AM205: Assignment 4 (due 5 PM, November 7)

*For this assignment, first complete problems 1 and 2, and then complete **either** problem 3 (on numerical PDEs) or problem 4 (on optimization). If you submit answers for both problems 3 and 4, the maximum score from the two will be taken when calculating your grade.*

Program files: A number of program and data files for this homework can be downloaded as a single ZIP file from the course website.

1. **Convergence and stability of a numerical scheme.** Consider the numerical scheme

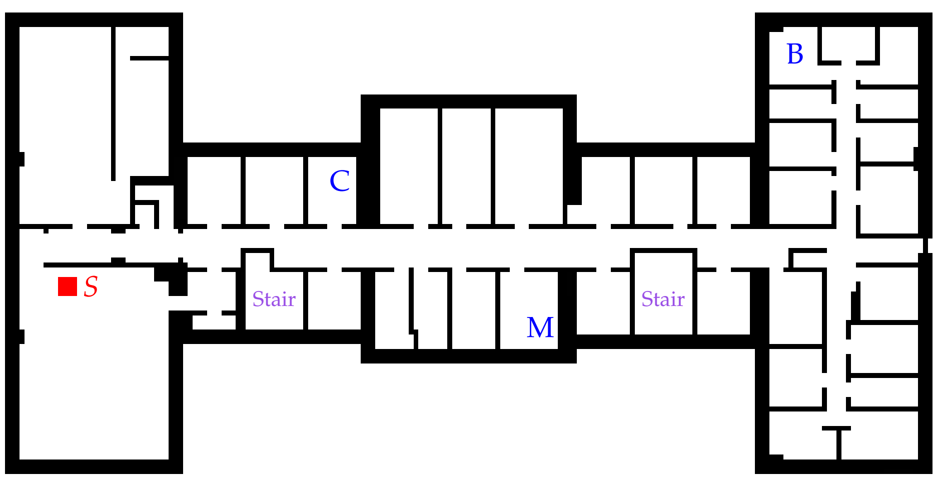
$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{\Delta t^2} - c^2 \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} = 0 \qquad (1)$$

to solve the one-dimensional wave equation $u_{tt} - c^2 u_{xx} = 0$. Here, $c \in \mathbb{R}$ and U_j^n is the numerical approximation of $u(j\Delta x, n\Delta t)$.

   (a) Show that the numerical scheme in Eq. 1 is second-order accurate.

   (b) Use Fourier stability analysis, by subsituting in the ansatz $U_j^n = \lambda(k)^n e^{ijk\Delta x}$, to show that the numerical scheme is stable. Because the wave equation is second-order in time, you will get two solutions $\lambda(k)$ for each k, and both must have magnitude less than or equal to 1 for stability.

2. **Totally rocking out in Pierce Hall.** The image below shows a map of the third floor of Pierce Hall. All the doors are open, apart from those to the main stairwells and the passageway to Maxwell–Dworkin.



A text file `pierce.txt` is provided that encodes this map as a 100×200 matrix using 1 for walls and 0 for empty space. Use the convention that $(j,k) = (0,0)$ is the top left of the matrix and $(j,k) = (99,199)$ is bottom right of the matrix. The grid spacing is $h = 36.6$ cm.

A group of students hold an event in Pierce 301, the large room in the bottom left of the map. They set up a loudspeaker shown in red, covering the region S over gridpoints (j,k) for

$57 \leq j \leq 60, 15 \leq k \leq 18$. When testing the speaker, they drive it with a 50 Hz sine wave,[1] creating sound pressure waves that travel throughout the building, disturbing the occupants. The pressure field $p(x,y,t)$ satisfies the wave equation

$$\frac{\partial^2 p}{\partial t^2} - c^2 \nabla^2 p = 0, \tag{2}$$

where $c = 3.43 \times 10^4$ cm s^{-1} is the speed of sound. At $t = 0$ when the speaker is switched on, the pressure satisfies the initial conditions $p(x,y,t) = p_t(x,y,t) = 0$. At each wall, the pressure satisfies the boundary condition

$$\mathbf{n} \cdot \nabla p = 0, \tag{3}$$

where $\mathbf{n}$ is a unit vector normal to the wall. In the region S the pressure is driven to satisfy

$$p(x,y,t) = f(t) = p_0 \sin \omega t, \tag{4}$$

where the angular frequency is $\omega = 100\pi$ s^{-1} and the pressure constant[2] is $p_0 = 10$ Pa.

(a) Write a program to solve for the pressure field inside the building, using the two-dimensional discretization

$$\frac{P^{n+1}_{j,k} - 2P^n_{j,k} + P^{n-1}_{j,k}}{\Delta t^2} - c^2 \frac{P^n_{j+1,k} + P^n_{j,k+1} - 4P^n_{j,k} + P^n_{j-1,k} + P^n_{j,k-1}}{h^2} = 0. \tag{5}$$

where $P^n_{j,k}$ is the numerical approximation of $p(jh, kh, n\Delta t)$. Use a timestep of $\Delta t = \frac{h}{2c}$ or smaller. As initial conditions, use

$$P^0_{j,k} = 0, \qquad P^1_{j,k} = \begin{cases} f(\Delta t) & \text{if } (j,k) \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

To account for the boundary condition in Eq. 3, use the ghost node approach: when considering a point (j,k) in Eq. 5 that references an orthogonal neighbor (j^*, k^*) that is a wall, treat $P^n_{j^*,k^*}$ as equal to $P^n_{j,k}$. As an example of this, suppose that at a particular (j,k), the points $(j, k-1)$ and $(j+1, k)$ are within walls. Then, after taking into account the boundary conditions, the appropriate finite-difference relation is

$$\frac{P^{n+1}_{j,k} - 2P^n_{j,k} + P^{n-1}_{j,k}}{\Delta t^2} - c^2 \frac{P^n_{j,k+1} - 2P^n_{j,k} + P^n_{j-1,k}}{h^2} = 0. \tag{7}$$

due to cancellation of some terms. To account for the condition in Eq. 4, enforce throughout the computation that $P^n_{j,k} = f(n\Delta t)$ for $(j,k) \in S$.

(b) Make two-dimensional plots of the pressure field in the building at $t = 0.015$ s, $t = 0.105$ s, $t = 0.505$ s, and $t = 1.005$ s. In the program files, there are some example programs in the `py_2dplot` and `matlab_2dplot` directories that you may find useful, which make plots of a two-dimensional field with the map overlaid. You should expect that your program may take a reasonable amount of wall clock time, possibly up to about three-quarters of an hour to simulate to $t = 1.005$ s. You may wish test your program over smaller intervals of t and consider possible code optimizations if necessary.

[1] The typical human auditory range is from 20 Hz to 20 kHz, so this sound is at the lower end of what is audible.

[2] This pressure is similar to what you might get at a rock concert.

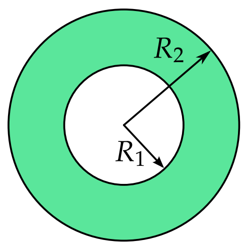
(c) Three people B, C, and M are trying to work at locations $(10, 167)$, $(35, 73)$, and $(67, 115)$. Find the time t in seconds, accurate to at least two decimal places, when the three people first hear the sound, defined as when $|p(t)|$ at their location exceeds 10^{-3} Pa. Discuss whether your results are reasonable, given the locations of the people in relation to the loudspeaker.

(d) On a single pair of axes, plot $p(t)$ at the three people's locations over the interval $0 \le t \le 1$ s. Which person is most likely to be disturbed by the loudspeaker?

(e) **Optional.** Make a movie of p over the time interval $0 \le t \le 2.5$ s. In addition, make a movie of the quantity $g = (p^2 + p_t^2 \omega^{-2})^{1/2}$ over the same interval.

(f) **Optional.** Estimate the sound level that B, C, and M hear in terms of decibels. Discuss what modifications could be made to the PDE in Eq. 2 and boundary condition in Eq. 3 to account for sound attenuation.

3. **Heat diffusion in a pipe.** Suppose a metal pipe is heated in an industrial oven as part of a manufacturing process, and we wish to model the temperature distribution in the pipe as a function of time. The pipe's cross-section is shown below; it has an inner radius of R_1 and an outer radius of R_2.



The temperature is assumed to be uniform along the pipe. Due to the axisymmetry, the problem can be expressed in polar coordinates,

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \tag{8}$$

for $r \in [R_1, R_2]$ and $t \in [0, t_f]$, with boundary conditions[3]

$$\left. \frac{\partial u}{\partial r} \right|_{r=R_1} = 0, \tag{9}$$

$$\alpha \left. \frac{\partial u}{\partial r} \right|_{r=R_2} = -u(R_2). \tag{10}$$

Here α is the thermal diffusivity of the pipe, and $u(r, t)$ is the non-dimensionalized temperature in the pipe, defined as

$$u(r, t) = \frac{T(r, t) - T_{\text{oven}}}{T_0 - T_{\text{oven}}}, \tag{11}$$

where $T(r, t)$ is the temperature measured in degrees Fahrenheit, and T_0 and T_{oven} are the initial temperature in the pipe and the temperature in the oven, respectively. As a result, the

[3]Note that for dimensional consistency a heat transfer coefficient is required in Eq. 10 but this is omitted for the sake of simplicity.

initial condition on u is $u_0(r) = 1$, and at steady state the pipe will be in thermal equilibrium with the oven so that $u(r,t) \to 0$ as $t \to \infty$. Thermal insulation on the interior of the pipe is modeled by Eq. 9, and Eq. 10 models heat transfer on the pipe's outer surface between the pipe and the air in the oven.

We will approximate this PDE using a finite-difference method. Suppose we have spatial nodes $r_j = R_1 + (\Delta r)j$ for $j = 0, 1, \ldots, n_r - 1$, where $n_r = (R_2 - R_1)/\Delta r + 1$. Also, suppose we have discrete time-levels $t^n = n\Delta t$ for $n = 0, 1, 2, \ldots, n_t - 1$, where $n_t = t_f/\Delta t + 1$.

(a) At time t^n and at an interior node r_j ($j \neq 0$, $j \neq n_r - 1$), write down a Backward Euler (in time) and centered difference (in space) finite-difference approximation for the above PDE.

(b) Using the ghost node approach described in the lectures, derive second-order finite-difference approximations at time t^n of the left boundary condition, Eq. 9, at $r = R_1$, and the right boundary condition, Eq. 10, at $r = R_2$.

(c) For the parameters $R_1 = 1.5$, $R_2 = 2$, $\alpha = 0.2$, and with the initial condition $u(0,r) = 1$, use your answers to parts (a) and (b) to compute an approximate solution to this IBVP for the time interval $t \in [0,2]$ (*i.e.* $t_f = 2$). Use a spatial step size of $\Delta r = 0.01$, and temporal step size $\Delta t = 0.01$, and plot the solution at $t = 0, 0.4, 0.8, 1.2, 1.6, 2$ in a single figure.

(d) **Optional.** Use the composite trapezoid rule on the finite difference grid from part (c) to compute the average (non-dimensional) temperature of the pipe at each time level in your calculation, and plot the results as a function of time. Recall that the average value of a function f on a domain $\Omega \subset \mathbb{R}^2$ is $\bar{f} = \frac{1}{|\Omega|} \int_\Omega f\, dx\, dy$, where $|\Omega|$ denotes the area of Ω.

(e) **Optional.** The industrial heating process should be stopped at the time, t_{stop}, when the average (non-dimensional) temperature in the pipe is 0.5. Use your average temperature plot to estimate t_{stop}.

4. **Solving a nonlinear boundary value problem (BVP).** Consider the nonlinear ordinary differential equation BVP
$$u''(x) = e^{u(x)}, \qquad x \in (-1, 1), \tag{12}$$
with zero Dirichlet boundary conditions $u(-1) = u(1) = 0$, and introduce an n-point grid $x_i = -1 + ih$ where $h = \frac{2}{n-1}$.

A finite-difference approximation gives the nonlinear system $F(U) = 0$, where $U \in \mathbb{R}^{n-2}$ is the finite-difference solution vector after the two boundary terms $U_0 = U_{n-1} = 0$ are dropped since they are already known. The components of the nonlinear function $F : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ are

$$F_1(U) = \frac{U_2 - 2U_1}{h^2} - e^{U_1} = 0, \tag{13}$$
$$F_i(U) = \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} - e^{U_i} = 0, \quad i = 2, 3, \ldots, n-3, \tag{14}$$
$$F_{n-2}(U) = \frac{-2U_{n-2} + U_{n-3}}{h^2} - e^{U_{n-2}} = 0. \tag{15}$$

(a) Derive the Jacobian $J_F \in \mathbb{R}^{(n-2)\times(n-2)}$ for the system $F(U) = 0$. Describe the sparsity pattern of J_F.

(b) Use Newton's method to solve this nonlinear ODE BVP for $n = 101$, using the Jacobian matrix derived in (a). Start with an initial guess $U^0 = 0$. Terminate Newton's method once a relative step size, $\|\Delta U^k\|_2 / \|U^k\|_2$, satisfies $\|\Delta U^k\|_2 / \|U^k\|_2 \leq 10^{-10}$, where k refers to the Newton iteration count. Plot the approximate solution U (padded with U_0 and U_{n-1}) of the ODE BVP above on the n-point grid, and report your approximation to $u(0)$ to three significant digits.