Differential Equations

MATH 204 Koç University
Duygu Sezen Islakoğlu
Instructor: Varga Kalantarov

MATH 204

Functional equation is where the unknown is a function.

DEF: A differential equation is an equation where the unknown is a function and it involves a derivative(s) of the unknown function.

▶ $y''(t) - 2y(t) = \sin t$

$y'(t) = \sin(y(t))$

$m\,a(t) = F$

$m\,x''(t) = F$

If the unknown function in a differential equation is a function of one variable then this kind of equation is called an ordinary differential equation. → ODE

If is a function of multivariable function → PDE (partial differential equation)

$$\begin{cases} y'(t) = y^2(t) \\ y(0) = 1 \end{cases}$$

$$y(t) = \frac{1}{1-t}$$

■ $\begin{cases} y'(t) = 0 & , \forall t \in \mathbb{R} \\ y(t_1) = y(t_2) & , \forall t_1, t_2 \end{cases}$

$y(t_2) - y(t_1) = y'(c)(t_2 - t_1) = 0$

MEAN VALUE THEOREM

▶ $y'(t) = 2$

$(y(t) - 2t)' = 0$

$y(t) - 2t = c \quad , \quad y(t) = 2t + c$

▶ $y''(t) + 4y(t) = 0$

$A \sin(2t)$

$B \cos(2t)$

▶ $y''(t) + 2y'(t) + \sqrt{2}\, y(t) = 0$

DEF: The order of highest derivative of the unknown function is called the order of a differential equation.

■ $y^{(5)}(t) - y^{6}(t) = \sin(y(t))$ → order: 5

▶ $y'(t) + 2y(t) = 0$

$y'(t) = -2y(t)$

$y'(t) + ay(t) = 0$

$y'(t) = -ay(t)$

$y(t) = e^{-at}$

$y(t) = ce^{-at}$

▶ $y'(t) + 2t\,y(t) = 0$

$y = c\,e^{-t^2}$

▶ $y'(t) + a(t)y(t) = 0$

$y(t) = c\,e^{-\int_0^t a(s)ds}$

▶ $y'(t) = -a(t)y(t)$

$\dfrac{y'(t)}{y(t)} = -a(t)$

$(\ln y(t))' = -a(t)$

$\ln y(t) = -\int a(t)dt$

GEÇİŞ

$y(t) = ce^{-\int a(t)dt}$

$y^{(n)}(t) + a_1(t)\,y^{(n-1)}(t) + \dots$

$+ a_{n-1}(t)y'(t) + a_n(t)y(t) = h(t)$

homogeneous, linear ODE

n^{th} order linear ODE

(parantez içinde) → is a function

solution $= y(t)$

• FIRST ORDER LINEAR EQUATIONS •

$y' + p(t)y = g(t) \qquad (1)$

▶ $\underbrace{(2+t^3)y'(t) + 3t^2 y(t)}_{\text{TRICK}} = 2t$

$[(2+t^3)\,y(t)]' = 2t$

$\Rightarrow (2+t^3)\,y(t) = t^2 + C$

$y(t) = \dfrac{t^2+C}{2+t^3}$

▶ $y'(t) + p(t)y(t) = g(t) \Big/ \mu(t) \qquad (1)$

TRY TO APPLY TO TRICK

$\mu(t)y'(t) + \mu(t)p(t)y(t) = \mu(t)g(t) \qquad (2)$

assume $\mu'(t) = p(t)\mu(t)$

-ÇARPIMIN TÜREVİ YAPMAK İÇİN-

$\dfrac{\mu'(t)}{\mu(t)} = p(t) \Rightarrow (\ln \mu)' = p$

$\mu(t) = \exp\left[\int p(t)dt\right]$

So taking μ (2) $\mu(t) = \exp\left[\int p(t)dt\right]$

$\exp\left[\int p(t)dt\right]y'(t) + p(t)\exp\left[\int p(t)dt\right]y(t)$

$= g(t)\exp\left[\int p(t)dt\right] \qquad (2')$

ÇARPIMIN TÜREVİ

$\dfrac{d}{dt}\left[\exp\left[\int p(t)dt\right]y(t)\right] = g(t)\exp\left[\int p(t)dt\right]$

$\exp\left[\int p(t)dt\right]y(t) = \int g(t)\exp\left[\int p(t)dt\right]dt + C \Big/ \exp\left[-\int p(t)dt\right]$

HER TARAFI BÖLER

$y(t) = C\cdot\exp\left[-\int p(t)dt\right] + \exp\left[-\int p(t)dt\right]\int g(t)\exp\left[\int p(t)dt\right]dt$

The function $\mu(t)$ here is called an integrating factor.

▶ $y'(t) + 2y(t) = t \Big/ e^{2t}$

$e^{2t}y'(t) + 2e^{2t}y(t) = t\cdot e^{2t}$

$(e^{2t}y(t))' = te^{2t}$

$e^{2t}y(t) = \int te^{2t}dt + C$

$y(t) = Ce^{-2t} + e^{-2t}\left(\int te^{2t}dt\right)$

INTEGRATION BY PARTS → DO IT

$= C e^{-2t} + t - \frac{1}{4}$

INITIAL VALUE PROBLEM

$$\begin{cases} y'(t) + p(t)\,y(t) = g(t) \\ y(0) = y_0 \end{cases}$$

$= C$ oluyor doğal olarak

$/ \exp\left[\int_0^t p(s)\,ds\right]$

$$\exp\left[\int_0^t p(s)\,ds\right] y'(t) + p(t)\exp\left[\int_0^t p(s)\,ds\right] y(t) = g(t)\exp\left[\int_0^t p(s)\,ds\right]$$

$$\frac{d}{dt}\left[\exp\left[\int_0^t p(s)\,ds\right] y(t)\right] = g(t)\exp\left[\int_0^t p(s)\,ds\right]$$

$$\exp\left[\int_0^t p(s)\,ds\right] y(t) - \underbrace{y(0)}_{y_0} = \int_0^t g(s)\exp\left[\int_0^s p(t)\,dt\right] ds$$

use this

$$\int_0^t f'(s)\,ds = f(t) - f(0)$$

$$y(t) = y(0)\exp\left[-\int_0^t p(s)\,ds\right] + \exp\left[-\int_0^t p(s)\,ds\right]\int_0^t g(s)\exp\left[\int_0^s p(t)\,dt\right] ds$$

▶ $\begin{cases} y'(t) - 3y(t) = 2 \\ y(0) = 1 \end{cases}$ $\quad p(t) = -3$

$y(t) = \underbrace{y(0)}_{\mu} + \frac{\quad}{\mu}$

$$y(t) = 1 \cdot e^{3t} + e^{3t}\int_0^t 2e^{-3s}\,ds$$

$$= e^{3t} + e^{3t} \cdot \frac{2}{3}\left(1 - e^{-3t}\right)$$

▶ Show that the solution of the problem

$$\begin{cases} y'(t) + y(t) = \frac{1}{1+t^2} \\ y(0) = y_0 \end{cases}$$

tends to 0 as $t \to \infty$

$$y'(t) = y_0 e^{-t} + e^{-t}\int_0^t \frac{e^s}{1+s^2}\,ds$$

$$\lim_{t\to\infty} e^{-t}\int_0^t \frac{e^s}{1+s^2}\,ds$$

$$\lim_{t\to\infty} \frac{\int_0^t \frac{e^s}{1+s^2}\,ds}{e^t}$$

$$= \lim_{t\to\infty} \frac{\frac{e^t}{1+t^2}}{e^t} = \lim_{t\to\infty} \frac{1}{1+t^2} = 0$$

SEPERABLE EQUATIONS

An equation of the form

$$A(x) = B(y)\,y' \qquad (1)$$

is called a seperable equation.

$$\mathcal{A}(x) = \int A(x)\,dx \Rightarrow \mathcal{A}'(x) = A(x)$$

$$\mathcal{B}(y) = \int B(y)\,dy \Rightarrow \mathcal{B}'(y) = B(y)$$

We can rewrite in the form:

$$[\mathcal{A}(x)]' = [\mathcal{B}(y(x))]' \qquad (2)$$

$A(x) = B(y(x))$

$$(2) \Rightarrow \frac{d}{dx}\left[A(x) - B(y(x))\right] = 0$$

$$\Rightarrow A(x) - B(y(x)) = C$$

PROBLEM 1

Solve the initial value problem.

$$\begin{cases} (1+e^x)\, y y' = e^x \\ y(0) = 1 \end{cases}$$

$$y y' = \frac{e^x}{1+e^x}$$

$$= \left[\frac{1}{2} y^2(x)\right]' = \frac{(1+e^x)'}{1+e^x} = \left[\ln(1+e^x)\right]'$$

$$\int_0^x \left[\frac{1}{2} y^2(s)\right]' ds = \int_0^x \left[\ln(1+e^s)\right]' ds$$

$$\frac{1}{2} y^2(x) - \frac{1}{2} y^2(0) = \ln(1+e^x) - \ln 2$$

$$\frac{1}{2} y^2 - \frac{1}{2} = \ln\left(\frac{1+e^x}{2}\right) \Rightarrow y^2(x) = 1 + \ln\left(\frac{1+e^x}{2}\right)^2$$

$$y(x) = \pm\sqrt{1+\ln\left(\frac{1+e^x}{2}\right)^2}$$

ama biz +'yı tercih edeceğiz çünkü initial value var.

EXACT EQUATION

An equation of the form

$$M(x,y) + N(x,y)\, y' = 0 \qquad (1)$$

is called exact equation if there exists a function $U(x,y)$ such that

$$U_x(x,y) = M(x,y)$$
$$U_y = N(x,y) \qquad (2)$$

$$\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$$

- $\vec{F}(x,y) = \nabla U(x,y)$

Suppose that (1) is conservative and (2) holds. Then using (2) in (1) we get

$$U_x(x, y(x)) + U_y(x, y(x))\, y'(x) = 0$$ → önce x'e göre sonra y'ye göre alıyor.

$$\frac{d}{dx}\left[U(x, y(x))\right] = 0 \Rightarrow U(x,y) = C$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad (*)$$

$(*)$ is conservative $\Leftrightarrow N_x(x,y) = M_y(x,y)$

$$\left.\begin{array}{ll} U_x = M & U_y = N \\ U_{xy} = M_y & U_{yx} = N_x \end{array}\right\} M_y = N_x$$

$$\underbrace{(2xy^2+x^3)}_{M}dx + \underbrace{(2yx^2-3y^2)}_{N}dy = 0$$

$M_y = 4xy$, $N_x = 4xy$, $M_y = N_x$

So the equation is an exact equation.

$U_x = M$ $\quad U_x(x,y) = 2xy^2 + x^3$

$$U(x,y) = \int (2xy^2 + x^3)\,dx$$

$$= x^2y^2 + \frac{1}{4}x^4 + \varphi(y)$$

take derivative w/ respect to y.

$U_y = N$ $\quad U_y(x,y) = 2yx^2 - 3y^2$

$$U_y(x,y) = 2x^2y + \varphi'(y)$$

$$2yx^2 - 3y^2 = 2x^2y + \varphi'(y)$$

$$\varphi(y) = -y^3 + C_1$$

$$U(x,y) = x^2y^2 + \frac{1}{y}x^4 - y^3 + C_1$$

$$x^2y^2 + \frac{1}{4}x^4 - y^3 = C$$

$$\varphi(0) = 1$$

$$x^2y^2 + \frac{1}{4}x^4 - y^3 = 1$$

INTEGRATING FACTOR

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad (1)$$

Suppose that (1) is not an exact equation.

$$M_y \neq N_x$$

TO MAKE IT EXACT

MULTIPLY w/ $\mu(x)$

$$\mu(x)\,M(x,y)\,dx + \mu(x)\,N(x,y)\,dy = 0 \qquad (2)$$

(2) is an exact equation iff

$$(\mu(x)N(x,y))_x = (\mu(x)\,M(x,y))_y$$

GARPIMIN TÜREVİ

$$\mu'(x)N(x,y) + \mu(x)N_x(x,y) = \mu(x)\,M_y(x,y)$$

MULTIPLY w/ $\frac{1}{\mu(x)N(x,y)}$

$$\mu(x)\left[M_y(x,y) - N_x(x,y)\right] = \mu'(x)N(x,y)$$

$$\frac{\mu'(x)}{\mu(x)} = \frac{M_y(x,y) - N_x(x,y)}{N(x,y)} \Rightarrow$$

(1) has an integrating factor $\mu(x)$ depends on x iff

condition

$$\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = H(x)$$

DEPENDS ONLY ON x.

$$\mu(x) = e^{\int H(x)dx}$$

PROBLEM 3

$$\underbrace{(x^2 - \sin^2 y)}_{M}\,dx + \underbrace{x\sin(2y)}_{N}\,dy = 0$$

$$\frac{M_y - N_x}{N} = \frac{-\sin(2y) - \sin(2y)}{x\sin(2y)} = \frac{-2}{x}$$

$$\mu(x) = e^{-\int \frac{2}{x}dx} = e^{-\ln x^2} = x^{-2}$$

- INTEGRATING FACTOR DEPENDING ON y:

$$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$$

$$(\mu(y)M(x,y))_y = (\mu(y)N(x,y))_x$$

$$\mu'(y)M(x,y) + \mu(y)M_y(x,y) = \mu(y)N_x(x,y)$$

$$\mu'(y)M(x,y) = \mu(y)\left[N_x(x,y) - M_y(x,y)\right]$$

$$\frac{\mu'(y)}{\mu(y)} = \frac{N_x(x,y) - M_y(x,y)}{M(x,y)}$$

BERNOULLI EQUATION

$$y'(x) + p(x)y(x) = q(x)[y(x)]^n \quad , \text{any } n. \qquad (B)$$

$$u = y^{1-n}$$

- MULTIPLY (B) by y^{-n}

$$\underbrace{y^{-n}y'}_{} + p(x)y^{1-n} = q(x)$$

$$\left(\frac{1}{1-n}y^{1-n}\right)' + P(x)y^{1-n} = q(x)$$

$(1-n)$

$$\left(\frac{1}{1-n}u\right)' + P(x)u(x) = q(x)$$

$$u'(x) + (1-n)p(x)u(x) = (1-n)q(x)$$

- $u(x) = Ce^{-(1-n)\int P(x)dx} + e^{-(1-n)\int P(x)dx}\int q(x)e^{(1-n)\int p(x)dx}dx$

$$y(x) = u^{\frac{1}{1-n}}$$

- $y' + 3y = e^{2x}y^3 \qquad u = y^{-2}$

$$y^{-3}y' + 3y^{-2} = e^{2x}$$

-2

$$\frac{-1}{2}(y^{-2})' + 3y^{-2} = e^{2x}$$

$$u'(x) - 6u(x) = -2e^{2x}$$

HOMOGENEOUS EQUATIONS

$y' = f\left(\frac{y}{x}\right)$, $u = \frac{y}{x}$, $y = xu$

$y' = u + xu' = f\left(\frac{y}{x}\right) = f(u)$

$xu' + u = f(u)$

$xu' = f(u) - u \Rightarrow \frac{du}{f(u)-u} = \frac{dx}{x}$

$u' = \frac{du}{dx}$

$\Rightarrow x\frac{du}{dx} = f(u) - u$

$$\begin{cases} y' = f(t,y) & [0,T] \\ y(0) = y & y(t) = ? \end{cases}$$

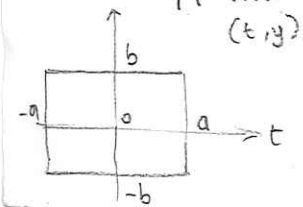
EXISTENCE AND UNIQUENESS THEOREM

Suppose that a function of two variables $f(t,y)$, $f_y(t,y)$ are continuous on a rectangle $R: |t| \le a$, $|y| \le b$ then the initial value problem

$$\begin{cases} y' = f(t,y), & (1) \\ y(0) = 0 & (2) \end{cases}$$

has a unique solution on the interval $[-h, h]$, where $h = \frac{b}{M}$

$M = \max_{(t,y)\in R} |f(t,y)|$



- $\text{sign } t = \begin{cases} 1, & t>0 \\ 0, & t=0 \\ -1, & t<0 \end{cases}$ ▶ NO SOLUTION

$y'(t) = \text{sign}(y(t))$

- $y'^2 = -5y^2 - 1$ ▶ NO SOLUTION

- $\begin{cases} y' = y^{\frac{2}{3}} \\ y(0) = 0 \end{cases}$ $y(t) \equiv 0$

$y^{-\frac{2}{3}} y' = 1$

$3\left(y^{\frac{1}{3}}\right)' = 1$

$3y^{\frac{1}{3}}(t) - 3\underbrace{y^{\frac{1}{3}}(0)}_{0} = t$

$y^{\frac{1}{3}}(t) = \frac{t}{3}$ $y(t) = \left(\frac{t}{3}\right)^3$

▶ NOT A UNIQUE SOLUTION

▶ $\frac{2}{3}y^{-\frac{1}{3}}$ → NOT CONTINUOUS CANNOT APPLY THE THEOREM

- PICARD'S ITERATION : $y_n(t) = \int_0^t f(s, y_{n-1}(s))\,d$, $n = 0, 1, \dots,$

PROOF

since $f_y(t,y)$ is continuous on R there exists $L > 0$ such that

$|f_y(t,y)| \le L$, $\forall (t,y) \in R$

WHAT IS THAT?

Thanks to MVT for derivative

$|f(t,y) - f(t,z)| = |f_y(t,y^*)(y-z)| \le |f_y(t,y^*)|\,|y-z| \le |y-z|$

- $\underbrace{|f(t,y) - f(t,z)| \le L|y-z|}_{\text{LIPSCHITZ CONDITION}}$, $\forall t \in [-a,a]$, $\forall y,z \in [-b,b]$

Integrating (1) over $(0,t)$

$$y(t)-\underbrace{y(0)}_{\text{given}} = \int_0^t f(s,y(s))\,ds$$

$$y(t) = \int_0^t f(s,y(s))\,ds \qquad (3)$$

Let us consider the sequence $\{y_n(t)\}$, $n=0,1,2,\dots$

$y_0(t)\equiv 0$

$y_1(t) = \int_0^t f(s,0)\,ds$

$y_2(t) = \int_0^t f(s,y_1(s))\,ds$

$y_3(t) = \int_0^t f(s,y_2(s))\,ds$

◆ $y_n(t) = \int_0^t f(s,y_{n-1}(s))\,ds$

$\vdots$

$y_n(t) \Rightarrow y(t)$ of (3)

will be a solution

CONTINUES

$$y_2(t) = \int_0^t f(s,0)\,ds$$

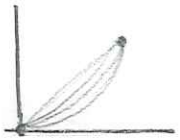
$$\Rightarrow |y_2(t)| \le \int_0^t |f(s,0)|\,ds \le Mt \le b$$

$\sum_{n=0}^{\infty} f_n(t)$, f_n continuous on $[a,b]$

$|f_n(t)| \le a_n$, $n=0,1,\dots$

$\sum_{n=0}^{\infty} a_n < \infty$ then $\sum_{n=0}^{\infty} f_n(t)$ converges.

▨ $\{y_n(t)$ $y_0+(y_1-y_0)+(y_2-y_1)+\dots$



▶ $\begin{cases} y'(t) = |y(t)|\,t^2 \\ y(0)=1 \end{cases}$

$$y(t) = 1 + \int_0^t |y(s)|\,s^2\,ds$$

$y_0(t)=1$

$y_1(t) = 1 + \int_0^t 1 s^2\,ds = 1 + \frac{t^3}{3}$ BIR ÖNCELİNİ KOYUYOR.

$y_2(t) = 1 + \int_0^t s^2\left(1+\frac{s^3}{3}\right)ds$

3RD WEEK

UNIQUENESS

Assume that (3) has another solution $z(t)$

$$z(t) = \int_0^t f(s,z(s))\,ds \qquad (3')$$

Subtract (3′) from (3)

$$y(t)-z(t) = \int [f(s,y(s)) - f(s,z(s))]\,ds$$

$$|y(t)-z(t)| \le \int [f(s,y(s)) - f(s,z(s))]\,ds$$

$$\le \underbrace{L}_{\text{LIPSCHITS}} \int_0^t \underbrace{|y(s)-z(s)|}_{w(t)}\,ds$$

$$\begin{cases} w'(t) \le Lw(t) \\ w(0)=0 \end{cases}$$

$$\begin{cases} w'(t) - Lw(t) \le 0 \\ w(0)=0 \end{cases} \Big/ e^{-Lt}$$

Let us subtract (1′) from (1)

$$\begin{cases} z' + p(t)z = q(t) & (1') \\ z(0)=\alpha & (2') \end{cases}$$

$y'-z'+p(t)y-p(t)z=0$

$y(0)-z(0)=0$

$w = y-z$

$$\begin{cases} w'(t)+p(t)w(t)=0 \\ w(0)=0 \end{cases} \Big/ e^{\int_0^t p(s)ds}$$

ÇARP. μ

$$\frac{d}{dt}\left[e^{\int_0^t p(s)ds}\,w(t)\right]=0 \qquad y=\frac{c}{\mu} \Rightarrow y\mu = c$$

$$\Rightarrow e^{\int_0^t p(s)ds}\,w(t) = w(0)=0$$

$w(t)\equiv 0$

$y\equiv z$ → identical

$$e^{-Lt} w'(t) - L e^{-Lt} w(t) \le 0$$

$$\frac{d}{dt}\left(e^{-LT} w(t)\right) \le 0 \quad , t \ge 0$$

$$\Rightarrow 0 \le e^{-LT} w(t) \le w(0) \le 0$$
$$\Rightarrow w(t) = 0$$

$$\int_0^t |y(s) - z(s)|\, ds = 0$$
$$|y(t) - z(t)| \equiv 0$$

LINEAR 2ND ORDER ODE'S

$f = m.a \qquad y(t), \; a(t) = y''(t)$

$\frac{1}{m} F(t, y, y') = y''$

General form

$$y'' + p(t) y' + q(t) y = g(t) \qquad (1)$$
$$y(0) = y_0 \; , \; y'(0) = y_1 \qquad (2)$$

GENERAL FORM
$y'' + p(t) y' + q(t) y = 0$

2nd order linear homogeneous equations with constant coefficients

$$y'' + a y' + b y = 0$$

- $y'' = 0 \quad (a=0, b=0)$

$(y')' = 0 \Rightarrow y' = C_1 \Rightarrow (y - c_1 t)' = 0$
$\Rightarrow y - c_1 t = C_2$
$y = c_1 t + c_2$

- $y'' + a y' = 0 \quad , y' = w \Rightarrow w' + a w = 0 \quad (b=0)$

- $y'' + b y = 0 \qquad b = -m^2$
  $y'' - m^2 y = 0 \qquad e^{mt}, e^{-mt}, \ldots$

- $y'' + m^2 y = 0 \qquad b = m^2$
  $\sin(mt), \cos(mt)$

- $y'' + a y' + b y = 0$ (H) $\qquad y = e^{rt}$
  $y' = r e^{rt}$
  $y'' = r^2 e^{rt}$

$r^2 e^{rt} + a r e^{rt} + b e^{rt}$
$(r^2 + a + b) e^{rt} = 0$

$r^2 + a r + b = 0$ (Ch)

So e^{rt} is a solution of (H) iff r satisfies the equation (Ch)

The equation (Ch) is called Characteristic equation for (H)

(i) Suppose that (Ch) has two distinct real roots $r_2 \ne r_1$. Then the functions $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are solutions of (H)

It is clear that for each C_1, C_2 - constants

$$C_1 e^{r_1 t} + C_2 e^{r_2 t} \qquad (G)$$

is also a solution of (H).

QUESTION

Is (G) a general solution?

▶ $y'' + 6y' + 5y = 0$

$r^2 + 6r + 5 = 0$

$r_{1,2} = -3 \pm 2$

$e^{-5t}, e^{-t} \quad \Big| \quad y = C_1 e^{-5t} + C_2 e^{-t}$

THEOREM (Existence and Uniqueness)

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) & (1) \\ y(0) = y_0, \quad y'(t_0) = y_1 & (2) \end{cases}$$

Suppose that $p(t), q(t), g(t)$ are continuous on some interval $[a,b]$ and $t_0 \in [a,b]$

Then the problem (1)(2) has a unique solution on $[a,b]$

▶ $$\begin{cases} y''(t) + \dfrac{1}{\cos t} y' + \dfrac{1}{2t^2-1} y = e^t \\ y(0) = 1, \quad y'(0) = 2 \end{cases}$$

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

a'nin sınırı: $2t^2 - 1 = 0$, $t = \frac{1}{\sqrt{2}}$

b'nin sınırı: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ → SMALLER!

$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ is a solution.

$$\begin{cases} y'' + ay' + by = 0 \\ y(0) = y_0, \quad y'(0) = y_0^1 \end{cases}$$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad y' = r_1 c_1 e^{r_1 t} + r_2 c_2 e^{r_2 t}$

$$\begin{cases} c_1 + c_2 = y_0 \\ c_1 r_1 + c_2 r_2 = y_0^1 \end{cases}$$

This system has a unique solution if

$$\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix} \neq 0$$

CRAMER

$r_2 - r_1 \neq 0$

$$C_1 = \frac{\begin{vmatrix} y_0 & 1 \\ y_0^1 & r_2 \end{vmatrix}}{r_2 - r_1} = \frac{r_2 y_0 - y_0^1}{r_2 - r_1} \qquad C_2 = \frac{\begin{vmatrix} 1 & y_0 \\ r_1 & y_0^1 \end{vmatrix}}{r_2 - r_1} = \frac{y_0^1 - r_1 y_0}{r_2 - r_1}$$

$$y(t) = \frac{r_2 y_0 - y_1^0}{r_2 - r_1} e^{r_2 t} + \frac{y_0^1 - r_1 y_0}{r_2 - r_1} e^{r_2 t}$$

SUPERPOSITION PRINCIPLE

$y'' + p(t)y' + q(t)y = 0 \qquad (H1)$

If y_1, y_2 are solutions of (H1) then for each constants C_1, C_2 the function

$$y = C_1 y_1 + C_2 y_2$$

is also a solution of (H_1)

$$y_1'' + p(t)y_1' + q(t)y_1 = 0$$

$$c_1 y_1'' + p(t)c_1 y_1' + q(t)c_1 y_1 = 0 \qquad (i)$$

$$c_2 y_2'' + p(t)c_2 y_2' + q(t)c_2 y_2 = 0 \qquad (ii)$$

ADDING (i) (ii)

$$\underbrace{(c_1y_1 + c_2y_2)}_{y}'' + p(t)\underbrace{(c_1y_1 + c_2y_2)}_{y}' + q(t)\underbrace{(c_1y_1 + c_2y_2)}_{y} = 0$$

$$L[\varphi] = \varphi'' + p(t)\varphi' + q(t)\varphi$$

$$L[\lambda\varphi] = \lambda L[\varphi], \quad L[\varphi_1 + \varphi_2] = L[\varphi_1] + L[\varphi_2]$$

suppose that y_1, y_2 are solutions of (H 1)

QUESTION 1

can we choose c_1, c_2 so as to satisfy also the initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0' \qquad (I1)$$

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

$$y'(t) = c_1 y_1'(t) + c_2 y_2'(t)$$

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \end{cases} \qquad (*)$$

initial value.

If $\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0$

then $(*)$ has a unique solution.

$$\begin{vmatrix} e^{2t} & 4e^{2t} \\ 2e^{2t} & 8e^{2t} \end{vmatrix} = 0 \qquad \begin{vmatrix} e^{t} & e^{2t} \\ e^{t} & 2e^{2t} \end{vmatrix} = e^{3t}$$

$$y'' - 3y' + 2y = 0$$

$$e^t, e^{2t}$$

For given functions y_1, y_2

$$W[y_1, y_2](t) := \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

The Wronski determinant

$$W[y_1, \dots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & & & \\ \vdots & & & \\ y_1^{(n-1)} & & & y_n^{(n-1)} \end{vmatrix}$$

THEOREM Let $p(t), q(t)$ be continuous on I, $y_1(t), y_2(t)$ are solutions of the equation:

$$y'' + p(t)y' + q(t)y = 0 \qquad (H1)$$

If $W[y_1, y_2](t_0) \neq 0$ for some $t_0 \in I$, the general solution of (H1) has the form;

- $y(t) = c_1 y_1(t) + c_2 y_2(t)$ Linear combination

PROOF

Let $\varphi(t)$ be an arbitrary solution of (H1)

Let us calculate $\varphi(t_0) = \varphi_0 \quad \varphi'(t_0) = \varphi_0'$

According to previous theorem, we can choose c_1, c_2 so that the function

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the initial conditions.

$$y(t_0) = \varphi_0 \ , \ y'(t_0) = \varphi_0^1 \quad *$$

so we have $\varphi(t)$ is a solution of H1 satisfying conditions

$$\varphi(t_0) = \varphi_0 \ , \ \varphi'(t_0) = \varphi_0^1$$

$y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution of (H1) satisfying

$$y(t_0) = \varphi_0 \ , \ y'(t_0) = \varphi_0^1 \quad *$$

$$\varphi \equiv y \ , \ \forall t \in I \quad (\text{uniqueness Theorem})$$

Hatanagi bir fonksiyon

- $y_1(t), y_2(t)$ defined on I are called linearly dependent if there exists k

$$y_1(t) = k\, y_2(t)$$

If y_1, y_2 are linearly dependent

$$W[y_1, y_2] \equiv 0$$

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} k y_2 & y_2 \\ k y_2' & y_2' \end{vmatrix} \equiv 0$$

- If $W[y_1, y_2](t_0) \neq 0$ then y_1, y_2 are linearly independent.

HW 2 examples indep but $W[y_1, y_2] = 0$

$f_1, f_2 \quad (0,2)$

$$f_1(t) = \begin{cases} 0 & \text{for } t \in (0,1) \\ (1-t)^2 & \text{for } t \in [1,2) \end{cases}$$

$$f_2(t) = \begin{cases} (1-t)^2 & \text{for } t \in (0,1) \\ 0 & \text{for } t \in [1,2) \end{cases}$$

They are linearly indep but $W[f_1, f_2] = 0$

DEF: A set of solutions y_1, y_2 of the equation (1) is called a fundamental set of solutions of (1) if for some $t_0 \in I$

$$W[y_1, y_2](t_0) \neq 0$$

THEO REM (Abel's Theorem)

suppose that y_1 and y_2 are solutions of (1), then the following formula holds true

- $W[y_1,y_2](t) = c\, e^{-\int p(t)dt}$ A
- $W[y_1,y_2](t) = W[y_1,y_2](t_0)\, e^{-\int_{t_0}^{t} p(s)ds}$ A_0

PROOF

$$y_1''(t) + p(t)y_1'(t) + q(t)y_1(t) = 0 \qquad (2)$$

$$y_2''(t) + p(t)y_2'(t) + q(t)y_2(t) = 0 \qquad (3)$$

MULTIPLY (2) by $-y_2(t)$ and (3) by $y_1(t)$ and add them up.

$$\underbrace{-y_2(t)y_1''(t) + y_1(t)y_2''}_{\frac{d}{dx}W[y_1,y_2](t)} \underbrace{-y_2(t)y_1'(t)p(t) + y_1(t)y_2'(t)p(t)}_{p(t)\underbrace{\left[y_1y_2' - y_1'y_2\right]}_{W[y_1,y_2](t)}} = 0$$

Then we have

$$\frac{d}{dx}W[y_1,y_2](t) + p(t)\,W[y_1,y_2](t) = 0 \Rightarrow A, A_0$$

THEO REM 3

If y_1, y_2 are solutions of the homogenous equation (1) and $W[y_1,y_2](t_0) = 0$ then y_1 and y_2 are dependent.

PROOF

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = 0 \quad \text{wronski}$$

Consider the system of equations

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = 0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = 0 \end{cases}$$

This system has a non zero solution c_1^*, c_2^* such that $c_1^{*2} + c_2^{*2} \neq 0$

We say like this to show they are not both 0.

Let us consider the function

$$\varphi(t) = c_1^* y_1(t) + c_2^* y_2(t)$$

- $c_1^* y_1(t_0) + c_2^* y_2(t_0) = \varphi(t_0) = 0$
- $c_1^* y_1'(t_0) + c_2^* y_2'(t_0) = \varphi'(t_0) = 0$

Since y_1 y_2 are solutions of (1),

$$(\phi_0)\begin{cases} \varphi''(t) + p(t)\varphi'(t) + q(t)\varphi(t) = 0 \\ \varphi(t_0) = 0, \ \varphi'(t_0) = 0 \end{cases} \Rightarrow \varphi(t) = 0, \ \forall t \in I$$

Since 0 is a solution of the problem ϕ_0 and this problem has a unique solution we have:

$\varphi(t) = 0, \forall t \Leftrightarrow y_1, y_2$ are linearly dep

$c_1^* y_1(t) + c_2^* y_2(t) = 0, \ \forall t$

REDUCTION OF ORDER

Consider the equation

$$y'' + p(t)y' + q(t)y = 0 \qquad (1)$$

Suppose that $y_1(t)$ is a solution of (1)

Look for a second solution $y_2(t)$ that is linearly indep. of $y_1(t)$ in the form

$$y_2(t) = v(t)\, y_1(t)$$

$$y_2'(t) = v' y_1 + v y_1' \qquad y_2'' = v'' y_1 + \underbrace{v' y_1' + v' y_1'}_{2v'y_1'} + v y_1''$$

$$v'' y_1 + 2v' y_1' + v y_1'' + p(t)(v' y_1 + v y_1') + q(t)\, v y_1 = 0$$

$$v\,\underbrace{(y_1'' + p(t) y_1' + q(t) y_1')}_{0} + y_1 v'' + (2y_1' + p(t) y_1) v' = 0$$

since it is solution of 1.

$$v' = w$$

$$w' + \left(\frac{2y_1'}{y_1} + p(t)\right) w = 0$$

We find w from last equation. Then $v = \int w\, dt$

Hence $y_2(t) = \int w(t)\, dt \cdot y_1(t)$

▶ $t^2 y'' + 2ty' - 2y = 0$

$$y'' + \frac{2}{t} y' - 2y = 0$$

$$y_1(t) = t$$

$$v' = z \qquad z' + \left(2\frac{1}{t} + \frac{2}{t}\right) z = 0$$

$$z' + \frac{4}{t} z = 0 \qquad \Big/ e^{\int \frac{4}{t} dt} = t^4$$

$$t^4 z' + 4t^3 z = 0$$

$$(t^4 z)' = 0 \quad , z = \frac{1}{t^4}$$

$$v = \int \frac{1}{t^4}\, dt = -\frac{1}{3} t^{-3}$$

$$y_2 = \frac{-1}{3} t^{-3} \cdot y_1 = \frac{-1}{3} t^{-2}$$

SECOND ORDER HOMOGENEOUS LINEAR ODE'S WITH CONSTANT COEFFICIENTS

$$y'' + ay' + b = 0 \qquad (1)$$

$$r^2 + ar + b = 0 \qquad (2)$$

If (2) has two distinct real roots $r_1 \neq r_2$, then the general solution of (1) has the form;

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

PAGE: 9

$e^{r_1 t} \quad e^{r_2 t}$

$$\begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0$$

(#) Suppose that the characteristic equation (2) has two complex conjugate roots.

$$r_{1,2} = \frac{-a}{2} \mp \frac{1}{2}\sqrt{a^2-4b} \quad , \quad a^2-4b<0$$

$$\underline{r_1 = \lambda + i\mu} \ , \ \underline{r_2 = \lambda - i\mu} \quad , \quad \lambda = \frac{-a}{2} \ , \ \mu = \sqrt{|a^2-4b|}$$

The function

$e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$ are solutions of (1).

But (1) is homogeneous equation. Therefore each linear combination of these solutions is also a solution.

$$e^{(\lambda+i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}\cos(\mu t) + e^{\lambda t}\sin(\mu t)\, i$$

(Because $e^{ix} = \cos x + i\sin x$)

$$e^{(\lambda-i\mu)t} = e^{\lambda t}\cos(\mu t) + e^{\lambda t}\sin(\mu t)\, i$$

Then the functions

$$y_2(t) = \frac{1}{2}\left[e^{(\lambda+i\mu)t} + e^{(\lambda-i\mu)t}\right] = e^{\lambda t}\cos(\mu t) \rightarrow \text{we get real solution}$$

$$y_2(t) = \frac{1}{2i}\left[e^{(\lambda+i\mu)t} - e^{(\lambda-i\mu)t}\right] = e^{\lambda t}\sin(\mu t)$$

These functions are linearly independent since

$$\begin{vmatrix} e^{\lambda t}\cos(\mu t) & e^{\lambda t}\sin(\mu t) \\ \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t) & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos(\mu t) \end{vmatrix} \neq 0$$

$$\lambda e^{2\lambda t}\cos(\mu t)\sin(\mu t) + \mu e^{2\lambda t}\cos^2(\mu t) - \lambda e^{2\lambda t}\cos(\mu t)\sin(\mu t) + \mu e^{2\lambda t}\sin^2(\mu t)$$

$$= \mu e^{2\lambda t} \neq 0$$

μ is not zero since we said it is complex.

Hence the general solution of (1) in this case has the form:

- $y(t) = e^{\lambda t}\left[c_1\cos(\mu t) + c_2\sin(\mu t)\right]$

▶ Find the solution

$$\begin{cases} y'' - 2y' + 10y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

$$r_{1,2} = \frac{2}{2} \mp \frac{1}{2}\sqrt{4-40} = 1 \mp 3i$$

$r_1 = \lambda + i\mu \qquad r_2 = \lambda - i\mu$

$$y(t) = e^{1.t}\left[c_1\cos(3t) + c_2\sin(3t)\right]$$

$y(0) = 0 \Rightarrow c_1 = 0$

$$y'(t) = (e^t . c_2\sin(3t)) = c_2(e^t\sin(3t) + 3e^t\cos(3t)$$

$c_1 = 0$

$$y'(0) = c_2 . 3 = 1$$

$$\Rightarrow c_2 = \frac{1}{3}$$

Hence the solution is

- $y(t) = \frac{1}{3}e^t\sin(3t)$

$r_1 = r_2$

$(2) \Leftrightarrow (r-r_1)^2 = 0$

$r^2 - 2r_1 r + r_1^2 = 0$

$y'' - 2r_1 y' + r_1^2 y = 0$

$$\underbrace{(y' - r_1 y)}_{z}' - r_1 \underbrace{(y' - r_1 y)}_{z} = 0$$

$z' - r_1 z = 0$

$z = c_1 e^{r_1 t}$

Then

$y' - r_1 y = c_1 e^{r_1 t}$ multiply by $e^{-r_1 t}$

$e^{-r_1 t} y'(t) - r_1 e^{-r_1 t} y(t) = c_1$

$\Leftrightarrow \left(e^{-r_1 t} y(t)\right)' = c_1$

$\left(e^{-r_1 t} y(t) - c_1 t\right)' = 0$

$e^{-r_1 t} y(t) - c_1 t = C_2$

$y(t) = e^{r_1 t}(c_1 t + C_2)$

▶ $y'' + 6y' + 9y = 0$

$r_1 = -3$

$y(t) = e^{-3t}(c_1 t + C_2)$

LINEAR NON-HOMOGENEOUS EQUATIONS

$y'' + p(t) y' + q(t) y = g(t) \qquad (1)$

Lemma: If y_1 and y_2 are solutions of (1) then the function

$y(t) = Y_1(t) - Y_2(t)$

is a solution of the corresponding homogeneous equation.

$Y_1'' + p(t) Y_1' + q Y_1 = g(t)$

$-\ Y_2'' + p(t) Y_2' + q Y_2 = g(t)$

$(Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(Y_1 - Y_2) = 0$

THEOREM If $Y_1(t)$ is some solution of (1), $y_1(t), y_2(t)$ are linearly independent solutions of the homogeneous equation

$y'' + p(t) y' + q(t) y = 0$

then the general solution of (1), has the form

$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y_1(t)$

▶ $y'' - 2y = e^t$

$Y_1 = -e^t$

▶ $y'' + 3y = e^{5t} \qquad Y_1 = A e^{5t}$

$5A e^{5t}$

$25A e^{5t} + 3A e^{5t} = e^{5t}$

$28A = 1$

$A = \frac{1}{28}$

How to find a particular solution $Y(t)$ of (1) if you know a fundamental set of solutions y_1, y_2 of (2)

METHOD OF VARIATION OF PARAMETERS

IDEA Look for solution of (1) in the form

$$Y(t) = u(t)\,y_1(t) + v(t)\,y_2(t)$$

Bu formda degil ama böyle yazilabilir.

$$Y'(t) = u(t)\,y_1'(t) + v(t)\,y_2'(t) + \underbrace{u'(t)\,y_1(t) + v'(t)\,y_2(t)}_{(i)}$$

$$Y''(t) = u(t)\,y_1''(t) + v(t)\,y_2''(t) + u'(t)\,y_1'(t) + v'(t)\,y_1'(t)$$

(i) $u'(t)\,y_1(t) + v'(t)\,y_2(t) = 0$ (Assumption)

NEXT STEP

Plug the values of Y'' and Y' into (1)

$$u(t)\,y_1''(t) + v(t)\,y_2'' + u'(t)\,y_1'(t) + v'(t)\,y_2'(t) + p(t)\left[u(t)\,y_1'(t) + v(t)\,y_2'(t)\right]$$
$$+ q(t)\left[u(t)\,y_1(t) + v(t)\,y_2(t)\right] = g(t)$$

$$u(t)\underbrace{\left[y_1''(t) + p(t)\,y_1'(t) + q(t)\,y_1(t)\right]}_{=0} + v(t)\underbrace{\left[y_2''(t) + p(t)\,y_2'(t) + q(t)\,y_2(t)\right]}_{=0}$$

$=0$ y_1 is the solution of (2) $=0$ y_2 is the solution of (2)

$$+ u'(t)\,y_1'(t) + v'(t)\,y_2'(t) = g(t)$$

Hence we have

$$u'(t)\,y_1'(t) + v'(t)\,y_2(t) = g(t)$$

So we have a system

$$\begin{cases} u'(t)\,y_1(t) + v'(t)\,y_2(t) = 0 \\ u'(t)\,y_1'(t) + v'(t)\,y_2'(t) = g(t) \end{cases}$$

$$\begin{vmatrix} 0 & y_1(t) \\ g(t) & (y_1'' \end{vmatrix}$$

Thanks to Cramer's Rule we have

$$u'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ g(t) & y_2'(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = \frac{-g(t)\,y_2(t)}{W[y_1, y_2](t)} \Rightarrow u(t) = -\int \frac{g(t)\,y_2(t)}{W[y_1, y_2](t)}\,dt$$

$$\Rightarrow v(t) = \int \frac{g(t)\,y_1(t)}{W[y_1, y_2]}\,dt$$

Therefore

$$Y(t) = -y_1(t)\int \frac{g(t)\,y_2(t)}{W[y_1, y_2](t)}\,dt + y_2(t)\int \frac{g(t)\,y_1(t)}{W[y_1, y_2](t)}$$

THIS IS ENOUGH

$$y(t) = c_1 y_1 + c_2 y_2 - y_1(t)\int \frac{g(t)\,y_2(t)}{W[y_1, y_2](t)}\,dt + y_2(t)\int \frac{g(t)\,y_1(t)}{W[y_1, y_2](t)}\,dt$$

general solution of (1)

$y'' - 2y' - 3y = te^{-t}$

Find a general solution.

$y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0$

$r_1 = -1,\ r_2 = 3$

$y_1(t) = e^{-t}$ $y_2(t) = e^{3t}$

are linearly independent solution of the homogeneous equation

$$Y(t) = -e^{-t}\int \frac{t\,e^{-t}e^{3t}}{4e^{2t}}\,dt + e^{3t}\int \frac{t\,e^{-t}\cdot e^{-t}}{4e^{2t}}\,dt$$

$$= -e^{-t}\frac{t^2}{8} + \frac{e^{3t}}{4}\int t\,e^{-4t}\,dt$$

$$\int t\,e^{-4t}\,dt = \int t\left(-\frac{1}{4}e^{-4t}\right)'dt$$

$$= -\frac{t}{4}e^{-4t} + \frac{1}{4}\int e^{-4t}\,dt$$

$$= -\frac{t}{4}e^{-4t} - \frac{1}{16}e^{-4t}$$

$$Y(t) = -e^{-t}\frac{t^2}{8} - \frac{e^{3t}}{4}\cdot\frac{t}{4}e^{-4t} - \frac{e^{3t}}{4}\frac{1}{16}e^{-4t}$$

$$= -e^{-t}\left(\frac{t^2}{8} + \frac{t}{16} + \frac{1}{64}\right)$$

FORCED VIBRATIONS: RESONANS EFECT

$$my'' + ky = F_0 \sin(wt)$$ different from book

- $y'' + \frac{k}{m}y = \frac{F_0}{m}\sin(wt)$ $\quad \frac{k}{m} = W_0^2$

$$y'' + w_0^2 y = \frac{F_0}{m}\sin(wt)$$

$r^2 + W_0^2 = 0$ $\qquad y_1(t) = \cos(W_0 t)$

$r_{1,2} = \pm i W_0$ $\qquad y_2(t) = \sin(W_0 t)$

$c_1\cos(W_0 t) + c_2\sin(W_0 t)$ = is the general solution of $y'' + w^0 y^2 = 0$

$y(t) = C_1\cos(w_0 t) + C_2\sin(w_0 t) + Y(t)$ → since it is not homogeneous

Let $w \neq W_0$. **case 1**

$$Y(t) = A\sin(wt)$$

$$-Aw^2\cancel{\sin(wt)} + Aw_0^2\cancel{\sin(wt)} = \frac{F_0}{m}\cancel{\sin(wt)}$$

$$A(w_0^2 - w^2) = \frac{F_0}{m} \Rightarrow A = \frac{F_0}{m(w_0^2 - w^2)} \qquad Y(t) = \frac{F_0}{m(w_0^2 - w^2)}\sin(wt)$$

Thus the general solution of (V) has the form

- $y(t) = c_1\cos(w_0 t) + c_2\sin(w_0 t) + \frac{F}{m(w_0^2 - w^2)}\sin(wt)$ different from book

Let $w = W_0$ **case 2**

$$Y(t) = -\cos(wt)\int \frac{\frac{F_0}{m}\sin^2(wt)}{W}\,dt + \sin(wt)\int \frac{\frac{F_0}{m}\sin(wt)\cos(wt)}{W}$$

$$\begin{vmatrix} \cos(wt) & \sin(wt) \\ -w\sin(wt) & w\cos(wt) \end{vmatrix} = W$$

$$= -\cos(wt)\frac{F_0}{mw}\int\left(\frac{1}{2} - \frac{1}{2}\cos(2wt)\right)dt + \sin(wt)\frac{F_0}{mw}\int \frac{1}{2}\sin(2wt)\,dt$$

$$= -\cos(wt)\frac{F_0}{2mw}\,t + \frac{F_0}{2mw}\cos(wt)\frac{1}{2}\sin(2wt) - \frac{F_0}{2mw}\sin(wt)\cos(2wt)$$

$$Y(t) = -\frac{F_0}{2mw}\,t\cos(wt) + \frac{F_0}{4mw}\sin(wt)$$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

General Solution of (V) is

- $y(t) = c_1\cos(wt) + c_2\sin(wt) - \frac{F_0}{2mw}\,t\cos(wt)$

REVIEW OF POWER SERIES

$\sum_{n=0}^{\infty} a_n (x-x_0)^n = f(x)$ $\quad \{a_n\} \to$ sequence

$\sum_{n=0}^{\infty} c_n = \lim_{m\to\infty} \sum_{n=0}^{m} c_n$ $\quad \sum_{n=0}^{m} c_n = S_m$: m^{th} Partial sum

If $\lim_{m\to\infty} \sum_{n=0}^{m} c_n$ exists, then we say that the series $\sum_{n=0}^{\infty} c_n$ is convergent. } If not exists, divergent.

$S = \lim_{m\to\infty} \sum_{n=0}^{m} c_n$ Sum of series

THEOREM If $\sum_{n=0}^{\infty} c_n$ is convergent, then $c_n \to 0$ as $n\to\infty$

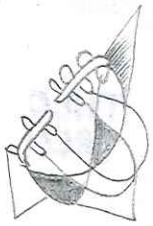
$$S_m = c_0 + c_1 + \dots + c_{m-1} + c_m$$
$$= S_{m-1} + c_m$$

$c_m = S_m - S_{m-1} \longrightarrow S - S = 0$ as $m \to \infty$

If the sequence $\{c_m\}$ is not convergent

or $\lim_{n\to\infty} c_n \neq 0$

then the series $\sum_{n=0}^{\infty} c_n$ is divergent.



- RATIO TEST

$\sum_{n=1}^{\infty} a_n$, $a_n > 0$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

If $L<1$, then the series is convergent

$L>1$, then the series is divergent

$L=1$ we do not have any conclusion.

- ROOT TEST

$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$

- COMPARISON TEST

$0 < a_n \leq b_n$

$\sum_{n=1}^{\infty} b_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$

notation: convergent

If $\sum_{n=1}^{\infty} a_n$ divergent then $\sum_{n=1}^{\infty} b_n$ divergent.

If $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum b_n < \infty$

then

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum a_n + \sum b_n$$

- $\underbrace{1 + q + q^2 + \dots + q^{n-1} + q^n}_{S_n} = \frac{1-q^{n+1}}{1-q}$

$$1 + qS_n = S_n + q^{n+1}$$

$$S_n = \frac{1-q^{n+1}}{1-q}$$

- $1 \,\big|\, \frac{1-x}{1+x+x^2+\dots}$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \lim_{m\to\infty} (1 + x + \dots + x^m)$$

$$= \lim_{m\to\infty} \frac{1-x^{m+1}}{1-x} = \frac{1}{1-x}$$

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x|<1}$$

$f(x)$ continuous

$$\frac{f(b)-f(a)}{b-a} = f'(c) \quad \Big\} \text{ Mean Value Theorem}$$

$$\frac{f(x)-f(x_0)}{x-x_0} = f'(c)$$

$$\boxed{f(x) = f(x_0) + f'(c)(x-x_0)}$$

TAYLOR'S FORMULA

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots}_{T_n(x)} + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

$$\rightarrow f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

$$\Rightarrow f(x) \approx T_n(x)$$

- f, g $[a,b]$ continuous differentiable

$g'(x) \neq 0$ on (a,b)

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(c)}{g'(c)}$$

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

- $a_n = \dfrac{f^{(n)}(x_0)}{n!}$

POWER SERIES

If a function $f(x)$ can be represented as a sum of power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \text{ on some interval}$$

then f is called a real analytic function $\longrightarrow$ if you know its a piece of graph you can form other parts

The Taylor series about the point $x_0 = 0$ is called the Mclaurent series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\sum_{n=0}^{\infty} \underbrace{a_n x^n}_{c_n} \qquad (*)$$

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$$

Hence the series $(*)$ is convergent if

$$|x| < \frac{1}{\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

If $|x| < L_1$ then series converges

If $|x| > L_1$ - series divergent

$$L_1 = \frac{1}{\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

THEOREM 2 If a power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is convergent at some point $x = x_1$ then it is convergent on the interval

$$(-|x_1|, |x_1|)$$

PROOF Since $\sum_{n=0}^{\infty} a_n x_1^n < \infty$ the general term of this series tends to 0 as $n \to \infty$

$$a_n x_1^n \longrightarrow 0 \quad \text{as } n \to \infty$$

Since $\{a_n x_1^n\}$ is convergent, it is bounded, i.e, there exists M_0 s.t.

$$|a_n x_1^n| \leq M_0 \quad , \forall n = 0, 1, \dots$$

$$|a_n x^n| = \left|a_n \frac{x^n}{x_1^n} x_1^n\right| = |a_n x_1^n| \left|\frac{x}{x_1}\right|^n \leq M_0 \left|\frac{x}{x_1}\right|^n$$

$$|a_n x^n| \leq M_0 \left|\frac{x}{x_1}\right|^n \quad , \forall |x| < |x_1|$$

It is clear that

$$\sum_{n=0}^{\infty} M_0 \left|\frac{x}{x_1}\right|^n = \frac{M_0}{1 - \left|\frac{x}{x_1}\right|}$$

Thus the series $\sum_{n=0}^{\infty} |a_n x^n|$ is also convergent

$$|x| < |x_1|$$

If a power series is convergent, then the set of convergence is a symmetric interval

$(-r, r)$ and divergent for $|x| > r$

r-radius of convergence.

THEOREM If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges on $(-r, r)$ to a function f, $\left(\longrightarrow f(x) = \sum_{n=0}^{\infty} a_n x^n\right)$

Then the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also convergent on $(-r, r)$

moreover

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

it is differentiable then $f''(x)$ $f'''(x)$ ...
(it can be represented as power series)

The series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ is also convergent on $(-r, r)$

and $\int_0^x f(s)\,ds = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

Recall

$|x - x_0| < L_1$ $\qquad L_1 = \lim_{n \to \infty} \left|\frac{a_n}{a_{n+1}}\right|$

$|x - x_0| > L_1$

$$\sum_{n=0}^{\infty} a_n x^n \quad (-r, r)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} \quad , \quad a_n = \frac{1}{\sqrt{n}}$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ divergent

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ convergent

$$\lim_{n \to \infty} \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n+1}}} = \lim_{n \to \infty} \frac{\sqrt{n+1}}{\sqrt{n}} = 1$$

$(-1, 1)$

This series is an alternating series

$\sum_{n \to \infty}^{\infty} (-1)^n a_n$ $\qquad \{a_n\}$ is decreasing

$a_{n+1} < a_n$ and $a_n \to 0$ as $n \to \infty$

$y' - ky = 0$ $\quad y = \sum_{n=0} a_n x^n \quad y' = \sum_{n=1} n\, a_n x^{n-1}$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - k \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - k \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - k a_n \right] x^n = 0 \Rightarrow (n+1) a_{n+1} - k a_n = 0$$

$$\sum_{n=0}^{\infty} b_n x^n = 0 \qquad (-r, r)$$

$b_n = 0$

$b_0 + b_1 x + \dots + b_n x^n + \dots = 0$ place $x=0$

$\Rightarrow b_0 = 0$

$(b_1 + b_2 x + \dots + b_n x^{n-1} + \dots) = 0$ Take $x=0$

$$\sum_{n=0}^{\infty} b_n x^n = 0 \qquad x=0$$

$b_0 = 0$

$$\sum_{n=1}^{\infty} n b_n x^{n-1} = 0$$

$a_{n+1} = \frac{k}{n+1} a_n \quad n = 0, 1, 2, \dots$ <u>recurrence relation</u>

$a_1 = \frac{k}{1} a_0$

$a_2 = \frac{k}{2} a_1 = \frac{k^2}{2} a_0$

$a_3 =$

$\vdots$

$a_n = \frac{k^{n!}}{n!} a_0$

$$y(x) = \sum_{n=0}^{\infty} \frac{k^n}{n!} a_0 x^n$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(kx)^n}{n!}$$

$$y(x) = a_0 e^{kx}$$

$y'' + y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\left[(n+2)(n+1) a_{n+2} + a_n \right]}_{0} x^n = 0$$

$$a_{n+2} = -\frac{1}{(n+2)(n+1)} a_n$$

- $n=0$ $\quad$ • $a_2 = -\frac{1}{1.2} a_0$ $\quad$ • $a_4 = \frac{-1}{4.3} a_2 = \frac{1}{4!} a_0 \longrightarrow a_{2n} = \frac{(-1)^n . 1}{2n!} a_0$
- $n=1$ $\quad$ • $a_3 = -\frac{1}{3.2} a_1$ $\quad$ • $a_5 = \frac{-1}{5.4} a_3 = \frac{1}{5!} a_1 \longrightarrow a_{2n+1} = (-1)^n \frac{1}{(2n+1)!} a_1$

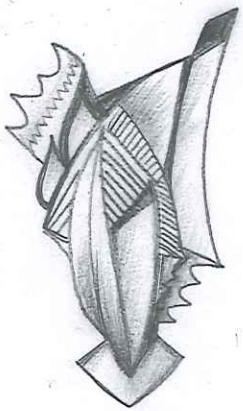
$$y(x) = a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}_{\cos x} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}_{\sin x}$$

▶ $y'' - xy' - y = 0$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\underbrace{\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x^n}_{\text{ilk elemanı}} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2.1\, a_2 - \underbrace{a_0}_{\text{son ifadenin ilk elemanı}} + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n+1)a_n\right] x^n = 0$$



⊛ $2a_2 - a_0 = 0$

⊛ $(n+2)(n+1)\, a_{n+2} - (n+1)a_n = 0$

• $a_{n+2} = \dfrac{1}{n+2}\, a_n \qquad n = 1, 2, \dots$ ✓

$a_2 = \frac{1}{2} a_0 \qquad a_3 = \frac{1}{3} a_1$

$a_4 = \frac{1}{2.4} a_0 \qquad a_5 = \frac{1}{5.3} a_1$

$a_{2n} = \frac{1}{2}\cdot\frac{1}{4}\cdots\frac{1}{2n} a_0 \qquad a_{2n+1} = \frac{1}{3}\cdot\frac{1}{5}\cdots\frac{1}{2n+1} a_1$

GENERAL SOLUTION

$$y(x) = a_0\left(\frac{1}{2} + \frac{1}{2}\frac{1}{4}x^2 + \dots + \frac{1}{2.4\cdots 2(n+1)} x^{2n} + \dots\right)$$

$$+ a_1\left(x + \frac{1}{3}x^3 + \dots + \frac{1}{1.3.5\cdots(2n+1)} x^{2n+1} + \dots\right)$$

DEF: A function f is called real analytic at the point x_0 if it has a power series expansion about this point.

$\sum a_n (x - x_0)^n$ which is convergent on some interval $(x_0 - d, x_0 + d)$

A function f is called real analytic on (a,b) if it is real analytic at each point $x_0 \in (a,b)$

THEOREM If p, q, g are real analytic functions at the point x_0 then the general solution of the equation

$$y'' + p(x)y' + q(x)y = 0 \qquad (1)$$

has the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$ of the power series which is convergent on some interval $(x_0 - d, x_0 + d)$, where p and q are real analytic

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

$y_1\ y_2$ – fundamental set of solutions on the form of power series.

x_0 is called a regular (ordinary point) for (1) if p, q are analytic at x_0.

LAPLACE TRANSFORM

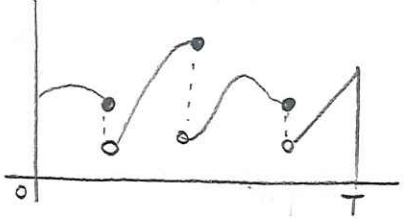
$[0, \infty)$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

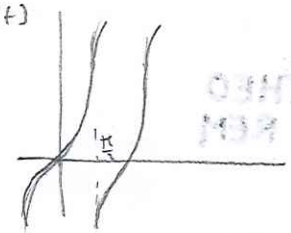
$F(t)$

DEF 1: A function f defined on interval $[0, T]$ is called piecewise continuous on this interval, if it is continuous on $[0, T]$ except of

$0 \le t_0 < t_1 < \dots < t_{n-1} < t_n \le T$ where f has one-sided limits.



$\tan t = f(t)$



DEF 2: A function $f(t)$ defined on $[0,\infty)$ is called of exponential order $a>0$ if there exist $M>0$ such that

$$|f(t)| \leq c\, e^{at} \quad , t \geq M$$

▶ $e^{2t} = 1.e^{2t}$ exponential order 2

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt}$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

▶ Let $f(t) = c$

$$\mathcal{L}\{c\} = \int_0^\infty e^{-st}\, c\,dt = c\int_0^\infty e^{-st}\,dt = \lim_{A\to\infty} c\int_0^A e^{-st}\,dt$$

$$= c \lim_{A\to\infty}\left[\frac{-1}{s}e^{-sA} + \frac{1}{s}\right] = \frac{c}{s}$$

$$\boxed{\begin{aligned} \bullet \int_0^1 \frac{dt}{t^p} &= \lim_{\varepsilon\to 0}\int_\varepsilon^1 t^{-p}\,dt = \lim_{\varepsilon\to 0}\left[\frac{t^{-p+1}}{-p+1}\Big|_{t=\varepsilon}^{1}\right] \\ &= \frac{1}{1-p} \quad \text{if} \quad p<1 \end{aligned}}$$

$$\boxed{\begin{aligned} \int_m^\infty f(t)\,dt &= \lim_{A\to\infty}\int_m^A f(t)\,dt \\ \bullet \int_1^\infty \frac{dt}{t^p} &= \lim_{A\to\infty}\int_1^A t^{-p}\,dt \\ &= \lim_{A\to\infty}\left[\frac{t^{-p+1}}{-p+1}\Big|_{t=1}^{t=A}\right] \\ &= \lim_{A\to\infty}\left[\frac{A^{-p+1}}{1-p} - \frac{1}{1-p}\right] \\ &= \frac{1}{p-1} \quad \text{if} \quad p>1 \end{aligned}}$$

$$\mathcal{L}\{c\} = \frac{c}{s}$$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\,dt = \int_0^\infty \frac{-1}{s}\left(e^{-st}\right)' t\,dt$$

$$= \underbrace{-\frac{1}{s}e^{-st}\, t\Big|_{t=0}^{t\to\infty}}_{0} + \frac{1}{s}\int_0^\infty e^{-st}$$

$$= \frac{1}{s}\left[\frac{-1}{s}e^{-st}\right]\Big|_{t=0}^{t\to\infty} = \frac{1}{s^2}$$

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2\,dt = \int_0^\infty \left(-\frac{1}{s}e^{-st}\right)' t^2\,dt$$

$$= -\frac{1}{s}e^{-st}\, t^2\Big|_{t=0}^{t\to\infty} + \frac{2}{s}\int_0^\infty e^{-st}\, t\,dt$$

$$= \frac{2}{s^3}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-(s-a)t}\,dt = \left[-\frac{1}{s-a}e^{-(s-a)t}\Big|_{t=0}^{t\to\infty}\right] = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin(bt)\} = \underbrace{\int_0^\infty e^{-st}\sin(bt)\,dt}_{I_1}$$

$$= \int_0^\infty e^{-st}\left(\frac{-1}{b}\cos(bt)\right)'dt$$

$$= \frac{1}{b}e^{-st}\cos(bt)\Big|_{t=0}^{t\to\infty} - \frac{s}{b}\int_0^\infty e^{-st}\cos(bt)\,dt$$

$$= \frac{1}{b} - \frac{s}{b}\int e^{-st}\left(\frac{1}{b}\sin(bt)\right)'dt = \frac{1}{b} - \frac{s}{b^2}e^{-st}\sin(bt)\Big|_{t=0}^{t\to\infty} - \frac{s}{b^2}\underbrace{\int_0^\infty e^{-st}\sin(bt)\,dt}_{I}$$

$$I_1 = \frac{1}{b} - \frac{s^2}{b^2} I_1$$

$$I_1 = \frac{b}{s^2+b^2}$$

since $I_1 = \mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}$

THEOREM

suppose that $f(t)$ is continuous and $f'(t)$ is piecewise continuous on $[0,\infty)$

Then

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

PROOF

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t) \Big|_{t=0}^{t\to\infty} + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

- $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ if $\mathcal{L}\{f(t)\} = F(s)$
- $\mathcal{L}\{f''(t)\} = \mathcal{L}\{(f'(t))'\} = s\mathcal{L}\{f'(t)\} - f'(0)$
$= s[sF(s) - f(0)] - f'(0)$

$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \dots$
$- s f^{(n-2)}(0)$
$- f^{(n-1)}(0)$ → bir adım geriden başladığı için

- $\mathcal{F}(t) = \int_0^t f(\tau)\,d\tau$

$\mathcal{L}\{\int_0^t f(\tau)\,dt\} = ?$

$\mathcal{F}'(t) = f(t)$

$\mathcal{L}\{\mathcal{F}'(t)\} = \mathcal{L}\{f(t)\} = F(s)$

$s\mathcal{L}\{\mathcal{F}(t)\} - \mathcal{F}(0) = F(s)$

$\Rightarrow \mathcal{L}\{\mathcal{F}(t)\} = \frac{F(s)}{s}$, $s > 0$

$F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$F'(s) = \int_0^\infty -te^{-st} f(t)\,dt = -\int_0^t e^{-st}\, t f(t)\,dt$

$F''(s) = \int_0^\infty e^{-st} t^2 f(t)\,dt$

▶ $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a)$

- $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

CONVOLUTION INTEGRAL

$f(t), g(t)$ $[0, \infty)$

$(f*g)(t) = \int_0^t f(t-\tau) g(\tau)\,d\tau$

- BILINEAR $(cf)*g) = c(f*g)(t)$
- $((f_1+f_2)*g)(t) = (f_1*g)(t) + (f_2*g)(t)$
- $((f*g)*h)(t) = (f*(g*h))(t)$
- $(f*g)(t) = (g*f)(t)$

COMMUTTATIVE

$(f*g)(t) = \int_0^t f(t-\tau) g(\tau)\,d\tau$

$$\left[\begin{array}{l} t-\tau = \eta \\ \text{If } \tau = 0,\ \eta = t \\ \text{If } \tau = t,\ \eta = 0 \\ d\eta = -d\tau \end{array}\right] \quad \begin{array}{l} = -\int_t^0 f(\eta)\, g(t-\eta)\,d\eta \\ = \int_0^t g(t-\eta) f(\eta)\,d\eta = (g*t)(f) \end{array}$$

- $\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$

If $h(t) = \int_0^t f(t-\tau) g(\tau)\,d\tau$, $H(s) = \mathcal{L}\{h(t)\}$
$F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$

Then
- $H(s) = F(s)G(s)$
- $h(t) = \mathcal{L}^{-1}\{F(s)G(s)\}$

▶ $\begin{cases} y''-4y=0 \\ y(0)=0,\ y'(0)=2 \end{cases}$

$$s^2 Y(s) - s\,y(0) - y'(0) - 4Y(s) = 0$$

$$(s^2-4)\,Y = 2$$

$$Y(s) = \frac{2}{s^2-4} = \frac{1}{2}\left(\frac{1}{s-2} - \frac{1}{s+2}\right)$$

$$y(t) = \frac{1}{2}\left(e^{2t} - e^{-2t}\right) \qquad y(t) = e^{2t} + e^{-2t}$$

▶ $\begin{cases} y''+9y=0 \\ y(0)=1,\ y'(0)=0 \end{cases}$

$$s^2 Y(s) - s\,y(0) - \cancel{y'(0)}^{\,0} + 9Y(s) = 0$$

$$(s^2+9)\,Y(s) - s = 0$$

$$\Rightarrow Y(s) = \frac{s}{s^2+9} \Rightarrow y(t) = \cos(3t)$$

● $\mathcal{L}\{\cos(bt)\} = \dfrac{s}{s^2+b^2}$

▶ $\begin{cases} y''+y=\sin t \\ y(0)=0 \quad y'(0)=0 \end{cases}$ $\qquad Y(s) = \dfrac{1}{(s^2+1)^2}$, $y(t)=?$

$$s^2 Y(s) + Y(s) = \frac{1}{1+s^2}$$

initial values

$$\mathcal{L}\{y(t)\} = \frac{1}{(1+s^2)^2} = \frac{1}{(1+s^2)}\,\frac{1}{(1+s^2)}$$

$$y(t) = \int_0^t \sin(t-\tau)\sin\tau\,d\tau$$

$$= \int_0^t [\sin t\cos\tau - \cos t\sin\tau]\sin\tau\,d\tau$$

$$= \sin t\int_0^t \sin\tau\cos\tau\,d\tau - \cos t\int_0^t \sin^2\tau\,d\tau$$

$$= \frac{1}{2}\sin t\int_0^t \sin(2\tau)\,d\tau - \frac{1}{2}\cos t\int_0^t [1-\cos(2\tau)]\,d\tau$$

HW

▶ $\begin{cases} y''+6y'+34y=0 \\ y(0)=3,\ y'(0)=1 \end{cases}$

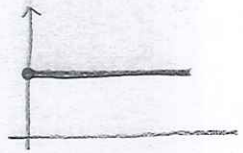
$$s^2 Y(s) - 3s - 1 + 6[sY(s) - 3] + 34Y(s) = 0$$

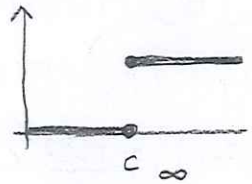
$$Y(s) = \frac{3s+19}{s^2+6s+34} = \frac{3s+19}{(s+3)^2+5^2} = 3\,\frac{s+3}{(s+3)^2+5^2} + 2\,\frac{5}{(s+3)^2+5^2}$$

$$y(t) = 3e^{-3t}\cos(5t) + 2e^{-3t}\sin(5t)$$

HEAVISIDE FUNCTION

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t\geq 0 \end{cases} \qquad u_c(t) = \begin{cases} 0, & t<c \\ 1, & t\geq c \end{cases}$$

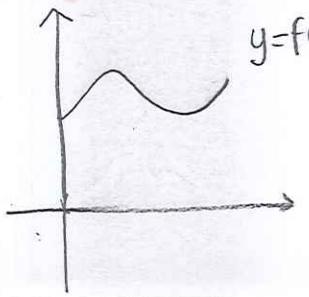


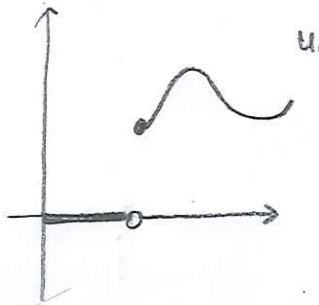


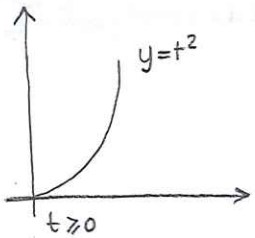
$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st}\,u_c(t)\,dt = \int_0^\infty e^{-st}\,dt$$

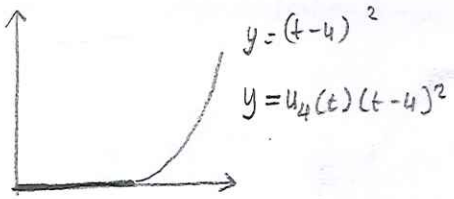
$$= -\frac{1}{s}\,e^{-st}\Big|_{t=c}^{t\to\infty} = \frac{1}{s}e^{-sc}$$

■ $\mathcal{L}\{u_c(t)\} = \dfrac{1}{s}\,e^{-sc}$







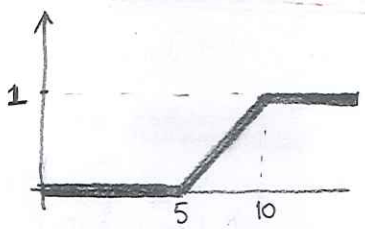


$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\,dt$$
$$= \int_0^\infty e^{-s(\tau+c)} f(\tau)\,d\tau$$
$$= \int_0^\infty e^{-sc} e^{-s\tau} f(\tau)\,d\tau$$
$$= e^{-sc}\int_0^\infty e^{-s\tau}\,d\tau = e^{-sc}F(s)$$

- $t-c=\tau$
  $t=c \Rightarrow \tau=0$
  $t\to\infty \Rightarrow t\to\infty$
  $t=\tau+c$
  $d\tau=dt$

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-sc}F(s)$$
$$\mathcal{L}^{-1}\{e^{-sc}F(s)\} = u_c(t)f(t-c)$$

$$\begin{cases} y''+4y=g(t) \\ y(0)=0,\ y'(0)=0 \end{cases} \qquad g(t)=\begin{cases} 0 & , 0\le t<5 \\ \frac{1}{5}(t-5) & , 5\le t<10 \\ 1 & , 10\le t \end{cases}$$



REWRITE

$$g(t)=\frac{1}{5}\left(u_5(t)(t-5)-u_{10}(t)(t-10)\right)$$

$$\mathcal{L}\{g(t)\} = \frac{1}{5}\left(e^{-5s}\frac{1}{s^2} - e^{-10s}\frac{1}{s^2}\right)$$

$$\underline{s^2Y(s)-y(0)t-y'(0)}+4Y(s) = \frac{1}{5s^2}\left(e^{-5s}-e^{-10s}\right)$$

$$Y(s) = \frac{1}{5}\left(e^{-5s}-e^{-10s}\right)\frac{1}{s^2(s^2+4)}$$

$$Y(s) = \frac{1}{5}\left(e^{-5s}H(s) - e^{-10s}H(s)\right)$$

$$y(t) = \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10)$$

where $h=\mathcal{L}^{-1}\{H(s)\}$

$$h(t) = \mathcal{L}^{-1}\underbrace{\left\{\frac{1}{s^2(s^2+4)}\right\}}_{\frac{1}{4}\left[\frac{1}{s^2}-\frac{1}{2}\cdot\frac{2}{s^2+4}\right]}$$

$$\mathcal{L}^{-1}\{H(s)\} = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

OR
CONVOLUTION

$$\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{2}{s^2+4}\right\} = \frac{1}{2}\int_0^t (t-\tau)\sin(2\tau)\,d\tau$$

SYSTEMS OF LINEAR FIRST ORDER ODE'S

$$(1)\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + g_1(t) \\ x_2' = a_{21}(t)x_1 + \dots + a_{2n}(t)x_n + g_2(t) \\ \dots \\ x_n' = a_{n1}(t)x_1 + \dots + a_{nn}(t)x_n + g_n(t) \end{cases}$$

a_{ij}, $ij=1,\dots,n$, g_i, $i=1,\dots,n$

$x_1, x_2, \dots, x_n$ are unknown functions

i) general solution

ii) Find solution of (1) that satisfies the initial conditions
$x_1(0)=x_1^0, \dots x_n(0)=x_n^0$

▶ $\begin{cases} x_1' = x_1(t) + x_2(t) \\ x_2' = x_1(t) - 2x_2(t) \end{cases}$

1- $\begin{cases} x_1 + x_2 = 0 \\ x_1 - 2x_2 = 0 \end{cases}$

2- $x_1 = x_2 = 0$

3- let $x_1 = v_1 e^{\lambda t}$ $\quad \lambda v_1 e^{\lambda t} = v_1 e^{\lambda t} + v_2 e^{\lambda t}$

$x_2 = v_2 e^{\lambda t}$ $\quad \lambda v_2 e^{\lambda t} = v_1 e^{\lambda t} - 2v_2 e^{\lambda t}$

$\begin{cases} v_1 + v_2 = \lambda v_1 \\ v_1 - 2v_2 = \lambda v_2 \end{cases}$ | $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$

VECTOR SPACES

Set where operations of addition of elements and multiplication by number is defined.

$C^1[0,1]$ $\quad f, g \in C^1[0,1] \longrightarrow$ not finite dimensional

$(f+g)(t) = f(t) + g(t)$

$(\lambda f)(t) = \lambda f(t)$

- $\vec{v}_1, \dots, \vec{v}_n \in V$

  $c_1 v_1 + c_2 v_2 + \dots = 0 \Rightarrow c_1 = c_2 = \dots = c_n = 0$

- Dimension of the space is the maximal number of linearly independent vectors of V.

Polynomials ≤ 2

$a_1 x^2 + a_2 x + a_3$

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$ $\quad \det A = |A|$

$\det A = a_{11}(-1)^{1+1} M_{11} + a_{12}(-1)^{1+2} M_{12} + \dots$

A^{-1} iff $|A| \neq 0$

$\begin{cases} a_{11}x_1 + \dots = b_1 \\ \vdots \\ a_{n1}x_n + \dots = b_n \end{cases}$ $\quad Ax = B$

IF $|A| \neq 0$

$A = \begin{bmatrix} a_{11} & \cdots \\ a_{1n} & a_{nn} \end{bmatrix}$ $\quad X_1 = \dfrac{\begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ b_n & a_{n1} & & a_{nn} \end{vmatrix}}{|A|}$ $\quad X_2 = \dfrac{\begin{vmatrix} a_{12} & & b_1 \\ \vdots & & \vdots \\ & & b_n \end{vmatrix}}{|A|}$

- $A\vec{x} = \vec{B}$

  $A\vec{x} = \Theta$ (* has a non-trivial solution $\Leftrightarrow |A| = 0$)

*** DEF** A number λ is called an eigenvalue of a matrix A if there exists a non-zero vector $\vec{x}$ s.t.

$$A\vec{x} = \lambda\vec{x} \qquad (1)$$

The nonzero X in (1) is called an eigenvector if A corresponding to the eigenvalue λ.

☒ Eigenvectors form an 1-dimensional vectorspace.

▶ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

$A\vec{x} = \lambda\vec{x}$ This system has a non-zero solution

$$\Leftrightarrow \begin{vmatrix} a_{11}-\lambda & & \\ a_{21} & a_{22}-\lambda & \\ a_{31} & \cdots & a_{33}-\lambda \end{vmatrix} = 0$$ <u>Characteristic equation</u>

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{vmatrix} = 0$$

$$(a_{11}-\lambda)\begin{vmatrix} a_{22}-\lambda & a_{23} \\ 0 & a_{33}-\lambda \end{vmatrix} = 0$$

$$(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) = 0$$

$\lambda_1 = a_{11}$

$\lambda_2 = a_{22}$

$\lambda_3 = a_{33}$

▶ $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$, $\begin{vmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{vmatrix} = 0$

$\lambda^2 - 1 - 3 = 0 \Rightarrow \lambda_1 = -2,\ \lambda_2 = 2$

$A\vec{x} = \lambda\vec{x} \Rightarrow (A - \lambda I)\vec{x} = 0$

$(a_{11}-\lambda)v_1 + a_{12}v_2 = 0$

$a_{21}v_1 + (a_{22}-\lambda)v_2 = 0$

$$\begin{cases} (1+2)v_1 + \sqrt{3}v_2 = 0 \\ \sqrt{3}v_1 + v_2 = 0 \end{cases}$$

$v_1 = -1$

$v_2 = \sqrt{3}$

$$\vec{v}^{(1)} = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$$

$(1-\lambda)v_1 + \sqrt{3}v_2 = 0$

$\sqrt{3}v_1 - (1+\lambda)v_2 = 0$

$-v_1 + \sqrt{3}v_2 = 0$

$\sqrt{3}v_1 - 3v_2 = 0$

$v_1 = 3,\ v_2 = \sqrt{3}$

$$\vec{v} = \begin{bmatrix} 3 \\ \sqrt{3} \end{bmatrix}$$

$\downarrow$

NOT UNIQUE

$$\left[\begin{array}{l} \alpha \mid A\vec{v} = \lambda\vec{v} \\ \alpha A\vec{v} = \alpha\lambda\vec{v} \\ A(\alpha\vec{v}) = \lambda(\alpha\vec{v}) \end{array}\right]$$

▨ $\vec{x}(t) = \cos t\, i + \sin t\, j$

$\vec{x}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix},\ t \in [0, 2\pi]$

$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$\vec{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$

▨ $(a_{11}-\lambda)v_1 + a_{12}v_2 + \dots + a_{1n}v_n = 0$ (2)

(i) $\dots$

$a_{n1}v_1 + a_{n2}v_2 + \dots + (a_{nn}-\lambda)v_n = 0$

<u>auxiliary equation</u>

(ii) Characteristic equation

After finding eigenvalues $\lambda_1, \dots, \lambda_n$, use (2) to find corresponding eigenvectors.

▶ PROBLEM 1

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

$$(1-\lambda)\begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = 0$$

$(1-\lambda)[(1-\lambda)^2 - 9] = 0$

$\lambda = -2 \quad \lambda = 1 \quad \lambda = 4$

$$(2)\begin{cases} (1-\lambda)v_1 - v_2 - v_3 = 0 \\ (1-\lambda)v_2 + 3v_3 = 0 \\ 3v_2 + (1-\lambda)v_3 = 0 \end{cases}$$

<u>$\lambda = -2$</u>

$$\begin{cases} 3v_1 - v_2 - v_3 = 0 \\ 3v_2 + 3v_3 = 0 \\ 3v_2 + 3v_3 = 0 \end{cases}$$

$v_2 - v_3 = 0$

$3v_1 = 0$

$v_1 = 0$

<u>$\lambda = 1$</u>

$$\begin{cases} -v_2 - v_3 = 0 \\ 3v_3 = 0 \\ 3v_2 = 0 \end{cases}$$

$v_2 = v_3 = 0$

$v_1 = 1$

$\downarrow$

take 1

$$V_{\lambda=1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

<u>$\lambda = 4$</u>

▶ PROBLEM 2: Show that if $v^{(1)}, v^{(2)}$ are eigenvectors of a matrix A corresponding to distinct eigenvalues λ_1, λ_2 then $v^{(1)}, v^{(2)}$ are linearly independent

We have to show that $c_1 v_1 + c_2 v_2 = 0 \Rightarrow c_1 = c_2 = 0$

$$A(c_1 \vec{v}^{(1)} + c_2 \vec{v}^{(2)}) = 0$$
$$c_1 A \vec{v}^{(1)} + c_2 A v^{(2)} = 0$$
$$c_1 \lambda_1 \vec{v}^{(1)} + c_2 \lambda_2 \vec{v}^{(2)} = 0$$

$$-\lambda_2 / \quad c_1 \vec{v}^{(1)} + c_2 \vec{v}^{(2)} = 0$$
$$\lambda_1 c_1 \vec{v}^{(1)} + \lambda_2 c_2 \vec{v}^{2} = 0$$

$$c_1(\lambda_1 - \lambda_2)\vec{v}^{1} = 0 \Rightarrow c_1 = 0$$

HW Show that if $A^T = A$ (that is A symmetric) Then eigenvectors corresponding eigenvalues are orthogonal.

$$\vec{V}_1^{(1)} \cdot \vec{V}_2^{(2)} = 0$$

$$A V_1 = \lambda_1 V_1 \Rightarrow (A - \lambda_1) V_1 = 0$$

$$A V_2 = \lambda_2 V_2 \Rightarrow (A^T - \lambda_2) V_2 = 0$$

▶ $y'' + p(t)y' + q(t)y = g(t)$ $\quad y = y_1$

$$\begin{cases} y_1' = y_2 \\ y_2' = -p(t)y_2 - q(t)y_1 + g(t) \end{cases} \qquad y_1' = y'$$

$$y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \dots + p_{n-1} y' + p_n y = g(t)$$

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \vdots \\ y_n' = -p_1 y_{n-1} - p_2 y_{n-2} - \dots - p_n y_1 + g(t) \end{cases}$$

SYSTEMS OF LINEAR FIRST ORDER ODE

$$P(t) = \begin{bmatrix} p_{11}(t) & & \\ & & \\ & & p_{nn}(t) \end{bmatrix}$$

$$\vec{x}'(t) = P(t)x(t) + g(t) \quad (1)$$
$$x'(t) = P(t)x(t) \quad (2)$$

corresponding homogeneous equation

THEO 1 REM 1- If p_{ij}, $i,j = 1, \dots, n$ $g_1, \dots, g_n$ are continuous on (a,b) and $t_0 \in (a,b)$ then the initial value problem

$$\begin{cases} x'(t) = P(t)\vec{x} + g(t)x \\ x(t_0) = \vec{x}^{\,0} \end{cases}$$

has a unique solution on (a,b)

2- If $\vec{x}^{(1)}(t)$, $\vec{x}^{(2)}(t)$ are solutions of the homogeneous equation (2) then for each c_1, c_2

$$x(t) = c_1 \vec{x}^{1}(t) + c_2 \vec{x}^{(2)}(t)$$

is also a solution of (2).

3- Let $\vec{x}^{(1)} = \begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \end{bmatrix}, \dots, \vec{x}^{(n)} = \begin{bmatrix} x_{1n} \\ x_{2n} \\ x_{nn} \end{bmatrix}$ be solutions of (2)

then the function

$$W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}](t) = \begin{vmatrix} x_{11} & x_{12} & \cdots \\ \vdots & & \\ & & x_{nn} \end{vmatrix}$$ is called the Wronskian of these solutions.

THEOREM 2 If $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ are solutions of (2) on (a,b), $t_0 \in (a,b)$ and $W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}](t_0) \neq 0$. Then the general solution of (2) has the form

$$\vec{X}(t) = c_1 \vec{x}^{(1)}(t) + \dots + c_n \vec{x}^{(n)}(t)$$

THEOREM 3 (ABEL'S) If $a_{ij}(t)$, $i,j = 1, \dots, n$, are continuous on (a,b) and $\vec{x}^{(1)}(t) = \begin{bmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{bmatrix}, \dots, \vec{x}^{(n)}(t) = \begin{bmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix}$ are solutions of (1),

then $W[\vec{x}^{(1)} \dots \vec{x}^{(n)}](t) = c e^{\int (p_{11}(t) + \dots p_{nn}(t))\, dt}$ (A)

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{bmatrix}$ $\quad trA = a_{11} + \dots + a_{nn}$ Trace.

$$\begin{cases} x_1' = P_{11}(t) x_1 + P_{12}(t) x_2 \\ x_2' = P_{21}(t) x_1 + P_{22}(t) x_2 \end{cases} \quad (2)$$

$\vec{x}^{(1)} = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$ $\quad \vec{x}^{(2)} = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}$ are the solutions

Then $W[\vec{x}^{(1)}, \vec{x}^{(2)}] = c e^{\int (P_{11}(t) + P_{22}(t))dt}$ (A_2)

Lemma

$$\frac{d}{dt} \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} = \begin{vmatrix} x_{11}'(t) & x_{12}'(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} + \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}'(t) & x_{22}'(t) \end{vmatrix}$$

$$\frac{d}{dt} \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} = \frac{d}{dt} \left(x_{11}(t) x_{22}(t) - x_{12}(t) x_{21}(t) \right)$$

$$= x_{11}'(t) x_{22}(t) - x_{12}'(t) x_{21}(t) + x_{11}(t) x_{22}'(t) - x_{12}(t) x_{21}'(t)$$

$$= \begin{vmatrix} x_{11}'(t) & x_{12}'(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} + \begin{vmatrix} x_{11} & x_{12} \\ x_{21}' & x_{22}' \end{vmatrix}$$

Suppose that

$\vec{X}^{(1)}(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix}$, $\vec{X}^{(2)}(t) = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix}$ are solutions of (2)

Then $\frac{d}{dt} W[\vec{x}^{(1)}, \vec{x}^{(2)}](t) = \frac{d}{dt} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$

Thanks to Lemma

$$= \underbrace{\begin{vmatrix} x_{11}'(t) & x_{12}'(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}}_{I_1} + \underbrace{\begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}'(t) & x_{22}'(t) \end{vmatrix}}_{I_2}$$

$$\begin{cases} x_{11}' = P_{11} x_{11} + P_{12} x_{12} \\ x_{21}' = P_{21} x_{11} + P_{22} x_{21} \end{cases} \quad (3)$$

$$\begin{cases} x_{12}' = P_{11} x_{12} + P_{12} x_{22} \\ x_{22}' = P_{21} x_{12} + P_{22} x_{22} \end{cases} \quad (4)$$

$$= I_1 + I_2 = \begin{vmatrix} P_{11}X_{11} + P_{12}X_{21} & P_{11}X_{12} + P_{12}X_{22} \\ X_{21} & X_{22} \end{vmatrix}$$

multiply second row and add to first row $\to$ w/ $-P_{12}$

This does not change the determinant

$$+ \begin{vmatrix} X_{11} & X_{12} \\ P_{21}X_{11} + P_{22}X_{21} & P_{21}X_{12} + P_{22}X_{22} \end{vmatrix}$$

multiply first row w/ $-P_{21}$ and add to second row.

$$= \begin{vmatrix} P_{11}X_{11} & P_{11}X_{12} \\ X_{21} & X_{22} \end{vmatrix} + \begin{vmatrix} X_{11} & X_{12} \\ P_{22}X_{21} & P_{22}X_{22} \end{vmatrix}$$

$$= P_{11} \begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{vmatrix} + P_{22} \begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{vmatrix} = (P_{11} + P_{22}) \begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{vmatrix}$$

DEF $\vec{X}^{(1)}, \dots, \vec{X}^{(n)}$ is called a fundamental set of solutions of

$$\vec{X}' = P(t)\vec{X} \qquad (1)$$

if the wronskian of these solutions is not zero.

So we have

$$\frac{d}{dt} W[\vec{X}^{(1)}, \vec{X}^{(2)}](t) = (P_{11}(t) + P_{22}(t))\, W[\vec{X}^{(1)}, \vec{X}^{(2)}](t) \quad \Big| \; e^{-\int (P_{11}(t) + P_{22}(t))dt}$$

$$\frac{d}{dt}\left[e^{-\int (P_{11}(t) + P_{22}(t))dt}\, W[\vec{X}^{(1)}, \vec{X}^{(2)}](t) \right] = 0$$

eger $\int_{t_0}^{t}$ yazarsan

$$W[\vec{X}^{(1)}, \vec{X}^{(2)}](t) = W[\vec{X}^{(1)}, \vec{X}^{(2)}](t_0)\, e^{\int_{t_0}^{t} (P_{11}(s) + P_{22}(s))ds}$$

PROPOSITION

If $\vec{X}^{(1)}, \dots, \vec{X}^{(n)}$ are solutions of

$$\vec{X}' = P(t)\vec{X},$$

then they are L.D. $\Leftrightarrow$

$$W[\vec{X}^{(1)}(t), \dots, \vec{X}^{(n)}(t)] = 0, \ \forall t.$$

THEOREM If $\vec{X}^{(1)}, \dots, \vec{X}^{(n)}$ is a fundamental set of solutions of (1), then the general solution of (1) has the form:

$$\vec{X}(t) = C_1 \vec{X}^{(1)}(t) + \dots + C_n \vec{X}^{(n)}(t)$$

SYSTEMS OF EQUATIONS W/ CONSTANT CO.

$$\vec{X}' = A\vec{X}$$

$$\vec{X}(t) = e^{rt}\vec{V}$$

$$\vec{X}'(t) = re^{rt}\vec{V} \Rightarrow re^{rt}\vec{V} = A(e^{rt}\vec{V})$$

$$re^{rt}\vec{V} = e^{rt}A\vec{V} \Rightarrow A\vec{V} = r\vec{V}$$

Suppose that A has n distinct real eigenvalue $r_1, r_2, \dots, r_n$ corresponding to eigenvectors $\vec{V}^{(1)}, \dots, \vec{V}^{(n)}$

Hence the vector functions

$$\vec{X}^{(1)} = e^{r_1 t}\vec{V}^{(1)}, \dots, \vec{X}^{(n)}(t) = e^{r_n t}\vec{V}^{(n)}$$

is a fundamental set of solutions

$\vec{X}(t) = C_1 \vec{X}^{(1)}(t) + \dots + C_n \vec{X}^{(n)}(t)$ is a general solution.

$A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ $\quad r_1 = -2 \quad r_2 = 2$

$\vec{V}^{(1)} = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \quad \vec{V}^{(2)} = \begin{bmatrix} 3 \\ \sqrt{3} \end{bmatrix}$

$$\begin{cases} x_1' = x_1 + \sqrt{3}\, x_2 \\ x_2' = \sqrt{3}\, x_1 - x_2 \end{cases} \qquad \vec{X}^{(1)}(t) = e^{-2t}\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \qquad \vec{X}^{(2)}(t) = e^{2t}\begin{bmatrix} 3 \\ \sqrt{3} \end{bmatrix}$$

$$\vec{X}(t) = C_1 \vec{X}^{(1)}(t) + C_2 \vec{X}^{(2)}(t) = \begin{bmatrix} -c_1 e^{-2t} + 3c_2 e^{2t} \\ \sqrt{3} c_1 e^{-2t} + \sqrt{3} c_2 e^{2t} \end{bmatrix}$$

$$\vec{X}(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix} =$$

$x_{11} = -c_1 e^{-2t} + 3c_2 e^{2t}$

$x_{21} = \sqrt{3} c_1 e^{-2t} + \sqrt{3} c_2 e^{2t}$

$\vec{X}' = A\vec{X} \quad (1)$

$\boxed{2 \times 2}$

suppose that A has two complex-conjugate eigenvalues

$r_1 = \lambda + i\mu \quad , \quad r_2 = \lambda - i\mu$

$\vec{V}^{(1)}$ and $\vec{V}^{(2)}$ are corresponding eigenvectors.

$\vec{V}^{(1)} = \vec{a} + i\vec{b} \leftarrow r_1 = \lambda + i\mu$

$\vec{X}(t) = e^{r_1 t}\vec{V}^{(1)} = e^{(\lambda + i\mu)t}(\vec{a} + i\vec{b})$ is a solution of (1)

$\vec{X}(t) = e^{\lambda t}\left(\cos(\mu t) + i\sin(\mu t)\right)(\vec{a} + i\vec{b})$

$= \underbrace{e^{\lambda t}\left(\cos(\mu t)\vec{a} - \sin(\mu t)\vec{b}\right)}_{u(t)} + i\,\underbrace{e^{\lambda t}\left(\sin(\mu t)\vec{a} + \cos(\mu t)\vec{b}\right)}_{v(t)}$

Hence the functions

$u(t) = e^{\lambda t}\left(\cos(\mu t)\vec{a} - \sin(\mu t)\vec{b}\right)$

$v(t) = e^{\lambda t}\left(\sin(\mu t)\vec{a} + \cos(\mu t)\vec{b}\right)$

are linearly independent solutions

$\vec{X}' = A\vec{X}$
$\vec{X} = u + iv$
is a solution, then
$c_1 u + c_2 v$
is the general solution

The general solution:

$$\vec{X}(t) = e^{\lambda t}\left(c_1\cos(\mu t) + c_2\sin(\mu t)\right)\vec{a} + e^{\lambda t}\left(-c_1\sin(\mu t) + c_2\cos(\mu t)\right)\vec{b}.$$

$$\begin{cases} x_1' = 4x_1 - 3x_2 \\ x_2' = 3x_1 + 4x_2 \end{cases}$$

Find the general solution

$A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$

$$\begin{cases} (4-r)v_1 - 3v_2 = 0 \\ 3v_1 + (4-r)v_2 = 0 \end{cases}$$

$\begin{vmatrix} 4-r & -3 \\ 3 & 4-r \end{vmatrix} = 0 \quad , \quad (r-4)^2 + 9 = 0$

$(r-4) = \pm 3i$

$r_1 = 4 - 3i \qquad r_2 = 4 + 3i$

Find eigenvector

$\checkmark \quad 3iv_1 - 3v_2 = 0$
$\quad\;\; 3v_1 + 3iv_2 = 0$

$v_1 = 1$
$v_2 = i$

$\rightarrow V = \begin{bmatrix} 1 \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{a}} + i\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}}$

$$\vec{X}(t) = e^{4t}\left(c_1\cos(3t) - c_2\sin(3t)\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{4t}\left(c_1\sin(3t) + c_2\cos(3t)\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= e^{4t}\begin{bmatrix} c_1\cos(3t) - c_2\sin(3t) \\ c_1\sin(3t) + c_2\cos(3t) \end{bmatrix}$$

$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_1(t) = e^{4t}\left(c_1\cos(3t) + c_2\sin(3t)\right)$
$x_2(t) = e^{4t}\left(c_1\sin(3t) + c_2\cos(3t)\right)$

$y'' + m^2 y = 0$

$r^2 + m^2 = 0 \Rightarrow r_{1,2} = \pm m i$

$y(t) = c_1 \cos(mt) + c_2 \sin(mt)$

$x_1 = y$, $x_2 = y'$, $\begin{cases} x_1' = x_2 \\ x_2' = -m^2 x_1 \end{cases}$

$A = \begin{bmatrix} 0 & 1 \\ -m^2 & 0 \end{bmatrix}$ $\quad A\vec{x} = \begin{bmatrix} x^2 \\ -mx^2 \end{bmatrix}$

$\begin{cases} -r v_1 + v_2 = 0 \\ -m^2 v_1 - r v_2 = 0 \end{cases}$

$\begin{vmatrix} -r & 1 \\ -m^2 & -r \end{vmatrix} = 0$

$r^2 + m^2 = 0$

$r_{1,2} = \pm i m$

$-im v_1 + v_2 = 0$

$-m^2 v_1 - i m v_2 = 0$

$v_1 = 1 \quad v_2 = i M$

$\vec{v}^{(1)} = \begin{bmatrix} 1 \\ im \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{a} + i \underbrace{\begin{bmatrix} 0 \\ m \end{bmatrix}}_{b}$ $\longrightarrow$ one eigenvector is enough

$\lambda = 0, \mu = m$

$\vec{x}(t) = (c_1 \cos(mt) + c_2 \sin(mt)) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-c_1 \sin(mt) + c_2 \cos(mt)) \begin{bmatrix} 0 \\ m \end{bmatrix}$

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \cos(mt) + c_2 \sin(mt) \\ -c_1 m \sin(mt) + c_2 m \cos(mt) \end{bmatrix}$

FUNDAMENTAL MATRIX

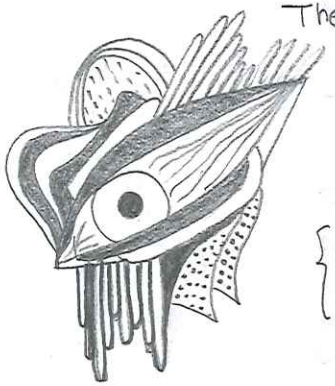
Let $\vec{x}^{(1)} = \begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \end{bmatrix}, \dots, \vec{x}^{(n)} = \begin{bmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{bmatrix}$ be a fundamental set

$(1): \vec{x}' = P(t)\vec{x}$ —— of solutions of (1)

The matrix

$$\Psi(t) = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ & & & \\ & & & x_{nn} \end{bmatrix}$$

is called a fundamental matrix of (1)



The general solution of (1) has the form

$\vec{x}(t) = \Psi(t)\vec{c}$ (2) $\qquad \vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$\begin{cases} x' = P(t)\vec{x} \\ \vec{x}(t_0) = \vec{x}^0 \end{cases}$

$x(t) = \Psi(t)\vec{c}$

$\boxed{\vec{x}(t_0) = \Psi(t_0)}$ $\boxed{\vec{x}^0 = \Psi(t_0)\vec{c}}$

$\boxed{\vec{c} = \Psi^{-1}(t_0)\vec{x}^0}$

Hence the solution of this initial value problem is

$\vec{x}(t) = \Psi(t)\Psi^{-1}(t_0)\vec{x}^0$

$\Psi'(t) = P(t)\Psi(t)$

$x_{11}' = P_{11}x_{11} + P_{12}x_{21}$ | $x_{12}' = P_{11}x_{12} + P_{12}x_{22}$

$x_{21}' = P_{21}x_{11} + P_{22}x_{21}$ | $x_{22} = P_{21}x_{12} + P_{22}x_{22}$

$\Psi' = \begin{bmatrix} x_{11}' & x_{12}' \\ x_{21}' & x_{22}' \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} = P(t)\Psi(t)$

EXPONENTIAL OF A MATRIX

$$e^{at} = 1 + \frac{t}{1}a + \frac{t^2}{2!}a^2 + \dots + \frac{t^n}{n!}a^n$$

$$e^{B} = I + \frac{1}{1!}B + \dots + \frac{1}{n!}B^n$$

$$e^{tA} = I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \dots + \frac{t^n}{n!}A^n$$

$$e^{B} = \lim_{n\to\infty}\left(I + \frac{1}{1!}B + \dots + \frac{1}{n!}B^n\right)$$

$$\frac{d}{dt}e^{tA} = Ae^{tA}$$

$$\frac{d}{dt}\left[I + \frac{t}{1!}A + \dots + \frac{t^n}{n!}A^n + \dots\right]$$

$$= \left[A + \frac{t}{1!}A^2 + \dots + \frac{t^{n-1}}{(n-1)!}A^n + \dots\right]$$

$$= A\left[I + \frac{t}{1!}A + \dots\right] = Ae^{tA}$$

RECALL

$$e^{A} = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \dots$$

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$e^{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{1!}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} + \dots + \frac{1}{n!}\begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{1}{1!}a + \frac{1}{2!}a^2 + \dots & 0 \\ 0 & 1 + \frac{1}{1!}b + \dots \end{pmatrix} = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$$

$$\frac{d}{dt}e^{tA} = Ae^{tA}$$

$$\phi(t) = e^{tA}$$

$$\left.\begin{aligned} &\phi(0) = I \\ &\frac{d}{dt}\phi(t) = A\phi(t) \quad \phi(0) = I \end{aligned}\right\}$$

DEF Two matrices are called similar matrices if there exists a non-singular matrix Q s.t. $A = QBQ^{-1}$

DEF A matrix A is called a diagonalizable matrix if A is similar to a diagonal matrix i.e. there exist a diagonal matrix D and non-singular matrix Q s.t.

- $A = QDQ^{-1}$

Lemma If A is diagonalizable, i.e.

$$A = QDQ^{-1}$$

then

- $e^{A} = Qe^{D}Q^{-1}$

$$A = QDQ^{-1},\ A^2 = QD^2Q^{-1}$$

THEOREM If $n\times n$ matrix A has n linearly independent eigenvectors then it is diagonalizable.

PROOF: Let $\vec{V}^{(1)} = \begin{bmatrix} v_{11} \\ \vdots \\ v_{n1} \end{bmatrix}, \dots, \begin{bmatrix} v_{1n} \\ \\ v_{nn} \end{bmatrix}$ be linearly independent eigenvectors of A.

Consider the matrix $Q = \begin{bmatrix} v_{11} & & \\ & & \\ v_{n1} & \dots & v_{nn} \end{bmatrix}$

Since all columns are linearly independent

$$|\phi| \neq 0 \Rightarrow Q^{-1} \text{ exists}$$

Let us denote by .

$$D=\begin{bmatrix}\lambda_1 & \cdots & 0\\ 0 & \lambda_2 & \vdots\\ \vdots & & \\ 0 & \cdots & \lambda_n\end{bmatrix}$$, where λ_k is an eigenvalue corresponding to the eigenvector $\vec{V}^{(k)}$, $k=1,\dots,n$

$$A\vec{V}^{(k)} = \lambda_k \vec{V}^{(k)}$$

Then $AQ=QD$ (*)

$$\vec{V}^{(1)}=\begin{bmatrix}v_{11}\\ v_{21}\end{bmatrix} \quad \vec{V}^{(2)}=\begin{bmatrix}v_{12}\\ v_{22}\end{bmatrix} \qquad A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$$

corresponding to λ_1, λ_2 respectively

$$Q=\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix} \qquad AQ=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}=\begin{bmatrix}(a_{11}v_{11}+a_{12}v_{21}) & \cdots\\ (a_{21}v_{11}+a_{22}v_{21}) & \cdots\end{bmatrix}$$

$$=QD$$

$$\begin{cases}x_1' = 4x_1+2x_2\\ x_2' = 3x_1-x_2\end{cases}$$

Find the fundamental matrix $\phi(t)$ which satisfies

$$\phi(0)=I$$

$$\begin{cases}\vec{X}' = A\vec{X} & (A)\\ \vec{X}(0)=\vec{X}^0 & (B)\end{cases} \qquad \vec{X}(t)=\phi(t)\vec{X}^0$$

$\vec{X}(t)=\Psi(t)\Psi^{-1}(0)\vec{X}^0$ is a solution of (A)(B)

- $e^{tA}=Qe^{tD}Q^{-1}$ $\quad Q=\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}$, $D=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$

$$A=\begin{bmatrix}4 & 2\\ 3 & -1\end{bmatrix} \qquad \phi=e^{tA} \qquad \begin{cases}(4-r)v_1+2v_2=0\\ 3v_1-(1+r)v_2=0\end{cases} \qquad \begin{vmatrix}4-r & 2\\ 3 & -(1+r)\end{vmatrix}$$

$$(r+1)(r-4)-6=0$$

$$r_1=-2 \quad r_2=5$$

$$\lambda_1=-2 \quad \lambda_2=5$$

$$6v_1+2v_2=0 \qquad \vec{V}^{(1)}=\begin{bmatrix}1\\ -3\end{bmatrix}$$
$$3v_1+v_2=0$$

$$-v_1+2v_2=0 \qquad \vec{V}^{(2)}=\begin{bmatrix}2\\ 1\end{bmatrix}$$
$$3v_1-6v_2=0$$

$$Q=\begin{bmatrix}1 & 2\\ -3 & 1\end{bmatrix}$$

$$\phi(t)=e^{tA}=Qe^{tD}Q^{-1}$$

$$e^{tA}=\begin{bmatrix}1 & 2\\ -3 & 1\end{bmatrix}\begin{bmatrix}e^{-2t} & 0\\ 0 & e^{5t}\end{bmatrix}\frac{1}{7}\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}$$

$$e^{tA}=\Psi(t)\Psi^{-1}(0)$$

$$e^{tA}\vec{X}^0=\vec{X}(t)$$

$$\Psi(t)\Psi^{-1}(0)\vec{X}^0=\vec{X}(t)$$

$$\Psi(t)=\begin{bmatrix}e^{-2t} & 2e^{5t}\\ -3e^{-2t} & e^{5t}\end{bmatrix}$$

$$\Psi^{-1}(0)=\frac{1}{7}\begin{bmatrix}1 & -2\\ 3 & 1\end{bmatrix} \qquad \frac{1}{\det}\begin{bmatrix}\cdot & \cdot\\ \cdot & \cdot\end{bmatrix}$$

$$e^{tA}=\frac{1}{7}\begin{bmatrix}e^{-2t} & 2e^{5t}\\ -3e^{-2t} & e^{5t}\end{bmatrix}\begin{bmatrix}1 & -2\\ 3 & 1\end{bmatrix}$$

MULTIPLE EIGENVALUES

$A\vec{V}^{(1)}=\lambda_1\vec{V}^{(1)}$ DOUBLE ROOT e^{r_1t}, te^{r_1t}

$\vec{X}'=A\vec{X}$, λ_1 - double eigenvalue of A corresponding to $\vec{V}^{(1)}$

$$\vec{X}^{(2)}(t)=[\vec{a}+t\vec{b}]e^{\lambda_1 t} \qquad \vec{a}=\begin{bmatrix}a_1\\ a_2\end{bmatrix} \quad \vec{b}=\begin{bmatrix}b_1\\ b_2\end{bmatrix}$$

$$\vec{X}^{(2)}(t) = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix} = \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) e^{\lambda_1 t}$$

$$x_{12}(t) = (a_1 + t b_1) e^{\lambda_1 t}$$

$$x_{22}(t) = (a_2 + t b_2) e^{\lambda t}$$

Carpman Türevi

$$\begin{cases} x_{12}' = 5x_{12} - x_{22} \\ x_{22}' = x_{12} + 3x_{22} \end{cases}$$

$$4(a_1 + t b_1) e^{4t} + b_1 e^{4t} = (a_1 + t b_1) e^{\lambda_1 t}$$

$$= 5(a_1 + t b_1) e^{4t} - (a_2 + t b_2) e^{4t}$$

$$4(a_2 + t b_2) e^{4t} + b_2 e^{4t}$$

$$= (a_1 + t b_1) e^{4t} + 3(a_2 + b_2) e^{4t}$$

- $4a_1 + b_1 = 5a_1 - a_2$
- $b_1 = 5b_1 - b_2$
- $4a_2 + b_2 = a_1 + 3a_2$
- $4b_2 = b_1 + 3b_2$

NON-HOMOGENEOUS LINEAR SYSTEMS

$$\vec{x}' = P\vec{x} + g \qquad (1)$$

$$\vec{x}' = P\vec{x} \qquad (2)$$

$\vec{x}^{(1)}, \vec{x}^{(2)}, \dots, \vec{x}^{(n)}$ fundamental set of solutions of (2), then the general solution of (1) has the form:

$$\vec{x}(t) = c_1\vec{x}^{(1)}(t) + \dots + c_n\vec{x}^{(n)}(t) + \vec{X}_p(t)$$

where $\vec{X}_p(t)$ is a particular solution of (1)

CASE 1

$$\vec{x}' = A\vec{x} + \vec{g}(t) \qquad (A_1)$$

Assume that A is diagonalizable.
(A has n linearly indep. eigenvectors)

$\vec{V}^{(1)}, \dots, \vec{V}^{(n)}$ corresponding to eigenvalues

$r_1, \dots, r_n$

$$A = QDQ^{-1}$$

We make the change

$$\vec{X} = Q\vec{y}$$

$$Q\vec{y}' = AQ\vec{y} + \vec{g}(t) \quad / Q^{-1}$$

$$Q^{-1}Q\vec{y}' = Q^{-1}AQ\vec{y} + Q^{-1}\vec{g}(t)$$

$$\vec{y}' = D\vec{y} + Q^{-1}\vec{g}(t)$$

$$\vec{h}(t) = Q^{-1}\vec{g}(t)$$

$$\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \vec{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \quad D\vec{y} = \begin{bmatrix} r_1 & & 0 \\ & r_2 & 0 \\ & & 0 \\ 0 & \cdots & r_n \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} r_1y_1 \\ \vdots \\ r_ny_n \end{bmatrix}$$

$$\begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} r_1y_1 \\ \vdots \\ r_ny_n \end{bmatrix} + \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \Rightarrow \begin{cases} y_1' = r_1y_1 + h_1 \\ y_n' = r_ny_n + h_n \end{cases}$$

Solving these equations you find

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \vec{X}_p = Qy$$

$\vec{x}(t)$ a solution of the initial value problem

$$\begin{cases} \vec{x}' = P(t)\vec{x} + \vec{g}(t) \\ \vec{x}(t_0) = \vec{x}^0 \end{cases}$$

is a sum of two vector functions

$\vec{X}_h(t)$ - solutions of

$$\begin{cases} \vec{x}_h' = P(t)\vec{x}_h \\ \vec{x}_h(0) = \vec{x}^0 \end{cases}$$

$\vec{X}_p(t)$ - solutions of

$$\begin{cases} \vec{x}_p' = P(t)\vec{x}_p + \vec{g}(t) \\ \vec{x}_p(t_0) = \theta \end{cases}$$

$$\vec{x}_h(t) = \Psi(t)\Psi^{-1}(t_0)\vec{x}^0$$

$$\begin{cases} x_1' = 2x_1 + x_2 + 2e^t \\ x_2' = x_1 + 2x_2 \\ x_1(0) = 2, \ x_2(0) = 1 \end{cases}$$

$\vec{x}^0$

HOMOGENEOUS

$$\rightarrow \begin{cases} x_1' = 2x_1 + x_2 \\ x_2' = x_1 + 2x_2 \\ x_1(0) = 2, \ x_2(0) = 1 \end{cases}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \begin{cases} (2-r)v_1 + v_2 = 0 \\ v_1 + (2-r)v_2 = 0 \end{cases}$$

$$\begin{vmatrix} 2-r & 1 \\ 1 & 2-r \end{vmatrix} = 0$$

$$(2-r)^2 = 1$$

$$r_1 = 1 \quad r_2 = 3$$

"Duhrer"

$$v_1 + v_2 = 0 \qquad v_1 - v_2 = 0$$

$$\vec{V}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \vec{V}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Psi(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}$$

$$X_{h:\text{homogeneous}}(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}\Psi^{-1}(0)\vec{x}^0$$

$$\Psi^{-1}(t) = \frac{1}{2e^{4t}}\begin{bmatrix} e^{3t} & -e^{3t} \\ e^t & e^t \end{bmatrix}$$

$$\Psi^{-1}(0) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\vec{x}^0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{X}_h(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so

$$\begin{cases} \vec{X}_p' = A X_p + \vec{g} \\ X_p(0) = \theta \end{cases}, \quad \vec{X}_p(t) = Q\vec{y}, \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \vec{y}' = D\vec{y} + Q^{-1}\vec{g}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \checkmark \quad \vec{g} = \begin{bmatrix} 2e^t \\ 0 \end{bmatrix} \checkmark \quad Q^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2e^t \\ 0 \end{bmatrix}$$

$$\begin{matrix} e^{-t} \\ e^{-3t} \end{matrix} \begin{cases} y_1' = y_1 + e^t & y_1(0) = 0 \\ y_2' = 3y_2 + e^t & y_2(0) = 0 \end{cases}$$

integrating factor

$$(e^{-t} y_1)' = 1$$

$$e^{-t} y_1(t) = t$$

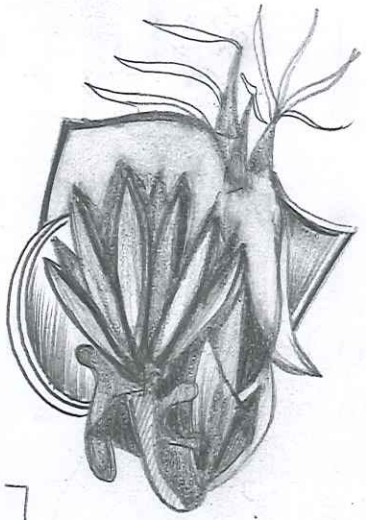
$$y_1(t) = te^t$$

$$(e^{-3t} y_2)' = e^{-2t}$$

$$e^{-3t} y_2 = \frac{1}{2}(1 - e^{2t})$$

$$y_2(t) = \frac{e^{3t}}{2}(1 - e^{2t})$$

$$\vec{X}_p = Q\vec{y} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} te^t \\ e^{3t} - \frac{1}{2}e^t \end{bmatrix}$$



METHOD OF VARIATION OF PARAMETERS

$$\vec{X}' = P\vec{X} + \vec{g} \qquad (1)$$

$$\vec{X}' = P(t)\vec{X}(t) \qquad (2)$$

$P_{ij}(t)$ $\quad P(t) = [P_{ij}]_{n \times n}$ continuous on (a,b)

$g_j(t)$, $j = 1, \dots, n$ and $t_0 \in (a,b)$

Then the problem (1),(2) has a unique solution on (a,b)

$$\vec{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

Our aim is to find a particular solution of the non-homogeneous equation (1) if we know a fundamental set of solutions of corresponding homogeneous equation (2)

Let

$$\vec{X}^{(1)} = \begin{bmatrix} x_{11} \\ \vdots \\ x_{n1} \end{bmatrix} \dots \vec{X}^{(n)} = \begin{bmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

be a fundamental set of solutions of (2) and

$$\Psi(t) = \begin{bmatrix} x_{11} & \dots & x_{n1} \\ \vdots & & \\ x_{n1} & & x_{nn} \end{bmatrix}$$

is the fundamental matrix.

Then the general solution of (2) has the form

$$\vec{X}(t) = \Psi(t)\vec{C}$$

we are going to look for solution of (1) in the form

$$X_p(t) = \Psi(t)\vec{u}(t).$$

inserting in (1) we get:

$$\underline{\Psi'(t)\vec{u}(t)} + \Psi'(t)\vec{u}'(t) = \underline{P(t)\Psi(t)\vec{u}(t)} + g(t)$$

same: because $\Psi'(t) = P(t)\Psi(t)$

We obtain

$$\Psi(t)\vec{u}'(t) = \vec{g}(t)$$

$$\begin{cases} \vec{u}'(t) = \Psi^{-1}(t)\vec{g}(t) \\ \vec{u}(0) = \theta \end{cases}$$

Recall

$x' = P(t)\vec{x} + \vec{g}(t)$ (1)

$x'(t) = P(t)\vec{x}(t)$ (2)

① find fundamental matrix of (2)

$\Psi(t)$ - fund. matrix

general solution of (2)

$$X_h(t) = \Psi(t)\vec{C}$$

② find particular solution of (1)

$$\vec{X}_p(t) = \Psi(t)\int \Psi^{-1}(t)\, g(t)\,dt$$

$$\vec{x}(t) = \Psi(t)\vec{c} + \Psi(t)\int \Psi^{-1}(t) g(t)\,dt$$

$$\vec{x}(t) = \vec{X}_h + \vec{X}_p$$

$$\vec{X}(t) = \Psi(t)\Psi^{-1}(t)\vec{x}_0 + \Psi(t)\int_0^t \Psi^{-1}(\tau)\vec{g}(\tau)\,d\tau$$

▶ Solve the initial value problem

$$x_1' = -2x_1 - 4x_2$$
$$x_2' = -x_1 + x_2 + 5e^t$$
$$x_1(0) = 2,\ x_2(0) = 1$$

$$\vec{x}' = A\vec{x} + \vec{g}$$
$$\vec{x}(0) = \vec{x}^0$$

$$A = \begin{bmatrix} -2 & -4 \\ -1 & 1 \end{bmatrix},\quad \vec{g} = \begin{bmatrix} 0 \\ 5e^t \end{bmatrix}$$

$$\vec{x}^0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

To find the fundamental matrix we find eigenvalues and eigenvectors of A.

$$\begin{cases} -(2+r)v_1 - 4v_2 = 0 \\ -v_1 + (1-r)v_2 = 0 \end{cases} \qquad \begin{vmatrix} -(2+r) & -4 \\ -1 & (1-r) \end{vmatrix} = 0$$

$$(r-1)(r+2) - 4 = 0$$
$$r^2 + r - 6 = 0$$
$$r_1 = -3,\ r_2 = 2$$

$$r_1 = -3 \rightarrow v_1 - 4v_2 = 0 \Rightarrow \vec{v}^{(1)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$r_2 = 2 \rightarrow -4v_1 - 4v_2 = 0 \Rightarrow \vec{v}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence we have the fundamental matrix

$$\Psi(t) = \begin{bmatrix} 4e^{-3t} & e^{2t} \\ e^{-3t} & -e^{2t} \end{bmatrix}$$

$$\Psi^{-1}(t) = \frac{-e^t}{5}\begin{bmatrix} -e^{2t} & -e^{2t} \\ -e^{-3t} & 4e^{-3t} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} e^{3t} & e^{3t} \\ e^{-2t} & -4e^{-2t} \end{bmatrix}$$

$$\Rightarrow \Psi^{-1}(0) = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$$

$$\vec{X}_h(t) = \begin{bmatrix} 4e^{-3t} & e^{2t} \\ e^{-3t} & -e^{2t} \end{bmatrix}\underbrace{\begin{bmatrix} 1/5 & 1/5 \\ 1/5 & -4/5 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\begin{bmatrix} 3/5 \\ -2/5 \end{bmatrix}} = \begin{bmatrix} \frac{12}{5}e^{-3t} - \frac{2}{5}e^{2t} \\ \frac{3}{5}e^{-3t} + \frac{2}{5}e^{2t} \end{bmatrix}$$

check initial values

solution of homogeneous problem.

$$\vec{X}_p(t) = \begin{bmatrix} 4e^{-3t} & e^{2t} \\ e^{-3t} & -e^{2t} \end{bmatrix}\int_0^t \Psi^{-1}(\tau)\vec{g}(\tau)\,d\tau$$

$$\Psi^{-1}(\tau)\vec{g}(\tau) = \frac{1}{5}\begin{bmatrix} e^{3\tau} & e^{3\tau} \\ e^{-2\tau} & -4e^{-2\tau} \end{bmatrix}\begin{bmatrix} 0 \\ 5e^{\tau} \end{bmatrix}$$

$$= \begin{bmatrix} e^{4\tau} \\ -4e^{-\tau} \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{-3t} & e^{2t} \\ e^{-3t} & -e^{2t} \end{bmatrix}\begin{bmatrix} \frac{1}{4}(e^{4t} - 1) \\ 4(e^{-t} - 1) \end{bmatrix}$$

FOURIER SERIES

$f(x)$ 2π - periodic function

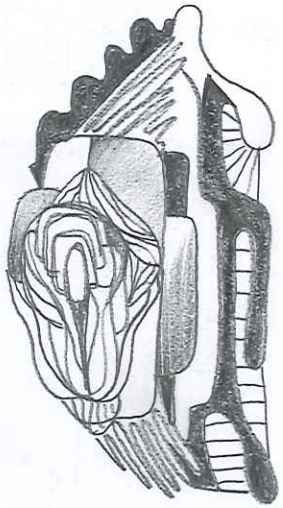
$u(x,t)$ temperature at the time t

removed heater and t times pa

$$\begin{cases} u_t = a^2 u_{xx}, \ x \in (0,\pi), t > 0 & (1) \\ u(0,t) = u(\pi,t) = 0, \ \forall t & (2) \\ u(x,0) = f(x), \ x \in [0,\pi] & (3) \end{cases}$$

(1) is called the heat equation

$a > 0$ a^2 - coefficient of heat conductivity



$u = e^{\alpha t} \sin(\beta x)$

$u_t = \alpha e^{\alpha t} \sin(\beta x)$

$u_{xx} = -\beta^2 e^{\alpha t} \sin(\beta x)$

$\cancel{\alpha e^{\alpha t} \sin(\beta x)} = -a^2 \beta^2 e^{\alpha t} \sin(\beta x)$

$\Rightarrow -\alpha = a^2 \beta^2$

$u(x,t) = e^{-a^2\beta^2 t} \sin(\beta x)$

$u(x,t) = e^{-a^2 n^2 t} \sin(nx)$

so we have seen that for each $n = 1, 2, \ldots$

$u_n(x,t) = e^{-n^2 a^2 t} \sin(nx)$

index: n

is a solution of (1) that satisfies (2)

$u^N(x,t) = \sum_{n=1}^{N} D_n e^{-n^2 a^2 t} \sin(nx)$ — solution of (1) that satisfies (2), but not (3)

for instance when $f = x(\pi - x)$

$$\boxed{u_n(x,0) = x(\pi - x)}$$

Let us try to find solution of (1) in the form

$$u(x,t) = X(x)\,T(t)$$

Insert into (1)

$$X(x)\,T'(t) = a^2 X''(x)\,T(t) \quad \Big/ \frac{1}{a^2 T(t) X(x)}$$

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda \quad (4)$$

$$\begin{cases} -X''(x) = \lambda X(x), \ x \in (0,\pi) & (5) \\ X(0) = X(\pi) = 0 & (5a) \end{cases}$$

$u(0,t) = 0$

$u(\pi,t) = 0$

$X(0)\,T(t) = 0$

$X(\pi)\,T(t) = 0$

$T'(t) = -\lambda a^2 T(t)$ (6) (*) (NEXT PAGE)

- STURM - LIOVILLE EIGENVALUE PROBLEM

$L = -\frac{d^2}{dx^2}$ $\quad Lx = \lambda x$

Let us solve the problem (5),(5a)

(i) $\boxed{\lambda=0} \Rightarrow X''(x)=0 \Rightarrow X(x)=C_1+C_2x$

$X(0)=C_1=0 \quad X(\pi)=C_2\pi=0 \Rightarrow C_2=0$

Hence $\lambda=0$ is not an eigenvalue.

(ii) $\boxed{\lambda<0}$ $\quad X''+\lambda X=0, \quad r^2+\lambda=0$

$r^2=-\lambda=|\lambda|, \quad r_{1,2}=\pm\sqrt{|\lambda|}$

$|x| = \begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$

In this case the general solution has the form

$$X(x)=C_1e^{-\sqrt{|\lambda|}x}+C_2e^{\sqrt{|\lambda|}x}$$

$X(0)=C_1+C_2=0 \qquad C_2=-C_1$

$X(\pi)=C_1e^{-\sqrt{|\lambda|}\pi}-C_1e^{\sqrt{|\lambda|}\pi}=C_1\left(e^{-\sqrt{|\lambda|}\pi}-e^{\sqrt{|\lambda|}\pi}\right)$

$=0=C_1=C_2=0$

So the problem (5),(5a) has not negative eigenvalues.

(iii) $\lambda>0$. In this case characteristic equation $r^2+\lambda=0$ has two complex conjugate roots $r_1=i\sqrt{\lambda}, r_2=-i\sqrt{\lambda}$

In this case a general solution has the form:

$$X(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$$

$X(0)=C_1=0, \quad X(\pi)=C_2\sin(\sqrt{\lambda}\pi)=0,$

$\sin(\sqrt{\lambda}\pi)=0$

$\sqrt{\lambda}\pi=\pm n\pi$

$n=1,2,\dots \qquad \boxed{\lambda_n=n^2}$

So we have seen that if $\lambda_n=n^2$, then $\sin(nx)$ satisfies (5),(5a)

In other words (5),(5a) has eigenvalues $\lambda_n=n^2$, $n=1,\dots,2$

and corresponding eigenfunctions $\sin(nx)$, $n=1,2,\dots$

$X_n(x)=\sin(nx)$ – eigenfunction corresponding to eigenvalue n^2.

$T_n'(t)=-\lambda_n a^2T_n(t) \qquad (*)$

$T_n(t)=D_ne^{-\lambda_na^2t}$ is a solution of $(*)$

Hence for each $n=1,2,\dots$

$$u_n(x,t)=D_ne^{-n^2a^2t}\sin(nx)$$

is a solution of (1), which satisfies (2)

$$u_n(x,0)=D_n\sin(nx)\overset{?}{=}f(x)$$

$$u_N(x,t)=\sum_{n=1}^{N}D_ne^{-n^2a^2t}\sin(nx)$$

is satisfying (1),(2), but not (3) for arbitrary f.

- $u(x,t)=\sum_{n=1}^{\infty}D_ne^{-n^2a^2t}\sin(nx)$
- $u(x,0)=\sum_{n=1}^{\infty}D_n\sin(nx)=f(x)$

$\left[\frac{a_n}{2}+\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))\right]$

TWO POINT BOUNDARY VALUE PROBLEMS

▶ Let $a(x)$ be a continuous function on $[a,b]$

$$\begin{cases} -y'' + a(x)y = \lambda y & (Y1) \\ y(0) = y(L_2) = 0 & (Y2) \end{cases}$$

Boundry • conditions

Mendeleyev

Show that each eigenvalue of the problem (Y1), (Y2) is a real number

Let us multiply (Y1) by corresponding eigenfunction $y(x)$ and integrate over $(0,L)$

$$-\int_0^L y''(x)y(x)dx + \int_0^L a(x)y^2(x)dx = \lambda \int_0^L y^2(x)dx \quad (**)$$

$$\int_0^L y''(x)y(x)dx = \cancel{y'y\Big|_0^L} - \int_0^L [y'(x)]^2 dx$$

Hence from $(**)$ we get

$$\lambda = \frac{\int_0^L [y'(x)]^2 dx + \int_0^L a(x)y^2(x)dx}{\int_0^L y^2(x)dx}$$

▶ $y'' + y = 0$

$y(0) = 0$, $y'(\pi) = 3$

$$y(x) = 3\sin x$$

▶ $$\begin{cases} y'' + 4y = x & (\alpha_1) \\ y(0) = y(\pi) = 0 & (\alpha_2) \end{cases}$$

Boundries

Multiply (α_1) by $\sin(2x)$ and integrate over $(0,\pi)$

$$J_1 = \int_0^\pi y''(x)\sin(2x)dx + 4\int_0^\pi y(x)\sin(2x)dx$$

$$= \int_0^\pi x\sin(2x)dx$$

$$\int_0^\pi y''(x)\sin(2x)dx = \cancel{y'\sin(2x)\Big|_{x=0}^{x=\pi}}_{0} - 2\int_0^\pi y'(x)\cos(2x)dx$$

$$= \cancel{-2y\cos(2x)\Big|_{x=0}^{x=\pi}}_{0} - 4\int_0^\pi y(x)\sin(2x)dx$$

$$\int_0^\pi y''(x)\sin(2x)dx = -4\int_0^\pi y(x)\sin(2x)dx$$

Replace it

$\Longrightarrow J = 0$

$u_t = a^2 u_{xx}, \quad x \in (0,\pi), \ t>0$

$u(0,t) = u(\pi,t) = 0 \qquad t>0$

$u(x,0) = f(x)$

$$u(x,t) = \sum_{n=1}^{\infty} D_n e^{-n^2a^2t} \sin(nx)$$

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin(nx) = f(x)$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \quad \text{Fourier Series}$$

Question

Given 2π-periodic function $f(x)$. Can we express f in form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

THEOREM If $f(x)$ is 2π-periodic function s.t. f and f' are piecewise continuous then the Fourier series of this function

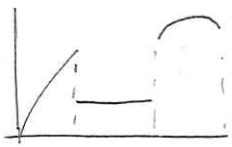
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \text{ where}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

converges to $f(x)$ at the points where f is continuous and converges to

$\frac{1}{2}[f(x^+) + f(x^-)]$ at the points x, where f is discontinuous



$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = \frac{1}{2}[f(x^+) + f(x^-)]$$

$f(x^+) = \lim_{x \to x^+} f(x) \qquad f(x^-) = \lim_{x \to x^-} f(x)$

Assume $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = f(x) \qquad (1)$

Multiplying (1) by $\cos(mx)$ and integrating over $(-\pi,\pi)$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mx)\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx + b_n\int_{-\pi}^{\pi}\cos(mx)\sin(nx)\,dx\right)$$

$$\int_{-\pi}^{\pi}\cos(mx)\,dx = \frac{1}{m}\sin(mx)\Big|_{-\pi}^{\pi} = 0$$

- $m \neq n$ $\quad \int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(m+n)x + \cos(m-n)x]\,dx = 0$

- $m = n$ $\quad \int_{-\pi}^{\pi}\cos^2(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[1+\cos 2mx]\,dx = \pi$

$\int_{-\pi}^{\pi}\cos(mx)\sin(nx) = 0$ $\quad$ ● $a_m = \frac{1}{2}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx$

Integrating (1) over $(-\pi,\pi)$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi}\frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi}\cos(nx)\,dx}_{0} + b_n \underbrace{\int_{-\pi}^{\pi}\sin(nx)\,dx}_{0}$$

$\int_{-\pi}^{\pi} f(x)\,dx = a_0\pi$ $\quad$ ● $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

Multiplying (1) by $\sin(mx)$ and integrating over $(-\pi,\pi)$

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\sin(mx)\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nx)\sin(mx)\,dx + b_n\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx\right)$$

$$= \frac{a_0}{2}\frac{1}{n}\cos(mx)\Big|_{-\pi}^{\pi}$$

- $n \neq m$ $\quad \int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx = 0$

- $n = m$ $\quad \int_{-\pi}^{\pi}\sin^2(mx)\,dx = \pi$

● $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx \qquad (\mathrm{II})$

$f(x) \sim \frac{a}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$, where a_0, a_n, b_n are defined by (I, II, III)

▶ $f(x) = x^2 \quad (-\pi,\pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\left[\frac{1}{n}\sin(nx)\right]dx$$

$$= \frac{1}{n\pi}x^2\sin(nx)\Big|_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

$$= \frac{2}{n\pi}\int_{-\pi}^{\pi} x\left(\frac{1}{n}\cos(nx)\right)dx = \frac{2}{n^2\pi}x\cos(nx)\Big|_{-\pi}^{\pi}$$

$$= \frac{4}{n^2\pi}\pi\cos(n\pi) = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\,dx = 0$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx) \qquad x \in (-\pi,\pi)$$

$$x = \pi \rightarrow \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n \qquad \boxed{\frac{2}{3}\pi^2 = 4\sum_{n=1}^{\infty}\frac{1}{n^2}} \Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

FOURIER SIN AND FOURIER COS SERIES

even functions $f(x)$ is 2ℓ-periodic function

Then
$$\int_{-\ell}^{\ell} f(x)\,dx = 2\int_{0}^{\ell} f(x)\,dx$$

If $f(x)$ is an odd function, then $\int_{-\ell}^{\ell} f(x)\,dx = 0$

THEOREM If f is continuous, f' is piecewise continuous 2π-periodic, then

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right), \text{ where}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

① f-even $\quad b_n = 0, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx, \; n = 0, 1, 2, \ldots$

- $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$ where

FOURIER COS SERIES

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

② f-odd $\quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

- $f \sim \sum_{n=1}^{\infty} b_n \sin(nx)$

FOURIER SIN SERIES

BESSEL INEQUALITY

THEOREM If $f(x)$ is 2π-periodic piecewise continuous function, then the following inequality holds true

- $$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx$$

$$S_N = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

- $$0 \le \int_{-\pi}^{\pi} \left[f(x) - S_N(x)\right]^2 = \int_{-\pi}^{\pi} \left[f(x) - \frac{a_0}{2} - \sum_{n=1}^{N} \left(a_n \cos(nx) + b_n \sin(nx)\right)\right]^2 dx$$

$$= \int_{-\pi}^{\pi} f^2(x)\,dx + \int_{-\pi}^{\pi} \left[\frac{a_0^2}{4} + \sum_{n=1}^{N} \left(a_n^2 \cos^2(nx) + b_n^2 \sin^2(nx)\right)\right.$$

$$- \int_{-\pi}^{\pi} f(x) a_0\,dx - \sum_{n=1}^{N} \left(2a_n \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx + 2b_n \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\right.$$

$$- a_0 \sum_{n=1}^{N} \left(a_n \underbrace{\int_{-\pi}^{\pi} \cos(nx)\,dx}_{0} + b_n \underbrace{\int_{-\pi}^{\pi} \sin(nx)\,dx}_{0}\right)$$

$$- 2\sum_{n=1}^{N} a_n b_n \int_{-\pi}^{\pi} \cos(nx)\sin(nx)\,dx$$

- $$0 \le \int_{-\pi}^{\pi} f^2(x)\,dx + \frac{a_0^2}{2}\pi + \pi \sum_{n=1}^{N} (a_n^2 + b_n^2)$$

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} (a_n^2 + b_n^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx$$

$\longrightarrow$ PASS TO THE LIMIT AS $N \to \infty$ AND GET THE EQUALITY

THEOREM If $f(x)$ is continuous, $f'(x)$ is piecewise continuous 2π-periodic

Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

and

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\,dx$$

PARSEVAL IDENTITY

$V, v_1, v_2, \dots, v_n$ mutually orthogonal $(v_1, v_k) = 0$, $v_1 \neq v_k$

$$\|v_1 + v_2 + \dots + v_n\|^2 = \|v_1\|^2 + \dots + \|v_n\|^2$$

$u = v_1 + \dots v_n$

$$\|u\|^2 = \sum_{n=} \|v_k\|^2$$

$$(f, g) = \int_{-\pi}^{\pi} f(x) g(x)\,dx$$

f is T-periodic

$$\int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx$$

$$F(x) = \int_x^{x+T} f(s)\,ds$$

$$F'(x) = f(x+T) - f(x) = 0$$

$\Rightarrow F(x) = \text{constant}$

$F(0) = F(a) \Rightarrow$

$$\int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx$$

$$F(x) = \int_{\varphi(x)}^{\psi(x)} f(s)\,ds$$

$$F'(x) = f(\psi(x))\,\psi'(x) - f(\varphi(x))\,\varphi'(x)$$

$$\begin{cases} u_t = a^2 u_{xx} \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} D_n e^{-n^2 a^2 t} \sin(nx)$$

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin(nx) = f(x) = \sum_{n=1}^{\infty} f_n \sin(nx)$$

$$f_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\,dx$$

$$D_n = f_n$$

$$u(x,t) = \sum_{n=1}^{\infty} f_n e^{-n^2 a^2 t} \sin(nx)$$

$f(x) = \sin x + 3\sin(3x)$

$$f_n = \frac{2}{\pi} \int_0^{\pi} \left(\sin x + 3\sin(3x)\right) \sin(nx)\,dx$$

$$\int_0^{\pi} \sin x \sin(nx)\,dx = \begin{cases} 0, & n \neq 1 \\ \frac{\pi}{2} \end{cases}$$

$$\int_0^{\pi} \sin^2 x\,dx = \frac{\pi}{2}$$

$$u(x,t) = \underbrace{e^{-t} \sin x + 3 e^{-9t} \sin(3x)}_{f_1 e^{-1 \cdot a^2 \cdot t} + f_3 e^{-9 \cdot a^2 \cdot t}}$$

$-n^2 a^2 t$

$f_1 \neq 0$

$f_3 \neq 0$

$$f_1 = \frac{2}{\pi} \int_0^{\pi} \sin x \sin(x)\,dx = \frac{2}{\pi} \int_0^{\pi} \frac{1}{2}\left[1 - \cos(2x)\right] dx = 1$$

$$f_3 = \frac{2}{\pi} \int 3\sin^2(3x)\,dx = \frac{3 \cdot 2}{\pi} \int_0^{\pi} \frac{1}{2}\left[1 - \cos(6x)\right] dx = 3$$

THEOREM (WEIRESHRASS)

If $f_n(x)$, $n=1,2,\dots$ are continuous on $[a,b]$,

$|f_n(x)| \le a_n$, $x \in [a,b]$ and $\sum_{n=1}^{\infty} a_n < \infty$,

Then $\sum_{n=1}^{\infty} f_n(x) = f(x)$, where $f(x)$ is continuous on $[a,b]$

$f'(x)$ - piecewise continuous

$$f' \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\alpha_n \cos(nx) + \beta_n \sin(nx)\right) \text{ we have}$$

$$\frac{\alpha_0^2}{2} + \sum_{n=1}^{\infty} (\alpha_n^2 + \beta_n^2) \le \frac{1}{\pi} \int_{-\pi}^{\pi} [f'(x)]^2 dx \quad (B1)$$

- $\alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos(nx) dx$

$$= \frac{1}{\pi} f(x) \cos(nx) \Big|_{x=-\pi}^{x=\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)(-n \sin(nx)) dx$$

$$= \frac{n}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = n b_n$$

$$\alpha_n = n b_n \Rightarrow b_n = \frac{1}{n} \alpha_n$$

- $\beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin(nx) dx$

$$= \frac{1}{\pi} f(x) \sin(nx) \Big|_{-\pi}^{\pi} - \frac{n}{\pi} \int$$

$$= -n a_n$$

- $a_n = -\frac{1}{n} \beta_n$

I would like to prove that

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty$$

$ab \le \frac{1}{2} a^2 + \frac{1}{2} b^2$

$(a-b)^2 \ge 0$

$$|a_n| + |b_n| = \frac{1}{n} |n a_n| + \frac{1}{n} |n b_n|$$

$$= \frac{1}{n} |\beta_n| + \frac{1}{n} |\alpha_n| \le \frac{1}{2n^2} + \frac{\beta^2}{2} + \frac{1}{2n^2} + \frac{\alpha_n^2}{2}$$

$$= \underbrace{\frac{1}{n^2} + \frac{1}{2}(\alpha_n^2 + \beta_n^2)}_{C_n}$$

From (B1) $\sum_{n=1}^{\infty} (\alpha_n^2 + \beta_n^2) < \infty$ then $\sum_{n=1}^{\infty} C_n < \infty$.

Hence $\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty$

Uniqueness of Solution

Assume that $V(x,t)$ is another solution of (1)-(3)

$$\begin{cases} V_t = a^2 V_{xx} & (1A) \\ V(0,t) = V(\pi,t) = 0 & (2A) \\ V(x,0) = f(x) & (3A) \end{cases}$$

Then the function $Z(x,t) = u(x,t) - V(x,t)$ is a solution of

$$\begin{cases} Z_t = a^2 Z_{xx} & (1B) \\ Z(0,t) = Z(\pi,t) = 0 & (2B) \\ Z(x,0) = 0, \ x \in (0,\pi) & (3B) \end{cases}$$

Multiply (1B) by Z

$$Z Z_t = a^2 Z Z_{xx}$$

$$\left(\frac{1}{2} Z^2\right)_t = a^2 (Z Z_x)_x - Z_x^2$$

Integrating last equality

$$\int_0^{\pi} \left(\frac{1}{2} Z^2(x,t)\right)_t dx = a^2 \left[Z Z_x \Big|_0^{\pi} - \int_0^{\pi} Z_x^2 dx \right]$$

$$\frac{1}{2} \underbrace{\frac{d}{dt} \int_0^{\pi} Z^2(x,t) dx}_{\le 0} = -a^2 \int_0^{\pi} Z_x^2(x,t) dx \le 0$$

Let $y(t) = \int_0^{\pi} Z^2(x,t) dx$ $\quad y'(t) \le 0 \Rightarrow y(t) \le y(0) \ \forall t > 0$

$$\int_0^{\pi} Z^2(x,t) dx \le 0$$

$$\int_0^{\pi} Z^2(x,t) dx = 0 \Longrightarrow \underline{Z = 0} \quad \forall x, t$$

HEAT EQUATION UNDER NON-ZERO BOUNDRY CONDITIONS

$$\begin{cases} u_t = a^2 u_{xx} & (1) \\ u(0,t) = T_1 \,,\, u(\pi,t) = T_2 & (2) \\ u(x,0) = f(x) & (3) \end{cases}$$

Idea

$$u(x,t) = v(x,t) + W(x)$$

where $W(x)$ is a solution of

$$\begin{cases} W''(x) = 0 \\ W(0) = T_1 \quad W(\pi) = T_2 \end{cases} \quad (W)$$

and V is a solution of

$$\begin{cases} V_t = a^2 V_{xx} \\ V(0,t) = V(\pi,t) = 0 \\ V(x,0) = f(x) - W(x) \end{cases}$$

$$V(x,t) = \sum_{n=1}^{\infty} g_n e^{-n^2 a^2 t} \sin(nx) \quad \text{where}$$

$$g_n = \frac{2}{\pi} \int_0^{\pi} (f(x) - W(x)) \sin(nx)\, dx$$

Solutions of (W)

$$W'' = 0 \Rightarrow W = C_1 + C_2 x$$

$$W(x) = T_1 + C_2 x$$

$$W(\pi) = T_1 + C_2 \pi = T_2 \Rightarrow c_2 = \frac{1}{\pi}(T_2 - T_1)$$

$$W(x) = T_1 + \frac{1}{\pi}(T_2 - T_1)\, x$$

$$u(x,t) = \sum_{n=1}^{\infty} g_n e^{-n^2 a^2 t} \sin(nx) + T_1 + \frac{1}{\pi}(T_2 - T_1)\, x$$

$$\begin{cases} u_t = a^2 u_{xx} - b u \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$$u = e^{-bt} v(x,t)$$

$$u_t = e^{-bt} v_t(x,t) - b e^{-bt} v(x,t) = a^2 e^{-bt} v_{xx}(x,t) - b e^{-bt} v(x,t)$$

$$e^{-bt} v_t = a^2 e^{-bt} v_{xx} \Rightarrow \begin{cases} V_t = a^2 V_{xx} \\ V(0,t) = V(\pi,t) = 0 \\ V(x,0) = f(x) \end{cases}$$

Recall

$$\begin{cases} -X'' = \lambda X \\ X(0) = X(\pi) = 0 \end{cases}$$

$\lambda_n = n^2$, corresponding eigenfunctions

$$X_n(x) = \sin(nx)$$

$$T_n'(t) = -n^2 a^2(t)\, T_n(t)$$

$$T_n(t) = D_n e^{-n^2 \int a(\tau)\, d\tau}$$

$$u(x,t) = \sum_{n=1}^{\infty} D_n e^{-n^2 \int_0^t a(\tau)\, d\tau} \sin(nx)$$

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin(nx) = f(x) = \sum_{n=1}^{\infty} f_n \sin(nx), \quad f_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

WAVE EQUATION

$$\begin{cases} u_{tt} = a^2 u_{xx} \,,\, x \in (0,\pi),\, t > 0 & (1) \\ u(x,0) = f(x),\; u_t(x,0) = g(x) & (2) \\ u(0,t) = u(\pi,t) = 0 & (3) \end{cases}$$

-SEPERATION OF VARIABLES-

$$u(x,t) = X(x)\, T(t)$$

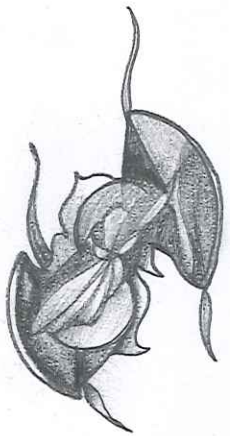
$$X(x) T''(t) = a^2 X''(x) T(t) \quad \Big/ \frac{1}{a^2 T(t) X(x)}$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda$$

① $\begin{cases} -X''(x) = \lambda X(x) \\ X(0) = X(\pi) = 0 \end{cases}$

We solved this problem

$\lambda_n = n^2$, $\sin(nx)$



$T''(t) = -\lambda a^2 T_n(t)$

$T_n'(t) = -n^2 a^2 T_n(t)$

$r^2 = -n^2 a^2$, $r_{1,2} = \pm i n a$

$T_n(t) = C_n \cos(nat) + D_n \sin(nat)$, $n = 1, 2, \ldots$

$u_n(x,t) = \left(C_n \cos(nat) + D_n \sin(nat)\right) \sin(nx)$. for each $n = 1, 2, \ldots$ $u_n(x,t)$ satisfies (1) and (3)

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos(nat) + D_n \sin(nat)\right) \sin(nx)$$

$$(2) \Leftrightarrow u(x,0) = \sum_{n=1}^{\infty} C_n \sin(nx) = f(x) = \sum_{n=1}^{\infty} f_n \sin(nx) \Rightarrow f_n = C_n$$

$$f_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-C_n na \sin(nat) + D_n na \cos(nat)\right) \sin(nx)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} D_n na \sin(nx) = g(x) = \sum_{n=1}^{\infty} g_n \sin(nx)$$

$$g_n = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx)\, dx$$

$$D_n na = g_n \Rightarrow \boxed{D_n = \frac{g_n}{na}}$$ where $g_n = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx)$

Hence the solution of the problem (1)-(3)

$$u(x,t) = \sum_{n=1}^{\infty} \left(f_n \cos(nat) + \frac{g_n}{na} \sin(nat)\right) \sin(nx)$$

uniqueness of a solution

Assume that $V(x,t)$ is another solution of (1)-(3)

Let $Z(x,t) = u(x,t) - V(x,t)$

It is clear that

$$\begin{cases} Z_{tt} = a^2 Z_{xx} \ , \ x \in (0,\pi), t > 0 & (1a) \\ Z(x,0) = 0 \ , \ Z_t(x,0) = 0, \ x \in (0,\pi) & (2a) \\ Z(0,t) = Z(\pi,t) = 0 & (3a) \end{cases}$$

$$Z_t Z_{tt} = a^2 Z_t Z_{xx}$$

$$\left(\frac{1}{2} Z_t^2\right)_t = a^2 \left((Z_t Z_x)_x - Z_{tx} Z_x\right)$$

We integrate last equality w/ respect to x over $(0,\pi)$

$$\frac{1}{2} \int_0^{\pi} (Z_t^2)_t\, dx = a^2 \left[\int_0^{\pi} (Z_t Z_x)_x\, dx - \frac{1}{2} \int_0^{\pi} (Z_x^2)_t\, dx \right]$$

$$\int_0^{\pi} (Z_t Z_x)_x\, dx = Z_t Z_x \Big|_0^{\pi} = Z_t(\pi,t) Z_x(0,t) - Z_t(0,t) Z_x(0,t) = 0$$

Since $Z(0,t) = Z(\pi,t) = 0 \ \forall t > 0$

$Z_t(0,t) = Z_t(\pi,t) = 0$, $\forall t > 0$

$$\frac{1}{2}\int_0^{\pi}(z_t^2)_t\,dx = -\frac{a^2}{2}\int_0^{\pi}(z_x^2)_t\,dx$$

$$\frac{d}{dt}\underbrace{\int_0^{\pi}\left[\frac{1}{2}z_t^2(x,t)+\frac{a^2}{2}z_x^2(x,t)\right]dx}_{E(t)}=0$$

$$E'(t)=0$$

$$E(t)=E(0)$$

$$\frac{1}{2}\int(z_t^2)_t\,dx = a^2\left[\int_0^{\pi}(z_t z_x)\,dx - \frac{1}{2}\int_0^{\pi}(z_x^2)_t\,dx\right]$$

$$\int_0^{\pi}\left[z_t^2(x,t)+a^2 z_x^2(x,t)\right]dx=0 \quad \forall t>0$$

$$z_t^2+a^2 z_x^2=0 \qquad z_t=z_x=0 \Rightarrow z(x,t)=\text{const}$$

since $z(0,t)=z(\pi,t)=0$

$$\Longrightarrow z(x,t)=0$$

H(W)

① $\begin{cases} u_{tt}-a^2u_{xx}+bu_t=0 \\ u(x,0)=f(x),\ u_t(x,0)=g(x) \\ u(0,t)=u(\pi,t)=0 \end{cases}$

② $\frac{d}{dt}\int_0^{\pi}\left[\frac{1}{2}u_t^2(x,t)+\frac{1}{2}u_x^2(x,t)\right]=-b\int_0^{\pi}u_t^2(x,t)\,dx$

$$E'(t)=-b\int_0^{t}u_t^2(x,t)\,dx$$

Initial value problem for the wave equation

Cauchy

$$\begin{cases} u_{tt}=c^2u_{xx},\ x\in(-\infty,\infty),\ t>0 & (1) \\ u(x,0)=f(x),\ u_t(x,0)=g(x),\ x\in(-\infty,\infty) & (2) \end{cases}$$

— d'Alembert's Formula —

We are looking for solution in the form $u(x,t)=u(x-ct,x+ct)$

$$\xi=x-ct$$

$$\eta=x+ct$$

$$u_t=u_\xi\xi_t+u_\eta\eta_t \qquad \xi_t=-c \quad \xi_x=1$$

$$u_x=u_\xi\xi_x+u_\eta\eta_x \qquad \eta_t=c,\ \eta_x=1$$

same for any function

$$u_t=-c\cdot u_\xi+c\,u_\eta$$

$$u_x=u_\xi+u_\eta$$

$$u_{tt}=\begin{bmatrix}-c\left(u_{\xi\xi}\xi_t+u_{\xi\eta}\eta_t\right)\\ +c\left(u_{\eta\xi}\xi_t+u_{\eta\eta}\eta_t\right)\end{bmatrix}=\begin{bmatrix}-c\left(-c\cdot u_{\xi\xi}+u_{\xi\eta}c\right)\\ +c\left(-c\cdot u_{\eta\xi}+u_{\eta\eta}c\right)\end{bmatrix}$$

$$=c^2\left(u_{\xi\xi}+u_{\eta\eta}\right)-2c^2u_{\xi\eta} \quad (3)$$

$$u_{xx}=u_{\xi\xi}\xi_x+u_{\xi\eta}\eta_x+u_{\eta\xi}\xi_x+u_{\eta\eta}\eta_x$$

$$=u_{\xi\xi}+u_{\xi\eta}+u_{\eta\xi}+u_{\eta\eta}$$

$$=\left(u_{\xi\xi}+u_{\eta\eta}\right)+2u_{\xi\eta} \quad (4)$$

Inserting the values of u_{tt} and u_{xx} from (3) and (4) into the equation (1) we get

$$c^2[u_{\xi\xi}+u_{\eta\eta}]-2c^2u_{\xi\eta}-c^2[(u_{\xi\xi}+u_{\eta\eta})+2u_{\xi\eta}]=0$$

$$-4c^2u_{\xi\eta}=0$$

$$u_{\xi\eta}=0 \qquad (5)$$

$$(u_\xi)_\eta=0 \Longrightarrow u_\xi=\varphi_1(\xi)$$

$$u=\int\varphi_1(\xi)d\xi+\psi(\eta)$$

$$u=\varphi(\xi)+\psi(\xi)$$

$$u(x,t)=\varphi(x-ct)+\psi(x+ct)$$

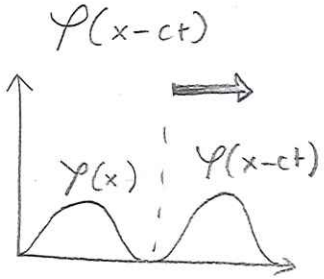
not same!

$$u_x=\varphi'(x-ct)+\psi'(x+ct)$$

$$u_{xx}=\varphi''(x-ct).1+\psi'(x+ct).1$$

$$u_t=\varphi'(x-ct)(-c)+\psi'(x+ct)(c)$$

$$u_{tt}=\varphi''(x-ct)c^2+\psi''(x+ct)c^2$$



We have to choose φ and ψ so that the initial conditions (2) be satisfied.

$$u(x,t)=\varphi(x-ct)+\psi(x+ct)$$

$$u(x,0)=\varphi(x)+\psi(x)$$

$$u_t=\varphi'(x-ct)(-c)+\psi'(x+ct)c$$

$$u_t(x,0)=-c\varphi'(x)+c\psi'(x)=g(x)$$

$$\int_0^x[-c\varphi'(s)+c\psi'(s)]ds=\int_0^x g(s)ds$$

$$-c\varphi(x)+c\psi(x)+c\varphi(0)-c\psi(0)=\int_0^x g(s)ds$$

$$-c\varphi(x)+c\psi(x)=-c\varphi(0)-c\psi(0)+\int_0^x g(s)ds$$

$$\begin{cases}\varphi(x)+\psi(x)=f(x) & (6)\\ -\varphi(x)+\psi(x)=\frac{1}{c}\int_0^x g(s)ds-\varphi(0)+\psi(0) & (7)\end{cases}$$

adding (6) and (7) we get

$$2\psi(x)=f(x)+\frac{1}{c}\int_0^x g(s)ds-\varphi(0)+\psi(0)$$

$$\psi(x)=\frac{1}{2}f(x)+\frac{1}{2c}\int_0^x g(s)ds-\frac{1}{2}\varphi(0)+\frac{1}{2}\psi(0)$$

subtracting (7) from (6) we get

$$2\varphi(x)=f(x)-\frac{1}{c}\int_0^x g(s)ds+\varphi(0)-\psi(0) \qquad (8)$$

$$\varphi(x)=\frac{1}{2}f(x)-\frac{1}{2c}\int_0^x g(s)ds+\frac{1}{2}\varphi(0)-\frac{1}{2}\psi(0) \qquad (9)$$

$u(x,t) = \varphi(x-ct) + \psi(x+ct)$

Thus employing (8) and (9) we arrive at the equality

$$u(x,t) = \frac{1}{2} f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)ds + \cancel{\frac{1}{2}\varphi(0)} - \cancel{\frac{1}{2}\psi(0)}$$

$$+\frac{1}{2} f(x+ct) + \frac{1}{2}\int_0^{x+ct} g(s)ds - \cancel{\frac{1}{2}\varphi(0)} + \cancel{\frac{1}{2}\psi(0)}$$

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

d'Alembert's Formula

▶ $u_{tt} - 2u_{xx} = 0$

$u(x,0) = \sin(2x)$

$u_t(x,0) = 0$

$$u(x,t) = \frac{1}{2}\left[\sin\left(2(x-\sqrt{2}t)\right) + \sin\left(2(x+\sqrt{2}t)\right)\right]$$