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## Final Project Introduction Draft

Partial Differential Equations, Spring 2023

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Partial differential equations are used to explain all kinds of models in nature. In this paper we discuss a partial differential equation that models the evolution in time of an age-structured population. Our equation is useful for age-structured models, demographic models where the population at time  $t$  has an age distribution superimposed on it. In other words, at time a given time  $t$ , the ages of the individuals in a population are also considered. Age-structured models are related to general physiologically-structured models, where any other variable (such as size or weight) can replace the age variable. Consequently, the ideas we discuss in this paper, though focused on age, can also be extended to other physiological structures.

Per Wikipedia, the **McKendrick–von Foerster equation** was first presented by McKendrick in 1926 as a "deterministic limit of lattice models applied to epidemiology", and, independently in 1959, by biophysics professor Heinz von Foerster for describing cell cycles.

The general form of the **McKendrick–von Foerster equation** is

$$u_t = -u_a - m(a)u.$$

This equation is a form of the advection equation where  $m(a)$  represents the maternity rate at some age and  $u(a, t)$  represents the density at age  $a$  and time  $t$ .

This means we can use a certain age,  $a$ , and a time,  $t$ , and then compute the density of the population of that particular age at that time.

In our project, we will derive the general age-structured model on Logan, page 231:

$$\begin{aligned} u_t &= -u_a - m(a)u, \quad a > 0, t > 0 \\ u(0, t) &= \int_0^\infty b(a, t)u(a, t) da, \quad t > 0 \\ u(a, 0) &= f(a), \quad a \geq 0 \end{aligned}$$

where the newly introduced terms  $b(a, t)$  and  $f(a)$  represent the average reproduction rate and the population of female organisms, respectively.

We will then discuss the stable age structure to see what happens over a long time. Finally, we plan on presenting the renewal equation (a simplified version of the age-structured model above where  $b = b(a)$  and  $m = \text{constant}$ ), which we will solve using the method of characteristics.

In conclusion, we will discuss various extensions of the McKendrick-von Foerster equation and the importance it plays in helping scientists understand population models. Our main resources will be Chapter 5.1 of Logan and other related papers, such as this [one](#), or other presentations, such as this [one](#).