Homework 11

Partial Differential Equations, Spring 2023 — David Yang

Chapter 4.7, Example 4.28

Consider the problem

$$
\begin{gathered}
u_t - 3u_{xx} = 0, \;\; 0 < x < 1, \, t > 0, \\
u(0,t) = 2e^{-t}, \;\; u(1,t) = 1 \\
u(x,0) = x^2, \;\; 0 < x < 1.
\end{gathered}
$$

Complete the calculation (Solve the w PDE using the eigenfunction method).

Solution. Following Example 4.28, we homogenize the boundary condition by defining

$$
w(x,t) = u(x,t) - \left(2e^{-t} + \left(1 - 2e^{-t}\right) x\right).
$$

Then w solves the problem

$$
\begin{gathered}
w_t - 3u_{xx} = 2e^{-t}(1-x), \;\; 0 < x < 1, \, t > 0, \\
w(0,t) = w(1,t), \;\; t > 0 \\
w(x,0) = x^2 + x, \;\; 0 < x < 1.
\end{gathered}
$$

■

Chapter 4.7, Exercise 7

Solve twice and check your answers match:

(a) **Method 1: Apply the eigenfunction method directly to the non-homogeneous PDE for u.**

(b) **Method 2: Observe that the source term is time-independent. Convert the PDE to a homogeneous PDE for $w = u - u_{ss}$ where u_{ss} is the steady state solution to the PDE. (see Remark 4.29 on page 212). Solve the homogeneous PDE for w and recover u as $u = w + u_{ss}$.**

Chapter 4.2, Exercise 5

For the SLP (Sturm-Liouville Problem)

$$-y'' = \lambda y, \;\; 0 < x < l; \;\; y(0) - ay'(0) = 0, \;\; y(l) + by'(l) = 0,$$

show that $\lambda = 0$ is an eigenvalue if and only if $a + b = -l$.

Solution. If $\lambda = 0$ is an eigenvalue, we have that $-y''(x) = 0$, so $y''(x) = 0$ and

$$y(x) = C_1 x + C_2.$$

Furthermore, plugging in $x = 0$, we find that $y(0) = C_2$ and $y'(0) = C_1$. Plugging these into the first boundary condition, we get that

$$C_2 - aC_1 = 0.$$

Similarly, plugging in $x = l$, we find that $y(l) = C_1 l + C_2$ and $y'(l) = C_1$. Plugging these into the second boundary condition, we get that

$$C_1 l + C_2 + bC_1 = (b + l)C_1 + C_2 = 0.$$

We are left with the system of equations

$$\begin{cases} -aC_1 + C_2 = 0 \\ (b + l)C_1 + C_2 = 0 \end{cases}.$$

Solving for C_1 by subtracting the two equations, we find that

$$C_1(-a - b - l) = 0.$$

Since we must have that C_1 and C_2 are not both 0, we know that the SLP has eigenvalue 0 if and only if $-a - b - l = 0$, or when $a + b = -l$, as desired. ∎

Chapter 4.2, Exercise 9

Find the eigenvalues and eigenfunctions for the following problem with *periodic* boundary conditions:

$$-y''(x) = \lambda y(x), \;\; 0 < x < l,$$
$$y(0) = y(l), y'(0) = y'(l).$$

Solution. We will split our work into three cases: when $\lambda = 0, \lambda < 0$, and $\lambda > 0$.

First, if $\lambda = 0$, then we find that $y''(x) = 0$, so $y = ax + b$ for constants a, b. However, if $y(0) = y(l)$, then we must have that $a = 0$. There are no further restrictions on the constant b, so our boundary conditions tell us that the eigenvalue $\boxed{\lambda = 0 \text{ corresponds to a constant eigenfunction}}$.

Next, if $\lambda < 0$, then $y''(x) + \lambda y(x) = 0$ has solution

$$y(x) = ae^{-\sqrt{\lambda}x} + be^{\sqrt{\lambda}x}.$$

As we've shown before, exponential solutions cannot satisfy periodic boundary conditions, and so we have a trivial solution in this case.

Finally, we consider the case when $\lambda > 0$. The ODE $y''(x) + \lambda y(x) = 0$ will then have solution

$$y(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x).$$

The boundary condition $y(0) = y(l)$ tells us that

$$b = a\sin(\sqrt{\lambda}l) + b\cos(\sqrt{\lambda}l)$$

and the boundary condition $y'(0) = y'(l)$ tells us that

$$a\sqrt{\lambda} = a\sqrt{\lambda}\cos(\sqrt{\lambda}l) - b\sqrt{\lambda}\sin(\sqrt{\lambda}l)$$

Thus, after simplification, our boundary conditions give us the following sytem of equations:

$$\begin{cases} a\sin(\sqrt{\lambda}l) + b(\cos(\sqrt{\lambda}l) - 1) = 0 \\ a\cos(\sqrt{\lambda}l - 1) - b(\sin(\sqrt{\lambda}l)) = 0 \end{cases}.$$

Rewriting this system as a matrix expression, we have that

$$\begin{bmatrix} \sin(\sqrt{\lambda}l) & \cos(\sqrt{\lambda}l) - 1 \\ \cos(\sqrt{\lambda}l) - 1 & -\sin(\sqrt{\lambda}l) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This system only has a nontrivial eigenfunction if a and b are not both 0. Equivalently, we must have that

$$\det\left(\begin{bmatrix} \sin(\sqrt{\lambda}l) & \cos(\sqrt{\lambda}l) - 1 \\ \cos(\sqrt{\lambda}l) - 1 & -\sin(\sqrt{\lambda}l) \end{bmatrix}\right) = 2\cos(\sqrt{\lambda}l) = 0.$$

Since $\cos(\lambda l) = 0$, we must have that $\lambda l = 2\pi n$ for integer n, and so we have that the eigenvalues

$$\boxed{\lambda = \left(\frac{2\pi n}{l}\right)^2}$$

correspond to eigenfunctions

$$\boxed{y(x) = a_n \sin\left(\frac{2\pi n}{l}x\right) + b_n \cos\left(\frac{2\pi n}{l}x\right)}.$$

in the given problem. ■

Chapter 4.4, Exercise 1