AM205: Take-home midterm exam

This exam was posted at 5 PM on November 13th. Answers are due at 5 PM on November 14th. They can either be submitted via the iSites dropbox, or in person to Pierce Hall 305.

For queries, contact the teaching staff using a private message on Piazza. Any clarifications will be posted on Piazza.

The exam is open book—any class notes, books, or online resources can be used. The exam must be completed by yourself and no collaboration with classmates or others is allowed.

The exam will be graded out of forty points. Point values for each question are given in square brackets.

1. (a) **Modified Simpson's rule [7].** Determine a formula for the integral $\int_a^b f(x)dx$ of a smooth function f using a variation on Newton–Cotes quadrature with control points at a, $(3a+b)/4$, and b.

   (b) Suppose the formula from part (a) is used as part of a composite integration rule where $[a, b]$ is divided into N subintervals of equal length h. Following similar steps to the composite trapezoid rule analysis discussed in the lectures, find a bound[1] for the absolute integration error in terms of h, a, b, and f'''.

2. **A class of Runge–Kutta methods [12].** Consider the Butcher tableau shown below for a three-step Runge–Kutta method to solve the differential equation $y' = f(t, y)$ for a scalar function $y(t)$.

$$\begin{array}{c|ccc} 0 & & & \\ \beta & \beta & & \\ \gamma & 0 & \gamma & \\ \hline & 0 & 0 & 1 \end{array}$$

   (a) Show that for the method to be second-order accurate, γ must take a specific value, but there is no restriction on β. Determine the specific value of γ.

   (b) Consider the restricted class of differential equations where $f(t, y) = pt + qy$ for $p, q \in \mathbb{R}$. Find the specific value of β that makes the method third-order accurate.[2]

   (c) Consider the boundary value problem

$$y'' + 16(y')^4 = 2 + 2t, \qquad y(0) = 0, \qquad y(1) = 0 \tag{1}$$

   for the scalar function $y(t)$ on the interval $t \in [0, 1]$. Solve this by reformulating it as an initial value problem,

$$y'' + 16(y')^4 = 2 + 2t, \qquad y(0) = 0, \qquad y'(0) = g, \tag{2}$$

   where g is an unknown constant to be determined. Write a program using the Runge–Kutta scheme and the specific values of β and γ from parts (a) and (b)[3] to solve Eq. 2 over

[1] For full credit, your bound should be as strong as that considered in the analysis of the trapezoid rule.

[2] For this restricted class of differential equations, the second derivatives f_{tt}, f_{ty}, and f_{yy} all vanish, which greatly simplifies the algebra.

[3] If you are unable to calculate β and γ in parts (a) and (b), you can alternatively solve this part using Ralston's method, discussed in lecture 13.

the interval $t \in [0, 1]$ using a step size of $h = \frac{1}{1000}$. Show that $y(1) < 0$ if $g = -0.6044$ and $y(1) > 0$ if $g = 1$. Hence, write a bisection search algorithm to determine a value of g accurate to at least six decimal places such that $y(1) = 0$. For this value of g, plot y and y' over the interval $t \in [0, 1]$.

3. (a) **Differentiation on unequal grids [13].** Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, and let b and c be non-zero constants. Using only three evaluations of f at $x, x+b, x-c$ determine a general second-order accurate formula[4] for $f'(x)$.

   (b) Consider the unequally-spaced grid

$$x_k = (1+\alpha)^k, \tag{3}$$

   for the range $k = -N, -N+1, \ldots, N$ where α is a positive non-zero constant, and the "k" superscript represents taking a power. Let $f_k = f(x_k)$ be the discretized function values. Write a program that numerically calculates f' on this grid, using the second-order accurate formula from part (a). For $|k| \neq N$, use the three points x_{k-1}, x_k, and x_{k+1}. For $k = N$, use the three points x_N, x_{N-1}, and x_{N-2}. For $k = -N$, use x_{-N}, x_{-N+1}, x_{-N+2}.

   (c) Test the program using $f(x) = \sin 3x$ for $N = 10, 20, 40, 80, 160, 320, 640$, where α is chosen so that $x_N = 3$ and $x_{-N} = \frac{1}{3}$. Make a log–log plot of absolute infinity-norm[5] error $||f'_{\text{exact}} - f'_{\text{numerical}}||_\infty$ as a function of α, and use the plot to show that the numerically computed derivative is second-order accurate in α.

   (d) An alternative way to obtain a second-order accurate derivative is to introduce a uniform grid y_k such that $y_k = \log x_k$, which has the fixed spacing $h = \log(1+\alpha)$. Consider an arbitrary function $F(y)$ with discretized values $F_k = F(y_k)$, and write down the standard second-order centered-difference formula for $F'(y_k)$ at $k = 0$. Now, by considering the relationship

$$F(y) = f(e^y) \tag{4}$$

   and making use of the chain rule, obtain a second-order accurate formula for $f'(x_k)$ at $k = 0$.

   (e) By direct calculation, show that the difference between the formulae for $f'(x_k)$ at $k = 0$ from parts (b) and (d) is $O(\alpha^2)$.

4. **Determining a hidden charge distribution [8].** In non-dimensionalized units, the electric potential for a single point charge q in three dimensions is

$$\phi(r) = \frac{q}{4\pi r}, \tag{5}$$

   where r is the distance from the charge. If many point charges are present, their potentials are added together. The electric field is given by $\mathbf{E} = -\nabla\phi$.

   Consider an enclosed box covering $|x| \leq 1$, $|y| \leq 1$ in a two-dimensional plane. The box contains 25 point charges[6] of size $q_{i,j}$ located on a grid at positions $(x, y) = (\frac{2i-4}{5}, \frac{2j-4}{5})$ for

[4] In this context, second-order accuracy means that the error in the formula is $O(b^2)$ as $b \to 0$ keeping the ratio b/c constant. Equivalently, the formula is $O(c^2)$ as $c \to 0$ keeping c/b constant.

[5] You should evaluate the infinity norm numerically using the function values at all of the x_k.

[6] Even though the charges and the box live in the two-dimensional xy-plane, you should assume that this plane is a cross-section through three-dimensional space, and hence the electric potential for each charge will be given by Eq. 5.

$i = 0, 1, 2, 3, 4$ and $j = 0, 1, 2, 3, 4$. The charges cannot be directly measured. However, a text file `efield.txt` is provided on the course website that contains a large number of measurements of the electric field at positions outside the box.

(a) By using any means necessary, determine the charges $q_{i,j}$.

(b) You should find that to within numerical error, the $q_{i,j}$ are integers in the range from 0 to 31, which can be interpreted as five-digit binary numbers. For each of the five binary digits corresponding to 1, 2, 4, 8, 16, plot the 5×5 grid of charges, coloring in the charge if the binary digit is 1 and leaving it blank otherwise.