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# 1 Chapter 1

## 1.2 Chapter 1, Section 2

**Exercise 7.** Solve the initial boundary value problem

$$\begin{aligned}u_t + cu_x &= \lambda u, \quad x, t > 0 \\u(x, 0) &= 0, \quad x > 0, \quad u(0, t) = g(t), \quad t > 0.\end{aligned}$$

*Solution.* We should find that  $\phi(t) = e^{-\lambda t/c} g(-t/c)$  (notice the negative in the exponent). This follows from the fact that if  $\lambda(-ct)e^{\lambda t} = g(t)$ , then substituting  $t = -t/c$  gives us the negative in the exponent. This gives us  $u(x, t) = g(t - x/c)e^{-\lambda x/c}$ , in  $0 \leq x < ct$ , which matches the solutions. ■

**Exercise 12.** Find a formula that implicitly defines the solution  $u = u(x, t)$  of the initial value problem for the reaction-advection equation

$$\begin{aligned}u_t + cu_x &= -\frac{\alpha u}{\beta + u}, \quad x \in \mathbb{R}, \quad t > 0, \\u(x, 0) &= f(x), \quad x \in \mathbb{R}\end{aligned}$$

Here,  $v, \alpha, \beta$  are positive constants. Show from the implicit formula that you can always solve for  $u$  in terms of  $x$  and  $t$ .

*Solution.*  $f(x)$  should be  $f(x - ct)$ , as we need to change back to  $x - t$  coordinates. ■

## 1.3 Chapter 1, Section 3

**Exercise 2.** Let  $u = u(x, t)$  satisfy the heat flow model

$$\begin{aligned}u_t &= ku_{xx}, \quad 0 < x < l, \quad t > 0 \\u(0, t) &= u(l, t) = 0, \quad t > 0, \\u(x, 0) &= u_0(x), \quad 0 \leq x \leq l.\end{aligned}$$

Show that

$$\int_0^l u(x, t)^2 dx \leq \int_0^l u_0(x)^2 dx, \quad t \geq 0.$$

*Hint:* Let  $E(t) = \int_0^l u(x, t)^2 dx$  and show that  $E'(t) \leq 0$ . What can be said about  $u(x, t)$  if  $u_0(x) = 0$ ?

*Solution.* We have that  $E(t) \leq E(0) = \int_0^l u_0(x)^2 dx$ . This does not affect the solution, though, as if  $u_0 \equiv 0$ , then  $E(0) = 0$  still. ■

**Exercise 6.** Heat flow in a metal rod with a unit internal heat source is governed by the problem

$$\begin{aligned}u_t &= ku_{xx} + 1, \quad 0 < x < 1, \quad t > 0, \\u(0, t) &= 0, \quad u(1, t) = 1, \quad t > 0.\end{aligned}$$

What will be the steady-state temperature in the bar after a long time? Does it matter that no initial condition is given?

*Solution.* The answer should be

$$u(x) = -\frac{1}{2k}x^2 + \left(1 + \frac{1}{2k}\right)x.$$

Originally, there is a  $=$  sign rather than a  $+$ . ■

## 1.5 Chapter 1, Section 5

**Exercise 5.** The total energy of the string governed by equation (1.37) with boundary conditions (1.40) is defined by

$$E(t) = \int_0^\ell \left( \frac{1}{2} \rho_0 u_t^2 + \frac{1}{2} \tau_0 u_x^2 \right) dx.$$

Show that the total energy is constant for all  $t \geq 0$ . Hint: Multiply (1.37) by  $u_t$  and note that  $(u_t^2)_t = 2u_t u_{tt}$  and  $(u_t u_x)_x = u_t u_{xx} + u_{tx} u_x$ . Then show that

$$\frac{d}{dt} \int_0^\ell \rho_0 u_t^2 dx = [2\tau_0 u_t u_x]_0^\ell - \frac{d}{dt} \int_0^\ell \tau_0 u_x^2 dx.$$

*Solution.* The hint should include an extra factor of 2 (colored in red). ■

**Exercise 8.** At the end ( $x = 0$ ) of a long tube ( $x \geq 0$ ) the density of air changes according to the formula  $\tilde{\rho}(0, t) = 1 - \cos 2t$  for  $t \geq 0$ , and  $\tilde{\rho}(0, t) = 0$  for  $t < 0$ . Find a solution to the wave equation in the domain  $x > 0, -\infty < t < \infty$ , in the form of a right-traveling wave that satisfies the given boundary condition. Take  $c = 1$  and plot the solution surface.

*Solution.* We should have that

$$\tilde{\rho}(0, t) = F(-ct) = 1 - \cos(2t).$$

This tells us that

$$F(t) = 1 - \cos(2(t - x/c)).$$

Notice that the factors of 2 are inside the cos term rather than outside (as a coefficient). ■

## 2 Chapter 2

### 2.5 Chapter 2, Section 5

**Exercise 3.** Using Duhamel's principle, find a formula for the solution to the initial value problem for the convection equation

$$u_t + cu_x = f(x, t), \quad x \in \mathbb{R}, \quad t > 0; \quad u(x, 0) = 0, \quad x \in \mathbb{R}.$$

*Solution.* The solution contains no typos, but should be split into two solutions (pertaining to Exercises 3 and 4.) ■

**Exercise 4.** Solve the problem

$$u_t + 2u_x = xe^{-t}, \quad x \in \mathbb{R}, \quad t > 0; \quad u(x, 0) = 0, \quad x \in \mathbb{R}.$$

*Solution.* The given answer of

$$u(x, t) = -(x - 2t)(e^{-t} - 1) - 2te^{-t} + 2(1 - e^{-t})$$

is correct, but may be more clear in its simplest form, which is the result one should get after integration by parts:

$$u(x, t) = (-x - 2)e^{-t} + x - 2t + 2.$$

■

## A Completed Problems

The following list consists of problems I have done. The list is non-exhaustive, but contains problems that I did not find typos in.

### A.1 Chapter 1

- Section 1.1: 1-11
- Section 1.2: 1-11
- Section 1.3: 2, 3, 4, 6, 9
- Section 1.5: 3, 4, 5, 8, 9
- Section 1.7: 6
- Section 1.9: 1, 2, 5

### A.2 Chapter 2

- Section 2.5: 1-5