Homework 7

Partial Differential Equations, Spring 2023 — David Yang

Chapter 2.6, Problem 7

Solve $u_t = u_{xx}$ on $x, t > 0$ with $u(0,t) = a$, $t > 0$ and $u(x,0) = b$, where a and b are constants.

Solution. We begin by taking the Laplace transform of the PDE $u_t = u_{xx}$. This gives us $\mathcal{L}(u_t) = \mathcal{L}(u_{xx})$. Equivalently, we have that

$$s\mathcal{L}[u(x,s)] - u(x,0) = \mathcal{L}[u_{xx}],$$

or

$$sU(x,s) - b = U_{xx}(x,s).$$

We now have a simple second-order linear ODE that we can solve: $U_{xx} - sU = -b$. To find the general solution for this ODE, we will add the solution to the homogeneous ODE with a particular solution.

We can first solve the homogeneous ODE. Note that the characteristic polynomial for the homogeneous ODE $U_{xx} - sU = 0$ is $r^2 - s = 0$, with solutions $r = \pm\sqrt{s}$. This tells us that the homogeneous solution is

$$U(x,s) = a(s)e^{-x\sqrt{s}} + b(s)e^{-x\sqrt{s}}.$$

However, since we want U to be bounded, we also know that $b(s) = 0$. Thus,

$$U(x,s) = a(s)e^{-x\sqrt{s}}$$

is our bounded solution for the homogeneous ODE.

We can now notice that $U(x,s) = \frac{b}{s}$ is a particular solution to the ODE, as $U_{xx} = 0$ and $-sU = -b$, so $U_{xx} - sU = -b$.

Thus, combining our homogeneous and particular solutions gives us our general solution:

$$U(x,s) = a(s)e^{-x\sqrt{s}} + \frac{b}{s}.$$

We can now apply our boundary condition $u(0,t) = a$. This condition tells us that

$$U(0,s) = \mathcal{L}(a) = \frac{a}{s}.$$

Plugging in $x = 0$ into the general solution we found earlier, we have that $U(0,s) = a(s) + \frac{b}{s}$. Thus, we must have that

$$a(s) + \frac{b}{s} = \frac{a}{s}$$

and so we know that

$$a(s) = \frac{a-b}{s}.$$

Plugging this back into our general equation gives us the solution in the transform domain:

$$\begin{aligned} U(x,s) &= a(s)e^{-x\sqrt{s}} + \frac{b}{s} \\ &= \frac{a-b}{s}e^{-x\sqrt{s}} + \frac{b}{s}. \end{aligned}$$

Applying the inverse Laplace transform to this solution gives us the solution to our original problem:

$$\boxed{u(x,t) = (a-b)\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right) + b}.$$

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Chapter 2.7, Problem 15

Solve the Cauchy problem for the advection-diffusion equation using Fourier transforms:

$$u_t = Du_{xx} - cu_x, \;\; x \in \mathbb{R}, \, t > 0; \;\; u(x,0) = \phi(x), x \in \mathbb{R}.$$

Solution. We begin by taking the Fourier transform of the PDE $u_t = Du_{xx} - cu_x$. Using the rule

$$\mathcal{F}\left[\frac{\partial^k u}{\partial x^k}\right] = (-i\xi)^k \hat{u},$$

we get that

$$\begin{aligned}\hat{u}_t &= D(-i\xi)^2\hat{u} - c(-i\xi)\hat{u}\\ &= (-D\xi^2 + ci\xi)\hat{u}.\end{aligned}$$

Solving this ODE, we get that

$$\hat{u}(\xi,t) = A(\xi)e^{(-D\xi^2+ci\xi)t}.$$

We are given the initial value $u(x,0) = \phi(x)$. Thus, we should have that $\hat{u}(\xi,0) = \hat{\phi}(\xi)$. Plugging this condition into our expression for $\hat{u}(\xi,t)$, we get that $A(\xi) = \hat{\phi}(\xi)$.

Thus, we know that

$$\hat{u}(\xi,t) = \hat{\phi}(\xi)e^{(-D\xi^2+ci\xi)t}.$$

Using the convolution property of Fourier transforms, we have that

$$u(x,t) = \mathcal{F}^{-1}\left(\hat{\phi}(\xi)\right) * \mathcal{F}^{-1}\left[e^{(-D\xi^2+ci\xi)t}\right].$$

By definition, $\mathcal{F}^{-1}\left(\hat{\phi}(\xi)\right) = \phi$, and we know from class that $\mathcal{F}^{-1}\left[e^{-D\xi^2 t}\right] = \sqrt{\frac{1}{4\pi Dt}}e^{-\frac{x^2}{4Dt}}$. Furthermore, formula 3(c) on page 123 of Logan tells us that

$$\mathcal{F}^{-1}\left[\hat{u}(\xi,t)e^{-ia\xi}\right] = u(x+a).$$

Applying formula 3(c) here, we get that

$$\mathcal{F}^{-1}\left[e^{(-D\xi^2+ci\xi)t}\right] = \mathcal{F}^{-1}\left[e^{-D\xi^2 t}e^{-i(-ct)\xi}\right] = \sqrt{\frac{1}{4\pi Dt}}e^{-\frac{(x-ct)^2}{4Dt}}.$$

Thus, going back to our result from the convolution property, we get that

$$\begin{aligned}u(x,t) &= \mathcal{F}^{-1}\left(\hat{\phi}(\xi)\right) * \mathcal{F}^{-1}\left[e^{(-D\xi^2+ci\xi)t}\right]\\ &= \phi(x) * \sqrt{\frac{1}{4\pi Dt}}e^{-\frac{(x-ct)^2}{4Dt}}\end{aligned}$$

In conclusion, the solution to the given Cauchy problem for the advection-diffusion equation is

$$\boxed{u(x,t) = \phi(x) * \sqrt{\frac{1}{4\pi Dt}}e^{-\frac{(x-ct)^2}{4Dt}}}.$$

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