Homework 7

Partial Differential Equations, Spring 2023 David Yang

Chapter 2.6, Problem 7

Solve $u_t = u_{xx}$ on $x, t > 0$ with $u(0,t) = a$, $t > 0$ and $u(x,0) = b$, where a and b are constants.

Solution. We begin by taking the Laplace transform of the PDE $u_t = u_x x$. This gives us $\mathcal{L}(u_t) = \mathcal{L}(u_{xx})$. Equivalently, we have that

$$s\mathcal{L}[u(x,s)] - u(x,0) = \mathcal{L}[u_{xx}],$$

or

$$sU(x,s) - b = U_{xx}(x,s).$$

We now have a simple second-order linear ODE that we can solve: $U_{xx} - sU = -b$. To find the general solution for this ODE, we will add the solution to the homogeneous ODE with a particular solution.

We can first solve the homogeneous ODE. Note that the characteristic polynomial for the homogeneous ODE $U_{xx} - sU = 0$ is $r^2 - s = 0$, with solutions $r = \pm\sqrt{s}$. This tells us that the homogeneous solution is

$$U(x,s) = a(s)e^{-x\sqrt{s}} + b(s)e^{-x\sqrt{s}}.$$

However, since we want U to be bounded, we also know that $b(s) = 0$. Thus,

$$U(x,s) = a(s)e^{-x\sqrt{s}}$$

is our bounded solution for the homogeneous ODE.

We can now notice that $U(x,s) = \frac{b}{s}$ is a particular solution to the ODE, as $U_{xx} = 0$ and $-sU = -b$, so $U_{xx} - sU = -b$.

Thus, combining our homogeneous and particular solutions gives us our general solution:

$$U(x,s) = a(s)e^{-x\sqrt{s}} + \frac{b}{s}.$$

We can now apply our boundary condition $u(0,t) = a$. This condition tells us that

$$U(0,s) = \mathcal{L}(a) = \frac{a}{s}.$$

Plugging in $x = 0$ into the general solution we found earlier, we have that $U(0,s) = a(s) + \frac{b}{s}$. Thus, we must have that

$$a(s) + \frac{b}{s} = \frac{a}{s}$$

and so we know that

$$a(s) = \frac{a-b}{s}.$$

Plugging this back into our general equation gives us the solution in the transform domain:

$$\begin{aligned} U(x,s) &= a(s)e^{-x\sqrt{s}} + \frac{b}{s} \\ &= \frac{a-b}{s}e^{-x\sqrt{s}} + \frac{b}{s}. \end{aligned}$$

Applying the inverse Laplace transform to this solution gives us the solution to our original problem:

$$\boxed{u(x,t) = (a-b)\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right) + b}.$$

$\blacksquare$

Chapter 2.7, Problem 15

Solve the Cauchy problem for the advection-diffusion equation using Fourier transforms:

$$u_t = Du_{xx} - cu_x, \;\; x \in \mathbb{R}, t > 0; \;\; u(x,0) = \phi(x), x \in \mathbb{R}.$$

We begin by taking the Fourier transform of the PDE $u_t = Du_{xx} - cu_x$. Applying the rule

$$\mathcal{F}\left[\frac{\partial^k u}{\partial x^k}\right] = (-i\xi)^k \hat{u},$$

we get that

$$\begin{aligned}\hat{u}_t &= D(-i\xi)^2 \hat{u} - c(-i\xi)\hat{u} \\ &= (-D\xi^2 + ci\xi)\hat{u}.\end{aligned}$$

Solving this ODE, we get that

$$\hat{u}(\xi, t) = A(\xi)e^{(-D\xi^2 + ci\xi)t}.$$

We are given the initial value $u(x,0) = \phi(x)$. Thus, we should have that $\hat{u}(\xi, 0) = \hat{\phi}(\xi)$.