Homework 2

Partial Differential Equations, Spring 2023 — David Yang

Logan Chapter 1.2, Problem 5

Solve the pure initial value problems in the region $x \in \mathbb{R}$, $t > 0$.

$$u_t + xtu_x = 0,\ u(x,0) = f(x)$$

and

$$u_t + xu_x = e^t,\ u(x,0) = f(x).$$

Part 1. Solve $u_t + xtu_x = 0$, $u(x,0) = f(x)$ in the region $x \in \mathbb{R}, t > 0$.

Solution. Note that this is a first-order quasi-linear partial differential equation, so we can apply the method of characteristics to solve it.

Setting up the characteristic ODEs, we have that

$$t_\tau = 1,\ x_\tau = xt,\ \text{and } U_\tau = e^t$$

with the initial conditions for $\tau = 0$ as

$$t(\tau = 0) = 0,\ x(\tau = 0) = \xi,\ \text{and } U(\tau = 0) = f(\xi).$$

Solving the first ODE, we have that

$$t = \tau.$$

Substituting this into the second ODE, we have that $x_\tau = x\tau$ and so $\frac{1}{x}dx = \tau d\tau$. Solving for x, we get that $x = Ce^{\frac{\tau^2}{2}}$.

Since the initial condition tells us that $x(\tau = 0) = \xi$, we know that $C = \xi$, so our general form for x is

$$x = \xi e^{\frac{\tau^2}{2}}.$$

Finally, solving the third ODE gives us

$$U = f(\xi).$$

Now, by inverting the coordinate transformation, we can get back to xt coordinates:

$$\xi = xe^{-\frac{t^2}{2}} \text{ and } \tau = t.$$

Since $u(x,t) = U(\xi,\tau) = f(\xi)$, where $\xi = xe^{-\frac{t^2}{2}}$ and $\tau = t$, our solution to the original PDE is

$$\boxed{u(x,t) = f\left(xe^{-\frac{t^2}{2}}\right)}.$$

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Part 2. Solve $u_t + xu_x = e^t$, $u(x, 0) = f(x)$ in the region $x \in \mathbb{R}, t > 0$.

Solution. Since this is also a first-order quasi-linear partial differential equation, we can apply the method of characteristics to solve it.

Setting up the characteristic ODEs, we have that

$$t_\tau = 1,\ x_\tau = x,\ \text{ and } U_\tau = e^t$$

with the initial conditions for $\tau = 0$ as

$$t(\tau = 0) = 0,\ x(\tau = 0) = \xi,\ \text{ and } U(\tau = 0) = f(\xi).$$

Solving the first ODE with the initial condition, we have that

$$t = \tau.$$

Solving the second ODE, we have that $x_\tau = x$ and so $\frac{1}{x}dx = d\tau$. Solving for x, we get that $x = Ce^\tau$. With the given initial condition, we must have that $C = \xi$, and so our general form for x is

$$x = \xi e^\tau.$$

Finally, substituting $t = \tau$ into our third ODE, we get $U_\tau = e^\tau$. Integrating and solving for U, we get that

$$U = e^\tau + C' + f(\xi).$$

Since we are given the initial condition $U(\tau = 0) = f(\xi)$, we must have that $C' = -1$. Thus, our general form for U is

$$U = e^\tau - 1 + f(\xi).$$

Now, by inverting the coordinate transformation, we can get back to xt coordinates:

$$\xi = xe^{-t} \text{ and } \tau = t.$$

Since $u(x, t) = U(\xi, \tau) = e^\tau - 1 + f(\xi)$, where $\xi = xe^{-t}$ and $\tau = t$, our solution to the original PDE is

$$\boxed{u(x, t) = (e^t - 1) + f\left(xe^{-t}\right)}.$$

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