Complex Analysis

MATH 401 Koç University
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MATH 401
COMPLEX ANALYSIS

· CHAPTER 1 $\boxed{\text{SECTION 1}}$

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

additive group; field $\times +$; complete; algebraic closure, any polynomial

field extension

$\mathbb{C}$: smallest field containing $\mathbb{R}$ and i → commutative

2-dimensional vector space over $\mathbb{R}$

a basis of $\mathbb{C}$ $\{i, 1\}$

isomorphic to $\mathbb{R}^2$

▨ $z.\bar{z} = |z|^2$

$z^{-1} = \dfrac{\bar{z}}{|z|^2}$

THEOREM $\alpha, \beta \in \mathbb{C}$

· $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$

· $\overline{\alpha+\beta} = \bar{\alpha}+\bar{\beta}$

Field homeomorphism

isomorphism

involution → $\bar{\bar{z}} = z$

conjugate : $\mathbb{C} \to \mathbb{C}$

▨ $Re(z)$ → x, $Im(z)$ → y not yi

THEOREM $\alpha, \beta \in \mathbb{C}$

a) $|\alpha\beta| = |\alpha||\beta|$ → $\mathbb{C} \to \mathbb{R}$

b) $|\alpha+\beta| \leq |\alpha| + |\beta|$

PROOF

a) $|\alpha\beta|^2 = (\alpha\beta)(\overline{\alpha\beta})$

$= \alpha\beta\bar{\alpha}\bar{\beta}$

$= \alpha\bar{\alpha}\beta\bar{\beta}$

$= |\alpha|^2|\beta|^2$

$\Rightarrow |\alpha\beta| = |\alpha||\beta|$

b) $|\alpha+\beta|^2 = (\alpha+\beta)(\overline{\alpha+\beta})$

$= (\alpha+\beta)(\bar{\alpha}+\bar{\beta})$

$= \alpha\bar{\alpha} + \beta\bar{\beta} + \alpha\bar{\beta} + \bar{\alpha}\beta$ $\quad \overline{\alpha\bar{\beta}}$

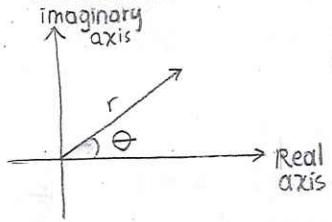
$= |\alpha|^2 + |\beta|^2 + 2Re(\alpha\beta)$

$\leq |\alpha|^2 + |\beta|^2 + 2|\alpha||\beta|$

$= (|\alpha| + |\beta|)^2$

Remark $|z_1 + z_2 + \dots| \leq |z_1| + |z_2| \dots$

$\boxed{\text{SECTION 2}}$ POLAR FORMS



$(r, \theta) \xrightarrow[\text{to go to}]{\text{we want}} z = x + yi$

$\| $

$re^{i\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$ → Power series extension

$x = r\cos\theta$

$y = r\sin\theta$

▨ Analytic functions

In $\mathbb{C}$, if one can take derivative of a $f \in \mathbb{C}$, then f is analytic.

$\mathbb{R} \xrightarrow{\text{exponential}} \mathbb{C}$

$\theta \longrightarrow e^{i\theta}$

$\boxed{z = r.e^{i\theta}}$ → $\arg(z)$

$w = \rho.e^{i\varphi}$

$zw = r\rho.e^{i(\theta+\varphi)}$

THEOREM $\theta, \varphi \in \mathbb{R}$

$e^{i\theta}.e^{i\varphi} = e^{i(\theta+\varphi)}$

PROOF

$e^{i(\theta+\varphi)} = \cos(\theta+\varphi) + i\sin(\theta+\varphi)$

$= \cos\theta\cos\varphi - \sin\theta\sin\varphi$

$+ i(\sin\theta\cos\varphi + \cos\theta\sin\varphi) = e^{i\theta}.e^{i\varphi}$

DEF $\mathbb{C} \xrightarrow{exp} \mathbb{C}$

$z \longmapsto e^z = e^x.e^{iy} = e^x.(\cos y + i\sin y)$

THEOREM $z, w \in \mathbb{C}$

Then $e^{z+w} = e^z.e^w$

▶ All complex numbers whose fifth power is -32

$z^5 + 32 = 0$

$$\begin{array}{l|l} z^5 + 32 & (z+2) \\ \hline z^5 + 2z^4 & z^4 \\ -2z^4 & \end{array} \quad \dots$$

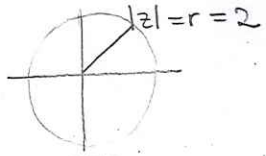
□ Write -32 in polar coordinates.

$-32 = 32.\boxed{e^{i\pi}}$ also $32.\boxed{e^{i3\pi}}$ also $32e^{i5\pi}$

root → $2e^{i\pi/5}$ root → $2e^{i\frac{3\pi}{5}}$ root → $2e^{i\pi}$ -2

root → $2e^{i11\pi/5}$

same (2π)



$\boxed{\text{SECTION 3}}$ (complex valued) Functions

$f : S \xrightarrow{cc} \mathbb{C}$

$z \longmapsto f(z) = u(z) + iv(z)$

function

$u, v : S \to \mathbb{R}$ → imaginary part

real valued function

real part

power functions

complex polynomials $z \longmapsto a_0 + a_1 z + a_2 z^2 + \dots$

$a_0, a_1 \dots a_n \in \mathbb{C}$

Power series

$z \longmapsto z^2$

$x+iy \longmapsto (x^2-y^2)+i(2xy)$

$\|$

$re^{i\theta} \longmapsto r^2 e^{i2\theta}$

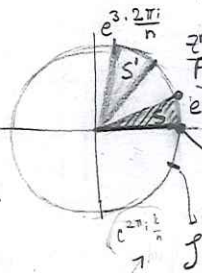
$z \longmapsto z^n$ = n to one function

$\bar{D} \longrightarrow \bar{D}$

closed disc of radius 1 centered at the origin

$\|$

$\{z \in \mathbb{C} \mid |z| \leq 1\}$



z^n is again in $\bar{D}$ because r is the same.

$e^{i0} = e^{i2\pi} = e^{i4\pi} \ldots$

$= e^{k2\pi i}$

$\zeta^{n-1} = e^{(n-1)\cdot\frac{2\pi i}{n}}$

$(\zeta^k)^n = 1$

$\underbrace{1, \zeta, \zeta^2, \ldots \zeta^{n-1}}_{n^{th} \text{ roots of unity}}$

$z \mapsto z^n$

$S \longrightarrow D$ surjective

or any slice where $\theta \geq \frac{2\pi}{n}$ would work

$\boxed{\text{SECTION 4}}$ LIMIT AND COMPACT SET IN $\mathbb{C}$ — same as $\mathbb{R}^2$

$D(z,r)$: open disc centered at $z \in \mathbb{C}$ with radius $r>0$. (like an open ball)

$\|$

$\{w \in \mathbb{C} \mid |w-z| < r\}$

$\bar{D}(z,r)$: closed " " " " "

$\|$

$\{w \in \mathbb{C} \mid |w-z| \leq r\}$

DEF $U \in \mathbb{C}$ is called open if $\forall z \in U \ \exists r>0$ s.t. $D(z,r) \subset U$

▶ upper half plane

$\{z \in \mathbb{C} \mid Im(z) > 0\}$ is open.

DEF $S \subset \mathbb{C}$

z is a boundary point of S if

$\forall r>0$ • $D(z,r) \cap S \neq \emptyset$

and • $D(z,r) \cap (\mathbb{C}/S) \neq \emptyset$

• z is adherent to S if

$\forall r>0$ • $D(z,r) \cap S \neq \emptyset$.

• z is an interior point of S if

$\exists r>0$ s.t. $D(z,r) \subset S$.

DEF $S \subset \mathbb{C}$ is closed if S contains all its boundary points.

Remark S is closed $\Leftrightarrow$ $\mathbb{C}/S$ is open

The closure of S ($\bar{S}$) is the set of all points adherent to S. ($\bar{S} = S + \{\text{boundary}\}$)

DEF $S \subset \mathbb{C}$ is called bounded if $\exists R>0$ s.t. $S \subset \bar{D}(0,R)$

$f : S \xrightarrow{\subset \mathbb{C}} \mathbb{C}$

α is adherent to S.

Then $\lim_{z \to \infty} f(z) = w$

means that

$\forall \varepsilon>0 \ \exists \delta>0$ s.t. $z \in D(\alpha,\delta) \cap S$

$\Longrightarrow f(z) \in D(w,\varepsilon)$

$f : S \longrightarrow \mathbb{C}$ is continous at $\alpha \in S$

if $\lim_{z \to \infty} f(z) = f(\alpha)$

DEF $\{z_n\}$ a sequence in $\mathbb{C}$

$\lim_{n \to \infty} z_n = w$ if $\forall \varepsilon>0 \ \exists N_0 \in \mathbb{N}$ s.t.

$n \geq N_0 \Rightarrow z_n \in D(w,\varepsilon)$

DEF $\{z_n\}$ is a sequence in $\mathbb{C}$

$\{z_n\}$ is called Cauchy seq. if $\forall \varepsilon>0$

$\exists N_0 \in \mathbb{N}$ s.t. $n,m \geq N_0 \Rightarrow |z_n - z_m| < \varepsilon$

□ Since $\mathbb{R}^2$ is complete, so is $\mathbb{C}$ i.e. any cauchy is convergent.

COMPACT SETS

$\{z_n\}$ sequence in $\mathbb{C}$.

w is a point of accumulation of $\{z_n\}$

$\forall \varepsilon>0$ $\exists$ infinitely many $n \in \mathbb{N}$ s.t.

$z_n \in D(w,\varepsilon)$

$S \subset \mathbb{C}$ (an infinite subset)

$\forall \varepsilon>0$, $S \cap D(w,\varepsilon)$ has infinitely many elements.

Weier Stross (If S is infinite bounded set of real numbers, then S has a point of accumulati

DEF $S \subset \mathbb{C}$ is called compact if every sequence in S has an accumulation point.

THEOREM For $S \subset \mathbb{C}$. TFAE → EXERCISE

i) S is compact

ii) every infinite subset of S has an accumulation point

iii) every seq in S has a convergent subsequence whose limit is in S.

THEOREM 4.1 $S \subset \mathbb{C}$ is compact $\Leftrightarrow$ if it is closed and bounded.

Heine Borel

THEOREM 4.2 S compact subset of $\mathbb{C}$

$\{S_n\}$ a sequence of non-empty, closed subsets of $\mathbb{C}$ with $S \supset S_1 \supset S_2 \dots$

Then $\bigcap_n S_n \neq \emptyset$

THEOREM 4.3 S: a compact set in $\mathbb{C}$

$f: S \longrightarrow \mathbb{C}$

If f is continous, then $f(S)$ is compact

THEOREM 4.4 S: compact set in $\mathbb{C}$.

$f: S \longrightarrow \mathbb{R}$

If f is continous then f has a max of on S

i.e. $\exists w \in S$ s.t. $f(z) \leq f(w)$ $\forall z \in S$

THEOREM 4.5 S: compact set

$f: S \longrightarrow \mathbb{C}$ continous.

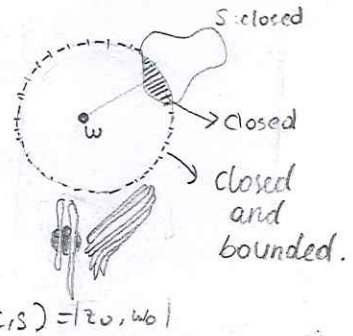
Then f is uniformly cont

$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $|z-w| < \delta \Longrightarrow |f(z) - f(w)| < \varepsilon$

$\hookrightarrow$ not depends on z

THEOREM 4.6 S: closed set in $\mathbb{C}$

$d(S, w) = \inf \{ |z-w| : z \in S \}$



THEOREM 4.7 K: compact set in $\mathbb{C}$

S: closed set in $\mathbb{C}$.

Then $\exists z_0 \in K, w_0 \in S$ s.t $d(K, S) = |z_0, w_0|$

THEOREM 4.8 S: compact set in $\mathbb{C}$

$\forall r > 0$ $\exists$ finitely many open discs of radius r whose union contains S.

THEOREM 4.9 S: compact set in $\mathbb{C}$

$\{U_i\}_{i \in I}$: a family of open sets in $\mathbb{C}$ covering S.

Then $\{U_i\}_{i \in I}$ has a finite subcover.

IS COMPLEX DIFFBLITY

U: open subset of $\mathbb{C}$

$f: U \longrightarrow \mathbb{C}$

$z \in U$

f is said to be complex diffble (holomorphic) at z if $\lim_{w \to 0} \frac{f(z+w) - f(z)}{w}$ exists.

Whenever it exists, the above limit is called the derivative of f at z and denoted by $f'(z)$.

Remark: There are many ways to go 0.

- $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $z + w \in D(z, \delta) \cap U \Longrightarrow \frac{f(z+w) - f(z)}{w} \in D(f'(z), \varepsilon)$

f, g diffble at z

- $f \pm g$ diffble
  $f \cdot / g$ diffble
- CHAIN
  $(g \circ f)'(z) = g'(f(z)) f'(z)$

An alternative characterization of the diff of f at z:

(What we want:
$\exists$ function φ of w and constant $c \in \mathbb{C}$ s.t.
- continous $f(z+w) - f(z) = cw + w\varphi(w)$
- $\lim_{w \to 0} \varphi(w) = 0$)

Suppose f is diffble at z. Let φ be the function of w defined by

$$\varphi(w) = \begin{cases} \frac{f(z+w) - f(z)}{w} - f'(z), & \text{if } w \neq 0 \text{ and } z + w \in U \\ \text{if } w = 0 \end{cases}$$

Observe that φ is continous at every w with $z+w \in U$

Moreover, $f(z+w) - f(z) = f'(z)w + w\,\varphi(w)$

and $\lim_{w\to 0} \varphi(w) = 0$

Conclusion

Conversely, suppose that there is a cont. function φ and a constant $c \in \mathbb{C}$ as above,

Then

$$\lim_{w\to 0} \frac{f(z+w)-f(z)}{w} = \lim_{w\to 0} \frac{c.w + w\,\varphi(w)}{w}$$

$$= \lim_{w\to 0} c + \varphi(w) = c$$

PROOF of the Chain Rule

since g is diff'ble at $f(z) = \mu$, $\exists$ continous func ψ of μ s.t.

$$g(\mu+\xi) - g(\mu)$$

$$= g'(\mu)\xi + \xi\,\psi(\xi)$$

and $\lim_{\xi\to 0} \psi(\xi) = 0$

$$\lim_{w\to 0} \frac{g(\overbrace{f(z+w)}^{\mu+\xi}) - \overbrace{g \circ f(z)}^{g(\mu)}}{w} = \lim_{w\to 0} \frac{g'(\mu).\xi + \xi\,\psi(\xi)}{w}$$

we can choose any. Then

$\xi = f(z+w) - f(z)$

$$= \lim_{w\to 0} \left(g'(\mu)\frac{\xi}{w} + \psi(\xi)\frac{\xi}{w} \right)$$

$\to 0$

$\to f'(z)$

$$= g'(\mu).f'(z)$$

▶ $f(z) = z^n$ $n \in \mathbb{Z}$ $f'(z) = n z^{n-1}$

Case $n \in \mathbb{Z}_+$; $n=1$: $\lim_{w\to 0} \frac{(z+w)-z}{w} = \lim_{w\to 0} 1 = 1$ ✓

Assume that $(z^k)' = k.z^{k-1}$

Then $(z^{k+1})' = (z^k.z)' = (z^k)'z + z^k(z)'$ ✓

$= (k.z^{k-1})z + z^k.1 = (k+1)z^k$

Proved by induction

Case $n=0$: $(1)' = \lim_{w\to 0} \frac{1-1}{w} = 0$

Case $n \in \mathbb{Z}$; $n = -m$ for some $m \in \mathbb{Z}_+$ ✓

So $(z^n)' = \left(\frac{1}{z^m}\right)' = \frac{(1)'.z^m - 1(z^m)'}{(z^m)^2}$

$$= \frac{-(m.z^{m-1})}{z^{2m}}$$

$$= -m.z^{-m-1} = n.z^{n-1}$$ ✓ □

I6 CAUCHY RIEMANN EQUATIONS

$f : U \longrightarrow \mathbb{C}$

$f(z) = u(x,y) + i\,v(x,y)$

u, v : real valued

□ A characterization of holomorphicity of f in terms of the partial derivatives of u and v

Suppose that f is diff'ble at $z \in U$ Hence we have a cont func φ of w s.t. $f(z+w) - f(z) = \underbrace{f'(z)}_{a+bi}\,\underbrace{w}_{h+ki} + w\,\varphi(w)$

and $\varphi(w) \xrightarrow[w\to 0]{} 0$ $x+y;$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{\overbrace{u(x+h,y) - u(x,y)}^{Re(f(z+h)-f(z))}}{h}$$

$$= \lim_{h\to 0} \frac{Re(f'(z).h + h\,\varphi(h))}{h}$$

$$= \lim_{h\to 0} \frac{ah + h\,Re\,\varphi(h)}{h}$$

$$= a + \lim_{h\to 0} (Re(\varphi(h))) = a$$

$\to 0$

$$\frac{\partial u}{\partial x} = Re(f'(z))$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{\overbrace{u(x,y+k) - u(x,y)}^{\mathrm{Re}(f(z+ik)-f(z))}}{k}$$

$$= \lim_{k\to 0} \frac{\mathrm{Re}\left(\overbrace{f'(z)\cdot(ik)}^{-bk+iak} + (ik)\cdot\varphi(ik)\right)}{k}$$

$$= \lim_{k\to 0} \frac{-bk - k\,\mathrm{Im}(\varphi(ik))}{k} = \lim_{k\to 0} -b + \cancel{\mathrm{Im}(\varphi(ik))}^{\to 0}$$

$$= -b = -\mathrm{Im}(f'(z))$$

similarly

$$\frac{\partial v}{\partial x} = b = \mathrm{Im}(f'(z)) \qquad \frac{\partial v}{\partial y} = a = \mathrm{Re}(f'(z))$$

so u and v satisfy Cauchy-Riemann equations : $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$

I7 ANGLES UNDER HOLOMORPHIC MAPS

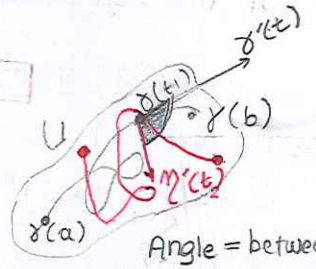
holomorphic $\longrightarrow$ $\boxed{\text{"conformal"}}$

preserves angles

U : open set in $\mathbb{C}$

$\gamma : [a,b] \xrightarrow{\text{cont}} U$

curve



Angle = between γ and η at $z_0 = \gamma(t_1) = \eta(t_2)$

Suppose that γ and η are diffble

THEOREM If f is hol at z_0 then the angle between γ and η at z_0 is equal to the angle between $f\circ\gamma$ and $f\circ\eta$ at z_0.

$\gamma(t_1) = \eta(t_2)$

Key Fact : The chain rule holds in this context as well.

$$(f\circ\gamma)'(t_1) = \underbrace{f'(\gamma(t_1))}_{= f'(z_0) = r_0 e^{i\varphi_0}}\gamma'(t_1)$$

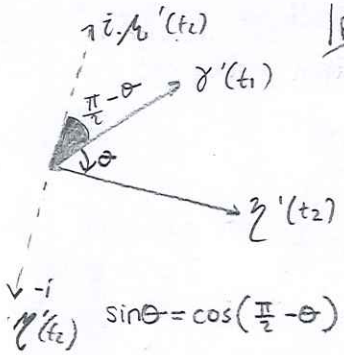
$$(f\circ\eta)'(t_2) = f'(\eta(t_2))\,\eta'(t_2)$$

$$\cos\theta = \frac{\langle \gamma'(t_1), \eta'(t_2)\rangle}{|\gamma'(t_1)|\,|\eta'(t_2)|}$$ where $\langle\cdot,\cdot\rangle$ denotes the standard inner-product on $\mathbb{R}^2$.

$\cos\psi$ angle between $f\circ\gamma$ and $f\circ\eta$ at $f(z_0)$

$$= \frac{\langle (f\circ\gamma)'(t_1), (f\circ\eta)'(t_2)\rangle}{|(f\circ\gamma)'(t_1)|\cdot|(f\circ\eta)'(t_2)|} = \frac{\langle f'(z_0)\gamma'(t_1), f'(z_0)\eta'(t_2)\rangle}{|f'(z_0)\gamma'(t_1)|\,|f'(z_0)\eta'(t_2)|}$$

$$\cos\theta = \frac{\cancel{|f'(z_0)|^2}\langle \gamma'(t_1), \eta'(t_2)\rangle}{\cancel{|f'(z_0)|}|\gamma'(t_1)|\cancel{|f'(z_0)|}|\eta'(t_2)|}$$



$$\sin\theta = \cos\left(\tfrac{\pi}{2}-\theta\right) = \frac{\langle \gamma'(t_1), -i\,\eta'(t_2)\rangle}{|\gamma'(t_1)|\,|-i\cdot\eta'(t_2)|}$$

$$\sin\psi = \cos\left(\tfrac{\pi}{2}-\psi\right) = \frac{\langle (f\circ\gamma)'t_1, -i\,(f\circ\eta)'t_2\rangle}{|(f\circ\gamma)'(t_1)|\,|-i\,(f\circ\eta)'(t_2)|} = \sin\theta$$

Hence $\theta = \psi$

CHAPTER II POWER SERIES

2.1. FORMAL POWER SERIES

$\sum_{n\geq 0} a_n T^n = a_0 + a_1 T + \dots$

$a_0, a_1, \dots \in \mathbb{C}$

Same info as the sequence $(a_0, \dots)$ of complex numbers.

$f+g = \sum_{n\geq 0} (a_n + b_n) T^n$

$f \cdot g$

cf

order of a power series : the index of the first non-zero coefficient.

e.g. if $f = a_r T^r + a_{r+1} T^{r+1} + \dots$ and $a_r \neq 0$ then $\mathrm{ord}(f) = r$

OBSERVATION $\mathrm{ord}(f+g) \geq \min\{\mathrm{ord}(f), \mathrm{ord}(g)\}$ ✓

$\mathrm{ord}(f \cdot g) = \mathrm{ord}(f) + \mathrm{ord}(g)$ ✓

inverse of $f = \sum_{n\geq 0} a_n T^n$ is a power series $g = \sum_{n\geq 0} b_n T^n$ such that $f \cdot g = 1$

The power series T does not have an inverse

Necessary condition to have an inverse $\mathrm{ord}(f) = 0$ (i.e. constant term of f is nonzero)

✓ CLAIM: If $\mathrm{ord}(f) = 0$, then f has an inverse.

PROOF : Suppose $a_0 \neq 0$. Finding an inverse of $a_0^{-1} f$ is equiv. to finding an inverse of f. $(1/(1-r) = 1 + r + r^2 + \dots)$

So w.l.o.g assume that $a_0 = 1$.

Let $h = 1 - f = -a_1 T - a_2 T^2 \dots = -\sum_{n\geq 1} a_n T^n$

✳ Since $(1-r)(1 + r + r^2 \dots) = 1$

$\underbrace{(1-h)}_{f}{}^{-1} = \underbrace{1 + h + h^2 + \dots}_{\text{is this a power series?}}$

- Observe that $\boxed{\mathrm{ord}(h) \geq 1}$ and $\mathrm{ord}(h^m) = m \cdot \mathrm{order}(h) \geq m$

Hence the $(n+1)^{st}$ coefficient of $1 + h + h^2 + \dots$ is the same as ——"—— $1 + h + h^2 + \dots + h^n$

Therefore we have a finite sum formula for each coefficient of $1 + h + h^2 + \dots$ verifying that it is really a well-defined (formal) power series.

For $f = \sum_{n\geq 0} a_n T^n$ and h some power series with $\boxed{\mathrm{order}(h) \geq 1}$

we can substitute h in f (composition $f \circ h = f(h(T))$)

$f(h(T)) = \sum_{n\geq 0} a_n h^n = a_0 + a_1 h + a_2 h^2 + \dots$

(Note that the nth coeff of $f(h(T))$ is the same as ——"—— $a_0 + a_1 h + \dots + a_{n-1} h^{n-1}$

- For $f = \sum_{n\geq 0} a_n T^n$ $g = \sum_{n\geq 0} b_n T^n$ if $a_k = b_k$ $\forall k = 0, 1, \dots, N-1$ not all

Then we say that $\boxed{f \text{ is conguent to } g \text{ modulo } T^N}$ and write $f \equiv g \pmod{T^N}$

- NOTE that this is an equivalent relation.

- Given any f and N, $\exists$ a $\boxed{\text{unique}}$ polynomial P of degree $\leq N-1$ s.t $f \equiv P \pmod{T^N}$

$\quad \| $

$a_0 + a_1 T + \dots + a_{N-1} T^{N-1}$

PROPERTIES OF THE COMPOSITION

f_1, f_2 : power series

h : power series with $\boxed{\mathrm{ord}(h) \geq 1}$

- $(f_1 + f_2)(h) = f_1(h) + f_2(h)$
- $(f_1 \cdot f_2)(h) = f_1(h) f_2(h)$
- $(f_1/f_2)(h) = f_1(h)/f_2(h)$ if $\mathrm{ord}(f_2) = 0$
- $f \circ (g \circ h) = (f \circ g) \circ h$

$\mathrm{ord}(0) = -\infty$

PROOF

Let P, Q and R be the unique polynomials conguent to f, g, h (mod T^N) Then

$f \circ (g \circ h) \equiv P \circ (Q \circ R) \equiv (P \circ Q) \circ R$
$\equiv (f \circ g) \circ h$

Since this congurence holds for every N, $f \circ (g \circ h) = (f \circ g) \circ h$

Extending the def of power series to include (formal) expressions of the form

$$a_{-m}T^{-m} + a_{-m+1}T^{-m+1} + \dots + a_{-1}T^{-1} + a_0 + \dots$$ → order $= -m$

We can make sense of $f^{-1} = 1/f$ even when $\mathrm{ord}(f) > 0$

In particular, for any non-zero P.S g, f/g is a P.S of the above kind.

Note that $\mathrm{ord}(1/g) = -\mathrm{ord}(g)$

and $\mathrm{ord}(f/g) = \mathrm{ord}(f) - \mathrm{ord}(g)$

In fact, if $f = a_m T^m + a_{m+1}T^{m+1} + \dots \quad a_m \neq 0$

then $\frac{1}{f} = T^{-m}(\tilde{f})^{-1}$ where $\tilde{f} = f \cdot T^{-m} = a_m + a_{m+1}T + \dots$

2.2 CONVERGENT POWER SERIES

Series of complex numbers: a sequence $z_1, z_2 \dots$

n^{th} partial sum of $\sum_{n=1}^{\infty} z_n = S_n = z_1 + z_2 + \dots + z_n$

We say that $\sum_{n=1}^{\infty} z_n$ is convergent if the sequence $s_1, \dots$ is a convergent one (namely if $\lim_{n\to\infty} S_n = w$, then $\sum_{n=1}^{\infty} z_n$ is convergent and it is equal to w sum of this series

SUM AND PRODUCT

If $A = \sum_{n=1}^{\infty} z_n$ and $B = \sum_{n=1}^{\infty} w_n$, then $A + B = \sum_{n=1}^{\infty}(z_n + w_n)$

and $A \cdot B = \lim_{n\to\infty}(s_n \cdot p_n)$ where s_n and p_n are the nth partial sums of the series $\sum z_n$ and $\sum w_n$

- Suppose c_n is a non-negative real number for every n. If the partial sums of $\sum_{n=1}^{\infty} c_n$ are bounded, then $\sum c_n$ is convergent and moreover $\sum c_n$ = least upper bound of the partial sums.

- n^{th} partial sum of $\sum z_n$: $s_n = z_1 + z_2 + \dots$

  We say that "$\sum z_n$ is absolutely convergent if $\sum |z_n|$ is convergent.

CLAIM: If a series is abs conv, then convergent.

PROOF: Let s_n be the partial sum. Then for $n \geq m$

$$|s_n - s_m| = \left|\sum_{k=m+1}^{n} z_k\right| \leq \sum_{k=m+1}^{n} |z_k|$$

$\leq S - (m^{th}$ partial sum of $\sum |z_k|)$ where S is the sum of $\sum |z_k|$.

By convergence of $\sum_{k=1}^{\infty} |z_k|$ for every $\varepsilon > 0$, there is N s.t. the above expression $< \varepsilon$ for $m > N$.

Therefore (s_n) is Cauchy hence convergent and hence $\sum_{n=1}^{\infty} z_n$ is convergent

- A VERSION OF - The Comparison Test

If $\sum c_n$ is a convergent series of non-neg real numbers and $|z_n| \leq c_n$ for every n, then the series $\sum z_n$ is abs conv.

- **FACTS**

① If $\sum z_n$ is abs conv then so is any "rearrangement series" of it. → with the same sum

② $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} z_{m,n}$ is abs conv, then so is $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}$

- $S \subset \mathbb{C}$ $\quad f: S \to \mathbb{C}$ bounded

  $\boxed{\text{supnorm}}$ of f: $\|f\|_S = \|f\| = \sup_{z \in S} |f(z)|$

- Let $\{f_n\}_{n=1}^{\infty}$ be a seq. of functions on S

  We say that $\{f_n\}$ converges uniformly on S if $\exists$ function f on S s.t. given $\varepsilon > 0$ $\exists N$ st. whenever $n \geq N$, $\|f - f_n\| < \varepsilon$

- We say that $\{f_n\}$ is a Cauchy sequence if $\forall \varepsilon > 0$ $\exists N$ s.t. $n, m \geq N \Longrightarrow \|f_n - f_m\| < \varepsilon$

THEOREM The normed space of functions on S is complete. (also converge uniformly)

PROOF Suppose $\{f_n\}$ is a Cauchy sequence of functions since $\{f_n(z)\}$ is a Cauchy seq of complex numbers for every $z \in S$

We can define a function $f: S \to \mathbb{C}$ s.t. $f(z) = \lim_{n\to\infty} f_n(z)$ $\forall z \in S$

→ Given $\varepsilon > 0$, $\exists N$ s.t. $m, n \geq N \Longrightarrow \|f_n - f_m\| < \varepsilon/2$

In particular $|f_n(z) - f_m(z)| < \varepsilon/2$ $\quad \forall z \in S$

Given $z \in S$, $\exists N_z$ st $|f(z) - f_n(z)| < \varepsilon/2$ whenever $n \geq N_z$

Hence for $m \geq N$ and any $z \in S$, choose $n \geq \max\{N, N_z\}$

$$|f(z) - f_m(z)| \leq \underbrace{|f(z) - f_n(z)|}_{< \varepsilon/3} + \underbrace{|f_n(z) - f_m(z)|}_{< \varepsilon/2}$$

But since this ineq. holds regardless of z, we get $\|f(z) - f_m(z)\| < 5\varepsilon/6 < \varepsilon/6$

proving $\boxed{\text{uniform convergence}}$

$\sum f_n$ is uniformly convergent if the sequence of partial sum sequence $\{\varphi_n\}$ is uniformly convergent

" absolute convergent " absolute convergent

" " " " absolute convergent

THEOREM (COMPARISON TEST)

$\sum c_n$ convergent series with $c_n \in \mathbb{R}_{\geq 0}$

$\sum f_n$: $\boxed{\text{series of functions}}$ with $f : S \to \mathbb{C}$

If $\|f_n\| \leq c_n$ for every n, then $\sum f_n$ converges absolutely and uniformly

PROOF For $m \leq n$, $\|S_n - S_m\| \leq \sum_{k=m+1}^{n} \|f_k\|$

$\leq \sum_{k=m+1}^{n} c_k$

By using the convergence of $\sum c_k$ and the above inequality we get both the uniform and abs convergence of $\sum f_n$ □

THEOREM ✓ The uniform limit of a seq of continous functions is continous.

THEOREM 2.4. Let $\{a_n\}$ be a seq of complex numbers and r be a positive real number such that $\sum |a_n| r^n$ is convergent Then $\sum a_n z^n$ is absolutely and uniformly convergent for $|z| \leq r$.

▶ $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converge absolutely and uniformly for every $z \in \mathbb{C}$

Given $z \in \mathbb{C}$, Let $\boxed{r = |z|}$ The series $\sum_{n=0}^{\infty} \frac{r^n}{n!}$ is conv by the Ratio Test from Calculus.

Because $\lim_{n\to\infty} \frac{r^{n+1}/(n+1)!}{r^n/n!} = \lim_{n\to\infty} \frac{r}{n+1} = 0 < 1$

So we can define a function

▶ $\mathbb{C} \to \mathbb{C}$

$z \longmapsto \sum_{n=0}^{\infty} \frac{z^n}{n!}$ This is well-defined continous function by abs & unif conv of the series above.

$roc = \infty$

Similarly we can define

$S : \mathbb{C} \to \mathbb{C}$ and $C : \mathbb{C} \to \mathbb{C}$ sine, cosine

using the power series.

$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ and $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$

$roc = \infty$ $roc = \infty$

Radius of convergence

THEOREM $\sum a_n z^n$: a power series which does not converge abs for some $z \in \mathbb{C}$.

Then $\exists r \in \mathbb{R}_{\geq 0}$ s.t. this PS conv abs for all $z \in \bar{D}(0,r)$

this PS does not conv abs for all $z \notin \bar{D}(0,r)$

PROOF

Comparison Test

$\{ s \in \mathbb{R}_{\geq 0} \mid \sum |a_n| s^n \text{ is convergent} \}$

is bounded since $\exists z \in \mathbb{C}$ s.t. $\sum a_n z^n$ is not absolutely conv.

Let r be the least upper bound of this set.

$|z| > r \Rightarrow \sum a_n z^n$ is not abs conv → since if it is, it should be in the set.

$|z| < r \Rightarrow \sum a_n z^n$ is abs conv → comparison test ■

DEF

$\{t_n\}$ sequence of complex numbers

t is a pt of accummulation :

$\forall \varepsilon > 0$ $\exists$ infinitely many n such that

$|t - t_n| < \varepsilon$

THEOREM Any bounded real seq. has a pt. of accummulation.

BOLZANO WEIRSTRASS

Limit superior of $\{t_n\}$:

$\boxed{\limsup t_n}$ = Least upper bound of the set of all points of accummulation

PROP Given $\varepsilon > 0$ * only finitely many n s.t. $t_n \geq (\limsup t_n) + \varepsilon$ and * infinitely many n s.t. $t_n \geq (\limsup t_n) - \varepsilon$

CHARACTERIZES LIMSUP t_n meaning that if λ is satisfying these, it needs to be Limsup t_n.)

PROP $\{t_n\}$ $\{s_n\}$ seq of real numbers ≥ 0

$t = \limsup t_n$

$s = \limsup s_n$

Then $\limsup (t_n + s_n) \leq t + s$

$\limsup (t_n \cdot s_n) \leq t \cdot s$ if one of them is not zero.

If $\lim t_n$ exist, then $\lim t_n = t$. ✓

THEOREM $\sum a_n z^n$: a (cx) power series (complex)

r : radius of conv of PS

Root Test

Then $r = \dfrac{1}{\limsup (|a_n|^{1/n})}$

PROOF Let $t = \limsup (|a_n|^{1/n})$ suppose that $t \neq 0, \infty$

Given $\varepsilon > 0$, $\exists$ only finitely many n s.t. $|a_n|^{1/n} \geq t + \varepsilon$ $(|a_n| \geq (t+\varepsilon)^n)$

so $\sum a_n z^n$ conv absolutely if $|z| < \frac{1}{t+\varepsilon}$

Just ignore these since they are finite!

$|a_n|^{1/n} |z| \leq (t+\varepsilon) \dfrac{1}{(t+\varepsilon)} = 1$

by the comparison test using geometric series

Hence $r \geq \dfrac{1}{t+\varepsilon}$ for every $\varepsilon > 0$, therefore $r \geq \dfrac{1}{t}$. (1)

Conversely $\forall \varepsilon > 0$ $\exists$ inf many n s.t. $|a_n|^{1/n} \geq t - \varepsilon$

$|a_n| \geq (t-\varepsilon)^n$

Hence $\sum a_n z^n$ does not conv abs if $|z| = 1/t-\varepsilon$ because

$\lim_{n\to\infty} |a_n||z|^n \neq 0$ (infinitely many)

So $r \leq \dfrac{1}{t-\varepsilon}$ for $\varepsilon > 0$ Therefore $r \leq \dfrac{1}{t}$. From (1), conclude $r = \dfrac{1}{t}$. □

Corollary 2.8 $\sum a_n z^n$: a P.S. w/ $r > 0$.

Then $\forall A > \frac{1}{r}$ $\exists C > 0$ s.t. $|a_n| \leq C \cdot A^n$

RATIO TEST $\{a_n\}$ a sequence of positive real numbers.

If $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = A \geq 0$, then $\lim_{n\to\infty} a_n^{1/n} = A$.

For $n \in \mathbb{Z}_+$

$(1+T)^n = \sum_{k=0}^{n} \binom{n}{k} T^k$

$\dfrac{n!}{k!(n-k)!}$

▶ BINOMIAL SERIES $n \in \mathbb{N}$

$\alpha \in \mathbb{C} \setminus \{0\}$

$\binom{\alpha}{n} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ ✓

$\sum_{n=0}^{\infty} \binom{\alpha}{n} T^n = (T+1)^\alpha$

The radius of convergence of this P.S. $= 1$ unless $\alpha \in \mathbb{Z}_+$

2.3

$f(T) = \sum a_n T^n \qquad g(T) = \sum b_n T^n$

✓ $(f+g)(T) = \sum (a_n + b_n) T^n$

$(f \cdot g)(T) = \sum c_n T^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

$(\alpha . f)(T) = \sum \alpha a_n T^n$

THEOREM If $f(T)$ and $g(T)$ are abs conv on $D(0,r)$, then $(f+g)(T)$, $(f \cdot g)(T)$, $(\alpha . f)(T)$ are also abs conv on $D(0,r)$

Moreover, for the functions $f(z), g(z), (f+g)(z), (f \cdot g)(z), (\alpha . f)(z)$ by these power series, we have

$(f+g)(z) = f(z) + g(z) \qquad (f \cdot g)(z) = f(z) . g(z) \qquad (\alpha f)(z) = \alpha f(z)$ for $z \in D(0,r)$

(*)

PROOF of (*)

Let $s \in (0,r)$. By cor 28 $\exists c > 0$ s.t.

$|a_n| \leq \frac{c}{s^n}$ and $|b_n| \leq \frac{c}{s^n}$ for all n.

Then $|c_n| \leq \sum_{k=0}^{n} |a_k| |b_{n-k}| \leq \sum_{k=0}^{n} \frac{c}{s^k} \frac{c}{s^{n-k}} = \frac{c^2 (n+1)}{s^n}$

$\Rightarrow |c_n|^{1/n} \leq \frac{c^{2/n} (n+1)^{1/n}}{s} \xrightarrow{n \to \infty} \frac{1}{s} \Rightarrow \limsup |c_n|^{1/n} \leq \frac{1}{s}$

Since the above ineq. holds for every $s \in (0,r)$ we have $\limsup |c_n|^{1/n} \leq 1/r$ and therefore by the root test

$\sum c_n z^n$ conv abs on $D(0,r)$

Note that we have proved conv of $\sum \left(\sum_{k=0}^{n} |a_k| |b_{n-k}| \right) |z|^n$ for $|z| < r$

Let $f_N(z) = a_0 + a_1 z + \dots + a_N z^N$ be the partial sum polynomial of $\sum a_n T^n$

$g_N(z) = b_0 + b_1 z + \dots$

$$\underbrace{|(fg)_N(z)}_{c_0 + c_1 z + \dots + c_N z^N} - f_N(z) g_N(z)| \leq \sum_{n=N+1}^{\infty} \left(\sum_{k=0}^{n} |a_k| |b_{n-k}| \right) |z|^n$$

$c_0 + c_1 z + \dots + c_N z^N = \sum_{k=0}^{N} a_k b_{N-k}$

This is the tail of a conv series hence can be made $< \varepsilon$ for choosing N suff. large.

Therefore $|(fg)(z) - f(z) g(z)| = 0$ proving $(fg)(z) = f(z) g(z)$ □

THEOREM a) $f(T) = \sum a_n T^n$: a nonconstant PS w/ positive ROC convergent.

If $f(0) = 0$, then $\exists s > 0$ s.t. $f(z) \neq 0 \ \forall z \in \overline{D}(0,s) \setminus \{0\}$

b) $\left. \begin{array}{l} f(T) = \sum a_n T^n \\ g(T) = \sum b_n T^n \end{array} \right\}$ convergent P.S

If $\{x \in \mathbb{C} \mid f(x) = g(x)\}$ has 0 as a point of accummulation then $a_n = b_n \ \forall n$.

each Power series corresponds to a function but the question is "is it possible for different PS's to have a same function?"

PROOF

$f(z) = a_m z^m + a_{m+1} z^{m+1} + \dots \qquad a_m \neq 0$

$= a_m z^m \underbrace{\left(1 + \frac{a_{m+1}}{a_m} z^M + \dots \right)}_{h(z) = b_1 z + b_2 z^2 + \dots}$

$h(z)$ has pos roc > 0 and 0 const term in particular $h(0) = 0$.

By continuity of the function defined by h on $D(0,r)$ $|h(z)| < 1$ for suff small $|z|$.

So for such $z \neq 0$, $f(z) \neq 0$.

(b) The P.S $(f-g)(T) = \sum (a_n - b_n)\, T^n$

is equal to 0 for all $x \in A$, and A has 0 as accum. point

By part (a) this P.S. has to be the constant 0 series.

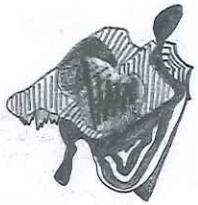
hence $a_n - b_n = 0 \ \forall n$ ■

As a consequence of this theorem,

$\exp : \mathbb{C} \to \mathbb{C}$ are the unique extensions (as power series) of the usual exponential sine and cosine func. on $\mathbb{R}$.

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}$$

$$\exp(zi) = \sum_{n=0}^{\infty} \frac{(zi)^n}{n!} = \underbrace{C(z)}_{\text{even ones}} + i\,\underbrace{S(z)}_{\text{odd ones}}$$



PROPOSITION

If $g(T)$ and $h(T)$ are P.S. w/ 0 constant terms, then

$$\exp(g(T)) \cdot \exp(h(T)) = \exp(g(T) + h(T))$$

$$\sum \frac{(g(T))^n}{n!} \quad \dots$$

In particular $\exp(z+w) = \exp(z) \cdot \exp(w) \ \forall z, w \in \mathbb{C}$

□ $\exp(iz - iz) = \exp(iz) \cdot \exp(-iz)$

$\exp(0) = 1$

$$[C(z) + iS(z)] \cdot [\underbrace{C(-z)}_{C(z)} + \underbrace{iS(-z)}_{-S(z)i}]$$

$$C(z)^2 + S(z)^2 = 1$$

NOTATION / — TERMINOLOGY

$f(T) = \sum a_n T^n$: formal P.S. w/ $a_n \in \mathbb{C}$

$\varphi(T) = \sum c_n T^n$: " " $c_n \in \mathbb{R}_{\geq 0}$

say "f is dominated by φ" if $|a_n| \leq c_n \ \forall n$

write "$f < \varphi$"

<u>Observation</u> if $f < \varphi$ and $g < \psi$ then $f \pm g < \varphi \pm \psi$

and $f \cdot g < \varphi \cdot \psi$

THEOREM f: a convergent power series with non-zero constant term

g: the formal inverse of f, i.e. $f \cdot g = 1 = g \cdot f$

Then g is convergent.

(it has positive radius of convergence)

SKETCH PROOF: "normalize f" and assume that $a_0 = 1$

$$f = \sum_{n=0}^{\infty} a_n T^n = 1 + a_1 T + \dots$$

Let $\boxed{h = 1 - f} = -a_1 T - a_2 T^2 - \dots$

$$g = \frac{1}{1-h} = 1 + h + h^2$$

By <u>Cor 2.8.</u>, $|a_n| \leq A^n \ \forall n$ hence

$$\text{so } h(T) < \sum_{n=1}^{\infty} A^n T^n$$

$$< \frac{AT}{1-AT}$$

hence

$$g(T) < 1 + \left(\frac{AT}{1-AT}\right) + \left(\frac{AT}{1-AT}\right)^2 + \dots = \frac{1}{1 - \frac{AT}{1-AT}} = \frac{1-AT}{1-2AT}$$

Therefore

$$g(T) < (1-AT) \underbrace{\sum_{n=0}^{\infty} (2A)^n T^n}_{\text{conv PS}}$$

$\Rightarrow g$ is conv. □

THEOREM

- $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $h(z) = \sum_{n=1}^{\infty} b_n z^n$: convergent PS's.
- $r, s > 0$ s.t.

  $r.o.c(f) > r$

  $\sum_{n=1}^{\infty} |b_n| s^n \leq r$

The formal PS obtained by composing f and h

$$g(T) = \sum_{n=0}^{\infty} a_n \left(\sum_{k=1}^{\infty} b_k T^k\right)^n$$

Converges w/ $roc > s$

MOREOVER

⊛ for $z \in \bar{D}(0,s)$, $g(z) = f(h(z))$

PROOF

NOTATION

$$g(T) < \sum_{n=0}^{\infty} |a_n| \underbrace{\left(\sum_{k=1}^{\infty} |b_n| T^k\right)^n}_{\text{conv on } \bar{D}(0,s) \text{ and } \leq r}$$

$$< \sum_{n=0}^{\infty} |a_n| r^n$$

$\Rightarrow g$ has roc $> s$, in particular g is convergent.

To prove the 2nd claim: for any $z \in \bar{D}(0,s)$ for $\forall \varepsilon > 0$, for sufficiently large N

$|g(z) - f_N(h(z))|$ where $f_N(z) = \sum_{n=0}^{N-1} a_n z^n$ partial sum

$$\leq \underbrace{\sum_{n=N}^{\infty} |a_n| \left(\sum_{k=1}^{\infty} |b_k| |z|^k\right)^n}_{\text{tail of a convergent series}} < \varepsilon$$

on the other hand, since $f_N \Longrightarrow f$ → partial sum of a function converges uniformly

$\exists N_0$ s.t. $|f(w) - f_N(w)| < \varepsilon$ for any $w \in \bar{D}(0,r)$ and $N > N_0$

So for any $\varepsilon > 0$ and suff. large N $|g(z) - f(h(z))| \leq \overbrace{|g(z) - f_N(h(z))|}^{\varepsilon} + \underbrace{|f_N(h(z)) - f(h(z))|}_{\substack{\| \\ w \in \bar{D}(0,r)}}$

for $z \in \bar{D}(0,s)$

Therefore $|g(z) - f(z)| \overset{\varepsilon}{=} 0$ i.e.

$g(z) = f(h(z))$

$\forall z \in \bar{D}(0,s)$ □

2.3 → 2.4

DEF f is a function defined on some n'hood of z_0

"f is <u>analytic at z_0</u>" if f has a P.S. expansion centered at z_0.

i.e. $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ <u>for every</u> $z \in D(z_0, r)$ for some $r > 0$.

– Since a_n is not changing, r.o.c does not change.

DEF • f is analytic on the open set U if f is analytic at z_0 $\forall z_0 \in U$.

• for a (not necessarily open) subset S of $\mathbb{C}$, f is analytic on S. if f is restriction of function $\bar{f}$ where $\bar{f}$ is analytic on a open set $U \supset S$.

• f, g : analytic $\Rightarrow f \pm g$, $f \cdot g$ analytic

f/g is analytic at any point z_0 s.t. $g(z_0) \neq 0$.

• $\left.\begin{array}{l} f: \overset{\text{open}}{U} \to \mathbb{C} \text{ analytic on } U \\ g: \underset{\text{open}}{V} \to \mathbb{C} \quad '' \quad '' \quad V \end{array}\right\}$ $f \circ g$ is analytic on V.

THEOREM f is <u>analytic at 0</u>

r is roc of $\sum_{n=0}^{\infty} a_n z^n$ $f(z)$

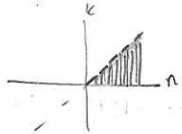
Then f is <u>analytic on $D(0,r)$</u>

PROOF

Let $z_0 \in D(0,r)$, Pick $s > 0$ s.t. $|z_0| + s < r$ $\Big\}$ $f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n (z_0 + (z - z_0))^n$ $\forall z \in D(z_0, s)$

$$= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} z^{n-k} (z - z_0)^k$$

Since $\sum |a_n| \sum \binom{n}{k} |z_0|^{n-k} |z-z_0|^k$ is conv,



$$= \sum_{k=0}^{\infty} \underbrace{\left[\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right]}_{b_k} (z-z_0)^k$$

Corollary

· f is analytic at z_0

r : roc of $\sum_{n=0}^{\infty} a_n (z-z_0)^n = f(z)$

Then f is analytic on $D(z_0, r)$

▶ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} T^n$ r.o.c $= \frac{1}{\limsup |1/n|^{\frac{1}{n}}} = 1$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$ abs conv on $D(0,1)$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n$ abs conv on $D(1,1)$ $\log z$

exp in complex is not injective.

exp in $\mathbb{R}$ is injective and has inverse

2.5

THEOREM f is analytic at z_0 $\left(f(z) = \sum_{n=0}^{\infty} a_n z^n \right)$ roc $= r > 0$

(1) The series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ ($z - z_0$) has roc $= r$

(2) f is holomorphic on $D(z_0, r)$ and

$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ for $z \in D(z_0, r)$

PROOF (1) Since $\limsup |a_n|^{1/n} = 1/r$

$$\Longrightarrow \limsup |n a_n|^{1/n} = \underbrace{\limsup |n|^{1/n}}_{1} \cdot \underbrace{\limsup |a_n|^{1/n}}_{1/r}$$

$$= 1/r$$

(2) Let $z \in D(0,r)$ Pick $\delta > 0$ s.t. $|z| + \delta < r$

$\frac{1}{w} \left[f(z+w) - f(z) \right]$ For $|w| < \delta$

$$= \frac{1}{w} \sum_{n=0}^{\infty} a_n \left((z+w)^n - z^n \right) = \frac{1}{w} \sum_{n=0}^{\infty} a_n \sum_{k=1}^{n} \binom{n}{k} z^{n-k} \cdot w^k$$

$$= \underbrace{\sum_{n=0}^{\infty} a_n \cdot n \cdot z^{n-1}}_{\text{first term}} + \underbrace{\sum_{n=0}^{\infty} \sum_{k=2}^{n} a_n \binom{n}{k} z^{n-k} w^{k-1}}_{\xrightarrow{w \to 0} 0}$$

since its absolute value $\leq |w| \sum_{n=0}^{\infty} \sum_{k=2}^{n} |a_n| \cdot \binom{n}{k} |z|^{n-k} |w|^{k-2}$

$$\leq |w| \underbrace{\sum_{n=0}^{\infty} \sum_{k=2}^{n} |a_n| \binom{n}{k} |z|^{n-k} \delta^{k-2}}_{\text{constant not depending on } w.}$$

So when $w \longrightarrow 0$ it goes to $\longrightarrow 0$

$$\Longrightarrow f'(z) = \lim_{w \to 0} \frac{f(z+w) - f(z)}{w} = \sum_{n=0}^{\infty} a_n n \cdot z^{n-1}$$

□

Remark

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \dots (n-k+1) a_n (z-z_0)^{n-k}$$

$$f^{(k)}(z_0) = k(k-1) \dots (k-k+1) \cdot a_k = k! \, a_k$$

$$\boxed{\Rightarrow a_k = \frac{f^{(k)}(z_0)}{k!}}$$ Taylor's formula

Under the same conditions

$$F(z) = \underbrace{\sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-z_0)^{n+1}}_{roc = r}$$ is a primitive of f.

i.e. $F'(z) = f(z)$ on $D(z_0, r)$

▶ e.g. on $D(1,1)$ $\sum_{n=0}^{\infty} (-1)^n (z-1)^n$ is the derivative of $\log z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (z-1)^n}{n}$

$$\log(z+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}$$

$$1/z = \frac{1}{1-(1-z)}$$

2.6

THEOREM 6.1 $f(T) = a_1 T + a_2 T^2 + \dots \quad a_1 \neq 0$

(a) f has a unique inverse as a formal power series,
i.e $\exists!\, g(T) = \sum b_n T^n$ s.t. $f(g(T)) = T$
(which also implies $g(f(T)) = T$)

(b) If $f(T)$ is conv, then so is $g(T)$

(c) If f is analytic function on an open set $U \ni z_0$ and $f'(z_0) \neq 0$, then f is a <u>local analytic isomorphism at z_0</u>.

THEOREM 6.2 U : open subset of $\mathbb{C}$
f : analytic on U

If f is not constant on any open subset of U then f is an $\underbrace{\text{open mapping}}_{f:\ \text{open} \to \text{open}}$

THEOREM 6.3 f is analytic at z_0 and not constant around z_0.
$\left(f(z) - f(z_0) = a_m (z - z_0)^m + \dots \quad \begin{matrix} m \geq 1 \\ a_m \neq 0 \end{matrix} \right)$

Then $\exists$ local analytic isomorphism φ at 0 s.t.
$f(z) - f(z_0) = \left((\varphi(z - z_0)) \right)$

THEOREM 6.4 U : open set $\subset \mathbb{C}$
f : analytic on U and injective
V : $f(U)$

Then $f : U \to V$ is an analytic isomorphism and $f'(z) \neq 0 \quad \forall z \in U$

2.7 MAXIMUM MODULUS PRINCIPLE

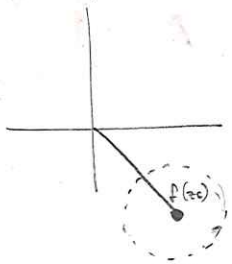
THEOREM U : open set in $\mathbb{C}$
f : analytic on U

$|f(z_0)| \geq |f(z)| \quad \forall z \in U$

If $|f|$ has a max on U at $z_0 \in U$, then
f is locally constant at z_0
[i.e. $\exists r > 0$ s.t. f is constant on $D(z_0, r)$]

<u>PROOF</u>

Assume that f is <u>not</u> locally constant at z_0.
So the P.S. expansion of f at z_0
$f(z) = \underset{\substack{\| \\ f(z_0)}}{a_0} + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$



is not constant. By Theorem 2.6.2,
f is an open mapping on some $D = D(z_0, \delta)$
so $f(D)$ contains an open disc $D(\underset{\substack{\| \\ f(z_0)}}{a_0}, s)$ $s > 0$

This contradicts with the maximality of $|f(z_0)|$

There are definitely bigger value for absolute value.

THEOREM 2.7.3 Fundamental Theorem of Algebra
Every non-constant complex polynomial has a zero

<u>PROOF</u> Let f be a non-constant polynomial.
$f(z) = a_0 + a_1 z + \dots + a_d z^d \quad a_d \neq 0 \quad d \geq 1$
Assume that f has no zero, i.e. $f(z) \neq 0 \quad \forall z \in \mathbb{C}$
Then $1/f$ is a well defined function which is analytic on $\mathbb{C}$
Observe that $\lim_{z \to \infty} \left| \frac{1}{f(z)} \right| = 0$ so $\left| \frac{1}{f(0)} \right| > 0 \quad \exists R > 0$ s.t.
whenever $|z| \geq R$, $\left| \frac{1}{f(z)} \right| < \left| \frac{1}{f(0)} \right|$

• • •

II

1. HOLOMORPHIC FUNCTIONS ON CONNECTED SETS

$\gamma : [a,b] \xrightarrow{C^1} U$ open subset of $\mathbb{C}$

$u, v : [a,b] \xrightarrow{C^1} \mathbb{R}$

$\gamma(t) = \underbrace{u(t)}_{\text{real}} + i\underbrace{v(t)}_{\text{imaginary}}$

γ has a continous derivative

$\gamma'(t) = u'(t) + i v'(t)$

- Given curves $\gamma, \mu : [a,b] \to U$

$$(\gamma + \mu)' = \gamma' + \mu'$$

$$(\gamma \cdot \mu)' = \gamma'\mu + \gamma\mu'$$

$$(\gamma/\mu)' = \frac{\gamma'\mu - \gamma\mu'}{\mu^2} \quad \text{where } \mu(t) \neq 0 \ \forall t \in [a,b]$$

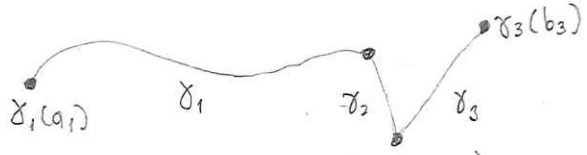
$[a,b] \xrightarrow{\gamma} U \xrightarrow{f} \mathbb{C}$ holomorphic

$$(f \circ \gamma)'(t) = f'(\gamma(t)) \cdot \gamma'(t)$$

<u>Path</u>: $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ path from $\gamma_1(a_1)$ to $\gamma_k(b_k)$

for each i, $\gamma_i : [a_i, b_i] \to U$ is a curve

- $\gamma_i(b_i) = \gamma_{i+1}(a_{i+1}) \quad \forall i = 1, \dots, k-1$



(path / pathwise)

- An open set U is <u>connected</u>:

$\forall z, w \in U$, $\exists$ path in U from z to w.

✓ ✓ ✗

THEOREM 1.1. U is open connected set in $\mathbb{C}$, f holomorphic function $f : U \to \mathbb{C}$

If $f' = 0$ on U, then f is constant on U.

PROOF Let $\alpha, \beta \in U$, and suppose that $\exists$ curve γ from α to β, $[a,b] \to U$ $\forall t \in [a,b]$,

$$(f \circ \gamma)'(t) = \underbrace{f'(\gamma(t))}_{\in U} \cdot \gamma'(t) = 0$$

$\Rightarrow f \circ \gamma : [a,b] \to U$ is constant

so $\underset{f(a)}{f \circ \gamma(a)} = \underset{f(b)}{f \circ \gamma(b)}$

easily generalize to the case where $\exists$ path in U from α to β.

DEF. $f : U \to \mathbb{C}$ "A primitive of f" is a holomorphic funct. $g : U \to \mathbb{C}$ s.t. $g'(z) = f(z)$ $\forall z \in U$.

THEOREM 1.2. U open connected set in $\mathbb{C}$.

① If f is analytic and non-constant on U, then $Z_f = \{z \in U : f(z) = 0\}$ is discrete. (every point z in the set is isolated.)

$= \forall z \in Z_f \quad \exists r > 0$

$D(z,r) \cap Z_f = \{z\}$

② $f, g : U \to \mathbb{C}$ analytic

S: a non discrete subset of U

If $f(z) = g(z)$ $\forall z \in S$, then $f = g$ on U

PROOF

(1) $\Rightarrow$ (2) apply (1) to $f-g$ in (2)

To prove (1) assume that Z_f is not discrete.

Let $S = \{z \in Z_f \mid z \text{ is not isolated in } Z_f\} \neq \emptyset$

For any $z \in S$, by Thm II.3.2, $\exists r > 0$ s.t. $f = 0$ on $D(z,r)$. Since this implies $D(z,r) \subset S$, S is open. On the other hand, if $w \in \bar{S}$, then $\exists$ seq. $(w_n) \subset S$ s.t. $\lim w_n = w$.

So $\lim \underbrace{f(w_n)}_{0} = f(w)$ by continuity of f.

So $w \in Z_f$. Moreover, since any $D(w, \varepsilon)$ contains some points $w_n \in S \subset Z_f$,

w is a non-isolated point of Z_f $\Rightarrow w \in S$ $\Rightarrow S = \bar{S}$ and hence S close

limit point and in S.

Since U is connected, $S \neq \emptyset$ and S open & S closed.

$\Longrightarrow S = U$. so $f = 0$ on U controdicting "f is not constant on U" □

DEF Topological connectedness:

S is connected $\Leftrightarrow$ S and $\emptyset$ are the only clopen subsets of S.

Topological Connectedness = Connectedness for two open subsets of $\mathbb{R}^n$.

THEOREM 1.3 – GLOBAL MAXIMUM MODULUS – – PRINCIPLE –

U open connected, $f : U \to \mathbb{C}$ analytic

If $|f|$ has a max on U, then f is constant on U.

PROOF

If $z_0 \in U$ satisfies $|f(z_0)| \geq |f(z)|$ $\forall z \in U$, then by the local m.m.P f is locally constant z_0. Hence z_0 is a non-isolated point of Z_f and Z_f is not discrete. By the previous theorem, f is constant.

2 INTEGRALS OVER PATHS

$f: U \xrightarrow{\text{cont}} \mathbb{C}$
open

$\gamma: [a,b] \xrightarrow{C^1} U$

$$\int_\gamma f = \int_a^b \underbrace{f(\gamma(t))\gamma'(t)}_{F:[a,b]\to\mathbb{C}} dt$$

$\int_\gamma f = \int_\gamma f(z)dz$

$F = u + iv$

$u, v: [a,b] \to \mathbb{R}$

$f \circ \gamma: [a,b] \to \mathbb{C}$

$\gamma': [a,b] \to U \subset \mathbb{C}$

- For cont $F: [a,b] \to \mathbb{C}$ primitive of F

$t \longmapsto \int_a^t F(s)ds$ is a diffble function whose derivative is F.

▶ $\int_{|z|=1} 1/z \, dz = \int_\gamma \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot i \cdot e^{it} dt = 2\pi i$

oriented counter clockwise

$\gamma: [0, 2\pi] \to \mathbb{C}$

$t \to e^{it}$

$[a,b] \xrightarrow{\gamma} U \xrightarrow{f} \mathbb{C}$

$\uparrow \psi: C^1$

$[c,d]$

$\psi(c) = a$

$\psi(d) = b$

$$\int_\gamma f = \int_{\gamma\circ\psi} f = \int_c^d f(\gamma\circ\psi(t))\underbrace{(\gamma\circ\psi)'(t)}_{\gamma'(\psi(t))\psi'(t)} dt$$

$= \int_a^b f(\gamma(s))\gamma'(s)ds$ $\quad s' = \psi'(t)dt$, $s = \psi(t)$

Path $\Gamma = \{\gamma_1, \dots, \gamma_k\}$

$\int_\Gamma f = \sum_{i=1}^k \int_{\gamma_i} f$

THEOREM 2.1 $f: U \to \mathbb{C}$ cont

$g: U \to \mathbb{C}$ is a primitive of f

γ: Path in U from α to β

Then $\int_\gamma f = g(\beta) - g(\alpha)$

PROOF

Special case $\gamma: [a,b] \to U$ curve

$$\int_\gamma f = \int_a^b \underbrace{g'(\gamma(t))}_{(g\circ\gamma)'(t)}\gamma'(t)dt$$

$\overset{\text{original FTC}}{=} g\circ\gamma(b) - g\circ\gamma(a) = g(\beta) - g(\alpha)$

General case $\gamma = \{\gamma_1, \dots, \gamma_k\}$ is a path from α to β.

$$\int_\gamma f = \sum_{i=1}^k \int_{\gamma_i} f \overset{\text{special case}}{=} \sum_{i=1}^k g(\gamma_i(b_i)) - g(\gamma_i(a_i))$$

$$= -g(\gamma_i(a_i)) + g(\gamma_k(b_k))$$

$$= -g(\alpha) + g(\beta)$$

- $\int_M dw = \int_{\partial M} w$ (Stokes Theorem)

Corollary U open set $f: U \to \mathbb{C}$ cont

f has a primitive on U

γ: $\underbrace{\text{closed}}_{\text{loop}}$ path in U

Then $\int_\gamma f = 0$

▶ $\int_{CR} 1/z = 2\pi i \implies 1/z$ has no primitive

$|z| = R$ with counter clockwise orientation

▶ $n \neq -1$ $\int_{C_R} z^n dz = 0$ $\quad \frac{z^{n+1}}{n+1}$ is a primitive of z^n.

THEOREM 2.2 U: Connected open set

$f: U \to \mathbb{C}$ continous

If $\int_\gamma f = 0$ $\forall$ closed path in U, then f has a primitive on U.

PROOF Pick $z_0 \in U$.

Let $g: U \to \mathbb{C}$, $z \longmapsto \int_{\substack{\text{a path} \\ \text{from } z_0 \\ \text{to } z}} f$

g is well defined since $\int_\gamma f = \int_\tau f$ for any two paths γ and τ in U from z_0 to z.

$\left(0 = \int_{\gamma \cup \tau^{-1}} f = \int_\gamma f - \int_\tau f\right)$

for $z \in U$ $g'(z) = \lim_{w\to 0} \frac{g(z+w) - g(z)}{w} = \lim_{w\to 0} \frac{1}{w}\left[\int_{z_0}^{z+w} f - \int_{z_0}^{z} f\right]$

$= \lim_{w\to 0} \frac{1}{w}\left[\int_{z_0}^{z+w} f + \int_z^{z_0} f\right] = \lim_{w\to 0} \frac{1}{w}\int_z^{z+w} f$

$= \lim_{w\to 0} \frac{1}{w}\int_\gamma f = \lim_{w\to 0} \frac{1}{w}\left(w\int_0^1 f(z+tw)dt\right)$

$= \left(\gamma: [0,1] \to U, \; t \longmapsto z + tw \quad (\text{for } |w| \text{ suff small})\right)$

$= \lim_{w\to 0} \int_0^1 f(z) + \varphi(t)dt$

where $\varphi: [0,1] \to \mathbb{C}$, $t \longmapsto f(z+tw) - f(z)$

$= f(z) + \lim_{w\to 0} \int_0^1 \varphi(t)dt$

$\left(\lim_{t\to 0} \varphi(t) = 0\right.$. moreover $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. where $|w| < \delta$, $\left.|\varphi(t)| < \varepsilon \text{ for every } t\right)$

But since $\left|\int_0^1 \varphi(t)dt\right| \leq \sup_{t\in[0,1]} |\varphi(t)|dt$

The above limit is equal to 0,

$g'(z) = f(z)$ $\square$

Previous Lecture

U: open set in $\mathbb{C}$

$f: U \xrightarrow[\text{cont}]{} \mathbb{C}$

$\gamma: [a,b] \xrightarrow[C^1]{} U$

or general case

path $\gamma = [\gamma_1, \dots, \gamma_k]$

$$\int_\gamma f = \sum_{i=1}^{k} \int_{\gamma_i} f$$

$$\int_\gamma f = \int_a^b f(\gamma(t)).\gamma'(t)\,dt$$

$$= \int_\gamma f(z)\,dz$$

- $\int_\gamma f = 0$ if γ is a closed path and f has a primitive

THEOREM 2.1 AND 2.2

THEOREM 2.3 U: open set in $\mathbb{C}$
$f: U \longrightarrow \mathbb{C}$ cont
γ: path in U

Then $\left|\int_\gamma f\right| \leq \underbrace{\|f\|_\gamma}_{\sup_{t\in[a,b]} |f(\gamma(t))|} \cdot L(\gamma)$ length of γ: $\int_a^b |\gamma'(t)|\,dt$

PROOF

Special case $\gamma: [a,b] \xrightarrow[\text{cont}]{} U$
(γ is curve)

$$\left|\int_\gamma f\right| \leq \int_a^b \underbrace{|f(\gamma(t))|}_{\leq \|f\|_\gamma} |\gamma'(t)|\,dt \leq \|f\|_\gamma \underbrace{\int_a^b |\gamma'(t)|\,dt}_{L(\gamma)}$$

The general case ($\gamma = (\gamma_1, \dots, \gamma_k)$) is obtained by using triangle ineq.

$$\left|\int_\gamma f\right| \leq \sum_{i=1}^{k} \left|\int_{\gamma_i} f\right| \leq \sum \|f\|_{\gamma_i} \cdot L(\gamma_i) = L(\gamma)$$

THEOREM 2.4. (a) If (f_n) is a unif conv sequence of cont functions on U, then

$$\int_\gamma \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\gamma f_n$$

(b) If $\sum_n g_n$ is a unif. conv series of cont functions on U, then

$$\int_\gamma \sum_n g_n = \sum_n \int_\gamma g_n$$

PROOF

(a) let $f = \lim_{n\to\infty} f_n$

$$\left|\int_\gamma f - \int_\gamma f_n\right| = \left|\int_\gamma (f - f_n)\right| \leq \|f - f_n\|_\gamma L(\gamma)$$

Since $\lim_{n\to\infty} f_n = f$ uniformly, $\forall \varepsilon > 0$ $\exists N_0 \in \mathbb{N}$ s.t. whatever $n \geq N_0$

$\|f - f_n\| < \varepsilon / L(\gamma)$ so whenever $n \geq N_0$ we have $\left|\int_\gamma f - \int_\gamma f_n\right| < \varepsilon$

Proving that $\boxed{\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f} = \int_\gamma \lim_{n\to\infty} f_n$

(b) This is conseq of part (a) since, for s_n = the nth partial sum of the series

$$\int_\gamma \sum_n g_n = \int_\gamma \lim_{n\to\infty} s_n \overset{(a)}{=} \lim_{n\to\infty} \int_\gamma s_n = \lim_{n\to\infty} \int_\gamma (g_1 + \dots + g_n)$$

$$= \lim_{n\to\infty} \left(\int_\gamma g_1 + \int_\gamma g_2 + \dots + \int_\gamma g_n\right) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \int_\gamma g_k\right)$$

$$= \sum_{n=1}^{\infty} \int g_n$$

$\square$

▶ For $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$\int_{C_R} f(z)\,dz = 0, \quad \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z}\,dz = a_0$$

$\frac{1}{z}$ has no primitive you have to exclude many things including 0 to make it primitive.

$$\frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z^2}\,dz = a_1$$

$$\int_{C_R} z^n\,dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

III 3 Constructing local primitive for a holomorphic function

U: connected open set in $\mathbb{C}$

$f: U \longrightarrow \mathbb{C}$ holomorphic

$z_0 \in U$

?? $g(z) = \int_{\gamma_{z_0 \to z}} f$

some path in U from $z_0 \to z$

Is this well-defined, i.e. does this integral depend on the choice of $\gamma_{z_0 \to z}$

$z \in D(z_0, r)$

rectangle: R :

boundary of R: ∂R : counter clockwise

$\hat{f}$: extension of f on $R = \hat{f}$

$\hat{f}$ is holomorphic on $U \supset R$

THEOREM 3.1 (Goursat)

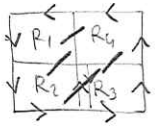
R: rectangle

f: holomorphic on R

$$\int_{\partial R} f = 0$$

PROOF

∂R: midpoints



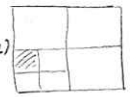
$$\left|\int_{\partial R} f\right| \leq \sum_{i=1}^{k} \left|\int_{\partial R_i} f\right| \leq 4 \left|\int_{\partial R^{(1)}} f\right|$$

$$\left(\int_{\gamma^-} f = -\int_{\gamma} f \right)$$

Let $R^{(1)}$ be the rectangle among $R_1, \dots, R_4$

s.t. $\left|\int_{\partial R} f\right| \geq \left|\int_{\partial R^{(i)}} f\right| \quad \forall i = 1, 2, 3 \dots$

Define $R^{(2)}, R^{(3)} \dots$ along similar lines



so for any n, we have

$$\left|\int_{\partial R} f\right| \leq 4^n \left|\int_{\partial R^{(n)}} f\right| \leq 4^n \cdot \|f\|_{\partial R^{(n)}} \cdot \overbrace{L(\partial R^{(n)})}^{L_n}$$

$$L_n = L_{n-1}/2 = L_1/2^{n-1} = L_0/2^n \quad \text{where } L_0 = L(\partial R)$$

Note that $\bigcap_{n=1}^{\infty} R^{(n)} = \{z_0\}$ is a single point

since $R^{(1)} \supset \dots \supset R^{(n+1)}$ and each $R^{(k)}$ is compact close and bounded.

since f is holomorphic at $z_0 \in R$,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\varphi(z)$$

for any z is some open disc $D(z_0, r) = D$ and some function φ s.t.

$$\lim_{z \to z_0} \varphi(z) = 0$$

For suff large n, $R^{(n)} \subset D$, For such n

$$\left|\int_{\partial R} f(z)\right| = \left|\int \left(\underbrace{f(z_0) + f'(z_0)(z - z_0)}_{0} + (z - z_0)\varphi(z) \right) dz \right|$$

$$= \left|\int_{\partial R^{(n)}} (z - z_0)\varphi(z)\, dz\right| < \varepsilon$$

for sufficiently large n since $\bigcap_n R^{(n)} = \{z_0\}$ and $\lim(\varphi) \to 0$

$$\leq \underbrace{\sup_{z \in R^{(n)}} |z - z_0|}_{\leq \frac{L_0}{2^{n+1}} \text{ diameter}} \cdot \|\varphi(z)\|_{\partial R^{(n)}} \cdot L_0/2^n$$

as $n \to \infty$ this tends to 0. $\square$

THEOREM 3.2 $U = D(z_0, r)$

$f : U \longrightarrow \mathbb{C}$ continous

suppose that $\int_{\partial R} f = 0$ for every rectangle R in U

Then the function $g : U \longrightarrow \mathbb{C}$ by

$$g(z_1) = \int_{\gamma_{z_0 \to z_1}} f$$

where $\gamma_{z_0 \to z_1}$: horizontal + vertical curves

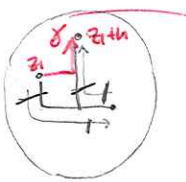
is holomorphic on U and $g'(z) = f(z)$ $\forall z \in U$

<u>PROOF</u> The condition that $\int_{\partial R} f = 0$ for every rectangle R in U,

implies that g is well-defined since <u>the definition is indep of the choice of $\gamma_{z_0 \to z_1}$</u>

For $z_1 \in U$ and $h \in \mathbb{C}$ s.t $z_1 + h \in U$

$$g(z_1+h) - g(z_1) = \int_{\gamma_{z_0 \to z_1+h}} f - \int_{\gamma_{z_0 \to z_1}} f = \int_{\gamma} f$$



since f is cont at z_1, for the function ψ defined by

$$\psi(z) = f(z) - f(z_1)$$

we have $\lim_{z \to z_1} \psi(z) = 0$

constant, has primitive

$$= \int_{\gamma} f(z)dz = \int_{\gamma} \big(f(z_1) + \psi(z)\big)dz = (z+h-z)\, f(z_1) + \int_{\gamma} \psi(z)dz$$

Hence $$\underbrace{\frac{g(z_1+h) - g(z_1)}{h}}_{g'(z_1)} = f(z_1) + \underbrace{\lim \frac{1}{h} \int \psi(z)dz}_{0}$$

since $\lim_{z \to z_1} \psi(z) = 0$

bounded by length which goes zero

$g'(z_1) = f(z_1)$

$\square$

THEOREM 3.3 (corollary of 3.1+3.2)

U : open $\boxed{\text{disc}}$ in $\mathbb{C}$

$f : U \longrightarrow \mathbb{C}$ <u>holomorphic</u>

Then f has a primitive on U.

II.4 ANOTHER DESCRIPTION OF INTEGRAL ALONG THE PATH

GOAL : To extend the def of $\int_{\gamma} f$ to the case where γ is not necessarily C^1 but only continous

Lemma 4.1

U : open set in $\mathbb{C}$

$\gamma : [a,b] \xrightarrow[\text{continous}]{} U$

Then $\exists\, r > 0$ s.t

$$|\gamma(t) - z| \geq r$$

$\forall t \in [a,b]$, $\forall z \in \mathbb{C} \setminus U$

<u>PROOF</u> . Special case of the distance between a compact set and a closed set.

(Read from text)

U open set in $\mathbb{C}$ → given, so we do not need connectedness

$\gamma : [a,b] \xrightarrow[\text{cont.}]{} U$

$f : U \xrightarrow[\text{hol.}]{} \mathbb{C}$

P a partition of $[a,b]$

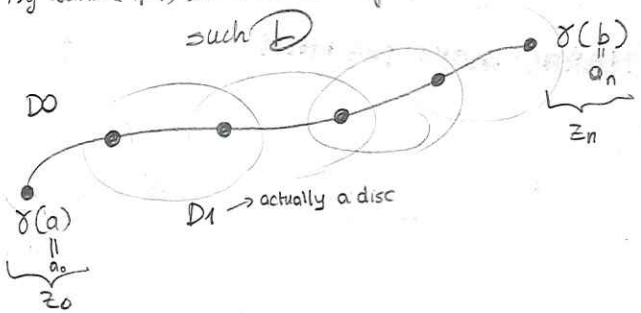
$$\underset{a}{\underset{\parallel}{a_0}} \le a_1 \le a_2 \le \dots \le a_{n-1} \le \underset{b}{\underset{\parallel}{a_n}}$$

$\mathcal{D} = \{D_0, D_1, \dots, D_{n-1}\}$ a set of open discs s.t.

$\gamma([a_i, a_{i+1}]) \subset D_i$

By Thm 3.3, f has a primitive g_i on D_i $\forall i = 0, \dots, n-1$

By lemma 4.1, we can always find such $\mathcal{D}$

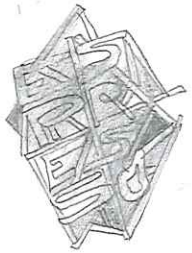


We define

$$\int_\gamma f = \sum_{i=0}^{n-1} \int_{\gamma_i} f \qquad \gamma_i = \gamma|_{[a_i, a_{i+1}]}$$

$$= \sum_{i=0}^{n-1} \left(g_i(z_{i+1}) - g_i(z_i) \right)$$

- Why is this well-defined? i.e. why is this independent of P and $\mathcal{D}$



Lemma 4.2 Given U, γ and f as above

$$\sum_{i=0}^{n-1} [g_i(z_{i+1}) - g_i(z_i)]$$

is indep of partition P, set of open discs $\mathcal{D}$ and choice of primitives $g_0, g_1, \dots, g_{n-1}$ for f on $D_0, D_1, \dots$

PROOF

Let us fix the P. Suppose that $\mathcal{D}'$ is another set of open discs s.t. $\gamma([a_i, a_{i+1}]) \subset D_i'$ and h_i is a primitive of f on D_i' for $i = 0, 1, \dots n$

In $D_i \cap D_i'$ (a non-empty open set) both g_i and h_i are primitives of f so they differ by a constant.

so let c_i be the constant

$g_i(z) - h(z) \quad \forall z \in D_i \cap D_i'$ hence

$g_i(z_{i+1}) - h_i(z_{i+1}) = c_i = g_i(z_i) - h_i(z_i)$

Therefore $\sum_{i=0}^{n-1} [g_i(z_{i+1}) - g_i(z_i)] = \sum_{i=0}^{n-1} [h_i(z_{i+1}) - h_i(z_i)]$

i.e. this sum is indep of choice of discs and primitives

$= h_i(z_{i+1}) - h_i(z_i)$

Refinement P'

Lemma 4.3 U open set in $\mathbb{C}$

γ, η : cont paths w/ the same beginning and end point

$f : U \longrightarrow \mathbb{C}$ holomorphic. Then $\int_\gamma f = \int_\eta f$ whenever γ and η are $\boxed{\text{close together}}$

Stronger than homotopy

$\gamma, \eta : [a,b] \to U$

$\exists P = [a_0, a_1, \dots]$ of $[a,b]$

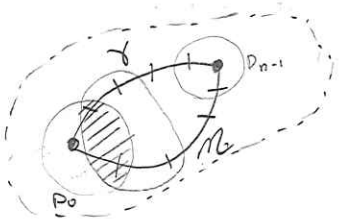
and $\mathcal{D} = \{D_0, \dots, D_{n-1}\}$ s.t.

$\gamma[a_i, a_{i+1}] \subset D_i \supset \eta[a_i, a_{i+1}]$

family of open discs

PROOF Pick a primitive f_i of f on each open disc D_i,

Then $\int_\gamma f = \sum_{i=0}^{n-1} [g_i(\gamma(a_{i+1})) - g_i(\gamma(a_i))]$



$\int_\eta f = \sum_{i=0}^{n-1} [g_i(\eta(a_{i+1})) - g_i(\eta(a_i))]$

On $D_i \cap D_{i+1}$ g_{i+1} and g_i differ by a constant C_i

II.5 THEOREM

$\mathcal{U}$: open set in $\mathbb{C}$

$f : \mathcal{U} \to \mathbb{C}$ holomorphic

γ, η : continous paths in $\mathcal{U}$ (same beginning and end point)

$\int_\gamma f = \int_\eta f$ if γ and η are homotopic (rel end points) will be defined.

THEOREM 5.2 (a special case of 5.1) (Two loops)

an even more special case ; $\gamma(a) = \gamma(b)$
$\quad\quad\quad\quad\quad\quad\quad\quad \| \quad\quad \|$
$\quad\quad\quad\quad\quad\quad\quad\quad \eta(a) = \eta(b)$

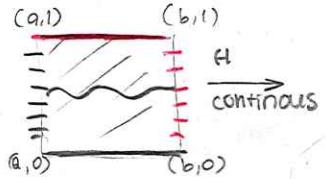
η to be the constant path at $\eta(a) = \eta(b)$

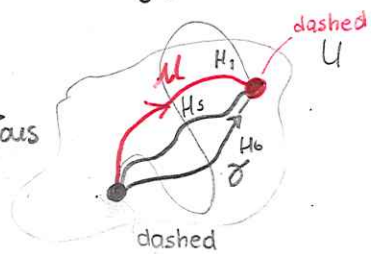
$\gamma \simeq \eta$ (rel ∂) $\Rightarrow \int_\gamma f = \int_\eta f = 0$ since η is constant

We say that "γ is homotopic to η (relative endpoints)" and write "$\gamma \simeq \eta$ (rel ∂)" if there is a continous map (called homotopy)

$H : [a,b] \times [0,1] \longrightarrow \mathcal{U}$ s.t.

- $H(t,0) = \gamma(t)$
- $H(t,1) = \eta(t)$
- $H(a,s) = \gamma(a) = \eta(a)$
- $H(b,s) = \gamma(b) = \eta(b)$





$\forall s \in [0,1]$ $H_s : [a,b] \xrightarrow{\text{cont}} \mathcal{U}$
$\quad\quad\quad\quad\quad\quad\quad\quad t \longmapsto H(t,s)$

SKETCH PROOF

The image $[a,b] \times [0,1]$ under H is compact

Domain is compact

Also H is uniformly continous.

$\exists \varepsilon > 0$ s.t. $|z - w| \geq \varepsilon$ $\forall z \in H([a,b] \times [0,1])$, $w \in \mathbb{C} \setminus \mathcal{U}$

By unif cont. of H, we can find partitions $\mathcal{P} = \{a_0, a_1, \dots\}$ and $\mathcal{P}'$ $\{c_0, c_1, \dots\}$

and a family of open discs $\mathcal{D} = \{D_{ij}\}_{\substack{0 \leq i \leq n-1 \\ 0 \leq j \leq m-1}}$

s.t. $H([a_i, a_{i+1}] \times [c_j, c_{j+1}]) \subset D_{ij}$ $\forall i \in \{0, \dots n-1\}$, $j \in \{0, \dots m-1\}$

Observe that

H_{c_0} is close together with H_{c_1}
H_{c_1} " " " " H_{c_2}
$H_{c_{m-1}}$ " " " H_{c_m}

using the partition $\mathcal{P}$ of $[a,b]$ and the family $\mathcal{D}$ in each case

Therefore by Lemma 4.3

$$\int_{H_{c_0} = H_0 = \gamma} f = \int f = \dots = \int_{H_1 = \eta} f$$

Lemma 5.3 $\mathcal{U}$: open and convex in $\mathbb{C}$ → $\forall z, w \in \mathcal{U}, t \in [0,1]$ We have $(1-t)z + tw \in \mathcal{U}$

$\gamma, \eta : [a,b] \xrightarrow{\text{cont}} \mathcal{U}$

If $\gamma(a) = \eta(a)$, $\gamma(b) = \eta(b)$,

Then $\gamma \simeq \eta$ (rel ∂)

we used it to say H is well-defined.

PROOF

Let $H : [a,b] \times [0,1] \longrightarrow \mathcal{U}$ be defined by

$H(t,s) = (1-s)\gamma(t) + s\eta(t)$

Check that H is a homotopy from γ to η

THEOREM 6.1

$\mathcal{U}$: open and simply connected (~~not necessarily simply connected~~) connected and any closed path in $\mathcal{U}$ is homotopic to a constant path because it is starshaped

$f : \mathcal{U} \xrightarrow{\text{holomorphic}} \mathbb{C}$

$z_0 \in \mathcal{U}$

Then the function $g : \mathcal{U} \longrightarrow \mathbb{C}$

$$z \longmapsto \int_{\gamma_{z_0 \to z}} f$$

is well-defined and is a primitive of f on $\mathcal{U}$, where $\gamma_{z_0 \to z}$ is any path in $\mathcal{U}$ from z_0 to z.

PROOF By Thm 5.1, g is well-defined, i.e. $g(z) = \int_{\gamma_{z_0 \to z}} f$ is independent of choice of $\gamma_{z_0 \to z}$.

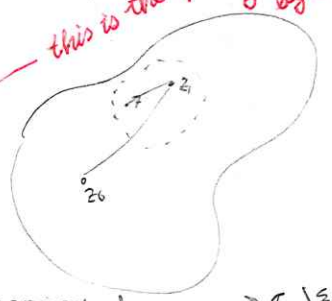
To prove that $g'(z) = f(z)$ on $\mathcal{U}$, let $z \in \mathcal{U}$.

Choose an open disc D in $\mathcal{U}$ containing z and let z_1 be the center of D.

$$g(z) = \int_{\gamma_{z_0 \to z}} f = \underbrace{\int_{\gamma_{z_0 \to z_1}} f}_{\text{indep of } z} + \underbrace{\int_{\gamma_{z_1 \to z}} f}_{c_0}$$

this is the primitive of f in D by Thm 3.2

$$\Rightarrow \frac{\partial g(z)}{\partial z} = 0 + f(z)$$



▶ A primitive of $1/z$ on a simply-connected open set $U \subset \mathbb{C} \setminus \{0\}$ (we need proper subset) → $\mathbb{C} \setminus \{0\}$ is not simply connected

Let $z_0 \in U$ $= r_0 e^{i\theta_0}$ $r_0 > 0$ $w_0 = \ln r_0 + i\theta_0$ $\ln : \mathbb{R}_{>0} \to \mathbb{R}$

$$\log : \mathcal{U} \to \mathbb{C}$$
$$z \to \int_{\gamma_{z_0 \to z}} \frac{1}{\xi} \, d\xi + w_0$$

This is a primitive of $1/z$ on $\mathcal{U}$ by Theorem 6.1

$\log z = \ln|z| + i\theta$

On the other hand, by exercise II.5.6,

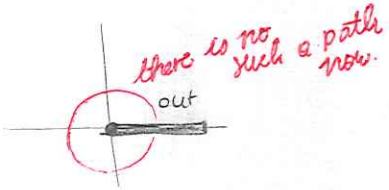
$$F(z) = w_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z - z_0)^n$$

is a primitive of $1/z$ on $D(z_0, |z_0|)$

So on some disc centered at z_0 both F and $\log$ are primitives of $\frac{1}{z}$, hence they differ by a constant. That constant is 0, since $F(z_0) = w_0 = \log z_0$

In particular, $\log$ is analytic on some disc D centered at z_0. Also by Ex. II.3.1 $e^{\log z} = z$ $\forall z \in D$.

Since z_0 is an arbitrary in $\mathcal{U}$, we can work around any other point $z_1 \in \mathcal{U}$ and obtain the analiticity of $\log$ around z_1. This way we get that $\log z$ is analytic on $\mathcal{U}$ hence $e^{\log z}$ is analytic on $\mathcal{U}$. Since $\mathcal{U}$ is connected and $e^{\log z} = z$ on $D \subset \mathcal{U}$, $e^{\log z} = z$ on $\mathcal{U}$ by Theorem II.2.1



THEOREM 7.1 LOCAL CAUCHY FORMULA

$\bar{D}$: a closed disc of positive radius.

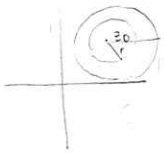
$f : \bar{D} \longrightarrow \mathbb{C}$ holomorphic on $\bar{D}$

$c := \partial \bar{D}$

Then $\forall z_0 \in \underset{\sim}{D}$, $f(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z - z_0} \, dz$

PROOF

Let $z_0 \in D$. For any $C_r = \partial D_r$ where $D_r \subset D$ is the open disc of radius r centered at z_0.



We have $C \simeq$ "C_r" $\Rightarrow \int_C f = \int_{\text{"}C_r\text{"}} f = \int_{C_r} f$

$\int_\alpha f + \int_{\alpha^-} f = 0$

Now let $g : \overset{\mathcal{U}_0}{\overset{\parallel}{\bar{D}\setminus\{z_0\}}} \to \mathbb{C}$

$z \to \dfrac{f(z)-f(z_0)}{z-z_0}$ exist since f is holomorphic

g is holomorphic on $\mathcal{U}_0$.

Moreover, $\int_C g = \int_{C_r} g$. We claim that $\int_{C_r} g = 0$

To prove this claim, note that

$$\left|\int_{C_r} g\right| \leq \|g\|_{C_r} \underbrace{L(C_r)}_{2\pi r} \leq \underbrace{\|g\|_{\bar{D}\setminus\{z_0\}}}_{} 2\pi r$$

this is finite because g can be extended to a continous function $h : \bar{D} \to \mathbb{C}$ $+z_0$

So $\exists B \geq 0$ st.

Compact $\to$ Bounded

$\left|\int_C g\right| = \left|\int_{C_r} g\right| \leq 2\pi B r$

$\Rightarrow \int_C g = 0$

$\forall r > 0$ sufficiently small e.g. $r = \varepsilon / 3\pi B$ for a given $\varepsilon > 0$.

Therefore $0 = \int_C \dfrac{f(z)-f(z_0)}{z-z_0}\, dz = \int_C \dfrac{f(z)}{z-z_0} - \int \dfrac{f(z_0)}{z-z_0}\, dz$

$$\Longrightarrow \int_C \frac{f(z)}{z-z_0}\, dz = \int_C \frac{f(z_0)}{z-z_0}\, dz = f(z_0) \int_C \frac{1}{z-z_0}\, dz$$

$$= f(z_0) \underbrace{\int_C \frac{1}{z-z_0}\, dz}_{2\pi i \text{ by a prev exercise}}$$

THEOREM 7.2 On an open set in $\mathbb{C}$ holomorphic $\Longrightarrow$ analytic

THEOREM 7.3 $z_0 \in \mathbb{C}$, $R > 0$

$f : \bar{D}(z_0, R) \xrightarrow[\text{holomorphic}]{} \mathbb{C}$

Then f has a power series expansion at z_0

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{for } a_n = \frac{1}{2\pi i} \int_{C_R} \frac{f(t)}{(t-z_0)^{n+1}}\, dt$$

Moreover

$$|a_n| \leq \sup_{t \in C_R} |f(t)| / R^n \quad \forall n,$$

consequently the R.O.C of this P.S $\geq R$

PROOF

By the L.C.F. (Theorem 7.1) we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t-z}\, dz \quad \forall z \in D(z_0, R)$$

On the other hand, $\forall t \in C$ circle

$$\frac{1}{t-z} = \frac{1}{(t-z_0)-(z-z_0)} = \frac{1}{(t-z_0)} \cdot \frac{1}{1-\left(\frac{z-z_0}{t-z_0}\right)} = \frac{1}{t-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{t-z_0}\right)^n$$

Therefore $\forall z \in D(z_0, R)$

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-z_0)} \cdot \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(t-z_0)^n}\, dt$$

$$\underset{\text{Comparison}}{\overset{\text{Thm III.2.4}}{=}} \sum_{n=0}^{\infty} (z-z_0)^n \underbrace{\left(\frac{1}{2\pi i} \int_C \frac{f(t)}{(t-z_0)^{n+1}}\, dt \right)}_{a_n}$$

Moreover,

$$|a_n| = \frac{1}{2\pi} \int_C \frac{|f(t)|}{|t-z_0|^{n+1}}\, dt$$

$$\leq \frac{1}{\cancel{2\pi}} \underbrace{L(C)}_{\substack{\text{length} \\ \cancel{2\pi} R}} \frac{\|f\|_C}{R^{n+1}} = \frac{\|f\|_C}{R^n}$$

Hence $\limsup |a_n|^{1/n} \leq \limsup \left(\frac{\|f\|_C^{1/n}}{R} \right) = \frac{1}{R} \overset{\text{Ratio Test}}{\Longrightarrow} \text{ROC} \geq R$

$\square$

▶ $f(z) = \dfrac{e^z}{z^2 - 3z + 2}$

f is analytic on $\mathbb{C} \setminus \{1, 2\}$

ROC of the PS exp of f

(a) at $z_0 = 0$ is 1 → smallest distance to problematic ones

(b) at $z_0 = i$ is $\sqrt{2}$

DEF Entire function : Holomorphic on $\mathbb{C}$

Corollary 7.14

f: entire function

If $\exists C > 0$, $k \in \mathbb{N}$ s.t. $\|f\|_R \leq C \cdot R^k$ for sufficiently large k

then f is a polynomial of degree $\leq k$

PROOF . is an exercise

THEOREM (Liouville's) (special case of Corollary)

Any bounded, entire function is constant.

PROOF

By Theorem 7.3, $\forall R > 0$, $f(z) = \sum a_n z^n$ on $\overline{D}(0,R)$

and $|a_n| \leq \|f\|_R / R^n$

$\leq C / R^n$

where C is $\sup_{z \in \mathbb{C}} |f(z)|$

Therefore $|a_n| = 0$ $\forall n \geq 1$.

(Because a_n does not depend on R)

Hence $a_n = 0$ $\forall n \geq 1$ and $f(z) = a_0$ $\forall z \in \mathbb{C}$.

$\square$

A proof of Fundamental Theo. Algebra based on Liouville's Theorem

Suppose that p is a complex polynomial with no roots.

Then $g : \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto \frac{1}{p(z)}$ is a well-defined, holomorphic function

Observe that $\lim_{|z| \to \infty} |g(z)| = 0$ unless p is a constant since $|p(z)| \xrightarrow[|z| \to \infty]{} \infty$

So $\exists R > 0$ s.t. $|g(z)| < 1$ for $z \in \mathbb{C} \setminus \overline{D}(0,R)$

On the other hand, since g is continous and $\overline{D}(0,R)$ is compact g is bounded on $\overline{D}(0,R)$. Therefore g is bold on $\mathbb{C}$.

By Liouville's Theorem . g is constant and so is p.

MORERA'S THEOREM
7.3

$\mathcal{U}$: open in $\mathbb{C}$

$f: \mathcal{U} \xrightarrow{\text{cont}} \mathbb{C}$

If $\int_{\partial R} f = 0$ for every rectangle R in $\mathcal{U}$, then f is analytic on $\mathcal{U}$.

CHAPTER IV *Winding Numbers and Cauchy's Theorem*

γ: closed path

$\alpha \notin \gamma$

$$W(\gamma,\alpha) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-\alpha}\,dz$$

e.g. $w(c, 0) = 1$ and $w(c, z_0) = 1$

circle centered at 0 counter clockwise

Lemma $W(\gamma,\alpha) \in \mathbb{Z}$ $\forall \gamma, \alpha$ as above

PROOF

(Special case: $\gamma : [a,b] \longrightarrow \mathbb{C}$ is a closed curve

HW modify the proof to get the general case.

Let $\gamma : [a,b] \xrightarrow{C^1} \mathbb{C}$, $\alpha \in \mathbb{C} \setminus \gamma$ actually the image of it

Define $F : [a,b] \to \mathbb{C}$

$$t \longmapsto \int_a^t \frac{\gamma'(s)}{\gamma(s)-\alpha}\,ds$$

Observe that

$F(b) = w(\gamma,\alpha)2\pi i$ | $F(a) = 0$

and

$$F'(t) = \frac{\gamma'(t)}{\gamma(-t)-\alpha} \quad \checkmark$$

Hence $\frac{d}{dt}\left(e^{-F(t)}\cdot(\gamma(t)-\alpha)\right)$

$$= e^{-F(t)}\left(-F'(t)\cdot(\gamma(t)-\alpha) + \gamma'(t)\right) = 0$$

so $e^{-F(t)}(\gamma(t)-\alpha)$ is indep of t.

$\Rightarrow \exists C \neq 0$ s.t. $C.e^{F(t)} = \gamma(t)-\alpha$ $\forall t \in [a,b]$

$$\Longrightarrow C.e^{F(b)} = \gamma(b)-\alpha = \gamma(a)-\alpha = C.e^{F(a)} = 0$$

$\Rightarrow F(b) = 2\pi i k$ for some $k \in \mathbb{Z} \Rightarrow W(\gamma,\alpha) \in \mathbb{Z}$

Lemma The function

$$\Omega : \mathbb{C}\setminus\gamma \longrightarrow \mathbb{Z}$$
$$\alpha \longrightarrow W(\gamma,\alpha)$$

is continous

PROOF

Given a point $\in \mathbb{C}\setminus\gamma$

Let $S > 0$ be the distance of α_0 to γ.

For $\alpha \in D(\alpha_0, S/2)$ we have

$$|\Omega(\alpha) - \Omega(\alpha_0)| = \frac{1}{2\pi}\left|\int_\gamma \frac{1}{z-\alpha} - \frac{1}{z-\alpha_0}\,dz\right|$$

$$\leq \frac{1}{2\pi} L(\gamma)|\alpha-\alpha_0| \underbrace{\left\|\frac{1}{z-\alpha}\right\|_\gamma}_{2/S} \left\|\frac{1}{z-\alpha_0}\right\|_\gamma = C|\alpha-\alpha_0|$$

for some $C > 0$

Lemma 1.3. Let $S \subset \mathbb{C}\setminus\gamma$ be connected. Then $W(\gamma,\alpha)$ is indep of $\alpha \in S$ and it is equal to zero if S is unbounded.

PROOF

The restriction of Ω in Lemma 1.2 to S

$\Omega : S \longrightarrow \mathbb{Z}$ is constant since $\Omega(S)$ is a connected subset of $\mathbb{Z}$, hence a singleton as $\mathbb{Z}$ is discrete.

If S is unbounded, then $\forall C > 0$ we can choose $\alpha \in S$ s.t. $|z-\alpha| > C$ $\forall z \in \gamma$ and hence $|W(\gamma,\alpha)| =$

$$\left|\frac{1}{2\pi i}\int_\gamma \frac{1}{z-\alpha}\,dz\right| \leq \frac{1}{2\pi} L(\gamma)\frac{1}{C}$$

i.e. we can make $|W(\gamma,\alpha)| < \varepsilon$ $\forall \varepsilon > 0$

Hence $W(\gamma,\alpha) = 0$ $\forall \alpha \in S$

IV.2. THE GLOBAL CAUCHY THM

$\mathcal{U}$: open in $\mathbb{C}$

γ: closed path in $\mathcal{U}$.

Q: What is sufficient condition on γ to guarantee that

$$\int_\gamma f = 0 \quad \forall f : \mathcal{U} \xrightarrow{\text{hol.}} \mathbb{C}$$

$\boxed{\text{NECESSARY}}$

$W(\gamma,\alpha) = 0 \quad \forall \alpha \notin \mathcal{U}$

DEF
- γ is homologous to 0 in $\mathcal{U}$, if $W(\gamma,\alpha) = 0$ $\forall \alpha \notin \mathcal{U}$
- γ is homologous to M in $\mathcal{U}$ if $W(\gamma,\alpha) = W(M,\alpha)$ $\forall \alpha \notin \mathcal{U}$.

THEOREM 2.1 $\mathcal{U}$: open in $\mathbb{C}$ COMMUTATIVITY

γ, μ closed paths in $\mathcal{U}$

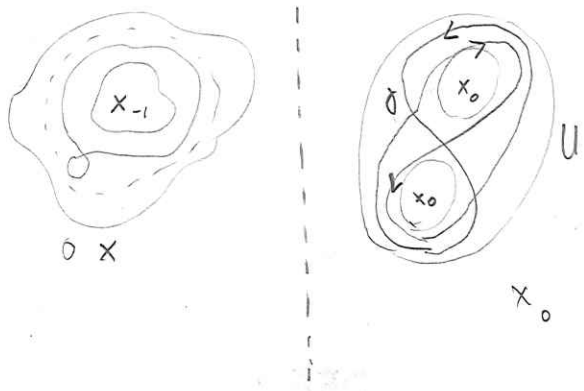
(a) γ is homotopic to μ in $\mathcal{U} \Rightarrow \gamma$ is homologous to μ in $\mathcal{U}$.

(b) γ and μ are close together in $\mathcal{U} \Longrightarrow$ // //

PROOF Let $\alpha \in \mathbb{C} \setminus \mathcal{U}$. Then $\frac{1}{z-\alpha}$ is hol on $\mathcal{U}$.

a) By III.5.2, $\int_\gamma \frac{1}{z-\alpha}\,dz = \int_\mu \frac{1}{z-\alpha}\,dz$ whenever

γ is homotopic to μ in $\mathcal{U}$. $\Longrightarrow W(\gamma,\alpha) = W(\mu,\alpha)$



$xyx^{-1}y^{-1}$

$x+y-x-y$

$=0$

THEOREM 4.2.2 Cauchy's Theorem

$\mathcal{U}$: open in $\mathbb{C}$

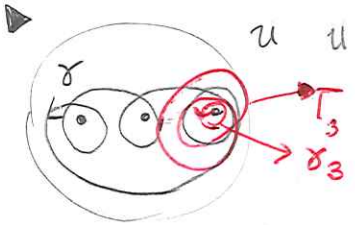
γ: closed "chain" in $\mathcal{U}$ — Formal linear combination of closed paths in $\mathcal{U}$.

If γ is homologous to 0 in $\mathcal{U}$,

then $\int_\gamma f = 0$ $\forall f$ holomorphic in $\mathcal{U}$.

If γ, μ are closed chains in $\mathcal{U}$ and γ, μ are homologous in $\mathcal{U}$,

then $\int_\gamma f = \int_\mu f$



$\mathcal{U}$ $\mathcal{U}$ = disc / $\{z_1, z_2, z_3\}$

$W = (\gamma, z_3) = W(\Gamma_3, z_3) = 2$

γ is homologous to the chain $2\gamma_1 + 2\gamma_2 + 2\gamma_3$ since

$W(\gamma,\alpha) = 2W(\gamma_1,\alpha) + 2W(\gamma_2,\alpha) + 2W(\gamma_3,\alpha)$ $\forall \alpha \notin \mathcal{U}$

~ ~~CHAPTER IV~~ APPLICATIONS OF CAUCHY'S THEOREM

THEOREM 4.2.4 $\mathcal{U}$: open in $\mathbb{C}$

γ: closed chain homologous to 0 in $\mathcal{U}$

$z_1, \dots, z_n$ distinct points in $\mathcal{U}$

$\gamma_1, \gamma_2, \dots$: boundaries of disjoint closed discs in $\mathcal{U}$ containg $z_1, \dots, z_n$ oriented counterclockwise.

$m_i = W(\gamma, z_i)$, $\mathcal{U}^* = \mathcal{U} \setminus \{z_1, \dots, z_n\}$

Then
- γ is homologous to $\sum_{i=1}^n m_i \gamma_i$ in $\mathcal{U}^*$
- $\int_\gamma f = \sum_{i=1}^n m_i \int_{\gamma_i} f$ $\forall f: \mathcal{U}^* \xrightarrow{hol} \mathbb{C}$

PROOF For a point $\alpha \notin \mathcal{U}^*$

$W(\gamma,\alpha) - W\left(\sum_{i=1}^n m_i \gamma_i, \alpha\right)$

$$= \begin{cases} 0 - 0, & \text{if } \alpha \notin \mathcal{U} \\ \underbrace{m_j - m_j}_{0}, & \text{if } \alpha = z_j \text{ for some} \end{cases}$$

Therefore γ is homologous to $\sum_{i=1}^n m_i \gamma_i$ in $\mathcal{U}^*$

As a consequence of Cauchy's Theorem,

$$\int_\gamma f = \int_{\left(\sum_{i=1}^n m_i \gamma_i\right)} f = \sum_{i=1}^n m_i \int_{\gamma_i} f \cdot, \ \forall f : \mathcal{U}^* \xrightarrow{hol} \mathbb{C}$$

CAUCHY'S FORMULA (TO GENERAL CASE)

$\mathcal{U}$: open in $\mathbb{C}$

γ : closed chain in $\mathcal{U}$

$z_0 : \mathcal{U} \setminus \gamma$

$f : \mathcal{U} \xrightarrow{hol} \mathbb{C}$

<u>Then</u> $\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0} dz = W(\gamma, z_0) \cdot f(z_0)$

<u>PROOF</u>

Let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ be the P.S.

expansion on f at z_0 with R.O.C $= R$

Let $r \in (0,R)$ So that $\overline{D}(z_0, r) \subset \mathcal{U}$.

By Theorem 2.4, γ is homologous to

$W(\gamma, z_0) \cdot C_r$ in $\mathcal{U} \setminus \{z_0\}$ where

C_r is $\partial \overline{D}(z_0, r)$ oriented counterclockwise.

Since $\frac{f(z)}{z-z_0}$ is hol. on $\mathcal{U} \setminus \{z_0\}$, by Theorem 2.4

both holomorphic

$$\int_\gamma \frac{f(z)}{z-z_0} dz = W(\gamma, z_0) \cdot \int_{C_r} \underbrace{\frac{f(z)}{z-z_0}}_{\frac{a_0}{z-z_0} + \underbrace{a_1 + a_2(z-z_0) + \dots}_{0}} dz = W(\gamma, z_0) \cdot 2\pi i \cdot \underbrace{a_0}_{f(z_0)}$$

$$\int_{\partial \overline{D}(0,R)} \frac{e^z}{z} dz = 2\pi i$$ because $z_0 = 0$, $f(z) = e^z$, $f(z_0) = 1$, $W(\gamma, 0) = 1$

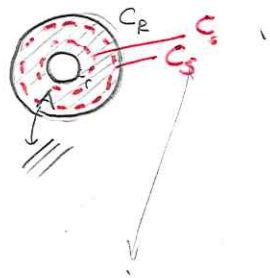
(counterclockwise)

CHAPTER V APPLICATIONS OF CAUCHY'S INTEGRAL FORMULA

2. LAURENT SERIES

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \underbrace{\underbrace{\sum_{n=0}^{\infty} a_n z^n}_{P.S!}}_{f^+(z)} + \underbrace{\sum_{n=-\infty}^{-1} a_n z^n}_{f^-(z)}$$

DEF Annulus: $A = \{ z \in \mathbb{C} \mid r \le |z| \le R$



THEOREM 2.1 $f : A \xrightarrow{hol} \mathbb{C}$

$s, S \in \mathbb{R}$: $r < s < S < R$

<u>Then</u> f has a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

which converges absolutely and uniformly on $\{ z \in \mathbb{C} \mid s \le |z| \le S \}$

Moreover, $a_n = \begin{cases} \frac{1}{2\pi i} \int_{C_R} \frac{f(a)}{a^{n+1}} da & n \ge 0 \\ \frac{1}{2\pi i} \int_{C_r} \frac{f(a)}{a^{n+1}} da & n < 0 \end{cases}$

<u>PROOF</u> $\exists \varepsilon > 0$ s.t. f is holomorphic on the open annulus

$\mathcal{U} = \{ z \in \mathbb{C} \mid r - \varepsilon < |z| < R + \varepsilon \} \supset A$

Observe that $C_R - C_r$ is homologous to 0 on $\mathcal{U}$.

Therefore $\forall z \in A$

$$2\pi i \ f(z) = \int_{C_R} \frac{f(a)}{a-z} da - \int_{C_r} \frac{f(a)}{a-z} da$$ by Cauchy's Formula

To evaluate $\int_{C_R} \frac{f(a)}{a-z}\,da$, proceed as in the proof of Thm III.7.3, i.e for $|z|<R$ (e.g. for $|z|\le S<R$), and $a\in C_R$ we have $\left|\frac{z}{a}\right|<1$, hence

$$\frac{1}{1-\frac{z}{a}} = \sum_{n=0}^{\infty}\frac{z^n}{a^n} \Rightarrow \frac{1}{a-z} = \sum_{n=0}^{\infty}\frac{z^n}{a^{n+1}}$$

$$\frac{1}{1-\frac{z}{a}} = \frac{a}{a-z}$$

Hence $\forall |z|\le S$, we have $\int_{C_R}\frac{f(a)}{a-z}\,da = \int_{C_R} f(a)\cdot\sum_{n=0}^{\infty}\frac{z^n}{a^{n+1}}\,da$

$$= \sum_{n=0}^{\infty}\left(\int_{C_R}\frac{f(a)}{a^{n+1}}\,da\right)z^n$$

- Read the rest

▷ (1) $f(z)=e^{1/z}$ on $A=\{z\in\mathbb{C} \mid r\le|z|\le R\}$

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{(1/z)^n}{n!} = 1+\frac{1}{z}+\frac{1}{2z^2}$$

(2) $f(z)=\frac{1}{z(z-1)} : \mathbb{C}\setminus\{0,1\} \xrightarrow{\text{hol}} \mathbb{C}$

(a) on $\{0<|z|<1\}$: $\frac{1}{z(z-1)} = -\sum_{n=0}^{\infty} z^{n-1}$

$$= -\frac{1}{z} - 1 - z - z^2 \cdots$$

(b) on $\{1<|z|<R\}$: $\frac{1}{z(z-1)} = \frac{1}{z^2}\cdot\frac{1}{1-\frac{1}{z}} = \frac{1}{z^2}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$

$$= \frac{1}{z^2}+\frac{1}{z^3}+\cdots$$

3. ISOLATED SINGULARITIES

f: holomorphic on $U = D(z_0,R)\setminus\{z_0\}$

"f has an isolated singularity at z_0"

$\Longrightarrow$ f has a Laurent series expansion on U

$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ at z_0 by the previous theorem.

3 POSSIBILITIES

(i) "f has a removable singularity at z_0":

$a_n = 0 \quad \forall n<0$

(ii) "f has a pole (of order m) at z_0":

$\exists m>0$ s.t $a_{-m}\neq 0$ and $a_n=0 \ \forall n<-m$

(iii) "f has an essential singularity at z_0":

$\exists\ \infty$-many $m>0$ s.t. $a_{-m}\neq 0$

THEOREM 3.1 If f is bounded on some neighborhood of its isolated singularity z_0, then f can be extended to z_0 as an analytic function, i.e f has a removable sing at z_0.

PROOF. For the Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, we have $a_n = \frac{1}{2\pi i}\int_{C_r}\frac{f(a)}{(a-z_0)^{n+1}}\,da \quad \forall n<0 \quad \forall r\in(0,R)$

So $\forall m>0$ $|a_{-m}| = \frac{1}{2\pi}\left|\int_{C_r}\frac{f(a)}{(a-z_0)^{-m+1}}\,da\right|$

$$\le \frac{1}{2\pi}\,2\pi r\,\frac{\|f\|_{C_r}}{r^{-m+1}} = \frac{\|f\|_{C_r}}{r^{-m}} \le \underbrace{\|f\|\cdot r^m}_{\xrightarrow{r\to 0}\ 0}$$

bounded on f, indep of r near z_0

$\Longrightarrow |a_{-m}|=0 \ \forall m>0$

Hence f has a rem. sing. at z_0 and moreover we can extend f to z_0 by $f(z_0)=a_0$ and $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is a p.s. exp. f analytic at z_0.

*If f has a pole of order m at z_0

$$f(z) = a_{-m}(z-z_0)^{-m} + \dots + a_{-1}(z-z_0)^{-1} + a_0 + \dots$$

and hence $(z-z_0)^m f(z)$ is analytic at z_0 (and non zero at z_0)

THEOREM Casorati–Weierstrass Theorem

$\mathcal{U} = D(0,R) \setminus \{0\}$ or z_0

$f : \mathcal{U} \xrightarrow{hol} \mathbb{C}$ w/an essential singularity at 0 (or z_0)

Then $f(\mathcal{U})$ is dense in $\mathbb{C}$

PROOF

Assume that $f(\mathcal{U})$ is not dense in $\mathbb{C}$

So $\exists w \in \mathbb{C}$ and $\varepsilon > 0$ s.t. $f(\mathcal{U}) \cap D(w,\varepsilon) = \emptyset$

i.e. $|f(z) - w| \geq \varepsilon \quad \forall z \in \mathcal{U}$

So the function $g : \mathcal{U} \to \mathbb{C}$, $z \mapsto \frac{1}{f(z)-w}$ is holomorphic and bounded (by $1/\varepsilon$)

Hence g has an isolated point at z_0, which is removable by Thm 3.1. i.e.

g can be extended to an analytic function on $D(0,R)$

So $\frac{1}{g(z)} = f(z) - w$ is a $\underbrace{\text{meromorphic function}}_{\text{holomorphic except for some isolated points which are poles}}$ on this disc.

which contradicts with f having an essential sing. at 0. $\square$

g holomorphic at 0

$\Rightarrow g(z) = z^k \cdot h(z)$ s.t. $h(0) \neq 0$

h is holomorphic

CHAPTER VI Calculus of Residues

DEF Residue of f at z_0 : $\mathrm{Res}_{z_0}(f) = a_{-1}$ where $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ (Laurent Series of f at z_0)

THEOREM (corollary of Thm VI.2.1)

1.1 If f has isolated singularity at z_0 and $C = \partial \bar{D}(z_0, r)$ oriented counter-clockwise and f is holomorphic on $\bar{D}(z_0,r) \setminus \{z_0\}$ then $\mathrm{Res}_{z_0}(f) = \frac{1}{2\pi i} \int_C f$

Residue Formula

U: open set in $\mathbb{C}$

γ: closed chain homologous to 0 in U

f: holomorphic on $U \setminus \{z_1, \dots, z_k\} \supset \gamma$

Then $\int_\gamma f = 2\pi i \sum_{j=1}^{k} W(\gamma_j, z_j) \cdot \mathrm{Res}_{z_j}(f)$ where γ_j is a small centered at z_j oriented counterclockwise.

(e.g. U is simply connected and γ is any closed chain)

Lemma (a) f has a simple pole at z_0

g is holomorphic at z_0

Then

$$\mathrm{Res}_{z_0}(f \cdot g) = g(z_0) \cdot \mathrm{Res}_{z_0}(f)$$

(b) If $f(z_0) = 0$ and $f'(z_0) \neq 0$, then $\mathrm{Res}_{z_0}(1/f) = \frac{1}{f'(z_0)}$

because otherwise residue $= 0$ ($1/f$ is holomorphic)

and $1/f$ has a $\underbrace{\text{simple}}_{\text{order } 1}$ pole at z_0.

PROOF (a) $f(z) = \sum_{-1}^{\infty} a_n (z-z_0)^n$ and $a_{-1} \neq 0$

$g(z) = b_0 + b_1 (z-z_0)^n$

$$(f \cdot g)(z) = \underbrace{a_{-1} b_0}_{\text{Residue at } z_0} \frac{1}{z-z_0} + (a_{-1} b_1 + a_0 b_0) + \dots$$

(b) $f(z) = a_0 + a_1(z-z_0)$

$a_0 = 0 \quad a_1 \neq 0$

$= a_1 (z-z_0)[1+h]$ where $h = \frac{a_2}{a_1}(z-z_0) + \dots$

$$\Rightarrow \frac{1}{f(z)} = \frac{1}{a_1(z-z_0)} \cdot \frac{1}{1+h} = \frac{1/a_1}{z-z_0} \cdot \underbrace{\sum_{n=0}^{\infty} (-h)^n}_{PS} = \frac{1/a_1}{z-z_0}\left(1-h+h^2-\dots\right)$$

$\Rightarrow$ $1/f$ has a simple pole at z_0 and $\mathrm{Res}_{z_0}(1/f) = 1/a_1$ $\square$

▶ $\mathrm{Residue}_{\pi} \frac{1}{\sin z} = \frac{1}{\cos \pi} = -1$

▶ $f(z) = \frac{z^2}{z^2-1}$ $\quad \mathrm{Res}_1(f) = \frac{1}{2} \cdot 1 = \frac{1}{2}$

$$\frac{z^2}{z+1} \cdot \frac{1}{z-1}$$

value at 1

Res 1

▶ $f(z) = \frac{\sin z}{z^2}$ $\quad \mathrm{Res}_0(f) = ?$

$$= \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots\right) = \frac{1}{z} - \frac{z}{3!}$$

Residue is 1.

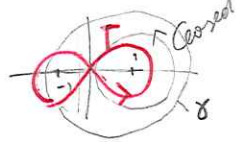
▶ $f(z) = \frac{z^2}{(z+1)(z-1)^2}$ $\quad \mathrm{Res}_1(f) = ?$

$= b_1$

where $\frac{z^2}{z+1} = \sum_{n=0}^{\infty} b_n (z-1)^n$

$$\frac{d}{dz}\left(\frac{z^2}{z+1}\right)\Big|_{z=1} = \frac{3}{4}$$

▶ $f(z) = \frac{z^2}{(z+1)(z-1)^2}$



$$\int_C f = 2\pi i \cdot \frac{3}{4} = \frac{3\pi i}{4}$$

$$\int_\Gamma f = 2\pi i\left(-1/4 + 3/4\right) = \pi i$$

$$\int_\gamma f = 2\pi i\left(\underbrace{\mathrm{Res}_{-1} f}_{= \frac{1}{4}} + \mathrm{Res}_1 f\right) = 2\pi i$$

Lemma 1.4 → at most a pole (not essential)

f: meromorphic at z_0, then

$$\mathrm{Res}_{z_0}(f'/f) = \mathrm{ord}_{z_0} f$$

PROOF

$$f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \dots, \quad a_m \neq 0$$

$$= a_m(z-z_0)^m[1+h] \qquad \mathrm{ord} \geq 1 \to h(z_0) = 0$$

$$f'(z) = m \cdot a_m(z-z_0)^{m-1}[1+h] + a_m(z-z_0)^m \cdot h'$$

$$\Longrightarrow \frac{f'}{f}(z) = \frac{m}{z-z_0} + \underbrace{\frac{h'}{h+1}}_{\text{Residue is } 0}$$

holomorphic at z_0

$$\mathrm{Res}_{z_0}\left(\frac{f'}{f}(z)\right) = \mathrm{Res}\left(m/_{z-z_0}\right) = m = \mathrm{ord}_{z_0} f \qquad \square$$

THEOREM 1.5

$\mathcal{U}$: open in $\mathbb{C}$

f: meromorphic on $\mathcal{U}$ with finite number of zeros and poles at $z_1, z_2, \dots, z_k$

γ: close chain homologous to 0 in $\mathcal{U}$ and $z_1, \dots, z_j \notin \gamma$

$$\text{Then } \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} = \sum_{j=1}^{k}\left(w(\gamma, z_j) \cdot \mathrm{ord}_{z_j} f\right)$$

In particular, when γ has an interior

if $w(\gamma, z) \in \{0,1\}$ $\forall z \in \mathbb{C} \setminus \gamma$

and f is meromorphic in the interior of γ w/no poles or zeros on γ

$$\frac{1}{2\pi i}\int_\gamma f'/f = \#\text{ of zeros of } f \text{ in the interior of } \gamma - \#\text{ poles of } f \text{ in the interior of } \gamma$$

THEOREM Rouché's Theorem

U: open in $\mathbb{C}$

γ: closed path hom to 0 in U with an interior

$f, g : U \xrightarrow{hol} \mathbb{C}$

If $|f(z)-g(z)| < |f(z)| \quad \forall z \in \gamma$, then

$$\#\text{ of zeros of } f \text{ in the interior of } \gamma = \#\text{ of zeros of } g \text{ in the interior of } \gamma$$

▶ How many zeros $P(z) = z^8 - 5z^3 + z - 2$ have inside the unit circle?

$$\underbrace{\underbrace{|C.z^d - (z^8 - 5z^3 + z - 2)|}_{|-z^8 - z + 2| \le 4} < \underbrace{|C.z^d|}_{-5.z^3 \text{ for } |z|=1} = |C| = 5}_{\text{valid}}$$

$-5.z^3$

$$\overset{\text{Rouché's}}{\Longrightarrow} \#\text{zeros of } P \text{ inside the unit circle } |z|=1 = \#\text{ of zeros } (-5z^3) = 3$$

inside the unit circle

PROOF

The inequality in the statement $\Longrightarrow \left|\frac{g(z)}{f(z)} - 1\right| < 1$ for $z \in \gamma$ for $z \in \gamma$ $\left[\begin{matrix} f(z) \neq 0 \\ \text{on } \gamma \end{matrix}\right]$

$F = g/f$ meromorphic on U w/poles at the zeros of f on U

The image of γ under $F \subset D(1,1)$

In particular, $F \circ \gamma$ is a closed path in $D(1,1)$ — inequality is used here

$$0 = W(F\circ\gamma, 0) = \int_{F\circ\gamma} \frac{1}{z}\,dz = \int_a^b \frac{1}{F\circ\gamma(t)} F'(\gamma(t)).\gamma'(t)\,dt$$

by the fact that $F\circ\gamma$ is hom to 0 in $D(1,1)$

assuming γ is $[a,b] \xrightarrow{C^1} U$

$$= \int_\gamma \frac{F'}{F}$$

$$= \int_\gamma g'/g - \int_\gamma f'/f$$

$$\Rightarrow \#\text{of zeros of } g - \#\text{of zeros } f = 0$$

□

VI.2. EVALUATION OF DEFINITE INTEGRALS

f: a real valued function of a single real variable

$$I = \int_a^b f(x)\,dx = ?$$

Suppose that the above function f can be extended to a complex-valued function (still denoted by f) on an open subset U of $\mathbb{C}$ containing $[a,b]$ then $I = \int_\gamma f$, where γ is the line on $\mathbb{R}$ from A to B.

$$\int_{\gamma\cup\Gamma} f = 2\pi i \sum_{j=1}^{k} W(\gamma\cup\Gamma, z_j).\mathrm{Res}_{z_j}(f)$$

Residue formula

$$\| $$

$$\int_\gamma f - \int_\Gamma f$$

▶ $\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx$ Let $f(z) = \frac{1}{1+z^4}$ on $\mathbb{C}$

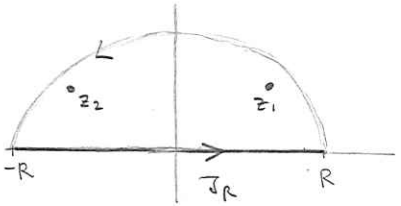
Observe that f is holomorphic on $\mathbb{C} \setminus \{z_1, \dots, z_4\}$

Where $z_j = e^{i(\pi/4 + j\pi/2)}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

$$= \lim_{A\to-\infty} \int_A^0 f(x)\,dx + \lim_{A\to\infty} \int_0^A f(x)\,dx$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4}\,dx$$



$$\int_{J_R} f + \int_{S_R} f = 2\pi i \underbrace{\left(\mathrm{Res}_{z_1}(f) + \mathrm{Res}_{z_2}(f)\right)}_{\text{We will compute it.}}$$

For $R > 1$:

$$\int_{S_R} f = \int f(Re^{it})\, iRe^{it}\, dt \leq \pi R \left\| \frac{1}{1+z^4} \right\|_{S_R} \leq \frac{\pi R}{\frac{15}{16} R^4} \xrightarrow{R \to \infty} 0$$

$S_R = Re^{it} : 0 \leq t \leq \pi$

On $S_R = |1+z^4| \geq R^4 - 1 \geq R^4 - \left(\frac{R}{2}\right)^4 = \frac{15}{16} R^4$ (APPROX)

Therefore $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx = 2\pi i \left(\mathrm{Res}_{z_1}(f) + \mathrm{Res}_{z_2}(f)\right)$

Since z_j are single poles of $1/1+z^4$ by a statement proved in the previous lecture,

$$\mathrm{Res}_{z_j}\left(1/1+z^4\right) = 1/g'(z_j) \quad \text{where } g(z) = 1+z^4, \quad g'(z) = 4z^3$$

THEOREM 2.1.

- f holomorphic everywhere except for a finite number of singularities none on $\mathbb{R}$.
- $M, a > 0$ s.t.

$$|f(z)| \leq \frac{M}{|z|^{1+a}} \quad \text{when } |z| \text{ suff large}$$

THEOREM 2.2

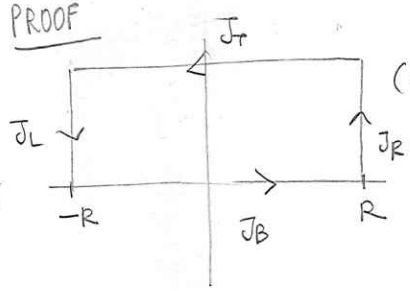
- f meromorphic on $\mathbb{C}$ w/ finite number of poles none on $\mathbb{R}$
- $M \in \mathbb{R}$ s.t.

$$|f(z)| \leq \frac{M}{|z|} \quad \text{when } |z| \text{ is suff large}$$

Then for any constant $c > 0$

$$\int_{-\infty}^{\infty} f(x)\, e^{icx}\, dx = 2\pi i \sum \text{residues of } f(z).e^{icz} \text{ on the upper half plane}$$

PROOF



For sufficiently big R, (all singularities in there)

$$\int_{J_B \cup J_R \cup J_T \cup J_L} f(z)\, e^{icz}\, dz = 2\pi i \sum \text{residues of } f(z)\, e^{icz} \text{ in the upper half plane}$$

- $\displaystyle\left| \int_{J_T} f(z) e^{icz}\, dz \right| = \left| \int_{-R}^{R} f(t+iR)\, e^{ic(t+iR)}\, dt \right|$

$$\leq \frac{M}{R}\, 2R \cdot e^{-cR} \xrightarrow{r \to \infty} 0$$

- $\displaystyle\left| \int_{J_R} f(z).e^{icz}\, dz \right| = \left| \int_0^R e^{-ct}\, dt \right| = \frac{M}{R} \cdot \frac{1}{c}\left(1 - e^{-cR}\right) \xrightarrow{R \to \infty} 0$

- Similarly, $\displaystyle\left| \int_{J_L} f(z)\, e^{icz}\, dz \right| \xrightarrow{r \to \infty} 0$ as well. $\square$

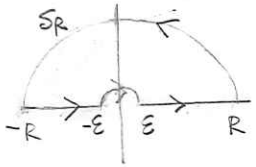
▶ $\int_0^\infty \frac{\sin x}{x}\,dx = ?$ (DIRICHLET'S INTEGRAL)

Consider $\int_{J_R} \frac{\sin z}{z}$ on $\mathbb{C}$ (imaginary part)

or consider $\int_{J_R} \frac{e^{iz}}{z}$ on $\mathbb{C}$ — isolated singularity on $\mathbb{R}$

where J_R is the straight line from $-R$ to R.

$\frac{e^{iz}}{z}$ has singularity at $z=0\in\mathbb{R}$



$\Gamma_{R,\varepsilon}$ (whole path)

$$\int_{\Gamma_{R,\varepsilon}} \frac{e^{iz}}{z}\,dz = 0$$

winding number is 0

- $$\left|\int_{S_R} \frac{e^{iz}}{z}\,dz\right| = \left|\int_0^\pi \frac{e^{i(Re^{it})}}{\cancel{Re^{it}}}\cdot i\cdot \cancel{Re^{it}}\,dt\right| = \left|i\int_0^\pi e^{iR\cos t - R\sin t}\right|$$

$\| \; Re^{it};\ 0\le t\le\pi$

$$\le \int_0^\pi e^{-R\sin t}\,dt = 2\int_0^{\pi/2} e^{-R\sin t}\,dt$$

$$\le 2\int_0^{\pi/2} e^{-R\,2/\pi\, t}\,dt = \frac{2\pi}{2R}\left(1-e^{-R}\right) \xrightarrow{R\to\infty} 0$$

- $$\int_{S_\varepsilon} \frac{e^{iz}}{z}\,dz = -\pi i\,\underbrace{\mathrm{Res}_0\left(\frac{e^{iz}}{z}\right)}_{1}$$

$\| \; -\pi i$

Therefore,

$$\int_{-\infty}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_\varepsilon^\infty \frac{e^{ix}}{x}\,dx = \pi i$$

TRIGONOMETRIC INTEGRALS

$$\int_0^{2\pi} Q(\cos\theta, \sin\theta)\,d\theta$$

where $Q(x,y)$: rational function of two variables

THEOREM $\int_0^{2\pi} Q(\cos\theta,\sin\theta)\,d\theta = 2\pi i \sum$ residues of f inside the unit circle

where $f(z) = \dfrac{Q\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)}{iz}$

$$\left(\begin{array}{l} \displaystyle\int_{z=e^{i\theta}} \frac{f(z)}{iz}\,dz \\ dz = i\cdot e^{i\theta}\,d\theta \\ \frac{dz}{iz} = d\theta \end{array}\right)$$

▶ For $a>1$ $$\int_0^{2\pi} \frac{1}{a+\sin\theta}\,d\theta = \int_C \frac{1}{a+\frac{z-1/z}{2i}}\,\frac{1}{iz}\,dz$$

C: unit circle oriented counter clockwise

$$= \int_C \frac{2}{2iaz+z^2-1}\,dz \overset{\text{Residue Formula}}{=} 2\pi i \sum \text{residues of } f \text{ inside the unit circle}$$

$$= 2\pi i\,\mathrm{Res}_{z_1}(f)$$

$$= 2\pi i\,\frac{1}{i\sqrt{a^2-1}}$$

$$= \frac{2\pi}{\sqrt{a^2-1}}$$

$$\left(\begin{array}{l} q(z) = z^2+2iaz-1 \\ z_\pm = \dfrac{-2ia\pm\sqrt{-4a^2+4}}{2} = -ia\pm i\sqrt{a^2-1} \\ \qquad \left|-ia-i\sqrt{a^2-1}\right| > 1 \\ \qquad \left|-ia+i\sqrt{a^2-1}\right| < 1 \end{array}\right)$$