MA-INF:2201: Statistics and Adjustment Theory WS 2021/2022

Exercise 6

Name: Saurabh Gupta & Dushko Klincharov

1 Theory Question : Product of Gaussians

$$
\begin{aligned}
LHS =& \int Norm_x[a, A] Norm_x[b, B] dx \\
=& \int \frac{1}{(2\pi)^{k/2}|A|^{1/2}} \exp\left(\frac{-1}{2}(x-a)^T A^{-1}(x-a)\right) \\
& \frac{1}{(2\pi)^{k/2}|B|^{1/2}} \exp\left(\frac{-1}{2}(x-b)^T B^{-1}(x-b)\right) dx \\
=& \frac{1}{(2\pi)^k(|A||B|)^{1/2}} \int \exp\left(\frac{-1}{2}(x-a)^T A^{-1}(x-a)\right) \exp\left(\frac{-1}{2}(x-b)^T B^{-1}(x-b)\right) dx \\
=& \frac{1}{(2\pi)^k(|A||B|)^{1/2}} \int \exp\left(\frac{-1}{2}\left[x^T A^{-1}x - 2x^T A^{-1}a + a^T A^{-1}a + x^T B^{-1}x - 2x^T B^{-1}b + b^T B^{-1}b\right]\right) dx \\
=& \frac{1}{(2\pi)^k(|A||B|)^{1/2}} \int \exp\left(\frac{-1}{2}\left[x^T(A^{-1}+B^{-1})x - 2x^T(A^{-1}a + B^{-1}b) + a^T A^{-1}a + b^T B^{-1}b\right]\right) dx \\
=& \frac{\exp\left(\frac{-1}{2}(a^T A^{-1}a + b^T B^{-1}b)\right)}{(2\pi)^k(|A||B|)^{1/2}} \int \exp\left(\frac{-1}{2}\left[x^T(A^{-1}+B^{-1})x - 2x^T(A^{-1}a + B^{-1}b)\right]\right) dx
\end{aligned}
$$

Now, Consider the integral term above

$$
\begin{aligned}
Int =& \int \exp\left(\frac{-1}{2}\left[x^T(A^{-1}+B^{-1})x - 2x^T(A^{-1}a + B^{-1}b)\right]\right) dx \\
=& \int \exp\left(\frac{-1}{2}\left[x^T\Sigma_*^{-1}x - 2x^T\Sigma_*^{-1}\Sigma_*(A^{-1}a + B^{-1}b)\right]\right) \\
& \exp\left(\frac{-1}{2}\left[(\Sigma_*(A^{-1}a + B^{-1}b))^T\Sigma_*^{-1}(\Sigma_*(A^{-1}a + B^{-1}b))\right]\right) \\
& \exp\left(\frac{1}{2}\left[(\Sigma_*(A^{-1}a + B^{-1}b))^T\Sigma_*^{-1}(\Sigma_*(A^{-1}a + B^{-1}b))\right]\right) dx \\
=& \exp\left(\frac{1}{2}\left[(\Sigma_*(A^{-1}a + B^{-1}b))^T\Sigma_*^{-1}(\Sigma_*(A^{-1}a + B^{-1}b))\right]\right) \\
& \int \exp\left(\frac{-1}{2}\left[x^T\Sigma_*^{-1}x - 2x^T\Sigma_*^{-1}\Sigma_*(A^{-1}a + B^{-1}b) + (\Sigma_*(A^{-1}a + B^{-1}b))^T\Sigma_*^{-1}(\Sigma_*(A^{-1}a + B^{-1}b))\right]\right) dx \\
=& exp\left(\frac{1}{2}\left[(\Sigma_*(A^{-1}a + B^{-1}b))^T\Sigma_*^{-1}(\Sigma_*(A^{-1}a + B^{-1}b))\right]\right) \\
& (2\pi)^{k/2}|\Sigma_*|^{1/2} \int Norm_x[\Sigma_*(A^{-1}a + B^{-1}b), \Sigma_*] dx
\end{aligned}
$$

Therefore, we have proved the integral part on the RHS of the required proof as follows:

$$LHS = \left[\frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} exp\left(\frac{1}{2}\left[(\Sigma_*(A^{-1}a+B^{-1}b))^T\Sigma_*^{-1}(\Sigma_*(A^{-1}a+B^{-1}b))\right]\right)\right]$$
$$\exp\left(\frac{-1}{2}(a^TA^{-1}a+b^TB^{-1}b)\right)\int Norm_x[\Sigma_*(A^{-1}a+B^{-1}b),\Sigma_*]dx$$

Now let us consider the residual term in the above equation which we need to simplify further

$$Res = \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} exp\left(\frac{1}{2}\left[(\Sigma_*(A^{-1}a+B^{-1}b))^T\Sigma_*^{-1}(\Sigma_*(A^{-1}a+B^{-1}b))\right]\right)$$
$$= \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} exp\left(\frac{1}{2}\left[(A^{-1}a+B^{-1}b)^T\Sigma_*(A^{-1}a+B^{-1}b)-(a^TA^{-1}a)-(b^TB^{-1}b)\right]\right)$$

Using the $\int Norm_x[a,A]Norm_x[b,B]dx(\mathrm{A}^{-1}+B^{-1})^{-1} = (A-A(A+B)^{-1}A) = (B-B(A+B)^{-1}B)$, and also the **symmetric property of covariance matrices**:

$$Res = \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|AB|)^{1/2}} exp\left(\frac{1}{2}\left[(a^TA^{-1}\Sigma_*A^{-1}a)+(b^TB^{-1}\Sigma_*B^{-1}b)+(a^TA^{-1}\Sigma_*B^{-1}b)+(b^TB^{-1}\Sigma_*A^{-1}a)-(a^TA^{-1}a+b^TB^{-1}b)\right]\right)$$
$$= \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} exp\left(\frac{1}{2}\left[(a^TA^{-1}\Sigma_*B^{-1}b)+(b^TB^{-1}\Sigma_*A^{-1}a)\right]\right)$$
$$exp\left(\frac{1}{2}\left[(a^TA^{-1}(A-A(A+B)^{-1}A)A^{-1}a)+(b^TB^{-1}(B-B(A+B)^{-1}B)B^{-1}b)-(a^TA^{-1}a+b^TB^{-1}b)\right]\right)$$
$$= \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} exp\left(\frac{1}{2}\left[(a^TA^{-1}\Sigma_*B^{-1}b)+(b^TB^{-1}\Sigma_*A^{-1}a)\right]\right)$$
$$exp\left(\frac{1}{2}\left[(a^TA^{-1}a)-(a^T(A+B)^{-1}a)+(b^TB^{-1}b)-(b^T(A+B)^{-1}b)-(a^TA^{-1}a+b^TB^{-1}b)\right]\right)$$
$$= \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} exp\left(\frac{-1}{2}\left[(a^T(A+B)^{-1}a)+(b^T(A+B)^{-1}b)\right]\right) exp\left(\frac{1}{2}\left[2(a^TA^{-1}\Sigma_*B^{-1}b)\right]\right)$$
$$= \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} exp\left(\frac{-1}{2}\left[(a^T(A+B)^{-1}a)+(b^T(A+B)^{-1}b)\right]\right) exp\left(\frac{1}{2}\left[2(a^TA^{-1}(B-B(A+B)^{-1}B)B^{-1}b)\right]\right)$$
$$= \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} exp\left(\frac{-1}{2}\left[(a^T(A+B)^{-1}a)+(b^T(A+B)^{-1}b)\right]\right) exp\left(\frac{1}{2}\left[2a^T(A^{-1}-A^{-1}B(A+B)^{-1})b\right]\right)$$

Using the **Woodbury matrix identity**: $(A+UCV)^{-1} = A^{-1}-A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1}$ and substituting $U = B,\ C = V = I$ we get $(A+B)^{-1} = A^{-1}-A^{-1}B(A+B)^{-1}$

$$Res = \frac{|\Sigma_*|^{1/2}}{(2\pi)^{k/2}(|A||B|)^{1/2}} exp\left(\frac{-1}{2}\left[(a^T(A+B)^{-1}a)+(b^T(A+B)^{-1}b)-2(a^T(A+B)^{-1}b)\right]\right)$$

Using the "**product of matrix determinants is equal to determinant of matrix products**" and "**determinant of matrix inverse is equal to the inverse of the determinant of the matrix**" identities along with the Woodbury matrix identity used above, we can write $\frac{|\Sigma_*|^{1/2}}{(|A||B|)^{1/2}}$ as $\frac{1}{|A+B|^{1/2}}$. Hence we have the final result of the proof as follows:

$$Res = \frac{1}{(2\pi)^{k/2}(|A+B|)^{1/2}} exp\left(\frac{-1}{2}\left[(a-b)^T(A+B)^{-1}(a-b)\right]\right)$$

$$\therefore \int Norm_x[a,A]Norm_x[b,B]dx = Norm_a[b,A+B]\int Norm_x[\Sigma_*(A^{-1}a+B^{-1}b),\Sigma_*]dx$$