Graph Theory

MATH 404 Koç University
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MATH 404

1) Assume you have 10 people in a room. Max. how many acquiantee can you have among these people if there are no group of 3 people who pairwise know each other?

2) Königsberg Bridge Problem
can you find a trail which travels each bridge exactly once and comes back to the original position?

DEF: A graph is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$ and a relation that associates with each edge to two vertices (not necessarily distinct) called its end points.

DEF

iki nokta arasında birden çok kenar olması

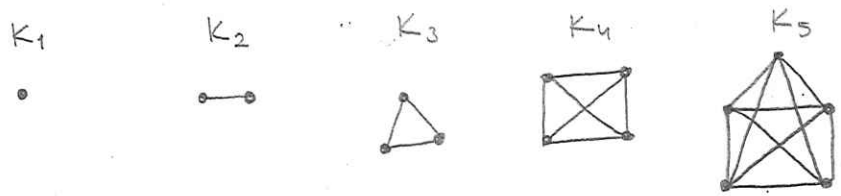
Multiple edges are edges having the same pair of endpoints.

A simple graph is a graph with no loops or multiple edges

$e = uv \Rightarrow$ vertices u and v are adjocent.

- A graph is finite if its vertex set and edge set are finite. (if empty : null)
- APPLICATIONS: Job assignment, max flow

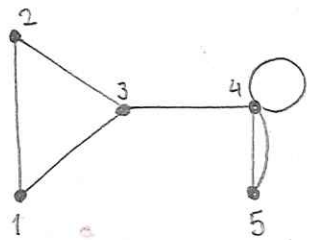
DEF: The complete graph on n vertices, K_n is the graph in which all pairs of vertices are adjocent.



# of edges of K_n is $\binom{n}{2} = \frac{n(n-1)}{2}$

bağlı

DEF: $d_G(i)$ is the number of edges incident with vertex i in G and is called the degree of i (each loop counting as two edges)



$d_G(5) = 2$
$d_G(4) = 5$
$d_G(3) = 3$

THEOREM If $G = (V, E)$ is a graph then $\sum_{v \in V} d(v) = 2|E|$

PROOF

Summing the degrees count each edge twice (one for each endpoint) So

$$\sum d(v) = 2|E|$$

Corollary

Every graph has an even number of vertices of odd degree.

DEF: $\Delta(G) = \max_{v \in V} \deg(v)$

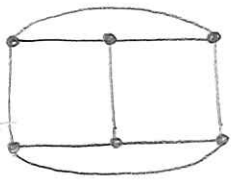
$\delta(G) = \min_{v \in V} \deg(v)$

<u>corollary</u>

$$\delta(G) \le \frac{2|E|}{n} \le \Delta(G)$$

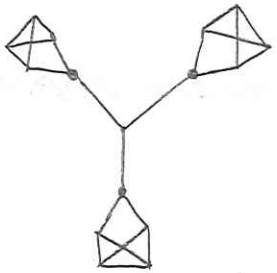
DEF: A graph is k-regular if degree of each vertex is k,

$\delta(G) = \Delta(G) = k$



3-regular

2-regular

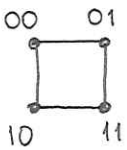


3-regular

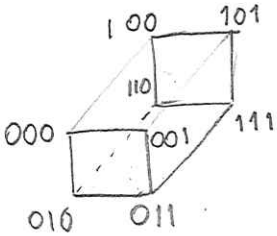
● counter-example of many things

DEF: The k-dimensional cube or hybercube Q_k is the simple graph whose vertices are the k-tuples with entries $\{0,1\}$ and whose edges are the pairs of k-tuples that differ in exactly one position.

Q_1



Q_2



Q_3

■ # of vertices $= 2^k$

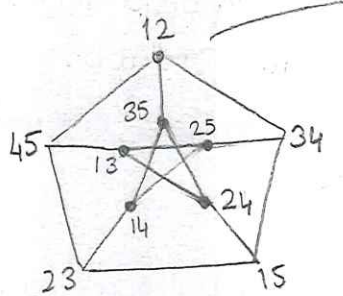
# of edges $= \frac{2^k \cdot k}{2}$

DEF: The Peterson graph is the simple graph whose vertices are the 2-element subsets of a 5 element set and whose edges are the pairs of disjoint 2-element subsets.

$V = \{ \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\} \}$

There are $\binom{5}{2} = 10$ vertices of the Peterson graph.

ab — cd where they are all different.

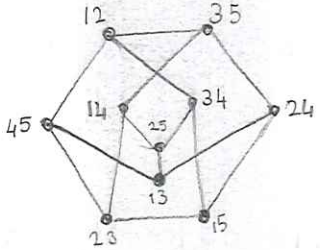


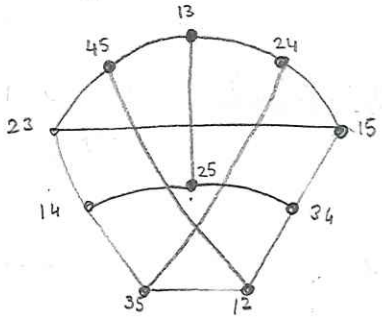
Peterson graph

3-regular

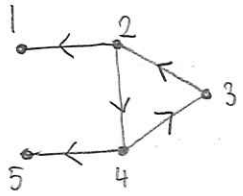
$\frac{3 \times 10}{2} = 15$ edges

HW Label these drawings.





DEF: A directed graph $G=(V,E)$ has ordered pairs as its edges.



$d^+(i)$ = in degree of i

$d^-(i)$ = out degree of i

$d^+(4) = 1$

$d^-(4) = 2$

DEF: A walk is a sequence $v_0 e_0 v_1 e_1 v_2 e_2 \cdots v_{L-1} e_{L-1}$ where $v_i \in V(G)$ and $e_i = v_i v_{i+1}$ for $0 \le i \le L-1$

This walk is defined to have length L. $(v_0\, v_L)$-walk

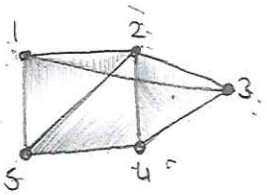
If no edge is repeated then the walk is called a trail.

If no vertex is repeated then the walk is called path

If G is simple then we can represent the walk $v_0 v_1 \cdots v_L$.

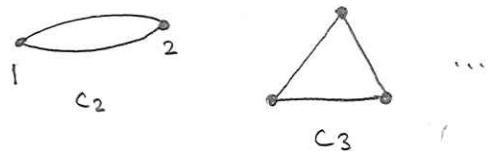
If the starting vertex and end vertex of a walk is the same then the walk is called closed.

If only the starting vertex is repeated on a closed walk then it is called a cycle.

▶ 

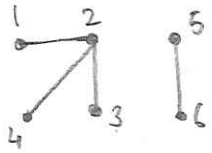
1234215	not a trail, not a path
1243325	trail, not a path
13254	trail, path
1231	cycle
2152432	closed walk, no cycle.

2 cycles

▶ 

C_n is the cycle with n vertices

DEF: The distance $d_G(u,v)$ between vertices u and v is the length of a shortest (u,v)-path in G, if such a walk exists and ∞ otherwise.

▶ G: 

$d(1,3) = 2$

$d(1,4) = 2$

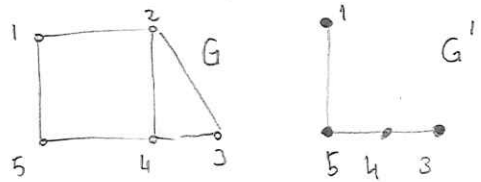
$d(1,5) = \infty$

$d(1,2) = 1$

DEF: The graph G is connected if $d_G(u,v) < \infty$ for all $u,v \in V(G)$.

DEF: The diameter of a connected graph G is defined to be $\max_{u,v \in V} d(u,v)$.

DEF: A subgraph of a graph G is a graph each of whose vertices belong to $V(G)$ and each of whose edges belong to $E(G)$



$G' \subseteq G$

DEF: A component in G is a maximal connected subgraph of G. A component is trivial if it has no edges. An isolated vertex is a vertex of degree 0.

G has 3 components $\{1,2,3,4\}, \{5,6\}, \{7\}$

Proposition Every graph with n vertices and k edges has at least n-k components.

PROOF:

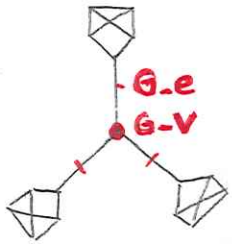
We may assume $n-k>0$ otherwise it is trivial.

An n_vertex group with no edges has n components. Each edge added reduces this by <u>at most one</u>, so when k edges added the number of components is <u>at least n-k</u>.

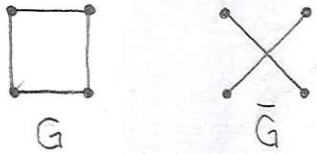
DEF: If e is an edge of G, G-e is the graph obtained from G by deleting the edge e. In general if $F \leq E(G)$ the Graph G-F is obtained by deleting the edges of F.

DEF: If v is a vertex of G we denote by G-v The graph obtained from G by deleting the vertex v and all edges incident with v, in general is $S \leq V(G)$, G-S is the graph obtained by deleting the vertex in S and all edges incident with any of them.

DEF: A cut_edge or cut_vertex of a graph is an edge or vertex whose deletion increases the number of components.

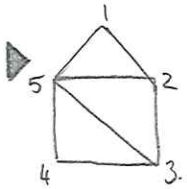


DEF: The complement of a graph $G=(V,E)$ is the graph $\bar{G}=(V,\bar{E})$ where $\{u,v\} \in \bar{E}$ iff $\{u,v\} \notin E$

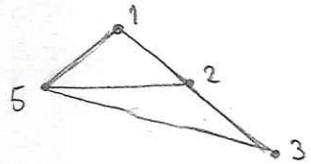


DEF: Let $G=(V,E)$ be a graph. The graph induced by $V'=\{v_1,\dots,v_k\} \subseteq V$ is the graph $G'=V',E'$ where $E'=\{\{u,v\} \in E : u,v \in V'\}$

$[V']$ represents the graph induced by V'



The graph induced by $\{1,5,2,3\}$



Fact: $[V'] = G - \overline{V'}$

$\overline{V'} = V(G) - V'$

THEOREM An edge is a cut-edge if and only if it belongs to no cycles.

PROOF

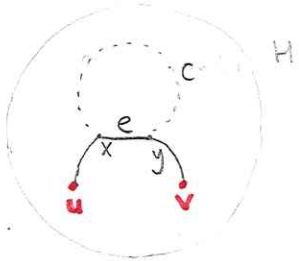
Let $e = xy$ be an edge in G and let H be the component containing e.

Suppose $H-e$ is connected. This implies that $H-e$ contains an x,y-path, and this path together with e completes a cycle.

Now suppose that e lies in a cycle C. Choose $u,v \in V(H)$. Since H is connected then H has a u,v-path P. If $e \notin P$ then $P \leq H-e$

Now assume $e \in P$.

WLOG assume x comes in the path before y.

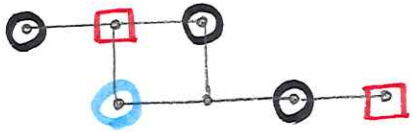


Now $H-e$ contains a v,x-path along P, an x,y path along C and y,v-path along P. Then there is a u-v-walk. By HW, every u,v-walk contains a u,v-path so $H-e$ is connected.

HW Every u-v-walk contains a u-v-path.

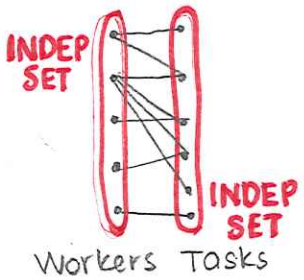
BIPARTITE GRAPHS

DEF: An independent set in a graph is a set of pairwise non-adjocent vertices.

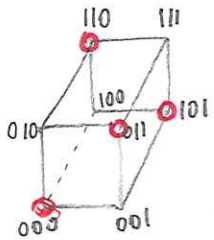


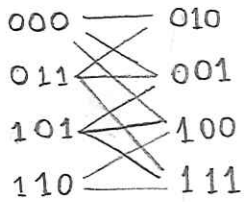
DEF: A graph G is bipartite if $V(G)$ is the union of two disjoint independent sets called partite sets of G.

▶



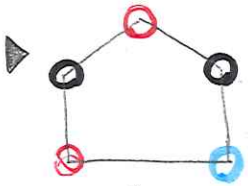
▶ Q_k is a bipartite graph.



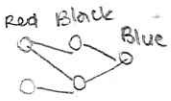


DEF: The chromatic number of a graph G, written $\chi(G)$ is the minimum number of colors needed to label the vertices so that adjocent vertices receive different colors.

▶ All bipertite graph have chromatic number 2. → related to independent sets



C_5



$\chi(C_5) = 3$

▶

$\chi(C_6) = 2$ → All even cycles have chromatic num is 2.

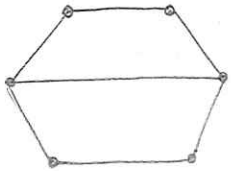
DEF: A graph is k-partite if $V(G)$ can be expressed by the union of k-independent sets.
↳ disjoint

Fact: Each color class in a proper coloring of G is an independent set, so if G is k-partite then $\chi(G) \leq k$.

▶

3-partite graph.

DEF: The girth of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite

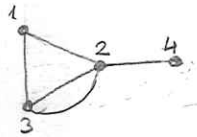


G has girth 4.

ISOMORPHISM IN GRAPHS

DEF: Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of a graph G, the adjocency matrix of G, $A(G)$ is the $n \times n$ matrix in which a_{ij} is the number of copies of the edge $\{v_i, v_j\}$ in G.
loopless

▶ G



$$\begin{array}{c|cccc} & v_1 & v_2 & & \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_2 & 1 & 0 & 2 & 1 \\ v_3 & 1 & 2 & 0 & 0 \\ v_4 & 0 & 1 & 0 & 0 \end{array}$$

Adjocency matrices are symmetric.

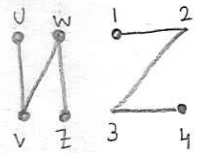
DEF: $M(G)$ is the $n \times m$ matrix in which entry m_{ij} is 1 if v_i is endpoint of e_i and otherwise 0 for a n-vertex graph G with m edges.

$$\begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 1 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

DEF: An isomorphism from a simple graph H is a bijection $f: V(G) \Rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$

$G \cong H$

▶

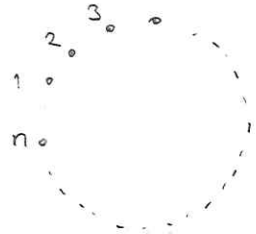


u	1
v	2
w	3
z	4

u	4
v	3
w	2
z	1

TEST bipartite / length of cycles /

DEF: The order of a graph G is the number of vertices $n(G)$
The size of the graph G, $e(G)$ is the # of edges.

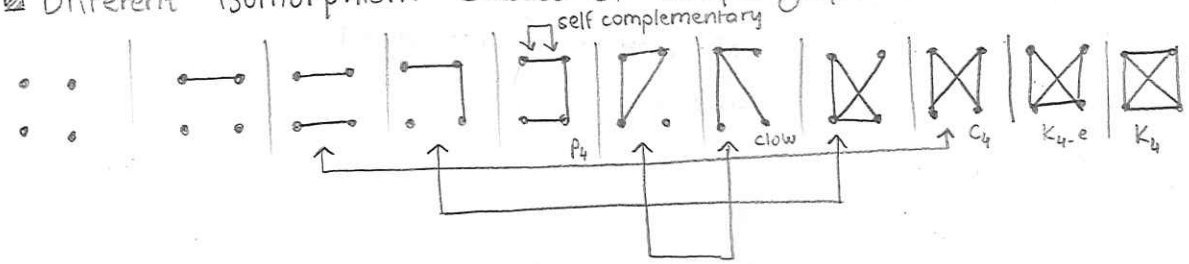


$\binom{n}{2}$ possible edges

$2^{\binom{n}{2}}$ different labelled graphs on n vertices.

There are $2^{\binom{4}{2}} = 64$ different graphs on 4 vertices.

▨ Different isomorphism classes of simple graphs of order 4



11 different isomorphism classes.

DEF: A graph is self-complementary if it is isomorphic to its complement.

▶

P_4

C_5

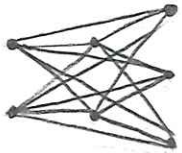
DEF: Isomorphism is an equivalence relation on graphs and an isomorphism class of graphs is an equivalence class of graphs under this relation.

DEF: P_n represents a path with n vertices.

DEF: A complete multipartite graph of type $K_{n_1 n_2 n_3 \dots n_x}$ is a x-partite graph with parts of size $n_1, \dots, n_x$ where any two vertices in different parts are adjocent.

▶ $K_{2,3}$

$K_{2,2,3}$



$$e\left(K_{n_1, \dots, n_x}\right) = \sum_{1 \le j < k \le x} n_j n_k$$

DEF: An automorphism of G is an isomorphism from G to G. A graph is vertex transitive if for every pair $u, v \in V(G)$ there is automorphism that maps u to v.

DEF: A graph is edge-transitive if for all $e, f \in E(G)$ there is an automorphism of G that maps the end points of e to the end points of f.

C_n

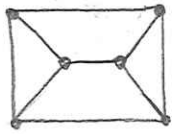
BOTH vertex and edge transitive

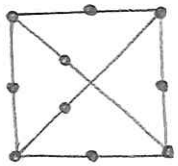
vertex and edge transitive

Peterson graph is vertex transitive.

REMARK: If G is vertex-transitive then G must be regular.

-
  edge-transitive
  not vertex-transitive

- 
  vertex-transitive
  not edge-transitive.

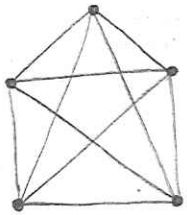
- 
  not vertex_transitive → degrees
  edge-transitive

REMARK: Vertex transitive graphs looks the same from every vertex, edge_transitive graphs looks the same from every edge.

GRAPH DECOMPOSITIONS

DEF: A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.

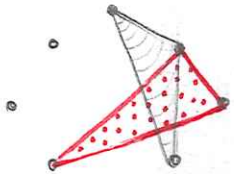
- C_5 decomposition of K_5



FACT

An n_vertex graph is self-complementary
$\Leftrightarrow$ K_n has a decomposition consisting of two copies of H.

- K_3 decomposition of K_7



$7 \times 3 = 21$
$\binom{7}{2} = 21$

K_7

DECOMPOSITION TEST (NECESSARY CONDITIONS)

- $\binom{a}{2}$ must be divided by b.
  a for bigger.

- Degree of bigger graph should be divided by degree of smaller.

FACT

If H decomposes G then
$e(H) \mid e(G)$

DEFINITION A clique in a graph G is a set of pairwise adjocent vertices.

BIPARTITE GRAPHS

Lemma: Every closed odd walk contains an odd cycle.

PROOF Induction

Basis Step $\ell = 1$ a loop also a cycle of length 1.

Induction step (Let $\ell > 1$) Assume every closed walk of length $r < \ell$ contains an odd cycle. If it has repeated vertex v, then W can be break into 2 closed walks of length less than ℓ. As the total # of edges is odd in W one of the smaller walks has an odd cycle. This cycle appears in W so W also containing an odd cycle

THEO REM (König)

A graph is bipartite if and only if it has no odd cycle.

PROOF

($\Rightarrow$) Let G be a bipartite graph. Every walk alternates between two sets of bipartitions, so every return to the original partite set happens after an even # of steps. Hence G has no odd cycle.

($\Leftarrow$) Let G be a graph with no odd cycle. We will construct a bipartition for each component.

Let u be the vertex in a non-trivial component H. Let $X = \{v \in V(H) : d(u,v) \text{ is even}\}$ and $Y = \{v \in V(H) : d(u,v) \text{ is odd}\}$ Furthermore any edge v_1v_2 with X or Y would create a closed walk with the shortest uv_1-path and uv_2-path. By the previous lemma such a walk should contain an odd cycle

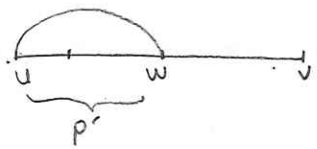
EULERIAN CIRCUITS $\longrightarrow$ start from any vertex.

DEF: A graph is Eulerian if it has a closed trail containing all edges

DEF: An even graph is a graph with vertex degree all even.

DEF: A maximal path in a graph G is a path P in G that is not contained in a longer path.

LEMMA If every vertex of a graph G has degree at least 2, then G contains a cycle.



Let P be a maximal path in G with end vertices u and v. As P cannot be extended, the second neighbor of u must be a vertex in P, say w. Then the path P' from u to w on P forms a cycle with the edge uw.

THEO REM A graph G is Eulerian $\Leftrightarrow$ it has at most one non-trivial component and its vertices all have even degree.

PROOF

($\Rightarrow$) Each time a Eulerian Circuit visits a vertex it uses up two edges and the first edge is paired up with the last edge so each vertex has even degree.

($\Leftarrow$) Use induction on the number of edges

<u>Basis</u> $m=0$. No edges to consider.

<u>Induction</u> $m>0$ with even degrees each vertex has degree at least 2 so the graph contains a cycle C by the above lemma. Let $G' = G.C$. G' is also even and have strictly less edges than G. So by induction hypothesis each component of G' contains on Eulerian Circuit.

G

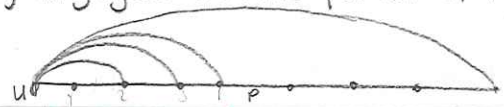
To find the circuit in G, we start by a vertex of C until we come to <u>a component of G'</u> for the first time we travel on the Eulerian tour of that component, this circuit ends at the vertex on C where we began the tour. When we complete the treversal of C, we have completed an Eulerian circuit in G.

Proposition

If G is a simple graph in which every vertex has degree at least k then G contains a path of at least k. If $k>2$, then G contains a cycle of length at least $k+1$.

<u>PROOF</u>

Let u be an endpoint of a maximal path P in G. Since P does not extend every neighbor of u is in $V(P)$. Since u has at least k neighbors and G is simple, P has at least k vertices other than u and has length at least k. If $k>2$, then the edge from u to its farest neighbor v along P completes a sufficiently long cycle with the portion of P from v to u.



EXTERNAL PROBLEMS

Proposition The minimum number of edges in a connected graph with n vertices is $(n-1)$

<u>PROOF</u>

HW

DEF: $N_G(u) = \{v \mid u \text{ is adjocent to } v \text{ in } G\}$

Proposition

If G is simple n-vertex graph with

$$\delta(G) \geq \frac{n-1}{2}$$

then G is connected.

<u>PROOF</u>

We will show that $\forall u,v \in V(G)$ if u and v are not adjocent then $|N_G(u) \cap N_G(v)| \geq 1$. So that if two vertices are not adjocent then they have a common neighboor.

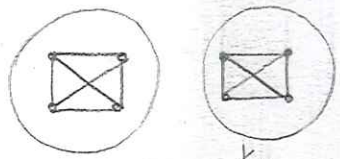
Since G is simple $|N_G(u)| \geq \delta(G) \geq \frac{n-1}{2}$

if u and v are not adjocent then

$$|N_G(u) \cup N_G(v)| \leq n-2$$

$$|N_G(u) \cap N_G(v)| = |N_G(u)| + |N_G(v)| - |N_G(u) \cup N_G(v)| \geq \frac{n-1}{2} + \frac{n-1}{2} - (n-2)$$

$$= 1$$

▶ 

K_4 K_4

$$\frac{n-1}{2} = \frac{8-1}{2} = \frac{7}{2}$$

$\delta(G) = 3$ $\quad G = K_4 + K_4$

$= 2K_4$

DEF: The graph obtained by taking the disjoint union of graph G and H is the disjoint union or sum, written by G+H. In general mG is the graph consisting of m pairwise disjoint copies of G.

THEOREM Every loopless graph G has a bipartite subgraph with at least $e(G)/2$ edges.

PROOF

We start by a random partition of $V(G)$ into sets X and Y when we delete the edges inside each part we have a bipartite subgraph H with bipartition X, Y. If H contains fewer than half the edges of G incident to a vertex v, then v has more edges to vertices in its own class than in the other class. Moving v to the other class gains more edges of G than it loses.

Switch v to the other class iteratively as long as the current subgraph captures less than half of the edges at some vertex. When this process terminates we will have

$$d_H(v) \geq \frac{d_G(v)}{2} \quad \text{so} \quad \sum_{v \in V} d_H(v) \geq \sum_{v \in V} \frac{d_G(v)}{2}$$

$$\text{so } 2e(H) \geq e(G) \quad \text{so } e(H) \geq \frac{e(G)}{2}$$

NOTE

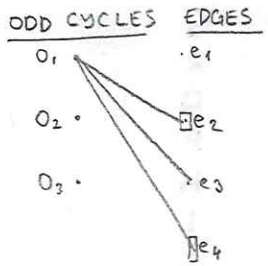
Observe that at each item then the number of edges in the new formed subgraph H strictly increases so we are sure that the process terminates.

REMARK

This process does not always forms the bipartite subgraph with most number of edges.

- $X = \{a, b, g, h\}$, $Y = \{c, d, e, f\}$ gives a 4-regular graph

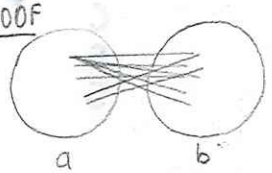
TELEGRAM



DEF: A graph is H-free if G has no induced subgraphs isomorphic to H.

THEOREM The maximum number of edges in a n-vertex triang-free graph is $\lfloor \frac{n^2}{4} \rfloor$

PROOF



$a+b=n$

$\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$

$G = K_{\lfloor \frac{n}{2} \rfloor / \lceil \frac{n}{2} \rceil}$ has $\lfloor \frac{n^2}{4} \rfloor$ edges. So such graph exists we need to show that it is the best possible.

Let G be an n-vertex triangle-free simple graph. Let $d_G(x) = D(G)$ Since G has no triangles there are no edges among neighborhood of G.

We sum up the degrees of X and vertices that are not adjocent to x, we are counting every edge at least once So $\sum_{v \in N(x)} d(v) \geq e(G)$

We sum up over $n - \Delta(G)$ vertices, each having degree at most $\Delta(G)$.

So $e(G) \leq (n - \Delta(G))\Delta(G)$

$(n - \Delta(G))\Delta(G)$ is maximized when $\Delta(G)$ is $\lceil \frac{n}{2} \rceil$ or $\lfloor \frac{n}{2} \rfloor$ Thus $e(G) \leq \lfloor \frac{n^2}{4} \rfloor$

GRAPHIC SEQUENCES

DEF: The degree sequence of a graph is the list of vertex degrees, usually written in non-increasing order $d_1 \geq d_2 \geq \dots \geq d_n$

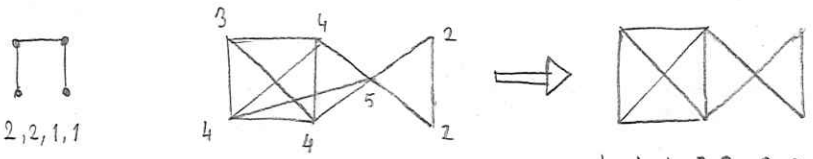
Proposition: The non-negative integers $d_1, d_2, \dots, d_n$ are the vertex degree of some graph if and only if $\sum d_i$ is even.

PROOF

We know that $\sum d_i = 2e(G)$ for any graph G so $\sum d_i$ being even is necessary.

Now suppose $\sum d_i$ is even. Then the number of odd vertices is even. We will construct a graph with $d(v_i) = d_i$. First pair up the vertices which needs to have odd degree. Put an edge between each pair. Now the remaining number of edges needed at each vertex is even. Put remaining degree loops at each vertex to get a graph with required degree distribution.

DEF: A graphic sequence is a list of non-negative numbers that is the degree sequence of some simple graph. A simple graph with degree sequence d "realizes" d.

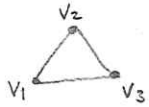
▶ 

▶ Start from a graphic sequence and built up.

☑ Not all sequences satisfying the obvious necessary conditions for graphic.

▶ $\begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\ 5 & 5 & 5 & 4 & 2 & 1 & 1 & 1 \end{array}$

The induced subgraph formed with $\{v_1, v_2, v_3\}$ has to be a triangle as total degree is 15 and $v_4 v_5 v_6 v_7 v_8$ has total degree 15. so 3 edges suppose to have both end points in $\{v_1, v_2, v_3\}$



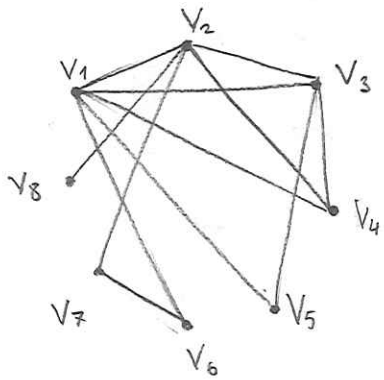
$v_4 \; v_5 \; v_6 \; v_7 \; v_8$ All the rest of the edges should have one endpoint in $\{v_1, v_2, v_3\}$ and the other endpoint in $\{v_4, v_5, \dots, v_8\}$ But v_4 has degree 4 but have 3 possible neighbors. ✗ So 5 5 5 4 2 1 1 1 is not graphic

▶ Which sequences are graphic?

THEOREM An integer list d is graphic if and only if d' is graphic where d' is obtained by deleting the largest element Δ in d and subtracting 1 from its Δ next largest elements.

▶ $\begin{array}{cccccccc} 5 & 5 & 4 & 3 & 2 & 2 & 2 & 1 \\ & 4 & 3 & 2 & 1 & 1 & 2 & 1 \\ & & 2 & 1 & 1 & 1 & 1 & 0 \\ & & & 0 & 0 & 1 & 1 & 0 \end{array}$

graphic

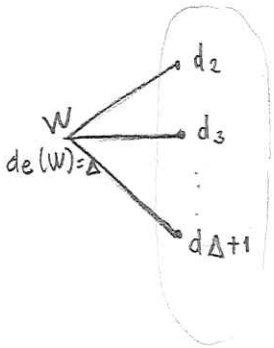


▶ $\begin{array}{cccccccc} 5 & 5 & 5 & 4 & 2 & 1 & 1 & 1 \\ & 4 & 4 & 3 & 1 & 0 & 1 & 1 \\ & & 3 & 2 & 0 & 0 & 0 & 1 \end{array}$ not graphic.

PROOF

Sufficiency: Given d with $d_1 \geq d_2 \geq \dots \geq d_n$ and a simple graph G' with degree sequence d'. We add a new vertex adjocent to vertices in G' with degrees $d_2 - 1, \dots, d_{\Delta+1} - 1$. These d_i are the Δ largest elements of d after Δ itself.

Necessity: We will start by a graph G realising d and produce a graph G' realising d'. Let w be a vertex of degree Δ in G. Let S a set of Δ vertices in G having the desired degrees $d_2, \dots, d_{\Delta+1}$. If $N_G(w) = S$, then we delete w to obtain G'. Otherwise some vertex of S is missing from $N(w)$. We will modify G to increase $|N(w) \cap S|$ without changing any vertex degree.



$S = N(w)$

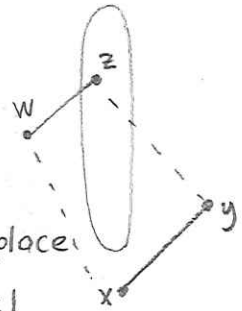
Then you are done.

continuing this process will yield a graph G^* such that $N_{G^*}(w) = S$

Choose $x \in S$ and $z \notin S$ so that w and z are adjocent ($z \in N(w)$) but w and x are not adjocent ($x \notin N(w)$)

Now since $d(x) > d(z)$

There must be a vertex y adjocent to x but not to z. Now delete $\{wz, xy\}$ and replace them with $\{wx, yz\}$ to increase $|N(w) \cap S|$



TREES

DEF (Acyclic Graph)

- A graph with no cycle is ocyclic.
- A forest is acyclic graph.
- A tree is connected acyclic graph.
- A spanning tree is a spanning subgraph that is a tree
- A leaf is a vertex of degree 1.

DEF (Spanning subgraph)

If $H \subseteq G$ and $V(H) = V(G)$, i.e. a subgraph that contains every vertex but $\boxed{\text{not necessarily each edge}}$, then H is called a spanning subgraph.

connected
$n-1$ edges
acyclic

PROPERTIES OF TREES

THEOREM For any n-vertex graph G, TFAE

(1) G is connected and has no cycles.

(2) Any two vertices of G are connected by exactly one path.

(3) G is connected and has $n-1$ edges.

(4) G has $n-1$ edges and acyclic.

PROOF

(1)→(2) Since G is connected, each pair of vertices are connected by a path. Let u and v be connected by more than one path P and Q then $(P \cup Q) = (P \cap Q)$ is a graph where degree of each vertex is even or 0. Hence $P \cup Q$ contains a cycle as the degrees of vertices in each component is at least 2.

(2)→(3) G is connected since any two vertices are connected by a path. Now suppose $xy \in E(G)$. Let P be a path between x and y that does not contain xy. This contradicts the fact that there is only one path between x and y. So $G - \{xy\}$ is disconnected. Hence every edge is a cut-edge.

INDUCTION

Basis step $(n=1)$ $|E(G)| = 0 = 1-1 = 0$

Induction step Suppose for all $r<n$, the statement is true. So graphs w/ $r<n$ vertices holding (2) has $n-1$ edges. then let $G - \{xy\}$ has two components w/ sizes $n_1 \leq n_2 \leq n$. Hence G has $n_1 - 1 + n_2 - 1 + 1 = n-1$ edges.

(3)→(4) If G has cycles, delete the edges of cycles one by one until the resulting graph G' is acyclic. Since no edges of acycle is a cut-edge G' is connected. We have proved that connected graphs w/ no cycles has $n-1$ edges. ((1)→(3)) So G' has $n-1$ edges. But G has $n-1$ edges. Hence $G = G'$, so no edge is deleted, meaning G has no cycles.

(4)→(1) Assume G is acyclic and has $n-1$ edges. We want to prove that G is connected. Let $G_1, \dots G_k$ be the components of G. Since every vertex appears in a component

$$\sum_i n(G_i) = n$$

Each component is connected and G is acyclic by ((1)→(3)), i.e

$$e(G_i) = n(G_i) - 1$$

So $e(G) = \sum_{i=1}^{k} e(G_i) = n-k$. But we know that $e(G) = n-1$. So $k=1$. Then there is only one component So G is connected.

Corollary

(i) Every edge of a tree is a cut-edge.

(ii) Adding one edge to a tree forms exactly one cycle.

(iii) Every connected graph contains a spanning tree.

Prop

If T is a tree w/ k-edges and G is a simple graph w/ $\delta(G) \geq k$, then T is a subgraph of G.

PROOF

By induction

(Induction basis) Every graph contains k.

(Induction step) ($k>0$) We assume that the claim holds for all trees fewer than k edges. Let v be a leaf $-$ T and u be its neighbor. Consider the smaller tree $T' = T - v$. G contains T' by induction hypothesis. Now because T' has only $k-1$ vertices other then u and $\deg_G(u) \geq k$. u has a neighbor x in G that is not in T'. Adding the edge $-$ expands T' be a copy of T in G.

DISTANCE

DEF: The eccentricity of a vertex written $\epsilon(u)$ is

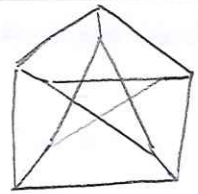
$$\max_{v \in V(G)} d_G(u,v)$$

DEF: The radius of a graph is

$$\text{rad } G = \min_{v \in V(G)} \epsilon(v)$$

Fact: $\max \epsilon(v) = \text{diam } G$.

▶ The Peterson graph has diameter 2
radius 2

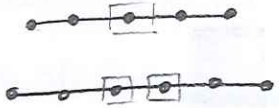


In a star, radius is 1
diameter is 2

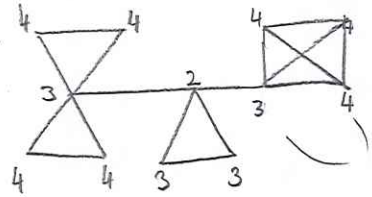
▶ Q_k has diameter k, radius k (Q_k -vertex transitive graph)
(k-cube)
↳ as $\epsilon(v) = G$ for each $v \in Q_k$

▶ $\text{diam}(P_n) = n-1$

$\text{rad}(P_n) = \left\lceil \frac{n-1}{2} \right\rceil$



▶



$\text{rad}(G) = 2$
$\text{diam}(G) = 4$

eccentricities of each vertices.

THEOREM (diam $G \geq 3 \Rightarrow$ diam $\bar{G} \leq 3$)

If G is a simple graph, then diam $G \geq 3$ implies diam $\bar{G} \leq 3$.

PROOF

When diam $G > 2$, there exists non-adjacent vertices $u, v \in V(G)$ w/ no common neighbor. Hence every $x \in V(G) - \{u,v\}$ has at least one of $\{u,v\}$ as a neighbor. This makes x adjacent in $\bar{G}$ be at least one of $\{u,v\}$ in $\bar{G}$. Since also $uv \in E(\bar{G})$ for every pair xy there is an x-y path of length at most 3 in $\bar{G}$ through $\{u,v\}$ Hence diam $\bar{G} \leq 3$.

DEF: The center of a graph G is the subgraph induced by the vertices of minimum eccentricity.

DEF: Wiener index of G $\quad D = \sum_{u,v \in V(G)} d_G(u,v)$

Fact: Average distance between two vertices in

$$G = \frac{D(G)}{\binom{n}{2}}$$

THEOREM Among trees with n vertices, the Wiener index $D(T) = \sum_{u,v} d(u,v)$ is minimized by stars and maximized by paths, both uniquely.

PROOF

Since a tree has $n-1$ edges, it has $n-1$ pairs of vertices of distance 1 and rest of the pairs has at least distance 2

So $\sum_{u,v} d(u,v) \geq n-1 + 2\left(\binom{n}{2} - (n-1)\right) = D$ (star with n vertices)

To show that no other tree achieves this, consider a leaf x in T and let v be its neighbor. If all other vertices have distance 2 from x, then they must be neighbors of v, and T is a star.

For maximization consider $D(P_n)$. This equals the sum of the distances from an end point u to the other vertices plus $D(P_{n-1})$

$D(P_{n-1})$

$$\sum_{v \in V(P_n)} d(u,v) = \sum_{i=0}^{n-1} i + D(P_{n-1}) = D(P_{n-1}) + \binom{n}{2}$$

$$\text{so } D(P_n) = \sum_{i=2}^{n} \binom{i}{2} = \binom{n+1}{3}$$

Now we will show that this maximizes $D(G)$ an n vertices

Basis step $(n=1)$ The only tree with one vertex is P

Induction Step $(n>1)$ Let u be a leaf on an n-vertex tree T.

Now $D(T) = D(T-u) + \sum_{u \in V(T)} d(u,v)$. By induction hypothesis $D(T-u) \leq D(P_{n-1})$

Thus it suffies to show that $\sum_{v \in V(T)} d(u,v)$ is maximized only when T is a path and v is an endpoint.

Consider the list of distances from u. In P_n this list is $1, 2, \dots, n-1$ all distinct. Shortest path from u to a vertex furthest from u contains vertices at all distances from u, so in any tree the set of distances from u to other vertices has no gaps. This repetition makes

$$\sum_{v \in V(T)} d(u,v)$$ smaller then when u is a leaf of a path. When T is not a path, such a repetition occurs.

1 2 3 4 5 6 7

1 2 2 3 3 4 5 smaller

THEOREM If G is connected n-vertex graph, then $D(G) \leq D(P_n)$

PROOF G will have a spanning tree H with n vertices and $D(H) \leq D(P_n)$. Also since H is a subgroup of G $D(H) \geq D(G)$ as every path in H is also a path in G.

SPANNING TREES AND NUMERATION

How many trees are there on the vertex set $\{1, 2, \dots, n\}$?

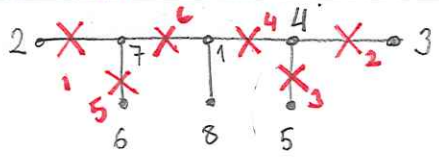
THEOREM There are exactly n^{n-2} trees on n vertices.

ALGORITHM (Prüfer Code)

Input: A tree T with vertex set $S \subseteq N$

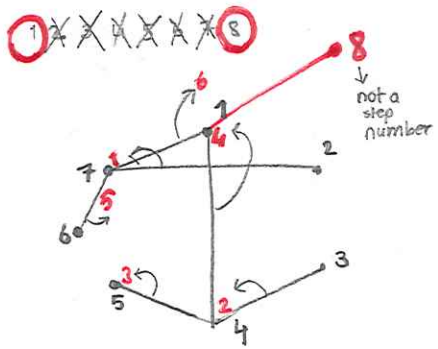
Iteration: At the ith step, delete the least remaining leaf and let a_i be the neighbor on this leaf.

After n-2 steps only one of the original edges remains and we have produced a sequence of length n-2 with entries in $\{1, \dots n\}$

▶ 2  3 T

7 4 4 1 7 1

TO GOING BACK TO THE GRAPH



NOT ISOMORPHISM

1 ~~2~~ ~~3~~ ~~4~~ ~~5~~ ~~6~~ ~~7~~ 8

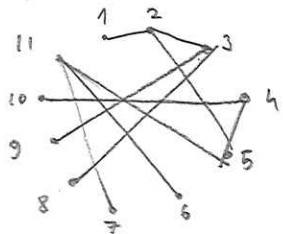
ALGORITHM:

To retrieve the tree we start by isolated vertices. At each step we create an edge and mark one vertex. There are at least two unmarked vertices that do not appear among the remaining entries of a.

Let x be the least of these, add $a_i x$ to the list of edges, and mark x. After repeating this n-2 times, two unmarked vertices remain we join them to form the final edge.

▶ 2 11 11 3 3 2 5 4 5

~~1~~ ~~2~~ ~~3~~ ~~4~~ 5 ~~6~~ ~~7~~ ~~8~~ ~~9~~ ~~10~~ 11



Corollary

Given positive integers $d_1, \dots, d_n$ summing to $2n-2$, there are exactly $\dfrac{(n-2)!}{\prod (d_i - 1)!}$ trees with vertex set $\{1, \dots, n\}$ s.t. vertex i has degree d_i for each i.

PROOF

A vertex x appears d_{x-1} times in the Prüfer code as we record x to the code, when we delete a neighbor of x, until we delete x itself or leave x among the last two vertices.

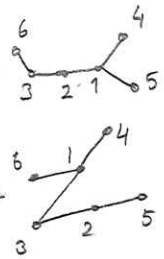
We will count lists of length $n-2$ that has d_{i-1} copies of i. There are $(n-2)!$ ways to permute $n-2$ distinct objects. Since copies of i are not distinguishable we have counted each desirable arrangement $\prod (d_i - 1)$ times.

1 2 3 4 5 6

3 2 2 1 1 1

1 1 2 3

1 2 3 1



$$\frac{4!}{2!} = 12.$$

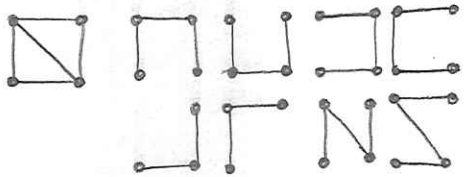
▶ Consider degree sequence $(3,1,2,1,3,1,1)$ for the vertices $\{1, \dots, 7\}$. There are

$$\frac{(7-2)!}{2!\,0!\,1!\,0!\,2!\,0!\,0!} = \frac{120}{4} = 30$$

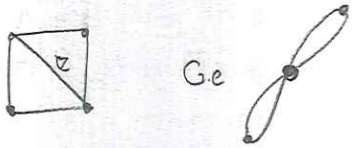
Such trees.

SPANNING TREES IN GRAPHS

Lets count the number of spanning trees in graph G.

▶ 

DEF In a graph G, contracting of edge e with endpoints u, v is the replacement of u and v with a single vertex whose incident edges are the edges other than e that were incident to u and v. The resulting graph $G.e$ has one less edge than G.

▶ 

PROPOSITION

Let $\tau(G)$ denote the number of spanning trees of a graph G. If $e \in E(G)$ is not along, then

$$\tau(G) = \tau(G-e) + \tau(G.e)$$

PROOF

A spanning tree of G either contains e or not. $\tau(G-e)$ counts the number of spanning trees of G that does not contain e. $\tau(G.e)$ counts the number of spanning trees of G that contains e. A spanning tree of $G.e$ with the edge e forms a spanning tree of G

▶ $\tau($ $) = \tau($ $) + \tau($ $) = 4+4=8$

NOTE We delete loops when they occur while counting the number of spanning trees.

$\tau($ $) = \tau($ $) + \tau($ $) = \tau($ $) + \tau($ $)$

since we don't count loops

$= 2+2=4$

▶ $\tau($ $) = \tau($ $) + \tau($ $)$

$= \tau($ $) + \tau($ $)$

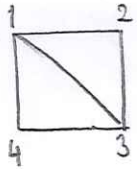
$+ \tau($ $) + \tau($ $)$

$= 1+2+4+4 = 11$

THEOREM (Matrix Tree Theorem)

Given a loopless graph G with vertex set $v_1, \dots, v_n$ let a_{ij} be the number of edges with endpoints $v_i v_j$. Let Θ be the matrix in which entry (i,j) is $-a_{ij}$ when $i \neq j$ and is $d(v_i)$ when $i=j$. When Θ^* is a matrix obtained by deleting row s and column t of Θ, then

$\tau(G) = (-1)^{s+t} \det \Theta^*$

▶ 

$$\Theta = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 3 & -1 & -1 & -1 \\ v_2 & -1 & 2 & -1 & 0 \\ & -1 & -1 & 3 & -1 \\ & -1 & 0 & -1 & 2 \end{array}$$

$$\Theta^* = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

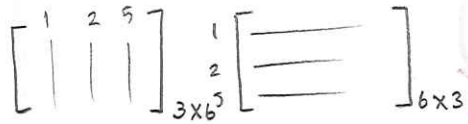
$\det \Theta^* = 2\times3\times2 + 0 + 0 - 0 - 2 - 2 = 8$ //

PROOF

Write $\Theta = MM^T$ where M is the incidence matrix of an orientation ± 1

BINET-COUCHY THEOREM

$$\det(AB) = \sum (\det A[s])(\det B[s])$$

▶ 

Choose S to have size $n-1$. When s are the edges of a spanning tree $\det[A]$ is ± 1. If not then it's 0. Observe that $A[s] = B^T[s]$

DECOMPOSITION AND GRACEFUL LABELINGS

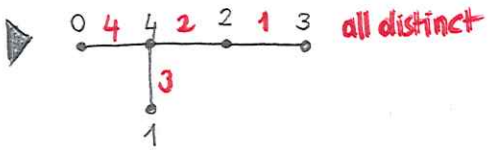
- When can we decompose a graph into isomorphic trees

CONJECTURE

If T is a fixed tree with m edges, then K_{2m+1} decomposes into $2m+1$ copies of T.

DEF: A graceful labeling of a graph G with m edges is a function $f: V(G) \to \{0, \dots, m\}$ s.t. distinct vertices receive distinct numbers and

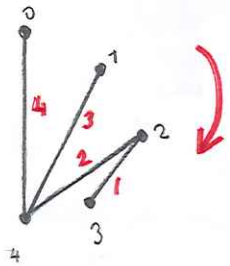
$$\{ |f(u) - f(v)| : uv \in E(G) \} = \{1, \dots, m\}$$



CONJECTURE

(Graceful Tree Conjecture) Every tree has a graceful labeling.

THEOREM (Rosa 1967) If a tree T with m edges has a graceful labeling then K_{2m+1} has a decomposition into $2m+1$ copies of T.



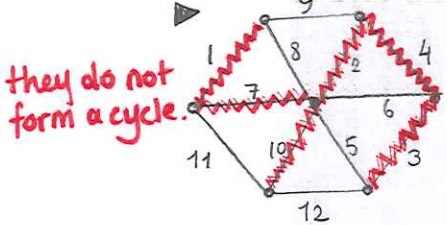
OPTIMIZATION AND TREES

Finding a minimum weight spanning tree (Kruskal's Algorithm)

DEF: A weighted graph is a graph where each edge is assigned a non-negative numerical label.

TARGET

Find a spanning tree where the sum of the labels of the edges used is the minimum possible.



ALGORITHM

Find a minimum weight spanning tree H.

INITIALIZATION

Set $E(H) = \emptyset$

ITERATION

If the next cheapest edge joins two components of H, then include it, otherwise discard it. Terminate when H is connected.

THEOREM In a connected weighted graph G, Kruskal's algorithm constructs a minimum weight spanning tree.

PROOF

The algorithm never chooses an edge that completes a cycle. So the algorithm produces an acyclic graph. If the final graph has more than once component then we have not considered any edge joining them if we did, we would have taken that edge so the original graph must be disconnected.

Therefore algorithm produces a tree for each connected graph.

Let T be the resulting tree, and let T^* be a spanning tree of minimum weight.

If $T = T^*$, then we are done. If $T \neq T^*$, then let u be the first edge chosen for T that is not in T^*

Adding e to T^* creates a cycle C. Since T has no cycle, C has an edge $e' \notin E(T)$. Consider the spanning tree $T^* + e - e'$

Since T^* contains e' and all edges of T chosen before e, both e' and e are available when the algorithm chooses e, and hence $w(e) \leq w(e')$. Thus $T^* + e - e'$ is a spanning tree with weight at most T^* that agrees with T for a longer initial list of edges then T^* does.

Repeating this argument eventually yields a minimum-weight spanning tree agrees completely with T.

SHORTEST PATHS

Observations: The u,v-portion of shortest u,z-path must be shortest u,v-path.

DIJKSTRA ALGORITHM

INITIALIZATION Set $S = \{u\}$ $t(u) = 0$

$t(z) = w(uz)$ for $z \neq u$ if uz is an edge
weight otherwise ∞

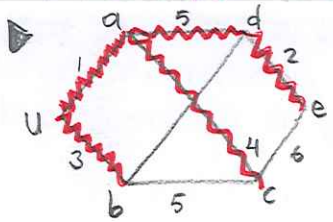
ITERATION Select a vertex v outside S s.t.

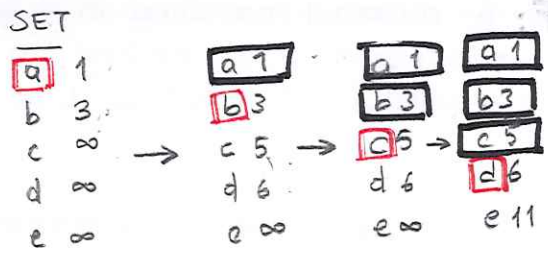
$t(v) = \min_{z \notin S} t(z)$ Add v to S. Explore edges from v to update tentative distances for each edge vz with $z \notin S$, update $t(z)$ to

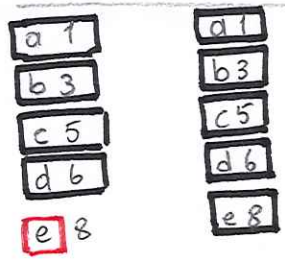
$\min\{t(z), t(v) + w(uz)\}$

The iteration continues until $S = V(G)$ or until $t(z) = \infty$ for every $z \notin S$ At the end

$d(u,v) = t(u)$ for all v.





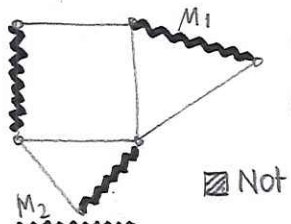


MATCHINGS AND FACTORS

DEF: A matching in a graph G is a set of non-loop edges with no shared endpoints. (set of independent edges) The vertices incident to the edges of a matching M are saturated by M, the other edges are unsaturated.

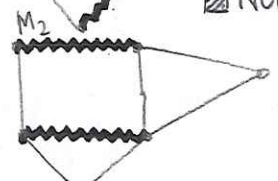
A perfect matching is a matching that saturates every vertex of a graph.

Fact: If a graph has a perfect matching then it has even number of vertices.



Perfect matching
Maximum matching

Not every maximal matching is a maximum matching.



Maximal matching

DEF: A maximal matching in a graph is a matching that cannot be enlarged by adding an edge. A maximum matching is a matching of maximum size among all matchings in the graph.

Fact: A matching M is maximal if every edge not in M is incident to an edge alredy in M.

HW Show that Q_k has a perfect matching.

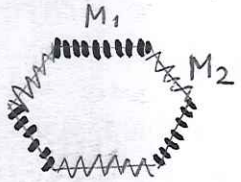
DEF: Given a matching M, <u>an M-alternating path</u> is a path that alternates between edges in M and edges not in M

DEF: An M-alternating path whose endpoints are unsaturated by M is an M-augmented path.

DEF: The symmetric difference of the sets A and B, denoted by $A \Delta B = (A/B) \cup (B/A)$

<u>Lemma</u>: Every component of the symmetric difference of two matchings M_1 and M_2 is a path or an even cycle.

PROOF: Since M_1 and M_2 are matchings, every vertex has at most one incident edge from each of them thus symmetric difference has at most two edges at each vertex. Since maximum degree is ≤ 2 every component is a path or cycle. Furthermore every path or cycle alternates between edges of $M_1 - M_2$ and $M_2 - M_1$. Thus each cycle has even length, with equal number of edges from M_1 and M_2.



THEOREM A matching M in a graph G is a maximum matching in G $\Longleftrightarrow$ G has no M-augmenting path.

<u>PROOF</u> An M-augmented path can be used to form a matching larger than M. So if G has an M-augmented path then M cannot be maximum.

Let M' be a matching larger than M. Let $F = M \Delta M'$. By the previous lemma every component of F is an even cycle or a path. The cycles have same numer of edges from M and M' and $|M'| > |M|$ so there exists a path P of odd length where the number of edges of M' on P is bigger than number of edges of M an P. Hence P starts and ends with an edge of M', thus P is an M-augmented path in G.

HALL'S THEOREM

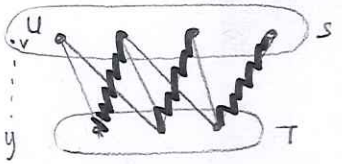
Let G be a bipartite graph with parts A and B, G has matching that saturates $A \iff |N(S)| \geq |S|$ for all $S \subseteq A$

PROOF

The $|S|$ vertices matched to S must lie in $N(S)$ so $|N(S)| \geq |S|$ is necessary for a perfect matching.

We will prove that Hall's condition is sufficient by showing that if there is a maximum matching M in G which does not saturate A then there is a set $S \subseteq A$ s.t. $|N(S)| < |S|$

Assume there is no matching saturating A. Let u be a vertex unsaturated M. Among all the vertices reachable from U by M-alternating paths in G, Let S consists of those in A and let T consist of those in B.



We claim that M matches T with $S - \{u\}$.

The M alternating paths from u reach B along edges not in M and return to A along edges in M. Hence every vertex of $S - \{u\}$ is reached by an edge in M from a vertex in T.

Since there is no M-augmented path, every vertex of T is saturated, thus an M-alternating path reaching $y \in T$ extends via M to a vertex of S. We have $|T| = |S - \{u\}|$

(There is a bijection from T to $S - \{u\}$)

The matching between T and $S - \{u\}$ yields $T \subseteq N(S)$. In fact $T = N(S)$. Suppose $y \in B - T$ has a neighbor $v \in S$. The edge vy cannot be in M, since u is unsaturated

Teleton

Hall's Theorem is Marriage Theorem if $|A| = |B|$

Corollary

For $k > 0$, every k regular bipartite graph has a perfect matching

PROOF

Let G be a k-regular graph with partition X and Y. Let $S \subseteq X$

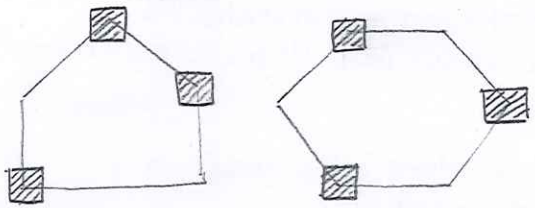
$|N(S)| \leq |S| . k$ as some neighbors maybe the same.

Also $|N(S)| . k \geq k|S| \Rightarrow |N(S)| \geq |S|$

total edges out of the neighborhood each vertex in $|S|$ is in the neighborhood of $N(S)$

MIN-MAX THEOREM

DEF: A vertex cover of a graph G is a set $\theta \subseteq V(G)$ that contains at least one end of every edge in G.



min vertex cover	$\beta(G)$
min edge cover	$\beta'(G)$
max independent set	$\alpha(G)$
max size of matching	$\alpha'(G)$

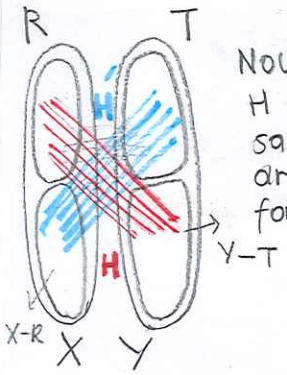
Fact: Since no vertex can cover two edges of a matching the size of every vertex cover is at least the size of every matching

$$\alpha'(G) \leq \beta(G)$$

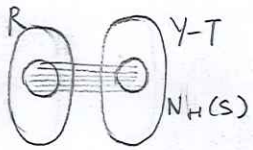
THEOREM If G is a bipartite graph, the maximum size of a matching in G equals the minimum size of a vertex cover of G.

Let G be a X-Y-bipartite graph and θ be a minimum size vertex cover and M be a matching. We have $|\theta| \geq |M|$. We will construct a matching with $|\theta|$ edges to prove the inequality.

Let $R = \Theta \cap X$ and $T = \Theta \cap Y$. Let H and H' be the subgroups of G induced by $R \cup (Y-T)$ and $T \cup (X-R)$



Now we will use Halls Theorem to show that H and H' have a perfect matching that saturates R and T respectively. Since H and H' are disjoint, the two matching together form a matching of size $|\Theta|$ in G.



Since $R \cup T$ is a vertex cover G has no edge from $Y-T$ to $X-R$. For each $S \subseteq R$ we consider $N_H(S)$ which is contained in $Y-T$. If $|N_H(S)| < |S|$ then we can substitute $N_H(S)$ for S in Θ to obtain a smaller vertex cover. Since $N_H(S)$ covers all edges incident to S that are not covered by T so $|N_H(S)| \geq |S|$ then Hall's condition is satisfied. Hence H has a matching that saturates R. A similar argument shows that H' has a matching that saturates T.

DEF An edge cover of G is a set L of edges such that every vertex of G is incident to same edge of L.

Remark: Only graphs without isolated vertices have edge covers

Facts: 1- A perfect matching forms an edge cover with $n(G)/2$ edges

2- $\beta'(G) \geq \alpha'(G)$

3- $\beta'(G) \geq \alpha(G)$ since no edge can cover two vertices of an independent set.

Lemma: In a graph G, $S \subseteq V(G)$ is an independent set $\iff \bar{S} = V(G) - S$ is a vertex cover. Hence $\alpha(G) + \beta(G) = n(G)$

min vertex cover

PROOF

Let S be an independent set then G does not have any edge that has both ends in S so all edges of G are incident to a vertex in $\bar{S}$. So $\bar{S}$ is a vertex cover. Conversely if $\bar{S}$ covers all edges, there are no edges joining vertices of S. Hence every maximal independent set is the complement of a minimal vertex cover. Hence $\alpha(G) + \beta(G) = n(G)$

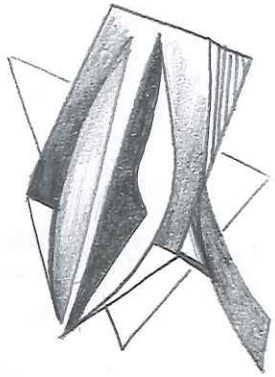
THEOREM If G is a graph without isolated vertices then $\alpha'(G) + \beta'(G) = n(G)$

MATCHINGS IN GENERAL GRAPHS

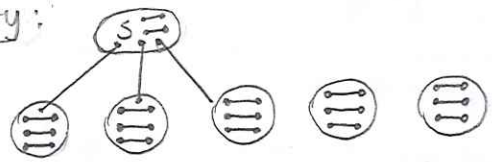
DEF: A k-factor is a spanning k-regular subgraph G. The edges of a 1-factor forms a perfect matching.

DEF: An odd component of a graph is a component of odd order. The number of odd components of H is $o(H)$.

THEOREM <Tutte [1947]> A graph G has a a 1-factor $\iff$ $o(G-S) \leq |S|$ for every $S \subseteq V(G)$



Necessity:



There must be a different vertex in S for each odd component odd component of G-S for G to have a 1-factor

Sufficiency

Adding an edge joining has components of G-S does not increase the # of odd components. Hence Tutte's condition is preserved by addition of edges.

Let G be a graph that satisfies Tutte's condition. If $S=\emptyset$ $o(G) \leq 0$ so G has an even number of vertices. Also we can assume that G is simple. Now odd edges to G to get the graph G' that has no 1-factor but adding any missing edge to G' forms a graph that has a 1-factor. In other words G' to has a 1-factor for every $e \notin E(G')$

Let $U = \{ v \in V(G') : d_{G'}(v) = n-1 \}$

case 1: Each component of $G'-U$ is a complete graph.

For each component take a maximal matching. This matching will saturate every vertex for even components and every vertex except one for each odd component since $o(G-U) \leq |U|$ and the vertices in U are adjacent to every other vertex in G' this left over vertices in odd components can be paired-up as G' has even # of vertices

case 2: Let $G-U$ has a component that is not a clique.

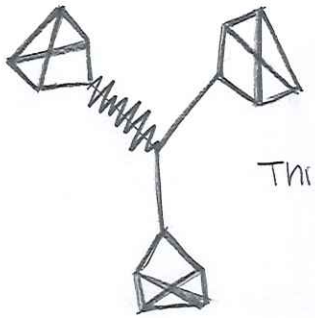
Tellon

Corollary

Every 3-regular graph with no cut edge has a 1 factor

Peterson graph has a one-factor but it does not have a 1-factorization

decomposes into 1-factors

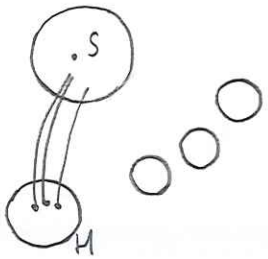


This graph is 3-regular with a cut-edge and it doesnot have a 1-factor.

PROOF for Corollary

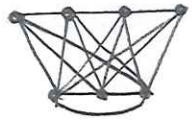
Let $S \subseteq V(G)$. Since G is 3-regular, each vertex of S is incident to three edges between S and the odd components of $G-S$. We will show that each odd component H of $G-S$ is incident to at least 3 such edge, then we will have $3o(G-S) \leq 3|S|$.

Let m be the edges between H and S. The sum of vertex degrees in H is $3n(H)-m$. Since H is a component $3n(H)-m$ is even. Now since $n(H)$ is odd, m must be odd. Since G has no cut-edge, m is not 1. So $m \geq 3$



DEF The join of simple graphs G and H written $G \vee H$ is the graph obtained from the disjoint union $G+H$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$

$P_4 \vee K_3$



$K_4 = K_2 \vee K_2$

Corollary

matchingin max. matching oldugunu dusundugumuzde kullanırız.

The largest number of vertices saturated by a matching in G is $\min_{S \subseteq V(G)} \{n(G) - d(S)\}$ where $d(S) = o(G-S) - |S|$

PROOF

Given $S \subseteq V(G)$, at most $|S|$ edges can match vertices of S to vertices in odd components of $G-S$, so every matching has at least $o(G-S)-|S|$ unsaturated vertices.

Let $d = \max\{o(G-S) - |S| : S \subseteq V(G)\}$. The case $S = \emptyset$ yields $d \geq 0$. Let $G' = G \vee K_d$

K_d

Since $d(S)$ has the same parity as $n(G)$ for each S, we know that $n(G')$ is even.

$(|S| + o(G-S) \equiv n(G) \mod 2$ so $o(G-S) - |S| \equiv n(G) - 2|S| \mod 2$

Hence $o(G-S) - |S| \equiv n(G) \mod 2$.

If G' satisfies Tutte's condition, then we can obtain a matching of desired size in G from a perfect matching in G', because deleting d added vertices eliminates edges that saturates at most d vertices in G.

The condition $o(G'-S') \leq |S'|$ holds for $S'=\emptyset$ because $n(G')$ is even. If S' is nonempty but does not contain all of K_d, Then $G'-S'$ has only one component, and $1 \leq |S|$. Finally when $K_d \subseteq S'$ we let $S = S' - V(K_d)$. We have $G'-S' = G-S$ so $o(G'-S') = o(G-S) \leq |S|+d = |S'|$ (*)

We have verified that G' satisfies <u>Tutte's Condition</u>.

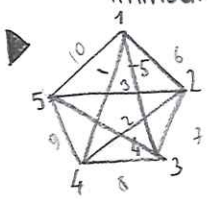
(* $d = \max\{o(G-S) - |S|\}$ but then $o(G-S) \leq |S|+d$)

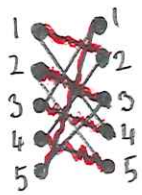
THEOREM (PETERSON 1891)

Every regular graph with positive even degree has a 2-factor.

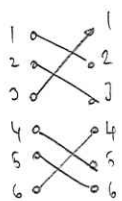
PROOF

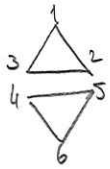
Let G be a $2k$-regular graph with vertices $v_1, \dots, v_n$. Every component of G is Eulerian. Take a Eulerian circuit c for each component. Define a bipartite graph H with parts X, Y where $X = \{w_1, \dots w_n\}$ and $Y = \{u_1, \dots, u_n\}$ and w_i is adjacent to u_i if v_i immediately follows v_i somewhere on the circuit.





As C enters and exists each vertex k-times. H is k-regular So H has a 1-factor since H is bipartite. The 1-factor in H can be transformed to a 2-factor of this component of G. Doing this for each component of G yields a 2-factor of G.





DEF: Two paths from u to v are internally disjoint if they have no common internal vertex.

THEOREM A graph G having at least three vertices is $\boxed{\text{2-connected}}$ $\iff$ for each pair $u,v \in V(G)$ There exists internally disjoint u-v paths in G.

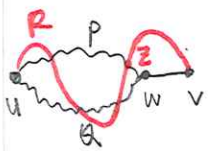
When G has internally disjoint u,v paths deleting one vertex cannot seperate u from v. Since this condition is given for every pair deleting just one vertex cannot seperate any two vertices.

Now suppose G is 2-connected. We will prove by induction on $d(u,v)$ that G has internally disjoint u,v paths.

$\boxed{\text{BASIS}}$ $d(u,v)=1$ As G is 2-connected. $G.uv$ connected, A u,v-path in $G.uv$ is disjoint in G from the edge uv

$\boxed{\text{INDUCTION}}$ Let $k=d(u,v)$. Let w be the vertex before v on a shortest u,v-path. We have $d(u,w)=k-1$. By induction hypothesis G has 2 disjoint u-w paths call P and Q.

If $v \in V(P) \cup V(Q)$ then we find the desired paths in the cycle $P \cup Q$



Now suppose not. Since G is 2-connected, $G-w$ is connected and contains a $\boxed{?}$ path R.

If R avoids P or Q, we are done.

So R shares vertices with both P and Q

Let z be the last vertex of R before v belonging to $P \cup Q$. WLOG assume $z \in P$ We combine uz-subpath of P with zv-subpath

? $\boxed{}$ $\boxed{}$? obtains a u-v path internally disjoint from $Q \cup wv$.

THEOREM For every graph G with at least three vertices the following are equivalent

1) G is 2-connected

2) For all $x,y \in V(G)$ there are internally disjoint x,y-paths

3) For all $x,y \in V(G)$ there is a cycle through x and y.

4) Every pair of edges lies on a common cycle and G has no isolated vertices.

THEOREM (Menger) If x and y are vertices of a graph G and $xy \in E(G)$, then the minimum size of a xy-cut equals the maximum number of pairwise internally disjoint xy-paths

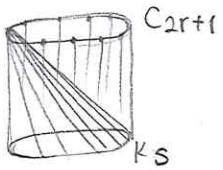
THEOREM G is $\left\{\begin{matrix} k\text{-connected} \\ k\text{-edge connected} \end{matrix}\right\} \iff$ for any pair of vertices $x,y \in V(G)$ $x \neq y$ there exists k $\left\{\begin{matrix} \text{internally} \\ \text{edge} \end{matrix}\right\}$ disjoint (x,y) paths in G.

Coloring

PHONE

▶ $\omega(G) \leq \chi(G)$

Let $G = C_{2r+1} \vee K_s$



$\omega(G) = s+2$

$\chi(G) \geq s+3$

$\chi(G) \geq \omega(G)$

For the join $\chi(G \cup H) = \chi(G) + \chi(H)$

The colors on G and H should be different.

GREEDY ALGORITHM FOR VERTEX COLORING

The greedy coloring is obtained by coloring vertices in the order $v_1, \dots, v_n$ assigning to v_i the smallest-indexed color not already used on its lower indexed neighbors.

PROPOSITION

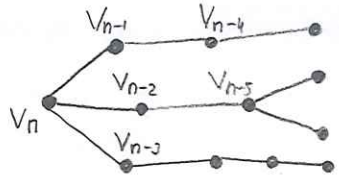
$\chi(G) \leq \Delta(G) + 1$

Color each vertex greedly as each vertex has at most $\Delta(G)$ earlier neighbors so greedy algorithm cannot need to use more than $\Delta(G)+1$ colors.

BROOK'S THEOREM

If G is connected graph other than a complete graph or odd cycle, then

$$\chi(G) \leq \Delta(G)$$



Let G be a connected graph, and let $K = \Delta(G)$ we may assume $k \geq 3$

if $k=1$, then G is K_2 graph

if $k=2$, then G is a cycle. If it is an even cycle then $\chi(G)=2$ else it is an odd cycle.

We will order the vertices so that each vertex has at most $k-1$ lower indexed neighbors when G is not k-regular, we can choose a vertex of degree less than k as v_n

G is connected so form a spanning tree of G from v_n, assigning indices in decreasing order as we reach vertices

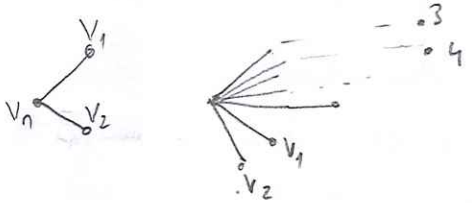
Each vertex other than v_n is the resulting ordering $v_1, \dots, v_n$ has a higher indexed neighbor along the path to v_n in the tree. Hence each vertex has at most $k-1$ lower indexed neighbors and the greedy algorithm uses at most k colors.

Now suppose G is regular

CASE 1 G has a cut vertex. Let x be a cut vertex and H be a component of G-x then let $G' = \langle V(H) \cup \{x\} \rangle$ The degree of x in G' is less than k so as above we can color G' by k colors. Apply the same process to all components of G-x and make sure the coloring agree for x in all of these graphs.

Case 2 G is 2-connected. We will try to make sure two neighbors of v_n got the same order.

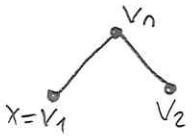
In particular, suppose that some vertex v_n has neighbors v_1, v_2 s.t. v_1 is not adjacent to v_n and $G-\{v_1, v_2\}$ is connected then index the vertices of a spanning tree of $G-\{v_1, v_2\}$ using $3, \dots, n$ such that labels increase along the path to the root v_n. The greedy algorithm uses at most $k-1$ colors on neighbors of v_n, since v_1 and v_2 receive the same color (we start by coloring v_1, v_2 with the same color.)



So we need to show that a k-regular 2-connected graph with $k \geq 3$ has three vertices v_1, v_2, v_3 such that v_1 and v_2 are adjacent to v_n, v_1 is not adjacent to v_2 and $G-\{v_1, v_2\}$ is connected.

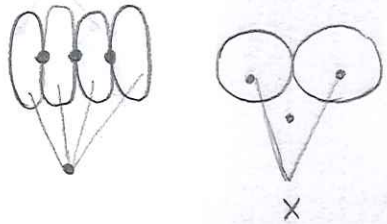
Choose a vertex x

If $K(G-x) \geq 2$ Let v_1 be x.



Let v_1 be x and v_2 be a vertex with distance 2 from x. Such a vertex v_2 exists because G is regular and is not a complete graph.

Let v_n be a common neighbor of v_1 and v_2



If $K(G-x)=1$, let $v_n = x$. Then $G-x$ has a cut-vertex y. Let H_1 and H_2 be 2-connected components of $G-\{xy\}$. Then G has no cut vertex so x is adjacent to 2 vertices v_1, v_2 s.t. $v_1 \in H_1$ and $v_2 \in H_2$

As y is a cut vertex of $G-x$, v_1 is not adjacent to v_2. Also $G-\{x, v_1, v_2\}$ is connected as H_1 and H_2 have no common cut vertex. Since $k \geq 3$ x has another neighbor and $G-\{v_1, v_2\}$ is connected.

since $i-1>j$ the number of edges increase so equalizing the sizes of the parts maximize the number of edges.

THEO (TURAN)

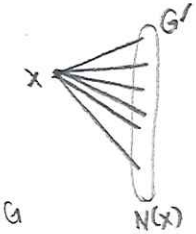
REM Among the n-vertex simple graphs with no $r+1$-clique $T_{n,r}$ has the max. number of edges.

PROOF

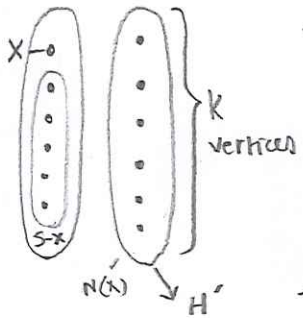
The Turan's graph has no $r+1$-clique we will prove that the maximum # of edges will be achieved with a r-partite graph with the help of the previous lemma, this will prove that T_{nr} has the max. # of edges among the graphs with no $r+1$ clique we will show that if G has no $r+1$ clique, then there is an r-partite graph H with the same vertex set as G and at least as many edges.

We will prove by induction on r. When $r=1$, G and H have no edges.

Consider $r>1$. Let G be an n-vertex graph with no $r+1$-clique. Let $x \in V(G)$ be a vertex degree $k=\Delta(G)$. Let G' be the subgraph of G induced by the neighbors of x. The graph G' has no r-clique Apply the induction hypothesis to G', this yields an $r-1$-partite graph H' with vertex set $N(x)$ such that $e(H') \geq e(G')$



Let H be the graph formed from H' by joining all of $N(x)$ to all of $S=V(G)-N(x)$



Since S is an independent set, H is r-partite. We will show that

$$e(H) \geq e(G)$$

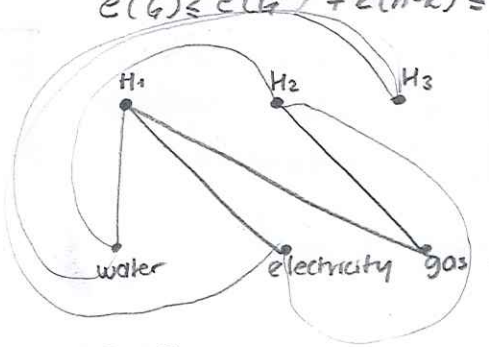
It is obvious that $e(H)=e(H')+k(n-k)$

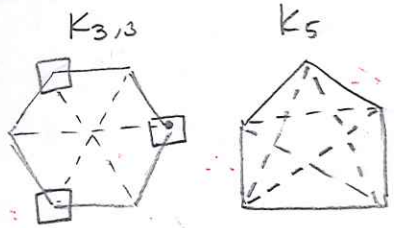
Also $e(G) \leq e(G') + \sum_{v \in S} d_G(v)$

Since $\Delta(G)=k$, we have $d_G(v) \leq k$ for each $v \in S$ and $|S|=n-k$

So $\sum_{v \in S} d_G(v) \leq k(n-k)$ so we have

$$e(G) \leq e(G') + k(n-k) \leq e(H') + k(n-k) = e(H)$$





PLANAR GRAPHS

4-color theorem: Can the regions of every map on a globe be colored with four colors so that regions share a non-trivial boundary have different colors.

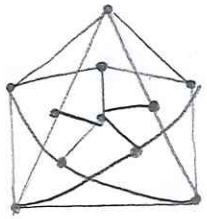
A graph is embedded on a surface if it can be drawn on that surface without edge crossings.

MYCIELSKI'S CONSTRUCTION

K-chromatic triangle-free graphs

$\omega(G)=2$

$\chi(G)=k$



Let G has vertex set $\{u_1,\dots,u_n\}$
add vertices $U=\{u_1,\dots,u_n\}$ and
w
Add edges to make u_i adjacent to all of $N_G(v_i)$ and finally $N(w)=U$

Assume G is triangle free. Show G' is triangle free. U is an independent set. So any triangle containing any vertex u_i has v_k, v_l as other vertices but then v_k, v_l has a common neighbor v_i hence $v_i v_k v_l$ is a triangle contradicts

- Any k-coloring of G results in a k-1 coloring of G.

PROOF HW

EXTREMAL PROBLEMS AND TURAN THEOREM

PROPOSITION Every k-chromatic graph with n vertices has at least $\binom{k}{2}$ edges.

PROOF An optimal coloring of a graph has an edge with endpoints of colors i j for each pair i,j of colors. Otherwise the vertices colored with i and j can be colored properly by i and we will have proper coloring with fewer number of colors. Since there are $\binom{k}{2}$ distinct pairs of colors, there must be $\binom{k}{2}$ distinct edges.

DEF: A complete multipartite graph is a simple graph G whose vertices can be partitioned into sets so that u and v are adjacent $\Leftrightarrow$ u and v belong to different sets of the partition. Equivalently every component of $\bar{G}$ is a complete graph. When $k\geq 2$ we write $K_{n_1,n_2,\dots,n_k}$ for the complete k-partite graph with partite sets of sizes $n_1,\dots,n_k$.

NOTE: A complete k-partite graph is k-chromatic where partite sets are the color classes.

THEOREM (Turan Graph) Turan graph $T_{n,r}$ is the complete r-partite graph with n vertices whose partite sets differ in size by at most 1 so the partite sets have sizes $\lceil \frac{n}{r} \rceil$ and $\lfloor \frac{n}{r} \rfloor$

other page

$T_{21,4}$ $\binom{3}{2}5^2+6.5.3=165$

Lemma: Among simple r partite graphs with n vertices, the Turan graph is the unique graph with the most edges.

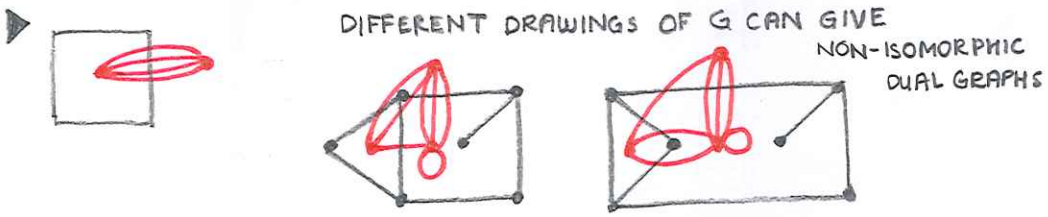
Given a complete r-partite graph with partite sets differing by more than 1 in size we move a vertex v from the largest size class i to smallest class of size j. Then v gains r-1 neighbors and loses j neighbors

THEOREM K_5 and $K_{3,3}$ cannot be embedded on the plane.

Let C be a spanning cycle. If the drawing does not have crossing edges, then C is drawn as a closed curve. Two chards conflict if their endpoints on C occur in alternating order. When two chards conflict, we can draw only one inside C and one outside C. $K_{3,3}$ has 3 pairs of conflicting chards, when C is a 5-cycle in K_5, there are 5 chards conflicting

DEF Faces of a planar graph G are the regions enclosed by edges of G. Each edge borders exactly two faces.

DEF The dual graph G^* of a plane graph G is a plane graph whose vertices correspond to the faces of G. If two faces in G share a common edge as a cite then the vertices corresponding to these faces in G^* are adjocent.



THEOREM If G is bipartite then $\chi'(G) = D(G)$

Every regular bipartite graph G has a 1-factor. By induction on $\Delta(H)$, we can show that H can be colored with $\Delta(H)$ colors if H is regular. We will show that for every bipartite graph G there exists a $\Delta(G)$-regular graph H that contains G. To construct H, add vertices to the parts with smaller # of vertices to equalize the sizes of the parts. If the resulting G' is not regular, then each partite set has a vertex with degree less than k. Add an edge between these two vertices continue adding each edges until the graph becomes k regular, the resulting graph is H.

$$\Delta(G) \leq \chi(G) \leq 2\Delta(G) - 1$$

Fat triangle

$$\chi'(G) = \frac{3\Delta(G)}{2}$$

$$\chi'(G) = \Delta(G) + \mu(G)$$

$\mu(xy)$ = is the multiplicity of the edge xy

$\mu(G)$ = is the max. of the edge multiplicities

THEOREM

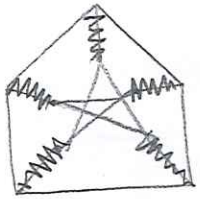
(Vizing) If G is simple graph, then $\chi'(G) \leq \Delta(G) + 1$

If G is any graph then $\chi'(G) \leq \Delta(G) + \mu(G)$

DEF: A decomposition of a regular graph G into 1-factors is a 1-factorization of G. A graph with a 1-factorization is 1-factorable.

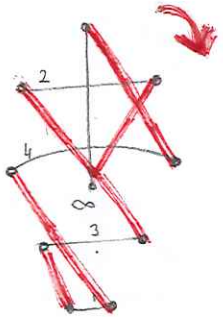
NOTE

A decomposition of G is an expression of G as a union of edge-disjoint subgraphs.

- 

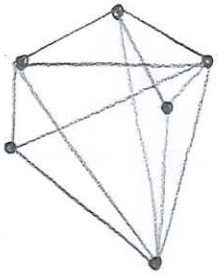
  Peterson graph does not have a 1-factorization.

- K_n where n is even



DEF. A hamiltonian graph is a graph w/ spanning cycle. A spanning cycle is called a Hamilton cycle. A spanning path is a Hamiltonian path.

We will strict our graphs to simple graphs.



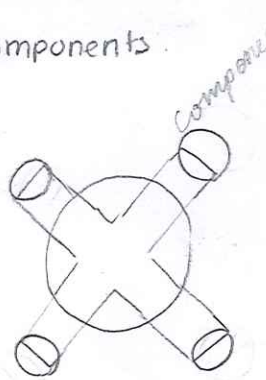
Remark: Every Hamiltonian graph is 2-connected. Removing an vertex leaves a spanning path.

NECESSARY CONDITION

Remark: $K_{m,n}$ is Hamiltonian only if $m=n$ and $n>1$.

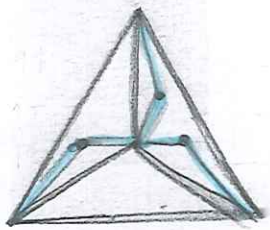
A spanning cycle visits two partite sets alternatingly.

Proposition If G has a Hamilton cycle, then for each nonempty set $S \subseteq V$ the graph $G-S$ has at most $|S|$ components.



When leaving a component of $G-S$ the cycle can only return to a distinct vertex of G. Hence the number of components of $G-S$ must be less than or equal to the # of components of S.

$c(G) = \#$ of components of G



This graph is not hamiltonian but satisfies the above necessary condition.

Peterson graph is not Hamiltonian