Homework 3

Partial Differential Equations, Spring 2023 David Yang

Logan Chapter 1.5, Problem 5

The total energy of the string governed by equation (1.37) with boundary conditions (1.40) is defined by

$$E(t) = \int_0^\ell \left(\frac{1}{2}\rho_0 u_t^2 + \frac{1}{2}\tau_0 u_x^2 \right) dx.$$

Show that the total energy is constant for all $t \geq 0$. Hint: Multiply (1.37) by u_t and note that $(u_t^2)_t = 2u_t u_{tt}$ and $(u_t u_x)_x = u_t u_{xx} + u_{tx} u_x$. Then show that

$$\frac{d}{dt} \int_0^\ell \rho_0 u_t^2 \, dx = [2\tau_0 u_t u_x]_0^\ell - \frac{d}{dt} \int_0^\ell \tau_0 u_x^2 \, dx.$$

Solution. We follow the hint provided in the book. We begin with equation (1.37):

$$\rho_0(x) u_{tt} = \tau_0 u_{xx}.$$

Multiplying both sides by u_t, we get that

$$\rho_0(x) u_t u_{tt} = \tau_0 u_t u_{xx}.$$

Since $(u_t^2)_t = 2u_t u_{tt}$ and $(u_t u_x)_x = u_t u_{xx} + u_{tx} u_x$, we have that

$$u_t u_{tt} = \frac{1}{2} \left(u_t^2 \right)_t \text{ and } u_t u_{xx} = (u_t u_x)_x - u_{tx} u_x.$$

Substituting these results into the equation $\rho_0(x) u_t u_{tt} = \tau_0 u_t u_{xx}$, we find that

$$\frac{1}{2} \left(u_t^2 \right)_t \rho_0 = \tau_0 \left((u_t u_x)_x - u_{tx} u_x \right).$$

Integrating both sides of the equation with respect to x and bounding the integral at $x = 0$ and $x = \ell$, we get that

$$\int_0^\ell \frac{1}{2} \left(u_t^2 \right)_t \rho_0 \, dx = \tau_0 \int_0^\ell (u_t u_x)_x - u_{tx} u_x \, dx.$$

Splitting the integral on the right-hand side into two separate integrals and simplifying, we get that

$$\int_0^\ell \frac{1}{2} \left(u_t^2 \right)_t \rho_0 \, dx = [\tau_0 u_t u_x]_0^\ell - \int_0^\ell \tau_0 u_{tx} u_x \, dx.$$

We can simplify further – note that we can move $\frac{d}{dt}$ out of the left integral and that

$$\int_0^\ell \tau_0 u_{tx} u_x \, dx = \frac{1}{2} \frac{d}{dt} \int_0^\ell \tau_0 u_x^2.$$

Substituting these results into the above equation, we get

$$\frac{d}{dt}\int_0^\ell \frac{1}{2}u_t^2\rho_0\,dx = [\tau_0 u_t u_x]_0^\ell - \frac{1}{2}\frac{d}{dt}\int_0^\ell \tau_0 u_x^2\,dx.$$

Thus, multiplying both sides by two gives us the desired result:

$$\frac{d}{dt}\int_0^\ell \rho_0 u_t^2\,dx = [2\tau_0 u_t u_x]_0^\ell - \frac{d}{dt}\int_0^\ell \tau_0 u_x^2\,dx.$$

Going back to the original question, we have that

$$E(t) = \int_0^\ell \left(\frac{1}{2}\rho_0 u_t^2 + \frac{1}{2}\tau_0 u_x^2\right)\,dx.$$

Thus, taking the derivative of both sides, plugging in our above results, and applying the boundary conditions from Equation 1.40 (since $u(0,t) = u(l,t) = 0$, we also have $u_t(0,t) = u_t(l,t) = 0$), we get that

$$E'(t) = \frac{1}{2}\frac{d}{dt}\int_0^\ell u_t^2\rho_0\,dx + \frac{1}{2}\frac{d}{dt}\int_0^\ell \tau_0 u_x^2\,dx = [\tau_0 u_t u_x]_0^\ell = 0.$$

Thus, the total energy $E(t)$ is constant for all $t \geq 0$. ■