left-to-right order of Figure 6.1, we can always be sure that by the time we get to a node v, we already have all the information we need to compute dist(v). We are therefore able to compute all distances in a single pass:

```
initialize all dist($\cdot$) values to $\infty$
dist($s$) $= 0$
for each $v \in V \backslash \{s\}$, in linearized order:
    dist($v$) $= \min_{(u,v)\in E}\{$dist($u$) $+ l(u,v)\}$
```

Notice that this algorithm is solving a collection of *subproblems*, $\{\texttt{dist}(u) : u \in V\}$. We start with the smallest of them, dist(s), since we immediately know its answer to be 0. We then proceed with progressively "larger" subproblems—distances to vertices that are further and further along in the linearization—where we are thinking of a subproblem as large if we need to have solved a lot of other subproblems before we can get to it.

This is a very general technique. At each node, we compute some function of the values of the node's predecessors. It so happens that our particular function is a minimum of sums, but we could just as well make it a *maximum*, in which case we would get *longest* paths in the dag. Or we could use a product instead of a sum inside the brackets, in which case we would end up computing the path with the smallest product of edge lengths.

Dynamic programming is a very powerful algorithmic paradigm in which a problem is solved by identifying a collection of subproblems and tackling them one by one, smallest first, using the answers to small problems to help figure out larger ones, until the whole lot of them is solved. In dynamic programming we are not given a dag; the dag is *implicit*. Its nodes are the subproblems we define, and its edges are the dependencies between the subproblems: if to solve subproblem B we need the answer to subproblem A, then there is a (conceptual) edge from A to B. In this case, A is thought of as a smaller subproblem than B—and it will always be smaller, in an obvious sense.

But it's time we saw an example.

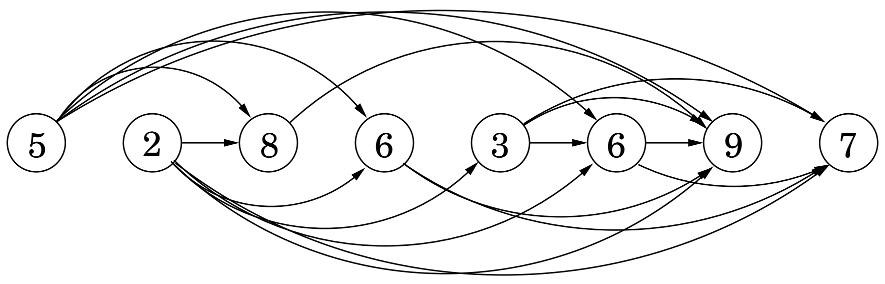
6.2 Longest increasing subsequences

In the *longest increasing subsequence* problem, the input is a sequence of numbers $a_1, \ldots, a_n$. A *subsequence* is any subset of these numbers taken in order, of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, and an *increasing* subsequence is one in which the numbers are getting strictly larger. The task is to find the increasing subsequence of greatest length. For instance, the longest increasing subsequence of $5, 2, 8, 6, 3, 6, 9, 7$ is $2, 3, 6, 9$:

$$\begin{tikzcd}
5 & 2 \arrow[rrr, bend right] & 8 & 6 & 3 \arrow[r, bend right] & 6 \arrow[r, bend right] & 9 & 7
\end{tikzcd}$$

In this example, the arrows denote transitions between consecutive elements of the optimal solution. More generally, to better understand the solution space, let's create a graph of *all* permissible transitions: establish a node i for each element a_i, and add directed edges (i, j) whenever it is possible for a_i and a_j to be consecutive elements in an increasing subsequence, that is, whenever $i < j$ and $a_i < a_j$ (Figure 6.2).

Figure 6.2 The dag of increasing subsequences.



Notice that (1) this graph $G = (V, E)$ is a dag, since all edges (i, j) have $i < j$, and (2) there is a one-to-one correspondence between increasing subsequences and paths in this dag. Therefore, our goal is simply to find the longest path in the dag!

Here is the algorithm:

```
for $j = 1, 2, \ldots, n$:
    $L(j) = 1 + \max\{L(i) : (i, j) \in E\}$
return $\max_j L(j)$
```

$L(j)$ is the length of the longest path—the longest increasing subsequence—ending at j (plus 1, since strictly speaking we need to count nodes on the path, not edges). By reasoning in the same way as we did for shortest paths, we see that any path to node j must pass through one of its predecessors, and therefore $L(j)$ is 1 plus the maximum $L(\cdot)$ value of these predecessors. If there are no edges into j, we take the maximum over the empty set, zero. And the final answer is the *largest* $L(j)$, since any ending position is allowed.

This is dynamic programming. In order to solve our original problem, we have defined a collection of subproblems $\{L(j) : 1 \leq j \leq n\}$ with the following key property that allows them to be solved in a single pass:

(*) There is an ordering on the subproblems, and a relation that shows how to solve a subproblem given the answers to "smaller" subproblems, that is, subproblems that appear earlier in the ordering.

In our case, each subproblem is solved using the relation

$$L(j) \;=\; 1 + \max\{L(i) : (i, j) \in E\},$$

an expression which involves only smaller subproblems. How long does this step take? It requires the predecessors of j to be known; for this the adjacency list of the reverse graph G^R, constructible in linear time (recall Exercise 3.5), is handy. The computation of $L(j)$ then takes time proportional to the indegree of j, giving an overall running time linear in $|E|$. This is at most $O(n^2)$, the maximum being when the input array is sorted in increasing order. Thus the dynamic programming solution is both simple and efficient.

There is one last issue to be cleared up: the L-values only tell us the *length* of the optimal subsequence, so how do we recover the subsequence itself? This is easily managed with the