

Exercise 6*Name: Saurabh Gupta & Dushko Klincharov***1 Theory Question : Product of Gaussians**

$$\begin{aligned}
LHS &= \int Norm_x[a, A] Norm_x[b, B] dx \\
&= \int \frac{1}{(2\pi)^{k/2} |A|^{1/2}} \exp\left(\frac{-1}{2} (x-a)^T A^{-1} (x-a)\right) \\
&\quad \frac{1}{(2\pi)^{k/2} |B|^{1/2}} \exp\left(\frac{-1}{2} (x-b)^T B^{-1} (x-b)\right) dx \\
&= \frac{1}{(2\pi)^k (|A||B|)^{1/2}} \int \exp\left(\frac{-1}{2} (x-a)^T A^{-1} (x-a)\right) \exp\left(\frac{-1}{2} (x-b)^T B^{-1} (x-b)\right) dx \\
&= \frac{1}{(2\pi)^k (|A||B|)^{1/2}} \int \exp\left(\frac{-1}{2} \left[x^T A^{-1} x - 2x^T A^{-1} a + a^T A^{-1} a + x^T B^{-1} x - 2x^T B^{-1} b + b^T B^{-1} b\right]\right) dx \\
&= \frac{1}{(2\pi)^k (|A||B|)^{1/2}} \int \exp\left(\frac{-1}{2} \left[x^T (A^{-1} + B^{-1}) x - 2x^T (A^{-1} a + B^{-1} b) + a^T A^{-1} a + b^T B^{-1} b\right]\right) dx \\
&= \frac{\exp\left(\frac{-1}{2} (a^T A^{-1} a + b^T B^{-1} b)\right)}{(2\pi)^k (|A||B|)^{1/2}} \int \exp\left(\frac{-1}{2} \left[x^T (A^{-1} + B^{-1}) x - 2x^T (A^{-1} a + B^{-1} b)\right]\right) dx
\end{aligned}$$

Now, Consider the integral term above

$$\begin{aligned}
Int &= \int \exp\left(\frac{-1}{2} \left[x^T (A^{-1} + B^{-1}) x - 2x^T (A^{-1} a + B^{-1} b)\right]\right) dx \\
&= \int \exp\left(\frac{-1}{2} \left[x^T \Sigma_*^{-1} x - 2x^T \Sigma_*^{-1} \Sigma_* (A^{-1} a + B^{-1} b)\right]\right) \\
&\quad \exp\left(\frac{-1}{2} \left[(\Sigma_* (A^{-1} a + B^{-1} b))^T \Sigma_*^{-1} (\Sigma_* (A^{-1} a + B^{-1} b))\right]\right) \\
&\quad \exp\left(\frac{1}{2} \left[(\Sigma_* (A^{-1} a + B^{-1} b))^T \Sigma_*^{-1} (\Sigma_* (A^{-1} a + B^{-1} b))\right]\right) dx \\
&= \exp\left(\frac{1}{2} \left[(\Sigma_* (A^{-1} a + B^{-1} b))^T \Sigma_*^{-1} (\Sigma_* (A^{-1} a + B^{-1} b))\right]\right) \\
&\quad \int \exp\left(\frac{-1}{2} \left[x^T \Sigma_*^{-1} x - 2x^T \Sigma_*^{-1} \Sigma_* (A^{-1} a + B^{-1} b) + (\Sigma_* (A^{-1} a + B^{-1} b))^T \Sigma_*^{-1} (\Sigma_* (A^{-1} a + B^{-1} b))\right]\right) dx \\
&= \exp\left(\frac{1}{2} \left[(\Sigma_* (A^{-1} a + B^{-1} b))^T \Sigma_*^{-1} (\Sigma_* (A^{-1} a + B^{-1} b))\right]\right) \\
&\quad (2\pi)^{k/2} |\Sigma_*|^{1/2} \int Norm_x[\Sigma_* (A^{-1} a + B^{-1} b), \Sigma_*] dx
\end{aligned}$$

Therefore, we have proved the integral part on the RHS of the required proof as follows:

$$LHS = \left[\frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} \exp\left(\frac{1}{2} \left[(\Sigma_*(A^{-1}a + B^{-1}b))^T \Sigma_*^{-1} (\Sigma_*(A^{-1}a + B^{-1}b)) \right] \right) \right] \\ \exp\left(\frac{-1}{2} (a^T A^{-1}a + b^T B^{-1}b)\right) \int Norm_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*] dx$$

Now let us consider the residual term in the above equation which we need to simplify further

$$Res = \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} \exp\left(\frac{1}{2} \left[(\Sigma_*(A^{-1}a + B^{-1}b))^T \Sigma_*^{-1} (\Sigma_*(A^{-1}a + B^{-1}b)) \right] \right) \\ = \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} \exp\left(\frac{1}{2} \left[(A^{-1}a + B^{-1}b)^T \Sigma_*(A^{-1}a + B^{-1}b) - (a^T A^{-1}a) - (b^T B^{-1}b) \right] \right)$$

Using the $\int Norm_x[a, A] Norm_x[b, B] dx (A^{-1} + B^{-1})^{-1} = (A - A(A+B)^{-1}A) = (B - B(A+B)^{-1}B)$, and also the **symmetric property of covariance matrices**:

$$Res = \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} \exp\left(\frac{1}{2} \left[(a^T A^{-1} \Sigma_* A^{-1} a) + (b^T B^{-1} \Sigma_* B^{-1} b) + (a^T A^{-1} \Sigma_* B^{-1} b) + (b^T B^{-1} \Sigma_* A^{-1} a) - (a^T A^{-1} a + b^T B^{-1} b) \right] \right) \\ = \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} \exp\left(\frac{1}{2} \left[(a^T A^{-1} \Sigma_* B^{-1} b) + (b^T B^{-1} \Sigma_* A^{-1} a) \right] \right) \\ \exp\left(\frac{1}{2} \left[(a^T A^{-1} (A - A(A+B)^{-1}A) A^{-1} a) + (b^T B^{-1} (B - B(A+B)^{-1}B) B^{-1} b) - (a^T A^{-1} a + b^T B^{-1} b) \right] \right) \\ = \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} \exp\left(\frac{1}{2} \left[(a^T A^{-1} \Sigma_* B^{-1} b) + (b^T B^{-1} \Sigma_* A^{-1} a) \right] \right) \\ \exp\left(\frac{1}{2} \left[(a^T A^{-1} a) - (a^T (A+B)^{-1} a) + (b^T B^{-1} b) - (b^T (A+B)^{-1} b) - (a^T A^{-1} a + b^T B^{-1} b) \right] \right) \\ = \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} \exp\left(\frac{-1}{2} \left[(a^T (A+B)^{-1} a) + (b^T (A+B)^{-1} b) \right] \right) \exp\left(\frac{1}{2} \left[2(a^T A^{-1} \Sigma_* B^{-1} b) \right] \right) \\ = \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} \exp\left(\frac{-1}{2} \left[(a^T (A+B)^{-1} a) + (b^T (A+B)^{-1} b) \right] \right) \exp\left(\frac{1}{2} \left[2(a^T A^{-1} (B - B(A+B)^{-1}B) B^{-1} b) \right] \right) \\ = \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} \exp\left(\frac{-1}{2} \left[(a^T (A+B)^{-1} a) + (b^T (A+B)^{-1} b) \right] \right) \exp\left(\frac{1}{2} \left[2a^T (A^{-1} - A^{-1}B(A+B)^{-1}B) b \right] \right)$$

Using the **Woodbury matrix identity**: $(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$ and substituting $U = B$, $C = V = I$ we get $(A+B)^{-1} = A^{-1} - A^{-1}B(A+B)^{-1}$

$$Res = \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} \exp\left(\frac{-1}{2} \left[(a^T (A+B)^{-1} a) + (b^T (A+B)^{-1} b) - 2(a^T (A+B)^{-1} b) \right] \right)$$

Using the "**product of matrix determinants is equal to determinant of matrix products**" and "**determinant of matrix inverse is equal to the inverse of the determinant of the matrix**" identities along with the Woodbury matrix identity used above, we can write $\frac{|\Sigma_*|^{1/2}}{(|A||B|)^{1/2}}$ as $\frac{1}{|A+B|^{1/2}}$. Hence we have the final result of the proof as follows:

$$Res = \frac{1}{(2\pi)^{k/2}(|A+B|)^{1/2}} \exp\left(\frac{-1}{2} \left[(a-b)^T (A+B)^{-1} (a-b) \right] \right)$$

$$\therefore \int Norm_x[a, A] Norm_x[b, B] dx = Norm_a[b, A+B] \int Norm_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*] dx$$