Homework 6

Partial Differential Equations, Spring 2023 David Yang

HW 6 Problems

1. **Find a solution to the diffusion PDE**

$$u_t - u_{xx} = 0 \text{ for } x \in \mathbb{R},\, t > 0$$

with initial value

$$u(x,0) = e^{-x^2/4} \text{ for } x \in \mathbb{R}.$$

Solution. We will use the solution formula for the diffusion equation on an unbounded domain. Recall that the solution to the diffusion PDE $u_t - ku_{xx} = 0$ on the unbounded domain with initial value $u(x,0) = \phi(x)$ is is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-(x-y)^2/(4kt)}\, dy.$$

Applying the solution formula to this problem (where $k = 1$ and $\phi(x) = e^{-x^2/4}$), we have that the solution to the given diffusion PDE on the unbounded domain is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-y^2/4} e^{-(x-y)^2/(4t)}\, dy.$$

Simplifying the integrand by combining the exponential terms, we get that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-ty^2/(4t)} e^{-(x-y)^2/(4t)}\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2 - ty^2}{4t}}\, dy \end{aligned}$$

Expanding the numerator and then completing the square, we get that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2 - ty^2}{4t}}\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x^2 - 2xy + (t+1)y^2)}{4t}}\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(t+1)\left(y - \frac{x}{t+1}\right)^2 + x^2 \cdot \frac{t}{t+1}}{4t}}\, dy. \end{aligned}$$

Factoring out a $e^{-\frac{x^2 \cdot \frac{t}{t+1}}{4t}} = e^{-\frac{x^2}{4(t+1)}}$ term from the integrand, we get that

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4(t+1)}} \int_{-\infty}^{\infty} e^{-\frac{(t+1)\left(y - \frac{x}{t+1}\right)^2}{4t}}\, dy$$

We will now make a substitution to transform the integrand into e^{-r^2}: let

$$r = \frac{\left(y - \frac{x}{t+1}\right)\sqrt{t+1}}{\sqrt{4t}}.$$

Then we also have that

$$dr = \frac{\sqrt{t+1}}{\sqrt{4t}}\, dy.$$

Making the substitution for r in our solution $u(x,t)$, we find that

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4(t+1)}} \int_{y=-\infty}^{y=\infty} e^{-\frac{(t+1)\left(y-\frac{x}{t+1}\right)^2}{4t}}\, dy \\
&= \frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}} \int_{r=-\infty}^{r=\infty} e^{-r^2}\, dr.
\end{aligned}$$

But we also know that $\int_{-\infty}^{\infty} e^{-r^2}\, dr = \sqrt{\pi}$, so plugging this back into our solution, we find that

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}} \int_{r=-\infty}^{r=\infty} e^{-r^2}\, dr \\
&= \left(\frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}}\right) \cdot \sqrt{\pi} \\
&= \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}}.
\end{aligned}$$

Thus, our solution to the given diffusion PDE with the given initial value is

$$\boxed{u(x,t) = \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}}.}$$

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2. **Show that your solution to # 1 satisfies the property that, for all $t > 0$,**

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u(x,0)\,dx.$$

In other words, $\int_{-\infty}^{\infty} u(x,t)\,dx$ is a *conserved quantity* (constant with respect to t).

To show that this property holds, we will show that the left and right hand sides simplify to the same expression. We will begin by working with the left-hand side.

Plugging in our solution for $u(x,t)$ into the left-hand side of the equation, we have that

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}}\,dx$$

3. (a) **If u solves the diffusion equation on the infinite domain ($x \in \mathbb{R}$), with bounded initial value $u(x,0) = \phi(x)$ that has the property that**

$$\lim_{x\to-\infty} \phi(x) = a \text{ and } \lim_{x\to\infty} \phi(x) = b \quad \textbf{(a, b constants)}.$$

What is the value of $\lim_{t\to\infty} u(x,t)$?

(b) **Review Eq 2.5 on page 82 of Logan, which is a solution for the PDE**

$$w_t = k w_{xx} \text{ for } x \in \mathbb{R},\, t > 0$$

$$w(x,0) = 0 \text{for } x < 0\,;\; w(x,0) = 1 \text{for } x > 0.$$

What is $\lim_{t\to\infty} w(x,t)$ and does this agree with your result in 3(a)?