AM205: Assignment 1 (due 5 PM, September 26)

1. (a) Derive the interpolating cubic polynomial for the data points $(0,0)$, $(1,0)$, $(2,1)$, and $(3,2)$ using the monomial basis and the Lagrange basis. Show that these two representations are equivalent. *You may need to use NumPy, MATLAB, or other software to solve the linear system for the four degrees of freedom.*

   (b) Find the interpolating polynomial for the data points $(0,2)$, $(2,0)$, $(4,0)$, and $(6,0)$ either by direct calculation or by using your result from (a).

2. (a) **Error bounds with Lagrange polynomials.** Let $f(x) = e^{4x} + e^{-2x}$. Write a program to calculate and plot the Lagrange polynomial $p_{n-1}(x)$ of $f(x)$ at the Chebyshev points $x_j = \cos((2j-1)\pi/2n)$ for $j = 1, ..., n$. For $n = 4$, over the range $[-1,1]$, plot $f(x)$ and Lagrange polynomial $p_3(x)$.

   (b) Recall from the lectures that the infinity norm for a function g on $[-1,1]$ is defined as $||g||_\infty = \max_{x\in[-1,1]} |g(x)|$. Calculate $||f - p_3||_\infty$ by sampling the function at 1,000 equally-spaced points over $[-1,1]$.

   (c) Recall the interpolation error formula from the lectures,

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\theta)}{n!}(x-x_1)(x-x_2)\dots(x-x_n) \tag{1}$$

   for some $\theta \in [-1,1]$. Use this formula to derive an upper bound for $||f - p_{n-1}||_\infty$ for any positive integer n. Your bound should be a mathematical formula, and should not rely on numerical sampling.

   (d) Find a cubic polynomal $p_3^\dagger$ such that $||f - p_3^\dagger||_\infty < ||f - p_3||_\infty$.

3. **Periodic cubic splines.** In the lectures we discussed the construction of cubic splines to interpolate between a number of control points. We found that it was necessary to impose additional constraints at the end points of the spline in order to have enough constraints to determine the cubic spline uniquely. Here, we examine the construction of cubic splines on a periodic interval $t \in [0,4)$, where $t = 0$ is equivalent to $t = 4$. By working in a periodic interval, this simplifies the spline construction and no additional end point constraints are required.

   (a) Consider the four points $(x,t) = (0,0), (1,1), (2,0), (3,-1)$. Construct a cubic spline $s_x(t)$ that is piecewise cubic in the four intervals $[0,1)$, $[1,2)$, $[2,3)$, and $[3,4)$. At $t = 0,1,2,3$ the cubics should match the control points, giving eight constraints. At $t = 0,1,2,3$ the first and second derivatives should match, giving and additional eight constraints and allowing $s_x(t)$ to be uniquely determined.

   (b) Plot $s_x(t)$ and $\sin(t\pi/2)$ on the interval $[0,4)$ and show that they are similar.

   (c) Construct a second cubic spline $s_y(t)$ that goes through the four points $(0,1)$, $(1,0)$, $(2,-1)$, and $(3,0)$. Plot $s_y(t)$ and $\cos(t\pi/2)$ on the interval $0 \le t < 4$ and show that they are similar.

(d) In the xy-plane, plot the parametric curve $(s_x(t), s_y(t))$ for $t \in [0, 4)$. Calculate the area enclosed by the parametric curve, and use it to estimate π to at least five decimal places, using the relationship $A = \pi r^2$ where r is taken to be 1.

(e) **Optional for the enthusiasts.** In parts (a) to (c), you constructed cubic spline approximations for sine and cosine using four equally-spaced control points. Generalize this to construct cubic spline approximations for sine and cosine using n equally-spaced control points. What is the rate of convergence of the approximation of π as n is increased?

4. **Fitting a planet's orbit.** A planet follows an elliptical orbit, which can be represented in a Cartesian (x, y) coordinate system by the equation

$$b_0 + b_1 x + b_2 y + b_3 xy + b_4 y^2 = x^2. \tag{2}$$

(a) Use a linear least-squares fit for the parameters b_0, b_1, b_2, b_3, b_4, given the following observations of the planet's position:

x	1.02	0.95	0.87	0.77	0.67
y	0.39	0.32	0.27	0.22	0.18
x	0.56	0.44	0.30	0.16	0.01
y	0.15	0.13	0.12	0.13	0.15

This data is provided in the file `q4a_data.txt` on the AM205 website.) Plot the resulting orbit with a continuous curve (making sure that you plot the entire ellipse) and the observations (marked with $\times$ symbols) on the same plot. What is the 2-norm of the residual for your best fit?

(b) The observation data is nearly rank-deficient, which implies that the matrix $A^\mathsf{T} A$ is nearly singular and hence the parameter fit will be sensitive to perturbations in the data (*i.e.* the least-squares fit is poorly conditioned in this case). To show this, compute the best fit to the perturbed data $\hat{x} = x + \Delta x$ and $\hat{y} = y + \Delta y$, where Δx, Δy are given below.

Δx	-0.0029	0.0007	-0.0082	-0.0038	-0.0041
Δy	-0.0033	0.0043	0.0006	0.0020	0.0044
Δx	0.0026	-0.0001	-0.0058	-0.0005	-0.0034
Δy	0.0009	0.0028	0.0034	0.0059	0.0024

This data is provided in the file `q4b_data.txt` on the AM205 website. Overlay the two sets of observations and corresponding orbits on the same plot.

5. **Approximate solution of a partial differential equation using least-squares fitting.** In the xy-plane, an elastic string is stretched along the horizontal axis between $x = 0$ and $x = \pi$. It undergoes small vibrations in the vertical direction, so that its vertical position is given by the function $y(x, t)$. The function satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}, \tag{3}$$

for some parameter c, subject to the boundary conditions that $y(0,t) = y(\pi,t) = 0$. One can verify that

$$y_k(x,t) = \sin kx \cos kct \tag{4}$$

is a solution to Eq. 3 for any integer k. Since Eq. 3 is linear in y, it follows that any linear combination of the y_k (*e.g.* $y_1 + 0.2y_2 - 0.3y_6$) will also be a solution.

(a) Suppose that the initial condition is

$$y(x,0) = 10^{-4}x^8(\pi - x)^4 \tag{5}$$

and $\partial_t y(x,0) = 0$. Consider the approximation

$$y(x,0) \approx \sum_{k=1}^{6} \alpha_k \sin kx \tag{6}$$

for some parameters $\alpha_1, \alpha_2, \ldots, \alpha_6$. By evaluating Eq. 5 at the 49 points $\frac{\pi}{50}$, $\frac{2\pi}{50}$, $\ldots$, $\frac{49\pi}{50}$, find the least-squares fit of the α_i.

(b) Using your calculation from part (a), write down an approximate solution $y_{\text{ap}}(x,t)$ to Eq. 3. Set $c = 1$. On the same axes, plot Eq. 5, and $y_{\text{ap}}(x,t)$ for $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$.

(c) **Optional for the enthusiasts.** If you are familiar with Fourier analysis, you will notice that Eq. 6 is similar to a Fourier sine series,

$$y(x,0) = \sum_{k=1}^{\infty} b_k \sin kx. \tag{7}$$

Evaluate the b_k using the standard Fourier integral formula—this can either be done numerically or with the aid of a symbolic maninupulation package such as Maple or Mathematica. Compare the values of the b_k with the α_k.