Linear Algebra

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MATH 320

Field.

$(F, +, \cdot)$

1) Addition is <u>commutative</u> ✓
2) Addition is associative $[(+)+ \; = \;]$ ✓
3) $\exists!$ a zero element $0 \in F$: $x+0=x \quad \forall x \in F$ ✓ (→ uniqueness)
4) $\forall x \in F, \; \exists! \, (-x) \in F \quad x+(-x)=0$ ✓
5) Multiplication $(\cdot)$ is commutative. ✓
6) Multiplication is associative. ✓
7) $\exists!$ a non-zero element 1 : $x.1=x \quad \forall x \in F$
8) $\forall x \neq 0, \; \exists! \, x^{-1}$: $x.x^{-1}=1$
9) Multiplication distributes over addition.

▶

Positive	No	$(\nexists 0)$
Non-negative	No	$(\nexists (-x) \; \forall x)$
All integers	No	$(\nexists (x^{-1}) \; \forall x)$
Rational numbers	Yes	
Complex numbers	Yes	

▶ $\{ x+y\sqrt{2} \; : x, y \in \mathbb{Q} \} \subseteq \mathbb{C}$

Systems of Linear Equations

$$\begin{aligned} A_{11} x_1 + A_{12} x_2 + \dots \quad &= y_1 \\ A_{21} x_1 + A_{22} x_2 + \dots \quad &= y_2 \\ \vdots \end{aligned}$$

$A_{ij} \in F, \; y_1, y_2, \dots, y_m \in F$

We are trying to find $(x_1, \dots, x_n) \in F^n$ satisfying the m eqs simultaneously.

▶ $F = \mathbb{Q}$;

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 0 \\ x_1 + 3x_2 + 4x_3 &= 0 \end{aligned}$$

$$\begin{aligned} -7x_2 - 7x_3 &= 0 \\ 7x_1 + 7x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_2 &= -x_3 \\ x_1 &= -x_3 \end{aligned}$$

All solutions must be of the form $(-c, -c, c)$ for some $c \in \mathbb{Q}$

observe $(-c, -c, c)$ satisfies the eqs

DEF A system of linear equations is called homogeneous if $y_k = 0$ for $k = 1, 2, \dots, m$.

For the general solution (1-1), choose m scalars $c_1, c_2, \dots, c_m$ and multiply the ith row with c_i and add them up.

$\vdots$

This is called a linear combination of the equations in (1-1)

Suppose we have another system of equations.

$$\mathbb{B}$$

such that every equation in $\mathbb{B}$ is a linear combination of the eqs of $\mathbb{A}$.

DEF If we can obtain $\mathbb{A}$ from $\mathbb{B}$ in the same fashion, then we say $\mathbb{A}$ and $\mathbb{B}$ are linearly equivalent.

THEO1 Equivalent systems of linear eqs have exactly the same solutions.

Represent (1-1) as

$$AX = Y$$

$\boxed{\text{Row operations}}$

1. Multiplication E_i^c
2. Replacement $c \in F$, $i \neq j$ E_{ij}^c
3. Interchanging i and j J_{ij}

NOTATION

Let e denote an elemantary row operation.

THEO2 To each elemantary row operation e, there corresponds on elemantary row operation e_1, of the same type as e, such that $e(e_1(A)) = e_1(e(A))$

PROOF :
- $e = E_i^c$

  Let $e_1 = E_i^{c^{-1}}$
- $e = E_{ij}^c$

  Let $e_1 = E_{ij}^{-c}$
- $e = J_{ij}$

  Let $e_1 = J_{ij}$

DEF If A and B are $m \times n$ matrices $/F$, $(A, B \in M_{m \times n}(F))$, we say that B is row equivalent to A if B can be obtained from A by a finite sequence of elemantary row operations.

NOTATION

$$A \leftrightarrow B$$

$$\left.\begin{array}{l} \text{Observe } A \leftrightarrow A \\ A \leftrightarrow B \Rightarrow B \leftrightarrow A \text{ (By Theorem 2)} \\ A \leftrightarrow B \text{ and } B \leftrightarrow C \Rightarrow A \leftrightarrow C \end{array}\right\} \text{equivalence relation}$$

THEO3 If $A \leftrightarrow B$, the homogeneous systems $Ax = 0$ and $Bx = 0$ have exactly the same solutions.

PROOF Suppose

$$A = A_0 \xrightarrow{e} A_1 \xrightarrow{e} \dots \xrightarrow{e} A_k = B$$

It is enough to show that A_j and A_{j+1} have exactly the same solutions for $j = 0, \dots, k-1$.

Consider $A_j X = 0$ and $A_{j+1} X = 0$

Note that we can pass between A_j and A_{j+1} via elementary row operations, which correspond to forming linear combination of rows.

$\Rightarrow A_j X = 0$ and $A_{j+1} X = 0$ are linearly equivalent.

$\Rightarrow$ Use Theorem 1.

▶ $F = \mathbb{Q}$

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \end{bmatrix} =: B$$

Solutions of $B = \{(11/3c, 5/3c, -17/3c, c) : c \in \mathbb{Q}\}$

DEF An $m \times n$ matrix A is called row-reduced if

(a) The first non zero entry in each row of A is 1.

(b) Each column of A which contains the leading non zero entry of some row has all its other entries 0.

THEO4 Every $m \times n$ matrix A/F is row eq. to a row reduced matrix.

Proof: HW

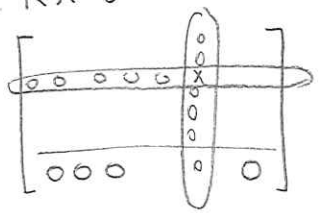
ROW REDUCED ECHELON FORM

- R is row-reduced.
- Every row of R which has all entries zero occurs below every row which has a nonzero entry.
- If rows $1, \dots, r$ are the non-zero rows of R, and if the leading non-zero entry of row i occurs in collusion k_i, then $k_1 < k_2 < \dots < k_i$

THEO5 $A \in M_{m \times n}(F) \Rightarrow A \longleftrightarrow A$ row reduced echelon matrix R.

SET UP: R: row reduced echelon matrix.

System: $RX = 0$



pivot columns + free variables $= \{x_1, \dots, x_n\}$

$$\left.\begin{aligned} x_{k_1} + \sum_{j=1}^{n-r} c_{1j} u_j &= 0 \\ &\vdots \\ x_{k_r} + \sum_{j=1}^{n} c_{rj} u_j &= 0 \end{aligned}\right\}$$

observation: By assigning random values to $u_1, \dots, u_{n-r}$ from F, we get solutions to $RX = 0$

<u>proposition</u>

If $r < n$, then $RX = 0$ has nontrivial solution.

THEOREM 6 $A \in M_{m\times n}(F)$ $m<n$

$\Rightarrow AX=0$ has a non-trivial solution.

Proof: $\exists R \in M_{m\times n}(F)$ R is RRE and $A \leftrightarrow R$

$AX=0$ and $RX=0$ have exactly the same solutions.

$r \leq m < n \Rightarrow RX=0$ has a non-trivial by the previous proposition.

THEOREM 7 If A is an $n\times n$ square matrix, then A is row equivalent to I_n iff $AX=0$ has only the trivial solution.

Proof: $(\Rightarrow)$ Suppose $A \leftarrow I_n$

$AX=0$ and $I_nX=0$ have the same solutions

$$\downarrow$$

$$\left.\begin{array}{l} x_1=0 \\ x_2=0 \\ x_3=0 \\ \vdots \\ \quad =0 \end{array}\right\} X=0$$

$(\Leftarrow)$ Suppose $AX=0$ has only the trivial solution.

Let R be the RRE : $A \leftrightarrow R$

We want to show that $R=I_n$

$RX=0$ has only the trivial solution.

$\Rightarrow r \geq n$ trivial olduğu için tam tersini alıyoruz.

But R is $n\times n \Rightarrow r=n$.

$\Rightarrow R=I_n$

Systems of the form $AX=Y$

Form the augmented matrix $A'=[A|Y]$ then row reduce A inside A'.

▶ $F=\mathbb{Q}$

$$A=\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix} \qquad AX=Y$$

$$\longrightarrow\longrightarrow \begin{bmatrix} 1 & 0 & 3/5 & 1/5(y_1+2y_2) \\ 0 & 1 & -1/5 & 1/5(y_2-2y_1) \\ 0 & 0 & 0 & (y_3-y_2+2y_1) \end{bmatrix}$$

$$\left.\begin{array}{l} y_3-y_2+2y_1=0 \\ x_2-\frac{1}{5}x_3=\frac{1}{5}(y_2-2y_1) \\ x_1+\frac{3}{5}x_3=\frac{1}{5}(y_1+2y_2) \end{array}\right\} \Rightarrow \text{Let } x_3=c \in \mathbb{Q}$$

$x_1=-3/5c+1/5(y_1+2y_2)$

$x_2=1/5c+1/5(y_2-2y_1)$

Matrix Multiplication

$AX=Y$ $(1-1)$

$A\cdot B=[AB_1 \; AB_2 \; \dots \; AB_p]$

$B=[B_1 \dots B_p]$ when $B_j=\begin{bmatrix} B_{1j} \\ \vdots \\ B_{nj} \end{bmatrix}$

THEOREM 8 $A(BC)=(AB)C$

DEF: An $m\times m$ matrix is said to be an elemantary matrix if it can be obtained from I_m by means of a single elementary row operation

$$e(I_m)$$

▶ $\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$

THEO REM9 $(e(A) = EA)$

Let e be an e.r.o and let E be the
↳ elementary row operations

'corresponding' elementary matrix $E = e(I)$

Then $e(A) = EA$

COROLLARY $A, B \in M_{m \times n}(F)$

$B \longleftrightarrow A \iff B = PA$ where P is a finite product of elementary matrices.

Proof $(\Leftarrow)$ $B = PA$. Let $P = E_r \cdots E_2 E_1$

$$A \longleftrightarrow (E_1 A) \longleftrightarrow E_2(E_1 A) \longleftrightarrow E_r \cdots E_2 E_1 A = PA = B$$

$(\Rightarrow)$ $A \xrightarrow[e_1]{} \cdots \xrightarrow[e_r]{} B$

$$B = e_r(\cdots e_1(A)) = \underbrace{E_r \cdots E_1}_{P} A$$

$$A \longleftrightarrow B \Rightarrow A = QB$$

$$A = Q(PA) = (QP)A$$

- P is invertible if P is a product of e.m
- ——————— iff ———————

DEF Let A be an $n \times n$ square matrix $/F$.

$B \in M_{n \times n}(F)$ $BA = I$, then B is called a left inverse of A.

$C \in M_{n \times n}(F)$: $AC = I$ then C is called a right inverse of A.

DEF If $B \in M_{n \times n}(F)$ $AB = BA = I$

B is called the inverse of A and denoted by A^{-1}.

LEMMA If A has a left inverse B and a right inverse C, then $B = C$.

- PROOF $BA = I$ and $AC = I$

$B = BI = B(AC) = (BA)C = IC = C$

THEO REM10 Let $A, B \in M_{n \times n}(F)$

(i) If A is invertible, so is A^{-1} and $(A^{-1})^{-1} = A$

(ii) If A and B are invertible, then so AB.

PROOF $(B^{-1}A^{-1})AB = AB(B^{-1}A^{-1})$

Corrolary A product of invertible matrices is invertible.

THEO REM11 An elementary matrix is invertible.

PROOF $(E_i^c)^{-1} = E_i^{1/c}$

$(E_{ij}^c)^{-1} = E_{ij}^{-c}$

$(J_{ij})^{-1} = J_{ij}$

THEOREM 12 Let $A \in M_{n\times n}(F)$ TFAE $\rightarrow$ (equivalent)

(i) A is invertible.

(ii) $A \leftrightarrow I$

(iii) A is product of elemantary matrices.

PROOF (i) $\rightarrow$ (ii) : Let $R \in M_{n\times n}(F)$: R rre $R \leftrightarrow A$ $\rightarrow$ row equivalent

$A = PR$, where $P = E_1 \dots E_k$, E_i e.m

$\Rightarrow A$ is invertible $\leftrightarrow R$ is invertible.

(ii) $\rightarrow$ (iii)

$A \leftrightarrow I$: $A = PI = P$

(iii) $\rightarrow$ (i)

Product of invertible matrices is invertible.

Corrolary* Let $A \in M_{n\times n}(F)$ be invertible

If a sequence of elementary row operation reduces A to I, then the same sequence applied to I yields A^{-1}.

PROOF $E_k \dots E_1 A = I$

$E_k \dots E_1 A = A^{-1}$

Corrolary Let $A, B \in M_{n\times n}(F)$

$B \leftrightarrow A \Leftrightarrow B = PA$ where P is an invertible $m\times m$ matrix.

THEOREM 13 $A \in M_{n\times n}(F)$ TFAE

(i) A is invertible

(ii) $AX = 0$ has only the trivial solution. $\rightarrow$ THEO 7 REM $A \leftrightarrow I$

(iii) $AX = Y$ has a solution for any Y $(n\times 1)$

PROOF • (i) $\rightarrow$ (ii):

Theorem 7 and Theorem 12

• (i) $\rightarrow$ (iii)

$X = A^{-1}Y$ is a solution of $AX = Y$

• (iii) $\rightarrow$ (ii)

Let $R \in M_{n\times n}(F)$: R is rre with $A \leftrightarrow R$.

$R = PA$

WTS $R = I$

Let $E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow RX = E$ has a solution.

$\Updownarrow$

$PAX = E$ " " " ✓

$\Updownarrow$

$AX = P^{-1}E$ " " " ✓

$\rightarrow AX = Y$ gibi

$\Rightarrow RX = E$ has a solution.

If the last row of R is zero, The last entry of E cannot be I.

$\Rightarrow R$ has no zero rows $\Rightarrow R = I$

Corrolary

A square matrix with either a left or a right inverse is invertible.

PROOF • Suppose A has a left inverse B. $BA = I$.

consider $AX = 0 \Rightarrow (BA)X = 0 \rightarrow X = 0$

$\Rightarrow AX = 0$ has only the trivial solution.

$\Rightarrow A$ is invertible by Thm 13.

• Suppose A has a right inverse C. $AC = I$.

$\Rightarrow C$ has a left inverse.

$\Rightarrow C$ is invertible.

$\Rightarrow A = C^{-1}$

$\Rightarrow A$ is invertible.

Corrolary

Let $A = A_1 A_2 A_3 \dots A_k$ where each $A_i \in M_{n\times n}(F)$

Then A is invertible $\Leftrightarrow$ Each A_i is invertible

PROOF

($\Leftarrow$) ✓

($\Rightarrow$) Suppose A is invertible. Let us show that A_k is invertible

Consider $A_k X = 0$. Now consider $A A_k^{-1} = A_1 \cdots A_{k-1}$

By the same argument as above, A_{k-1} is invertible.

APPLICATION

Finding A^{-1} using Corollary*.

$$[A \mid I] \longrightarrow [I \mid A^{-1}]$$

VECTOR SPACE

DEF A vector space V over a field F is a set together with operations addition and vector multiplication satisfying.

(A) $\forall x, y, z \in V$

(1) $x+y = y+x$

(2) $x+(y+z) = (x+y)+z$

(3) $\exists! 0 \in V : \forall x \in V \quad x+0 = 0+x = x$

(4) $\forall x. \ \exists (-x) \quad x+(-x) = 0$

$\Big\}$ $(V,+)$ is an Abelian group.

(B) $\alpha \in F, x \in V, \ \exists \underbrace{\alpha . x}_{\text{scalar multiplication}} \in V$

(1) $\alpha . (\beta x) = (\alpha \beta) x$

(2) $1 . x = x \quad \forall x \in V$

(C) (1) $\alpha (x+y) = \alpha x + \alpha y$

(2) $(\alpha + \beta) x = \alpha x + \beta x$

$\Big\}$

$F \xhookrightarrow[\text{ring injection}]{} \mathrm{End}(V)$ $\mathbb{Q}/\mathbb{C}$ desek olmazdı çünkü çarpımlar $\mathbb{C}$'ye düşüyor.

$R \longrightarrow \mathrm{End}(M)$; M is an R-module.

✓ ▶ (1) $\mathbb{C}^1 = \mathbb{C}/\mathbb{C}$

F^1 is a vector space = Every field is a vector space over itself.

✓ ▶ $\mathbb{C}[t]/\mathbb{C}$

Polynomials in the variable t with coefficients in $\mathbb{C}$

$\parallel$

$\{ \sum_n a_n x^n : a_n \in \mathbb{C}, a_n = 0$ for all but finitely many $n \}$

✓ ▶ $\mathbb{C}^n / \mathbb{C}$

Define $+$ on $\mathbb{C}^n$ as

$x = (x_1, x_2, \dots, x_n), \ y = (y_1, y_2, \dots, y_n) \in \mathbb{C}^n$

$x+y = (x_1+y_1, \dots, x_n+y_n)$

$\alpha . x = (\alpha x_1, \dots, \alpha x_n)$ for $\alpha \in \mathbb{C}, x \in \mathbb{C}^n$

$\theta = (0, 0, 0, \dots, 0) \in \mathbb{C}^n$

$-x = (-x_1, \dots, -x_n)$ for $x \in \mathbb{C}^n$

■ $\mathbb{R}^2$

$\cdot \ (a,b)(c,d) = (ac, bd)$ ✗ → örneğin $(1,0)$ ile neyi çarparsak çarpalım $(1,1)$ olmayacak.

$\cdot \ (a,b)(c,d) = (ac-bd, ad+bc)$ ✓ Field $\simeq \mathbb{C}$

$(a+bi)(c+di)$

■ Every finite dimensional v.s $/F$

$\simeq F^n$

isomorphic

▶ $\mathbb{C}/\mathbb{Q}$ ✓

▶ $\mathbb{C}/\mathbb{R}$ ✓ $\quad (\mathbb{C} \simeq \mathbb{R}^2)$

▶ $\theta = \{0\}$ Trivial v.s

LINEAR DEPENDENCE

$\{x_i\}$ is a set of vectors (might be empty)

$\sum_i x_i$ is summation of vectors in $\{x_i\}$

DEF A finite set $\{x_i\}$ of vectors is linearly dependent if there exist scalars $\{\alpha_i\}$, not all zero, such that

$\sum \alpha_i x_i = 0$

Conversely, if $\sum \alpha_i x_i = 0$ implies $\alpha_i = 0$ for all i, $\{x_i\}$ is linearly independent.

- $\{\theta\}$ is linearly dependent.
- $\emptyset$ is linearly independent.
- $x, y \in \mathbb{C}^1$

  $x = y = 0 \rightarrow \{x, y\} = \{0\}$ linearly dependent

  If not, $y(x) + -x(y) = 0$

  $\Rightarrow \{x, y\}$ is linearly dependent.

  $\Rightarrow$ Any two vectors in $\mathbb{C}$ are linearly dependent.

DEF If X is an infinite set of vectors, we say that X is linearly independent if every finite subset of X is linearly independent.

- $P = \mathbb{C}[t]$ Suppose $\{t^{k_1}, \dots, t^{k_r}\}$ is linearly dependent

  $\{1, t, t^2 \dots\}$

  $\alpha_1 t^{k_1} + \alpha_2 t^{k_2} + \dots = \theta$

  $\alpha_i = 0 \quad \forall i$

- $\mathbb{C}/\mathbb{R}$ $\{1, i\}$ linearly indep.

  $\mathbb{C}/\mathbb{C}$ $\{1, i\}$ linearly dep.

- $P = \mathbb{C}[t]$

  $\{1-t, t(1-t), 1-t^2\}$ is linearly dependent.

  Since $1-t+t-t^2 = 1-t^2 = 1-t+t(1-t)$

DEF (a) We say that $x \in V$ is a linear combination of $\{x_i\}_{i=1}^n$ if $x = a_1x_1 + \dots + a_nx_n$ for some $a_i \in F$. [F: scalar field for V]

(b) If x is a linear combination of $\{x_i\}_{i=1}^n$, we say that x is linearly dependent on $\{x_i\}_{i=1}^n$

(c) $\{x_i\}_{i=1}^n$ are linearly dependent: x_k is not L.D. on $\{x_i\}_{\substack{i=1 \\ i \neq k}}^n$ for any $1 \leq k \leq n$.

- $0 = a_1x_1 + a_2x_2 + \dots + a_nx_n \Rightarrow a_1 = a_2 = \dots = a_n = 0$

- $\{(1,0), (0,1)\}$ LI

  $\{(1,0), (0,0)\}$ LD

  $-2(1,0) + (2,0) = (0,0) (= 0_V \in V)$ (Linear Dependence Relation)

THEOREM (6) $\{v_1, \dots, v_n\} \subseteq$ is L.D iff for some $2 \leq k \leq n$, v_k is a linear combination of $\{v_1, \dots, v_{k-1}\}$

section

PROOF ($\Leftarrow$) $v_k = a_1v_1 + \dots + a_{k-1}v_{k-1} \Rightarrow -v_k + a_1v_1 + \dots + v_{k-1} = 0$

$\underset{-1 \neq 0 \text{ in } F}{\Rightarrow}$ $\{v_1, \dots, v_n\}$ are LD.

($\Rightarrow$) $b_1v_1 + b_2v_2 + \dots + b_nv_n = 0$

if $b_n \neq 0$, then $v_n = \frac{-b_1}{b_n} v_1 + \frac{-b_2}{b_n} v_2 + \dots$

$\Rightarrow k = n$ works

PROOF Let $\mathcal{B} = \{x_1, \dots, x_n\}$ be a basis of V

(applying Theorem 6)

Consider: $y_1, \dots, y_m, x_1, \dots, x_n$

$x_i = a_1 y_1 + \dots + a_m y_m + \dots$

(x but not y : $\{y_i\}_{i=1}^{m}$ are LI)

Key Point: $\text{span}(\{y_1, \dots y_m, x_1, \dots, x_n\} \setminus \{x_i\}) = V$

suppose $i = n$

- $y_1, \dots, y_m, x_1, \dots x_{n-1}$ ~~x_n~~

1. THEOREM ↑

THEOREM (weaker form)

Suppose V is a finite dim. vector space (i.e., V has a basis $\mathcal{B}$ with finite cardinality.)

Let B' be any other basis of V.

Then $|B| = |B'|$

DEF: $\dim V = |\mathcal{B}|$ where $\mathcal{B}$ any basis.

THEOREM 1 ($\Rightarrow$ weak Theorem)

- $X \subseteq V$ (a subset of finite cardinality) such that $\text{span}\, X = V$
- $Y \subseteq V$ such that Y is L.I.

$\Rightarrow |X| \geq |Y|$

PROOF Suppose on the contrary $\underset{n}{|X|} < \underset{m}{|Y|}$

$X = \{x_1, \dots, x_n\}$ $\quad Y = \{y_1, \dots, y_m\}$ L.I

$y_1, x_1 \dots x_n$ span V $\rightarrow$ Find an x_i s.t $x_i = a_1 y_1 + b_1 x_1 + \dots + b_{i-1} x_{i-1}$ Remember x's so that $i = n$

$y_2, y_1 x_1 \dots x_{n-1}$ span V (*)

$y_3, y_2 y_1 x_1 \dots x_{n-2}$ $\quad \vdots$

(*) $\forall v \in V$

$v = \alpha_1 x_1 + \dots + \alpha_n x_n$

$\hookrightarrow \beta_1 y_1 + \dots + \beta_n x_{n-1}$

$= \theta_0 y_1 + \theta_1 x_1 + \dots + \theta_{n-1} x_{n-1}$

$y_n, y_{n-1} \dots y_1 x_1$ span V

$y_{n+1}, y_n y_{n-1} \dots$ span V

$\rightarrow\leftarrow \Rightarrow m \leq n$.

(5.8) **THEOREM 2** (i) Every set of $n+1$ vectors in an n-dimensional vector space V is L.D (Theorem 1 ⇑)

- Basis (in an n-dim space)
  - L.I + spanning
  - L.I + # = n
  - Spanning + # n

(ii) Let B be a set of n vectors in an n-dim. Vector space V.

TFAE:

(a) $\mathcal{B}$ is a basis (L.I + spans V)

(b) $\mathcal{B}$ is L.I

(c) $\mathcal{B}$ spans V.

(b) ⇒ (a) ⇒ (c) ⇒ (b)

PROOF (b) ⇒ (a)

$\mathcal{B} = \{v_1, \dots, v_n\}$ is L.D but not a basis then by Theorem 5.7.

$\exists v_{n+1} \dots v_{n+k}$; $\{v_1, \dots v_n, v_{n+1} \dots v_{n+k}\}$ is a basis of V.

$\Rightarrow \dim V = n + k > n$ } contradiction.

(c) ⇒ (b):

$\mathcal{B} = \{v_1, \dots v_n\}$ $\quad \exists_i : v_i = a_1 v_1 + \dots + a_{i-1} v_{i-1}$

span $\mathcal{B} = V$, $\mathcal{B}$ is L.D $\quad \Rightarrow \underbrace{\mathcal{B} \setminus \{v_i\}}_{\text{size } n-1} \xrightarrow{\to X}$ spans V.

In fact, in general $k = \max_{1 \le \ell \le n} \{\ell : b_\ell \neq 0\}$ works

$[b_1 v_1 + \dots + b_{k-1} v_{k-1} + \underset{\overset{\shortparallel}{0}}{b_k} v_k + 0 = 0]$

DEF: [Basis] A basis (or a coordinate system) in a vector space V is a set $\mathcal{B}$ of linearly independent vectors, s.t. every vector in V is a linear combination of the vectors in $\mathcal{B}$.

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[Side Definition: $X \subseteq V$ any subset. $\operatorname{Span} X = \langle X \rangle$
$= \{a_1 v_1 + \dots + a_n x_n : v_i \in X, \ a_i \in F\}$
= all possible linear combinations on X.

DEF: (basis, bis): $\mathcal{B}$ basis:
- $\mathcal{B}$ - LI
- $\operatorname{Span} \mathcal{B} = V$

▶ $\mathbb{C}^n$: $\{\mathbb{E}_1, \mathbb{E}_2, \dots, \mathbb{E}_n\}$ is a <u>$\mathbb{C}$-basis</u> for $\mathbb{C}^n$.

$\mathbb{E}_i = (\delta_{ij})_{j=1}^n \quad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & \text{if } i = j \end{cases}$

$= [0, 0, \dots, 0, 1, 0, 0, \dots, 0]$

→ cardinality ↑ ith position

$\{(x_1, \dots, x_n) : x_i \in \mathbb{C}\}$

▶ $P = F[x] = \left\{ \sum_{i=0}^{n} a_i T^i : \begin{array}{l} a_i \in F \\ n \in \mathbb{Z}_{\geq 0} \end{array} \right\}$ = F-vector space of polynomials.

$B = \{p_i(T) = T^n : n \in \mathbb{Z}_{\geq 0}\}$ is a basis of $P = F[T]$

<u>EXERCISE</u> $F[T]$ has no basis of finite cardinality.

DEF: (coordinate)

Suppose V is a vector space and $\mathcal{B}$ is a basis of V
$\mathcal{B} = \{v_1, \dots, v_n\}$

Suppose $|\mathcal{B}| < \infty$. Given $v \in V$, we define <u>its coordinates $[v|\mathcal{B}]$ with respect to $\mathcal{B}$</u> is given by $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ where $\left.\begin{array}{l} v = a_1 v_1 + \dots + a_n v_n \\ v = b_1 v_1 + \dots + b_n v_n \end{array}\right\} a_i = b_i$?

Remark: This notion is well-defined.

$v = a_1 v_1 + \dots + a_n v_n$

$\Rightarrow (a_1 - b_1) v_1 + \dots + (a_n - b_n) v_n = 0 \Rightarrow a_i - b_i = 0 \ \forall i$
$\mathcal{B}$ is L.I. $\Rightarrow a_i = b_i$

$\Rightarrow$ If $\mathcal{B}$ is a basis, any element $v \in V$ may be expressed in a unique manner as a linear combination of $\mathcal{B} = \{v_i\}_{i=1}^n$

▶ $f(T) = T^3 + T - 5 \in \mathbb{Q}[T]_{(3)}$

$[f(T)|\mathcal{B}] = \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (Standart basis $= \{T^0, T^1, T^2, T^3\}$)

DEF V is a finite dimensional vector space: V admits a F-basis $\mathcal{B}$ of finite cardinality.

THEOREM V is f.d. vector space and if $\{y_1, \dots, y_m\}$ is a L.I set then
(i) $\{y_1, \dots, y_m\}$ is a basis.
OR
(ii) $\exists\, y_{m+1}, \dots, y_n \in V$ s.t. $\{y_1, \dots, y_n\}$ is a basis of V.

[any L.I set may be completed to a basis]

$|\mathcal{B}| = n$, $\mathcal{B}$ is LI
$\parallel$
y

Contradicts (Strong) Theorem 1.

$\boxed{\text{ISOMORPHISM}}$ *(relabelling)*

Goal: Every n-dim F-vector space is essentially same as $F^n = F \times F \times \dots \times F$

- $L: V \to W$ "$\longleftrightarrow$" $F^n \longrightarrow F^m$
  depend on the choice of bases

- $\mathcal{P}_n$ = Polynomials of deg $\leq n-1 \simeq \mathbb{R}^n$.
  0 da var.

DEF Suppose U and V are two F-vector spaces.
A morphis (linear transformation) from U to V.

$T: U \to V$
is a map such that.
- $T(x+y) = T(x) + T(y)$
- $T(\alpha x) = \alpha T(x)$

$$\begin{pmatrix} U \times U \longrightarrow U \\ (x,y) \longrightarrow x+y \\ \downarrow \qquad \downarrow \\ T(x,y) \qquad T(x)+T(y) \\ V \times V \longrightarrow V \end{pmatrix} \qquad \begin{pmatrix} U \times U \xrightarrow{add_U} U \\ T \downarrow \qquad \downarrow T \\ V \times V \xrightarrow[add_V]{} V \end{pmatrix}$$

If T is 1-1 and onto
(bijection)

Then we say that T is an isomorphism.

Remark $T^{-1}: V \to U$ (exists as a map)

HW: T^{-1} is a linear map.

LEMMA U is a vector space with basis $\mathcal{B}$ and V is another vector space s.t.
$T: U \xrightarrow{\sim} V$ is an isom.
Then $T(\mathcal{B})$ is a basis of V.

PROOF $\mathcal{B} = \{v_i\}_{i \in I}$, $T(\mathcal{B}) = \{T(v_i)\}_{i \in I}$
any indexing set $\subseteq V$

- Suppose $0 = a_1 T(v_1) + \dots + a_n T(v_n)$
  $\underset{T \text{ is linear}}{=} T(a_1 v_1 + \dots + a_n v_n)$
  $T(0_U) = 0_V$

REMARK
$T(0_U) = T(0_U + 0_U) = T(0_U) + T(0_U)$

T is 1-1.
$\Rightarrow a_1 v_1 + \dots + a_n v_n = 0$
$\Rightarrow a_1 = \dots = a_n = 0$
$\mathcal{B}$ is a basis.

- $\text{Span}(T(\mathcal{B})) \overset{?}{=} V$

$\forall v \in V$
$\parallel \to T$ is onto.
$\sum a_i T(v_i) = T(\sum a_i v_i) = T(u)$ $\quad \mathcal{B}$ is a basis for U
$\downarrow$
$u = \sum a_i v_i$

$\Rightarrow T(\mathcal{B})$ spans V.

Isomorphism sends a basis to a basis.

THEOREM Prop: If U and V are finite dim. vector spaces and
$T: U \xrightarrow{\sim} V$ is an isom. then
$\dim U = \dim V$

PROOF $|\mathcal{B}| = \dim U$
$\parallel$
T is 1-1 $\quad |T(\mathcal{B})| = \dim V$
$\hookrightarrow$ Lemma

THEO REM Any n-dimensional F-vector space V is isom. to F^n.

<u>PROOF</u> Choose a basis $\beta = \{v_1, \dots, v_n\}$ of V.

$$V \xrightarrow{T_\beta} F^n$$

$$v \longmapsto [v|\beta]$$

coordinates W.R.T β

$a_1 v_1 + \dots + a_n v_n$

$$T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)$$

$\boxed{\text{COROLLARY}}$ $\dim U = \dim V$

$$\Rightarrow U \xrightarrow{\sim} V$$

$$\begin{tikzcd} U \arrow[rr, "\sim"] & & V \\ & F^n \arrow[ul, "\sim"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

§10

DEF A non-empty subset M of a vector space V is called a subspace if $\forall x, y \in M$ and $\alpha \in F$ we have $\alpha x + y \in M$.

Remarks

- $0_V \in M$ any subspace

  $0_V = -x + x \in M$

- $(M, +_V)$ is an F-vector space alone.

▶ $\mathcal{P}_n$ vector subspace of $\mathcal{P}$.

$\mathcal{P}_n$ = (poly of degree $\leq n-1$)

▶ $t_1, \dots, t_m \in F$

$$V_m = \{p(T) \in \mathcal{P}_n : p(t_1) = \dots = p(t_m) = 0\}$$

$\subseteq \mathcal{P}_n$ is a subspace.

Question : $\dim V_m = ?$

THEO 1 REM Intersection of any number of subspaces of a given vector space V is a subspace

$U_i \subseteq V$

$i \in I$ is a subspace

→ herhangi bir indeksleme kümesi

<u>PROOF</u> $\forall v_1, v_2 \in \bigcap_{i \in I} U_i$, $\forall \alpha, \beta \in F$

$\alpha v_1 + \beta v_2 \in \bigcap U_i$ as well.

THEO 2 REM Let $S \subseteq V$ be any subset of a vector space V. Then the intersection of all subspaces of V that contains S is a subspace.

Smallest subspace of V that contains S $= \bigcap U \subseteq V$ subspace

$\cap\!\shortmid$

$m(S)$

$\shortparallel$

any other vector space containing S.

$S \subseteq U$: Subspace of V

$$\mathrm{Span}_F S = \left\{ \sum_{i=1}^{n} a_i v_i : \begin{array}{l} a_i \in F \\ v_i \in S \\ n \in \mathbb{Z}^+ \end{array} \right\}$$

- $\mathrm{Span}_F S$ is indeed a subspace:

  For any $x_1 = \sum_{i=1}^{n} a_i v_i \in \mathrm{Span}_F S$

  $x_2 = \sum_{i=1}^{n} b_i v_i \in \mathrm{Span}_F S$

  and any $\alpha \in F$: $\alpha x_1 + x_2 = \sum_{i=1}^{n} (\alpha a_i + b_i) v_i$

  $\in \mathrm{Span}_F S$.

$\Rightarrow \bigcap U \subseteq \mathrm{Span}_F S \longrightarrow S \in U$: subspace of V. } Smallest subspace of V containing S.

If W is any subspace with $S \subseteq W$ then $\mathrm{Span}_F S \leq W$.

Indeed, if $v_1, v_2, \dots, v_n \in S$ and $\alpha_1, \dots, \alpha_n \in F$, then

(W is a vector space) $\alpha_1 v_1 + \dots + \alpha_n v_n \in W$

$\Rightarrow \mathrm{Span}_F S \subseteq W$.

- Correct def. of Span S.

$\mathrm{span}_F S$ = smallest subspace of V containing S.

THEO 3 REM If U_1 and U_2 are two subspaces of V, then (remark: $U_1 \cup U_2$ is not (in general) a vector subspace.)

$\longrightarrow \mathrm{span}(U_1 \cup U_2) = \{x + y : x \in U_1, y \in U_2\}$

$\|$

Smallest subspace of V containing U_1 and U_2.

(NOTATION: $U_1 + U_2$)

PROOF $RHS \subseteq LHS$

($x+y$ is a linear combination of elements from $U_1 \cup U_2$)

$LHS \subseteq RHS$

Let $\sum_{i=1}^{n} a_i v_i \in \mathrm{Span}_F(U_1 \cup U_2)$

be any element, so that $v_i \in U_1 \cup U_2$.

Reorder v_i's in a way that

$\{v_1, \dots, v_k\} \subseteq U_1$

$\{v_{k+1}, \dots, v_n\} \subseteq U_2$, then

$$\sum a_i v_i = \underbrace{\sum_{i=1}^{k} a_i v_i}_{\in U_1} + \underbrace{\sum_{i=k+1}^{n} a_i v_i}_{\in U_2}$$

$\mathbb{R}^2 = \{y = 0\} + \{x = 0\}$

DEF $U_1 \subseteq V$ is a subspace

We say that $U_2 \subseteq V$ is a complement of U_1 in V

$U_1 + U_2 = V$

$U_1 \cap U_2 = \{0\}$

THEO 12.1 REM $U \subseteq V$ is a subspace of a finite dim. vector space V, (of dim n)

U is finite dim. itself and

Then $\dim U \leq \dim V$.

PROOF

The main point is that U has a basis.

- $U = \{0\}$ and $\dim U = 0$ ✓
- If not, $u_1 \in U$ ($u_1 \neq 0$) and if $U = \mathrm{span}\{u_1\}$ ✓

  if not u_2 which belongs to U but not $\mathrm{span}\{u_1\}$ and consider $\mathrm{Span}\{u_1, u_2\}$. If $= U$ ✓

... $\{u_1, \dots, u_{i-1}\} \in U$, LI such that $\emptyset = U \setminus \mathrm{Span}\{u_1, u_2, \dots, u_{i-1}\}$ ($u_i \neq 0$)

Then $\{u_1, \dots, u_i\}$ is LI.

This process stops at the latest in the nth step, since we cannot have $n+1$ L.I. vectors in V.

THEO 12.2 REM $U \leq V$ (n dimensional), (m dimensional)

Then $\exists$ a basis $\{x_1, x_2, \dots, x_m, x_{m+1}, \dots x_n\}$ of V such that $\{x_1, \dots, x_m\}$ is a basis of U.

PROOF Let $\{x_1, \dots, x_m\}$ be my basis of U. $\{x_1, \dots, x_m\} \in V$ is L.I and and we know that we may complete it to a basis.

DEF A linear functional

$$y : V \to F \ (= F')$$

is a linear map.

▶ $y : \mathbb{C}^n \to \mathbb{C}$ is a linear function.

$$(x_1, \dots, x_n) \longmapsto x_i$$
$$\longmapsto x_1^2 + \dots + x_n \longrightarrow \text{NOT}$$

WHY? $(2,0) = 2(1,0) \not\to 2$
$\longrightarrow 4$

▶ $y = F^n \to F$

$$x = (x_1, \dots, x_n) \mapsto a_1 x_1 + \dots + a_n x_n$$

where $a_1, a_2, \dots, a_n \in F$

Remark

$$y(0_V) = 0$$
$$y(0 \cdot 0_V) = 0 \cdot y(0_V) = 0$$

▶ $y : \mathcal{P} \longrightarrow F$
$\mathcal{P} = F[t]$

$$y(p(t)) = p(0)$$

More generality

$$y(p(t)) = \alpha_1 p(t_1) + \dots + \alpha_n p(t_n)$$

where $t_i \in F$, $\alpha_i \in F$ is a linear functional

$\to (y(p(t)) = 2p(0) + 3p(1))$

▶ $\Theta : V \to F$ (zero functional)
$v \mapsto 0$ is a linear functional.

▶ $C^0[a,b] = \{$ $\mathbb{R}$-vector space of continuous functional on $[a,b]$ $\}$

▶ $y : C^0[a,b] \to \mathbb{R}$

$$f \longmapsto a_1 f(t_1) + a_2 f(t_2) + \dots$$

$y_{a_1, \dots, a_n}$
$t_1, \dots, t_n$

$a_i \in \mathbb{R}$
$t_i \in [a,b]$

▶ $y : C^0[a,b] \to \mathbb{R}$
$y = y_g$

$$f \longmapsto \int_a^b f(t) g(t)\, dt$$

is also a linear functional.

$g \in C^0[a,b]$

$\to$ measures

▶ $y_1, y_2 : V \to F$ linear functionals

$$\alpha y_1 + \beta y_2 : V \to F$$
$$v \mapsto \alpha y_1(v) + \beta y_2(v)$$

is also a linear functional, then so is αy

$$(\alpha y)(v_1 + v_2) = \alpha(y(v_1 + v_2))$$
$$= \alpha y(v_1) + \alpha y(v_2)$$
$$= (\alpha y)(v_1) + (\alpha y)(v_2)$$
$$(\alpha y)(\lambda v) = \alpha y(\lambda v) = \alpha \lambda y(v)$$
$$= \lambda(\alpha y)(v)$$

• y_1+y_2 is also linear

$$(y_1+y_2)(v_1+v_2)$$
$$\| \text{def}$$
$$y_1(v_1+v_2)+y_2(v_1+v_2)$$
$$y_1(v_1)+y_1(v_2)+y_2(v_1)+y_2(v_2)$$
$$\|$$
$$(y_1+y_2)(v_1)+(y_1+y_2)(v_2)$$

DEF: $V' = \{f : V \to F \text{ linear function}\}$ is an F-vector space. (Dual space of V)

§14 (Brackets: "Notations duality")

$y : V \to F$
$\in$
x

$y(x) =: [x, y]$
$\|$
$x(y) := ev_x(y)$

Reason:

$V' \xrightarrow{ev_x} F$
$\downarrow$
$f \longmapsto f(x)$
is linear functional

Dual Bases §15

THEO 1 REM V: Finite dim. v.s $\{x_1, x_2, \dots x_n\}$ a basis. Then for any given $\{a_1, \dots a_n\} \subseteq F$. There is one and only one linear functional $\varphi : V \to F$ such that $\varphi(x_i) = a_i$.

PROOF

Existence Given $x = \beta_1 x_1 + \dots + \beta_n x_n \in V$

$$\varphi(x) = \beta_1 a_1 + \dots + \beta_n a_n$$

Then $\varphi : V \to F$ is indexed linear.

$$\varphi(x+x') = \varphi(x) + \varphi(x')$$
$$\varphi(\alpha x) = \alpha \varphi(x)$$

Uniqueness $\varphi_1(x_i) = \varphi_2(x_i) \ \forall i$

$$\varphi_1(x) = \sum_{i=1}^{n} \beta_i \varphi_1(x_i) = \sum_{i=1}^{n} \beta_i \varphi_2(x_i) = \varphi_2(x)$$

THEO 1

To know a linear functional on $V \Leftrightarrow$ to know its values on a given basis.

PROP $1\frac{1}{2}$ $\forall j \in \{1, 2, \dots, n\}$ There exists unique linear functional

functional

$y_j : V \to F$
$x_i \to 0 \quad i \neq j$
$x_j \to 1$

$y_j(x_i) = \delta_{ij}$
$\|$
$[x_i, y_j] \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

3. koordinatı nedir gibi.

▶ $y_1(x_1) = 1$
$y_1(x_2) = \dots = y_1(x_n) = 0$

PROP $1\frac{1}{2}$

THEO 2 REM Given a basis $B = \{x_1, \dots, x_n\}$ of V, $\exists$ unique basis $B' = \{y_1, \dots, y_n\}$ of V' s.t. $[x_i, y_j] = \delta_{ij}$

PROOF We only need to prove that $\{y_1, \dots, y_n\}$ from Prop $1\frac{1}{2}$ is indexed a basis.

• L.I = Suppose that $a_1 y_1 + \dots + a_n y_n = 0$
Then $\in 0.V'$ zero functional on V

$[x_i, a_1y_1 + \dots + a_ny_n]$

$\quad \| \text{ def } a_1y_1 + \dots + a_ny_n$

$[x_i, a_1y_1] + \dots + [x_i, a_ny_n]$

$\quad \| \text{ def } \alpha y$

$a_1[x_i, y_1] + \dots + a_n[x_i, y_n]$

$\quad \| \text{ def } y_j\text{'s}$

$a_i \; ; \; \forall i$ → çünkü kalanlar 0 yapacak.

- SPANS V':

Let $y \in V'$ be any element

$\hat{y} = y(x_1) \cdot y_1 + \dots + y(x_n) \cdot y_n =$

Claim : $\hat{y} = y$

PROOF First of all, observe that $\hat{y} \in V'$.

$\forall i$

$[x_i, \hat{y}] = [x_i, y(x_1)y_1 + \dots + y(x_n)y_n]$

$= y(x_i) \underbrace{[x_i, y_i]}_{1} = y(x_i)$

$= [x_i, y]$

ARA ADIM

$\underbrace{y(x_1)[x_i, y_1)}_{0} + \underbrace{\quad}_{0} + \dots$

Theorem 1

$\Longrightarrow \quad y = \hat{y}$

$= y(x_1)y_1 + \dots + y(x_n)y_n$

is a linear combination of $y_1, \dots, y_n$

DEF Given a basis B of V, the basis

$\| $

$\{x_1, \dots, x_n\}$

B' of Theorem 2 is called the dual basis (of V')

Observations

$[x, y] : V \times V' \to F$

is linear in each coordinate.

$[x + x', y] = [x, y] + [x', y]$

$[x, y + y'] = [x, y] + [x, y']$

THEO 3
REM

(1) If $u \neq v \in V$ then $\exists$

$\varphi \in V'$ s.t

$[u, \varphi] \neq [v, \varphi]$

$u - v$

(2) If $0 \neq u \in V$, then $\exists$

$\varphi \in V'$ s.t

$[u, \varphi] \neq 0$

PROOF $u = a_1x_1 + \dots + a_nx_n$

Then since $u \neq 0$, $\exists i$ with $a_i \neq 0$

Then $\varphi = y_i$ works.

REFLEXIVITY

linear maps from $V' \to F$

$$V \quad V' \quad (V')' = V''$$

$$\xrightarrow{\quad L \quad}$$

$$\cup$$

$$x_0 \longmapsto y \mapsto [x_0, y]$$

evaluation map

is a linear functional on V' $(\in V'')$

THEOREM (all finite dim v.s. are reflexive)

L is an isomorphism of F-vector spaces.

(in particular, given any linear functional $z = V' \to F$, $\exists!\, x_0$ such that $z(y) = [x_0, y]$

PROOF:

0 • STEP 0: L is linear.

$L(x_0 + x_0') \overset{?}{=} L(x_0) + L(x_0')$

$y \mapsto [x_0 + x_0', y]$

$y \mapsto [x_0, y] + [x_0', y]$

1 • STEP 1: $L : V \longrightarrow V''$ is injective.

$\dim n$

2 • $V \xleftrightarrow{\sim} L(V) = V''$

$\dim n \quad \dim n \quad \dim n$

$\| $

$\mathrm{im}(L)$ is a subspace of V''.

PROOF OF STEP 1:

Shorthand: $L : V \to W$ is a linear map.

L is 1-1 $\Leftrightarrow \ker(L) = \{x \in V : L(x) = 0\} = \{0\}$

$\big(L(x_1) = L(x_2) \Leftrightarrow L(x_1 - x_2) = 0$

$\Leftrightarrow x_1 - x_2 \in \ker(L)$

So L is 1-1 $\Leftrightarrow \ker(L) = \{0\} \big)$

$\ker(L) = \{x_0 \in V : [x_0, y] = 0 \ \forall y \in V'\}$

$= \{0_V\}$

Theorem 3 Sec 15 } only 0 goes to 0.

$\forall x \in V, \exists y \in V'$

$\neq 0$ s.t. $y(x) \neq 0$

$x = a_1 v_1 + \dots + a_n v_n$

▶ $a_3 \neq 0$

$[x, v_3'] = a_3 \neq 0$

$L(x_0)\, y := [x_0, y]$

is linear in y

DEF: $= [y, L(x_0)]$

$V \xleftrightarrow{\sim} V''$

$x_0 \longleftrightarrow L(x_0)$

in book, it will drop

§17 ANNIHILATORS

DEF: $S \subseteq V$ (any subset)

Let $S^\circ \subseteq V'$ be the set

$$S^\circ = \{y \in V' : [s, y] = 0, \forall s \in S\}$$

Observation: $S^\circ \subseteq V'$ is always a subspace.

THEOREM $W \subseteq V$ = n-dimensional

$\|$

m-dim

• $\dim W^\circ = n - m$

▶ $\{0_V\}^\circ = V' \quad V^\circ = 0_{V'}$

PROOF

HW: (1) $z \in L(S)$ (2) $\| \; S^\circ$

$\{x_1, \dots, x_m, x_{m+1}, \dots x_n\} = \mathcal{B}$ basis of V

$\{x_1', x_2', \dots, x_m', x_{m+1}' \dots x_n'\}$ Dual Basis of $\mathcal{B}'$ of V'

OBSERVE THAT

$\left.\begin{array}{l}\forall j \geq 1 \\ 1 \leq i \leq m\end{array}\right\}$ $\left[\sum_{i=1} x_i, x_{m+j}'\right] = 0$ her zaman 0.

$\delta_{i, m+j} = 0$

$\{x_{m+1}', \dots, x_n'\} \in W^\circ \to$ kusacı her y.

$\Longrightarrow N = \text{Span}_F \{x_{m+1}', \dots, x_n'\}$

want: $W^\circ \subseteq N$

Let $y \in W^\circ \subseteq V'$

$\| \; b_1 x_1' + b_2 x_2' + \dots + b_n x_n'$

$y(x_i) = 0$ $\forall i = 1, \dots, m$

$\Updownarrow$

$b_1 = 0 = b_2 = 0 \dots b_m = 0$

• $N \to b_{m+1} x_{m+1}' + \dots + b_n x_n'$

THEO 2 REM

V	V'	V''
$\cup\!\!\mid$	$\cup\!\!\mid$	$\cup\!\!\mid$
W	W°	$(W^\circ)^\circ = W^{\infty}$

• STEP 1: $\dim W = m$

$\dim W^\circ = \dim V' - m = n - m$

$\dim (W^\circ)^\circ = (\dim V'') - (n-m) = m$

• STEP 2. W maps to W^{∞} under ι. $\square$

$\forall x \in W$, want: $\iota(x) \in (W^\circ)^\circ$ $\subseteq V''$

$y(x) = [x, y] \underset{def}{=} [\iota(x), y] = 0$

$\forall y \in W^\circ$

$\square$

DEF U, V two F-spaces.

The (external) <u>direct sum</u> of U and V

$U \oplus V = \{(u, v) : u \in F, v \in F\}$ with coordinate-wise scalar multiplication and vector addition.

$\|$

F-Vector space

$\alpha_1 (u_1, v_1) + \alpha_2 (u_2, v_2)$

$(\alpha_1 u_1 + \alpha_2 u_2, \alpha_1 v_1 + \alpha_2 v_2)$

R^1	R^2
$(y=0) \hookrightarrow$	$(xy\text{-plane})$
$x \longmapsto$	$(x, 0)$

<u>Observation</u>

Consider

$$\begin{tikzcd} U \arrow[r, hook, bend left, "\iota_1"] \arrow[r, leftrightarrow, bend right, "\sim (u,0)"'] & U \oplus V & V \arrow[l, hook, bend left, "\iota_2"] \end{tikzcd}$$

Ψ $\qquad$ Ψ

$u \longmapsto (u, 0_V)$

$(0_V, v) \longleftarrow\!\shortmid\; v$

which vectors go to Zero?

(only zero vector goes $\to$ injective)

THEOREM $U, V \subseteq W$, then TFAE equivalent

→ KITABA BAKMA

(1) $\underset{\substack{\cup \\ U+V}}{W} \xleftarrow{\sim} \underset{(U,V)}{U \oplus V}$

(2) $W = U + V$ and $U \cap V = \{0\}$ → Boş kume degil. → W is the (internal) direct sum of U and V.

(3) Every vector w in W may be represented uniquely

$$w = u + v, u \in U, v \in V$$

(1) ⇒ (2) (surjective ⇒ $\forall w \in W, \exists u \in U, \exists v \in V$

$$\underline{u + v = w}$$

$$\Rightarrow W \subseteq U + V \subseteq W$$

$$\Rightarrow U, V \subseteq W$$

injective: Suppose $a \in (U \cap V)$

$$\left.\begin{array}{l}(a,0) \longmapsto a \\ (0,a) \longmapsto a\end{array}\right\} x = 0$$

(2) ⇒ (3) Suppose $w = u_1 + v_1 = u_2 + v_2$

$$\Rightarrow (u_1 - u_2) = (v_2 - v_1) \in (U \cap V) = \{0\}$$

$$\Rightarrow u_1 = u_2, v_1 = v_2$$

(3) ⇒ (1) (surjective: $w \in W$, find U, V with $u + v = w$ then $(u,v) \longmapsto w$

injective: $(u,v) \longmapsto 0 = u + v = 0 + 0$

NOTATION $U, V \subseteq W$, $\underbrace{U + V = W, \ U \cap V = 0}_{U \oplus V = W}$

V

THEOREM 19.1 $\dim(U \oplus V) = \dim(U) + \dim(V)$

$$\left[\dim(U+V) = \dim U + \dim V - \dim(U \cap V)\right]$$

- $\mathbb{R}^2 = (x=0) \oplus (y=0)$
- $(x=0, y=0) \oplus (x=0, z=0) = yz$-plane in $\mathbb{R}^3$
- $\underbrace{\mathcal{P}_{\leq 1}}_{\simeq \mathbb{R}^2} \oplus \underbrace{(x=0, y=0)}_{\mathbb{Z}} \simeq \mathbb{R}^3$

HW

$U_1 \oplus U_2 \oplus \dots \oplus U_n$

When

$U_1 + \dots + U_n$ is a direct sum?

When $U_1 \oplus U_2 \oplus \dots \rightsquigarrow W$

PROOF

$\{u_1, \dots, u_n\}$ a basis of U
$\{v_1, \dots, v_m\}$ " " V

then $\left\{\begin{array}{ll}(u_i, 0) & 1 \leq i \leq n \\ (0, v_j) & 1 \leq j \leq m\end{array}\right.$ is a basis of $U \oplus V$

→ HW

THEOREM 19.2 $\dim_F(W) = m + n$,

$U \subseteq W$ is a subspace of dim m.

Then $\exists V \subseteq W$ s.t

$\dim V = n$ and $U \oplus V = W$

PROOF $\{\underbrace{u_1, u_2, \dots, u_m}_{\text{basis of } U}, \underbrace{v_1, v_2, \dots v_n}_{\text{span of this set is } V}\}$

(can complete to a basis of W)

and this V works

$U \cap V = \{0\}$

$U + V = W$

THEOREM 20.1 $U_1, U_2 \subseteq V$

$U_1 \oplus U_2 = V$

Then $U_2' \xrightarrow{\sim} U_1^\circ \subseteq V'$

$U_1' \xrightarrow{\sim} U_2^\circ \subseteq V'$

and $U_1^\circ \oplus U_2^\circ = V'$ Claim: this is an isomorphism

Bu iki tarafın dualı alındığında nolur? Bunu sorgulıyor.

extend v_1 to a linear functional

$$\begin{array}{l} V \xrightarrow{\tilde{u}_1} F \quad \text{via } \tilde{u}_1|_{U_1} = u_1 \\ \| \\ U_1 \oplus U_2 \qquad\qquad |_{U_2} = 0 \end{array}$$

PROOF

Given $z \in V'$ $(z : V \to F)$

$z(v) = z(u_1 + u_2) = \underbrace{z(u_1)}_{z_2^\circ(v)} + \underbrace{z(u_2)}_{z_1^\circ(v)}$

Claim: $z_1^\circ : V \xrightarrow{\text{proj}_{U_1}} F$ are linear functionals

and $z_i^\circ \in U_i^\circ$

$\Rightarrow U_2^\circ + U_1^\circ = V'$

(HW — prove the rest of the theorem
incorporate the projections in the picture)

$U_1^\circ \cap U_2^\circ \overset{?}{=} \{0\} \longrightarrow$ ✓

$z(u_1) = 0 \quad \forall u_1 \in U_1$

$z(u_2) = 0 \quad \forall u_2 \in U_2$

Then for any $v \in V$, $v = u_1 + u_2$, $\exists u_i \in U_i$: $z(v) = z(u_1) + z(u_2) = 0$

$\Rightarrow z = 0$ is the zero map.

THEOREM $V = M \oplus N$

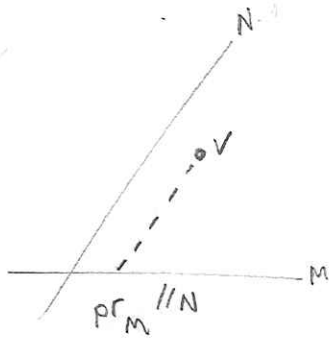
Then there are "natural" isomorphism.

$$\begin{tikzcd} V \arrow[d] \arrow[dr, bend left, "pr_M^{/\!/N}"] & \\ (M \cap V \arrow[r, "f"] & F) \end{tikzcd}$$

$M' \xleftrightarrow{\sim} N^\circ \subseteq V'$

$N' \xleftrightarrow{\sim} M^\circ$

$\longrightarrow M^\circ \oplus N^\circ \xleftrightarrow{\sim} V'$



PROOF Given $V = M \oplus N$

DEF: $V \longmapsto M$
$v \longmapsto m$

is called proj. of V to M, parallel to N; and we denote it by

$pr_M^{/\!/N}(v) = m$

PROOF: consider

$$\left.\begin{array}{ccc} M' & \xrightarrow[pr^*]{} & V' \supseteq N^\circ \\ & & \cup\!\!\mid \\ f & \longmapsto & f \circ pr_M^{/\!/N} \end{array}\right\} \text{This map is linear.}$$

- We claim that the image of pr^* lands in N°, so we have a map.

$$\begin{array}{ccc} M' & \xrightarrow{pr^*} & N^\circ \\ f & \longmapsto & f \circ pr_M^{/\!/N} \end{array}$$

To prove the claim, we only need to check that

$$0 = f(0) = f \circ pr_M^{/\!/N}(n) = 0 \quad \forall n \in N.$$

$$\hookrightarrow 0 + n$$

pr^* is an isomorphism.

STEP 1 $\quad \dim M' = \dim M$

$$\| $$

$$\dim N^\circ = \dim V - \dim N$$

$$= \dim M.$$

$m + n = v \Rightarrow \dim M + \dim N = \dim V$

STEP 2 pr^* is an injection.

$$pr^*(f) = 0_{V'} \iff f \circ pr_M^{/\!/N}(v) = 0 \quad \forall n \in V$$

$$\Rightarrow \underbrace{f \circ pr_M^{/\!/N}(m)}_{M} = 0 \quad \forall m \in M$$

Observation = coset that contains an element of M.

$$0_{V/M} = 0_V + M = M$$

$$(v+M) + (0+M) = (v+M)$$

$$M^\circ \oplus N^\circ \xleftrightarrow{\sim} V'$$

$$(g \circ pr_N^{/\!/M}, g \circ pr_M^{/\!/N}) \longleftarrow\!\shortmid g$$

DEF (QUOTIENT SPACE)

V : F-vector space

$\cup\!\mid$

M : subspace

equivalence class of v [coset of v]

$$V/M = \{ v+M : v \in V \}$$

$\|$

Quotient space

Question: when is $v+M = v'+M$?

$\Updownarrow$

$v - v' \in M$

THEOREM $(V/M \quad \alpha(v+M) = \alpha v + M$, $(v_1+M)+(v_2+M) = (v_1+v_2)+M$

is an F-vectorspace.

PROOF Homework.

- Fun complementler isomorphic

THEO 22.1 REM Suppose

$$V = M \oplus N$$

Then

$$\begin{array}{ccc} N & \xrightarrow[pr/M]{\sim} & V/M \\ n & \longmapsto & n+M \end{array}$$

$pr_N^{/\!/M}$

$\longrightarrow M$ modunda projection gibi.

is an isomorphism of F-vectorspace.

Linearity $\quad pr/_M(n_1+n_2) \overset{?}{=} pr/_M(n_1) + pr/_M(n_2)$

$\forall n \in N \quad pr/_M(\alpha n) = \alpha\, pr/_M(n)$

$\forall \alpha \in F \qquad = \alpha(n+M)$

$\| \longrightarrow$ def

$\alpha n + M$..

Injective $n+M = 0_{V/M}$
$= 0+M$ $\Big\}$ so $n \in M$. $\Rightarrow n+M=M$
↳ exists element $n \in M$

$\Rightarrow n \in M \cap N = \{0\}$ $(V = M \oplus N)$

Surjective

$(v+M) \in V/M$
$\downarrow$
$(n+m+M)$
$\downarrow$
$(n+M)$

$n \longrightarrow n+M$
$n_0 \longrightarrow n_0+M = v+M$

$$\left(\begin{tikzcd} V/M \arrow[d] & \\ F & \end{tikzcd} \longleftrightarrow \begin{tikzcd} V \arrow[d, "f"] & \supseteq M \arrow[d] \\ F & 0 \arrow[l] \end{tikzcd} \quad f \in M^\circ \in V'\right)$$

$$\left(\begin{tikzcd} \text{commutative diagram} \\ A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d] \\ P \arrow[r] & C \end{tikzcd}\right)$$

Corollary

$\dim(V/M) = \dim V - \dim M$ → Think about a proof without choosing N. → Hint $(V/M)' \text{ vs } M^\circ$

BILINEAR more general bracket

DEF: U, V F-vector space

A bilinear form is a map

$$w = U \oplus V \longrightarrow F$$

such that w is linear in each coordinate

$$w(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 w(x_1, y) + \alpha_2 w(x_2, y)$$

▶ $U \oplus U' F$

$(x,y) \longmapsto [x,y] = y(x)$

is a bilinear form.

[n linear forms also make sense: we'll see them and they will lead us to the direct(?) definition of determinants]

THEO 23.1 REM Suppose U, V are finite dimensional.

$\mathcal{B}(U,V) : \{\text{bilinear forms on } U \oplus V\}$

Let $\mathcal{B}_U = \{x_1, \dots, x_m\}$ is a basis of U,
Let $\mathcal{B}_V = \{y_1, \dots, y_n\}$ " " " V.

Then $w_{ij}(x_{i'}, y_{j'}) = \alpha_{ij}$
$\exists! \, w_{ij} \in \mathcal{B}(U,V)$

$$\left(\begin{array}{l} \mathcal{B}(U,V) = \{U \oplus V \to F\} \text{ bilinear} \\ \neq (U \oplus V)' \end{array}\right) \to \text{HW } \dim \neq m+n$$

PROOF: Let $x \in U$, $y \in U$
$x = \sum_{i=1}^{n} a_i x_i$, $y = \sum_{j=1}^{m} b_j y_j$

$$w(x,y) = w\left(\sum_{i=1}^{n} a_i x_i, \sum_{j=1}^{m} b_j y_j\right)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j w(x_i, y_j)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \alpha_{ij}$$

This proves uniqueness

Existence :

Set $w(x,y) = \sum_{i,j} a_i b_j x_{ij}$

where $x = \sum a_i x_i$, $y = \sum b_i y_j$

Check that w is bilinear.

THEO 23.2 REM $\mathcal{B}(u,v)$ is a vector space of <u>dimension m.n</u>.

More precisely, the bilinear forms

$$\{w_{pq}\}_{p=1}^{n}{}_{q=1}^{m} \subseteq B(u,v)$$

given by

$$w_{pq}(x_i, y_j) = \delta_{pi}\,\delta_{qj}$$

$\vee$

Kronecker delta

$\begin{cases} 0 & \text{at } (x_p, y_q) \\ 1 & \text{at } (x_i, y_j) \end{cases}$

form a basis of $\mathcal{B}(U,V)$.

PROOF

L.I $\sum_{p,q} c_{pq} \cdot w_{pq} = 0$

$\xRightarrow[\text{at } (x_i, y_j)]{\text{evaluate}}$ $0 = \sum_{p,q} c_{pq} \cdot w_{pq}(x_i, y_j)$

$= c_{ij} \quad \forall i,j$

Span $\mathcal{B}(u,v)$: For any $w \in \mathcal{B}(u,v)$ need to find $\{d_{pq}\}$ s.t.

$$w = \sum_{p,q} d_{pq} w_{pq}$$

Interlude

If $w = \sum_{p,q} d_{pq} w_{pq}$ then

$$w(x_i, y_j) = \sum_{p,q} d_{pq}\, w_{pq}(x_i, y_j)$$
$$= d_{ij}$$

Guess

$$w = \sum_{p,q} w(x_p, y_q) w_{pq}$$

To check this

$$w(x,y) = \sum_{p,q} w(x_p, y_q)\, w_{pq}(x,y)$$

$\forall x \in U$, $\forall y \in V$

$\Longleftrightarrow$ HW $w(x_i, y_j) = \sum_{p,q} w(x_p, y_q)\, w_{pq}(x_i, y_j)$

$\Updownarrow$

$$w(x_i, y_j) \overset{\checkmark}{=} w(x_i, y_j) \underbrace{w_{ij}(x_i, y_j)}_{1}$$

Corollary

$\dim \mathcal{B}(U,V) = \dim U \cdot \dim V$

$\mathcal{B}(\mathbb{R}[x], \mathbb{R}[y]) \cong \mathbb{R}[x,y]'$

$$(f,g) \longmapsto \int_0^1 f(t) g(t)\, dt = \sum a_{ij} x^i y^j$$

is a bilinear form.

$\boxed{\text{TENSOR PRODUCTS}}$

$$\mathbb{R}[x,y] \xleftrightarrow{\sim} \mathbb{R}[x] \overset{\otimes ?}{\otimes} \mathbb{R}[y]$$

$$x^i y^j \longleftarrow\!\shortmid\; x^i \otimes y^j$$

Question: How can I form a vector space W out of U and V s.t.

$\mathcal{B}(u,v) \xleftrightarrow{\sim} W'$

$\boxed{\text{DEF}}$ $(U \otimes V) = B(U,V)'$

reflexivity $\updownarrow$ S U, V are finite dimensional.

$(U \otimes V)$

$$U \otimes V = B(U,V)'$$

$$\sum a_{xy}\, x \overset{\vee}{\otimes} y \longmapsto z(w) = w(x,y)$$

$$a_x \otimes b_y \longleftrightarrow abz$$

$$x_1 \otimes y + x_2 \otimes y = (x_1 + x_2) \otimes y \longleftrightarrow z_{x_1 + x_2, y}$$

$$z_{x_1 y} + z_{x_2 y}$$

THEOREM $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$ is a basis of U,

$\mathcal{Y} = \{y_1, y_2, \dots, y_m\}$ is a basis of V,

Then the set of vectors

$$z_{ij} = x_i \otimes y_j \quad \begin{pmatrix} i = 1, 2, \dots n \\ j = 1, 2, \dots m \end{pmatrix}$$

form a basis of $U \otimes V = B(U,V)'$

<u>PROOF</u> $w_{pq} \in B(U,V)$ s.t.

$$w_{pq}(x_i, y_j) = \delta_{pi}\, \delta_{qj}$$

($\{w_{pq}\}$ is the "standard basis" of $B(U,V)$)

Let $\{w_{pq}'\} \subseteq B(U,V)'$ be the dual basis of

$\{w_{pq}\}$ so that $[w_{ij}, w_{pq}'] = \delta_{ip}\delta_{jq}$

(Goal: $z_{ij} = w_{ij}'$) $\Leftrightarrow [w, z_{ij}] = [w, w_{ij}']$ $\forall w \in B(U,V)$

Suppose $w = \sum_{p,q} \alpha_{pq} w_{pq}$ is any bilinear form.

$$w_{ij}'(w) = \sum_{p,q} \alpha_{pq} [w_{ij}', w_{pq}]$$

$$= \alpha_{ij} = w(x_i, y_j)$$

$$= z_{ij}(w)$$

$\forall w \in B(U,V)$

$\Rightarrow z_{ij} = w_{ij}'$

(evaluation maps)

$B(U,V)' =$ evaluation maps (x,y)

$(\neq U \oplus V)$

$=$ which vector space is this (describe it in terms of U and V)

What is it then $\longrightarrow U \otimes V$

Bilinear form on $U \oplus V$ $\longleftrightarrow$ Linear functional on $U \otimes V$

★ <u>Permutations</u>: A permutation of $S = \{1, \dots, n\}$ is a <u>bijection</u>.

DEF $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$

$\left.\begin{matrix} 1\text{-}1 \\ \Updownarrow \\ \text{onto} \end{matrix}\right\}$ bijective

- $S_n :=$ Set of all permutations of $\{1, \dots, n\}$

  $|S_n| = n(n-1)\dots = n!$ *nasıl dizecegim*

- $\sigma, \tau \in S_n$, $\sigma \circ \tau = \sigma\tau \in S_n$

- $\exists!\, e \in S_n : e(k) = k$ (id function)

- $\sigma e = \sigma$
  $e\sigma = \sigma$

- $\exists \sigma^{-1} : \{1,\dots,n\} \to \{1,2,\dots,n\}$
  s.t. $\sigma^{-1}\sigma = \sigma\cdot\sigma^{-1} = e$

Indeed $\sigma^{-1}(k) = j$. s.t. $\sigma(j) = k$ $\to$ given
$\downarrow$
$\exists!$

- For $\sigma_1, \sigma_2, \sigma_3 \in S_n$, $\sigma_1(\sigma_2\sigma_3) = (\sigma_1\sigma_2)\sigma_3$
  $\Rightarrow S_n$ is a group of size $n!$
  (symmetric group on n-elements)

DEF Let $p \neq q \in \{1,\dots,n\}$
consider τ given by
$\tau(p) = q$
$\tau(q) = p$
$\tau(i) = i \quad \forall i \neq p, q$

Such a permutation, τ is called transposition.

DEF A k-cycle $\sigma \in S_n$ is a permutation such that $\exists\, i_1, \dots, i_k$ with

$$i_1 \xrightarrow[\sigma]{} i_2 \xrightarrow[\sigma]{} i_3 \cdots \xrightarrow[\sigma]{} i_k$$

σ

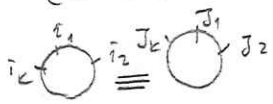
$\sigma(i_j) = i_{j+1}$ if $1 \le j \le k-1$
$\sigma(i_k) = i_1$
$\sigma(r) = r \quad \forall r \notin \{i_1, \dots, i_k\}$

<u>NOTATION</u> $(i_1, i_2, \dots i_k)$ denotes the k-cycle introduced above.

▶ Any transposition (p,q) is a 2-cycle.

$$(3,5,1) = (1,3,5)(r) = \begin{cases} 3 & r=1 \\ 2 & r=2 \\ 5 & r=3 \\ 4 & r=4 \\ 1 & r=5 \end{cases}$$

<u>Observation</u>
$(i_1, \dots, i_k) = (j_1, \dots, j_k)$
$\Updownarrow$
$(k = k')$ and



<u>Observation</u>
Suppose $\sigma = (i_1, \dots, i_r)$ and $\tau = (j_1, \dots, j_s)$ be the cycles.
We say that σ and τ are disjoint to mean that

$$\{i_1, \dots, i_r\} \cap \{j_1, \dots, j_s\} = \emptyset$$

If σ and τ are disjoint, then

$$\sigma\tau = \tau\sigma$$

<u>Notation</u> $\pi \in S_n$

$$\pi^i = \begin{cases} e & , i = 0 \\ \pi \circ \dots \circ \pi & , i > 0 \\ (\pi^{-1})^i & , i < 0 \end{cases} \longrightarrow i \text{ times self composition}$$

THEOREM Every permutation π is a product of disjoint cycles (which are uniquely determined up to reordering)

Idea: Start with any element i such that $\pi(i) \neq i$.

$(i \ \pi(i) \ \pi^2(i) \dots$

$\exists \alpha > \beta \in \mathbb{Z}^+$:

$\pi^{\alpha}(i) = \pi^{\beta}(i)$

DEF Let $\binom{m}{>1}$ be the smallest positive integer with $\pi^m(i) = i$

Then $\# \underbrace{\{i, \pi(i), \dots, \pi^{m-1}(i)\}}_{\text{Distinct elements}} = m$

$\Rightarrow (i \ \pi(i) \dots \pi^{m-1}(i))$

SKETCH

① If $m = n \Rightarrow \pi = (i, \pi(i) \dots \pi^{n-1}(i))$

If $m < n$ then $\exists j \notin \{i, \dots \pi^{m-1}(i)$

$\pi = (i, \dots, \pi^{m-1}) \Leftarrow \forall$ such that $j : \pi(j) = j$

$\exists j : \pi(j) \neq j$

$> (j \ \pi(j) \dots \pi^{n-1}(j))$ → observation $= \pi^{\alpha}(i) = \pi^{\beta}(j)$ $(\alpha > \beta)$

$\vdots$

π is then the product of these cycles that appear in this process.

(When the process ends, for any $k \in \{1, \dots, n\}$ either $\pi(k) = k$, or else k appeared in a cycle of the form $(j_0, \pi(j_0) \dots \pi^{s-1}(j_0))$ and in no other cycle.)

DEF $\pi \in S_n$: fixed

$i, j \in \{1, \dots, n\}$

related

$i \sim j$ if $\exists \alpha : \pi^{\alpha}(i) = j$

For equivalence class

- $i \sim i$: $\pi^0(i) = i$
- $i \sim j \Rightarrow j \sim i$: $\pi^{\alpha}(i) = j \Rightarrow \pi^{-\alpha}(j) = i$
- $i \sim j, j \sim k \Rightarrow i \sim k$

$\Updownarrow \quad \Updownarrow \quad \Uparrow$

$j = \pi^{\alpha}(i) \quad k = \pi^{\beta}(j) \Rightarrow \pi^{\alpha+\beta}(i) = k$

$\Rightarrow \{1, 2, \dots n\} = \coprod (\text{Equivalence Classes})$

$= \coprod_{\eta=1}^{s} C_{\eta}$

Given π, choose any representative $i_{\eta} \in C_{\eta}$ and set $\pi_{\eta} = (i_{\eta}, \pi(i_{\eta}) \dots)$

$\Rightarrow \pi = \prod \pi_{\eta}$

THEOREM 2 Every cycle is a product of transpositions → non unique

PROOF $(1\ 2\ 3\ 4\ 5)$

$= (1\ 5)(1\ 4)(1\ 3)(1\ 2)$

corollary Every permutation is a product of transpositions.

$(1\ 3)(1\ 2) = (1\ 2)(2\ 3) = (1,2,3)$

- # of transposition in the decomposition is not unique.

THM/HW: $\sigma, \tau \in S_n$, then $\exists u \in S_n$ s.t. the cycle decomps of σ and τ had the same "shape" $u^{-1}\sigma y = z$ iff

THEOREM $\sigma = \tau_1^{(1)}, \dots \tau_n^{(1)}$

$= \tau_1^{(2)} \dots \tau_m^{(2)}$

is a product of transposition in two different ways

Then $m \equiv n \pmod 2$

DEF σ is odd $\Leftrightarrow$ n is odd $(-1)^n$

is even $\Leftrightarrow$ n is even

THEOREM $\sigma \in S_k$

$\sigma = \tau_1 \tau_2 \dots \tau_n$ (a product of transpositions)

Then $(-1)^n$ depends only on σ. ✓

PROOF $f = \prod_{i<j} (t_i - t_j)$

$(k=4: f = (t_1-t_2)(t_1-t_3)(t_1-t_4)$
$(t_2-t_3)(t_2-t_4)$
$(t_3-t_4))$

Consider $\sigma(\pm f) = \pm \prod_{i<j} (t_{\sigma(i)} - t_{\sigma(j)})$

(1) $\tau \in S_k$, $(\sigma\tau)(f) = \pm \prod_{i<j} (t_{\sigma\tau(i)} - t_{(\sigma\tau(j))})$

$= "\pm" \sigma \prod_{i<j} (t_{\tau(i)} - t_{\tau(j)})$

$= \sigma(\tau(f))$ $\quad * \quad T_{\sigma(i)} - T_{\sigma(j)} \neq T_{\sigma(i')} - t_{\sigma(j')}$

$\xRightarrow[\sigma \text{ is a bijection}]{} i = i', j = j'$

Observation: $\sigma(f) = \pm f$
$= \text{sign}(\sigma) f$

THEOREM $(-1)^n = \text{sign}(f)$

(2) τ is a transformation, then ✓
$\text{sign}(\tau) = -1$

$\tau = (2,4)$

$(-1) \cdot (t_1-t_2)(t_1-t_3)(t_1-t_4)$
$(t_2-t_3)(t_2-t_4)(t_3-t_4)$

$(t_1-t_2) \leftrightarrow (t_1-t_4)$

$(t_2-t_3) \to -(t_3-t_4)$

$(t_3-t_4) \to -(t_2-t_3)$

$(t_2-t_3)(t_3-t_4)$
$\updownarrow$
$(t_2-t_3)(t_3-t_4)$

$(i<j)$ $\quad t_i - t_j \longrightarrow -(t_i - t_j)$ $\qquad (\alpha,\beta) f = -f$

$\alpha < i$

$$\begin{cases} t_\alpha - t_i \longrightarrow t_\alpha - t_j \\ t_\alpha - t_j \longrightarrow t_\alpha - t_i \end{cases}$$

$t_\alpha - t_i$

$i<j<\beta$

$$\begin{cases} t_i - t_\beta \longrightarrow t_j - t_\beta \\ t_j - t_\beta \longrightarrow t_i - t_\beta \end{cases}$$

$i<\theta<j$

$$\begin{cases} t_i - t_\theta \longrightarrow -(t_\theta - t_j) \\ t_\theta - t_j \longrightarrow -(t_i - t_\theta) \end{cases}$$

(1)

$$(\sigma\tau)(f) = \sigma \prod_{i<j} (t_{\tau(i)} - t_{\tau(j)})$$

$$= \sigma\left(\operatorname{sign}(\tau) \cdot \prod_{i<j} (t_i - t_j)\right)$$

$$\left.\begin{aligned} &= \operatorname{sign}(\tau) \cdot \operatorname{sign}(\sigma) \cdot f \\ &= \operatorname{sign}(\sigma\tau) f \end{aligned}\right\}$$

● $\operatorname{sign}(\sigma\tau) = \operatorname{sign}(\sigma)\operatorname{sign}(\tau)$

$\Longrightarrow \operatorname{sign} : S_k \longrightarrow \{\pm 1\}$ is group hom.

COMPLETION OF THE PROOF

If $\sigma = \tau_1 \tau_2 \dots \tau_k$

then $\operatorname{sign}(\sigma) = \operatorname{sign}(\tau_1) \cdots \operatorname{sign}(\tau_k)$
$= (-1)^n$

DEF $\mathcal{A}_k = A_k = \ker(\operatorname{sign})$

$= \left\{ \begin{array}{l} \text{permutation that are } n \\ \text{product of even } \# \text{ transpositions} \end{array} \right.$

▶ $\operatorname{sign}((i_1, \dots, i_p)) = (-1)^{p-1}$

$\quad \| $

$(i_1, i_p) \dots (i_1, i_2)$

DEF σ is even if $\operatorname{sign}(\sigma) = +1$
odd

§29 MULTILINEAR ALGEBRA

$V_1, \dots, V_k$: F-vector spaces

$f : V_1 \oplus \dots \oplus V_k \longrightarrow F$ is called multilinear if

$\forall i$ $\quad w(\dots, x_{i-1}, x_i, x_{i+1}, \dots)$ is a linear functional on V_1.

$\Longleftrightarrow$

$$w(\dots, x_{i-1}, \alpha x + \beta y, x_{i+1}, \dots)$$
$$= \alpha w(\dots, x_{i-1}, x, x_{i+1}, \dots)$$
$$+ \beta w(\dots, x_{i-1}, y, x_{i+1}, \dots)$$

THEOREM The set of multiple forms form n. F-vector space.

- (αw)
- (w_1+w_2) and the space of multilinear forms have dimension $\to n_1 \cdot n_2 \cdot n_3 \cdots n_k$

<u>PROOF</u> $\mathcal{B}_i = \{ v_1^{(i)}, \dots, v_{n_i}^{(i)} \}$ is a basis of V:

Claim: $\exists!$ multilinear map

$$w_{(i_1,\dots,i_k)} : V_1 \oplus V_2 \cdots \oplus V_k \to F$$

given by

$$w_{(i_1,i_2,\dots,i_k)} \left(v_{j_1}^{(1)}, v_{j_2}^{(2)}, \dots, v_{j_k}^{(k)} \right)$$

$\to \in \{1,\dots,n_1\} \times \dots \times \{1,\dots,n_k\}$

$$= \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_k j_k}$$

▶ Suppose $V_i = V$ $i = 1, \dots k$ a multilinear form

$$w : V \oplus V \oplus \dots \oplus V \to F$$

is called a bilinear form on V.

dim (k-linear form on V) $= n^k$

(1: linear form $\equiv$ linear functional)
(2: linear form $\equiv$ bilinear form on $V \oplus V \to F$)

▶ $V = F^n$ $\quad F^n \oplus \dots \oplus F^n \to F$

$$\| $$

$$\mathrm{mat}_{n \times n}(F)$$

$$w : V \oplus V \dots \oplus V \to F$$

$$(v_1, \dots, v_k) \mapsto w(v_1, \dots v_k)$$

Let $\pi \in S_k$, then we can consider

$$(\pi w)(v_1, \dots v_k) = w(v_{\pi(1)}, \dots v_{\pi(k)})$$

and πw is still a k-linear form.

- $(\pi w)(\dots, \alpha v_i, \dots)$

$$= w(\dots, \alpha v_{\pi(i)}, \dots)$$

$$= \alpha w(\dots, v_{\pi(i)} \dots)$$

$$= \alpha (\pi w)(\dots, v_i, \dots)$$

DEF A k-linear form w is called symmetric if $\pi w = w$ $\forall \pi \in S_k$

$\Updownarrow$ $(\pi_1 \pi_2)(w) = \pi_1(\pi_2 w)$

$\forall \pi = \tau_1, \dots \tau_n$ is a proof of transpositions.

$\tau w = w$ $\forall \tau$: transpositions

$\Updownarrow$

$$w(\dots, x_i, \dots, x_j, \dots) = w(\dots, x_j \dots, x_i \dots)$$

$\forall i, j$

▶ $\underset{\uparrow y_1}{F} \oplus \underset{\uparrow y_2}{F} \to F$

$$y_1(v_1) \cdot y_2(v_2) + y_1(v_2) y_2(v_1)$$

$$\underbrace{V\oplus\cdots\oplus V}_{k}\oplus\underbrace{V\oplus\cdots\oplus V}_{k-h}$$

$k \downarrow F$ u_1: h-linear form

$k-h \downarrow F$ u_2: $(k-h)$ linear form

$$\sum_{\pi\in S_k} \pi\left(u_1(x_1,\dots,x_h)\cdot u_2(x_{h+1},\dots,x_k)\right)$$

DEF A k-linear form

$w: V\oplus V\cdots\oplus V \longrightarrow F$

is called skew-symmetric if

$$\tau w = \operatorname{sign}(\tau)\, w \quad \forall \pi\in S_k$$

$$\Updownarrow$$

$$\left.\begin{array}{l}\forall \tau: \text{transposition} \quad \tau w = -w\end{array}\right\} \text{by Def}$$

$$\Updownarrow$$

$$w(\dots, x_i, \dots, x_j, \dots)$$

$$-w(\dots, x_j, \dots, x_i, \dots)$$

DEF _____ alternating if

$$w(\dots, x, \dots, x, \dots) = 0$$

ith position $\neq$ jth position

Remark

$\operatorname{Char}(F)\neq 2$

skew $\Leftrightarrow$ alternating

<u>IN GENERAL</u> alternating $\Rightarrow$ skew symmetric.

PROOF: w is an alternating k-linear form,

$1\le i<j\le k$,

$$w_0(x_i, x_j) = w(\dots, x_i, \dots, x_j, \dots)$$

other than x_i, x_j fixed.

fixed.

$$= w_0(x_i+x_j, x_i+x_j)$$

$$= \underbrace{w_0(x_i,x_i)}_{0} + \underbrace{w_0(x_i,x_j)}_{0} + w_0(x_j,x_i) + \underbrace{w_0(x_j,x_j)}_{0}$$

w: alternating

FROM NOW ON, W IS ALTERNATING K-LINEAR FORM ON V.

THEO 2 REM If $x_1, x_2, \dots, x_k$ are L.D.

Then $w(x_1,\dots,x_k)=0$

(If $k>n$, there are no non-zero alternating k-forms)

[So: WLOG $k\le n$, if we are only considering alternating forms.]

THEO 3 REM If $w\neq 0$ is an alternating n-form $(k=n)$ and $\{x_1, x_2, \dots, x_n\}$ is L.I in V then $w(x_1,\dots,x_n)\neq 0$.

THEO 4 REM $\dim_F \operatorname{Alt}_n(V)\le 1$

More precisely, any two alternating forms w_1 and w_2 are linearly dependent. $\left(\exists \alpha,\beta\in F \quad \alpha w_1 = \beta w_2\right)$

THEO 5 REM $\dim_F \operatorname{Alt}_n(V) = 1$

$(\exists w\neq 0 \ n\text{-form})$

DEF $L: V \to V$ is called a linear transformation

if $L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$

$\forall \alpha, \beta \in F$

$v_1, v_2 \in V$

$w: V \oplus \dots \oplus V \longrightarrow F$ is a non-zero alternating form.

$$(Lw)(x_1, \dots, x_n) = w(Lx_1, \dots, Lx_n)$$
$$\in Alt_n(V)$$

$\Rightarrow \exists\, \delta = \delta_L$ s.t

$Lw = \delta w$

DEF: $\det(L) = \delta$

▶ $A \in Mat_{n\times n}(\mathbb{R})$

$L_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$x \longmapsto Ax$

$w(Ax_1, \dots, Ax_n) = \det(A) \cdot w(x_1, x_2 \dots x_n)$

set $x_i = \mathbb{E}_i$ $\Downarrow$

$$w(A_1, \dots, A_n) = \det(A) \cdot w(\mathbb{E}_1, \dots, \mathbb{E}_n)$$

A_1: first column of A; $w(\mathbb{E}_1, \dots, \mathbb{E}_n) \neq 0$

$\Rightarrow \{A_1, \dots, A_n\}$ = columns of A are LI iff $\det(A) \neq 0$

▷ HW

- Check the well-known - - Prop of det. -

□ Alternatinglerin boyutu 1.

PROOF (2)

$\exists i > 1 : x_i = a_1 x_1 + \dots + a_{i-1} x_{i-1}$

Then

$w(x_1, \dots, x_{i-1}, x_i, \dots) =$

$w(x_1, \dots, a_1 x_1 + a_2 x_2 + \dots + a_{i-1} x_{i-1}, \dots)$

$$\left.\begin{aligned} &= a_1 w(x_1 \dots, x_{i-1}, x_1 \dots) \\ &\quad a_2 w(x_1, \dots x_{i-1}, x_2 \dots) + \\ &\quad \vdots \\ &\quad a_{i-1} w(x_1 \dots x_{i-1}, x_{i-1}) \end{aligned}\right\} 0$$

▶ $x_3 = x_1 + 2x_2$

$w(x_1, x_2, x_3, x_4) = w(x_1, x_2, x_1 + 2x_2, x_4)$

$$\underbrace{w(x_1, x_2, x_1, x_4)}_{0} + 2\underbrace{w(x_1, x_2, x_2, x_4)}_{0}$$

PROOF (3)

Want: $\underbrace{w(x_1, \dots, x_n)}_{\text{Basis of } V} = 0$

$\Rightarrow w = 0$

$\Leftrightarrow w(y_1, \dots, y_n) = 0 \quad \forall y_1, \dots, y_n \in V$

suppose $w(x_1, x_2, \dots x_n) = 0$

$y_i = \sum_{j=1}^{n} a_{ij} x_j$, then

$$w(y_1 \dots y_n) = w\left(\sum_{j=1}^{n} a_{1j} x_j, \dots \sum_{j=1}^{n} a_{nj} x_j\right)$$
$$= \sum * \, w(z_1, z_2 \dots, z_n)$$

$\{z_1, \dots, z_n\} \subseteq \{x_1, \dots x_n\}$

$= \sum * \, w(z_1, \dots, z_n)$

$\{z_1, \dots, z_n\} = \{x_1, \dots x_n\}$

$= \left(\sum * (-1)^{\square} \right) w(x_1, x_2, \dots, x_n) = 0$

<u>PROOF (4)</u> $\underset{\neq 0}{w_1}, \underset{\neq 0}{w_2}$ Theorem 3

$\underbrace{w_1(x_1, \dots, x_n)}_{\text{Basis of } V} \neq 0 \neq w_2(x_1, x_2, \dots, x_n)$

Claim: Set $\lambda = \dfrac{w_2(x_1, \dots, x_n)}{w_1(x_1, \dots, x_n)} \in F$.

Then $w_2 = \lambda w_1$.

$\left(\Longleftrightarrow w_2(y_1, y_2, \dots, y_n) = \lambda w_1(y_1, \dots, y_n) \right.$

$\forall y_1, \dots y_n \in V$

$w_2(y_1, \dots, y_n) =$

$= \left(\sum * (-1)^{\square} \right) \cdot w_2(x_1, \dots, x_n)$

$= \left(\sum * (-1)^{\square} \right) \lambda w_1(x_1, x_2, \dots, x_n) = \lambda w_1(y_1, y_2 \dots y_n)$

So linearly Dependent

<u>PROOF 5</u>

We will prove by induction that for all $1 \leq k \leq n$, $\exists$ non zero k-linear form.

$k = 1$: Any non-zero linear functional works.

Suppose $\underset{\neq 0}{v} \in \mathrm{Alt}_k(V)$ for $k < n$. by theorem

$v(x_1^\circ, \dots, x_k^\circ) \neq 0$

$u(x_{k+1}^\circ) \neq 0$

$u(x_i^\circ) = 0 \quad \forall i = 1, 2, \dots, k$

$\underset{\neq 0}{u} \in V'$ (= 1-linear form)

consider $w = \sum_{i=1}^{k} (i, k+1) \, v(t_1 \dots t_k) u(t_{k+1}) - v(t_1, \dots t_k) u(t_{k+1})$

BU YAPIYOR

$= \sum_{i=1}^{k} v(t_1, \dots, t_{i-1}, t_{k+1}, t_{i+1}, \dots, t_k) u(t_i)$

$- v(t_1, \dots, t_k) u(t_{k+1})$

$= \sum_{i=1}^{k} (-1)^{k-1} v(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{k+1}) \, u(t_i)$

$\underbrace{- v(t_1, \dots, t_k) u(t_{k+1})}$

is a $(k+1)$-linear form. Sabit.

- $w \neq 0$?
- w alternating ?
- $w(x_1^\circ, \dots, x_k^\circ, x_{k+1}^\circ)$

$\sum_{i=1}^{k} (\dots) \cdot \underbrace{u(x_i^\circ)}_{0 \text{ by choice}}$

not zero diye basta sayıedik.

$\underbrace{- v(x_1^\circ, \dots, x_k^\circ) \cdot u(x_{k+1}^\circ)}_{\boxed{\text{LAST COMPONENT}}} \neq 0$

- $w(\dots, x_\alpha, \dots, x_\alpha, \dots) = 0 \quad \alpha < \beta$

case 1 $\beta \le k$

$$w(\dots, x_\alpha, \dots, x_\beta, \dots) = \Big(v(\dots, x_{\alpha-1}, x_{k+1}, x_{\alpha+\beta}, \dots x_{\beta_j} \dots) \Big)$$

$u(x_\alpha) = x_\beta$

only surviving terms are for $i = \alpha$ and $i = \beta$

$$+ \; v(\dots, x_\alpha, \dots, x_{\beta-1}, x_{k+1}, x_{\beta+1})$$

swapping αth term with βth.

$$= 0$$

- $k = 3$ $\quad V \in Alt_3(V)$, $u \in V'$.

$$w(t_1, t_2, t_3, t_4) = \sum_{i=1}^{3} (i, 4)\, v(t_1, t_2, t_3)\, u(t_4) - v(t_1, t_2, t_3)\, u(t_4)$$

$$= v(t_4, t_2, t_3)\, u(t_1) + v(t_1, t_4, t_3)\, u(t_2) + v(t_1, t_2, t_4)\, u(t_3) - v(t_1, t_2, t_3)\, u(t_4)$$

$(k+1)$_linear

HW

$x_1 = x_2$

$$w(x_1, x_1, x_3, x_4) = v(x_4, x_1, x_3)\, u(x_1) + v(x_1, x_4, x_3)\, u(x_1) = 0$$

LINEAR TRANSFORMATIONS

(1) $0 : V \to V$

$v \mapsto 0$

$v \mapsto 0 . v$

$id = 1 : V \to V$

$v \mapsto 1 . v = v$

(2) scaling Transformation

$\alpha : V \to V \quad (\alpha \in F)$

$v \mapsto \alpha v$

- $V \to V$ $\qquad x_0$: fixed

$x \mapsto y_0(x) . x_0 \qquad y_0 \in V'$

is a linear transformation

- $V \to V$ $\qquad$ Fix $x_1, \dots, x_n \in V$

$y_1, \dots, y_n \in V'$

$x \mapsto y_1(x) . x_1 + \dots + y_n(x) x_n$

is also a L.Transformation

In an n-dimensional vector space, these are all linear transformations!

$$L(V, V) = \{\text{Linear Transformation } V \to V\}$$

$$= \underbrace{End(V)}_{\text{"F_algebra"}}$$

vector space

Endomorphism

$\simeq Mat_{n \times n}(F)$

linear

V is finite dim'l of dim n.

▶ $\pi \in S_n$

$\mathbb{C}^n \to \mathbb{C}^n$

$(x_1, x_2, \dots, x_n) \longmapsto (x_{\pi(1)}, \dots, x_{\pi(n)})$

is a linear transformation.

▶ $P(T) \in \mathcal{P}$, $a_n \in F$

$$P(T) = \sum_{n=0}^{d} a_n T^n$$

$$D : \mathcal{P} \to \mathcal{P}$$

$$P(T) \longmapsto \sum_{n=1}^{d} n a_n T^{n-1} = \frac{d}{dt} P(T)$$

▶ $S : \mathcal{P} \to \mathcal{P}$

$$P(T) \longmapsto \sum_{n=0}^{d} \frac{a_n}{n+1} T^{n+1}$$

Observation

① $\mathcal{P} \xrightarrow{S} \mathcal{P} \xrightarrow{D} \mathcal{P}$, $\mathcal{P} \xrightarrow{\mathrm{id}} \mathcal{P}$

② $\mathcal{P} \xrightarrow{D} \mathcal{P} \xrightarrow{S} \mathcal{P}$, $\mathcal{P} \xrightarrow{\neq \mathrm{id}} \mathcal{P}$

THEOREM $(L(V,V), +)$ is an F-vector space.

PRE PROOF

- $(L_1 + L_2)(v) = L_1(v) + L_2(v)$
- $\forall \alpha \in F \quad (\alpha L)(v) = \alpha L(v)$
- $0 : V \to V$ (zero map)

  $(L + 0) = (0 + L) = L$
- $\forall L \in L(V,V)$, $\exists (-L) : L + (-L) = 0$
- $L_1 L_2 := L_1 \circ L_2$

$$\begin{tikzcd} V \arrow[rr, "L_1L_2"] \arrow[dr, "L_2"'] & & V \\ & V \arrow[ur, "L_1"'] & \end{tikzcd}$$

OBSERVE:

(1) $0 \cdot L_1 = L_1 \cdot 0 = 0$

(2) $L_1 \cdot 1 = 1 \cdot L_1 = L_1$

(3) $L_1(L_2 + L_3) = L_1 L_2 + L_1 L_3$ → linear

(4) $(L_1 + L_2) L_3 = L_1 L_3 + L_2 L_3$

(5) $= (L_1 L_2) L_3 = L_1 (L_2 L_3)$

PROOF of 4

$(L_1 + L_2) L_3 (v)$

$= L_1(L_3(v)) + L_2(L_3(v))$

$= L_1 L_3 + L_2 + L_3$

INVERTIBLE LINEAR TRANSFORMATIONS AND INVERSES

DEF: $L: V \to V$ is said to be invertible when L is 1-1 and onto.

When L is invertible, consider

$$V \longleftarrow V : L^{-1}$$

$$\begin{array}{ccccc} & & \Psi & & \Psi \\ L^{-1}(y) & \xrightarrow{\ni} & x & \xmapsto[\text{onto}]{} & y \\ & & \exists! & \longrightarrow & 1-1 \end{array}$$

<u>Claim</u> L^{-1} is linear

$$L^{-1}(\alpha x + \beta y) \overset{?}{=} \alpha L^{-1}(x) + \beta L^{-1}(y)$$

$\forall x, y \in V$

$\alpha, \beta \in F$

<u>PROOF</u> $(A = B \Leftrightarrow L(A) = L(B)$

L is 1-1

$$L \circ L^{-1}(\alpha x + \beta y) \overset{?}{=} L(\alpha L^{-1}(x) + \beta L^{-1}(y))$$

$\|$ L is linear

$$\left.\begin{array}{l} \alpha L(L^{-1}(x)) + \\ \beta L(L^{-1}(y)) \end{array}\right\} \alpha x + \beta y$$

- L is invertible $\Leftrightarrow L$ is 1-1 onto

$\Leftrightarrow \exists L^{-1}: V \to V$ s.t

$L^{-1} \circ L = L \circ L^{-1} = 1$

▶ $T^{-1} = 1$ $\quad O$ is neither 1-1 nor onto $(\cancel{O^{-1}})$

THEOREM 1: Suppose A, B, C are L.T. and

$AB = CA = I$

onto $\quad$ one-to-one

A is invertible and $B = C = A^{-1}$

<u>PROOF</u> If A is invertible.

$A^{-1}AB = A^{-1} \cdot 1 \Rightarrow B = A^{-1}$

$C = CAA^{-1} = 1 \cdot A^{-1} \Rightarrow A^{-1}$

- $CA = 1 \Rightarrow A$ is 1-1.

$Ax = Ay \Rightarrow \underbrace{CA}_{1}x = \underbrace{CA}_{1}y$

$\Rightarrow x = y$

$\Rightarrow A$ is 1-1.

- $AB = 1 \Rightarrow A$ is onto.

Given $y \in V$, set $x = By$. Then $Ax = ABy = y \Rightarrow A$ is onto.

▶ $D: \mathcal{P} \to \mathcal{P}$ is not one to one

$S: \mathcal{P} \to \mathcal{P}$ is not onto.

but $DS = 1$. $(SD \neq 1)$

<u>LEMMA</u> $A: V \to V$ L.T.

A is 1-1 $\Leftrightarrow \ker(A) = \{x \in V : Ax = 0\}$

$= \{0\}$

THEO 2 REM Suppose V is finite dimensional. Then

A is invertible $\Leftrightarrow A$ is 1-1

$\Leftrightarrow \ker(A) = \{0\}$

$\Leftrightarrow A$ is onto.

<u>"PROOF"</u>

$\operatorname{im}(A) \subseteq V$ is a subspace: $\to$ right

$\ker(A) \subseteq V$ is a subspace. $\to$ left

<u>Rank Nullity</u>

$\dim(\operatorname{im}(A)) + \dim(\ker(A)) = \dim V$

A is 1-1 $\Leftrightarrow \ker(A) = \{0\} \Leftrightarrow \dim(\ker(A)) = 0 \Leftrightarrow \dim \operatorname{im}(A)$

A is onto $\Leftrightarrow \operatorname{im}(A) = V \Leftrightarrow$?

Let $\{x_1, \dots, x_n\}$ be a basis of V.

Claim: $\{Ax_1, \dots, Ax_n\}$ is L.I. $\Rightarrow$ is also a basis

Indeed $\sum \alpha_i A x_i = 0 \Rightarrow A(\sum \alpha_i x_i) = 0$

$\underset{A \text{ is } 1\text{-}1}{\Rightarrow} \sum \alpha_i x_i = 0 \underset{X \text{ is a basis}}{\Rightarrow} \alpha_i = 0 \ \forall i$

Let $y \in V$, then $\exists \beta$

$y = \sum \beta_i A x_i$
$= A(\beta_1 x_1 + \dots + \beta_n x_n)$
$\in \operatorname{im}(A)$

$\Rightarrow A$ is onto.

(Onto $\Rightarrow$ 1-1): Fix a basis $\{y_1, \dots, y_n\}$ of V, so that $y_i = Ax_i$ for some $x_i \in V$ (A is onto)

$\underset{A \text{ is linear}}{\Longrightarrow} \{x_1, \dots, x_n\}$ is L.I

It is a basis of V.

If $Ax = 0$, write $x = \sum \alpha_i x_i$ then

$0 = A(\sum \alpha_i x_i) = \sum \alpha_i A x_i = \sum \alpha_i y_i$

$\Rightarrow \alpha_i = 0$ ($\{y_1, \dots, y_n\}$ is a basis)

$\Rightarrow x = 0 \Rightarrow A$ is 1-1.

THEO REM 1 $\frac{1}{2}$ V is finite dim. then

A is invertible $\Leftrightarrow \exists B : AB = 1 \longrightarrow A$ is onto

$\Leftrightarrow \exists C : CA = 1 \longrightarrow A$ is 1-1.

And in other case $B = C = A^{-1}$

THEO REM 3 A, B are invertible. Then AB is invertible (THEO REM 1) and in fact $(AB)^{-1} = B^{-1}A^{-1}$

If $\alpha \neq 0$ Then $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$. Moreover A^{-1} is invertible and $(A^{-1})^{-1} = A$.

NOTATION

$$A^n = \begin{cases} \underbrace{A \cdot A \cdot A \cdots A}_{n \text{ times}} & n \in \mathbb{Z}_{>0} \\ 1 & n = 0 \\ (A^{-1})^{-n} & n < 0 \end{cases}$$

DEF $A : V \to V$ is a L.T where V: finite dim. vector space.

$$\left(M \in \mathrm{Mat}_{n \times n}(F) \rightsquigarrow \underset{\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \to M\alpha}{F^n \to F^n} \right)$$

and Let $X = \{x_1, \dots, x_n\}$ be a basis of V. Consider $Ax_j = \sum \alpha_{ij} x_i$ (ordered)

$V \ni$ — Bunların sütun olmasını istiyor.

$$[Ax_j \mid X] = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$$

Then the matrix of A WRT X, denoted by $[A \mid X] = [A]_X = [A]$

And it is given by

$[A \mid X] = (\alpha_{ij})$

$= \underbrace{[Ax_1 \mid X]}_{\text{first column}} \underbrace{[Ax_2 \mid X]}_{\text{Second column}} \dots$

"Equivalence class of" $\underset{\mathrm{Mat}_{n \times n}(F)}{\text{matrices}} \longleftrightarrow$ "Linear Transformation"

$D : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ poly of deg $\leq n-1$

Consider the "standard basis" $X = \{x_1(t) = 1, x_2(t) = t, \dots, x_n(t) = t^{n-1}\}$ of $\mathcal{P}_n$.

$Dx_1 = 0 \cdot x_1 + \dots + 0 \cdot x_n \Rightarrow [Dx_1 \mid X] = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$Dx_2 = 1 \cdot x_1 + 0 \cdot x_2 + \dots + 0 x_n \Rightarrow [Dx_1 \mid X] = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$Dx_i = (i-1) x_{i-1}$

$\Rightarrow [Dx_i \mid X] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (i-1) \\ 0 \\ 0 \end{bmatrix} \to i-1$

$$\Rightarrow [D \mid X] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & \vdots \\ \vdots & \vdots & 0 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \end{bmatrix}$$

$A: V \longrightarrow V$ a linear transformation (bases are ordered)

$\|$

fin.dim.

F vector space with a basis X

$[A] = [A \mid X]$ = matrix of A WRT X

Let $M, N \in Mat_{n\times n} F$ $\quad \alpha \in F$, $[\alpha \mid X] = \begin{pmatrix} \alpha & & & 0 \\ & \alpha & & \\ & & \alpha & \\ 0 & & & \alpha \end{pmatrix} = [\alpha]$ = scalar matrix

$M+N$, αM ($\alpha \in F$)

MN (tricky, we'll see why that funny definition is our choice)

<u>FACT</u> $(Mat_{n\times n}(F), +, \cdot)$ is a "ring" with $0 = [0]$ is additive identity, $1_{n\times n} = [1]$ is the multi identity.

$(L(V,V) = End_F(V), +, \circ)$

$= \{$Linear transformation $V \to V\}$

is a ring, 0 = additive identity

1 = mult. identity

THEOREM Fix a basis X of V.

$$L(V,V) \xrightarrow{\Psi_X} Mat_{n\times n}(F)$$

$\Psi_X \longrightarrow$ consistent for X

$$A \longmapsto [A \mid X]$$

is an isomorphis of rings.

<u>PROOF</u> $[A+B \mid X] \overset{?}{=} [A \mid X] + [B \mid X]$

$[AB \mid X] \overset{?}{=} [A \mid X][B \mid X]$

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[dr] & V \arrow[d, "x_i"] \\ & F \end{tikzcd}$$

- $\dim_F(L(V,V)) = n^2$ **HW**

(reduce it to the fact that $\dim_F V' = n$)

$\|$

$\dim_F Mat_{n\times n}(F)$

- Ψ_X is injective: If $\Psi_X(A) = 0$

$\Rightarrow [A \mid X] = [0]$

$\Rightarrow A x_i = 0 \quad \forall i = 1, \dots, n$

$\Rightarrow Ax = 0 \quad \forall x \in V \Rightarrow A = 0$

⊛ <u>useful fact</u> $L: V \to W$ is a linear map: $L(\alpha v + w) = \alpha L(v) + L(w)$

Suppose $\dim V = \dim W$ and L is an isom.

<u>PROOF</u>

$$V \xrightarrow[L]{\sim} im(L) \subseteq W$$

$\dim = n \qquad n \qquad n$

$$[A+B \mid X] = \begin{bmatrix} [(A+B)x_1 \mid X] & [\cdots &][(A+B)x_n \mid X] \\ & \vdots & \\ [Ax_1 + Bx_1 \mid X] \cdots & & \end{bmatrix}$$

$$[A \mid X] + [B \mid X] = \begin{bmatrix} [Ax_1 \mid X] + [Bx_1 \mid X] \cdots \end{bmatrix}$$

$[u+v \mid X] = [u \mid X] + [v \mid X]$

<u>Observation</u>

$\forall x \in V$

$$[Ax \mid X] = [A \mid X][x \mid X]$$

★★ (i.e., if V if "enhanced" with the "frame" (coordinate system $\equiv$ basis) then any linear trans. reduces to matrix mult.)

<u>Question</u>

A is given, how to choose X?

★ ★★

$$\begin{tikzcd} x \arrow[rr, mapsto] & & {[x \mid X]} \\ V \arrow[r, "X", "\sim"'] \arrow[d, "A"'] & F^n \arrow[d, "{[A \mid X]}"] \\ V \arrow[r, "X", "\sim"'] & F^n \\ Ax \arrow[rr, mapsto] & & {[Ax \mid X]} \end{tikzcd}$$

$[Ax \mid X] = [A \mid X][x \mid X]$

▶ V: 3-dim

$$[A \mid X] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$\| \quad \{x_1, x_2, x_3\}$

$[A x_1 \mid X]$

$$\left.\begin{aligned} Ax_1 &= 1 \cdot x_1 \\ Ax_2 &= 2 \cdot x_2 \\ Ax_3 &= 3 \cdot x_3 \end{aligned}\right\} X \text{ consists of eigenvectors}$$

DEF

A is diagonalizable iff $\exists X: [A \mid X]$ is diagonal

iff $\exists X$ that consist entirely of eigenvectors of A.

$$\begin{pmatrix} 1 & e & \pi \\ 0 & 2 & \pi^2 \\ 0 & 0 & 3 \end{pmatrix}$$

$A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x,y,z) \longmapsto (x, ex+2y, \pi x + \pi^2 y + 3z)$

CLAIM (HW/107): $\exists \mathcal{X} : [A|\mathcal{X}] = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$

$\mathcal{X} = (x_1, x_2, x_3)$

$A : \left(\mathbb{R}^3 \longrightarrow \mathbb{R}^3 \text{ with a suitable coord change} \right)$ is a 3-parameter change.

DEF

$W \subseteq V$ is a subspace,

$A : V \to V$ is a L.T

We say that W is invariant under A to mean that $AW \subseteq W$.

▶ $\mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \longmapsto (x, x+2y)$

Find a invariant subspace.

Answer: Span $\{(0,1)\}$

HW

$D : \mathcal{P}_3 \longrightarrow \mathcal{P}_3$

Find an invariant subspace.

$$\begin{pmatrix} \underbrace{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}}_{\mathcal{P}_2} & \begin{matrix} 0 \\ 2 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{0} \end{pmatrix}$$

→ derecesi en çok 2 olanlar

→ $\mathcal{P}_1$ = constant poly.

$A|_W : W \to W$ is a L.T (on W!)

(restriction)

Key Point

What can we say about the matrix of A (or A itself) with the aid W?

CHOOSE a basis $\mathcal{X}_m = \{x_1, x_2, \dots, x_m\}$ of W and complete it to a basis $\mathcal{Y} = \{x_i, \dots, x_m, y_{m+1}, \dots, y_n\}$ of V

$\longrightarrow \underset{W}{\underset{\cap}{A x_i}} = a_{1i} x_1 + \dots + a_{mi} x_m + 0 \cdot y_{m+1} + 0 \dots + 0$

$$[A|\mathcal{X}] = \left[\begin{array}{c|c} \left[A|_W \,\middle|\, \mathcal{X}_W\right] & [B_0] \\ \hline \underbrace{\begin{matrix} 0 & \cdots & 0 \end{matrix}}_{m} & [A_2] \end{array}\right] \Big\} m$$

$$= \left(\begin{array}{c|c} [A_1] & [B_0] \\ \hline & [A_2] \end{array}\right)$$

▶ NEED NOT BE ZERO. WHEN ZERO, WE SAY THAT A IS <u>REDUCABLE</u>

$A x_i = A|_W x_i = \dots$

Reducability

DEF: Suppose W_1 and W_2 are two complementary subspaces of V such that both are invariant under $A : V \to V$.

$(AW_1 \subseteq W_1, AW_2 \subseteq W_2)$

Then we say that (W_1, W_2) reduces A.

$V = W_1 \oplus W_2$

$\mathcal{X} = \{ \underbrace{x_1, \dots, x_m}_{\mathcal{X}_1}, \underbrace{y_{m+1}, \dots, y_n}_{\mathcal{B}_2} \}$

$$[A|\mathcal{X}] = \left(\begin{array}{c|c} [A|_{W_1}|\mathcal{B}_1] & 0 \\ \hline 0 & [A|_{W_2}|\mathcal{B}_2] \end{array}\right)$$

= block diagonal matrix.

▶ Diagonal matrix = "fully reduced"

Reverse Engineering all this process

$A_1 : W_1 \longrightarrow W_1$

$A_2 : W_2 \longrightarrow W_2$

$\longrightarrow C : W_1 \oplus W_2 \to W_1 \oplus W_2$

$\langle x, y \rangle \mapsto \langle A_1 x, A_2 y \rangle$

Choose $\mathcal{B}_1 = \{x_1, \dots, x_m\}$ of W_1

$\mathcal{B}_2 = \{y_{m+1}, \dots, y_n\}$ of W_2.

$\Rightarrow \{\langle x_1, 0\rangle, \dots, \langle x_m, 0\rangle, \langle 0, y_{m+1}\rangle, \dots \langle 0, y_n\rangle\} \to \mathcal{B}$

is a basis of $W_1 \oplus W_2$.

$$[C|\mathcal{B}] = \left(\begin{array}{c|c} [A_1|\mathcal{B}_1] & 0 \\ \hline 0 & [A_2|\mathcal{B}_2] \end{array}\right)$$

- $W \subseteq V$, W: invariant under A, although there are many ways to complement W ($\exists U \subseteq V : U \oplus W = V$) it may happen that $\nexists$ invariant complementary space.

▲▶ $\mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \longmapsto (x+y, y)$

$[A|\mathcal{E}] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

indirgenemez

Suppose span$\{(a,b)\}$ is invariant

$\Rightarrow A(a,b) \in \text{Span}\{(a,b)\}$

$\quad \| $

$(a+b,b) = (a,b) + (b,0)$

$\Leftrightarrow \alpha(a,b) = (b,0)$

$\Leftrightarrow \left.\begin{array}{l}\alpha b = 0 \\ \alpha a = b\end{array}\right\}$

$b \neq 0$: $\alpha = 0$, $b = 0$ $\times$ we first assumed that it is non zero.

$b = 0$: $\Rightarrow$ span$\{(a,0)\}$ = span$\{(1,0)\}$ is the only invariant subspace.

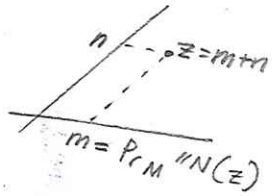
PROJECTIONS $\rightarrow$ morfizm dünyasında $\oplus$ demek oluyor.

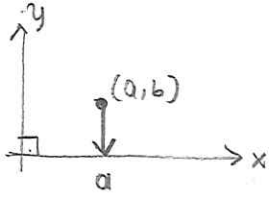
$V = M \oplus N$ (given $z \in V$ $\exists!\, m \in M$, $n \in N$)

$z = m + n$

DEF: Projection of V onto M along (parallel to) N

$$Pr_M^{\,/\!/N} = E : V \longrightarrow M$$
$$z \longmapsto m$$





THEOREM Any projection is a linear transformation. Moreover, a linear transformation $E: V \to V$ is a projection (i.e. $\exists\, M, N \subseteq V$ with $V = M \oplus N$, and $E = Pr_M^{\,/\!/N}$)

[redacted] $E^2 = E$

$(E(Ez) = E^2 z = Ez)$

PROOF:

($\Rightarrow$) If $E: V \to V$ is a projection, say to M along N, then given $z \in V$,

Let $m \in M$, $n \in N$ be such that $z = m + n$, and

$Ez = m$

$E^2 z = Em = m$

$m = m + 0$.

$\Rightarrow E^2 = E$

<u>Observation</u>

$n = 0 + n$ $\quad E = Pr_M^{\,/\!/N}$, then for any $n \in N$

$En = 0$

($\Leftarrow$) Suppose

Define $N := \ker(E)$

$M := \{z \in V : \underbrace{Ez = z}_{(1-E)z = 0}\}$

$M = \text{Ker}(1-E)$ (is also a subspace)

HW: Check that any projection is linear.

We claim that $E = Pr_M^{\,/\!/N}$.

- $V \overset{?}{=} M \oplus N \Leftrightarrow V = M + N$, $M \cap N = \{0\}$

- If $z \in M \cap N \Rightarrow$

  $z = Ez = 0 \Rightarrow z = 0$ (because of K)

- $z = Ez + (1-E)z$

  $\quad \in V \quad \in M \quad \in N$

$\underbrace{E((1-E)z)}_{EN = 0} = (E - E^2)z = 0$

If $z = m \in M$ then $Em = m$

Moreover if $\left.\begin{array}{l} Ez = z \\ \| \\ m \end{array}\right.$ $z = m + n$ $\Big\}$ $n = 0$ and $z = m \in M$

Conversely: if $Ez = 0$ then writing $z = m + n$, we see that

$\left.\begin{array}{l} Ez = 0 \\ \| \\ m \end{array}\right\}$ $z = n \in N$.

• $z \in V$ is any element.
$\quad \| $
$\quad m+n$

$$Ez = E(m+n) = Em + En$$

Corrolary of the proof

① If $E = Pr_{M}^{/\!/N}$ then

$$M = \ker(1-E)$$
$$N = \ker(E)$$

② E is a projection to M along N
iff $(1-E)$ is a projection to N along M.

PROOF

E is a projection $\Leftrightarrow E^2 = E$

$(1-E)$ " " " $\Leftrightarrow (1-E)^2 = (1-E) \Leftrightarrow 1 - 2E + E^2 = 1 - E$
$\Leftrightarrow E^2 = E \Leftrightarrow E$ is projection

$E = Pr_{M}^{/\!/N} \Rightarrow M = \ker(1-E)$
$N = \ker(E)$

$\Longrightarrow 1 - E = Pr_{N}^{/\!/M}$

$1-(1-E) = E$

Remark (extreme case)
$N = 0 \Leftrightarrow M = V \Leftrightarrow E = 1$
$\ker(1-E)$

THEOREM $E_1 = Pr_{M_1}^{/\!/N_1}$ $\quad V = M_1 \oplus N_1$
$E_2 = Pr_{M_2}^{/\!/N_2}$ $\quad = M_2 \oplus N_2$

Suppose $1+1 \neq 0$ → Characteristic ihi olepilse

(i) $E_1 + E_2$ is a projection iff
$E_1E_2 = E_2E_1 = 0$

When this is the case,

$$E = E_1 + E_2 = Pr_{\underbrace{M_1 \oplus M_2}_{\Rightarrow M_1 \cap M_2 = 0}}^{/\!/N_1 \cap N_2}$$

(ii) $E_1 - E_2$ is a projection iff
$E_1E_2 = E_2E_1 = E_2$, and when
this is the case

$$E = E_1 - E_2 = Pr_{M_1 \cap N_2}^{/\!/N_1 \oplus M_2}$$

(iii) If $E_1E_2 = E_2E_1 = E$ then E is
a projection and
$$E = Pr_{M_1 \cap M_2}^{/\!/N_1 + N_2}$$

(i) ($\Rightarrow$) $E_1 + E_2$ is a proj.

$\Leftrightarrow (E_1+E_2)^2 = E_1^2 + E_1E_2 + E_2E_1 + E_2^2$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = E_1 + E_2$
by assumption

$\Longleftrightarrow E_1E_2 + E_2E_1 = 0 \qquad (*)$

E_1, E_2 proj
$E_i^2 = E$

So Normally E_1^2

$\Longrightarrow$ Multiply A from the left/right by E_1.

$E_1E_2 + E_1E_2E_1 = 0$
$\underline{E_1E_2E_1 + E_2E_1}$

$\Longrightarrow E_1E_2 = E_2E_1$

$1+1 \neq 0 \Longrightarrow (*)\ E_1E_2 = E_2E_1 = 0$

$\Rightarrow E_1E_2 = E_2E_1 = 0$
$\Rightarrow E_1E_2 + E_2E_1 + E_1^2 + E_2^2$
$\qquad \| $
$\qquad (E_1+E_2)^2$
$\Rightarrow E_1 + E_2$ is a proj.

$= E_1 + E_2$
$(E_i^2 = E_i)$

want : When $E = E_1 + E_2$ is a proj.

$$M_1 \oplus M_2 = \ker(1-(E_1+E_2)) = M$$

$$N_1 \cap N_2 = \ker(E_1+E_2) = N$$

• Suppose $z \in M$ $\left(\underset{E_1 z + E_2 z}{\underset{\parallel}{(E_1+E_2)z}} = z \right)$

$z = m_1 + n_1$
$= m_2 + n_2$

$m_1 = E_1 z$; $m_2 = E_2 z$
$n_1 = (1-E_1)z$, $n_2 = (1-E_2)z$

$$\left(\begin{array}{l} E_1 m_1 = m_1 \\ 0 = E_2 E_1 m_1 = E_2 m_1 \\ E_2 m_2 = m_2 \Rightarrow E_1 m_2 = 0 \end{array} \right)$$

We assume this at the beginning

$$z = E_1(m_2+n_2) + E_2(m_1+n_1)$$
$$= E_1(n_2) + E_2(n_1)$$
$$\in M_1 + M_2$$

$$\left. \begin{array}{l} E_1(E_1(n_2)) \\ \quad \parallel \\ E_1(n_2) \end{array} \right\} E_1(n_2) \in M_1$$

and similarly, for the other pair.

• Suppose $z = m_1 + m_2 \in M_1 + M_2$

$$(E_1+E_2)(z) = (E_1+E_2)(m_1+m_2)$$
$$= m_1 + 0 + 0 + m_2$$
$$= z$$

$\Longrightarrow z \in M$

$\Longrightarrow M_1 + M_2 \subseteq M$

Finally If $z \in M_1 \cap M_2 \Rightarrow E_1 z = E_2 z = z$

apply ?

• If $z \in N_1 \cap N_2 \Rightarrow \left. \begin{array}{l} E_1 z = 0 \\ E_2 z = 0 \end{array} \right\} (E_1+E_2)z = 0$

$\Longrightarrow N_1 \cap N_2 \subseteq \ker(E_1+E_2)$

If $z \in \ker(E_1+E_2)$ $\overset{\Rightarrow (E_1+E_2)z}{E_1 z + E_2 z = 0} \Longrightarrow E_1^2 z + E_1 E_2 z = 0$

$\overset{\text{its a projection}}{\Longrightarrow} E_1 z + 0 = 0$

$\Longrightarrow z \in N_1$

$\Longrightarrow z \in N_2$

apply by E_1 from the left. (by E_2)

$\Longrightarrow \ker(E_1+E_2) \subseteq N_1 \cap N_2$

(ii) $E_1 - E_2$ is a proj. iff

$$1-(E_1-E_2) = (1-E_1)+E_2$$

is a proj.

$$\underset{i}{\Longleftrightarrow} (1-E_1)E_2 = E_2(1-E_1) = 0$$

$$\Updownarrow$$

$$E_1 E_2 = E_2 E_1 = E_2$$

When this is the case, by (i),

$$1-(E_1-E_2) = \underset{\underset{N_1 \oplus M_2}{\parallel}}{Pr_{\underset{\parallel M_1 \cap N_2}{N_1 \oplus M_2}}}$$

$$\Longrightarrow E_1 - E_2 = Pr_{M_1 \cap N_2}$$

$$E_2 + (1-E_1)$$
$$\downarrow Pr_{N_1} \quad \parallel M_1$$

(b) $E_1E_2 = E_2E_1 = E$

Then $E = Pr_{M_1\cap M_2}^{/\!/ N_1+N_2}$

PROOF $E^2 = (E_1E_2)^2 = E_1^2E_2^2$
$= E_1E_2 = E$

$\Rightarrow E$ is a projection

(a) $M_1\cap M_2 \overset{?}{=} \{z\in V : E_2 = z\}$

(b) $N_1+N_2 \overset{?}{=} \{z\in V : E_2 = 0\}$

(a) If $Ez = z \Rightarrow E_1z = \underbrace{E_1}_{E_1}\underbrace{E_1E_2}_{E}z$
$= E_1E_2z$
$= Ez = z$

$\Rightarrow z\in M$

Likewise, $z\in M_2$

$\Rightarrow z\in M_1\cap M_2 \longrightarrow (\supseteq)$

conversely, $E_1z = z = E_2z$ $(z\in M_1\cap M_2)$

then $\underset{\overset{\shortparallel}{E_2}}{E_2E_1}z = E_2z = z \Rightarrow z\in RHS$ $(\subseteq)$

(b) If $Ez = 0$ $(E_1(E_2z) = 0)$

$\Rightarrow E_2z\in N_1$, likewise $E_1z\in N_2$

$z = \underset{\overset{\cap}{N_1}}{E_2z} + \underset{\overset{\cap}{N_2}}{(1-E_2)z}$

$\Rightarrow z\in N_1+N_2 \longrightarrow (\supseteq)$

If $z = n_1+n_2$, $n_i\in N_i$

$E(z) = E(n_1+n_2) = En_1+En_2$
$= E_2E_1n_1 + E_1E_2n_2$
$= 0$

$E = E_1E_2 = E_2E_1$

$\Rightarrow z\in RHS$

PROJECTIONS AND INVARIANCE

THEOREM 1 $A:V\to V$ is a linear transformation and $M\subseteq V$ is an invariant subspace. $(AM\subseteq M)$

then $EAE = AE$ for any projection to M. Conversely, if $EAE = AE$ for some projection to M, then M is invariant under A.

PROOF $V = M\oplus N$, $AM\subseteq M$

Let $\underset{\overset{\cap}{V}}{z} = m+n$, then be any element

$EAE = AE$ $\left\{ \begin{array}{l} EAEz = EAm = Am \\ \quad \in M \qquad E = Pr_M^{/\!/ N} \\ AEz = Am \end{array} \right.$

$Ez = m$
AEz

Choosing the complementary space N amounts to the choice of $E = Pr_M^{/\!/ N}$

Suppose $EAE = AE$ for some projection $E = Pr_M^{/\!/ N}$. Then if $m\in M$ so that $Em = m$

$Am \overset{?}{=} EAm = EAEm \overset{*}{=} AEm$
$= Am$

criteria for $Am\in M$

(Projection kendine gidiyor mu, gitmiyor mu bunu kontrol ediyoruz)

Δ • Reducing a linear transformation A amounts to finding a projection E that computes with A.

THEOREM 2 $V = M\oplus N$, $A:V\to V$. Then M and N are both invariant under A ((M,N) reduces A) iff

$EA = AE$ where $E = Pr_M^{/\!/ N}$
(or $= Pr_N^{/\!/ M} = 1 - Pr_M^{/\!/ N}$)

Observe that $EA = AE \Rightarrow \underbrace{EAE = AE^2 = AE}_{\Rightarrow M \text{ is invariant}}$

$\Downarrow$

$(1-E)A = A(1-E) \Rightarrow$
$(1-E)A(1-E) = A(1-E)(1-E)$
$\Rightarrow N$ is also invariant.

$(\Leftarrow)$ The proof has ended.

Suppose now $AN\subseteq N$, $AM\subseteq M \Rightarrow EAE = AE$ $\quad (1-E)A(1-E) = A(1-E)$

Theorem 1

$\left(\overset{\text{if}}{E = Pr_M^{/\!/ N}} \Leftrightarrow 1-E = Pr_N^{/\!/ M} \right)$

$\left. \begin{array}{l} EAE = EA \\ \overset{\shortparallel}{AE} \end{array} \right\} EA = AE$

ADJOINTS (and transpose of a matrix)

SET UP: V any vector space

V' dual space

$\parallel$

$\{\text{linear functionals } V \to F\}$

DEF Let $A' : V' \to V'$ be the "linear transformation" given by

$$[Ax, y] = [x, A'y]$$

A' is given by $(A'y)(x) = y(Ax)$

- $x \longmapsto y(Ax)$ is a linear functional.

$\Longrightarrow A'y \in V'$

Why is $A' : V' \to V'$ is a linear transformation.

$$A'(y_1 + y_2) \overset{?}{=} A'y_1 + A'y_2$$

$$\Updownarrow$$

$$[x, A'(y_1+y_2)] \overset{?}{=} [x, A'y_1 + A'y_2]$$

$\parallel$ by def

$$[Ax, y_1 + y_2]$$

$$\parallel$$

$$[Ax, y_1] + [Ax, y_2]$$

$$\parallel$$

$$[x, A'y_1] + [x, A'y_2]$$

$$\parallel$$

$$[Ax, y_1] + [Ax, y_2]$$

PROPERTIES A' is called the adjoint of A $(A^*\ldots)$

② $0' = 0$

③ $1' = 1$

④ $(A+B)' = A' + B'$

⑤ $(\alpha A)' = \alpha A'$

⑥ $(AB)' = B'A'$

⑦ $(A^{-1})' = (A')^{-1}$

PROOF ⑥ $[ABx, y] = [x, (AB)'y]$

$$\parallel$$

$$[A(Bx), y]$$

$\parallel$ def A'

$$[Bx, A'y]$$

$\parallel$ def of B'

$$[x, (B'A')y]$$

$$\Longrightarrow B'A' = (AB)'$$

⑦ $A'(A^{-1})' = (A^{-1}A)' = 1' = 1$

$$(A^{-1})'A' = (AA^{-1})' = 1' = 1$$

$\boxed{\text{PROP}}$

Suppose V is finite dim, and $A : V \to V$ is a L.T. Then

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "\wr"] \\ V'' \arrow[r, "A''"] & V'' \end{tikzcd}$$

$\boxed{\text{PROOF}}$

$$[Ax, y] = [x, A'y] = [A''x, y]$$

$$[Ax, y] = [x, A'y] \quad \Rightarrow A'' = A \ (*)$$

$$[\iota(x), A'y] = [A''\iota(x), y]$$

THEOREM $L(V,V) \longrightarrow L(V', V')$

$$A \longmapsto A'$$

$\mathcal{T} = \{x_1, \ldots, x_n\}$ is a basis of V.

$\mathcal{T}' = \{y_1, \ldots, y_n\}$ is the dual basis of V'.

$$\begin{tikzcd} L(V,V) \arrow[r, leftrightarrow, "1-1"] \arrow[d, leftrightarrow] \arrow[dr, phantom, "(*)"] & L(V',V') \arrow[d, leftrightarrow] \\ \mathrm{mat}_{n\times n}(F) & \mathrm{mat}_{n\times n}(F) \end{tikzcd}$$

$[A|x]$ $\qquad$ $\boxed{[A'|x']} = \boxed{{}^t[A|x]}$

PROOF $[A'|x'] = (\alpha'_{ij})$

$[A|x] = (\alpha_{ij})$

where $\alpha_{ij} = [Ax_j, y_i]$

$$\sum_k \alpha_{kj} x_k$$

$$= [x_j, A'y_i]$$

$$\sum \alpha'_{ki} y_k$$

$$= \alpha'_{ji}$$

$$\Longrightarrow [A|x] = {}^t[A'|x']$$

$\boxed{\text{ADJOINTS OF PROJECTIONS}}$

THEOREM $E = Pr_{M \parallel N}$ then $E' = Pr_{N^\circ \parallel M^\circ}$

PROOF

$(E')^2 \overset{?}{=} E'$ and

$[x, (E')^2 y] \overset{?}{=} [x, E'y]$

$\parallel \qquad\qquad \parallel$

$[E^2x, y] \qquad [Ex, y]$

$\parallel$

$[Ex, y] \Rightarrow E'$ is a proj. use (b)

(i) If $y \in M^\circ$ then $\forall x \in V$

$$[x, E'y] = [\underset{\in M}{Ex}, \underset{\in M^\circ}{y}] = 0$$

$$\Longrightarrow E'y = 0$$

$$\Longrightarrow y \in \ker(E')$$

(ii) If $y \in \ker(E')$ so that $E'y = 0$

then $\forall x \in M$,

$$0 = [x, E'y] = [Ex, y] \underset{x \in M}{=} [x, y]$$

$\Rightarrow y \in M^\circ$

$\ker(E') = M^\circ$

(iii) If $y \in N^\circ$, then $\forall x \in V$

$$[x, y] = [Ex, y] + [\overbrace{(1-E)x}^{0}, y]$$

$$\qquad\qquad\qquad\qquad\quad \underset{N}{\uparrow} \qquad \underset{N^\circ}{\uparrow}$$

$$= [Ex, y] = [x, E'y]$$

$\Rightarrow E'y = y \Rightarrow y \in \ker(1-E')$

$y \in \ker(1-E')$

(iv) If $E'y = y$ $\forall x \in N$

$$[x, y] = [x, E'y] = [Ex, y] \underset{x \in N}{=} 0$$

$\Rightarrow y \in N^\circ$

$\Rightarrow \ker(1-E') = N^\circ$

THEOREM (§ 45.2)

$A: V \longrightarrow V$

$\cup\!\!\mid$

M: invariant under A.

Then M° is invariant under A'.

PROOF

If $y \in M^\circ$, $\forall x \in M$

$[x, A'y] = [\underset{M}{\underset{\cap}{Ax}}, \underset{M^\circ}{\underset{\cap}{y}}] = 0 \Rightarrow A'y \in M^\circ$

Corollary

If (M,N) reduces

$A: V \to V$ then (N°, M°) reduces A'.

PROOF

$V = M \oplus N$

$\circlearrowleft A \quad \circlearrowleft A$

$V' = M^\circ \oplus N^\circ$

$\circlearrowleft A' \quad \circlearrowleft A'$ (Theorem)

preserves

QUESTION I

$V, \mathcal{X}, \mathcal{Y}$ two bases

(f.d. vector space)

Given $v \in V$ $[v|\mathcal{X}]$ vs $[v|\mathcal{Y}]$?

Answer

$\mathcal{X} = \{x_1, \dots, x_n\}$ $\quad y_j = \sum \alpha_{ij} x_i \quad [y_j|\mathcal{X}] = \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{nj} \end{bmatrix}$

$\mathcal{Y} = \{y_1, \dots, y_n\}$

Then set $T(\mathcal{Y}|\mathcal{X}) = \begin{bmatrix} [y_1|\mathcal{X}] \cdots [y_n|\mathcal{X}] \end{bmatrix}$

$[v|\mathcal{X}] \underset{\circledast}{=} T(\mathcal{Y}|\mathcal{X})\,[v|\mathcal{Y}]$

$\circledast$ When $v = y_j$

$[y_j|\mathcal{X}] = \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{nj} \end{bmatrix}$

$$\underbrace{\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1j} & \cdots & \alpha_{1n} \\ & & \alpha_{2j} & & \\ & & \vdots & & \\ \alpha_{n1} & \cdots & \alpha_{nj} & \cdots & \alpha_{nn} \end{pmatrix}}_{T(\mathcal{Y}|\mathcal{X})} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{[y_j|\mathcal{Y}]} \to j^{th} \text{ coordinate}$$

$= \begin{bmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{nj} \end{bmatrix} = [y_j|\mathcal{X}]$

$\circledast$ is linear in V.

THEOREM $\mathcal{X}, \mathcal{Y} \subseteq V$ two bases

$[v|\mathcal{X}] = T(\mathcal{Y}|\mathcal{X})[v|\mathcal{Y}]$

THEOREM 11 $T(\mathcal{Y}, \mathcal{X}) = [A|\mathcal{X}] = ([Ax_1|\mathcal{X}] \cdots [Ax_n|\mathcal{X}])$

$= T(\mathcal{Y}|\mathcal{X})$

$\| $

(α_{ij}) where $A: V \longrightarrow V$

$x_j \longmapsto y_j = \sum \alpha_{ij} x_i$

"linear change of variables"

• Observation / Convention

$v \in V$ with basis $\mathcal{X} = \{x_1, \dots, x_n\}$

$v' \in V' \longrightarrow \mathcal{X}'$ (dual basis)

$\| $

$\{x_1', \dots, x_n'\}$

fully convention

$[v|\mathcal{X}] = \begin{bmatrix} \vdots \end{bmatrix} \in Mat_{n \times 1}$

$[v'|\mathcal{X}'] = [\dots \quad \dots]$

$[v, v'] = [v|\mathcal{X}]^T \cdot [v'|\mathcal{X}']$

$\underbrace{\overset{\|}{\sum a_i x_i} \; \overset{\|}{\sum b_j x_j'}}_{a_1b_1 + \dots + a_nb_n}$

QUESTION 3:

$B: V \to V$

$[B|\mathcal{X}]$ vs $[B|\mathcal{Y}]$

QUESTION 4:

If $[B|\mathcal{X}] = [C|\mathcal{Y}]$

then B vs C?

$L(V,V) \xrightarrow{\text{"}\times\text{" (varies)}} Mat_{n \times n}(F)$

$L \longmapsto [L|\mathcal{X}]$

depends on the choice of $\mathcal{X}$.

Answer to 3

$[B|X] = T(Y|X)[B|Y]T(X|Y)$

Corollary to 11: $T(Y|X)$ is invertible.

PROOF Since $T(Y|X) = [A|X]$, $T(Y|X)$ is invertible iff A is.

But A is clearly onto, (since $\mathrm{im}(A) \supseteq Y$) so it has to be an isom.

$$\left[\begin{array}{l} T(Y|X)^{-1} = [A^{-1}|X] \quad \text{where } A^{-1}: V \to V,\ y_j \mapsto x_j \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = T(X|Y) \end{array}\right]$$

$$[v|Y] = T(X|Y)[v|X]$$
$$= \underbrace{T(X|Y)\,T(Y|X)}_{\text{It has to be } I.}\,[v|Y]$$

$$[B|X] = \underbrace{T(Y|X)}_{[A|X]}[B|Y]\underbrace{T(X|Y)}_{[A|X]^{-1}}$$

PROOF

$\forall v \in V$

$[Bv|X] = [B|X][v|X]$ → def of the mat. of B WRT X.

$\|$

$T(Y|X)[Bv|Y]$

$T(Y|X)[B|Y] \cdot [v|Y]$

$*$

$T(X|Y) \cdot [v|X]$

$*$

Answer to 4

$$[B|X] = [C|Y] = T(X|Y)[C|X]T(Y|X)$$
$$= [A|X]^{-}[C|X][A|X]$$
$$= [A^{-1}CA|X]$$

X respects multiplication on both sides

$\Longrightarrow B = A^{-1}CA$

DEF Matrices $[B]$ and $[C]$ are similar if $\exists [A]$ invertible such that $[B] = [A^{-1}][C][A]$

How to decide $[B]$ is similar to $[C]$? They need to have same "Jordan block form"

Given $[B]$ and $[C]$, when are they the matrices of the same L.T. $L: V \to V$

Any invertible matrix $[A]$ is a transition matrix: (α_{ij})

Set $A: V \to V$

$x_y \longmapsto y_j = \sum \alpha_{ij} x_i$

Then $\{y_1, \dots, y_n\} = Y$ is a basis of V (since A is invertible) and

$T(Y|X) = [A|X] = [A]$

▶ Goal in its simple form

For any linear transformation $L: \mathbb{R}^3 \to \mathbb{R}^3$ such that $L^3 = 0$ and $\dim \ker(L) = 1$ then $\exists$ basis X of $\mathbb{R}^3$ s.t

$$[L|X] = \begin{pmatrix} 0 & 1 & \blacksquare \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

HW $L: \mathbb{R}^2 \to \mathbb{R}^2$ Do it

$$[L|X] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

• REMINDER

$B: V \to V$, X, Y (bases)

$$\begin{tikzcd}
{[v|Y] \in F^n} \arrow[r, "{[B|Y]}"] \arrow[d, leftrightarrow] & F^n \arrow[d, leftrightarrow] & \ni [Bv|Y] = [B|Y][v|Y] \\
v \in V \arrow[r, "B"] \arrow[d, leftrightarrow] & V \arrow[d, leftrightarrow] & \ni Bv \arrow[d, leftrightarrow] \\
{[v|X] \in F^n} \arrow[r, "{[B|X]}"] & F^n & \ni [Bv|X] = [B|X][v|X]
\end{tikzcd}$$

$T(X|Y)$

$T(Y|X)$

$$\underbrace{T(X|Y)[B|X]T(Y|X)}_{[B|Y]}$$

$[B|X] = [C|Y]$, B vs C ?

$$\begin{tikzcd}
V|X \arrow[r, "B"] \arrow[d, "A"] & V|X \arrow[d, "A"] \\
V|Y \arrow[r, "C"] & V|Y
\end{tikzcd}$$

Recall

$[A|X] = T(\mathcal{Y}|X)$ "$\mathcal{Y}$ with the frame $\mathcal{Y}$"

"Transition morphism"

where $A x_j = y_j \Rightarrow A^{-1} C A = B$

Quotient transformation

$A : V \longrightarrow V$

$\cup|$

M: invariant subspace

$\bar{A} : V/M \longrightarrow V/M$ → depends on M

$x + M \longmapsto Ax + M$

$\overset{\shortparallel}{x' + M} \longmapsto \overset{\shortparallel ?}{Ax' + M}$

$\Updownarrow$

$A(x - x') \in M$

(M is invariant) → well defined

başka şekilde translate edilebilir.

(1, 1) ile (0, 1) aynı yere götürüyor.

$\bar{A}$ is called the quotient transformation

NOTATION

$\bar{x} := x + M$

- $\bar{A}\,\bar{x} = \overline{Ax}$ rewrite the notation

- $V = M \oplus N$ (M, N) reduces A.

$\circlearrowleft A \quad \circlearrowleft A$

- $N \xrightarrow[\mathrm{Red}_M = (\bar{\ }) = \mathrm{Bar}]{\sim} V/M$

$n \longmapsto n + M$

→ linearity

– injective because $M \cap N = \{0\}$

– $x + M$, $x \in V$ (surjective)

$x + M = n + m + M = n + M = \bar{n} = \mathrm{Red}_M(n)$, $x = m + n$

Question: $A/N : N \xrightarrow{A|N} N$

$$\begin{tikzcd} N \arrow[r, "A|N"] \arrow[d, "\mathrm{Red}_M"'] \arrow[dr, phantom, "\circlearrowleft"] & N \arrow[d] \\ V/M \arrow[r, "\bar{A}"'] & V/M \end{tikzcd}$$

commute eder mi?

$\bar{A} \circ \mathrm{Red}_M = \mathrm{Red}_M \circ A|N$ $\xrightarrow{\text{PROOF}}$ $\bar{A} \circ \mathrm{Red}_M(n) = \bar{A}\bar{n} = \overline{An} = \mathrm{Red}_M \circ A(n)$

RANGE-NULL SPACE / RANK NULLITY

DEF $M \leq V$ any subspace

$A : V \longrightarrow V$ L.T

$AM = \{Am : m \in M\}$

$R(A) = A'V$ görüntü yer

$\mathrm{Null}(A) = \ker(A)$

Properties

i) A is invertible

$\Updownarrow$

$R(A) = V$ and $\mathrm{Null}(A) = \{0\}$

ii) If V is fin.dim, A is invertible

$\Updownarrow$

$\mathrm{Null}(A) = \{0\}$

$\Updownarrow$

$R(A) = V$

iii) M is invariant $\Leftrightarrow AM \subseteq M$

iv) (M,N) reduces $A \Leftrightarrow AM \subseteq M,\ AN \subseteq N$

(v) $* R(Pr_M^{//N}) = \mathbf{M}$

$\overset{\shortparallel}{E}$

$M = \{Ex = x\}$ for every x

- If $x \in M \Rightarrow Ex = x \Rightarrow x \in R(E)$
- If $x \in R(E)$, so that $x = Ey$ (image)

then $Ex = E^2y = Ey = x$

hence $x \in M$.

$\text{NULL}_*\left(Pr_M^{//N}\right) = \mathbf{N}$

Is $\text{Null}(A) + R(A)$ a direct sum?

A non-example $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbb{R}^2 \xrightarrow[A]{} \mathbb{R}^2$

$A^2 = 0$

$A \neq 0$

$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in R(A)$ $*$

$\in \text{Null}(A)$

$A^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$

(so $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is in both $R(A)$ and $\text{Null}(A)$ since it is not zero, it is not direct sum.)

THEOREM $R(A)^\circ = N(A')$

If V is finite dim. $\rightarrow$ $N(A)^\circ = R(A')$

PROOF $y \in R(A)^\circ \Leftrightarrow [Ax, y] = 0 \quad \forall x \in V$

$\Leftrightarrow [x, A'y] = 0 \quad \forall x \in V$

$\Leftrightarrow A'y = 0$

$\Leftrightarrow y \in N(A')$

- $R(A')^\circ = N(A'') = N(A)$ under $V = V''$

$R(A') = R(A')^{\circ\circ} = N(A)^\circ$

RANK-NULLITY THEOREM

DEF: $A: V \longrightarrow V$

rank of A $= \rho(A) = \dim(R(A))$

nullity of A $= \nu(A) = \dim(N(A))$

V: n-dim vector space

THEOREM $\rho(A) = \rho(A')$

and

$\nu(A) = n - \rho(A)$

- $V/N(A) \xrightarrow{\sim} R(A)$

(1st isom. theorem)

$x + N(A) \longmapsto Ax$

$\bullet\ \nu(A') = n - \rho(A)$

$\left(= \dim(V) - \dim(R(A))\right)$

$X = \underbrace{\{x_1, \dots, x_m, \dots, x_n\}}_{\text{basis of } N(A)}$ is a basis V.

$\Rightarrow R(A) \subseteq \operatorname{Span}\{Ax_{m+1}, \dots, Ax_n\}$

Given $x \in V$, $Ax = a_{m+1}x_{m+1} + \dots + a_n x_n$

$x = a_1 x_1 + \dots + a_n x_n$

$\Rightarrow \underbrace{\dim(R(A))}_{\rho(A)} \leq n - \nu(A)$

$\Rightarrow \rho(A') \leq n - \nu(A')$

$= \rho(A)$

$\Rightarrow \rho(A) = \rho(A'') \leq \rho(A') \leq \rho(A)$

just like squeezing

$\Rightarrow \rho(A) = \rho(A')$

$\Rightarrow \nu(A') = n - \rho(A')$

$\bullet$

$\Rightarrow \nu(A'') = n - \rho(A'')$

set A' in place of A

$\nu(A'') = \nu(A)$, $n - \rho(A'') = n - \rho(A)$

$$\begin{tikzcd} V \arrow[d, "A"'] & V'' \arrow[l, equal] \arrow[d, "A''"] \\ V & V' \arrow[l, equal] \end{tikzcd}$$

If $\sum_{i=1}^{p} \alpha_i y_i = 0$ then

$$0 = A\left(\sum_{i=1}^{n} \alpha_i y_i\right) = \sum_{i=1}^{p} \alpha_i x_i$$

$A y_i = x_i \ (i=1,\dots,p)$
$A y_i = 0 \ (i=p+1,\dots,n)$

$$\Rightarrow \sum_{i=p+1}^{n} \alpha_i y_i = 0 \Rightarrow \alpha_i = 0 \quad i = p+1,\dots,n$$

$\Rightarrow \alpha_i = 0$

In other words $\underbrace{\text{span}\{y_1,\dots,y_p\} \oplus N(A)}_{=V}$

Set $Px_i = y_i$, P is invertible since $R(P) = V$

$PAy_i = Px_i = y_i$, if $i = 1,\dots,p$

$Ay_i = x_i \quad i=1,\dots,p$
and

$PAy_i = P0 = 0 \qquad i = p+1,\dots,n$

$Ay_i = 0$

$$\Rightarrow PA = P_{\substack{N(A) \\ \text{span}\{y_1,\dots,y_p\}}}$$

$E^2 = E^2 E = I \Rightarrow E = I$

(E is an invertible proj $\Leftrightarrow E = Id$)

DETERMINANTS

$A: V \to V$

$0 \neq w \in Alt_n(V) = \text{dimension } 1 = \text{Span}_F\{w_0\}$

$\bar{A}: Alt_n(V) \to Alt_n(V)$

$$w \longmapsto w(Ax_1,\dots,Ax_n)$$
$$\| $$
$$w(x_1,\dots,x_n)$$

$$w_0 \longmapsto w_0(Ax_1,\dots,Ax_n) \to \text{Some other Alternating } n \text{ form}$$
$$\| $$
$$\delta(A)\cdot w_0(x_1,\dots,x_n)$$

DEF $\delta(A) = \det(A)$

▶ If $A = \alpha$, then $\bar{A}w(x_1,\dots,x_n)$
$= w(Ax_1,\dots,Ax_n)$
$= w(\alpha x_1,\dots,\alpha x_n)$
$= \alpha^n w(x_1,\dots,x_n)$
$\Rightarrow \det(\alpha I) = \alpha^n$

$(\det(I) = 1 \quad \det(0) = 0)$

PROP $A: V \to V$
$B: V \to V$

$$\det(AB) = \det(A)\det(B) = \det(BA)$$

PROOF $\bar{AB}w(x_1,\dots,x_n) = \det(AB)w$
$\|$
$w(ABx_1,\dots,ABx_n)$
$\|$
$\bar{A}w(Bx_1,\dots,Bx_n)$
$\|$
$\det(A)\,w(Bx_1,\dots,Bx_n)$
$\|$
$\det(A)\bar{B}w(x_1,\dots,x_n) = \det(A)\det(B)w$

DEF A is singular if $\det(A) = 0$, non-singular if $\neq 0$.

Corrolary

A is non-singular iff A is invertible

($\Leftarrow$) If A is invertible,

$1 = \det(AA^{-1}) = \det(A)\det(A^{-1}) \neq 0$

($\Rightarrow$) Suppose $\det(A) \neq 0$
Let $0 \neq w \in Alt_n(V)$
and $\{x_1,\dots,x_n\}$ be a basis of V.

$$\underbrace{\bar{A}w(x_1,\dots,x_n) = \underset{\neq 0}{\det(A)}\,\underset{\neq 0}{w(x_1,\dots,x_n)}}_{\neq 0 \text{ from prev. classes}}$$

$$\underbrace{w(Ax_1,\dots,Ax_n)}_{L.I}$$

$\Rightarrow \rho(A) = n$
$\Rightarrow A$ is invertible

$M: F^n \to F^n$
$(M \in Mat_{n\times n})$ } $\det(M)$ is defined by considering M as a linear transf on F^n.

Transformations of rank 1

$$A : V \longrightarrow V, \quad \rho(A) \leq 1$$
$$\rho(A) = \dim(\mathcal{R}(A)) = \dim(\mathrm{im}(A))$$

THEOREM $\rho(A) \leq 1 \Leftrightarrow$
$$[A \mid \mathcal{X}] = (\alpha_{ij} = \beta_i \gamma_j) \quad 1 \leq i, j \leq n$$

PROOF

$\rho(A) = 0 \Leftrightarrow A = 0$
$\Leftrightarrow \alpha_{ij} = 0$

$\exists x_0 \in V$

Wlog, $\rho(A) = 1 \Leftrightarrow \forall x \in V$, $Ax = y_0(x) x_0$
$\Updownarrow$
$\mathcal{R}(A) = \mathrm{span}_F \{x_0\}$

Observe that y_0 is a linear functional (since A is linear)
$$A(x_1 + x_2) = y_0(x_1 + x_2) \cdot x_0$$
$$\| $$
$$Ax_1 + Ax_2 = (y_0(x_1) + y_0(x_2)) x_0)$$

(HW) Check the scaling property.

Let now $\{x_1, \dots, x_n\} = \mathcal{X}$ be any basis of V and let $\{y_1, \dots, y_n\} = \mathcal{X}'$ be the dual basis.

For $[A \mid \mathcal{X}] = (\alpha_{ij})$, then jth column in $[A \mid \mathcal{X}]$

$A x_j = \sum \alpha_{ij} x_i$

$([x_i, y_j] = \delta_{ij}) \longleftrightarrow \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{nj} \end{bmatrix}$

$$\Longleftrightarrow \alpha_{ij} = [Ax_j, y_i]$$
$$= [y_0(x_j) x_0, y_i]$$
$$= \underbrace{[x_j, y_0]}_{\gamma_j} \underbrace{[x_0, y_i]}_{\beta_i}$$

$y_0(x_j) = [x_j, y_0]$

$y_0 = \sum \gamma_j y_j$ $\quad x_0 = \sum \beta_i x_i$

Suppose now $[A \mid \mathcal{X}] = (\beta_i \gamma_j)$

Question What is the appropriate x_0, y_0 such that $A(x) = y_0(x) x_0$?

Set $x_0 = \sum \beta_i x_i$ $\quad$ Define $\tilde{A}(x) = y_0(x) x_0$

$y_0 = \sum \gamma_j y_j$

Check that $[\tilde{A} \mid \mathcal{X}] = [A \mid \mathcal{X}] \Rightarrow A = \tilde{A}$

THEO 2 REM If A is a linear transformation of rank ρ, then $A = A_1 + \dots + A_\rho$ with $\rho(A_i) = 1$

PROOF

$\mathcal{R}(A) = \mathrm{span}_F \{x_1, \dots, x_\rho\}$

and $\{x_1, \dots, x_\rho, x_{\rho+1}, \dots, x_n\} = \mathcal{X}$ is a basis for V, $\{y_1, \dots, y_n\} = \mathcal{X}'$ (the dual basis)

Given $x \in V$ $\quad Ax = \sum_{i=1}^{\rho} \alpha_i x_i = \sum_{i=1}^{\rho} \alpha_i(x) x_i$

Observe that $\alpha_i(x) x_i = [Ax, y_i]$ determines a linear map $V \to V$. Its rank is 1.

Now set $A_i : V \to V$
$$x \mapsto [Ax, y_i] x_i$$
and clearly $A = A_1 + \dots + A_\rho$

THEO 3 REM Given any $A : V \to V$ $\exists$ invertible $P : V \to V$ such that PA is a projection

PROOF

$\{x_1, \dots, x_\rho\}$ is a basis of $\mathcal{R}(A)$

$\{x_1, \dots, x_\rho, x_{\rho+1}, \dots, x_n\}$ is a basis of V.

Choose $y_i \in V$ such that
$$A y_i = x_i \quad (i = 1, \dots, \rho)$$

Choose a basis $\{y_{\rho+1}, \dots, y_n\} = N(A) = \ker(A)$

CLAIM: $\{y_1, \dots, y_n\}$ is a basis of V.

$$\begin{tikzcd} V \arrow[r, "A"'] & V \\ N(A) \arrow[u, phantom, "\cup"] & \mathcal{R}(A) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\mathrm{Span}\{y_1, \dots, y_\rho\} \cap N(A)$ claim

$PA = Pr_{\mathcal{R}(A)}^{\parallel N(A)}$

PA = Projection to $\mathrm{Span}\{y_1, \dots, y_\rho\}$ $\parallel$ to $N(A)$

Let X be any basis of V.

consider $[A|X] = (\alpha_{ij})$

(Then $\det(A) = \det([A|X])$

$= \det\left(T(X|Y)^{-1}[A|Y]\,T(X|Y)\right)$

$\underset{\text{PROP}}{=} \det([A|Y])$

PROP 2

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sign}(\pi)\, \alpha_{1\pi(1)} \alpha_{2\pi(2)} \cdots \alpha_{n\pi(n)}$$

PROOF

$$w(Ax_1, \dots, Ax_n) = w\left(\sum_{i=1}^{n} \alpha_{i1} x_i, \dots, \sum_{i=1}^{n} \alpha_{in} x_i\right)$$

$=$

w is alternating

(HW: 3x3)

Try it

PROP3

$\det(A) = \det(A')$

proof: $\det(A) = \sum_{\pi \in S_n} \operatorname{sign}(\pi)\, \alpha_{\pi(1)1} \cdots \alpha_{\pi(n)n}$

$$? = \sum_{\pi \in S_n} \operatorname{sign}(\pi)\, \alpha_{1\pi(1)} \cdots \alpha_{n\pi(n)} = \det(A')$$

$[A'|X'] = (\alpha_{ji})$

PROPERTY

- $M \subseteq V$ $\quad AM \subseteq M$

A

Then: $\det(A) = \det(A|_M) \cdot \det(\mathrm{Red}_M(A))$

$\mathrm{Red}_M A : V/M \to V/M$

$x + M \mapsto Ax + M$

$$\det\left(\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right) = \det A \det B$$

DEF

$P_A(\lambda) = \det(A - \lambda I)$ is a poly of degree n, and leading coeff. $(-1)^n$ and it is called the char. poly.

$$w((A-\lambda)x_1, \dots, (A-\lambda)x_n)$$

$$\|$$

$$(-1)^n \lambda^n w(x_1, \dots, x_n)$$

$$+$$

PROPER VALUES

V/F , $A \in End(V)$

DEF: $\lambda \in F$ is an eigenvalue of A if $\exists x \neq 0 \in V$

$Ax = \lambda x$

Such an x is called an eigenvector of A for λ.

$M = \{x \in V : x$ is an eigenvector of A for $\lambda\}$ $[A(x_1+x_2) = \lambda(x_1+x_2)]$

$M\lambda = M \cup \{0\} \subset V$ is a subspace.

$m\lambda =$ (geometric) multiplicity of $\lambda = \dim_F M\lambda$

Spectrum of $A = \{\lambda \in F : \lambda$ is an eigenvalue of $A\}$

$[Ax = \lambda x \quad x \neq 0$
$\Leftrightarrow (A-\lambda)x = 0$
$\Leftrightarrow (A-\lambda)$ is not invertible$]$

$\overset{*}{=} \{\lambda \in F : A - \lambda$ is not invertible$\}$

▶ $0 \in End(V)$ has the only eigenvalue $\lambda = 0$ with $m\lambda = \dim_F M\lambda = \dim_F V$

A

▶ $1 \in End(V)$ " $\lambda = 1$ with $m\lambda = \dim_F M\lambda = \dim_F V$

Observe

$Ax = \lambda x \Leftrightarrow \underbrace{(A-\lambda)}_{\in End(V)} x = 0$

↳ çünkü ikisi de operator

$\Rightarrow (x \in M_\lambda \Leftrightarrow x \in N(A-\lambda))$

$\Rightarrow m\lambda = \nu(A-\lambda)$

↳ null space'in boyutu

PROP Let $A, B \in End(V)$ B is invertible

(i) • λ is an eigenvalue of $A \Leftrightarrow \lambda$ is an eigenvalue of BAB^{-1}.

(ii) • x is an eigenvector of A for $\lambda \Leftrightarrow Bx$ is an eigenvector of BAB^{-1} for λ.

Byerine B^{-1} koy.

$B^{-1}(BAB^{-1})B = A$

$**$

$B^{-1}Bx = x$

$(\Rightarrow) \underset{**}{\Rightarrow} (\Leftrightarrow)$ $\exists x = 0$ $Ax = \lambda x$

$BAB^{-1}y = BAB^{-1}Bx = BAx = B\lambda x = \lambda Bx = \lambda y$

Remark

Similar matrices (endomorphisms) have the same eigenvalues. (and eigenvectors are given by a coordinate change)

Observe

☒ Suppose $Ax = \lambda x$

$\underline{A^2 x} = A(Ax) = A\lambda x = \lambda A(x) = \underline{\lambda^2 x}$

In general for p a polynomial if λ is an eigenvalue of A, then $p(\lambda)$ is an eigenvalue for $p(A)$ (The converse is not true)

☒ λ_0 is an eigenvalue of $A \Leftrightarrow N(A-\lambda I) \neq 0$
$\Leftrightarrow \det(A-\lambda) = 0 \Leftrightarrow \lambda_0$ is a zero of $char(A)$
$= \det(A-\lambda)$

☒ A has eigenvalue $\lambda \in F \Leftrightarrow char(A)$ has a zero $\lambda \in F$

kökü var

▶ $\mathbb{C}$

$\Longrightarrow$ If F is algebraically closed, A always has an eigenvalue.

↓ kökün olması

MULTIPLICITY

(For 320) F is alg. closed.

$\Updownarrow$

$p(x) \in F[x] = (x-\lambda_1)(x-\lambda_2)\ldots$

λ_i is not necessarily distinct.

• V/F $A \in End(A)$, $\lambda \in$ Spectrum of A

<u>DEF</u> a_λ = Alg. mult. of λ = Mult. of λ as a zero of $char(A)$

In general, $m_\lambda \neq a_\lambda$

$D : P_n \to P_n$

Let $x(t) \in P_n$ and suppose x is an eigenvector of A.

$\Rightarrow Dx = \frac{dx(t)}{dt} = \lambda x(t) \to x = c \cdot e^{\lambda t}$ for some $c \in \mathbb{C}$

$\downarrow$ $\in P_n$

204 $\to \lambda = 0 \Rightarrow char(A) = \lambda^n$

$\downarrow$ not the same $\downarrow$ variable

$x(t) = $ constant

$m_\lambda = \dim_F M_\lambda = 1$

$a_\lambda = n$

<u>Remark</u>

V/F, $\dim(V) = n$, F only closed

$A \in End(V)$

$\to$ $char(A)$ has exactly n zeros, counted with multiplicities.

<u>Remark</u>

$a_\lambda \geq m_\lambda$

Indeed. If λ_0 is an eigenvalue of A, M_{λ_0} is invariant under A.

Let $A_0 = A|_{M_{\lambda_0}}$ $\to (\lambda_0 - \lambda)^{m_{\lambda_0}} \mid (\lambda_0 - \lambda)^{a_{\lambda_0}} q(\lambda)$ $\to$ as polynomials. $((\lambda_0 - \lambda), q(\lambda)) = 1$

$\Updownarrow$

$q(\lambda_0) \neq 0$

$\det(A_0 - \lambda) \mid \det(A - \lambda)$ [bölecek]

▶ $(x-1)$ ise $(x-3)$ olamaz.

$\Longrightarrow m_{\lambda_0} \leq a_{\lambda_0}$

$A : V \to V$: f.d vectorspace $/_F$

$F \supseteq$ all the roots of $P_A(\lambda)$

THEO 1 REM $\exists$ $n+1$ vector subspaces

$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$

such that each V_i is invariant under A.

(Observation : $\dim V_i = i$)

Why does $V_1 = span_F\{x\}$ exist?

$A \circlearrowright$ dim 1

$\Leftrightarrow Ax \in V_1 = Span\{x\}$

$\Leftrightarrow Ax = \lambda x$

$\Leftrightarrow \exists$ an eigenvector.

Suppose $V_0 \subsetneq \cdots \subsetneq V_n = V$

Choose a basis (by induction) s.t.

$\{x_1, \dots, x_i\}$ is a basis of V_1.

$\mathcal{X} = \{x_1, \dots, x_n\}$ is a basis of V_n.

$$[A|\mathcal{X}] = \begin{pmatrix} a_{11} & & & & & \\ 0 & a_{22} & & & & \\ 0 & & \bigcirc & \ddots & & \\ \vdots & & & & a_{jj} & \\ 0 & 0 & 0 & 0 & 0 & \cdots a_{nn} \end{pmatrix}$$

(upper triangular)

$$Ax_j = \sum_{i=1}^{n} \alpha_{ij} x_i = \sum_{i=1}^{j} \alpha_{ij} x_i$$

$\cap\!\!\shortmid$

$V_j = span\{x_1, \dots, x_j\}$

$(\Rightarrow \alpha_{ij} > 0$ if $i > j)$

• $char([A|\mathcal{X}]) = (x - a_{11})(x - a_{nn})$

$= P_A(x) = \det(xI - A)$

PROOF

$n = 0, 1$ ✓

Let V be a vector space of $\dim = n$

Let x' be an eigenvector for A'.

[(Observation) **HW** $P_A(t) = P_{A'}(t)$ lets say for a_{nn}

$M \mapsto {}^tM$ preserves determinants]

common spectrum

$A' \circlearrowleft$ invariant

$\mathrm{Span}_F\{x'\} = M \subseteq V'$

$\Longrightarrow M^\circ =$ annihilator of x' is also invariant under A and has dimension $n-1$.

$= V_{n-1}$

By induction hypothesis,

$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subseteq V_n = V_n$

$\circlearrowleft A \quad \circlearrowleft A$

THEO 2 REM $M \in \mathrm{Mat}_{n\times n}(F)$, $P_M(T)$ splits in F.

Then $\exists T$ such that

TMT^{-1} is uppertriangular.

Corollary

If λ is an eigenvalue of A and $P \in F[t]$ is any poly. then $P(\lambda)$ is an eigenvalue of $P(A)$

$[P(A)|*]_{jj} = P(a_{jj})$ when $[A|*]$ is upper triangular.

DEF: A is nilpotent if $\exists q : A^q = 0$

Smallest such q is nilpotency index.

THEO 1 REM: A is nilpotent of index q

Choose $x_0 \in V : A^{q-1} \cdot x_0 \neq 0$.

Then

$\{A^{q-1}x_0, A^{q-2}x_0, \ldots, Ax_0, x_0\}$

are L.I. Moreover, if M is the space spanned by these, $\exists N$ s.t. (M, N) reduces A.

PROOF

$$\sum_{i=0}^{q-1} \alpha_i A^i x_0 = 0$$

Say $\alpha_j \neq 0$, j smallest such

$$\Longrightarrow A^j x_0 = \sum_{i \geq j+1} \beta_i A^{j+1} A^{i-j-1} x_i$$

$$= A^{j+1}(y)$$

$$\Longrightarrow A^{q-j-1} A^j x_0 = A^q y = 0$$

$$\| $$

$$0 \neq A^{q-1} x_0$$

Easy to see: $AM \subseteq M$

$(A x_i = x_{i+1}$ (if $i < q-1$)

$\in M$

$(x_i = A^i x_0 \quad i = 0, \ldots, q-1)$

Reading **HW** 110-111

THEO2 REM: (Jordan form for nilpotent matrices)

A is nilpotent of deg q.

Then $\exists$ integers $r, q_1, \ldots, q_r$ vector $x_1, \ldots, x_r$ such that

$q_1 + q_2 + \cdots q_r = n$

(i) $q = q_1 \geq \cdots \geq q_r$

(ii) the vectors

$$\mathcal{X} = \begin{cases} A^{q_1-1}x_1, \ldots, Ax_1, x_1 \\ A^{q_2-1}x_2, \ldots, Ax_2, x_2 \\ \vdots \\ A^{q_r-1}x_r, \ldots, Ax_r, x_r \end{cases}$$

form a basis of V.

(iii) $A^{q_i} x_i = 0$

Moreover

The data $(r, q_1, \dots, q_r)$ determines A uniquely up to isom:

B is any other nilpotent linear transformation with the same properties, then

$$\exists T: V \to V : \quad TAT^{-1} = B$$

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ V \arrow[r, "B"] & V \end{tikzcd} \iff \begin{array}{c} [A \mid \mathcal{X}] \\ \| \\ [B \mid \mathcal{Y}] \end{array}$$

$$[A \mid \mathcal{X}] = \begin{pmatrix} \overbrace{\begin{matrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{matrix}}^{N_{q_1}} & & \\ & N_{q_2} & \\ & & N_{q_3} \\ & & & \ddots \end{pmatrix}$$

$A(A^{q_1-2} x_0) = A^{q_1-1} x_0$

LAST THEOREM

$A^{q-1} x_0 \neq 0$

$$\Longrightarrow \underbrace{\operatorname{span}\{x_0, \dots, A^{q-1} x_0\}}_{\dim = q} \oplus N = V$$

A $\quad$ A

The inductive procedure to determine $x_1, \dots, x_r$ and $q_1, \dots, q_r$

Use the theorem to find a q-dim. subspace

$M = \operatorname{SPAN}\{A^{q_1-1} x_1, \dots, A x_1, x_1\} \quad (q_1 = q)$

invariant under A, and a complementary invariant subspace N.

$A|_N : N \to N$ is still nilpotent, say of index $q_2 \leq q_1$ $(N \leq V)$

Apply the theorem again on N to find

$M_1 = \operatorname{Span}_F\{A^{q_2-1} x_2, \dots, A x_2, x_2\}$ of dim $q_2 - 1$, invariant under A, and a complementary space N_1.

When the procedure stops, we arrive at the desired basis.

MAIN THEOREM (Jordan Block Form)

$A: V \to V$

$P_A(t) = (t-\lambda_1)^{m_1} \cdots (t-\lambda_k)^{m_k}$

Then $\exists$ basis $\mathcal{X}$ s.t.

$$[A \mid \mathcal{X}] = \begin{pmatrix} J_{\lambda_1}(m_{11}) & & & \\ & \ddots & & \\ & & J_{\lambda_k} & \\ & & & \ddots \end{pmatrix}$$

Tek tek indirgenir.

- $J_{\lambda_i}(m_{ij}) = \underbrace{\begin{pmatrix} \lambda_i & 1 & & & 0 \\ & \ddots & 1 & & 0 \\ & & \ddots & 1 & 0 \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{pmatrix}}_{m_{ij} \times m_{ij}}$

$-\lambda_i$ yapınca nilpotent olur.

- $\sum_j m_{ij} = m_i$
- Then data $(\lambda_i, (m_{ij})_j^{r_2})$ determines A upto isomorphism. $\quad m_{ij+1} \leq m_{ij}$

$m \longleftrightarrow q$ (in the prev. theorem)