AM205: Assignment 3 (due 5 PM, October 24)

1. **Convergence rates of two integrals.** Consider the function

$$f(x) = \frac{1}{\frac{5}{4} - \cos x}. \tag{1}$$

   (a) It can be shown that

$$I_A = \int_0^{\pi/3} f(x)dx = \frac{8\pi}{9}. \tag{2}$$

   Write a program to numerically evaluate I_A using the composite trapezoid rule with n intervals of equal size h, for $n = 1, 2, \ldots, 50$, and make a log–log plot of the absolute error as a function h. On the same axes, overlay the composite trapezoid error bound from the lectures, $E(h) = \frac{h^2\pi}{36}||f''||_\infty$, and show that your numerically computed results are smaller than the bound.

   (b) It can be shown that

$$I_B = \int_0^{2\pi} f(x)dx = \frac{8\pi}{3}. \tag{3}$$

   Write a program to numerically evaluate I_B using the composite trapezoid rule with n points, for $n = 1, 2, \ldots, 50$. Make a log–log plot of the absolute error as a function h. Does the absolute error scale like h^m for some m?

   (c) **Optional.** Use residue calculus to prove the integral in Eq. 3.

2. (a) **Adaptive integration.** Given that the cubic Legendre polynomial is $P_3(x) = \frac{1}{2}x(5x^2 - 3)$, derive the 3-point Gauss quadrature rule on the interval $[-1, 1]$ by evaluating the relevant integrals by hand. Demonstrate that this quadrature rule integrates all polynomials up to the expected degree exactly.

   (b) In the lectures we discussed a method of evaluating integrals $\int_a^b f(x)dx$ by adaptively refining the calculation in regions where the function varies rapidly. To begin, a tolerance level T is introduced, and the calculation starts using a single integration interval $[a, b]$. Let $I_{a,b}$ be the integral of f over this interval using the three-point quadrature rule from part (a). In addition, define $c = \frac{a+b}{2}$ and calculate $\hat{I}_{a,b} = I_{a,c} + I_{c,b}$ as a more refined estimate of the integral. Then $E_{a,b} = |I_{a,b} - \hat{I}_{a,b}|$ is an estimate of the error of $I_{a,b}$. If $E_{a,b} < Tl$, where l is the length of the interval, then the error is acceptable, and the method can terminate. Otherwise this interval must be subdivided into $[a, c]$ and $[c, b]$, and the above procedure must be applied to these two intervals. The procedure must be applied recursively until the errors become smaller than the tolerance.

   Write a function to implement this adaptive integration scheme. Using $T = 10^{-6}$, apply it to the integrals

$$\int_{-1}^{3/4} (x^m - x^2 + 1)dx \tag{4}$$

   for $m = 4, 5, 6, 7, 8$. For each case, report the value of the integral, the total estimated error (by summing the relevant $E_{\alpha,\beta}$ terms), and the total number of intervals that are used.

(c) Use your adaptive integration routine from part (a) and $T = 10^{-6}$ to evaluate the four integrals

$$\int_{-1}^{1} |x| dx, \qquad \int_{-1}^{2} |x| dx,$$

$$\int_{-1}^{1} (500x^6 - 700x^4 + 245x^2 - 3)\sin^2(2\pi x) dx, \qquad \int_{0}^{1} x^{3/4} \sin\frac{1}{x}\, dx.$$

For each case, report the value of the integral, the total estimated error (by summing the relevant $E_{\alpha,\beta}$ terms), and the total number of intervals that are used.

(d) One of your integrals from part (c) is incorrect, and the true answer is significantly outside the estimated error bound. Determine which integral this is, and state why this happens. Find a method of altering your integration routine to correctly calculate the integral to five decimal places of accuracy.

3. **Error analysis of a numerical integration rule.** Applying the midpoint quadrature rule (*i.e.* $n = 0$ Newton–Cotes with the quadrature point at the midpoint) on the interval $[t_k, t_{k+1}]$ to $y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(t, y(t))dt$ leads to the implicit *midpoint method*,

$$y_{k+1} = y_k + hf(t_{k+1/2}, (y_k + y_{k+1})/2), \tag{5}$$

where $t_{k+1/2} = t_k + \frac{h}{2}$.

(a) Use Taylor series expansions to show that the order of accuracy of this method is 2.

(b) What is the stability region of the method for the equation $y' = \lambda y$? In other words, for what values of $\bar{h} = h\lambda \in \mathbb{C}$ is the method stable?

4. (a) **A multi-step method.** Consider solving the differential equation $y' = f(t, y)$ at time-points t_k with corresponding numerical solutions y_k. The multi-step Nyström numerical method is based upon the integral relation

$$y(t_{k+1}) = y(t_{k-1}) + \int_{t_{k-1}}^{t_{k+1}} f(t, y)dt. \tag{6}$$

Derive a multi-step numerical method by approximating the integrand $f(t, y)$ with the polynomial interpolation using the function values at t_{k-2}, t_{k-1}, and t_k. Your method should have the form

$$y_{k+1} = y_{k-1} + h(\alpha f_{k-2} + \beta f_{k-1} + \gamma f_k) \tag{7}$$

where α, β, and γ are constants to be determined, h is the timestep interval size, and $f_l = f(t_l, y_l)$ for an arbitrary l.

(b) Find the exact solution to the second-order differential equation

$$y''(t) + 2y'(t) + 5y(t) = 0. \tag{8}$$

subject to the initial conditions $y(0) = 1$, $y'(0) = 0$.

(c) Write Eq. 8 as a coupled system of two first-order differential equations for $\mathbf{y} = (y, v) = (y, y')$. Solve the system over the interval $0 \leq t \leq 3$ with a timestep of $h = 0.02$ using your multi-step method from part (a).

Before Eq. 7 can be applied, $\mathbf{y}_1$ and $\mathbf{y}_2$ must be calculated accurately. Use one of the following two approaches:

i. set them based on the exact solution from part (b),

ii. calculate them using the classical fourth-order Runge–Kutta method.

Make a log–log plot of the absolute error between the numerical and exact values of y at $t = 3$ as a function of h, over the range from $h = 10^{-4}$ to $h = 10^{-1}$. Show that your method is third-order accurate.

(d) **Optional.** Suppose that instead of setting $\mathbf{y}_1$ and $\mathbf{y}_2$ accurately, you instead make use of forward Euler steps. Create a log–log plot of the absolute error as a function of h for this case and determine the order of accuracy.

5. **Asteroid collision.** An asteroid is detected near the Earth and the Moon in the plane of their orbits. Use your ODE-solving skills to determine if there is danger of collision.

This problem can be treated as a circular restricted three-body problem, which considers the motion of an object of negligible mass in the presence of two massive gravitating bodies that circularly orbit each other. The motion of the object is assumed to be restriced to the 2D plane of the circular orbit of the two massive bodies.

The Earth and Moon have circular orbits about each other. If we work in a co-rotating frame, we can consider the Earth and Moon to be at fixed locations and then we only need to determine the trajectory of the asteroid. In dimensionless units, fix the Earth at position $(x, y) = (0, 0)$ and the Moon at position $(x, y) = (1, 0)$.

The asteroid (which has negligible mass) will move according to the gravitational forces it feels from the Earth and the Moon. The motion of the asteroid is described by its position $x(t)$, $y(t)$, and velocity $u(t)$, $v(t)$ as functions of time. Throughout the asteroid's movement, the *Jacobi integral*, J, is a constant of motion, and is given by

$$J(x, y, u, v) = (x + \mu)^2 + y^2 + \frac{2(1 - \mu)}{\sqrt{x^2 + y^2}} + \frac{2\mu}{\sqrt{(x - 1)^2 + y^2}} - u^2 - v^2 \tag{9}$$

where $\mu = 0.01$ is the mass ratio of the Moon to Earth. The asteroid's equations of motion are given by the system of ODEs,

$$\begin{aligned} x' &= -\frac{1}{2}\frac{\partial J}{\partial u}, & \qquad y' &= -\frac{1}{2}\frac{\partial J}{\partial v}, \\ u' &= v + \frac{1}{2}\frac{\partial J}{\partial x}, & \qquad v' &= -u + \frac{1}{2}\frac{\partial J}{\partial y}. \end{aligned} \tag{10}$$

(a) Write the system of ODEs to be solved in terms of x, y, u, and v.

(b) Suppose that based on observations we know that at $t = 0$ the asteroid's position and velocity are given by $(x(0), y(0)) = (1.08, 0)$ and $(u(0), v(0)) = (0.5 \cos(\theta), 0.5 \sin(\theta))$

where $\theta \in [\frac{3\pi}{2} - 0.3, \frac{3\pi}{2} + 0.3]$, so there is some uncertainty in the asteroid's initial velocity. Use your favorite ODE solver[1] to integrate this ODE from $t = 0$ to $t = 7$, using 5 different initial conditions determined by sampling θ at evenly spaced points in the interval $[\frac{3\pi}{2} - 0.3, \frac{3\pi}{2} + 0.3]$ (your samples should include the endpoints of the interval). On a single figure, plot the five trajectories $(x(t), y(t))$ of the asteroid corresponding to the initial conditions you considered. The Earth has radius $R_{\text{Earth}} = 0.02$ and the Moon has radius $R_{\text{Moon}} = 0.005$. Add circles of the appropriate radii on your plot to indicate the regions occupied by the Moon and the Earth. Do any of the five trajectories collide with the Earth or Moon between $t = 0$ and $t = 7$? If yes, which initial condition (or initial conditions) produce a collision?

(c) If this ODE system were solved exactly, the Jacobi integral would be exactly constant as a function of time, since it is a conserved quantity of the system. We can therefore use the value of the Jacobi integral to check the robustness of our numerical approximation. Plot the relative error in the Jacobi integral, $\frac{|J(t)-J(0)|}{J(0)}$, for $t \in [0, 7]$ for the $\theta = \frac{3\pi}{2}$ trajectory and explain what you observe.

(d) **Optional.** Try some alternative ODE solution methods and compare the errors in J that occur over the integration interval.

[1] In Python, a good choice would be the `odeint` routine. In MATLAB, the `ode45` routine could be used, using the additional command `odeset('RelTol',1e-10,'AbsTol',1e-10)` to improve accuracy.