Homework 11

Partial Differential Equations, Spring 2023 David Yang

Chapter 4.7, Example 4.28

Consider the problem

$$u_t - 3u_{xx} = 0, \;\; 0 < x < 1,\, t > 0,$$
$$u(0,t) = 2e^{-t}, \;\; u(1,t) = 1$$
$$u(x,0) = x^2, \;\; 0 < x < 1.$$

Complete the calculation (Solve the w PDE using the eigenfunction method).

Solution. Following Example 4.28, we homogenize the boundary condition by defining

$$w(x,t) = u(x,t) - \left(2e^{-t} + \left(1 - 2e^{-t}\right) x\right).$$

Then w solves the problem

$$w_t - 3u_{xx} = 2e^{-t}(1-x), \;\; 0 < x < 1,\, t > 0,$$

$$w(0,t) = w(1,t), \;\; t > 0$$
$$w(x,0) = x^2 + x, \;\; 0 < x < 1.$$

To solve this PDE using the eigenfunction, we begin by determining the eigenfunctions for the homogeneous PDE

$$w_t - 3w_{xx} = 0$$
$$w(0,t) = w(1,t) = 0.$$

We will use the separation of variables method and let $w = X(x)T(t)$. Plugging this into our PDE and simplifying to the desired form, we get

$$\frac{X_{xx}}{X} = \frac{T_t}{3T} = -\lambda.$$

Solving the X ODE, we find that $\lambda = 0$ and $\lambda < 0$ yield trivial solutions, with $\lambda > 0$ giving the solution

$$X = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x).$$

Plugging in the initial conditions $X(0) = X(1) = 0$, we find that

$$C_2 \sin(\sqrt{\lambda}) = 0,$$

meaning that $\sqrt{\lambda} = \pi n$ for integer n.

Thus, we find that the eigenfunction $X_n = \sin(\pi n x)$ corresponding to eigenvalues $\lambda_n = (\pi n)^2$.

We now have the eigenfunctions to the homogeneous PDE. Defining our solution $w(x,t)$ with the eigenfunctions, we have

$$w(x,t) = \sum_{n=0}^{\infty} C_n(t) \sin(\pi n x)$$

Furthermore, our initial source term $f(x,t) = 2e^{-t}(1-x)$ can also be expressed with the eigenfunctions:

$$f(x,t) = 2e^{t}(1-x) = \sum_{n=1}^{\infty} f_n(t) \sin(\pi n x)$$

with $f_n(t) = 2\int_0^1 f(x,t)\sin(\pi n x)$.

We can do the same for the initial condition $w(x,0) = g(x) = x^2 + x$ where

$$g(x) = \sum_{n=1}^{\infty} g_n(x) \sin(\pi n x)$$

with $g_n(t) = 2\int_0^1 g(x)\sin(\pi n x)$.
Plugging in our work into the initial PDE $w_t - 3w_{xx} = 2e^t(1-x) = f(x)$, we get that

$$\sum_{n=1}^{\infty} C_n'(t)\sin(\pi n x) + 3C_n(n\pi)^2 \sin(\pi n x) = \sum_{n=1}^{\infty} f_n(t)\sin(\pi n x).$$

Matching both sides term-by-term, we arrive at the ODE

$$C_n'(t) + 3(n\pi)^2 C_n = f_n(t).$$

Solving this ODE using integrating factors, we get the solution

$$C_n(t) = C_n(0)e^{-3n^2\pi^2 t} + \int_0^t f_n(\tau)e^{-3n^2\pi^2(t-\tau)}d\tau.$$

Note that $w(x,0) = \sum\limits_{n=1}^{\infty} C_n(0)\sin(n\pi x)$, so we know $C_n(0) = g_n = 2\int_0^1 g(x)\sin(\pi n x)\,dx$ by definition. Thus, we have a formal expression for C_n, our Fourier sine coefficients for w.

Thus, our solution to the w PDE is

$$\begin{aligned}
w(x,t) &= \sum_{n=1}^{\infty} C_n(t)\sin(n\pi x) \\
&= \sum_{n=1}^{\infty} C_n(0)e^{-3n^2\pi^2 t} + \int_0^t f_n(\tau)e^{-3n^2\pi^2(t-\tau)}d\tau \\
&= \sum_{n=1}^{\infty} \left[\left(2\int_0^1 g(x)\sin(\pi n x)\,dx\right)\cdot e^{-3n^2\pi^2 t} + \int_0^t f_n(\tau)e^{-3n^2\pi^2(t-\tau)}d\tau\right]\sin(n\pi x).
\end{aligned}$$

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Chapter 4.7, Exercise 7

Solve twice and check your answers match:

$$u_t = ku_{xx} + \sin(3\pi x), \;\; 0 < x < 1, t > 0$$
$$u(0,t) = u(1,t) = 0 \;\; t > 0$$
$$u(x,0) = \sin(\pi x) \;\; 0 < x < 1$$

(a) **Method 1: Apply the eigenfunction method directly to the non-homogeneous PDE for u.**

Solution. We first solve the homogeneous PDE to determine the eigenfunctions. The homogeneous PDE is $u_t - ku_{xx} = 0$, which has eigenfunctions $\sin(n\pi x)$ corresponding to eigenvalues $\lambda = (n\pi)^2$.

Thus, we can construct our solution as

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin(n\pi x),$$

where we will determine our $C_n(t)$ coefficients from the ODE and its initial values.

We can also expand the initial condition $u_t - ku_{xx} = \sin(3\pi x) = f(x)$ using our eigenfunctions:

$$f(x) = \sin(3\pi x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x).$$

We see that $f_3 = 1$ and $f_n = 0$ otherwise. Similarly, we expand our initial condition $u(x,0) = \sin(\pi x) = g(x)$ using our eigenvalues:

$$g(x) = \sin(\pi x) = \sum_{n=1}^{\infty} g_n \sin(n\pi x).$$

We find that $g_1 = 1$ and $g_n = 0$ otherwise.

We can now plug in our constructed solution $u(x,t) = \sum\limits_{n=1}^{\infty} C_n(t)\sin(n\pi x)$ into the original PDE $u_t - ku_{xx} = f(x) = \sin(3\pi x)$. This gives us

$$\sum_{n=1}^{\infty} \left[C_n'(t) + k(n\pi)^2 C_n(t)\right]\sin(n\pi x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x).$$

Matching term-by-term and focusing on the coefficients, we get the ODE

$$C_n'(t) + k(n\pi)^2 C_n(t) = f_n.$$

Solving this ODE using integrating factors, we get that

$$C_n(t) = C_n(0)e^{-kn^2\pi^2 t} + \int_0^t f_n(\tau)e^{-kn^2\pi^2(t-\tau)}d\tau.$$

Using the initial condition $u(x,0) = sin(\pi x)$, we have that

$$u(x,0) = \sum_{n=1}^{\infty} C_n(0)\sin(n\pi x) = \sin(\pi x).$$

This tells us that $C_1(0) = 1$ and $C_n(0) = 0$ otherwise. Combining all of our work together by plugging it back into $u(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin(n\pi x)$, we get that

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} C_n(t)\sin(n\pi x) \\ &= e^{-k\pi^2 t}\sin(\pi x) + \left(\int_0^t f_3(\tau)e^{-k(3)^2\pi^2(t-\tau)}d\tau\right)\sin(3\pi x) \\ &= e^{-k\pi^2 t}\sin(\pi x) + \left(\int_0^t e^{-9k\pi^2(t-\tau)}d\tau\right)\sin(3\pi x) \end{aligned}$$

Simplifying further, we arrive at the solution

$$\boxed{u(x,t) = e^{-k\pi^2 t}\sin(\pi x) + \frac{1}{9\pi^2 k}\left(1 - e^{-9k\pi^2 t}\right)\sin(3\pi x)}.$$

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(b) **Method 2: Observe that the source term is time-independent. Convert the PDE to a homogeneous PDE for $w = u - u_{ss}$ where u_{ss} is the steady state solution to the PDE. (see Remark 4.29 on page 212). Solve the homogeneous PDE for w and recover u as $u = w + u_{ss}$.**

Solution. We follow the steps laid out in Remark 4.29. Remark 4.29 tells us that the equilibrium solution must satisfy

$$-kU'' = \sin(3\pi x), \;\; U(0) = U(1) = 0.$$

Solving using separation of variables, we get that

$$U(x) = \frac{1}{9\pi^2 k}\sin(3\pi x).$$

Now, we define $w(x,t) = u(x,t) - U(x) = \sin(\pi x) - \frac{1}{9\pi^2 k}\sin(3\pi x)$. Then w solves the PDE

$$\begin{gathered} w_t - kw_{xx} = 0 \\ w(0,t) = w(1,t) = 0 \\ w(x,0) = \sin(\pi x) - \tfrac{1}{9\pi^2 k}\sin(3\pi x) \end{gathered}$$

We can now solve this using the separation of variables method. We define $w = X(x)T(t)$. As we know, the eigenfunctions for this system are $X_n = \sin(n\pi x)$ corresponding to eigenvalues $\lambda = (n\pi)^2$.

Similarly, solving the T ODE which we get from substituting into the PDE, we get that $T = e^{-\lambda t} = e^{-(n\pi)^2 t}$.

Using linear superposition, we express our solution as

$$w(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

for coefficients a_n. Using the initial condition $w(x,0)$, we should have that

$$w(x,0) = \sum_{n=1}^{\infty} C_n \sin(n\pi x) = \sin(\pi x) - \frac{1}{9\pi^2 k} \sin(3\pi x).$$

Matching the coefficients, we get that $C_1 = 1$ and $C_3 = -\frac{1}{9\pi^2 k}$, with all other coefficients 0. Thus, we know that

$$\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty} C_n \sin(n\pi x) \\ &= \sin(\pi x) e^{-k\pi^2 t} - \frac{1}{9\pi^2 k} \sin(3\pi x) e^{-9k\pi^2 t}. \end{aligned}$$

Since our original solution is $u(x,t) = w(x,t) + U(x)$ with $U(x) = \frac{1}{9\pi^2 k} \sin(3\pi x)$, we have that

$$u(x,t) = \sin(\pi x) e^{-k\pi^2 t} - \frac{1}{9\pi^2 k} \sin(3\pi x) e^{-9k\pi^2 t} + \frac{1}{9\pi^2 k} \sin(3\pi x).$$

Rearranging and grouping terms, we get the same solution as part (a):

$$\boxed{u(x,t) = e^{-k\pi^2 t} \sin(\pi x) + \frac{1}{9\pi^2 k} \left(1 - e^{-9k\pi^2 t}\right) \sin(3\pi x)}.$$

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Chapter 4.2, Exercise 5

For the SLP (Sturm-Liouville Problem)

$$-y'' = \lambda y, \;\; 0 < x < l; \;\; y(0) - ay'(0) = 0, \;\; y(l) + by'(l) = 0,$$

show that $\lambda = 0$ is an eigenvalue if and only if $a + b = -l$.

Solution. If $\lambda = 0$ is an eigenvalue, we have that $-y''(x) = 0$, so $y''(x) = 0$ and

$$y(x) = C_1 x + C_2.$$

Furthermore, plugging in $x = 0$, we find that $y(0) = C_2$ and $y'(0) = C_1$. Plugging these into the first boundary condition, we get that

$$C_2 - aC_1 = 0.$$

Similarly, plugging in $x = l$, we find that $y(l) = C_1 l + C_2$ and $y'(l) = C_1$. Plugging these into the second boundary condition, we get that

$$C_1 l + C_2 + bC_1 = (b + l)C_1 + C_2 = 0.$$

We are left with the system of equations

$$\begin{cases} -aC_1 + C_2 = 0 \\ (b + l)C_1 + C_2 = 0 \end{cases}.$$

Solving for C_1 by subtracting the two equations, we find that

$$C_1(-a - b - l) = 0.$$

Since we must have that C_1 and C_2 are not both 0, we know that the SLP has eigenvalue 0 if and only if $-a - b - l = 0$, or when $a + b = -l$, as desired. ■

Chapter 4.2, Exercise 9

Find the eigenvalues and eigenfunctions for the following problem with *periodic* boundary conditions:

$$-y''(x) = \lambda y(x), \;\; 0 < x < l,$$
$$y(0) = y(l), y'(0) = y'(l).$$

Solution. We will split our work into three cases: when $\lambda = 0, \lambda < 0$, and $\lambda > 0$.

First, if $\lambda = 0$, then we find that $y''(x) = 0$, so $y = ax + b$ for constants a, b. However, if $y(0) = y(l)$, then we must have that $a = 0$. There are no further restrictions on the constant b, so our boundary conditions tell us that the eigenvalue $\boxed{\lambda = 0 \text{ corresponds to a constant eigenfunction}}$.

Next, if $\lambda < 0$, then $y''(x) + \lambda y(x) = 0$ has solution

$$y(x) = ae^{-\sqrt{\lambda}x} + be^{\sqrt{\lambda}x}.$$

As we've shown before, exponential solutions cannot satisfy periodic boundary conditions, and so we have a trivial solution in this case.

Finally, we consider the case when $\lambda > 0$. The ODE $y''(x) + \lambda y(x) = 0$ will then have solution

$$y(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x).$$

The boundary condition $y(0) = y(l)$ tells us that

$$b = a\sin(\sqrt{\lambda}l) + b\cos(\sqrt{\lambda}l)$$

and the boundary condition $y'(0) = y'(l)$ tells us that

$$a\sqrt{\lambda} = a\sqrt{\lambda}\cos(\sqrt{\lambda}l) - b\sqrt{\lambda}\sin(\sqrt{\lambda}l)$$

Thus, after simplification, our boundary conditions give us the following sytem of equations:

$$\begin{cases} a\sin(\sqrt{\lambda}l) + b(\cos(\sqrt{\lambda}l) - 1) = 0 \\ a\cos(\sqrt{\lambda}l - 1) - b(\sin(\sqrt{\lambda}l)) = 0 \end{cases}.$$

Rewriting this system as a matrix expression, we have that

$$\begin{bmatrix} \sin(\sqrt{\lambda}l) & \cos(\sqrt{\lambda}l) - 1 \\ \cos(\sqrt{\lambda}l) - 1 & -\sin(\sqrt{\lambda}l) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This system only has a nontrivial eigenfunction if a and b are not both 0. Equivalently, we must have that

$$\det\left(\begin{bmatrix} \sin(\sqrt{\lambda}l) & \cos(\sqrt{\lambda}l) - 1 \\ \cos(\sqrt{\lambda}l) - 1 & -\sin(\sqrt{\lambda}l) \end{bmatrix}\right) = 2\cos(\sqrt{\lambda}l) = 0.$$

Since $\cos(\lambda l) = 0$, we must have that $\lambda l = 2\pi n$ for integer n, and so we have that the eigenvalues

$$\boxed{\lambda = \left(\frac{2\pi n}{l}\right)^2}$$

correspond to eigenfunctions

$$\boxed{y(x) = a_n \sin\left(\frac{2\pi n}{l} x\right) + b_n \cos\left(\frac{2\pi n}{l} x\right)}.$$

in the given problem. ■

Chapter 4.4, Exercise 1

Consider the pure boundary value problem for Laplace's equation given by

$$\begin{gathered} u_{xx} + u_{yy} = 0,\ \ 0 < x < l,\, 0 < y < 1 \\ u(0,y) = 0,\ \ u(l,y) = 0,\ \ 0 < y < 1 \\ u(x,0) = 0,\ \ u(x,1) = G(x),\ \ 0 < x < l. \end{gathered}$$

Use the separation of variables method to find an infinite series representation of the solution. Here, take $u(x,y) = \phi(x)\psi(y)$ and identify a boundary value problem for $\phi(x)$; proceed as in other separation of variables problems.

Solution. We will use the separation of variables method to find the solution to the pure boundary value problem. We take $u(x,y) = \phi(x)\psi(y)$. Plugging this into Laplace's equation, we get

$$\phi_{xx}\psi + \phi\psi_{yy} = 0.$$

Simplifying, we get that

$$\frac{\phi_{xx}}{\phi} = -\frac{\psi_{yy}}{\psi} = -\lambda.$$

We begin by solving the ϕ ODE: $-\phi'' = \lambda\phi$ with $x(0) = x(l) = 0$. The eigenfunctions of this ODE are $\phi_n = \sin\left(\frac{n\pi x}{l}\right)$ corresponding to eigenvalues $\lambda_n = \left(\frac{n\pi}{l}\right)^2$.

We will similarly solve the ψ ODE: $-\psi_{yy} + \lambda\psi = 0$, which has eigenfunctions

$$\begin{aligned} \psi_n(y) &= c_n e^{-\frac{n\pi}{y}l} + d_n e^{\frac{n\pi}{y}l} \\ &= a_n \cosh\left(\frac{n\pi y}{l}\right) + b_n \sinh\left(\frac{n\pi y}{l}\right). \end{aligned}$$

Thus, we can construct our solution $u(x,y) = \phi(x)\psi(y)$ using linear superposition:

$$u(x,y) = \sum_{n=1}^{\infty} \left[a_n \cosh\left(\frac{n\pi y}{l}\right) + b_n \sinh\left(\frac{n\pi y}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right).$$

We are given that $u(x,0) = 0$, so we know $a_n = 0$. On the other hand, $u(x,1) = G(x)$, so we have

$$u(x,1) = G(x) = \sum_{n=1}^{\infty} \left[b_n \sinh\left(\frac{n\pi y}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right).$$

Thus, we know that the $b_n \sinh\left(\frac{n\pi}{l}\right)$ coefficients correspond to Fourier sine coefficients, so

$$b_n = \frac{2}{l\sinh\left(\frac{n\pi}{l}\right)} \int_0^l G(x) \sin\left(\frac{n\pi x}{l}\right)\, dx.$$

This gives us the solution

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi y}{l}\right) \sin\left(\frac{n\pi y}{l}\right)}$$

with the b_n coefficients defined above.

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