Advanced Calculus

MATH 208 Koç University
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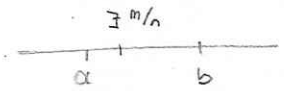
MATH 208
•CHAPTER 1•

THEOREM $\underbrace{\text{Rationals}}$ are dense in $\mathbb{R}$.

subset of $\mathbb{R}$ consisting of rational numbers $\frac{m}{n}$ where $m, n \in \mathbb{Z}$ = set of integers, $n \neq 0$

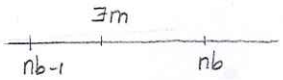
i.e. for any given open interval (a,b) there is a rational number $r \in (a,b)$



How do we think about a proof?

$$a < \frac{m}{n} < b \iff an < m < nb$$

What if $an < nb - 1 \leq m < nb$



$$an < nb - 1$$
$$1 < n(b-a)$$
$$\frac{1}{n} < b - a$$

PROOF:

Since $b-a$ is a positive real number, there is a positive integer n s.t. $\frac{1}{n} < b-a$. Also there is an integer m which satisfies $nb - 1 \leq m < nb$ Therefore

$$na < nb - 1 \leq m < nb \Rightarrow na < m < nb$$
$$\Rightarrow a < \frac{m}{n} < b$$

COMPLETENESS AXIOM OF REAL NUMBERS

Every nonempty subset of real numbers which is bounded above has a $\underbrace{\text{least upper bound}}_{\text{supremum}}$.

THEOREM Irrationals are dense in $\mathbb{R}$.

PROOF:

We want to show that $\exists$ an irrational number $\alpha \in (a,b)$

$\exists$ a rational number $r \in \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right) \Rightarrow \frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$

$$\Rightarrow a < \underbrace{r\sqrt{2}}_{\alpha} < b$$

CLAIM

α is irrational because $r\sqrt{2}$ cannot be rational

$\square$

• A few formulas

- geometric sum: $1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r}$ for any $r \neq 1$, $r \in \mathbb{R}$

  PROOF:

  $$(1-r)(1 + r + r^2 + \dots + r^n) = 1 - r^{n+1}$$

- binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

- $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + b^{n-1})$

CHAPTER 2

A sequence is a function $f: \mathbb{N} \to \mathbb{R}$
$\{1, 2, \dots\}$

$f(n) = a_n$

$\{a_n\}$ a sequence of real numbers.

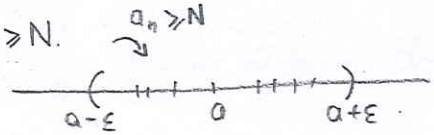
This may or may not converge

$\frac{1}{n} \to 0 \quad \lim_{n\to\infty} \frac{1}{n} = 0$

$(-1)^n$ does not converge.

DEF We say that a sequence a_n converges to a if for <u>every real positive ε</u> there is a corresponding $N \in \mathbb{N}$ s.t.

$|a_n - a| < \varepsilon$ for all $n \geq N$.



▶ Show that $\frac{1}{n} \to 0$

<u>PROOF</u> given $\varepsilon > 0$, we need to <u>find $N \in \mathbb{N}$</u> so that

$$\left|\frac{1}{n} - 0\right| < \varepsilon \text{ for all } n \geq N.$$

This means we need $N \in \mathbb{N}$ so that $\frac{1}{n} < \varepsilon \quad \forall n \geq N$.

$$\Updownarrow$$

$$\frac{1}{\varepsilon} < n$$

Just pick any natural number N so that $\frac{1}{\varepsilon} < N$

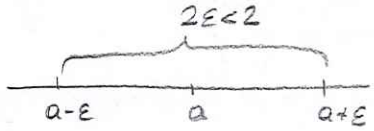
The Completeness Axiom $\Rightarrow$ (Archimedean Property)

▶ Show that $(-1)^n$ does not converge

<u>PROOF</u> Suppose $(-1)^n$ converges to $a \in \mathbb{R}$.

Choose $\varepsilon > 1$. There is $N \in \mathbb{N}$ so that

$|a_n - a| < \varepsilon \quad \forall n \geq N$



So (-1) and $+1$ cannot belong to $(a-\varepsilon, a+\varepsilon)$ at the same time.

Distance is 2

THEOREM Suppose $a_n \to a$ and $b_n \to b$ Then

(1) $a_n + b_n \to a + b$

(2) $c a_n \to ca$

(3) $a_n b_n \to ab$

(4) $\frac{a_n}{b_n} \to \frac{a}{b}$

(assume $b_n \neq 0$, $b \neq 0$)

<u>PROOF</u>

(1) Suppose $a_n \to a$ and $b_n \to b$ Then we would like to show that $\forall \varepsilon > 0 \ \exists \ N \in \mathbb{N}$ so that

$$|a_n + b_n - a - b| < \varepsilon \quad \forall n \geq N.$$

We know that <u>there is $N_1 \in \mathbb{N}$</u> so that $|a_n - a| < \frac{\varepsilon}{2}$ for all $n \geq N_1$.

There is $N_2 \in \mathbb{N}$ so that $|b_n - b| < \frac{\varepsilon}{2}$ for all $n \geq N_2$

*

Let $N = \max\{N_1, N_2\}$ For all $n \geq N$, we have

$$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

Triangle Inequality

Therefore $\lim_{n\to\infty} (a_n + b_n) = a + b$

(2) → exercise (use $\frac{\varepsilon}{c}$)

(3) Informally, we think as follows:

$$|a_n b_n - ab| < \varepsilon$$

$$a_n b_n - ab = \cancel{(a_n - a)}^{\to 0}\cancel{(b_n - b)}^{\to 0} + \cancel{ab_n}^{\to ab} + \cancel{ba_n}^{\to ba} - 2ab$$

Formally, we prove a simple lemma first

Lemma: $c_n \to 0$, $d_n \to 0 \Rightarrow c_n d_n \to 0$

Assuming that the lemma is true

$$\lim_{n\to\infty} (a_n b_n - ab) = \underbrace{\lim_{n\to\infty} \overbrace{(a_n - a)}^{c_n}\overbrace{(b_n - b)}^{d_n}}_{0} + \underbrace{\lim_{n\to\infty} a_n b}_{ab} + \underbrace{\lim_{n\to\infty} b_n a}_{ba} - 2ab$$

PROOF of the Lemma

*

$\forall \varepsilon\ \exists N_1 \in \mathbb{N}$ such that $|c_n - 0| < \sqrt{\varepsilon}$ for all $n \geq N_1$

Moreover $\exists N_2 \in \mathbb{N}_2$ such that $|d_n - 0| < \sqrt{\varepsilon}$ for all $n \geq N_2$.

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$ we have

$$|c_n d_n| = |c_n||d_n| < \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon$$

Therefore $c_n d_n \to 0$

(4) $a_n \to a$, $b_n \to b$ $\Rightarrow \frac{a_n}{b_n} \to \frac{a}{b}$ provided that $b \neq 0$, $b_n \neq 0$

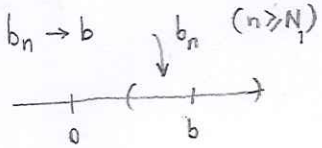
PROOF First we prove $\frac{1}{b_n} \to \frac{1}{b}$ (once we prove this proposition $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} \to a \cdot \frac{1}{b} = \frac{a}{b}$)

* Reciprocal

PROOF OF PROP What do we need?

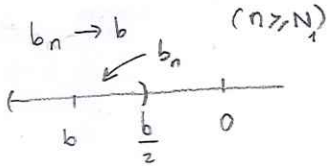
$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b b_n}\right| = \frac{1}{|b||b_n|}|b - b_n|$$

We need to get rid of b_n.



$|b_n| > \frac{b}{2}$

$\varepsilon = \frac{b}{2}$



$|b_n| > \frac{|b|}{2}$

In either case $\exists N_1 \in \mathbb{N}$ so that

$$|b_n| > \frac{|b|}{2} \Rightarrow \frac{1}{|b_n|} < \frac{2}{|b|} \quad \text{for } n \geq N_1$$

Thus

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{1}{|b||b_n|}|b_n - b| < \underbrace{\frac{2}{|b|^2}}_{c}|b_n - b| \quad \text{for each } n \geq N_1$$

Given $\varepsilon > 0$ $\exists N_2$ so that $|b_n - b| < \frac{\varepsilon}{c}$ for all $n \geq N_2$.

Let $N = \max\{N_1, N_2\}$ Therefore, for each $n \geq N$

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < c\,|b_n - b| < c \cdot \frac{\varepsilon}{c} = \varepsilon$$

2.2

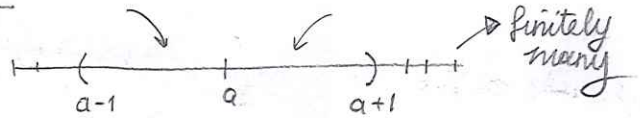
DEF: A sequence $\{a_n\}$ is <u>bounded</u> if there is a real number $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Prop If $\{a_n\}$ is convergent, then it is bounded

(The converse is not true ▶ $-1, 1, -1, 1$)

PROOF

idea



Formally

Suppose that a_n converges to the real number a.

For $\varepsilon = 1$, there is a natural number N so that $|a_n - a| < 1$ for all $n \geq N$. Then we observe

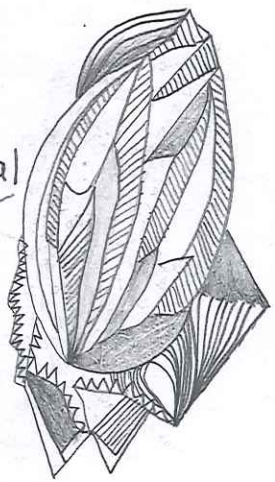
$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| \leq 1 + |a|$$

for all $n \geq N$.

Let $M = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, 1 + |a|\}$

Therefore $|a_n| \leq M$ for all $n \in \mathbb{N}$

$\Rightarrow \{a_n\}$ is bounded.



There are bounded sequences which are not convergent.

▶ $(-1)^n$ bounded but not convergent

DEF: We say that <u>a subset $S \subset \mathbb{R}$ is dense</u> in $\mathbb{R}$ if any open interval (a, b) contains an element of S.

▶ $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.

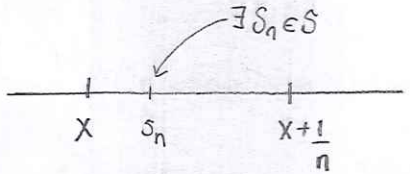
$\mathbb{Z}$ is not dense in $\mathbb{R}$.

<u>Prop</u>

A subset $S \subset \mathbb{R}$ is dense $\Leftrightarrow$ $\forall x \in \mathbb{R}$, there is a sequence in S converging to x.

<u>PROOF</u>

($\Rightarrow$) Assume that $S \subset \mathbb{R}$ is dense.



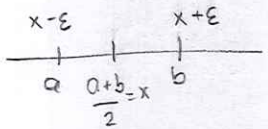
Given a real number x

$\exists S_n \in S$ so that $S_n \in \left(x, x + \frac{1}{n}\right)$ We claim that $\{S_n\}$ converges to x.

$\Longrightarrow |S_n - x| < \frac{1}{n}$ $\forall n$

$\Longrightarrow$ as $n \to \infty$ $S_n \to x$

($\Leftarrow$) Suppose for all $x \in \mathbb{R}$, there is a sequence $\{S_n\} \subset S$ s.t. $\lim_{n \to \infty} S_n = x$



$\varepsilon = b - \frac{a+b}{2} = \frac{b-a}{2} \Rightarrow \exists N \in \mathbb{N}$ s.t.

$$|s_n - x| < \varepsilon \text{ for all } n \geq N$$

$$\Rightarrow s_N \in S \cap (a,b)$$

SEQUENTIAL DENSITY OF RATIONALS

For any given $x \in \mathbb{R}$, there is a sequence of <u>rational numbers</u> converging to x.

CLOSED SETS

A subset $K \subset \mathbb{R}$ is said to be $\boxed{\text{closed}}$ if $\{a_n\}$ is a sequence in K converging to a, then $a \in K$.

- $\mathbb{Q}$ is not closed.
- There is a sequence of rational numbers converging to $\sqrt{2} \notin \mathbb{Q}$
- $[a,b]$ is a closed set in $\mathbb{R}$.
  - <u>PROOF</u>

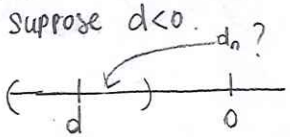
<u>Lemma</u>

Suppose $\{d_n \geq 0\}$ is a sequence converging to $d \in \mathbb{R}$.

Then $d \geq 0$.

<u>PROOF</u>

Suppose $d < 0$.



Let $\varepsilon = -\frac{d}{2} > 0$. Consider the interval $(d-\varepsilon, d+\varepsilon)$

epsilon needs to be positive

negative region

$\exists N \in \mathbb{N}$ s.t. $d_n \in (d-\varepsilon, d+\varepsilon)$ for all $n \geq N$.

But this is a contradiction to $d_n \geq 0$ for all $n \in \mathbb{N}$.

Suppose $\{c_n\}$ is a sequence in $[a,b]$ converging to a point $c \in \mathbb{R}$.

Then the sequence $\{(c_n - a) \geq 0\}$ converges to $c - a \geq 0$

by the lemma

$\Longrightarrow c \geq a$

Also $\{(b - c_n) \geq 0\} \geq 0$

converges to $b - c \geq 0$ by lemma

$\Rightarrow b \geq c$

$\Rightarrow c \in [a,b]$

2.3. Monotone Convergence Theorem

A sequence $\{a_n\}$ is called monotonically ~~increasing~~ **decreasing**

if $a_n \leq a_{n+1}$ for all $n \leq N$.

$a_n \geq a_{n+1}$

"Monotone means either increasing or decreasing"

THEOREM A monotone sequence $\{a_n\}$ is convergent $\Longleftrightarrow$ bounded.

<u>PROOF</u>

We only need to prove that <u>monotone + bounded $\Rightarrow$ conv.</u>

Suppose $\{a_n\}$ is monotonically increasing and bounded.

completeness Axiom

$S = \{a_n \mid n \in \mathbb{N}\}$ is bounded above $\Rightarrow S$ must have a least upper bound $= \ell$.

We claim that $a_n \to \ell$.

<u>PROOF</u> given $\varepsilon > 0$

Observe that $a_n \leq \ell < \ell + \varepsilon$

Since ℓ is the least upper bound, $\ell-\varepsilon$ is not an upper bound

$\exists N \in \mathbb{N}$ so that $\ell-\varepsilon < a_N \leq a_n$ for all $n \geq N$

$\Longrightarrow \ell-\varepsilon < a_n < \ell+\varepsilon$ for all $n \geq N$.

$\Longrightarrow |a_n - \ell| < \varepsilon \quad \forall n \geq N$

$\Longrightarrow \lim a_n = \ell$

$\square$

2.3. MONOTONE CONVERGENCE THEOREM

bdd + monotone $\Longrightarrow$ convergent ✓

An application

THEOREM (Nested Intervals Theorem)

Suppose there are real numbers $a_n < b_n$ for all $n \in \mathbb{N}$

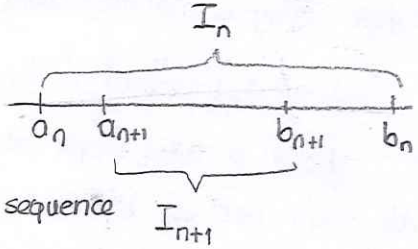
Let $I_n = [a_n, b_n]$ Suppose $I_{n+1} \subseteq I_n$ and also assume that $\lim_{n\to\infty} (b_n - a_n) = 0$

Then there is exactly one real number x which belongs to all of the intervals I_n for all $n \in \mathbb{N}$

In fact, $x = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

PROOF

$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$



$\{a_n\}$ is monotonically increasing sequence which is bounded above by b_1.

Therefore, by Monotone Convergence Theorem, $\{a_n\}$ is convergent

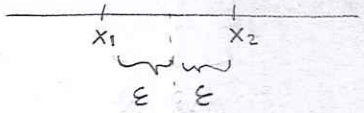
In fact, $\lim_{n\to\infty} a_n = a = \sup\{a_n \mid n \in \mathbb{N}\}$

Similarly, $\{b_n\}$ is monotonically decreasing and bounded below by a_1.

By M.C.T., $\lim_{n\to\infty} b_n = b = \inf\{b_n \mid n \in \mathbb{N}\}$

Since we know that $\lim_{n\to\infty}(b_n - a_n) = 0 \Rightarrow a = b = x \in I_n$ for all $n \in \mathbb{N}$.

Such a point x is unique. Suppose $\exists x_1 \neq x_2 \in I_n \quad \forall n \in \mathbb{N}$.



Let $\varepsilon = \dfrac{x_2 - x_1}{2}$. Since $\lim_{n\to\infty}(b_n - a_n) = 0$

We know that $\exists N \in \mathbb{N}$ s.t. $b_N - a_N < \varepsilon$

This contradicts with $x_1, x_2 \in I_N = [a_N, b_N]$ (Distance between x_1 and x_2 is 2ε)

2.4 SEQUENTIAL COMPACTNESS THEOREM

Goal: to prove that $[a,b]$ is sequentially compact

Let $\{a_n\}$ be a sequence.

DEF Let $n_1 < n_2 < n_3 \cdots < n_k < n_{k+1}$ be a sequence of strictly increasing natural numbers.

Then $\{b_k\} = \{a_{n_k}\}$ is called a subsequence of $\{a_n\}$

▶ Let $\{a_n\}$ be some sequence like $\underset{a_1}{5}, \underset{a_2}{1}, \underset{a_3}{\overset{b_1}{\frac{1}{2}}}, \frac{1}{3}, \overset{b_2}{\frac{2}{5}}, 100, \sqrt{2}, \overset{b_3}{\sqrt{3}}$

$b_1 = a_3 = a_{n_1}$ $\quad b_2 = a_5 = a_{n_2}$ $\quad \Rightarrow n_{k=2} = 5$

- $n_k \geq k$

Prop

If a sequence $\{a_n\}$ converges to a, then all of its subsequences converges to a

PROOF

Since $\lim_{n\to\infty} a_n = a$, we know that $\forall \varepsilon > 0 \ \exists n \in \mathbb{N}$ so that $|a_n - a| < \varepsilon \ \forall n \geq N$. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$

We observe $|a_{n_k} - a| < \varepsilon \ \forall k \geq N$.

$\forall k \geq N$
$\forall n_k \geq k$

This follows from the fact that $n_k \geq k$

THEOREM Every sequence $\{a_n\}$ has a monotone subsequence.
decreasing or increasing

PROOF

We say that $m \in \mathbb{N}$ is a peak index if $a_n \leq a_m$ for all $n \geq m$. Given any sequence $\{a_n\}$. There are two cases.

CASE I

$\{a_n\}$ has infinitely many peak indices.

CASE II

$\{a_n\}$ has finitely many peak indices.

Suppose case II holds, then

Let N be any natural number larger than all the peak indices.

$n_1 = N$. By definition of a peak index,

$\exists n_2 > n_1$ so that $a_{n_1} < a_{n_2}$

$\exists n_3 > n_2$ so that $a_{n_2} < a_{n_3}$

By induction, we can construct a sequence $\{a_{n_k}\}$ which strictly increasing.

DO PROPERLY

Suppose case I holds. Call the kth peak index as n_k.

$a_{n_{k+1}} \leq a_{n_k} \Rightarrow \{a_{n_k}\}$ is monotonically decreasing subsequence.

THEOREM Every bounded sequence has a convergent subsequence.

PROOF

Let $\{a_n\}$ be a bdd sequence. It has a monotone subsequence which is also bdd. Therefore that subsequence is convergent.

DEF We say that $S \subset \mathbb{R}$ is sequentially compact provided that every sequence $\{a_n\}$ in S has a convergent subsequence converging to a point in S.

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THEOREM SEQUENTIAL COMPACTNESS THEOREM $[a,b]$ is sequentially compact.

PROOF Let $\{a_n\}$ be any sequence in $[a,b]$ Then $\{a_n\}$ is bounded. So $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$

Since $[a,b]$ is closed, the limit of $\{a_{n_k}\}$ must belong to $[a,b]$ S

SECTION 3.1 CONTINUITY

DEF Suppose that $f: D \to \mathbb{R}$. Let $x_0 \in D$.
domain = subset of $\mathbb{R}$.

We say that f is continuous at x_0, if for every sequence $\{x_n\}$ in D converging to x_0, we have $\{f(x_n)\}$ converging to $f(x_0)$ We say that f is continuous on D if it is continuous at every point in D.

$f(x) = \begin{cases} 1 & ; x \in \mathbb{Q} \\ 0 & ; x \notin \mathbb{Q} \end{cases}$ $\quad f: \mathbb{R} \to \mathbb{R}$ Dirichlet

CLAIM f is not continuous at every point $\in \mathbb{R}$.

PROOF Let x any real number.

Seq. density of rationals → $\exists$ sequence $\{u_n\}$ where $u_n \in \mathbb{Q}$ and $u_n \to x$

Seq. density of irrat → $\exists$ sequence $\{v_n\}$ where $v_n \notin \mathbb{Q}$ and $v_n \to x$

$$u_n \to x \overset{\text{Assuming } f \text{ is}}{\underset{\text{continuous}}{\Longrightarrow}} f(u_n) \to f(x) = 1$$

$\#$ contradiction $\Rightarrow f$ is discontinuous at x.

$$v_n \to x \overset{\text{Assuming } f \text{ is}}{\underset{\text{continuous}}{\Longrightarrow}} \underbrace{f(v_n)}_{0} \to f(x) = 0$$

Basic Properties

$f, g: D \to \mathbb{R}$. Suppose f and g are continuous at $x_0 \in D$.

Then

① $f \pm g$ is continuous at $x_0 \in D$

② $f \cdot g$ is cont. at $x_0 \in D$

③ $\frac{f}{g}$ is cont. at $x_0 \in D$. $g \neq 0$

PROOF (HW) in the book

THEOREM Suppose $f: D \to \mathbb{R}$ and $g: U \to \mathbb{R}$ s.t. $f(D) \subset U$. Assume that f is cont. at $x_0 \in D$ and g is cont. at $f(x_0) \in U$.

Then $g \circ f: D \to \mathbb{R}$ is continuous at $x_0 \in D$.

PROOF Suppose $\{x_n\}$ is a sequence in D converging to x_0.

Since f is cont. at x_0, $f(x_n) \to f(x_0)$

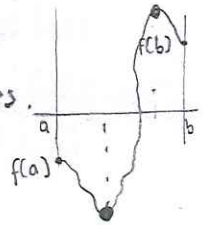
Since g is cont. at $f(x_0)$, and $\{f(x_n)\}$ is a sequence in U.

$$g(f(x_n)) \to g(f(x_0))$$

$\Rightarrow g \circ f$ is continuous at $x_0 \in D$.

3.2 EXTREME VALUE THEOREM

THEOREM Suppose $f: [a,b] \to \mathbb{R}$ is continuous. Then f attains its max and min values.



PROOF

STEP 1

We will show that $\exists M \in \mathbb{R}$ s.t. $f(x) \leq M$ for all $x \in [a,b]$. This implies $S = f([a,b])$ is bounded above. Therefore, S has a supremum $= c$

STEP 2

Show that $\exists x_0 \in [a,b]$ s.t. $f(x_0) = c$

PROOF OF STEP 1 Suppose f is not bounded above

$\forall n \in \mathbb{N}, \exists x_n \in [a,b]$ so that $f(x_n) \geq n$.

So we have a sequence $\{x_n\}$ in $[a,b]$

By Sequential compactness of $[a,b]$, there is a subsequence

$\{x_{n_k}\}$ converging to $x_0 \in [a,b]$

Thus, $f(x_{n_k}) \to f(x_0)$ by the continuity of f

Hence $\{f(x_{n_k})\}$ is a bounded sequence. $\longrightarrow$ $\boxed{\text{This leads a contradiction.}}$

$f(x_{n_k}) \geq n_k \geq k$ for any $k \in \mathbb{N}$

PROOF OF STEP 2

Let $c = \sup S = \sup f([a,b])$

Given $n \in \mathbb{N}$, $c - \frac{1}{n}$ is not an upper bound of S which means

$\exists x_n \in [a,b]$ s.t. $c - \frac{1}{n} < f(x_n) \leq c \Rightarrow \lim_{n\to\infty} f(x_n) = c$

Since $[a,b]$ is seq. compact, $\exists$ subsequence $\{x_{n_k}\}$ converging to some $x_0 \in [a,b]$

$\lim_{n\to\infty} f(x_{n_k}) = c$

but we know that $\lim_{n\to\infty} f(x_{n_k}) = f(x_0)$ $\longrightarrow$ continuity

$\Rightarrow f(x_0) = c$

For attaining min value

- $f: [a,b] \longrightarrow \mathbb{R}$ is a continuous function.

There is $x_0 \in [a,b]$ s.t. $-f(x_0)$ is max

$\Rightarrow f(x_0)$ is the min value.

$\left(\begin{array}{l} \forall x \in [a,b] \\ -f(x_0) \geq -f(x) \\ \Downarrow \\ f(x_0) \leq f(x) \end{array} \right)$

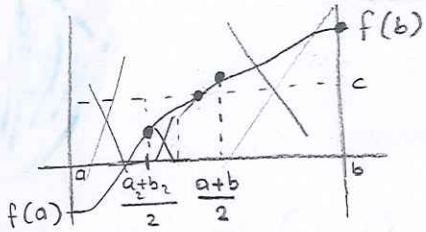
3.3 INTERMEDIATE VALUE THEOREM

intermediate value

THEOREM $f: [a,b] \to \mathbb{R}$ cont. and let $f(a) < c < f(b)$.

Then $\exists x_0 \in (a,b)$ s.t. $f(x_0) = c$

PROOF Use Nested Intervals Theorem + Bisection method.



$[a_1, b_1] = [a,b]$

$[a_2, b_2] = [a_1, \frac{a_1+b_1}{2}]$

$[a_3, b_3] = [\frac{a_2+b_2}{2}, b_2]$

We have a sequence $[a_n, b_n]$ of nested intervals so that

$\lim_{n\to\infty} b_n - a_n = 0$. By the Nested Intervals Theorem,

$\exists x_0 \in [a,b]$ s.t. $x_0 \in \bigcap_{i=1}^{\infty} [a_i, b_i]$

$f(a_n) \leq c \leq f(b_n)$

$\lim_{n\to\infty} f(a_n) = c = \lim_{n\to\infty} f(b_n)$

$c = f(x_0)$

3.4 UNIFORM CONTINUITY

DEF A function $f: D \to \mathbb{R}$ is <u>uniformly continuous</u> provided that if $\{u_n\}$ and $\{v_n\}$ are sequences in D s.t.

$\lim_{n\to\infty} (u_n - v_n) = 0$, then $\lim_{n\to\infty} (f(u_n) - f(v_n)) = 0$

not necessarily convergent

PROP If f is uniformly continuous then it is cont. on D.

PROOF Let $x_0 \in D$ and let $\{x_n\}$ be a seq converging to x_0.

Define $u_n = x_n$, $v_n = x_0$

Then $\lim_{n\to\infty} (u_n - v_n) = \lim_{n\to\infty} (x_n - x_0) = 0$

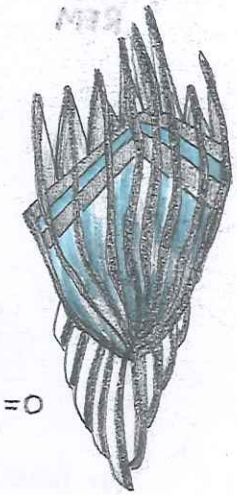
Since f is uniformly cont. on D,

$$\lim_{n\to\infty} (f(u_n) - f(v_n)) = 0 \Rightarrow \lim_{n\to\infty} (f(x_n) - f(x_0)) = 0$$

$$\Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0)$$

$\Rightarrow f$ is continuous at $x_0 \in D$.

$\Rightarrow f$ is cont. on D.



[Converse is not true

▶ $f(x) = \frac{1}{x}$ on $(0,1)$ — continuous but not uniformly]

▶ Let $u_n = \frac{1}{n}$, $v_n = \frac{1}{2n}$. $\lim_{n\to\infty}(u_n - v_n) = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{2n}\right) = \lim_{n\to\infty} \frac{1}{2n} = 0$

but $\lim_{n\to\infty}(f(u_n) - f(v_n)) = \lim_{n\to\infty}(n - 2n) = \lim_{n\to\infty} -n \neq 0$

THEOREM If $f:[a,b] \to \mathbb{R}$ is continuous, then it is uniformly continuous.

PROOF Suppose f is continuous but not uniformly

Then there are sequences $\{u_n\}$ and $\{v_n\}$ in $[a,b]$ (not necessarily convergent) so that $(u_n - v_n) \to 0$

But $(f(u_n) - f(v_n)) \not\to 0$

(Exercise 12)

└ possibly passing to a subsequence (for avoiding n_{k_s} notation)

there is $\varepsilon > 0$ so that $|f(u_n) - f(v_n)| \geq \varepsilon \quad \forall n \in \mathbb{N}$

Since $[a,b]$ is sequentially compact, there is a subsequence $\{u_{n_k}\}$ converging to $x_0 \in [a,b]$

Since $(u_n - v_n) \to 0$, we know $(u_{n_k} - v_{n_k}) \to 0$

But $u_{n_k} \to x_0 \Rightarrow v_{n_k} \to x_0$

By continuity of f, $f(u_{n_k}) \to f(x_0)$ and $f(v_{n_k}) \to f(x_0)$

Therefore $(f(u_{n_k}) - f(v_{n_k})) \to 0$. This contradicts to the fact that $\exists \varepsilon > 0$ s.t. $|f(u_{n_k}) - f(v_{n_k})| \geq \varepsilon \quad \forall k \in \mathbb{N}$

3.5 ε-δ CRITERION FOR CONTINUITY

Suppose that $f: D \to \mathbb{R}$ and let $x_0 \in D$.

The ε-δ criterion holds for f at x_0: $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

THEOREM Let $f: D \to \mathbb{R}$, and $x_0 \in D$ TFAE

① f is continuous at x_0 (using seq.)

② the ε-δ criterion holds for f at x_0.

PROOF

($\Rightarrow$) Suppose f is continuous at x_0. Assume that ε-δ Criterion does not hold for f at x_0.

There is some $\varepsilon > 0$ for which no δ satisfies the criterion. Let $\delta = \frac{1}{n}$

There must be $x \in D$ w/ $|x - x_0| < \frac{1}{n}$ but

$\rightarrow |f(x) - f(x_0)| \geq \varepsilon$

Call $x = x_n$. The sequence $\{x_n\}^{\in D}$ converges to x_0.

Then by continuity of f, we know $f(x_n) \to f(x_0)$

CONTRADICTION!

($\Leftarrow$) Suppose ε-δ criterion holds at x_0. to show that f is continuous at x_0, Let $\{x_n\}$ be any sequence in D converging to x_0 we would like to prove that

(*) $f(x_n) \to f(x_0)$

We want to show that $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ so that $|f(x_n) - f(x_0)| < \varepsilon \quad \forall n \geq N$

$\exists \delta > 0$ such that $\underbrace{|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon}_{\text{By Assumption}}$

(*) Since $x_n \to x_0$, $\exists N \in \mathbb{N}$ s.t. $|x_n - x_0| < \delta$ for all $n \geq N$

so $|x_n - x_0| < \delta \Rightarrow |f(x_n) - f(x_0)| < \varepsilon$

▨ ε-δ criterion for a domain D:

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t.

$|u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon$

Statement about whole domain

THEOREM Let $f: D \to \mathbb{R}$ be a function. TFAE

① f is uniformly cont. on D

② the ε-δ criterion holds for f on its domain D.

Proof is an exercise in the book.

BACK TO 3.3

DEF A subset $D \subset \mathbb{R}$ is called <u>convex</u> provided that $u, v \in D$

$u, v \Rightarrow [u, v] \subseteq D$

There is a characterization of convex subset of $\mathbb{R}$

(EXERCISE) any convex subset = an interval in $\mathbb{R}$.

▶ $(-\infty, 5)$, $(-3, 3)$

$\mathbb{R}$, $(-5, 6]$, $[5, \infty)$

THEOREM If I is an interval and $f: I \to \mathbb{R}$ is a continuous function then $f(I)$ is an interval.

<u>PROOF</u> To show that $f(I)$ is an interval, we need to verify that it is convex. Let $y_1 < y_2$ be points in $f(I)$

Let $c \in [y_1, y_2]$. $\exists x_1$ and x_2 s.t. $f(x_1) = y_1$ and $f(x_2) = y_2$

$J = [x_1, x_2] \subset I \Rightarrow f$ is continuous on J.

By the Intermediate Value Theorem applied to f on J,

there is $x \in J$ s.t. $f(x) = c \Rightarrow c \in f(I) \Rightarrow [y_1, y_2] \subset f(I)$ convex

$\Rightarrow f(I)$ is an interval.

3.6 MONOTONE FUNCTIONS

DEF $f: D \to \mathbb{R}$ is called monotonically increasing (decreasing) provided that

$u, v \in D$, $u < v \Rightarrow f(u) \leq f(v)$ ($\geq$)

THEOREM $\left\{ \begin{array}{l} f: D \to \mathbb{R}, \boxed{\text{monotone}} \\ f(D) \text{ is an interval} \end{array} \right\} \Rightarrow f$ is continuous.

<u>PROOF</u> Suppose $f: D \to \mathbb{R}$ monotonically increasing (if dec. use $-f$ trick)

To show that f is continuous on D, Let $x_0 \in D$, Let $\{x_n\}$ is a sequence in D converging to x_0, We need to prove that $f(x_n) \to f(x_0)$

This means $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $|f(x_n) - f(x_0)| < \varepsilon \quad \forall n \geq N$.

Equivalently need $-\varepsilon < \underbrace{f(x_n)}_{y_n} - \underbrace{f(x_0)}_{y_0} < \varepsilon$ focus on this

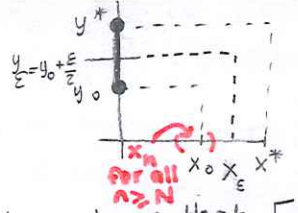
$y_n < y_0 + \varepsilon$ we are trying to show

If $y_n \leq y_0 \quad \forall n \in \mathbb{N}$, then we are done.

Otherwise, for some index $y_0 < y_n = y^*$

$\exists x^* > x_0$ in D so that

$f(x^*) = y^*$



Since $f(D)$ is an interval, we know that $[y_0, y^*] \in f(D)$

$y_\varepsilon \in [y_0, y^*] \Rightarrow \exists x_\varepsilon > x_0$ s.t. $f(x_\varepsilon) = y_\varepsilon$

Since $x_n \to x_0$, $\exists N \in \mathbb{N}$ s.t. $x_n < x_\varepsilon$ $\forall n \geq N$

$\Rightarrow \underbrace{f(x_n)}_{y_n} \leq f(x_\varepsilon) = y_\varepsilon = y_0 + \frac{\varepsilon}{2} < y_0 + \varepsilon$ $\forall n \geq N$

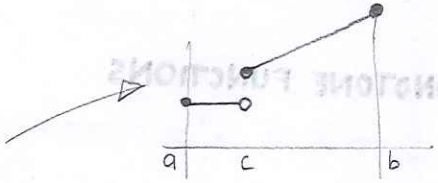
Corollary

$f: I \xrightarrow[\text{monotone}]{} \mathbb{R}$

f cont $\Leftrightarrow f(I)$ is an interval.

$\Rightarrow$ 1st Theorem

$\Leftarrow$ 2nd Theorem



monotone

$f: [a,b] \to \mathbb{R}$

but not cont.

CONTINUITY OF INVERSE FUNCTIONS

DEF: $f: D \to \mathbb{R}$ is called strictly increasing (decreasing) provided that

$u, v \in D$ $u < v \Rightarrow f(u) < f(v)$

Fact

Strictly monotone functions are 1-1

THEOREM $f: I \xrightarrow[\text{strictly monotone}]{\text{interval}} \mathbb{R} \Rightarrow f^{-1}$ is cont.

PROOF

$f^{-1}: f(I) \to I$

monotone, image is interval } continuous

3.7 LIMITS

Let $D \subseteq \mathbb{R}$ and let $x_0 \in \mathbb{R}$

We say that x_0 is a limit point of D if $\exists$ a sequence $\{x_n\}$ in $D \setminus \{x_0\}$ converging to x_0.

- $D = (a,b)$ a is limit point in D
  b " " " " "
  $c \in (a,b)$ " " " " "
- $\mathbb{Q} \subseteq \mathbb{R}$. What are the limit points of $\mathbb{Q}$?
  all real numbers by the density of $\mathbb{Q}$.
- $\mathbb{N} \subseteq \mathbb{R}$ limit points of $\mathbb{N} = \emptyset$

$f: D \to \mathbb{R}$ $x_0 \in \mathbb{R}$ x_0 is a limit point of D.

$\lim_{x \to x_0} f(x) = \ell \Leftrightarrow \forall \{x_n\} \subseteq D \setminus \{x_0\}$
$x_n \to x_0 \Rightarrow f(x_n) \to \ell$

Exercise 8

$f: D \to \mathbb{R}$ $x_0 \in D$ and x_0 is a limit point of D.

f cont at $x_0 \Leftrightarrow \boxed{\lim_{x \to x_0} f(x) = f(x_0)}$

The points which are not limit points are isolated points.

CHAPTER 4

4.1

DEF An open interval $I=(a,b)$ containing x_0 is called <u>a neighborhood of x_0</u>.

Suppose that $f: I \to \mathbb{R}$. Then we say that f is <u>differentiable</u> at x_0 if the following limit exists

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

If this is the case, then it is called <u>the derivative of f at x_0</u> and we use $f'(x_0)$ for the derivative.

Observation

f is diff'ble at $x_0 \Longrightarrow f$ is continuous at x_0.

<u>PROOF</u>

$$\lim_{x \to x_0} (f(x)-f(x_0)) = \lim_{x \to x_0} \left[\frac{f(x)-f(x_0)}{x-x_0} \cdot (x-x_0) \right]$$

$$= f'(x_0) \cdot 0 = 0$$

$\square$

Suppose $f: I \to \mathbb{R}$. We say that f is diff'ble on I if it is diff'ble at every point of I.

If this is the case, then $f': I \to \mathbb{R}$ is the derivative function.

THEOREM $f, g: I \to \mathbb{R}$ on an open interval $I=(a,b)$ including x_0. Suppose f and g are diff'ble at x_0.

a) $f+g$ is diff'ble at x_0 and $(f+g)'(x_0) = f'(x_0)+g'(x_0)$

b) $f \cdot g$ " " " " " $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

c) $\frac{1}{g \neq 0}$ " " " " " $\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{(g(x_0))^2}$ *reciprocal*

d) $\frac{f}{g \neq 0}$ " " " " " $\left(\frac{f}{g}\right)'(x_0) = \frac{f'g - fg'}{g^2}$

<u>PROOF</u>

(b) $$\lim_{x \to x_0} \frac{(fg)(x)-(fg)(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}$$

$$\lim_{x \to x_0} \frac{f(x)g(x) \overset{\shortparallel}{-} f(x)g(x) + f(x_0)g(x) \overset{\shortparallel}{-} f(x_0)g(x_0)}{x-x_0}$$

$$= \lim_{x \to x_0} g(x) \frac{f(x)-f(x_0)}{x-x_0} + \lim_{x \to x_0} f(x_0) \frac{g(x)-g(x_0)}{x-x_0} = g(x_0)f'(x_0) + f(x_0)g'(x_0)$$

(c) $$\lim_{x \to x_0} \frac{(1/g)(x)-(1/g)(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{1/g(x) - 1/g(x_0)}{x-x_0}$$

$$= \lim_{x \to x_0} \frac{g(x_0)-g(x)}{(x-x_0)\, g(x)\, g(x_0)} = \frac{-g'(x_0)}{(g(x_0))^2}$$

4.2

THEOREM $f: I \to \mathbb{R}$, $x_0 \in I=(a,b)$, strictly monotone and continuous. Suppose that f is diff'ble at x_0 and $f'(x_0) \neq 0$. Let $J=f(I)$. Then $f^{-1}: J \to \mathbb{R}$ is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

PROOF

Since f is a continuous function on an interval I, then $J = f(I)$ is an open interval around $y_0 = f(x_0)$

Therefore J is a nhbd of y_0. Also $f^{-1}: J \to \mathbb{R}$ is cont. on J. Let $x = f^{-1}(y)$, $x_0 = f^{-1}(y_0)$

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{x - x_0}{f(x) - f(x_0)} = \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}$$

$$\lim_{y \to y_0} x \equiv \lim_{y \to y_0} f^{-1}(y) \underset{\substack{\text{continuity} \\ \text{of } f^{-1}}}{=} f^{-1}(y_0) = x_0$$

THEOREM *Chain Rule*

$f: I \to \mathbb{R}$, $x_0 \in I = (a,b)$ Let J be an open interval containing $f(I)$ and let $g: J \to \mathbb{R}$. If f is diff'ble at x_0 and g is diff'ble at $f(x_0) = y_0$, then $(g \circ f)$ is diff'ble at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

PROOF

$$\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

$$\lim_{x \to x_0} \underbrace{\frac{g(f(x)) - g(f(x_0))}{y - y_0}}_{g'(y_0) = g'(f(x_0))} \cdot \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{f'(x_0)}$$

This can be done if $f(x) \neq f(x_0)$ in a nbhd of x_0. Otherwise the proof needs to be fixed.

Introduce a new function

$$h: J \to \mathbb{R}$$

$$h(y) = \begin{cases} \dfrac{g(y) - g(y_0)}{y - y_0} & \text{if } y \neq y_0 \\ g'(y_0) & \text{if } y = y_0 \end{cases}$$

Observe that h is continuous at y_0.

Also $g(y) - g(y_0) = h(y)(y - y_0)$ $\forall y \in J$.

(if $y = y_0$, both sides give 0)

$$\lim_{x \to x_0} \frac{g(f(x)) - g(f'(x_0))}{x - x_0} = \lim_{x \to x_0} h(f(x)) \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{f'(x_0)}$$

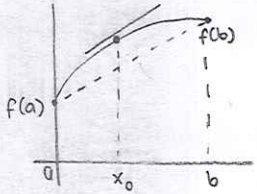
$$= h(\underbrace{f(x_0)}_{y_0}) \cdot f'(x_0)$$

$$= g'(f(x_0)) f'(x_0)$$

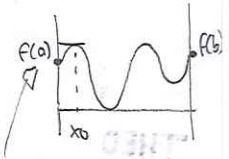
4.3

THEOREM (Mean Value Theorem)

$f: [a,b] \to \mathbb{R}$ continuous, f is diff'ble on (a,b) $\Rightarrow$ $\exists x_0 \in (a,b)$ s.t. $f'(x_0) = \dfrac{f(b) - f(a)}{b - a}$



PROOF The proof uses Rolle's Theorem.



THEOREM (Rolle's) $f: [a,b] \to \mathbb{R}$ cont, f is differentiable on (a,b), $f(a) = f(b)$ $\Rightarrow$ $\exists x_0 \in (a,b)$ s.t. $f'(x_0) = 0$

PROOF OF ROLLES

Lemma

$f: I \to \mathbb{R}$. Suppose x_0 is either a maximizer or minimizer of f.

open interval containing x_0

Assume that f is diff.ble at x_0

$\Rightarrow f'(x_0) = 0$

PROOF OF MEAN VALUE THEOREM

$$\underbrace{h(x) = f(x) - mx}_{\text{Apply Rolle's Theorem to } h(x)}$$

$$h(a) = f(a) - ma = f(b) - mb = h(b)$$

$$\frac{f(b) - f(a)}{b-a} = m$$

$\exists x_0 \in (a,b)$ so that

$$h'(x_0) = 0$$
$$\parallel$$
$$f'(x_0) - m = 0$$

$$f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

PROOF OF ROLLES

Since $f: [a,b] \to \mathbb{R}$ is a continuous function by Extreme Value Thm, f attains its min and max

Suppose this occurs at the end points.

But since $f(a) = f(b)$

$\Rightarrow f$ must be constant. Therefore for any $x_0 \in (a,b)$, $f'(x_0) = 0$

If either the max and min occurs at a point

$x_0 \neq a \quad x_0 \neq b$

Since $f'(x_0)$ exists $\underset{\text{By Lemma}}{\Rightarrow} f'(x_0) = 0$

PROOF OF LEMMA Assume that x_0 is maximizer.

$$\lim_{\substack{x \to x_0 \\ x < x_0}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\substack{x \to x_0 \\ x > x_0}} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

$\boxed{x < x_0}$

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0 \Rightarrow \lim_{\substack{x \to x_0 \\ x < x_0}} \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \Rightarrow f'(x_0) \geq 0$$

$\boxed{x > x_0}$

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0 \Rightarrow \lim_{\substack{x \to x_0 \\ x > x_0}} \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \Rightarrow f'(x_0) \leq 0$$

$\Big\}\ f'(x_0) = 0$

similarly for minimizer.

APPLICATIONS

$f: I \to \mathbb{R}$ diff.ble. Then f is constant on $I \iff f'(x) = 0 \ \forall x \in I$

open interval

PROOF ($\Rightarrow$) is clear.

($\Leftarrow$) Let $x_0 \in I$. $f(x_0) = c$

Pick any point $x \neq x_0$. Either $x > x_0$ or $x < x_0$. Suppose $x > x_0$.

Then apply MVT to $[x_0, x] \Rightarrow \exists z \in (x_0, x)$ s.t $\underset{\substack{\parallel \\ 0}}{f'(z)} = \dfrac{f(x) - f(x_0)}{x - x_0} = 0$

$$\Rightarrow f(x) = f(x_0)$$

suppose $x < x_0 \underset{\text{similarly}}{\Rightarrow} f(x) = f(x_0)$

IDENTITY CRITERION

$f, g: I \to \mathbb{R}$ diff.ble. Then f and g differ by a constant $\iff f'(x) = g'(x) \ \forall x \in I$

PROOF

($\Rightarrow$) clear.

($\Leftarrow$) Suppose $f'(x)=g'(x)$ $\forall x\in I$.

Then $(f-g)'(x)=0$ $\forall x\in I \Rightarrow f-g$ must be constant.

i.e., $f(x)-g(x)=C$

$\boxed{\text{MORE APPLICATIONS}}$

- $f: I \longrightarrow \mathbb{R}$ diff'ble $f'(x) \overset{<0}{>} 0$ $\forall x\in I \Rightarrow f$ is strictly increasing.
  open int. decreasing

PROOF Pick $x<y$ in I. Apply MVT to $[x,y]$

$\Rightarrow \exists z\in[x,y]$ s.t. $f'(z)=\dfrac{f(y)-f(x)}{y-x}>0 \Rightarrow f(y)>f(x)$

- **THEOREM** $f: I\to\mathbb{R}$ has a second derivative. Suppose $f'(x_0)=0$ at some $x_0\in I$.

  a) If $f''(x_0)>0$, then x_0 is local minimizer.

  b) If $f''(x_0)<0$, then x_0 is local maximizer.

PROOF

a) $\lim\limits_{x\to x_0}\dfrac{f'(x)-f'(x_0)}{x-x_0}>0 \underset{\text{By EX.18}}{\Longrightarrow} \exists\delta>0$ s.t.

$$\frac{f'(x)-f'(x_0)}{x-x_0}>0$$

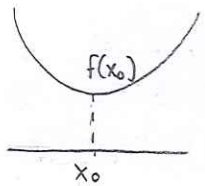
provided that

$x\in(x_0-\delta, x_0+\delta)$

If $x<x_0 \Rightarrow f'(x)<0$

If $x>x_0 \Rightarrow f'(x)>0$

$x=x_0 \Rightarrow f'(x_0)=0$

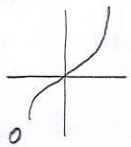


$\square$

If $f'(x_0)=0=f''(x_0)$

Then we cannot say anything, the theorem is inconclusive.

▶ $f(x)=x^3$ $f'(0)=f''(0)=0$



4.4. CAUCHY MVT

$\left.\begin{array}{l} f,g:[a,b]\longrightarrow\mathbb{R} \text{ cont.} \\ f,g \text{ are diff'ble on } (a,b)\end{array}\right\} \Rightarrow \exists x_0\in(a,b)$ s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(x_0)}{g'(x_0)}$$

$(g'(x)\neq 0 \ \forall x\in(a,b))$

PROOF Apply Rolle's to $h(x)=f(x)-mg(x)$

$h(b)=f(b)-mg(b)=f(a)-mg(a)=h(a)$

$\exists x_0\in(a,b)$ s.t.

$0=h'(x_0)=f'(x_0)-mg'(x_0) \Rightarrow f'(x_0)=mg'(x_0)$

$$\Rightarrow \frac{f'(x_0)}{g'(x_0)}=\frac{f(b)-f(a)}{b-a}$$

10.1

- $\mathbb{R}^n=\underbrace{\mathbb{R}\times\mathbb{R}\times\dots\times\mathbb{R}}_{n}$ $\quad u=(u_1,\dots,u_n)\in\mathbb{R}^n$

- $\mathbb{R}^n$ is a vector space.

  $u+v=(u_1+v_1,\dots,u_n v_n)$

  $ru=(ru_1,\dots,ru_n)$

  scalar $\in\mathbb{R}$ vector

- $\mathbb{R}^n$ is an $\boxed{\text{inner product}}$ space (dot product / scalar product) space

  $\langle u,v\rangle=u_1v_1+\dots+u_nv_n\in\mathbb{R}$

  $\langle u,v\rangle=\langle v,u\rangle$

- $\mathbb{R}^n$ is a normed space

  $\|u\|=\sqrt{\langle u,u\rangle}\geq 0$

  the norm of the vector $v\in\mathbb{R}^n$

- $\mathbb{R}^n$ is a $\boxed{\text{metric}}$ space.

$$\underbrace{d(u,v)}_{\text{distance}} = \|u-v\| \geq 0$$

$=$ if and only if $u=v$

- $\mathbb{R}^n$ is a $\boxed{\text{topological}}$ space.

We have a notion of an open set.

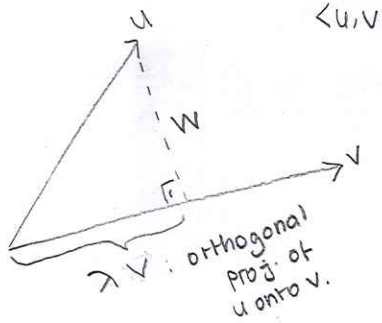
CAUCHY SCHWARTZ INEQUALITY

$u,v \in \mathbb{R}^n \quad |\langle u,v\rangle| \leq \|u\|\|v\|$

To prove this, we need to define orthogonal projection

We say that u and v are <u>orthogonal</u> if

$$\langle u,v\rangle = 0$$



$$u = \lambda v + \underbrace{w}_{\text{orthogonal to } v.}$$

$$w = u - \lambda v$$

$$\langle v,w\rangle = 0 \Rightarrow \langle v, u-\lambda v\rangle = 0 \quad \triangleright \text{ linear}$$

$$= \langle v,u\rangle - \lambda\langle v,v\rangle$$

$$= \langle u,v\rangle - \lambda\|v\|^2$$

$$\Rightarrow \lambda = \frac{\langle u,v\rangle}{\|v\|^2}$$

PROOF of Cauchy Schwarz

$$0 \leq \|u-\lambda v\|^2 = \langle u-\lambda v, u-\lambda v\rangle$$

$$= \langle u-\lambda v, u\rangle - \underbrace{\cancel{\langle u-\lambda v, \lambda v\rangle}}_{0}$$

because $w = u-\lambda v$ is orthogonal to v.

$$= \langle u,u\rangle - \lambda\langle v,u\rangle$$

$$0 \leq \|u\|^2 - \frac{\langle u,v\rangle^2}{\|v\|^2} \quad \text{multiply by } \|v\|^2$$

$$\langle u,v\rangle^2 \leq \|u\|^2\|v\|^2 \Leftarrow 0 \leq \|u\|^2\|v\|^2 - \langle u,v\rangle^2$$

TRIANGLE INEQUALITY $\quad u,v \in \mathbb{R}^n$

$$\|u+v\| < \|u\| + \|v\|$$

<u>PROOF</u> follows from Cauchy-Schwarz

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle$$

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$= (\|u\| + \|v\|)^2$$

10.2 Sequences in $\mathbb{R}^n$

A sequence in $\mathbb{R}^n$ is a function $\mathbb{N} \to \mathbb{R}^n$

$\{u_k\}$ is a sequence in $\mathbb{R}^n$, meaning that each $u_k \in \mathbb{R}^n$.

DEF We say that a sequence $\{u_k\}$ in $\mathbb{R}^n$ converges to a vector $v \in \mathbb{R}^n$ if $\forall \varepsilon > 0 \ \exists K \in \mathbb{N}$ such that

$$d(u_k, u) < \varepsilon \quad \text{for all } k \geq K.$$

A version of triangle inequality

$u, v, w \in \mathbb{R}^n$

$\text{dist}(u,v) \leq \text{dist}(u,w) + \text{dist}(v,w)$

PROOF

$\|u-v\| = \|u-w+w-v\| \leq \|u-w\| + \|w-v\|$

If a seq. $\{u_k\}$ in $\mathbb{R}^n$ converges, then the limit is unique.

PROOF

Suppose $u_k \to u$ and $u_k \to u'$

$\underbrace{\text{dist}(u,u')}_{\text{equal to zero}} \leq \cancel{\text{dist}(u,u_k)}^{\,0} + \cancel{\text{dist}(u_k,u')}^{\,0}$

$\Updownarrow$

$u = u'$

DEF: Let $p_i : \mathbb{R}^n \to \mathbb{R}$ projection onto ith component

$p_i(u_1, \dots, u_n) = u_i$

We say that a sequence $\{u_k\}$ in $\mathbb{R}^n$ <u>converges componentwise</u> to u if $\forall 1 \leq i \leq n$, $\{p_i(u_k)\}$ converges to $p_i(u) = u_i$

THEOREM A sequence $\{u_k\}$ converges to $u \iff$ it converges to u componentwise.

PROOF

($\Rightarrow$) We want to show that $p_i(u_k) \to p_i(u)$ for each $1 \leq i \leq n$

$|p_i(u_k - u)| = |p_i(u_k) - p_i(u)| \leq \underbrace{\|u_k - u\|}_{\text{dist}(u_k,u)} < \varepsilon \quad \forall k \geq K$

($\Leftarrow$) $\|u_k - u\|^2 = |p_1(u_k - u)|^2 + \dots + |p_n(u_k - u)|^2$

$= \cancel{|p_1(u_k) - p_1(u)|^2}^{\,0} + \dots + \cancel{|p_n(u_k) - p_n(u)|^2}^{\,0}$

Thus $u_k \to u$.

10.3

DEF <u>open ball of radius around u</u>

$B_r(u) = \{ v \in \mathbb{R}^n \mid \text{dist}(u,v) < r \}$

DEF Let $A \subseteq \mathbb{R}^n$. We say that $u \in A$ is an <u>interior point</u> of A if $\exists \varepsilon > 0$ s.t. $B_\varepsilon(u) \subseteq A$

The <u>interior of A</u> (int A) is the set of all interior points of A

- $\text{Int } Q = \emptyset$ in $\mathbb{R}$
  $\text{Int}(a,b) = (a,b)$

DEF $A \subseteq \mathbb{R}^n$ is called open iff $A = \text{int } A$

DEF $A \subseteq \mathbb{R}^n$ is called closed if for any seq. $\{u_k\} \subset A$ converging to a point $u \in \mathbb{R}^n$, we have that $u \in A$.

- $\mathbb{R}^n$ itself both open and closed.

THEOREM A is open $\iff \mathbb{R}^n \setminus A$ is closed.

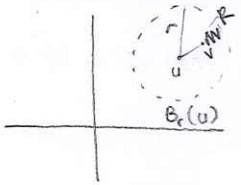
$\subseteq \mathbb{R}^n$

PROP: open ball is open.

PROOF Let $B_r(u)$ be an open ball in $\mathbb{R}^n$.

Let $v \in B_r(u)$. Let $R = r - \text{dist}(u,v)$

We claim that $B_R(v) \subset B_r(u)$



Let $w \in B_R(v) \Rightarrow \text{dist}(v,w) < R$

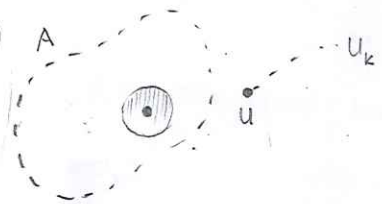
$\text{dist}(u,w) \leq \text{dist}(u,v) + \text{dist}(v,w) < R + r - R < r$

COMPLEMENTING CHARACTERIZATION

THEOREM $A \subseteq \mathbb{R}^n$ A is open $\Longleftrightarrow$ $\mathbb{R}^n \setminus A$ closed.

PROOF

($\Rightarrow$)



($\Leftarrow$) Suppose $\mathbb{R}^n \setminus A$ is closed. Assume A is not open. Then $\exists u \in A$ such that every open ball around u contains a point of $\mathbb{R}^n \setminus A$. For $r = \frac{1}{k}$, call the point $u_k \in \mathbb{R}^n \setminus A$

we cannot say A is closed.

We get a sequence $\{u_k\}$ in $\mathbb{R}^n \setminus A$, such that $u_k \to u$. $u \in \mathbb{R}^n \setminus A$ since it is closed but $u \in A$ a contradiction.

PROP

1) $\bigcup_{\text{arbitrary}}$ open sets = open

2) $\bigcap_{\text{arbitrary}}$ closed sets = closed

3) $\bigcap_{\text{finite}}$ open = open

4) $\bigcup_{\text{finite}}$ closed = closed

▶ ③ does not work arbitrary intersections

$$\bigcap_{k=1}^{\infty} \underbrace{\left(-\frac{1}{k}, \frac{1}{k}\right)}_{\text{open}} = \underbrace{\{0\}}_{\text{not open}}$$

PROOFS

① $u \in \bigcup_{s \in S} \underset{\text{open}}{O_s} \Rightarrow u \in O_s$ for some $s \in S$

$\Rightarrow \exists r > 0$ so that $B_r(u) \subseteq O_s \subseteq \bigcup_{s \in S} O_s$

② $C = \bigcap_{s \in S} \overset{\text{closed}}{C_s}$ To prove C is closed, it sufficies to prove $\mathbb{R}^n \setminus C$ is open.

$$\mathbb{R}^n \setminus C = \mathbb{R}^n \setminus \bigcap_{s \in S} C_s = \bigcup_{s \in S} \underbrace{(\mathbb{R}^n \setminus C_s)}_{\text{open}} \text{ is open}$$

De Morgan's Law

③ $u \in \bigcap_{i=1}^{n} \overset{\text{open}}{O_i} \Rightarrow u \in O_i \quad \forall i \in \{1, \dots, n\}$

$\Rightarrow \exists r_i > 0$ so that $B_{r_i}(u) \subseteq O_i$

$\Rightarrow$ Let $r = \min\{r_1, \dots, r_n\}$

$\Rightarrow B_r(u) \subseteq O_i \quad \forall i \in \{1, \dots, n\}$

$\Rightarrow B_r(u) \subseteq \bigcap_{i=1}^{n} O_i$

④ Just use De Morgan's Law and ③

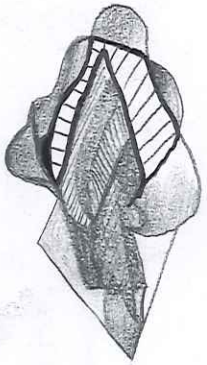
EXTERIOR AND BOUNDARY OF $A \subseteq \mathbb{R}^n$

We say that $u \in \mathbb{R}^n$ is an <u>exterior point of A</u> if there is an open ball around u which is contained in $\mathbb{R}^n \setminus A$.

<u>ext A</u> = the set of all exterior points of A in $\mathbb{R}^n$.

We say that $u \in \mathbb{R}^n$ is a <u>boundary point</u> of A if for all open balls around u, there is a point of A and a point of $\mathbb{R}^n \setminus A$ that belongs to

bd A = the set of boundary points of A in $\mathbb{R}^n$



For any subset A of $\mathbb{R}^n$, we have

disjoint union $\quad$ int A $\cup$ ext A $\cup$ bd A $= \mathbb{R}^n$

<u>PROP</u> $\quad A \subseteq \mathbb{R}^n \quad$ A open $\Leftrightarrow$ A $\cap$ bd A $= \emptyset$

A closed $\Leftrightarrow$ bd A $\subseteq$ A

Proof is exercise (Read the book)

11 CONTINUOUS MAPPINGS

11.1 Let $A \subseteq \mathbb{R}^n$. Let $F : A \to \mathbb{R}^m$ be a mapping.

Then we say that F is cont. at $u \in A$ if for every sequence $\{u_k\}$ in A converging to u, we have

$$F(u_k) \to F(u)$$

We say that F is cont if it is cont at each point of its domain.

▶ The projection function $P_i : \mathbb{R}^n \to \mathbb{R}$ is cont.

<u>PROOF</u>

Let $u \in \mathbb{R}^n$. Let $\{u_k\}$ be a seq. converging to u.

$\xrightarrow{\text{componentwise convergence}} P_i(u_k) \to P_i(u)$

$\Longrightarrow P_i$ is continuous for all $1 \le i \le n$

sums, products and quotients of cont functions.

<u>PROP</u> Let $f, g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$

suppose that f and g are cont at $u \in A$

1) $f+g$ is cont at u.
2) $f \cdot g$ is cont at u
3) f/g is cont at u, provided that $g \neq 0$ on A

<u>PROP</u> Let $G : A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$

$H : B \to \mathbb{R}^k$, where $G(A) \subseteq B \subseteq \mathbb{R}^m$

Assume that G is cont at $u \in A$, H is cont at $G(u) \in B$

Then $H \circ G$ is cont at $u \in A$.

<u>PROOF</u> Let $\{u_k\}$ be a seq. in A converging to u.

Since G is cont at u, we know that $G(u_k) \to G(u)$

Since H is cont at $G(u)$, we have

$$H(G(u_k)) \to H(G(u))$$

THEOREM $\quad F : A \xrightarrow{\subseteq \mathbb{R}^n} \mathbb{R}^m$

F is cont. at $u \in A \Leftrightarrow$ Each compenent of F is cont at u.

$\hookrightarrow F = (F_1, \dots, F_m)$

$F_i : A \to \mathbb{R}$

PROOF

same as $\Leftrightarrow$ in the theorem

$\{u_k\}$ seq in A $\quad u_k \to u$

$F(u_k) \to F(u)$

$\Updownarrow$ componentwise convergence

$\underbrace{p_i\, F(u_k)}_{F_i(u_k)} \longrightarrow \underbrace{p_i\, F(u)}_{F_i(u)}$

ε - δ CHARACTERIZATION OF CONTINUITY

$F : A \overset{\subseteq \mathbb{R}^n}{\longrightarrow} \mathbb{R}^m$, $u \in A$

TFAE

① F is cont at u.

exercise

② $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $v \in A$ $\operatorname{dist}(u,v) < \delta \Rightarrow \operatorname{dist}(F(u), F(v)) < \varepsilon$

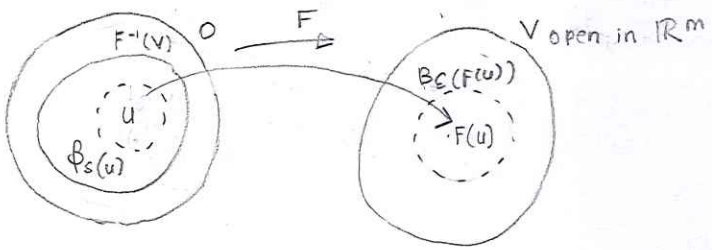
11.1

THEOREM $F : \underset{\text{open}}{O} \overset{\subseteq \mathbb{R}^n}{\longrightarrow} \mathbb{R}^m$ is cont $\Longleftrightarrow$ for each open set $V \subseteq \mathbb{R}^m$, $F^{-1}(V)$ is open in $\mathbb{R}^n$

(set wise (Does not mean that it is 1-1))

PROOF

($\Longrightarrow$) Let V be an open set in $\mathbb{R}^m$.



Prove that $F^{-1}(V)$ is open. Take any $u \in F^{-1}(V) \subseteq O$

since $F(u) \in V$ open, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(F(u)) \subseteq V$

since F is cont at every point in its domain, $\exists \delta > 0$

$$F(B_\delta(u)) \subseteq B_\varepsilon(F(u))$$

($\Longleftarrow$) To prove continuity of F on O, we take a random point $u \in O$. Given $\varepsilon > 0$. Consider $B_\varepsilon(F(u)) \subseteq \mathbb{R}^m$

$F^{-1}(B_\varepsilon(F(u)))$ is open by assumption.

not necessarily ball

$u \in F^{-1}(B_\varepsilon(F(u))) \Rightarrow \exists \delta > 0 \quad B_\delta(u) \subseteq F^{-1}(B_\varepsilon(F(u)))$

$\Rightarrow F(B_\delta(u)) \subseteq B_\varepsilon(F(u))$

$\Rightarrow$ at u, we satisfied ε-δ criterion of continuity of F.

Corollary

$f : \mathbb{R}^n \to \mathbb{R}$ continuous, let $c \in \mathbb{R}$

$A = \{u \in \mathbb{R}^n \mid f(u) < c\}$ $\quad B = \{u \in \mathbb{R}^n \mid f(u) > c\}$

$C = \{u \in \mathbb{R}^n \mid f(u) \geq c\}$ $\quad D = \{u \in \mathbb{R}^n \mid f(u) \leq c\}$

A : OPEN

$A = f^{-1}(-\infty, c)$

B : OPEN

$B = f^{-1}(c, \infty)$

C : CLOSED

$C = \mathbb{R}^n \setminus A$ closed

D : CLOSED

$D = \mathbb{R}^n \setminus B$ closed

11.2 Sequential Compactness / extreme value theorem / Uniform continuity

(BOLZANO - WEIERSTRASS = SEQUENTIAL COMPACTNESS THEOREM)

$A \subset \mathbb{R}^n$ $\underbrace{\text{seq. compact}}$ $\Leftrightarrow$ A bounded and closed

$\exists M>0$ s.t. $\|u\|<M$ $\forall u \in A$.

If $\{u_k\} \subset A$ is convergent, then its limit must belong to A.

Every sequence $\{u_k\}$ in A has a convergent subsequence converging to a point in A.

PROOF

($\Rightarrow$) $A \subseteq \mathbb{R}^n$ seq. compact. To prove that A is closed

Let $\{u_k\}$ be a sequence in A converging to $u \in \mathbb{R}^n$.

There is a subsequence $\{u_{k_j}\}$ which converges to a point in A since every subsequence of $\{u_k\}$ converges to the same limit, $u \in A$. To prove that A is bounded, assume otherwise. This gives us that there is $u_k \in A$ s.t. $\|u_k\| > k$.

By sequential compactness, $\exists$ subsequence $\{u_{k_j}\}$ which is convergent to $u \in A$.

convergent $\rightarrow$ bounded

$\|u_{k_j}\| \rightarrow \|u\|$, then $\|u_{k_j}\|$ must be bounded but $\|u_{k_j}\| \geq k_j > j \Rightarrow \|u_{k_j}\|$ is unbounded (contradiction)

($\Leftarrow$) For this direction we need a theorem.

THEOREM Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

PROOF induction on dim=n.

for n=1, we know the theorem holds.

Suppose it holds for n. We will show that it holds for n+1

Let $\{u_k\}$ be a bounded seq. in $\mathbb{R}^{n+1}$ $u_k = (\underset{\mathbb{R}^n}{v_k}, \underset{\mathbb{R}}{x_k})$

so $\{v_k\}$ is bounded in $\mathbb{R}^n \Rightarrow \exists \{v_{k_j}\}$ is convergent to $v \in \mathbb{R}^n$.

Take corresponding subsequence $\underbrace{\{x_{k_j}\}}_{\text{Bounded}} \Rightarrow \exists \{x_{k_{j_i}}\}$ is convergent to $x \in \mathbb{R}$.

Since $\{v_{k_j}\}$ is convergent to $v \in \mathbb{R}^n$, $\{v_{k_{j_i}}\}$ is convergent to $v \in \mathbb{R}^n$.

so $u_{k_{j_i}} = (v_{k_{j_i}}, x_{k_{j_i}}) \longrightarrow (v, x) \in \mathbb{R}^{n+1}$

Back To Boltzano-Weierstrass

A bdd and closed $\longrightarrow$ seq. comp.

PROOF

Let $\{u_k\}$ be any sequence in A $\Rightarrow$ Since A is bounded $\{u_k\}$ is bdd and therefore $\exists \{u_{k_{j_i}}\} \in A$ converging to a point $u \in \mathbb{R}^n$.

But A is closed $\Rightarrow u \in A$

$\boxed{\text{PROP}}$

$F: A \subset \mathbb{R}^n \xrightarrow[\text{cont}]{} \mathbb{R}^m$, A seq. compact $\Rightarrow F(A)$ is seq. compact.

PROOF

Let $\{v_k\}$ be a sequence in F(A). Let $\{u_k\}$ be a seq compact in A so that $F(u_k) = v_k$. Since A is seq. compact $\exists \{u_{k_j}\}$ of $\{u_k\}$ such that $u_{k_j} \longrightarrow u \in A \Rightarrow F(\underbrace{u_{k_j}}_{v_{k_j}}) = F(u) = v \in F(A)$

By continuity of F

THEOREM **Extreme Value Theorem**

$f: A^{\subseteq \mathbb{R}^n} \xrightarrow[\text{cont}]{} \mathbb{R}$, A seq. compact

$\Rightarrow f$ attains its max and min value on A.

PROOF

Lemma: If A is a seq. compact subset of $\mathbb{R}$, then A has a largest number and smallest number.

Proof of Lemma: A is seq. comp. $\Leftrightarrow$ A is closed and bounded.

A is bdd above, therefore it has a least upper bound b,

$k \in \mathbb{N}$ $\exists x_k \in A$ so that $b - \frac{1}{k} < x_k \leq b \Rightarrow \lim_{k\to\infty} x_k = b \in A$

because A is closed.

Then b is the largest member of A.
(similarly, A has a smallest member.) $\square$

Since A is seq. compact, f cont $\rightarrow f(A)$ seq compact.

But lemma says that $f(A)$ has a largest number and a smallest member.

DEF $A \subseteq \mathbb{R}^n$ We say that A has <u>Extreme Value Property</u> if every cont $f: A \to \mathbb{R}$ has a maximal and minimal functional value.

THEOREM $A \subseteq \mathbb{R}^n$ seq. compact $\Longleftrightarrow$ A has EVP.

PROOF $(\Rightarrow)$ ✓

$(\Leftarrow)$ Let $A \subseteq \mathbb{R}^n$ with EVP. We will show that A is closed and bounded.

Let $\left.\begin{array}{l} f: A \xrightarrow{\text{cont}} \mathbb{R} \\ u \to \|u\| \end{array}\right\} \Rightarrow f$ must attain its max value.

$\Rightarrow \{\|u\| \mid u \in A\}$ is bdd $\Rightarrow A$ is bdd.

Let $\{v_k\}$ be a seq in A, such that $v_k \to v \in \mathbb{R}^n$

$g: A \xrightarrow{\text{cont}} \mathbb{R}$, $u \to \|u - v\|$ — g has a minimal value attained at a point $u_* \in A$

but minimal value of g is 0

$\Rightarrow u_* = v \in A$

UNIFORM CONTINUITY

DEF: $F: A^{\subseteq \mathbb{R}^n} \to \mathbb{R}^m$ We say that F is uniformly continuous if for any two sequences $\{u_k\}, \{v_k\}$ in A s.t.

$\|u_k - v_k\| \to 0$ then $\|F(u_k) - F(v_k)\| \to 0$

Observe, Just like in dim 1

uniform continuity $\Rightarrow$ continuity

($\Leftarrow$ not true in general)

($\Leftarrow$ true if domain is seq. comp.)

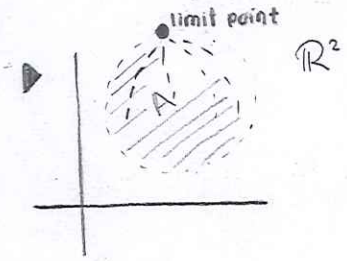
There is also <u>ε-δ criterion of uniform continuity</u>.

13 DERIVATIVES

13.1

DEF $A \subseteq \mathbb{R}^n$, we say that $u \in \mathbb{R}^n$ is a limit point of A if $\exists$ sequence $\{x_k\}$ in $A \setminus \{u\}$ converging to u.

may or not $\in A$



DEF: $A \subseteq \mathbb{R}^n$, $f: A \to \mathbb{R}$, u is a limit point of A

$$\boxed{\lim_{x \to u} f(x) = \ell}$$

$\Updownarrow$

For any sequence $\{x_k\}$ in $A \setminus \{u\}$ converging to u, we have

$f(x_k) \to \ell$

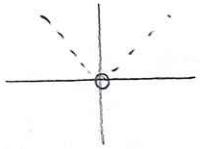
▶ $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$

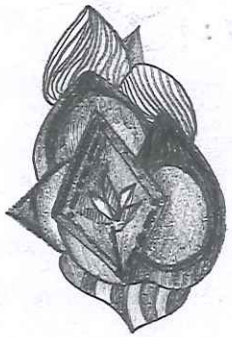
$A=\mathbb{R}^2\setminus\{(0,0)\}$

$f: A \longrightarrow \mathbb{R}$

$(x,y) \to \dfrac{xy}{x^2+y^2}$

$(0,0)$ is the limit point on A.





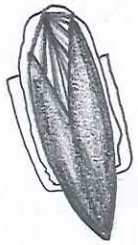
Suppose $(x_k, y_k) = (1/k, 1/k) \longrightarrow (0,0)$

$(x_k, y_k) = (0, \frac{1}{k}) \longrightarrow (0,0)$

$\lim\limits_{k\to\infty} \dfrac{\frac{1}{k}\cdot\frac{1}{k}}{\frac{1}{k^2}+\frac{1}{k^2}} = \dfrac{1}{2}$ $\qquad$ $\lim\limits_{k\to\infty} \dfrac{0\cdot\frac{1}{k}}{0+\frac{1}{k^2}} = 0$

they are not same

▶ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3}{x^2+y^2} = 0$



PROOF

$$\left|\frac{x^3}{x^2+y^2}\right| < \left|\frac{x^3}{x^2}\right| = |x|$$

comparison lemma.

13.2 PARTIAL DERIVATIVES

DEF $f: \underset{\text{open}}{O} \xrightarrow{\subseteq \mathbb{R}^n} \mathbb{R}$ We say that f has ith first partial derivative at $x \in O$ if $\lim\limits_{t\to 0} \dfrac{f(x+te_i)-f(x)}{t}$ exists.

unit vector in ith direction

If this limit exist, we denote it by

$$\frac{\partial f}{\partial x_i}(x)$$

DEF $f: \underset{\text{open}}{O} \xrightarrow{\subseteq \mathbb{R}^n} \mathbb{R}$ We say that f has first order partial derivatives if $\dfrac{\partial f}{\partial x_i}$ exists at each point $x \in O$, $\forall\ 1 \leq i \leq n$.

Question: If f has first order partial derivatives, is it true that it is continuous?

NO!

$\boxed{1}$ ▶ $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$

f is not cont at $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$$

DEF $f: \underset{\text{open}}{O} \xrightarrow{\subseteq \mathbb{R}^n} \mathbb{R}$

We say that f is of class C^1 if all first partial derivatives $\dfrac{\partial f}{\partial x_i}: O \to \mathbb{R}$ is continuous for each $1 \leq i \leq n$

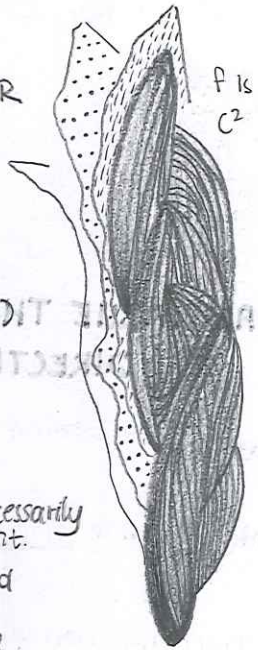
We say that f is of class C^2 if all second order partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j} : O \to \mathbb{R}$ is cont. for all $1 \le i, j \le n$

Summary

$$C^2 \Rightarrow C^1 \Rightarrow C^0$$

class of continuous functions $O \to \mathbb{R}$

converse is not necessarily true.



THEOREM $C^2 \Rightarrow$ equality of mixed partial derivatives

If $f : O \xrightarrow{\subseteq \mathbb{R}^n} \mathbb{R}$ is C^2, then $\frac{\partial^2 f}{\partial x \partial y}(x) = \frac{\partial^2 f}{\partial y \partial x}(x)$ for all $x \in O$

PROOF

We will write a proof for $n=2$.
(similar argument for $n \ge 3$)

Lemma: $f : \underset{\text{open}}{U} \xrightarrow{\subseteq \mathbb{R}^2} \mathbb{R}$. Suppose f has all second order partial derivatives at each point in U (not necessarily cont.)
Then $\exists (x_1, y_1)$ and (x_2, y_2) in U so that

$$\frac{\partial^2 f}{\partial x \partial y}(x_1, y_1) = \frac{\partial^2 f}{\partial y \partial x}(x_2, y_2)$$

PROOF OF THEOREM USING LEMMA

$f : O \xrightarrow{\subseteq \mathbb{R}^2} \mathbb{R}$ of class C^2

Let (x_0, y_0) be any point in $O \Rightarrow \exists r > 0$ s.t.

$B_r(x_0, y_0) \subseteq O$

Let $U_k = B_{\frac{r}{k}}(x_0, y_0)$ for $k \in \mathbb{N}$, by the Lemma

$\exists (x_k, y_k)$ and (u_k, v_k) so that

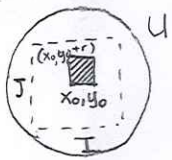
$$\frac{\partial^2 f}{\partial x \partial y}(x_k, y_k) = \frac{\partial^2 f}{\partial y \partial x}(u_k, v_k)$$

$(x_k, y_k) \to (x_0, y_0)$, $(u_k, v_k) \to (x_0, y_0)$

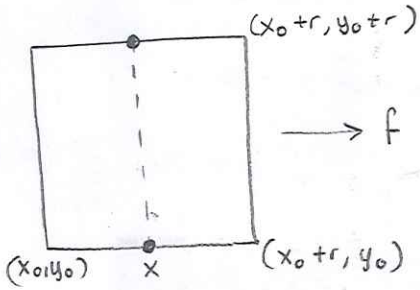
Since $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous functions

$$\left.\begin{aligned} &\lim_{(x_k, y_k) \to (x_0, y_0)} \frac{\partial^2 f}{\partial x \partial y}(x_k, y_k) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\ &\lim_{(u_k, v_k) \to (x_0, y_0)} \frac{\partial^2 f}{\partial y \partial x}(u_k, v_k) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) \end{aligned}\right\}$$

PROOF OF LEMMA



$I = (x_0 - 2r, x_0 + 2r)$
$J = (y_0 - 2r, y_0 + 2r)$



$\varphi(x) = f(x, y_0 + r) - f(x, y_0)$
$\varphi : I \to \mathbb{R}$ diff.ble

since y_0 is fixed, it is function of x.

Therefore we can use MVT on $\varphi : [x_0, x_0 + r]$

$$\frac{\varphi(x_0 + r) - \varphi(x_0)}{r} = \varphi'(x_1)$$

for some $x_1 \in (x_0, x_0 + r)$

$$= \frac{\partial f}{\partial x}(x_1, y_0 + r) - \frac{\partial f}{\partial x}(x_1, y_0)$$

$\alpha : J \to \mathbb{R}$ diff'ble

$\alpha(y) = \frac{\partial f}{\partial x}(x_1, y)$

Apply MVT to α on $[y_0, y_0 + r]$

$\exists y_1 \in (y_0, y_0 + r)$ s.t. $\frac{\alpha(y_0 + r) - \alpha(y_0)}{r} = \alpha'(y_1)$

$$= \frac{\partial^2 f}{\partial y \partial x}(x_1, y_1)$$

$$\underbrace{\varphi(x_0+r)-\varphi(x_0)=r^2\frac{\partial^2 f}{\partial y\partial x}(x_1,y_1)}$$

$$= f(x_0+r,y_0+r)-f(x_0+r,y_0)-f(x_0,y_0+r)+f(x_0,y_0)$$

$$= f(x_0+r,y_0+r)-f(x_0,y_0+r)-[f(x_0+r,y_0)-f(x_0,y_0)]$$

$\psi(y) = f(x_0+r,y)-f(x_0,y)$

$\psi: J\to\mathbb{R}$ diff ble

Apply MVT $\psi:[y_0,y_0+r]$

$\exists\, y_2\in(y_0,y_0+r)$ so that

$$\frac{\psi(y_0+r)-\psi(y_0)}{r}=\psi'(y_2)=\frac{\partial f}{\partial y}(x_0+r,y_2)-\frac{\partial f}{\partial y}(x_0,y_2)$$

$\beta: I\xrightarrow{\text{diff ble}}\mathbb{R}$ $\quad \beta(x)=\frac{\partial f}{\partial y}(x,y_2)$

Apply MVT to β on $[x_0,x_0+r]$

$\exists\, x_2\in(x_0,x_0+r)$ such that $\frac{\beta(x_0+r)-\beta(x_0)}{r}=\frac{\partial^2 f}{\partial x\partial y}(x_2,y_2)$

$$\beta(x_0+r)-\beta(x_0)=r\frac{\partial^2 f}{\partial x\partial y}(x_2,y_2)$$

$$\psi(y_0+r)-\psi(y_0)=r^2\frac{\partial^2 f}{\partial x\partial y}(x_2,y_2)$$

$$\frac{\partial^2 f}{\partial x\partial y}(x_2,y_2)=\frac{\partial^2 f}{\partial y\partial x}(x_1,y_1)$$

▶ $f(x,y)=\begin{cases}(xy)\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$

f is C^1, but not C^2.

PROOF: To show not C^2

$$\underbrace{\frac{\partial^2 f}{\partial x\partial y}(0,0)}_{1}\neq\underbrace{\frac{\partial^2 f}{\partial y\partial x}(0,0)}_{-1}$$

13.3 MEAN VALUE THEOREM AND DIRECTIONAL DERIVATIVES

Mean value Lemma

$\Downarrow$

Mean Value Prop

$\Downarrow$

Directional derivative theorem

$\Downarrow$

Mean Value Theorem $\Longrightarrow (C^1\Rightarrow C^0)\Rightarrow(C^2\Rightarrow C^1)$

$C^2\Rightarrow C^1\Rightarrow C^0$.

Lemma (MEAN VALUE)

Let $f: O\to\mathbb{R}$ be a function where O is open $O\subseteq\mathbb{R}^n$.

Suppose $\frac{\partial f}{\partial x_i}$ exists for some index i at each point in O.

Assume that $x\in O$ and the line connecting x to $x+ae_i$ lies in O, where $a\in\mathbb{R}$. Then, there exists $\theta\in\mathbb{R}$ such that $0<\theta<1$ and $f(x+ae_i)-f(x)=ae_i\frac{\partial f}{\partial x_i}(x+\theta ae_i)$

PROOF

Define $\phi(t) = f(x+te_i)$, where $\phi: I \to \mathbb{R}$ and I is an open set containing $[0,a]$. Observe that ϕ is continuous and differentiable on I. Since $\frac{\partial f}{\partial x_i}$ exists, $\phi'(t) = \frac{\partial f}{\partial x_i}(x+te_i)$

Restrict ϕ to $[0,a]$ and apply 1-dimensional MVT

It follows that there exists $\theta \in (0,1)$ s.t.

$$\frac{\phi(a) - \phi(0)}{a-0} = f(x+ae_i) - f(x) = \phi'(\theta a)$$

where $\theta \in (0,1)$

$$\Rightarrow f(x+ae_i) - f(x) = a \cdot \frac{\partial f}{\partial x_i}(x+\theta a e_i)$$

as desired.

MEAN VALUE PROPOSITION

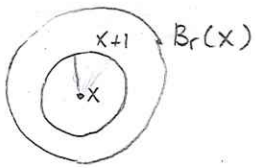
Let $f: B_r(x) \to \mathbb{R}$ have all first order partial derivatives where $x \in \mathbb{R}^n$.

If $x+h \in B_r(x)$, then there exists $z_1, \dots, z_n \in B_r(x)$ such that

$$f(x+h) - f(x) = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(z_i)$$

and

$$\|x - z_i\| < \|h\|$$



PROOF. We prove for $n=3$. For higher dimensions, the proof is similar to this. Observe that

$$\begin{aligned} f(x+h) - f(x) &= f(x_1h_1, x_2h_2, x_3h_3) - f(x_1, x_2, x_3) \\ &= f(x_1+h_1, x_2+h_2, x_3+h_3) - f(x_1+h_1, x_2+h_2, x_3) \\ &\quad + f(x_1+h_1, x_2+h_2, x_3) - f(x_1+h_1, x_2, x_3) \\ &\quad + f(x_1+h_1, x_2, x_3) - f(x_1, x_2, x_3) \end{aligned}$$

By MV Lemma, there exists $\theta_1, \theta_2, \theta_3 \in (0,1)$ such that

$$h_3 \cdot \frac{\partial f}{\partial x_3}(x+h_1, x_2+h_2, \theta_3 h_3) = f(x_1+h_1, x_2+h_2, x_3+h_3) - f(x_1+h_1, x_2+h_2, x_3)$$

$$h_2 \frac{\partial f}{\partial x_2}(x_1+h_1, x_2+\theta_2 h_2, x_3) = f(x_1+h_1, x_2+h_2, x_3) - f(x_1+h_1, x_2, x_3)$$

$$h_1 \frac{\partial f}{\partial x_1}(x_1+\theta_1 h_1, x_2, x_3) = f(x_1+h_1, x_2, x_3) - f(x_1, h_1, h_2)$$

Observe that $z_1, z_2, z_3 \in B_r(x)$ and $\|x - z_i\| = \theta|h| < \|h\|$ for each i with $i = 1,2,3$. Also,

$$f(x+h) - f(x) = \sum_{i=1}^{3} h_i \frac{\partial f}{\partial x_i}(z_i) \quad \text{as desired.}$$

DEF Let $f: O \to \mathbb{R}$ be a function where O is an open subset of $\mathbb{R}^n$, $x \in O$ and $p \in \mathbb{R}^n$ where $p \neq 0$. Then, if the limit

$$\lim_{t \to 0} \frac{f(x+tp) - f(x)}{t} \quad \text{exists,}$$

We call this limit the directional derivative of f at x in the direction p.

We denote this by $\frac{\partial f}{\partial p}(x)$

THEOREM Suppose $f: O \to \mathbb{R}$ is C^1 where $O \subseteq \mathbb{R}^n$ is open. Then all directional derivatives of f exist and

Directional Derivative Theorem

$$\frac{\partial f}{\partial p}(x) = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(x) \quad \text{for } \forall p \in O \text{ and } p \neq 0$$

$$= \langle \nabla f(x), p \rangle$$

PROOF

Consider the restriction of f to $B_r(x)$ where $x \in O$ and $B_r(x) \subseteq O$. Let $x + tp \in B_r(x)$ where $t\|p\| < r$. Then, by MVT proposition, there exist $z_1, \dots, z_n \in B_r(x)$ such that

$$\|x - z_i\| < \|tp\| \text{ for each } i \text{ with } 1 \leq i \leq n$$

and

$$f(x+tp) - f(x) = \sum_{i=1}^{n} t p_i \frac{\partial f}{\partial x_i}(z_i)$$

$$\Rightarrow \frac{f(x+tp) - f(x)}{t} = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(z_i)$$

Since $\frac{\partial f}{\partial x_i}$ is continuous for each index i with $1 \leq i \leq n$,

$$\lim_{t \to 0} \frac{f(x+tp) - f(x)}{t} = \sum_{i=1}^{n} \lim_{t \to 0} p_i \frac{\partial f}{\partial x_i}(z_i)$$

observe that as $t \to 0$, $z_i \to x$ since $\|x - z_i\| < \|tp\|$

It follows that $\lim_{t \to 0} \frac{f(x+tp) - f(x)}{t} = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(x)$

DEF The gradient of a function $f: O \to \mathbb{R}$ where $O \subseteq \mathbb{R}^n$ is open is defined by

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$$

if all the partials of f exists

MVT Let $f: O \to \mathbb{R}$ where $O \subseteq \mathbb{R}^n$ is open, f is C^1 and $x \in O$ and the line connecting x to $x+h$ lies in O.

Then $\exists \theta \in \mathbb{R}$ st $0 < \theta < 1$ and

$$f(x+h) - f(x) = \langle \nabla f(x + \theta h), h \rangle$$

PROOF By Directional Derivative Theorem,

$$\lim_{t \to 0} \frac{f(x+tp) - f(x)}{t} \overset{\text{def}}{=} \frac{\partial f}{\partial p}(x) \overset{C^1}{=} \langle \nabla f(x), p \rangle$$

for $\forall p \in \mathbb{R}^n$ w/ $p \neq 0$.

Observe that

$$\boxed{\frac{d}{dt}(f(x+tp))\Big|_{t=0}} = \frac{\partial f}{\partial p}(x) = \langle \nabla f(x), p \rangle$$

$$\Rightarrow \left(\frac{d}{dt} f(x+tp) \right)(t) = \langle \nabla f(x+tp), p \rangle \quad \forall t \in I$$

Define $\phi: I \to \mathbb{R}$ where $I \subset \mathbb{R}$ and $[0,1] \subset I$ by $\phi(t) = f(x+th)$, for $\forall t \in I$.

Then, there exists $\theta \in (0,1)$ s.t.

$$f(x+h) - f(x) = \frac{\phi(1) - \phi(0)}{1 - 0} = \phi'(0)$$

$$= \langle \nabla f(x + \theta h), h \rangle$$

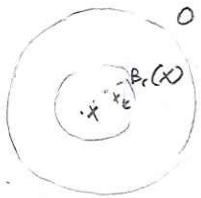
because

$$\frac{d}{dt}(f(x+th))\Big|_{t=0} = \frac{\partial f(x)}{\partial h}$$

THEOREM $(C^1 \Rightarrow C^0)$

$f: O \xrightarrow[C^1]{\subseteq \mathbb{R}^n} \mathbb{R} \Rightarrow \mathbb{R}$ is continuous on O.

PROOF



x_k sequence in $B_r(x)$ converging to x

Let $h_k = x_k - x$

By Mean Value Theorem, $\exists\ 0 < \theta_k < 1$

such that $\underbrace{f(x+h_k)}_{f(x_k)} - f(x) = \langle \nabla f(x+\theta_k h_k), h_k \rangle$

$$\lim_{k\to\infty} \left(f(x_k) - f(x)\right) = \lim_{k\to\infty} \langle \underbrace{\nabla f(x+\theta_k h_k)}_{\substack{\text{converges}\\ \text{to}\\ \nabla f(x)}}, \underbrace{h_k}_{\text{converges to zero}} \rangle$$

Therefore f is cont at $x \in O \Rightarrow f$ is cont on O.

Corollary $C^2 \Rightarrow C^1$

PROOF

suppose f is $C^2 \underset{\text{def.}}{\Longleftrightarrow} \frac{\partial f}{\partial x_i}$ are $C^1 \Rightarrow \frac{\partial f}{\partial x_i}$ are C^0.

$\Updownarrow$ def

f is C^1.

▶ $f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

f is not C^2.

(Because mixed partials at $(0,0)$ are different)

Is f C^1?

$$f_x(x,y) = y\,\frac{(x^4-y^4)+4x^2y^2}{(x^2+y^2)^2}$$
$(x,y) \neq (0,0)$

$$f_x(0,0) = \lim_{h\to\infty} \frac{f(h,0)-f(0,0)}{h} = 0$$

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0) = 0 \qquad m = \frac{y}{x}$$

$$\lim_{(x,y)\to(0,0)} y\,\frac{x^4-y^4+4x^2y^2}{(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} \frac{y\cancel{x^4}(1-m^4+4m^2)}{\cancel{x^4}(1+2m^2+m^4)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{y(1-m^4+4m^2)}{(1+m)^2} = 0$$

for exam

In m variable $\frac{(1-m^4+4m^2)}{(1+m^2)^2}$ bounded.

Because

$$\lim_{m\to\pm\infty} \frac{1-m^4+4m^2}{(1+m^2)^2} = -1$$

(Continuity implies that the function can go infinity if and only if at its endpoints.)

Similarly

$$\lim_{(x,y)\to(0,0)} f_y(x,y) = f_y(0,0) = 0$$

$$f\ C^1 \Rightarrow C^0$$

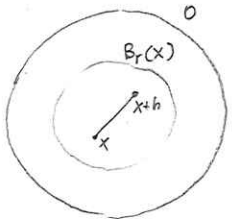
14.1

THEOREM First Order Appr. Theorem

$f: \underset{\text{open}}{O} \xrightarrow[C^1]{\subseteq \mathbb{R}^n} \mathbb{R}$, $x \in O$

Then $\lim_{h\to 0} \dfrac{f(x+h) - [f(x) + \overbrace{\langle \nabla f(x), h \rangle}^{\text{a linear func. of } h}]}{\|h\|} = 0$

PROOF



Apply MVT to $f(x+h)-f(x) = \langle \nabla f(x+\theta h), h\rangle$ for some $0<\theta<1$

$$\| f(x+h)-f(x)-\langle \nabla f(x), h\rangle \| = \| \langle \nabla f(x+\theta h)-\nabla f(x), h\rangle \|$$

$$\leq \| \nabla f(x+\theta h)-\nabla f(x)\| \|h\|$$

Cauchy-Schwarz Ineq.

$$\lim_{h\to 0} \frac{\| f(x+h)-f(x)-\langle \nabla f(x), h\rangle \|}{\|h\|} \leq \lim_{h\to 0} \| \nabla f(x+\theta h)-\nabla f(x)\| = 0$$

When $n=2$, $f: O \xrightarrow[C^1]{\subseteq \mathbb{R}^2} \mathbb{R}$, $(x_0,y_0)\in O$

Statement in the theorem becomes

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \left[f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)\right]}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

$$= 0.$$

DEF Tangent Plane

$\underset{\text{open}}{O} \subset \mathbb{R}^2$ $(x_0,y_0)\in O$

$f: O \to \mathbb{R}$ cont. at (x_0,y_0). The tangent plane to the graph of f at $(x_0, y_0, f(x_0,y_0))$ is a the graph of function Ψ

$\Psi: \mathbb{R}^2 \to \mathbb{R}$

$\Psi(x,y) = a + b(x-x_0) + c(y-y_0)$

for some $a,b,c \in \mathbb{R}$ s.t

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \Psi(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Tangent plane may not exist or more than one.

Corollary (of first approx.)

$f: O \xrightarrow[C^1]{\subseteq \mathbb{R}^2} \mathbb{R}$. Then there is a tangent plane to the graph of f at each $(x_0,y_0)\in O$.

This tangent plane is of the form

$$\Psi(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

PROOF $\Psi: \mathbb{R}^2 \to \mathbb{R}$

Take $h = (x,y) - (x_0,y_0) = (x-x_0, y-y_0)$,

APPLY the first order approx. theorem.

15.1 LINEAR ALGEBRA Skipped

15.2 THE DERIVATIVE MATRIX AND THE DIFFERENTIAL

First goal. $C^1 \Rightarrow C^0$ for mappings $\mathbb{R}^n \to \mathbb{R}^m$

DEF Let $F: O \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a vector valued function where O is open and let $x\in O$. F is said to <u>have first order partial derivatives at x</u> if for each i with $1\leq i\leq m$, F_i has first-order partial derivatives at x, where $F = (F_1, \dots, F_m)$

DEF F is said to have first-order partial derivatives if for each F_i has first-order partial derivatives

DEF F is said to be continuously differentiable (C^1) if F_i is continuously differentiable for each i.

THEOREM If $F: O \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is C^1 where O is open, then F is continuous (C^0)

PROOF Since F is C^1, by definition, F_i is C^1 for each i. Then F_i is C^0. Since F_i is continuous (C^0) for each i, F is C^0

DEF Let $F: O \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a mapping where O is open. Suppose F has first order partial derivatives at $x \in O$. Then the derivative matrix is an $m \times n$ matrix of the following form:

$$(DF(x))_{ij} = \frac{\partial F_i}{\partial x_j}(x) \qquad \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$$

(i th row is $\nabla F_i(x)$)

MEAN VALUE THEOREM FOR GENERAL MAPPINGS

THEOREM Let $F: O \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 mapping where O is open. Let $x \in O$, $x + h \in O$ and the line containing x to $x+h$ be in O. Then, there exist $\theta_1, \theta_2, \dots, \theta_m \in (0,1)$ s.t.

$$F_i(x+h) - F_i(x) = \langle \nabla F_i(x + \theta_i h, h) \rangle$$

we proved for real-valued functions

$$\Rightarrow F(x+h) - F(x) = Ah$$

where A is the matrix whose ith row is $\nabla F_i(x + \theta_i h)$

FIRST ORDER APPROX THEOREM FOR GENERAL MAPPINGS

Recall the theorem for scalar mappings:

$$f: O \subseteq \mathbb{R}^n \xrightarrow[C^1]{} \mathbb{R} \Rightarrow \lim_{h \to 0} \frac{f(x+h) - [f(x) + \langle \nabla f(x), h \rangle]}{\|h\|} = 0$$

THEOREM Let $F: O \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be C^1. Then,

$$\lim_{h \to 0} \frac{F(x+h) - [F(x) + DF(x)h]}{\|h\|} = 0$$

PROOF Define $R(h) = F(x+h) - [F(x) + DF(x)h]$

We want to show that

$$\lim_{h \to 0} \frac{\|R(h)\|}{\|h\|} = 0.$$

Observe that

$$R_i(h) = F_i(x+h) - [F_i(x) + \langle \nabla F_i(x), h \rangle]$$

Since $R_i(h)$ is real valued, we know that

$$\lim_{h \to 0} \frac{R_i(h)}{\|h\|} = 0$$

Since $\dfrac{\|R(h)\|}{\|h\|} = \dfrac{\sqrt{\sum_{i=1}^{m} [R_i(h)]^2}}{\|h\|}$ and since $R_i(h) \to 0$

For each index i with $1 \le i \le m$,

$$\lim_{h \to 0} \frac{\|R(h)\|}{\|h\|} = 0 \quad \text{as desired.}$$

THEOREM Let $F: O \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a mapping where O is open and let $x \in O$. Suppose that A is an $m \times n$ matrix satisfying

$$\lim_{h \to 0} \frac{F(x+h) - [F(x) + Ah]}{\|h\|} = 0$$

Then F has first-order partial derivatives at x and $A = DF(x)$

PROOF

Let $(A)_{ij} = a_{ij} \in \mathbb{R}$. We show that

$$(DF(x))_{ij} = a_{ij}$$

i.e.

$$(DF(x))_{ij} = \frac{\partial F_i}{\partial x_j}(x) = a_{ij}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. Observe that

$$P_i(F(x+h) - [F(x) + Ah]) = F_i(x+h) - [F_i(x) + \langle A_i, h \rangle]$$

where A_i is the ith row A. Again, observe that

$$\lim_{h \to 0} \frac{|F_i(x+h) - [F_i(x) + \langle A_i, h \rangle]|}{\|h\|} \leq \lim_{h \to 0} \frac{\|-[F(x) + Ah] + F(x+h)\|}{\|h\|} = 0$$

Thus,

$$\lim_{h \to 0} \frac{F_i(x+h) - [F_i(x) + \langle A_i, h \rangle]}{\|h\|} = 0$$

We parametrize h by $h = t \cdot e_j$ where $t \in \mathbb{R}$ and $t \neq 0$.

Then,

$$\lim_{t \to 0} \frac{F_i(x + te_j) - [F_i(x) + \langle A_i, te_j \rangle]}{\|te_j\| \to t}$$

$$\Rightarrow \lim_{t \to 0} \frac{F_i(x + te_j) - F_i(x)}{t}$$

$$= \lim_{t \to 0} \frac{t \langle A_i, e_j \rangle}{t}$$

$$= a_{ij}$$

Thus, we showed that

$$\frac{\partial F_i}{\partial x_j} = a_{ij}, \text{ as desired}$$

DEF Let $F: O \subseteq \mathbb{R}^n \to \mathbb{R}^m$ have first-order partial derivatives at x. Then the linear map $dF(x): \mathbb{R}^n \to \mathbb{R}^m$ is called the differential of F at x defined by

$$\underset{\text{vector}}{dF(h)} = DF(x) \cdot \underset{\text{vector}}{h}$$

15.3 CHAIN RULE

Notation: $g : \underset{\text{open}}{U} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ assume that g has 1st order partial derivatives at each point $p \in U$.

$$\underbrace{\frac{\partial g}{\partial x_i}(p)}_{D_i g(p)} = \lim_{t \to 0} \frac{g(p + te_i) - g(p)}{t} \qquad 1 \leq i \leq m$$

New notation

ADVANTAGE: We do not use coordinates like x_i.

THEOREM (Chain Rule)

$$F : \underset{\text{open}}{O} \subseteq \mathbb{R}^n \xrightarrow[C^1]{} \mathbb{R}^m$$

$$g : \underset{\text{open}}{U} \subseteq \mathbb{R}^m \xrightarrow[C^1]{} \mathbb{R}$$

$$F(O) \subseteq U$$

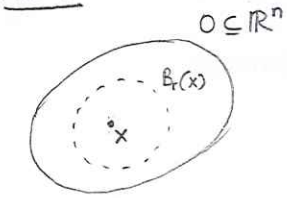
Then $g \circ F : O \longrightarrow \mathbb{R}$ is C^1.

$$\frac{\partial}{\partial x_i}(g \circ F)(x) = \sum_{j=1}^{m} D_j g(F(x)) \frac{\partial F_j}{\partial x_i}(x) \quad (*)$$

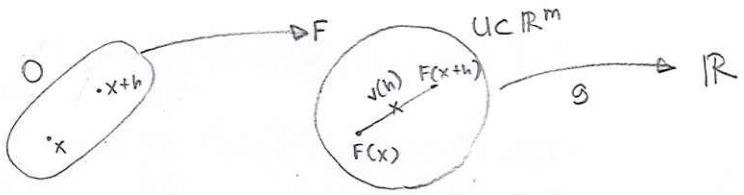
One can write this $(*)$ as follows

$$\nabla(g \circ F)(x) = \underbrace{\nabla g(F(x))}_{\substack{\text{gradient of } g \text{ is a vector of dim } m}} \cdot \underbrace{DF(x)}_{\substack{\text{derivative matrix of } F : \mathbb{R}^n \to \mathbb{R}^m}} \quad (m \times n)$$

PROOF



Observe that $F : O \to \mathbb{R}^m$ is continuous. The line segment connecting $F(x)$ and $F(x+h)$ is contained in U (by choosing r small enough)



Let $R(h) = F(x+h) - F(x) - DF(x)h$ $\Big\}$ $F(x+h) - F(x) = R(h) + DF(x)h$

First order Approx. Theorem $\Longrightarrow \lim_{h \to 0} \frac{\|R(h)\|}{\|h\|} = 0$

Apply MVT to $g : U \to \mathbb{R}$ at points $F(x)$ and $F(x+h)$

$$g(F(x+h)) - g(F(x)) = \langle \nabla g(v(h)), F(x+h) - F(x) \rangle$$
$$= \langle \nabla g(v(h)), R(h) + DF(x)h \rangle$$
$$= \langle \nabla g(v(h)), R(h) \rangle + \langle \nabla g(v(h)), DF(x)h \rangle$$

Lets take $h = te_i$ t is small real number

$$g(F(x + te_i) - g(F(x)) = \langle \nabla g(v(te_i), R(te_i) \rangle + \langle \nabla g(v(te_i), DF(x)te_i)$$

$$\frac{\partial(g \circ F)}{\partial x_i} = \lim_{t \to 0} \frac{(g \circ F)(x + te_i) - (g \circ F)(x)}{t} = \lim_{t \to 0} \langle \nabla g(v(te_i)), \frac{R(te_i)}{t} \rangle + \lim_{t \to 0} \langle \nabla g(v(te_i)), DF(x)e_i \rangle$$

Since $\lim_{h \to 0} \frac{\|R(h)\|}{\|h\|} = 0 \Rightarrow \lim_{t \to 0} \frac{R(te_i)}{t} = 0$

Moreover $\lim_{t \to 0} \nabla g(v(te_i)) = \nabla g(F(x))$

$$= \langle \nabla g(F(x)), DF(x)e_i \rangle$$

$[\ \ ][\ ] = [\ ]$

$m \times n$ $n \times 1$ $m \times 1$

ith column $\frac{\partial}{\partial x_i}$

$$= \sum_{j=1} \underbrace{D_j g(F(x))}_{C^1} \underbrace{\frac{\partial F_j}{\partial x_i}(x)}_{C^1} \Rightarrow D_j g \text{ and } \frac{\partial F_j}{\partial x_i} \text{ are continuous}$$

▶ $\underset{\text{psi}}{\psi} : \mathbb{R}^2 \xrightarrow[C^1]{} \mathbb{R}$, $\underset{\text{phi}}{\varphi} : \mathbb{R}^2 \xrightarrow{C^1} \mathbb{R}$, $f : O^{\subseteq \mathbb{R}^2} \xrightarrow[C^1]{} \mathbb{R}$

$$\frac{\partial}{\partial x} f(\psi(x,y), \varphi(x,y)) = D_1 f(\psi(x,y), \varphi(x,y)) \frac{\partial \psi}{\partial x}(x,y)$$ — Change it by ∂y for ∂y case

$$+ D_2 f(\psi(x,y), \varphi(x,y) \frac{\partial \varphi}{\partial x}(x,y)$$ — Change it by ∂y for ∂y case.

▶ $g : \mathbb{R}^3 \xrightarrow{C^1} \mathbb{R}$

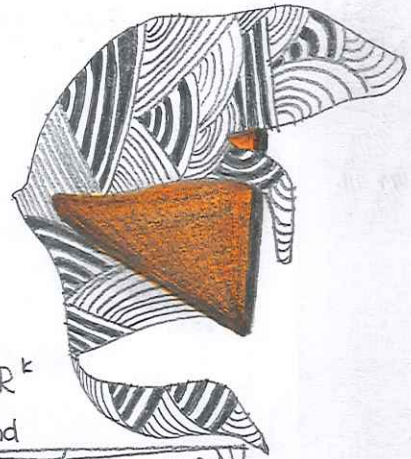
$\psi : \mathbb{R} \to \mathbb{R}$

$\psi(x) = g(x^2, \sin x, e^x)$

$\psi'(x) = D_1 g(x^2, \sin x, e^x)\, 2x$

$\quad + D_2 g(x^2, \sin x, e^x) \cos x$

$\quad + D_3 g(x^2, \sin x, e^x)\, e^x$



CHAIN RULE FOR GENERAL MAPPINGS

$F : \underset{\text{open}}{O} \subseteq \mathbb{R}^n \xrightarrow[C^1]{} \mathbb{R}^m$, $G : U^{\subseteq \mathbb{R}^m} \xrightarrow[C^1]{} \mathbb{R}^k \Rightarrow G \circ F : O^{\subseteq \mathbb{R}^n} \longrightarrow \mathbb{R}^k$ is C^1 and

$F(O) \subseteq U$

$$\boxed{\underset{(k \times n)}{D(G \circ F)(x)} = \underset{(k \times m)}{DG(F(x))} \cdot \underset{(m \times n)}{DF(x)}}$$

PROOF Reduce the proof to the previous theorem

$G = (G_1, \dots, G_k)$ $\quad (G \circ F)_j = \underbrace{G_j \circ F}_{\text{Since this is } C^1}$ and we can apply our previous theorem

$$\underbrace{\nabla(G_j \circ F)(x)}_{j^{th} \text{ row of } D(G \circ F)(x)} = \underbrace{\nabla G_j(F(x)) \cdot DF(x)}_{j^{th} \text{ row of the product } DG(F(x))\, DF(x)}$$

CHAPTER 16 INVERSE FUNCTION THEOREM

16.1

$F : O^{\subseteq \mathbb{R}^n} \xrightarrow{C^1} \mathbb{R}^n$, $x_0 \in O$

Suppose that $DF(x_0)$ is invertible ($\det DF(x_0) \neq 0$)

Then there is an open set $U \subset O$ containing x_0 and $\exists$ an open set $V \subset \mathbb{R}^n$ containing $F(x_0)$ s.t.

$F : U \to V$ is 1-1 and onto.

Moreover, $F^{-1} : V \longrightarrow U$ is C^1

$DF^{-1}(y) = [DF(x)]^{-1}$

where $y = F(x)$, $x \in U$, $y \in V$

$\boxed{\text{dim } 1}$

$f : O^{\subseteq \mathbb{R}} \xrightarrow{C^1} \mathbb{R}$, $f'(x_0) \neq 0$ for some $x_0 \in O$

Then $\exists$ an open interval $I \ni x_0$ and $\exists$ an open interval $J \ni f(x_0)$ s.t.

$f : I \to J$ is 1-1 and onto. Moreover

$f^{-1} : J \to I$ is C^1 and

$(f^{-1})'(y) = \dfrac{1}{f'(x)}$ where $y = f(x)$, $x \in I$, $y \in J$

PROOF

Assume that $f'(x_0) > 0$ ($f'(x_0) < 0$ is similar)

From the assumption that f is C^1 we know

$f' : O \longrightarrow \mathbb{R}$ is continuous

There is a positive $r > 0$ so that $f' > 0$ on $[x_0 - r, x_0 + r]$

$\overset{\text{MVT}}{\Longrightarrow}$ f is strictly increasing on $[x_0 - r, x_0 + r]$

$\Longrightarrow$ f is 1-1

We need to verify $f: \underbrace{(x_0-r, x_0+r)}_{I} \longrightarrow \underbrace{(f(x_0-r), f(x_0+r))}_{J}$ is onto

Given y satisfying $f(x_0-r) < y < f(x_0+r)$

$\exists x \in (x_0-r, x_0+r)$ s.t $f(x) = y$ (IVT)

This proves f is onto.

$f: I \longrightarrow J$ 1-1 and onto.

In chapter 4, we proved f^{-1} is differentiable and

$$(f^{-1})'(y) = \frac{1}{f'(x)} \quad \text{where } y \overset{\in J}{=} f(\overset{\in I}{x})$$

From this we conclude that $(f^{-1})'$ is continuous $\Rightarrow f^{-1}$ is C^2.

$\boxed{\text{dim } 2}$

$F: \overset{\subseteq \mathbb{R}^2}{O} \xrightarrow{C^1} \mathbb{R}^2$, Assume that $DF(x_0, y_0)$ is invertible.

for some $(x_0, y_0) \in O$. Then $\exists$ a neighborhood U of (x_0, y_0) and a neighborhood V of (x_0, y_0) st $F: U \longrightarrow V$ is 1-1 and onto. Moreover,

F^{-1} is C^1 and $DF^{-1}(u,v) = [DF(x,y)]^{-1}$

where $F(x,y) = (u,v) \in V$

▶ $F: O \longrightarrow \mathbb{R}^2$

$F(x,y) = (\psi(x,y), \phi(x,y))$

$$DF(x,y) = \begin{bmatrix} \frac{\partial \psi}{\partial x}(x,y) & \frac{\partial \psi}{\partial y}(x,y) \\ \frac{\partial \phi}{\partial x}(x,y) & \frac{\partial \phi}{\partial y}(x,y) \end{bmatrix}$$

$\det DF(x_0, y_0) \neq 0 \iff$

$$\frac{\partial \psi}{\partial x}(x_0,y_0) \cdot \frac{\partial \phi}{\partial y}(x_0,y_0) - \frac{\partial \psi}{\partial y}(x_0,y_0) \frac{\partial \phi}{\partial x}(x_0,y_0) \neq 0$$

$F^{-1}: V \longrightarrow U$

$F^{-1}(u,v) = (g(u,v), h(u,v))$

$$DF^{-1}(u,v) = \begin{bmatrix} \frac{\partial g}{\partial u}(u,v) & \frac{\partial g}{\partial v}(u,v) \\ \frac{\partial h}{\partial u}(u,v) & \frac{\partial h}{\partial v}(u,v) \end{bmatrix}$$

$\underbrace{J(x,y)}_{\text{Jacobian Determinant}} = \det \underbrace{DF(x,y)}_{\text{Jacobian Matrix}}$

$$\frac{\partial g}{\partial u}(u,v) = \frac{1}{J(x,y)} \cdot \frac{\partial \phi}{\partial y}(x,y)$$

$$\frac{\partial g}{\partial v}(u,v) = \frac{-1}{J(x,y)} \frac{\partial \psi}{\partial y}(x,y)$$

$$\frac{\partial h}{\partial u}(x,y) = \frac{-1}{J(x,y)} \frac{\partial \phi}{\partial x}(x,y) \qquad \frac{\partial h}{\partial v}(x,y) = \frac{1}{J(x,y)} \frac{\partial \psi}{\partial x}(x,y)$$

(*) $\exists x$ $\forall m$

▶ $F(x,y) = (e^{x-y} + x^2 y + x(y-1)^4, 1 + x^2 + x^4 + (xy)^5)$

$F: \mathbb{R}^2 \xrightarrow{C^1} \mathbb{R}^2$, $(x_0, y_0) = (1,1)$

$$DF(x,y) = \begin{bmatrix} e^{x-y} + 2xy + (y-1)^4 & -e^{x-y} + x^2 + 4x(y-1)^3 \\ 2x + 4x^3 + 5y^5 x^4 & 5x^5 y^4 \end{bmatrix}$$

$$DF(1,1) = \begin{bmatrix} 3 & 0 \\ 11 & 5 \end{bmatrix}$$

$\det DF(1,1) = 15 \neq 0$

We can apply Inverse Function Theorem

$F(1,1) = (2,4)$

$F^{-1}(u,v) = (g(u,v), h(u,v))$

(It is hard to find the inverse but we can still find the derivative.)

$$\frac{\partial g}{\partial u}(2,4) = \underbrace{\frac{1}{15}}_{\frac{1}{J}} \cdot 5 = \frac{1}{3} \qquad \frac{\partial g}{\partial v}(2,4) = \frac{-1}{15} \cdot 0 = 0$$

$$\frac{\partial h}{\partial u}(2,4) = \frac{-1}{15}\cdot 11 = \frac{-11}{15}$$

$$\frac{\partial h}{\partial v}(2,4) = \frac{1}{15}\cdot 3 = \frac{1}{5}$$

$\exists X$
$\forall M$

▶ $F(x,y) = (x^2-y^2, 2xy)$ $\quad F: \mathbb{R}^2 \xrightarrow{C^1} \mathbb{R}^2$

$(x_0,y_0) \neq (0,0)$ $\quad DF(x_0,y_0) = \begin{bmatrix} 2x_0 & -2y_0 \\ 2y_0 & 2x_0 \end{bmatrix}$

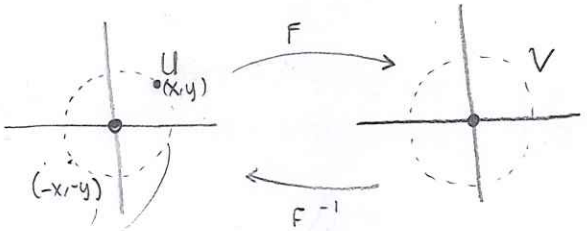
$\Downarrow$

$$J(x_0,y_0) = \det DF(x_0,y_0) = 4(x_0^2+y_0^2) \neq 0$$

Inverse function theorem applies for any point other than the origin.

At $(0,0) \Rightarrow DF(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ not invertible.

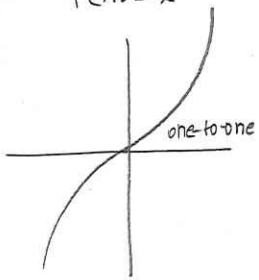
Is F is invertible around $(0,0)$?



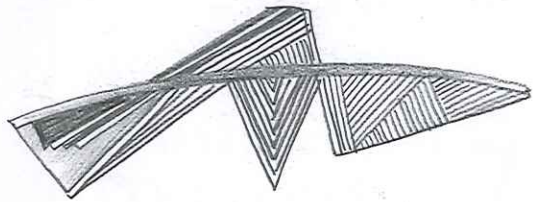
They both go to the same point

$\Rightarrow$ not one-to-one

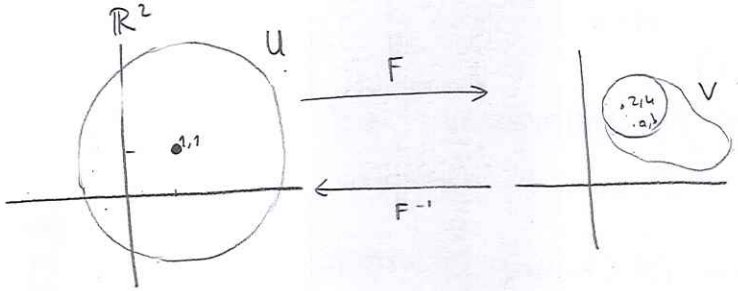
▶ $f(x) = x^3$



$f'(0) = 0$



▶ $\left.\begin{array}{l} e^{x-y} + x^2y + x(y-1)^4 = 2 \\ 1 + x^2 + x^4 + (xy)^5 = 4 \end{array}\right\} \Rightarrow \exists$ a unique sol? $(1,1)$ is a sol.



e any neighborhood $F(\mathcal{U})$

$$F(x,y) = \left(e^{x-y} + x^2y + x(y-1)^4,\ 1 + x^2 + x^4 + (xy)^5\right)$$

IFT says $\exists r > 0$ s.t. $|a-2|^2 + |b-4|^2 < r^2 \underset{1-1}{\Rightarrow} \underbrace{\begin{array}{l} e^{x-y} + x^2y + x(y-1)^4 = a \\ 1 + x^2 + x^4 + (xy)^5 = b \end{array}}_{\text{has a unique solution}}$

CHAPTER 17 IMPLICIT FUNCTION THEOREM AND APPLICATIONS

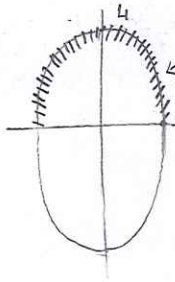
17.1 (Case dim = 2)

$f: \underset{\text{open}}{O} \subseteq \mathbb{R}^2 \xrightarrow{C^1} \mathbb{R}$, $f(x,y) = 0$ $\quad (x,y) \in O$

This set is not necessarily the graph of a function.

▶ $f: \mathbb{R}^2 \xrightarrow[C^1]{} \mathbb{R}^1$ $\quad f(x,y) = \frac{x^2}{9} + \frac{y^2}{16} - 1$

$f(x,y) = 0 \iff$



(upper part) is the graph of a C^1-function

$g: (-3,3) \to \mathbb{R}$

Solve y in terms of x.

$$\underbrace{\frac{y^2}{16} = 1 - \frac{x^2}{9} \Rightarrow y = 4\sqrt{1 - \frac{x^2}{9}}}$$

This is possible because

$$\frac{\partial f}{\partial y}(0,4) \neq 0$$

$$\frac{\partial f}{\partial y}(0,4) = \frac{1}{2} \neq 0$$

THEOREM (Dini's theorem) (n=2)

$f: O \xrightarrow[C^1]{} \mathbb{R}$ $\subseteq \mathbb{R}^2$ Suppose that $\frac{\partial f}{\partial y}(x_0,y_0) \neq 0$

for some $(x_0,y_0) \in O$ ($f(x_0,y_0)=0$). Then $\exists\, r>0$ such that

$g: I \xrightarrow{C^1} \mathbb{R}$ where $I = (x_0 - r, x_0 + r)$ and $f(x, g(x)) = 0 \ \forall x \in I$

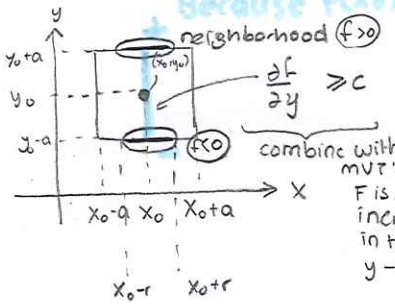
and whenever $|x - x_0| < r$, $|y - y_0| < r$, $f(x,y) = 0 \Rightarrow y = g(x)$

moreover, g is C^1 and $\frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x)) \cdot g'(x) = 0$

PROOF

$\frac{\partial f}{\partial y}(x_0,y_0) > 0$ (<0 is similar)

Because $f(x_0,y_0)=0$



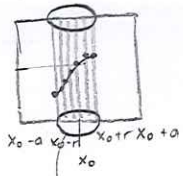
$\frac{\partial f}{\partial y} \geq c$

combine with MVT: F is strictly increasing in the positive y-direction.

$R = [x_0 - a, x_0 + a] \times [y_0 - a, y_0 + a]$ sequentially compact

$\exists\, a > 0, \exists\, c > 0$ such that

$\frac{\partial f}{\partial y}(x,y) \geq c$ on $\mathbb{R}$.



Ball around it, it stays negative

fix $x \in (x_0 - r, x_0 + r)$ Since $f(x, y_0 - a) < 0$ and $f(x, y_0 + a) > 0$ and f is C^0 (because C^1), we conclude by IVT that $\exists\, y$ s.t. $f(x,y) = 0$ where $y \in (y_0 - a, y_0 + a)$

Let $g(x) = y \Rightarrow f(x, g(x)) = 0 \ \forall x \in I = (x_0 - r, x_0 + r)$

Why is g uniquely determined?

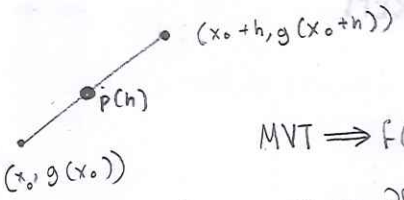
Because f is strictly increasing in the y-direction.

Let $x_0 + h \in I$ Then we know that $f(x_0 + h, g(x_0 + h)) = 0$

We also know $f(x_0, g(x_0)) = 0$

Then

$$f(x_0 + h, g(x_0 + h)) - f(x_0, g(x_0)) = 0$$



MVT $\Longrightarrow f(x_0 + h, g(x_0 + h)) - f(x_0, g(x_0))$

$$0 = \frac{\partial f}{\partial x}(p(h))h + \frac{\partial f}{\partial y}(p(h))[g(x_0 + h) - g(x_0)]$$

$$\frac{-\frac{\partial f}{\partial x}(x_0, g(x_0))}{\frac{\partial f}{\partial y}(x_0, g(x_0))} = g'(x_0) = \lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h} = \lim_{h \to 0} \frac{-\frac{\partial f}{\partial x}(p(h))}{\frac{\partial f}{\partial y}(p(h))}$$

Since R is sequentially compact, and

$\frac{\partial f}{\partial x}$ cont on $\mathbb{R}$

$\left|\frac{\partial f}{\partial x}(x,y)\right| \leq M$ for some $M > 0$

$$|g(x_0 + h) - g(x_0)| = \left| \frac{-\frac{\partial f}{\partial x}(p(h))}{\frac{\partial f}{\partial y}(p(h))} h \right| \leq \frac{M}{c}|h|$$

$\Longrightarrow g$ is cont at x_0.

All assumptions are true for any $x \in I$

$$\Rightarrow g'(x) = \frac{-\partial f/\partial x\,(x, g(x))}{\partial f/\partial y\,(x, g(x))} \quad \forall x \in I$$

INVERSE FUNCTION THEOREM

$F : O \subseteq \mathbb{R}^n \xrightarrow{C^1} \mathbb{R}^n$ Suppose $\boxed{DF(x_0)}$ is invertible (n×m derivative Matrix)

for some point $x_0 \in O$. Then $\exists$ a neighborhood U of x_0 and a nbhd V of $F(x_0)$ so that $F : U \to V$ is 1-1 and onto. Moreover $F^{-1} : V \to U$ is C^1 and

$$DF^{-1}(y) = [DF(x)]^{-1} \quad \text{where } y = F(x) \in V$$

PROOF

16.28 16.3

17.2 IMPLICIT FUNCTION THEOREM

$$F : O \subseteq \mathbb{R}^{n+k} \xrightarrow{C^1} \mathbb{R}^k \quad (n = k = 1 \Rightarrow \text{Dini's Theorem})$$

We would like to study the solution set

$$F(u) = 0 \quad u \in O$$

$u = (x, y) \quad x \in \mathbb{R}^n \quad y \in \mathbb{R}^k$

$F = (F_1, \dots, F_k)$

$$F(u) = 0 \iff \left.\begin{array}{l} F_1(x_1, \dots, x_n, y_1, \dots y_k) = 0 \\ \vdots \\ F_k(x_1, \dots, x_n, y_1, \dots, y_k) = 0 \end{array}\right\}$$

Can we solve y variables in terms of x variables

k equations
$n+k$ variables
undetermined system
unlikely to find explicit solutions

THEOREM (IMPLICIT FUNCTION)

$\boxed{(x_0, y_0) \in \mathbb{R}^{n+k}}$

$F : O \subseteq \mathbb{R}^{n+k} \xrightarrow{C^1} \mathbb{R}^k$

Suppose $D_y F(x_0, y_0)$ is invertible at some point $(x_0, y_0) \in O$ for which (partial derivative matrix.)

$F(x_0, y_0) = 0$ Then $\exists r > 0$ and $G : B \xrightarrow{C^1} \mathbb{R}^k$, $B = B_r(x_0) \subset \mathbb{R}^n$

such that $F(x, G(x)) = 0 \quad \forall x \in B$.

Moreover $\|x - x_0\| < r \quad \|y - y_0\| < r$, $F(x, y) = 0 \Rightarrow y = G(x)$ uniqueness of a solution

Furthermore, $\boxed{D_x F(x, G(x)) + D_y F(x, G(x))\, D\, G(x) = 0}$ (*)

$$D_y F(x, y) = \begin{bmatrix} \frac{\partial F_1}{\partial y_1}(x, y) & \cdots & \frac{\partial F_1}{\partial y_k}(x, y) \\ \vdots & & \\ \frac{\partial F_k}{\partial y_1}(x, y) & \cdots & \frac{\partial F_k}{\partial y_k}(x, y) \end{bmatrix}_{k \times k}$$

$$D_x F(x, y) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(x, y) & \cdots & \frac{\partial F_1}{\partial x_n}(x, y) \\ \vdots & & \frac{\partial F_k}{\partial x_n}(x, y) \\ \frac{\partial F_k}{\partial x_1}(x, y) & \cdots & \end{bmatrix}_{k \times n}$$

PROOF OF IMPLICIT F.T.

Let $H(x, y) = (x, F(x, y))$ $\quad x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$

$$H : O \subseteq \mathbb{R}^{n+k} \xrightarrow{C^1} \mathbb{R}^{n+k}$$

$$D H(x, y) = \left[\begin{array}{c|c} I_{n \times n} & \overset{\text{zero}}{0_{n \times k}} \\ \hline D_x F(x, y) & D_y F(x, y) \end{array}\right]_{(n+k) \times (n+k)}$$

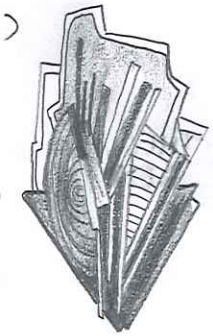
invertible $\iff$ $D_y F(x, y)$ invertible

($I_{n \times n}$ is invertible)

Since, by assumption $D_y F(x_0, y_0)$ is invertible at a solution $F(x_0, y_0) = 0$, $(x_0, y_0) \in \mathcal{O}$, we have $DH(x_0, y_0)$ is invertible.

Therefore by inverse function theorem $\exists$ a nbhd $\mathcal{U}$ of (x_0, y_0) and a nbhd V of $\underset{(x_0, F(x_0, y_0)) = (x_0, 0)}{H(x_0, y_0)}$ such that

$$H : \mathcal{U} \xrightarrow[\text{onto}]{1-1} V$$



Therefore

$$H^{-1} : V \longrightarrow \mathcal{U} \text{ is } C^1$$

$$H^{-1}(x, y) \longrightarrow (\underbrace{M(x,y)}_{n}, \underbrace{N(x,y)}_{k})$$

$$(x, y) = H \circ H^{-1}(x, y) = H(M(x,y), N(x,y))$$

$$= (M(x,y), F(M(x,y), N(x,y)))$$

$$\Rightarrow M(x,y) = x \text{ and } \left.\begin{aligned} y &= F(M(x,y), N(x,y)) \\ &= F(x, N(x,y)) \end{aligned}\right\}$$

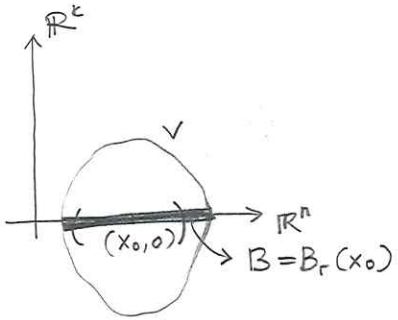
Observe $(x_0, 0) \in V \subset \mathbb{R}^{n+k}$

Hence $\exists r > 0$ s.t. $(x, 0) \in V \quad \forall x \in B_r(x_0)$

$$G : B \xrightarrow{C^1} \mathbb{R}^k$$

$$G(x) = N(x, 0)$$

$$F(x, G(x)) = F(x, N(x,0)) = 0$$



To prove uniqueness we use $H : \mathcal{U} \longrightarrow V$ 1-1

Suppose $(x, y) \in \mathcal{U}$ $F(x, y) = 0$

$$\Rightarrow (x, N(x,0)) = H^{-1}(x, 0)$$

and $H(x, y) = (x, 0) = H(x, N(x,0))$

$$\Rightarrow y = N(x, 0) = G(x)$$

$$F(x, G(x)) = 0 \iff F_i(x, G(x)) = 0 \quad \forall i$$

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n + \mathbb{R}^k \longrightarrow \mathbb{R}$$

$$\begin{tikzcd} x \arrow[r] \arrow[rr, bend right, "F_i \circ K"'] & (x, G(x)) \arrow[r, "F_i"] & F_i(x, G(x)) \end{tikzcd}$$

$$\underset{1 \times n}{\nabla(F_i \circ K)(x)} = \underset{1 \times (n+k)}{\nabla F_i(K(x))} \cdot \underset{(n+k) \times n}{DK(x)}$$

$$(*) \quad D_x F(x, G(x)) + D_y F(x, G(x)) \, D \cdot G(x) = 0$$

$$\Rightarrow \left.\begin{matrix} i,j \\ \text{th} \\ \text{position} \end{matrix}\right\} \quad \frac{\partial F_i}{\partial x_j}(x, G(x)) + \sum_{\ell=1}^{k} \frac{\partial F_i}{\partial y_\ell} \cdot \frac{\partial G_\ell}{\partial x_j} = 0$$

: If sum of the two matrices is zero then sum of the two positions of two matrices needs to be zero.

$$\nabla F_i(K(x)) \cdot DK(x)$$

$$\left(\frac{\partial F_i}{\partial x_1}(K(x)), \dots, \frac{\partial F_i}{\partial x_n}(K(x)), \frac{\partial F_i}{\partial y_1}(K(x)) \dots \frac{\partial F_i}{\partial y_k}(K(x)) \right)$$

$$DK(x) = \begin{array}{c} \\ \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{c} j \end{array} \\ \left[\begin{array}{c|c} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \frac{\partial G_1}{\partial x_j} \\ \vdots \\ \frac{\partial G_1}{\partial x_j} \end{matrix} \\ \hline \cdots & \end{array} \right] \end{array} (n+k) \times n$$

$\boxed{\text{Differentiable}} \overset{\not\Leftarrow}{\Rightarrow} C^0$: continuous

$\Downarrow$ $\quad \not\Downarrow \quad \Uparrow$

First partials exist and derivative map is given by Jacobian matrix $Df(a)$ $\quad C^1 \quad$ continuously diff'ble (first partials are cont)

DEF · OF · DIFFERENTIABILITY

$$f : \underset{\text{open}}{O} \overset{\subseteq \mathbb{R}^n}{\longrightarrow} \mathbb{R}^m$$

We say that f is diff'ble at $a \in O$ if there exists a linear map $T_a : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T_a(h)}{\|h\|} = 0$$

we cannot divide a vector with a vector.

Why is this a generalization of dim=1?

$n = m = 1 \qquad f : \underset{\text{open}}{I} \overset{\subseteq \mathbb{R}}{\longrightarrow} \mathbb{R}$

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T_a(h)}{h} = 0$$

☒ usual 1-dim'l calculus defn

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

$$\iff \lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0$$

$$T_a(H) = f'(a)h$$

If f is diff'ble at a, then we say that $T_a : \mathbb{R}^n \to \mathbb{R}^m$ is derivative of f at a.

NOTATION : $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ represents derivative at $a \in O \subseteq \mathbb{R}^n$

normally we use this notation for matrix. Here with taking basis as $\{e_1, e_2, \dots\}$ (standard basis) we can change it to linear map.

THEOREM (Uniqueness of Derivative)

$f : O \overset{\subseteq \mathbb{R}^n}{\longrightarrow} \mathbb{R}^m$ Suppose f is diff'ble at $a \in O$.

Suppose that $\lambda, \mu : \mathbb{R}^n \to \mathbb{R}^m$ are derivatives of f at a.
lambda, mu

PROOF

$$\lim_{h \to 0} \frac{\|\lambda(h) - \mu(h)\|}{\|h\|}$$

$$= \lim_{h \to 0} \frac{\|\lambda(h) - f(a+h) + f(a) - f(a) + f(a+h) - \mu(h)\|}{\|h\|}$$

Triangle inequality $\leq$

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} + \lim_{h \to 0} \frac{\|f(a+h) - f(a) - \mu(h)\|}{\|h\|}$$

Take $h = tx$, $x \overset{\neq 0}{\in} O \subset \mathbb{R}^n$ $\quad t \in \mathbb{R}$

$$\lim_{t \to 0} \frac{\|\lambda(tx) - \mu(tx)\|}{\|tx\|} = 0 = \frac{\|\lambda(x) - \mu(x)\|}{\|x\|}$$ Due to Linearity

Since $\|x\| \neq 0$ by assumption,

$$\Rightarrow \|\lambda(x) - \mu(x)\| = 0$$

$$\Rightarrow \lambda(x) - \mu(x) = 0$$

$$\Rightarrow \lambda(x) = \mu(x)$$

THEOREM (Diff'ble $\Rightarrow$ cont) $f: O \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

f diff'ble at $a \in O \Longrightarrow f$ cont at a.

PROOF

$$\lim_{x\to a} f(x)-f(a) = \lim_{x\to a}\left(Df(a)\overset{\text{plug-in}}{(x-a)} + \|x-a\| \frac{f(x)-f(a)-Df(a)(x-a)}{\|x-a\|}\right)$$

$$\|f(x)-f(a)\| \overset{\text{Triangle Inequality}}{\leq} \|\underbrace{Df(a)(x-a)}_{\text{linear map } \mathbb{R}^n\to\mathbb{R}^m}\| + \|x-a\| \frac{\|f(x)-f(a)-Df(a)(x-a)\|}{\|x-a\|}$$

$$\underset{\text{Lemma}}{\leq} M\|x-a\| + \text{rest} \to 0$$

By definition of derivative

Lemma

If $T: \mathbb{R}^n \to \mathbb{R}^m$ (linear map), then $\exists M>0$ s.t.

$$\|T(h)\| < M\|h\|$$

$$\forall h \in \mathbb{R}^n$$

PROOF OF Lemma

T can be represented by a matrix $T(h) = A\cdot h$

A is an $(m\times n)$ matrix.

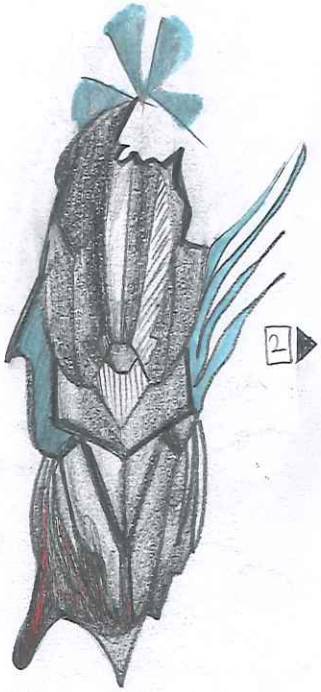
$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \leftarrow b_1 \\ \vdots \\ \leftarrow b_m \end{matrix}$$

$$T(h) = A\cdot h = (b_1\cdot h, \dots, b_m\cdot h)$$

$$\|T(h)\| = \|(b_1\cdot h, \dots, b_m\cdot h)\|$$

$$\underset{\text{like Pisagor}}{\leq} \sum_{i=1}^{m} |b_i\cdot h| \underset{\text{Cauchy Schwartz}}{\leq} \sum_{i=1}^{m} \|b_i\|\|h\|$$

$$\Rightarrow M = \sum_{i=1}^{m} \|b_i\|$$



1 ▶ (Of a function which has both partials but not cont!)

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0 = \frac{\partial f}{\partial y}(0,0)$$

f is not cont at $(0,0)$ hence f is not diff'ble at $(0,0)$

$$\boxed{\begin{aligned} &(x_k, y_k) = (1/k, 1/k) \\ &(x_k, y_k) \to (0,0) \\ &f(x_k, y_k) = \frac{1/k^2}{2/k^2} = \frac{1}{2} \\ &\lim_{k\to\infty} f(x_k, y_k) = \frac{1}{2} \neq f(0,0) = 0 \end{aligned}}$$

2 ▶ (Cont but not diff'ble function)

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Trick: Polar coordinates

$$x = r\cos\theta \quad y = r\sin\theta$$

$$f(r,\theta) = \frac{r^2\cos^2\theta\, r\sin\theta}{r^2} = \frac{r}{2}\cos\theta\sin 2\theta$$

$\lim_{r\to 0} \frac{r}{2}|\cos\theta\sin\theta| = 0$ because $\sin\theta, \cos\theta$ bounded functions.

$\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$. Suppose f is diff'ble at $(0,0)$

$$\Rightarrow Df(0,0) = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-\cancel{f(0,0)}^{0}-\cancel{Df(0,0)}^{0}(x,y)}{\sqrt{x^2+y^2}} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{(x^2+y^2)^{3/2}} = 0 \longrightarrow 1/2\cos\theta\sin 2\theta$$

Contradiction.

Limit does not exist.

$$\begin{cases}(x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)\\ 0\end{cases} \quad \Nearrow \quad \text{Diff.ble} \quad \Nearrow \quad \begin{cases}\frac{xy}{x^2+y^2}\\ 0\end{cases}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$C^2 \Longrightarrow C^1 \Longrightarrow C^0$$

$$\begin{cases}\frac{xy}{x^2+y^2}(x^2-y^2)\\ 0\end{cases} \qquad \begin{cases}\frac{x^2y}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$$F: O^{\subseteq\mathbb{R}^n} \longrightarrow \mathbb{R}^m$$

C^0 ✓

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0 \qquad \left|\frac{x^2y}{x^2+y^2}\right| \le \frac{|x^2y|}{|x^2|} = y$$

alternate : use polar coordinates

NOT-DIFF'BLE AT (0,0) : Suppose it is diffble $\Longrightarrow Df(0,0)=0$

$$\lim_{h\to 0} \frac{f(h)-\cancel{f(0)}^{0}-\cancel{Df(0)}^{0}}{\|h\|} = 0$$

$$\lim_{(x,y)\to 0} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2y}{(x^2+y^2)^{3/2}} = 0$$

DOES NOT EXIST

not true

$\frac{1}{2}\cos\theta\sin 2\theta$

$\boxed{3}$ • f is not C^1

PROOF

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases}\frac{2xy}{x^2+y^2} - \frac{2x^3y}{(x^2+y^2)^2}\\ 0\end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} - \frac{2x^3y}{(x^2+y^2)^2}$$

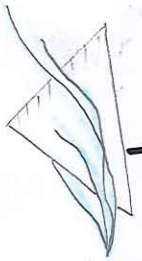
Plug in $y=mx \Longrightarrow \frac{2m^3}{(1+m^2)^2}$ Depends on m!

$m=1 \Rightarrow \frac{1}{2}$

$m=-1 \Rightarrow \frac{-1}{2}$

$\frac{\partial f}{\partial x}$ is not cont at (0,0)

$\frac{\partial f}{\partial y}$ is not cont at (0,0)



THEOREM $C^1 \Rightarrow$ Diffble

PROOF (15.11)

we did this already First Approx. Theorem for mappings.

$$F: O^{\subseteq\mathbb{R}^n} \xrightarrow[C^1]{} \mathbb{R}^m \Longrightarrow \lim_{h\to 0} \frac{\|F(x+h)-F(x)-Df(x)h\|}{\|h\|} = 0$$

$T_x : h \to Df(x)h$

$\mathbb{R}^n \to \mathbb{R}^m$

$\boxed{4}$ ▨ $\begin{cases}(x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)\\ 0\end{cases}$ is diff'ble.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h-0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h^2\sin\left(\frac{1}{|h|}\right)}{h}$$

$$= \lim_{h\to 0} h\sin\left(\frac{1}{\|h\|}\right) = 0$$

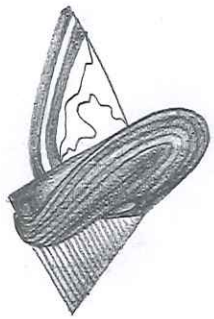
Only candidate for derivative at $(0,0)$ is the zero linear map.

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \cancel{f(0,0)}^{\,0} - \cancel{Df(0,0)(x,y)}^{\,0}}{\|(x,y)\|}$$

$$= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\,\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$= 0$$

$\Longrightarrow f$ is diff'ble at $(0,0)$ (everywhere)



CLAIM

$\boxed{5}$ f is not C^1 $\longrightarrow$ $\frac{\partial f}{\partial x}$ not cont at $(0,0)$, $\frac{\partial f}{\partial y}$ not cont at $(0,0)$

$$\frac{\partial f}{\partial x}(x,y) = 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

$$\frac{\partial f}{\partial x}(\underset{\neq 0}{x}, 0) = 2x\sin\left(\frac{1}{|x|}\right) - \underbrace{\frac{x}{|x|}}_{\text{does not goes } 0}\underbrace{\cos\left(\frac{1}{|x|}\right)}_{\text{does not converge}}$$

$$\lim_{x\to 0}\frac{\partial f}{\partial x}(x,0) = \text{DOES NOT EXIST}$$

CHAIN RULE

$f: \mathbb{R}^n \to \mathbb{R}^m$ diff'ble at a

$g: \mathbb{R}^m \to \mathbb{R}^k$ diff'ble at $f(a)$

Then $g\circ f: \mathbb{R}^n \to \mathbb{R}^k$ is diff'ble at a.

Moreover, $D(g\circ f) = Dg(f(a)) \circ Df(a)$

composition of linear maps

COMPONENTWISE DIFFERENTIATION

$f: \mathbb{R}^n \to \mathbb{R}^m$ diff'ble at $a\in\mathbb{R}^n \Longleftrightarrow f_i: \mathbb{R}^n \to \mathbb{R}$ is diff'ble at a.

components of f.

PROOF

Exercise. (Hint: Chain Rule)

(*) Final Exam

Differentiability VS Partial Derivatives

$f: \mathbb{R}^n \to \mathbb{R}$ diff'ble at $a\in\mathbb{R}^n \Longrightarrow \forall i\ \frac{\partial f}{\partial x_i}(a)$ exists

and

$$\frac{\partial f}{\partial x_i}(a) = Df(a)(e_i)$$

STD Basis Vector of $\mathbb{R}^n$

PROOF

$$\lim_{\substack{h\to 0\\ \mathbb{R}}} \frac{|f(a+he_i) - f(a) - Df(a)he_i|}{\|he_i\| = \|h\|\|e_i\| = \|h\| = h} = 0$$

$$\frac{\partial f(a)}{\partial x_i} = \lim_{h\to 0}\frac{f(a+he_i)-f(a)}{|h|} = Df(a)(e_i)$$

THEOREM $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff. at $a \Rightarrow \frac{\partial f_i}{\partial x_j}(a)$ exists

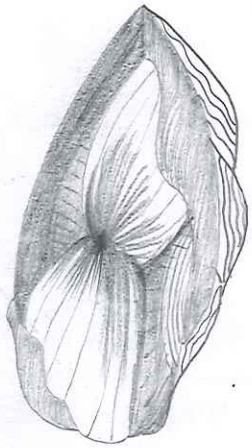
$\forall\ 1\le i\le m$

$\forall\ 1\le j\le n$

Moreover $[Df(a)]_{m\times n} = \left[\frac{\partial f_i}{\partial x_j}(a)\right]_{m\times n}$

Jacobian Matrix = Derivative Matrix

2 $\begin{cases} \frac{y^3}{x^2+y^2} \\ 0 \end{cases}$ has all directional derivatives at $(0,0)$

is cont at $(0,0)$ but not diff.ble at $(0,0)$

Exercise.