Homework 6

Partial Differential Equations, Spring 2023 David Yang

HW 6 Problems

1. **Find a solution to the diffusion PDE**

$$u_t - u_{xx} = 0 \text{ for } x \in \mathbb{R},\, t > 0$$

with initial value

$$u(x,0) = e^{-x^2/4} \text{ for } x \in \mathbb{R}.$$

Solution. We will use the solution formula for the diffusion equation on an unbounded domain. Recall that the solution to the diffusion PDE $u_t - ku_{xx} = 0$ on the unbounded domain with initial value $u(x,0) = \phi(x)$ is is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-(x-y)^2/(4kt)}\, dy.$$

Applying the solution formula to this problem (where $k = 1$ and $\phi(x) = e^{-x^2/4}$), we have that the solution to the given diffusion PDE on the unbounded domain is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-y^2/4} e^{-(x-y)^2/(4t)}\, dy.$$

Simplifying the integrand by combining the exponential terms, we get that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-ty^2/(4t)} e^{-(x-y)^2/(4t)}\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2 - ty^2}{4t}}\, dy \end{aligned}$$

Expanding the numerator and then completing the square, we get that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2 - ty^2}{4t}}\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x^2 - 2xy + (t+1)y^2)}{4t}}\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(t+1)\left(y - \frac{x}{t+1}\right)^2 + x^2 \cdot \frac{t}{t+1}}{4t}}\, dy. \end{aligned}$$

Factoring out a $e^{-\frac{x^2 \cdot \frac{t}{t+1}}{4t}} = e^{-\frac{x^2}{4(t+1)}}$ term from the integrand, we get that

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4(t+1)}} \int_{-\infty}^{\infty} e^{-\frac{(t+1)\left(y - \frac{x}{t+1}\right)^2}{4t}}\, dy$$

We will now make a substitution to transform the integrand into e^{-r^2}: let

$$r = \frac{\left(y - \frac{x}{t+1}\right)\sqrt{t+1}}{\sqrt{4t}}.$$

Then we also have that

$$dr = \frac{\sqrt{t+1}}{\sqrt{4t}}\,dy.$$

Making the substitution for r in our solution $u(x,t)$, we find that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4(t+1)}} \int_{y=-\infty}^{y=\infty} e^{-\frac{(t+1)\left(y-\frac{x}{t+1}\right)^2}{4t}}\,dy \\ &= \frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}} \int_{r=-\infty}^{r=\infty} e^{-r^2}\,dr. \end{aligned}$$

But we also know that $\int_{-\infty}^{\infty} e^{-r^2}\,dr = \sqrt{\pi}$, so plugging this back into our solution, we find that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}} \int_{r=-\infty}^{r=\infty} e^{-r^2}\,dr \\ &= \left(\frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}}\right) \cdot \sqrt{\pi} \\ &= \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}}. \end{aligned}$$

Thus, our solution to the given diffusion PDE with the given initial value is

$$\boxed{u(x,t) = \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}}.}$$

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2. **Show that your solution to # 1 satisfies the property that, for all $t > 0$,**

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u(x,0)\,dx.$$

In other words, $\int_{-\infty}^{\infty} u(x,t)\,dx$ is a *conserved quantity* (constant with respect to t).

3. (a) **If u solves the diffusion equation on the infinite domain ($x \in \mathbb{R}$), with bounded initial value $u(x,0) = \phi(x)$ that has the property that**

$$\lim_{x\to-\infty} \phi(x) = a \ \textbf{ and } \ \lim_{x\to\infty} \phi(x) = b \quad \textbf{(a, b constants)}.$$

What is the value of $\lim_{t\to\infty} u(x,t)$?

(b) **Review Eq 2.5 on page 82 of Logan, which is a solution for the PDE**

$$w_t = k w_{xx} \ \text{ for } x \in \mathbb{R},\ t > 0$$

$$w(x,0) = 0 \text{for } x < 0\,;\ w(x,0) = 1 \text{for } x > 0.$$

What is $\lim_{t\to\infty} w(x,t)$ and does this agree with your result in 3(a)?