Deriving LP Position Value from Price Feeds

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Abstract

We derive a formula for the relative growth in a constant product AMM's liquidity (the "Liquidity Growth Factor") given a trade that moves the price of token α in terms of token β from P_t to P_{t+1}. The formula relies only on the ratio of prices and the γ value (i.e. one minus the pool fee), with no price modeling assumptions. We also re-derive the spot value of constant product liquidity in terms of the prices of α and β. Finally, we combine these results to calculate the value of an LP position of initial liquidity L_0 at the end of a complete series of prices that the AMM pool holds.

1 Spot Value Derivation

We first compute the value of some amount of liquidity L of our AMM pool where L^2 is our constant product, i.e. $R_\alpha \cdot R_\beta = L^2$. R_α and R_β are respectively the reserves of token α and β. P_α and P_β are respectively the prices of token α and token β denominated in our numeraire of choice. $P_{\alpha,\beta}$ is the price of token α denominated in token β. We assume no arbitrage.

$$P_{\alpha,\beta} = \frac{R_\beta}{R_\alpha} = \frac{P_\alpha}{P_\beta}$$

$$R_\alpha = \frac{L}{\sqrt{P_{\alpha,\beta}}} \qquad R_\beta = L\sqrt{P_{\alpha,\beta}}$$

$$V(R_\alpha, R_\beta) = R_\alpha P_\alpha + R_\beta P_\beta$$

$$V(L) = \frac{L}{\sqrt{P_{\alpha,\beta}}} P_{\alpha,\beta} P_\beta + L\sqrt{P_{\alpha,\beta}} P_\beta$$

$$V(L) = 2LP_\beta\sqrt{P_{\alpha,\beta}} = 2L\sqrt{P_\alpha P_\beta} \tag{1}$$

2 Liquidity Growth Factor Derivation

We now derive a growth factor g_t that represents the factor by which the pool liquidity has grown after a trade at time step t. Let $(1-\gamma)$ be the pool fee charged to the trader. Our goal will be to put the growth factor solely in terms of γ and token prices before and after the trade. We begin with the constant product trade formula for the new reserves after the trade at time step t:

$$(R_\alpha + \gamma\Delta_\alpha)(R_\beta - \Delta_\beta) = L^2$$

Here Δ_α is the amount of token α paid by the trader and Δ_β is the amount of token β received by the trader.

Let P_t and P_{t+1} denote the value of $P_{\alpha,\beta}$ before and after the trade:

$$P_t = \frac{R_\beta}{R_\alpha} \qquad P_{t+1} = \frac{R_\beta - \Delta_\beta}{R_\alpha + \Delta_\alpha}$$

solving for $R_\beta - \Delta_\beta$ and plugging into our original equation we get:

$$P_{t+1}(R_\alpha + \gamma\Delta_\alpha)(R_\alpha + \Delta_\alpha) = L_t^2$$

multiplying through:

$$P_{t+1}\gamma\Delta_\alpha^2 + P_{t+1}R_\alpha(1+\gamma)\Delta_\alpha + P_{t+1}R_\alpha^2 - L_t^2 = 0$$

plugging into quadratic formula solving for Δ_α, and substituting R_α with $\sqrt{\frac{L_t^2}{P_t}}$

$$\begin{aligned}\Delta_\alpha &= \frac{-P_{t+1}(1+\gamma)\sqrt{\frac{L_t^2}{P_t}} \pm \sqrt{P_{t+1}^2(1+\gamma)^2\frac{L_t^2}{P_t} - 4\gamma P_{t+1}L_t^2(\frac{P_{t+1}}{P_t}-1)}}{2\gamma P_{t+1}} \\ &= \frac{-P_{t+1}(1+\gamma)\sqrt{\frac{L_t^2}{P_t}} \pm \sqrt{P_{t+1}^2(1-\gamma)^2\frac{L_t^2}{P_t} + 4\gamma P_{t+1}L_t^2}}{2\gamma P_{t+1}}\end{aligned}$$

and given that all of Δ_α, γ, P_t, and P_{t+1} are positive we can resolve to

$$\Delta_\alpha = \frac{1}{2\gamma}\sqrt{\frac{L_t^2}{P_t}}\left(\sqrt{\gamma(4\frac{P_t}{P_{t+1}} + \gamma - 2) + 1} - (1+\gamma)\right) \tag{2}$$

Now that we have Δ_α, we can solve for our proportional increase in L from t to $t+1$ (our "Liquidity Growth Factor") resulting from the trade:

$$g_t = \frac{L_{t+1}}{L_t} = \sqrt{\frac{(R_\alpha + \Delta_\alpha)(R_\beta - \Delta_\beta)}{(R_\alpha + \gamma\Delta_\alpha)(R_\beta - \Delta_\beta)}} = \sqrt{\frac{(R_\alpha + \Delta_\alpha)}{(R_\alpha + \gamma\Delta_\alpha)}}$$

plugging in for R_α and Δ_α, we can pull out and cancel $\sqrt{\frac{L_t^2}{P_t}}$ terms, simplifying to:

$$g_t = \sqrt{\frac{1}{\gamma}\cdot\frac{\sqrt{\gamma(4\frac{P_t}{P_{t+1}} + \gamma - 2) + 1} - (1-\gamma)}{\sqrt{\gamma(4\frac{P_t}{P_{t+1}} + \gamma - 2) + 1} + (1-\gamma)}}$$

We have arrived at an equation for g_t that only relies on γ and the price ratio $\frac{P_t}{P_{t+1}}$ for a trade from token α to token β. Next, we'd like to generalize this to trades in either direction. To do this, we can swap which token we define as α and β, which will result in replacing our prices P_t and P_{t+1} with their reciprocals. If $P_t > P_{t+1}$, we are trading from α to β, and otherwise from β to α.

With this information, we can replace the price ratio in our equation with a general ϕ_t defined as:

$$\phi_t = \max\left(\frac{P_t}{P_{t+1}}, \frac{P_{t+1}}{P_t}\right)$$

Resulting in our final equation:

$$g_t = \sqrt{\frac{1}{\gamma}\cdot\frac{\sqrt{\gamma(4\phi_t + \gamma - 2) + 1} - (1-\gamma)}{\sqrt{\gamma(4\phi_t + \gamma - 2) + 1} + (1-\gamma)}} \tag{3}$$

3 End Valuation of an LP Position

For simplicity, we will consider LP shares of an ETH/USDC pool, denominating in USDC. Assume we open an LP position of L_0 liquidity at time t_0, and have at our disposal the spot price feed of the AMM over the next T time steps. We assume all price transitions over this time can be seen in the price feed, i.e. no trades occur between time steps, as otherwise we will not be able to determine an exact result for L_T.

Our end liquidity after growth can be expressed as:

$$L_T = L_0 \prod_{t=0}^{T-1} g_t$$

Multiplying by the current value of liquidity from result (1), treating β as our numeraire USDC, gives us our final result:

$$V_T = 2L_T\sqrt{P_T} \tag{4}$$