Homework 4

Partial Differential Equations, Spring 2023 David Yang

Logan Chapter 1.7, Problem 6

(Dirichlet's Principle) Suppose that u satisfies the Dirichlet problem

$$\Delta u = 0,\ (x,y,z) \in \Omega,$$

$$u = f,\ (x,y,z) \in \delta\Omega.$$

Show that

$$\int_\Omega |\operatorname{grad} u|^2\, dV \le \int_\Omega |\operatorname{grad} w|^2\, dV$$

for any other function w that satisfies $w = f$ on $\delta\Omega$. Thus, the solution to the Dirichlet problem *minimizes the energy*

$$\int_\Omega |\operatorname{grad} w|^2\, dV$$

Hint: Let $w = u + v$, where $v = 0$ on $\delta\Omega$, and expand $\int_\Omega |\operatorname{grad} w|^2\, dV$ using Green's identity.

Solution. We will begin with the hint: let $w = u + v$ where $v = 0$ on $\delta\Omega$. Then, plugging in $w = u + v$ into $\int_\Omega |\operatorname{grad} w|^2\, dV$ gives

$$\int_\Omega |\operatorname{grad} w|^2\, dV = \int_\Omega |\operatorname{grad}(u+v)|^2\, dV.$$

Note that $\operatorname{grad}(u+v) = \operatorname{grad} u + \operatorname{grad} v$. Using this fact and expanding the above equation, we find that

$$\begin{aligned}\int_\Omega |\operatorname{grad} w|^2\, dV &= \int_\Omega |\operatorname{grad} u + \operatorname{grad} v|^2\, dV \\ &= \int_\Omega |\operatorname{grad}(u)|^2\ dV + 2\int_\Omega \operatorname{grad}(u)\cdot\operatorname{grad}(v)\, dV + \int_\Omega |\operatorname{grad}(v)|^2\ dV\end{aligned}$$

By Equation 4.62 (Green's First Identity) in Logan, however, we have that

$$\int_\Omega \operatorname{grad}(u)\operatorname{grad}(v)\, dV = \int_\Omega v\operatorname{grad}(u)\cdot\vec{n}\, dA - \int_\Omega v\Delta u\, dV.$$

Furthermore, we've defined $v = 0$ on $\delta\Omega$, and $\Delta u = 0$ on Ω. Thus, plugging in these values to the above equation, we have that

$$\int_\Omega \operatorname{grad}(u)\operatorname{grad}(v)\, dV = \int_\Omega v\cdot 0\cdot\vec{n}\, dA - \int_\Omega v\cdot 0\, dV = 0.$$

Since $\int_\Omega \operatorname{grad}(u)\operatorname{grad}(v)\, dV = 2\int_\Omega \operatorname{grad}(u)\operatorname{grad}(v)\, dV = 0$, we can plug this into our equation for $\int_\Omega |\operatorname{grad} w|^2\ dV$ to get

$$
\begin{aligned}
\int_\Omega |\operatorname{grad} w|^2\, dV &= \int_\Omega |\operatorname{grad}(u)|^2\ dV + 2\int_\Omega \operatorname{grad}(u)\cdot\operatorname{grad}(v)\, dV + \int_\Omega |\operatorname{grad}(v)|^2\ dV \\
&= \int_\Omega |\operatorname{grad}(u)|^2\ dV + \int_\Omega |\operatorname{grad}(v)|^2\ dV
\end{aligned}
$$

Finally, since

$$\int_\Omega |\operatorname{grad}(v)|^2\ dV \geq 0$$

by inspection, we have that

$$\int_\Omega |\operatorname{grad}(u)|^2\ dV \leq \int_\Omega |\operatorname{grad}(w)|^2\ dV$$

as desired. This tells us that the solution to the Dirichlet problem *minimizes the energy*

$$\int_\Omega |\operatorname{grad} w|^2\, dV.$$

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Dirichlet's Principle Follow-Up

Use Dirichlet's principle (in particular, the inequality half-way down page 65 in #6, Chapter 1.7) to prove that solutions are unique (assuming they exist) for Laplace's equation with Dirichlet boundary conditions:

$$\Delta u = 0,\ (x,y,z) \in \Omega,$$
$$u = f,\ (x,y,z) \in \delta\Omega.$$

Solution. Let us assume for the sake of contradiction that there are two distinct solutions u, w to Laplace's equation with Dirichlet boundary conditions. By Dirichlet's principle, we know that

$$\int_\Omega |\operatorname{grad} u|^2\, dV \le \int_\Omega |\operatorname{grad} w|^2\, dV \text{ and } \int_\Omega |\operatorname{grad} w|^2\, dV \le \int_\Omega |\operatorname{grad} u|^2\, dV$$

Equivalently, we must have that

$$\int_\Omega |\operatorname{grad} u|^2\, dV = \int_\Omega |\operatorname{grad} w|^2\, dV.$$

From the hint to the previous problem, note that since w satisfies $w = f$ on $\delta\Omega$, we can write $w = u + v$, where $v = 0$ on $\delta\Omega$. Following the same procedure as before, we have that

$$\begin{aligned}\int_\Omega |\operatorname{grad} w|^2\, dV &= \int_\Omega |\operatorname{grad}(u+v)|^2\, dV \\ &= \int_\Omega |\operatorname{grad}(u)|^2\ dV + 2\int_\Omega \operatorname{grad}(u)\cdot\operatorname{grad}(v)\, dV + \int_\Omega |\operatorname{grad}(v)|^2\ dV.\end{aligned}$$

As we showed in the previous problem with Green's Identity, $2\int_\Omega \operatorname{grad}(u)\cdot\operatorname{grad}(v)\, dV = 0$, and so we have that

$$\int_\Omega |\operatorname{grad} w|^2\, dV = \int_\Omega |\operatorname{grad}(u)|^2\ dV + \int_\Omega |\operatorname{grad}(v)|^2\ dV.$$

However, from Dirichlet's Principle, we also know that

$$\int_\Omega |\operatorname{grad} u|^2\, dV = \int_\Omega |\operatorname{grad} w|^2\, dV.$$

Combining this with the above equation, we know that

$$\int_\Omega |\operatorname{grad}(v)|^2\ dV = 0.$$

Since $|\operatorname{grad}(v)|^2 \ge 0$, we know that for this equation to hold, we must have that $\operatorname{grad}(v) = 0$ in Ω. However, we also know by definition that $v = 0$ on $\delta\Omega$. Since $v = 0$ on $\delta\Omega$ and $\nabla v = 0$ in Ω, we know that $v = 0$ for $\Omega \cup \delta\Omega$.

Thus,

$$w = u + v = u + 0 = u,$$

and we find that w, u are not distinct solutions.

Thus, by contradiction, solutions are unique (if they exist) for Laplace's equation with Dirichlet boundary conditions. ∎