Ex X is C.R.V. and its PDF is

$$f_X(x) = \begin{cases} c.(1-x^2) & , \quad -1 < x < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

a) $c = ?$ b) CDF ?

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

$$\int_{-1}^{1} c.\,(1-x^2)\,dx = 1$$

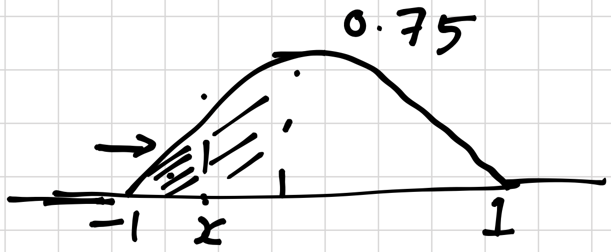
$$c\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = 1$$

$$c\left[\left(1-\frac{1}{3}\right) - \left(-1 - \frac{(-1)^3}{3}\right)\right] = 1$$

$$c\left[\frac{2}{3} - \left(-\frac{2}{3}\right)\right] = 1$$

$$c.\ \frac{4}{3} = 1 \Rightarrow c = \frac{3}{4} \quad \therefore\ c = 0.75$$

b) $$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$



- $x < -1 \qquad F_X(x) = 0$
- $-1 \leq x < 1 \qquad F_X(x) = \underbrace{\int_{-\infty}^{-1} 0.\,dx}_{0} + \int_{-1}^{x} 0.75(1-u^2)\,du$

$$F_X(x) = 0.75\left[u - \frac{u^3}{3}\right]_{-1}^{x}$$

$$= 0.75\left[\left(x - \frac{x^3}{3}\right) - \left(-1 + \frac{1}{3}\right)\right]$$

$$= 0.75\left[x - \frac{x^3}{3} + \frac{2}{3}\right]$$

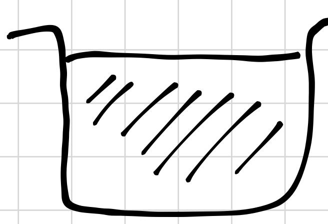
- $x \geq 1 \qquad F_X(x) = 1$

$$F_X(x) = \begin{cases} 0, & x < -1 \\ 0.75x - 0.25x^3 + 0.5, & -1 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

Ex A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of liters is a R.V. with PDF

$$f_X(x) = \begin{cases} 5(1-x)^4 & , \ 0 < x < 1 \\ 0 & , \ \text{otherwise} \end{cases}$$

what must be the capacity of the tank so that the probability of the supply being exhausted in a given week is 0.01.



α : deponun büyüklüğü

X: satılan benzin miktarı (1 ton cinsinden)

$$P(X \geq \alpha) = 0.01$$

$$= \int_{\alpha}^{1} 5(1-x)^4 \, dx = 0.01$$

$$5 \cdot \left(-\frac{1}{5}\right) \left[(1-x)^5\right|_{\alpha}^{1} = 0.01$$

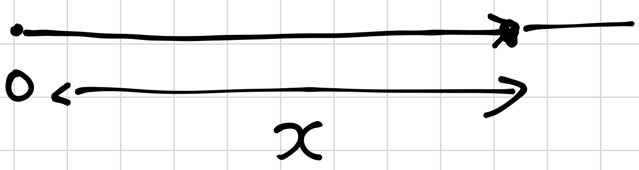
$$-\left[0 - (1-\alpha)^5\right] = 0.01$$

$$1-\alpha = \sqrt[5]{0.01}$$

$$\alpha = 1 - \sqrt[5]{0.01}$$

$$\alpha = 0.6019 \text{ ton} = \underline{\underline{601.9}} \text{ litre}$$

Exponential Distribution



The duration from any starting point until a "success" occurs.

N : a Poisson R.V, the number of successes along x

If the number of success is $\underline{\underline{\lambda}}$ per unit time

$$E(N) = x \cdot \lambda \quad \text{per duration } x$$

The first "success" must happen beyond X

$$\underbrace{P(X > x)}_{1 - F_X(x)} = P(N=0) = \frac{e^{-\lambda x} (\lambda x)^0}{0!} = e^{-\lambda x}$$

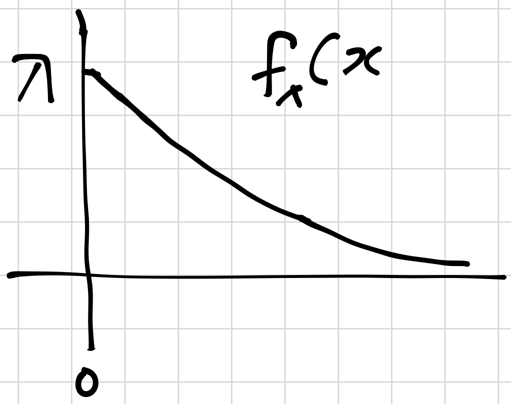
CDF $F_X(x) = 1 - P(X > x) = 1 - e^{-\lambda x} \quad , \; x \geq 0$

PDF $f_X(x) = \dfrac{d\, F_X(x)}{dx} = \begin{cases} \lambda \cdot e^{-\lambda x} , & x \geq 0 \\ 0 , & x < 0 \end{cases}$

Definition

X is an exponential (λ) R.V., if the PDF of X is $[\lambda > 0]$

$$f_X(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & , \ x \geq 0 \\ 0 & , \ x < 0 \end{cases}$$



Mean and Variance of Exponential R.V.

$$E[X^n] = \int_0^\infty \underbrace{x^n}_{u} \cdot \underbrace{\lambda \cdot e^{-\lambda x} \cdot dx}_{dv}$$

$$dv = \lambda \cdot e^{-\lambda x} \cdot dx \qquad u = x^n$$

$$v = -e^{-\lambda x} \qquad du = n \cdot x^{n-1} \cdot dx$$

$$\int_0^\infty u \cdot dv = uv \Big|_0^\infty - \int_0^\infty v \cdot du$$

$$= \underbrace{-x^n \cdot e^{-\lambda x}}_{0-0} \Big|_0^\infty \overset{-}{\cancel{+}} \int_0^\infty -e^{-\lambda x} \cdot n \cdot x^{n-1} dx$$

$$/* \lim_{x \to \infty} x^n \cdot e^{-\lambda x} = 0 \ */$$

$$E(x^n) = \underbrace{\frac{n}{\lambda} \int_0^\infty x^{n-1} \cdot \underbrace{e^{-\lambda x}}_{F_X(x)} \cdot dx}_{E(X^{n-1})}$$

$$E(x^n) = \frac{n}{\lambda} E(x^{n-1})$$

$$\mu_X(x) = E(x^1) = \frac{1}{\lambda} E(x^0) = \frac{1}{\lambda}$$

$$E(X^2) = \frac{2}{\lambda} E(X) = \frac{2}{\lambda^2}$$

$$Var(X) = E(X^2) - \mu_X^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Ex Suppose that the length of a phone call in minutes is an exponential R.V. with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that

a) You have to wait more than 10 minutes

b) between 10 and 20 minutes.

$$F_X(x) = \begin{cases} 0, & x<0 \\ 1-e^{-\lambda x}, & 0 \leq x \end{cases}$$

a) $P(X>10) = 1 - \underbrace{P(X \leq 10)}_{F_X(10)} = 1-(1-e^{-1}) \cong 0.368$

b) $P(10<X<20) = F_X(20)-F_X(10)$
$= e^{-1} - e^{-2} = 0.233$

Memoryless Property

We say that a non-negative R.V., X is memoryless if

$$P(X>s+t \mid X>t) = P(X>s) \quad \forall s,t>0$$

- We can start measuring the outcome at any given point.

- Exponential R.V is a "memoryless" R.V.

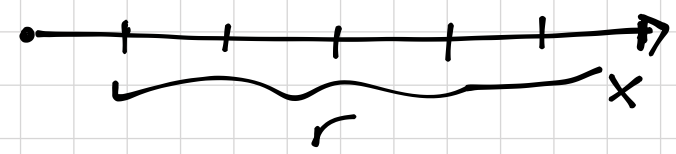
$$P(X>s+t \mid X>t) = \frac{\overbrace{P(X>s+t, X>t)}^{P(X>s+t)}}{P(X>t)}$$

$$P(X>s+t) = P(X>s) \cdot P(X>t)$$

Gamma and Erlang Random Variables

~ The time until $\underline{\underline{r}}$ "successes" occur in a Poisson Process

Ex :



X : time until $\underline{\underline{r}}$ successes occur

N : Poisson , λ successes per unit time

$$E(N) = \lambda x \quad \text{succeses per } x$$

$$P(X > x) = P(N \leq r) = \sum_{k=0}^{r} \frac{e^{-\lambda x} \cdot (\lambda x)^k}{k!}$$

CDF $\quad F_X(x) = P(X \leq x) = 1 - P(X > x)$

$$= 1 - \sum_{k=0}^{r} \frac{e^{-\lambda x} \cdot (\lambda x)^k}{k!}$$

PDF

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{\lambda^r \cdot x^{r-1} \cdot e^{-\lambda x}}{(r-1)!}$$

Definition

X is an Erlang R.V. $(\lambda, \overset{r}{n})$ if the PDF of X

$$f_X(x) = \begin{cases} \dfrac{\lambda^n \cdot x^{n-1} \cdot e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & , x < 0 \end{cases}$$

- If n is an integer we also call X a "Gamma R.V.".
- $n = 1 \Rightarrow X$ exponential R.V.

Gamma Function

$$\boxed{\Gamma(r) = \int_0^{\infty} x^{r-1} \cdot e^{-x} \cdot dx \quad , \; r>0}$$

Using integration by parts

$$u = x^{r-1} \qquad\qquad dv = e^{-x} dx$$

$$du = (r-1) \cdot x^{r-2} dx \qquad\qquad v = -e^{-x}$$

$$\int_0^{\infty} u \cdot dv = uv \Big|_0^{\infty} - \int_0^{\infty} v \cdot du$$

$$\Gamma(r) = \underbrace{-e^{-x} \cdot x^{r-1} \Big|_0^{\infty}}_{0-0} + \int_0^{\infty} e^{-x} \, \underline{(r-1)} \, x^{r-2} dx$$

$$\Gamma(r) = (r-1)\,\Gamma(r-1)$$

If r is an integer

$$\Gamma(r) = (r-1)!$$

~~A~~

Also $\Gamma(1) = 0! = 1$

The PDF of the Gamma R.V. can be written as

$$f_X(x) = \frac{\lambda^n \cdot x^{n-1} \cdot e^{-\lambda x}}{\Gamma(n)} \qquad n \in \mathbb{Z}^+, \quad \lambda, x > 0$$

Mean (Expected Value)

$$E(x) = \frac{1}{\Gamma(n)} \cdot \int_0^{\infty} x \cdot \lambda^n \cdot x^{n-1} \cdot e^{-\lambda x} dx$$

$$= \frac{1}{\Gamma(n)} \cdot \int_0^{\infty} \lambda^n \cdot x^n \cdot e^{-\lambda x} \cdot dx$$

$$y = \lambda x \qquad dy = \lambda dx \qquad dx = dy/\lambda$$

$$= \frac{1}{\Gamma(n)} \int_0^{\infty} y^n \cdot e^{-y} \cdot \frac{dy}{\lambda} = \frac{1}{\Gamma(n)} \cdot \frac{\Gamma(n+1)}{\lambda} = \frac{n}{\lambda}$$

Variance

$$Var(X) = \frac{n}{\lambda^2}$$

Ex

The lifetime (in years) of an electronic component is random variable, X, with a PDF

$$f(x) = \begin{cases} x \cdot e^{-x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

What is the expected lifetime of this component?

$$E(X) = \int_0^\infty x \cdot x \cdot e^{-x} \, dx = \int_0^\infty x^2 \cdot e^{-x} \, dx$$

We know that $\Gamma(r) = \int_0^\infty x^{r-1} \cdot e^{-x} \cdot dx$

$$E(X) = \Gamma(3) = (3-1)! = 2 \text{ years.}$$

4.5.3● Radars detect flying objects by measuring the power reflected from them. The reflected power of an aircraft can be modeled as a random variable Y with PDF

$$f_Y(y) = \begin{cases} \frac{1}{P_0} e^{-y/P_0} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $P_0 > 0$ is some constant. The aircraft is correctly identified by the radar if the reflected power of the aircraft is larger than its average value. What is the probability $P[C]$ that an aircraft is correctly identified?

C: aircraft correctly identified

$P[Y > E(Y)] = ?$

$\lambda = 1/P_0 \rightarrow$ Y is an Exponential R.V.

$E(Y) = P_0$

$$P(Y > P_0) = \int_{P_0}^{\infty} \frac{1}{P_0} \cdot e^{-y/P_0}\, dy$$

$$= -e^{-y/P_0} \Big|_{P_0}^{\infty} = 0 - (-e^{-P_0/P_0})$$

$$= e^{-1}$$

Ex Y is an exponential R.V. with variance $Var(Y) = 25$. a) Find the PDF.

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda x} , & x > 0 \\ 0 , & \text{otherwise} \end{cases}$$

$$Var(Y) = \frac{1}{\lambda^2} = 25 \Rightarrow \lambda = 1/5$$

$$f_Y(y) = \begin{cases} 1/5 \cdot e^{-y/5} , & x > 0 \\ 0 , & \text{otherwise} \end{cases}$$

b) $E(Y^2) = ?$

$E(Y^2) = Var(X) + \mu_X^2$

$E(Y) = \mu_X = \frac{1}{\lambda} = 5$ $\quad E(Y^2) = 25 - 25 = 0$

Ex The C.D.F. of C.R.V, V is

$$F_V(v) = \begin{cases} 0, & v < -5 \\ (v+5)^2/144, & -5 \leq v < 7 \\ 1, & v \geq 7 \end{cases}$$

a) $E(V)$ and $Var(V) = ?$

PDF of V is $f_V(v) = \frac{d}{dv} F_V(v)$

$$f_V(v) = \begin{cases} \frac{v+5}{72} & -5 \leq v < 7 \\ 0 & , \text{otherwise} \end{cases}$$

$$E(V) = \int_{-5}^{7} v \cdot \frac{v+5}{72} \cdot dv = \frac{1}{72} \left[\frac{v^3}{3} + 5 \frac{v^2}{2} \right]_{-5}^{7}$$

$$= \frac{1}{72} \left(\frac{343}{3} + \frac{245}{2} + \frac{125}{3} - 5 \cdot \frac{25}{2} \right)$$

$$= 3$$

The second moment, $E(V^2)$

$$E(V^2) = \frac{1}{72} \int_{-5}^{7} v^2 \cdot (v+5)\, dv = \dots = 15.55$$

$$Var(V) = E(V^2) - \mu_V^2 = 15.55 - 9 = \underline{\underline{6.55}}$$

c) Find the third moment of V

$$E(V^3) = \frac{1}{72} \int_{-5}^{7} v^3 (v+5)\, dv = \dots = \underline{\underline{86.2}}$$

Ex C.R.V, X has a PDF of

$$f_X(x) = \frac{1}{2} f_1(x) + \frac{1}{2} f_2(x) \quad \text{where}$$

$$f_1(x) = \begin{cases} c_1 , & 0 \leq x \leq 2 \\ 0 , & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} c_2 \cdot e^{-x} , & x \geq 0 \\ 0 , & \text{otherwise} \end{cases}$$

$c_1 , c_2 = ?$

1) For $x > 2$

$$f_X(x) = \frac{1}{2} f_2(x) = \frac{c_2}{2} e^{-x} , \quad \underline{c_2 \geq 0}$$

2) For $0 \leq x \leq 2$

$$f_X(x) = \frac{c_1}{2} + \frac{c_2}{2} e^{-x} > 0 \quad 0 \leq x \leq 2$$

→ homework ←

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$