Applied Math 205

- Website now moved to permanent location at
  http://iacs-courses.seas.harvard.edu/courses/am205/
- Last time: gave motivation for data fitting
- Today: polynomial interpolation for:
  1. A discrete set of points
  2. Continuous functions

The Problem Formulation

Let $\mathbb{P}_n$ denote the set of all polynomials of degree n on $\mathbb{R}$

i.e. if $p(\cdot; b) \in \mathbb{P}_n$, then

$$p(x; b) = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n$$

for $b \equiv [b_0, b_1, \ldots, b_n]^T \in \mathbb{R}^{n+1}$

The Problem Formulation

Suppose we have the data $\mathcal{S} \equiv \{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$, where the $\{x_0, x_1, \ldots, x_n\}$ are called interpolation points

Goal: Find a polynomial that passes through every data point in $\mathcal{S}$

Therefore, we must have $p(x_i; b) = y_i$ for each $(x_i, y_i) \in \mathcal{S}$, i.e. $n + 1$ equations

For uniqueness, we should look for a polynomial with $n + 1$ parameters, *i.e.* look for $p \in \mathbb{P}_n$

Vandermonde Matrix

Then we obtain the following system of $n+1$ equations in $n+1$ unknowns

$$\begin{aligned} b_0 + b_1 x_0 + b_2 x_0^2 + \ldots + b_n x_0^n &= y_0 \\ b_0 + b_1 x_1 + b_2 x_1^2 + \ldots + b_n x_1^n &= y_1 \\ &\vdots \\ b_0 + b_1 x_n + b_2 x_n^2 + \ldots + b_n x_n^n &= y_n \end{aligned}$$

Vandermonde Matrix

This can be written in Matrix form $Vb = y$, where

$$b = [b_0, b_1, \ldots, b_n]^T \in \mathbb{R}^{n+1},$$

$$y = [y_0, y_1, \ldots, y_n]^T \in \mathbb{R}^{n+1}$$

and $V \in \mathbb{R}^{(n+1)\times(n+1)}$ is the Vandermonde matrix:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}$$

Existence and Uniqueness

Let's prove that if the $n + 1$ interpolation points are distinct, then $Vb = y$ has a unique solution

We know from linear algebra that for a square matrix A
if $Az = 0 \implies z = 0$, then $Ab = y$ has a unique solution

If $Vb = 0$, then $p(\cdot; b) \in \mathbb{P}_n$ vanishes at $n + 1$ distinct points

Therefore we must have $p(\cdot; b) = 0$, or equivalently $b = 0 \in \mathbb{R}^{n+1}$

Hence $Vb = 0 \implies b = 0$, so that $Vb = y$ has a unique solution for any $y \in \mathbb{R}^{n+1}$

Vandermonde Matrix

This tells us that we can find the polynomial interpolant by solving the Vandermonde system $Vb = y$

In general, however, this is a bad idea since V is ill-conditioned

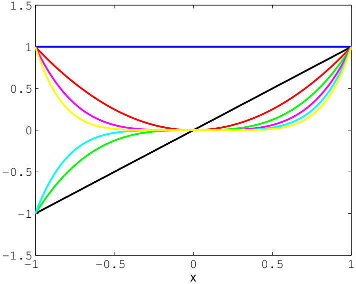
Monomial Interpolation

The problem here is that Vandermonde matrix corresponds to interpolation using the monomial basis

Monomial basis for $\mathbb{P}_n$ is $\{1, x, x^2, \ldots, x^n\}$

Monomial basis functions become increasingly indistinguishable

Vandermonde columns become nearly linearly-dependent
$\Longrightarrow$ ill-conditioned matrix!



Monomial Basis

Question: What is the practical consequence of this ill-conditioning?

Answer:

- We want to solve $Vb = y$, but due to finite precision arithmetic we get an approximation $\hat{b}$
- $\hat{b}$ will ensure $\|V\hat{b} - y\|$ is small (in a rel. sense),[1] but $\|b - \hat{b}\|$ can still be large! (see II.2 for details)
- Similarly, small perturbation in $\hat{b}$ can give large perturbation in $V\hat{b}$
- Large perturbations in $V\hat{b}$ can yield large $\|V\hat{b} - y\|$, hence a "perturbed interpolant" becomes a poor fit to the data

[1] This "small residual" property is because we use a stable numerical algorithm for solving the linear system

Monomial Basis

These sensitivities are directly analogous to what happens with an ill-conditioned basis in $\mathbb{R}^n$, e.g. consider a basis $\{v_1, v_2\}$ of $\mathbb{R}^2$:

$$v_1 = [1, 0]^T, \qquad v_2 = [1, 0.0001]^T$$

Then, let's express $y = [1, 0]^T$ and $\tilde{y} = [1, 0.0005]^T$ in terms of this basis

We can do this by solving a 2x2 linear system in each case (see II.2), and hence we get

$$b = [1, 0]^T, \qquad \tilde{b} = [-4, 5]^T$$

Hence the answer is highly sensitive to perturbations in y!

Interpolation

We would like to avoid these kinds of sensitivities to perturbations... How can we do better?

Try to construct a basis such that the interpolation matrix is the identity matrix

This gives a condition number of 1, and as an added bonus we also avoid inverting a dense $(n+1) \times (n+1)$ matrix

Lagrange Interpolation

Key idea: Construct basis $\{L_k \in \mathbb{P}_n, k = 0, \ldots, n\}$ such that

$$L_k(x_i) = \begin{cases} 0, & i \neq k, \\ 1, & i = k. \end{cases}$$

The polynomials that achieve this are called Lagrange polynomials[2]

See Lecture: These polynomials are given by:

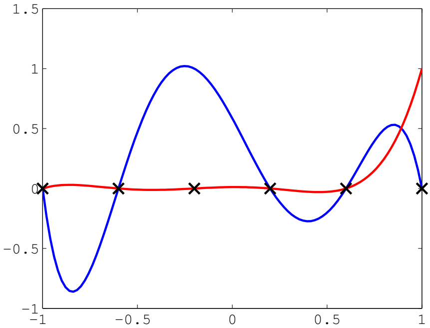
$$L_k(x) = \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}$$

and then the interpolant can be expressed as
$p_n(x) = \sum_{k=0}^{n} y_k L_k(x)$

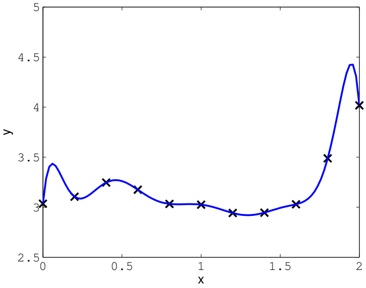
[2] Joseph-Louis Lagrange, 1736–1813

Lagrange Interpolation

Two Lagrange polynomials of degree 5



Hence we can use Lagrange polynomials to interpolate discrete data (recall plot from I.1)



We have essentially solved the problem of interpolating discrete data perfectly!

With Lagrange polynomials we can construct an interpolant of discrete data with condition number of 1

Interpolation for Function Approximation

Interpolation for Function Approximation

We now turn to a different (and much deeper) question: Can we use interpolation to accurately approximate continuous functions?

Suppose the interpolation data come from samples of a continuous function f on $[a, b] \subset \mathbb{R}$

Then we'd like the interpolant to be "close to" f on $[a, b]$

The error in this type of approximation can be quantified from the following theorem due to Cauchy[3]:

$$\boxed{f(x) - p_n(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0)\ldots(x - x_n) \text{ for some } \theta \in (a, b)}$$

[3] Augustin-Louis Cauchy, 1789–1857

Polynomial Interpolation Error

We prove this result in the case $n = 1$

Let $p_1 \in \mathbb{P}_1[x_0, x_1]$ interpolate $f \in C^2[a, b]$ at $\{x_0, x_1\}$

For some $\lambda \in \mathbb{R}$, let

$$q(x) \equiv p_1(x) + \lambda(x - x_0)(x - x_1),$$

here q is quadratic and interpolates f at $\{x_0, x_1\}$

Fix an arbitrary point $\hat{x} \in (x_0, x_1)$ and set $q(\hat{x}) = f(\hat{x})$ to get

$$\lambda = \frac{f(\hat{x}) - p_1(\hat{x})}{(\hat{x} - x_0)(\hat{x} - x_1)}$$

Goal: Get an expression for λ, since then we obtain an expression for $f(\hat{x}) - p_1(\hat{x})$

Polynomial Interpolation Error

Now, let $e(x) \equiv f(x) - q(x)$

- e has 3 roots in $[x_0, x_1]$, i.e. at $x = x_0, \hat{x}, x_1$
- Therefore e' has 2 roots in (x_0, x_1) (by Rolle's theorem)
- Therefore e'' has 1 root in (x_0, x_1) (by Rolle's theorem)

Let $\theta \in (x_0, x_1)$ be such that[4] $e''(\theta) = 0$

Then

$$\begin{aligned} 0 &= e''(\theta) = f''(\theta) - q''(\theta) \\ &= f''(\theta) - p_1''(\theta) - \lambda \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}(\theta - x_0)(\theta - x_1) \\ &= f''(\theta) - 2\lambda \end{aligned}$$

hence $\lambda = \frac{1}{2} f''(\theta)$

[4]Note that θ is a function of $\hat{x}$

Polynomial Interpolation Error

Hence, we get

$$f(\hat{x}) - p_1(\hat{x}) = \lambda(\hat{x} - x_0)(\hat{x} - x_1) = \frac{1}{2} f''(\theta)(\hat{x} - x_0)(\hat{x} - x_1)$$

for any $\hat{x} \in (x_0, x_1)$ (recall that $\hat{x}$ was chosen arbitrarily)

This argument can be generalized to $n > 1$ to give

$$\boxed{f(x) - p_n(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0)\ldots(x - x_n) \text{ for some } \theta \in (a, b)}$$

Polynomial Interpolation Error

For any $x \in [a, b]$, this theorem gives us the error bound

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!} \max_{x \in [a,b]} |(x - x_0) \ldots (x - x_n)|,$$

where $M_{n+1} = \max_{\theta \in [a,b]} |f^{n+1}(\theta)|$

If $1/(n+1)! \to 0$ faster than

$$M_{n+1} \max_{x \in [a,b]} |(x - x_0) \ldots (x - x_n)| \to \infty$$

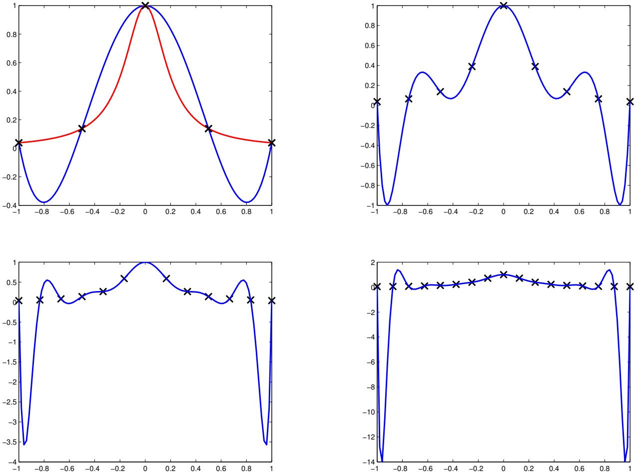
then $p_n \to f$

Unfortunately, this is not always the case!

Runge's Phenomenon

A famous pathological example of the difficulty of interpolation at equally spaced points is Runge's Phenomenon

Consider $f(x) = 1/(1 + 25x^2)$ for $x \in [-1, 1]$



Runge's Phenomenon

Note that of course p_n fits the evenly spaced samples exactly

But we are now also interested in the maximum error between f and its polynomial interpolant p_n

That is, we want $\max_{x\in[-1,1]} |f(x) - p_n(x)|$ to be small!

This is generally referred to as the "infinity norm" or the "max norm":

$$\|f - p_n\|_\infty \equiv \max_{x\in[-1,1]} |f(x) - p_n(x)|$$

Runge's Phenomenon

Interpolating Runge's function at evenly spaced points leads to exponential growth of infinity norm error!

We would like to construct an interpolant of f such that this kind of pathological behavior is impossible

Minimizing Interpolation Error

To do this, we recall our error equation

$$f(x) - p_n(x) = \frac{f^{n+1}(\theta)}{(n+1)!}(x - x_0)\ldots(x - x_n)$$

We focus our attention on the polynomial $(x - x_0)\ldots(x - x_n)$, since we can choose the interpolation points

Intuitively, we should choose $x_0, x_1, \ldots, x_n$ such that $\|(x - x_0)\ldots(x - x_n)\|_\infty$ is as small as possible

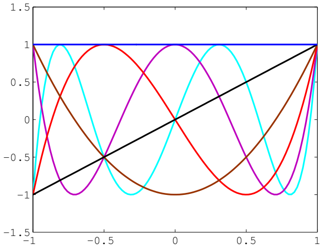
Interpolation at Chebyshev Points

Result from Approximation Theory:

For $x \in [-1, 1]$, the minimum value of $\|(x - x_0) \ldots (x - x_n)\|_\infty$ is $1/2^n$, achieved by the polynomial $T_{n+1}(x)/2^n$

$T_{n+1}(x)$ is the Chebyshev poly. (of the first kind) of order $n + 1$ (T_{n+1} has leading coefficient of 2^n, hence $T_{n+1}(x)/2^n$ is monic)

Chebyshev polys "equi-oscillate" between -1 and 1, hence it's not surprising that they are related to the minimum infinity norm



Interpolation at Chebyshev Points

Chebyshev polynomials are defined for $x \in [-1, 1]$ by
$T_n(x) = \cos(n \cos^{-1} x), n = 0, 1, 2, \ldots$

Or equivalently[5], the recurrence relation,
$T_0(x) = 1$,
$T_1(x) = x$,
$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, $n = 1, 2, 3, \ldots$

To set $(x - x_0) \ldots (x - x_n) = T_{n+1}(x)/2^n$, we choose interpolation points to be the roots of T_{n+1}

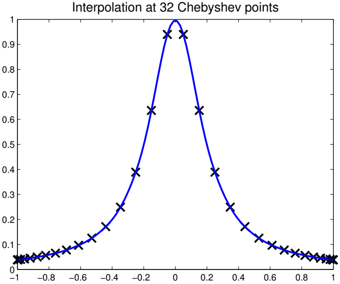
Exercise: Show that the roots of T_n are given by
$x_j = \cos((2j - 1)\pi/2n)$, $j = 1, \ldots, n$

[5]Equivalence can be shown using trig. identities for T_{n+1} and T_{n-1}

Interpolation at Chebyshev Points

We can combine these results to derive an error bound for interpolation at "Chebyshev points"

Generally speaking, with Chebyshev interpolation, p_n converges to any smooth f very rapidly! e.g. Runge function:



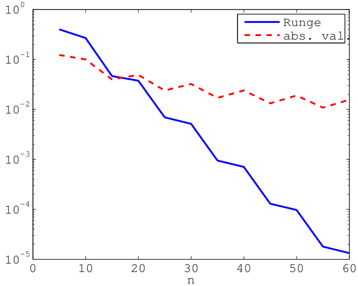
If we want to interpolate on an arbitrary interval, we can map Chebyshev points from $[-1, 1]$ to $[a, b]$

Interpolation at Chebyshev Points

Note that convergence rates depend on smoothness of f—precise statements about this can be made, outside the scope of AM205

In general, smoother $f \implies$ faster convergence[6]

e.g. compare convergence of Chebyshev interpolation of Runge's function (smooth) and $f(x) = |x|$ (not smooth)



[6] For example, if f is analytic, we get exponential convergence!

Another View on Interpolation Accuracy

We have seen that the interpolation points we choose have an enormous effect on how well our interpolant approximates f

The choice of Chebyshev interpolation points was motivated by our interpolation error formula for $f(x) - p_n(x)$

But this formula depends on f — we would prefer to have a measure of interpolation accuracy that is independent of f

This would provide a more general way to compare the quality of interpolation points... This is provided by the Lebesgue constant

Lebesgue Constant

Let $\mathcal{X}$ denote a set of interpolation points,
$\mathcal{X} \equiv \{x_0, x_1, \ldots, x_n\} \subset [a, b]$

A fundamental property of $\mathcal{X}$ is its Lebesgue constant, $\Lambda_n(\mathcal{X})$,

$$\Lambda_n(\mathcal{X}) = \max_{x \in [a,b]} \sum_{k=0}^{n} |L_k(x)|$$

The $L_k \in \mathbb{P}_n$ are the Lagrange polynomials associated with $\mathcal{X}$, hence Λ_n is also a function of $\mathcal{X}$

$\Lambda_n(\mathcal{X}) \geq 1$, why?

Lebesgue Constant

Think of polynomial interpolation as a map, $\mathcal{I}_n$, where
$\mathcal{I}_n : C[a,b] \to \mathbb{P}_n[a,b]$

$\mathcal{I}_n(f)$ is the degree n polynomial interpolant of $f \in C[a,b]$ at the interpolation points $\mathcal{X}$

Exercise: Convince yourself that $\mathcal{I}_n$ is linear (e.g. use the Lagrange interpolation formula)

The reason that the Lebesgue constant is interesting is because it bounds the "operator norm" of $\mathcal{I}_n$:

$$\sup_{f \in C[a,b]} \frac{\|\mathcal{I}_n(f)\|_\infty}{\|f\|_\infty} \leq \Lambda_n(\mathcal{X})$$

Lebesgue Constant

Proof:

$$
\begin{aligned}
\|\mathcal{I}_n(f)\|_\infty &= \|\sum_{k=0}^{n} f(x_k)L_k\|_\infty = \max_{x\in[a,b]}\left|\sum_{k=0}^{n} f(x_k)L_k(x)\right| \\
&\le \max_{x\in[a,b]}\sum_{k=0}^{n}|f(x_k)||L_k(x)| \\
&\le \left(\max_{k=0,1,\ldots,n}|f(x_k)|\right)\max_{x\in[a,b]}\sum_{k=0}^{n}|L_k(x)| \\
&\le \|f\|_\infty \max_{x\in[a,b]}\sum_{k=0}^{n}|L_k(x)| \\
&= \|f\|_\infty \Lambda_n(\mathcal{X})
\end{aligned}
$$

Hence

$$
\frac{\|\mathcal{I}_n(f)\|_\infty}{\|f\|_\infty} \le \Lambda_n(\mathcal{X}), \quad \text{so} \quad \sup_{f\in C[a,b]}\frac{\|\mathcal{I}_n(f)\|_\infty}{\|f\|_\infty} \le \Lambda_n(\mathcal{X}).
$$