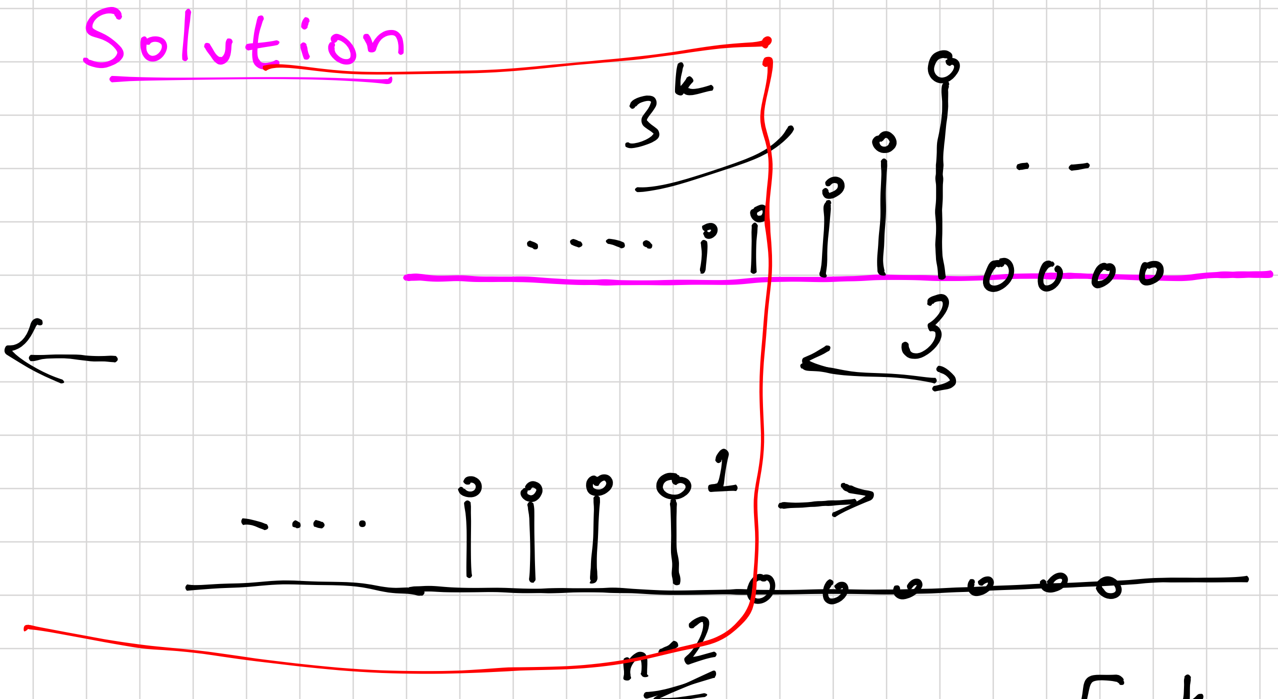
$$y(n) = \begin{cases} \frac{n+1}{4}, & 0 \leq n \leq 3 \\ 1, & 3 < n \leq 9 \\ \frac{13-n}{4}, & 9 < n \leq 12 \\ 0, & \text{otherwise} \end{cases}$$

Example

$$\left.\begin{aligned} x[n] &= 3^n\, u[3-n] \\ h[n] &= u[n-2] \end{aligned}\right\} \quad x[n] * h[n] = ?$$

Solution

$$x[k] = \begin{cases} 3^k, & k \le 3 \\ 0, & k > 3 \end{cases}$$



$$h[n-k] = \begin{cases} 1, & k \le n-2 \\ 0, & k > n-2 \end{cases}$$

(1) $\left.\begin{aligned} & n-2 \le 3 \\ \to\ & n \le 5 \end{aligned}\right\}$ $\quad w_n[k] = \begin{cases} 3^k, & k \le n-2 \\ 0, & \text{otherwise} \end{cases}$

$$y[n] = \sum_{k=-\infty}^{n-2} 3^k = 3^{n-2} \cdot \left(\frac{3}{2}\right) = \frac{1}{6} \cdot 3^n$$

(2) $n > 5 \qquad w_n[k] = \begin{cases} 3^k, & k \le 3 \\ 0, & \text{otherwise} \end{cases}$

$$y[n] = \sum_{k=-\infty}^{3} 3^k = 3^3 \cdot \left(\frac{3}{2}\right) = \frac{81}{2}$$

$$y[n] = \begin{cases} 3^n/6 & n \le 5 \\ 81/2 & n > 5 \end{cases}$$

Convolution Integral (CT convolution)

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

Let $\mathcal{H}$ be an LTI system

$$x(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(t)$$

$$y(t) = \mathcal{H}\{x(t)\}$$

$$= \mathcal{H}\left\{ \int_{-\infty}^{+\infty} x(\tau)\, \delta(t-\tau)\, d\tau \right\}$$

Using Linearity propery

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, \underbrace{\mathcal{H}\{\delta(t-\tau)\}}\, d\tau$$

$$h(t-\tau) \triangleq \mathcal{H}\{\delta(t-\tau)\}$$

Since $\mathcal{H}$ is T.I $\Rightarrow$ $\underline{\underline{h(t)}} = \mathcal{H}\{\delta(t)\}$

The Impulse Response

$$\delta(t) \longrightarrow \boxed{\mathcal{H}} \longrightarrow h(t)$$

$$y(t) = \int_{-\infty}^{+\infty} x(t)\, h(t-\tau)\, d\tau = x(t) * y(t)$$

is called "convolution".

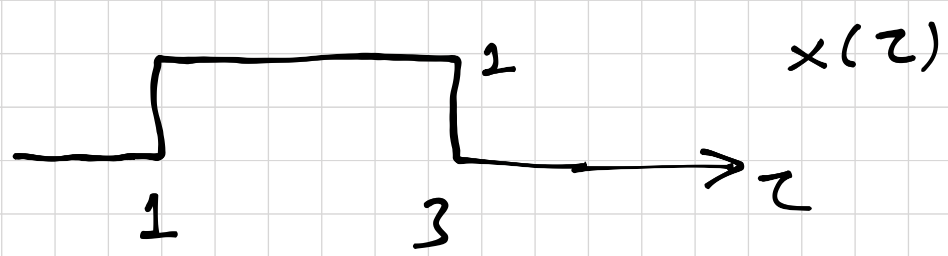
Procedure 2.2: Reflect-and-Shift Convolution Integral Evaluation

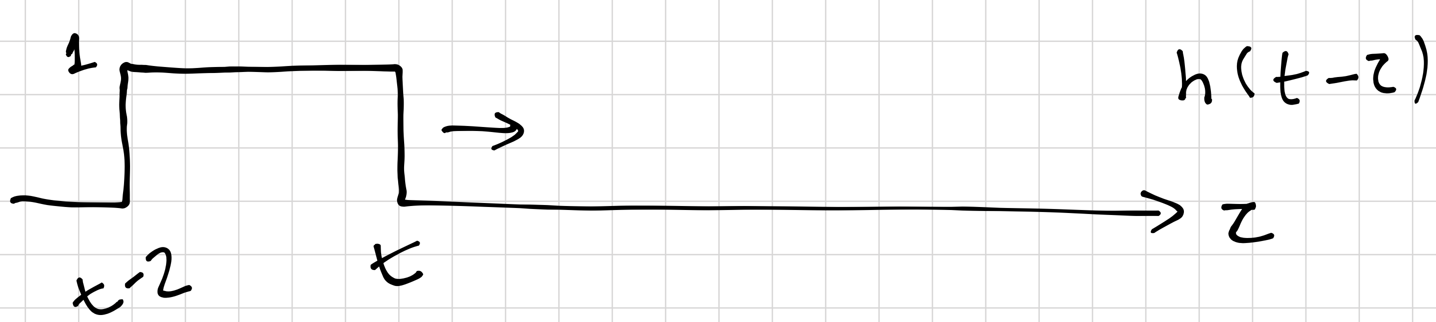
1. Graph $x(\tau)$ and $h(t-\tau)$ as a function of the independent variable τ. To obtain $h(t-\tau)$, reflect $h(\tau)$ about $\tau = 0$ to obtain $h(-\tau)$, and then shift $h(-\tau)$, by $-t$.
2. Begin with the shift t large and negative, that is, shift $h(-\tau)$ to the far left on the time axis.
3. Write the mathematical representation of $w_t(\tau)$. $w_t(\tau)$
4. Increase the shift t by moving $h(t-\tau)$ towards the right until the mathematical representation of $w_t(\tau)$ changes. The value t at which the change occurs defines the end of the current set of shifts and the beginning of a new set.
5. Let t be in the new set. Repeat steps 3 and 4 until all sets of shifts t and the corresponding representations of $w_t(\tau)$ are identified. This usually implies increasing t to a large positive value.
6. For each set of shifts t, integrate $w_t(\tau)$ from $\tau = -\infty$ to $\tau = \infty$ to obtain $y(t)$.

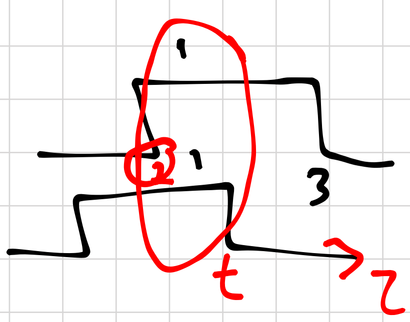
Ex

$$x(t) = u(t-1) - u(t-3)$$
$$h(t) = u(t) - u(t-2)$$
$$y(t) = x(t) * h(t) = ?$$

Sol







① $t < 1$ $\quad w_t(\tau) = 0 \quad y(t) = 0$

② $\left.\begin{matrix} t \geq 1 \\ t-2 \leq 1 \end{matrix}\right\}$ $\quad 1 \leq t < 3 \quad w_t(\tau) = \begin{cases} 1, & 1 \leq \tau \leq t \\ 0, & \text{oth.} \end{cases}$

$$y(t) = \int_1^t 1.\ d\tau = t - 1$$

③ $\left.\begin{matrix} t \geq 3 \\ t-2 < 3 \end{matrix}\right\}$ $\quad 3 \leq t \leq 5 \quad w_t(\tau) = \begin{cases} 1, & t-2 \leq \tau < 3 \\ 0, & \text{oth.} \end{cases}$

$$y(t) = \int_{t-2}^{3} 1 = 5 - t.$$

$$y(t) = \begin{cases} t-1, & 1 \leq t \leq 3 \\ 5-t, & 3 \leq t \leq 5 \\ 0, & \text{oth.} \end{cases}$$

④ $t \geq 5 \quad w_t(\tau) = 0 \Rightarrow y(t) = 0$