
Homework 6

Partial Differential Equations, Spring 2023

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HW 6 Problems

1. Find a solution to the diffusion PDE

$$u_t - u_{xx} = 0 \text{ for } x \in \mathbb{R}, t > 0$$

with initial value

$$u(x, 0) = e^{-x^2/4} \text{ for } x \in \mathbb{R}.$$

Solution. We will use the solution formula for the diffusion equation on an unbounded domain. Recall that the solution to the diffusion PDE $u_t - ku_{xx} = 0$ on the unbounded domain with initial value $u(x, 0) = \phi(x)$ is

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-(x-y)^2/(4kt)} dy.$$

Applying the solution formula to this problem (where $k = 1$ and $\phi(x) = e^{-x^2/4}$), we have that the solution to the given diffusion PDE on the unbounded domain is

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-y^2/4} e^{-(x-y)^2/(4t)} dy.$$

Simplifying the integrand by combining the exponential terms, we get that

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-ty^2/(4t)} e^{-(x-y)^2/(4t)} dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2 - ty^2}{4t}} dy \end{aligned}$$

Expanding the numerator and then completing the square, we get that

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2 - ty^2}{4t}} dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x^2 - 2xy + (t+1)y^2)}{4t}} dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(t+1)\left(y - \frac{x}{t+1}\right)^2 + x^2 \cdot \frac{t}{t+1}}{4t}} dy. \end{aligned}$$

Factoring out a $e^{-\frac{x^2 \cdot \frac{t}{t+1}}{4t}} = e^{-\frac{x^2}{4(t+1)}}$ term from the integrand, we get that

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4(t+1)}} \int_{-\infty}^{\infty} e^{-\frac{(t+1)\left(y - \frac{x}{t+1}\right)^2}{4t}} dy$$

We will now make a substitution to transform the integrand into e^{-r^2} : let

$$r = \frac{\left(y - \frac{x}{t+1}\right) \sqrt{t+1}}{\sqrt{4t}}.$$

Then we also have that

$$dr = \frac{\sqrt{t+1}}{\sqrt{4t}} dy.$$

Making the substitution for r in our solution $u(x, t)$, we find that

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4(t+1)}} \int_{y=-\infty}^{y=\infty} e^{-\frac{(t+1)\left(y - \frac{x}{t+1}\right)^2}{4t}} dy \\ &= \frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}} \int_{r=-\infty}^{r=\infty} e^{-r^2} dr. \end{aligned}$$

But we also know that $\int_{-\infty}^{\infty} e^{-r^2} dr = \sqrt{\pi}$, so plugging this back into our solution, we find that

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}} \int_{r=-\infty}^{r=\infty} e^{-r^2} dr \\ &= \left(\frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}} \right) \cdot \sqrt{\pi} \\ &= \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}}. \end{aligned}$$

Thus, our solution to the given diffusion PDE with the given initial value is

$$\boxed{u(x, t) = \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}}.}$$

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2. Show that your solution to # 1 satisfies the property that, for all $t > 0$,

$$\int_{-\infty}^{\infty} u(x, t) dx = \int_{-\infty}^{\infty} u(x, 0) dx.$$

In other words, $\int_{-\infty}^{\infty} u(x, t) dx$ is a **conserved quantity** (constant with respect to t).

Solution. To show that this property holds, we will show that the left and right hand sides simplify to the same expression. We will begin by working with the left-hand side.

Plugging in our solution for $u(x, t)$ when $t > 0$ into the left-hand side of the equation, we have that

$$\int_{-\infty}^{\infty} u(x, t) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}} dx.$$

Let us make the substitution $y = \frac{x}{\sqrt{4(t+1)}}$. We also have that $dy = \frac{1}{\sqrt{4(t+1)}} dx$. Substituting this into our integral, we get that

$$\begin{aligned} \int_{-\infty}^{\infty} u(x, t) dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}} dx \\ &= 2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{4(t+1)}} e^{-\frac{x^2}{4(t+1)}} dx \\ &= 2 \int_{-\infty}^{\infty} e^{-y^2} dy. \end{aligned}$$

We will now show that the right-hand side $\int_{-\infty}^{\infty} u(x, 0) dx$ can be rewritten in the same form. Plugging in the initial condition $u(x, 0) = e^{-x^2/4}$ into this integral, we get

$$\int_{-\infty}^{\infty} u(x, 0) dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{4}} dx.$$

Let us now make the substitution $y = \frac{x}{2}$. This also gives us $dy = \frac{1}{2} dx$. Substituting this back into our integral, we get

$$\begin{aligned} \int_{-\infty}^{\infty} u(x, 0) dx &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{4}} dx \\ &= 2 \int_{-\infty}^{\infty} \frac{1}{2} e^{-\frac{x^2}{4}} dx \\ &= 2 \int_{-\infty}^{\infty} e^{-y^2} dy. \end{aligned}$$

Thus, we see that

$$\int_{-\infty}^{\infty} u(x, t) dx = \int_{-\infty}^{\infty} u(x, 0) dx = 2 \int_{-\infty}^{\infty} e^{-y^2} dy = 2\sqrt{\pi}.$$

and so we find that $\int_{-\infty}^{\infty} u(x, t) dx$ is a *conserved quantity* (constant with respect to t), with $\int_{-\infty}^{\infty} u(x, t) dx = 2\sqrt{\pi}$ for all $t \geq 0$, as desired. ■

3. (a) **If u solves the diffusion equation on the infinite domain ($x \in \mathbb{R}$), with bounded initial value $u(x, 0) = \phi(x)$ that has the property that**

$$\lim_{x \rightarrow -\infty} \phi(x) = a \text{ and } \lim_{x \rightarrow \infty} \phi(x) = b \quad (\mathbf{a, b \text{ constants}}).$$

What is the value of $\lim_{t \rightarrow \infty} u(x, t)$?

Solution. u is a solution to the diffusion equation on the infinite domain with bounded initial value $u(x, 0) = \phi(x)$. Consequently, the solution (in the Poisson Integral representation form) is

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} \phi(x - r\sqrt{4kt}) dr.$$

Splitting the integral up into two separate integrals, we get that

$$u(x, t) = \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^0 e^{-r^2} \phi(x - r\sqrt{4kt}) dr + \int_0^{\infty} e^{-r^2} \phi(x - r\sqrt{4kt}) dr \right]$$

Thus,

$$\lim_{t \rightarrow \infty} u(x, t) = \lim_{t \rightarrow \infty} \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^0 e^{-r^2} \phi(x - r\sqrt{4kt}) dr + \int_0^{\infty} e^{-r^2} \phi(x - r\sqrt{4kt}) dr \right].$$

Per the given hint, we can pass the limit with respect to t inside the integral, so we get that

$$\lim_{t \rightarrow \infty} u(x, t) = \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^0 \lim_{t \rightarrow \infty} e^{-r^2} \phi(x - r\sqrt{4kt}) dr + \int_0^{\infty} \lim_{t \rightarrow \infty} e^{-r^2} \phi(x - r\sqrt{4kt}) dr \right].$$

We can evaluate each of the limits separately: note that when $r < 0$ (corresponding to the first integral), we have that

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-r^2} \phi(x - r\sqrt{4kt}) dr &= e^{-r^2} \lim_{t \rightarrow \infty} \phi(x - r\sqrt{4kt}) \\ &= e^{-r^2} \lim_{y \rightarrow \infty} \phi(y) \\ &= e^{-r^2} b. \end{aligned}$$

Similarly, we have that when $r > 0$ (corresponding to the second integral),

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-r^2} \phi(x - r\sqrt{4kt}) dr &= e^{-r^2} \lim_{t \rightarrow \infty} \phi(x - r\sqrt{4kt}) \\ &= e^{-r^2} \lim_{y \rightarrow -\infty} \phi(y) \\ &= e^{-r^2} a. \end{aligned}$$

Plugging these results back into our equation, we find that

$$\begin{aligned} \lim_{t \rightarrow \infty} u(x, t) &= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^0 \lim_{t \rightarrow \infty} e^{-r^2} \phi(x - r\sqrt{4kt}) dr + \int_0^{\infty} \lim_{t \rightarrow \infty} e^{-r^2} \phi(x - r\sqrt{4kt}) dr \right] \\ &= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^0 e^{-r^2} b dr + \int_0^{\infty} e^{-r^2} a dr \right] \end{aligned}$$

Factoring out the constants a and b and using the fact that $\int_{-\infty}^0 e^{-r^2} dr = \int_0^{\infty} e^{-r^2} dr = \frac{\sqrt{\pi}}{2}$, we have that

$$\begin{aligned}\lim_{t \rightarrow \infty} u(x, t) &= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^0 e^{-r^2} b dr + \int_0^{\infty} e^{-r^2} a dr \right] \\ &= \frac{1}{\sqrt{\pi}} \left[b \int_{-\infty}^0 e^{-r^2} dr + a \int_0^{\infty} e^{-r^2} dr \right] \\ &= \frac{1}{\sqrt{\pi}} \left[b \frac{\sqrt{\pi}}{2} + a \frac{\sqrt{\pi}}{2} \right] \\ &= \frac{1}{2} (a + b).\end{aligned}$$

Thus, we have that

$$\boxed{\lim_{t \rightarrow \infty} u(x, t) = \frac{1}{2} (a + b)}.$$

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(b) **Review Eq 2.5 on page 82 of Logan, which is a solution for the PDE**

$$w_t = k w_{xx} \text{ for } x \in \mathbb{R}, t > 0$$

$$w(x, 0) = 0 \text{ for } x < 0; w(x, 0) = 1 \text{ for } x > 0.$$

What is $\lim_{t \rightarrow \infty} w(x, t)$ and does this agree with your result in 3(a)?

Solution. Eq 2.5 on page 82 of Logan gives us a solution to the given PDE:

$$w(x, t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/(\sqrt{4kt})} e^{-r^2} dr.$$

Taking the limit of $w(x, t)$ as t approaches ∞ , we find that

$$\lim_{t \rightarrow \infty} w(x, t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \lim_{t \rightarrow \infty} \int_0^{x/(\sqrt{4kt})} e^{-r^2} dr.$$

Since $\lim_{t \rightarrow \infty} \frac{x}{\sqrt{4kt}} = 0$, we have that

$$\begin{aligned}\lim_{t \rightarrow \infty} w(x, t) &= \frac{1}{2} + \frac{1}{\sqrt{\pi}} \lim_{t \rightarrow \infty} \int_0^{x/(\sqrt{4kt})} e^{-r^2} dr \\ &= \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^0 e^{-r^2} dr \\ &= \frac{1}{2} + \frac{1}{\sqrt{\pi}} \cdot 0 \\ &= \frac{1}{2}.\end{aligned}$$

Since $\lim_{x \rightarrow -\infty} w(x, 0) = 0$ and $\lim_{x \rightarrow \infty} w(x, 0) = 1$, and

$$\lim_{t \rightarrow \infty} w(x, t) = \frac{1}{2} = \frac{1}{2} (0 + 1),$$

we see that our answer does match up with our result from 3(a), as desired.

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