Homework 8

Partial Differential Equations, Spring 2023 — David Yang

HW 8 Problem

Consider the 1st order linear initial value PDE problem:

$$tu_t + u_x = 0 \ \textbf{ for } t > 0, x \in \mathbb{R}$$

$$u(x,0) = f(x) \ \textbf{ for } x \in \mathbb{R}.$$

(a) **Apply the Method of Characteristics. Your goal is to find a characteristic value $\xi(x,t)$ so that any function of the form $u(x,t) = f(\xi)$ satisfies the PDE.**

Tip: In most problems of this type that you have worked on in before, you have set $t = \tau$ and used the ξ variable to parameterize the values of x along the x axis. For this problem, set $x = \tau$ and use the ξ variable to parameterize the values of t along the t-axis.

Solution. We begin by applying the Method of Characteristics. We get that

$$t_\tau = t, \ \ x_\tau = 1, \ \text{and } U_\tau = 0.$$

Since, as the tip says, we set $x = \tau$ and use the ξ variable to parameterize the values of t along the t-axis, we know $t(\tau = 0) = \xi$.

Solving for t and x by integrating with respect to τ and using these initial values, we find that

$$t = \xi e^\tau \quad \text{and} \quad x = \tau.$$

Solving for U, we get that

$$U = f(\xi)$$

which also matches the given initial condition $U(\xi, 0) = f(\xi)$.

Now, inverting our change of coordinates by expressing ξ and τ in terms of x and t, we get that

$$\tau = x \quad \text{and} \quad \xi = te^{-x}.$$

Plugging this change of variables back into our solution, we get that

$$u(x(\xi,\tau), t(\xi,\tau)) = f(\xi(x,t)) = f(te^{-x}).$$

Thus, our solution is

$$\boxed{u(x,t) = f(te^{-x})}.$$

■

(b) **Set $f(x) = x$ as the initial value for the PDE given above. Show that the form of the solution you found in (a) does not satisfy this initial value.**

Remark: In fact, no solutions exist for this PDE that solve the initial value $u(x, 0) = x$. This PDE is ill-posed for $u(x, 0) = x$.

Solution. We set $f(x) = x$ as per the instructions. We will now check whether the solution $u(x, t) = f(te^{-x})$ satisfies the initial value

$$u(x, 0) = x \text{ for } x \in \mathbb{R}.$$

Plugging in $t = 0$ to our solution, we get that

$$u(x, 0) = 0 \text{ for } x \in \mathbb{R}$$

However, notice that to satisfy the initial condition, we must have that $u(x, 0) = f(x) = x$ for $x \in \mathbb{R}$.

Clearly,

$$u(x, 0) = 0 \neq x \text{ for } x \in \mathbb{R},$$

and so the solution we found in (a) does not satisfy the initial value for the PDE. ■

(c) **Set $f(x) = 1$ as the initial value for the PDE given above. Now let ξ be the characteristic variable you found in (a). For what values of the constants a and b does the function $u(x, t) = a + b\xi$ also solve the PDE and satisfy the initial value $u(x, 0) = 1$?**

Solution. Let $u(x, t) = a + b\xi$, where $\xi = te^{-x}$ as we found in part (a). Equivalently, we have that

$$u(x, t) = a + bte^{-x}.$$

Calculating the partial derivatives, we get that

$$u_t = be^{-x} \quad \text{and} \quad u_x = -bte^{-x}.$$

Plugging these partials back into our PDE, we get

$$\begin{aligned} tu_t + u_x &= t\left(be^{-x}\right) + \left(-bte^{-x}\right) \\ &= bte^{-x} - bte^{-x} \\ &= 0. \end{aligned}$$

Thus, every value of a, b will solve the initial PDE problem $tu_t + u_x = 0$ for all $t > 0, x \in \mathbb{R}$.

On the other hand, to satisfy the initial value $u(x, 0) = 1$, we must have

$$\begin{aligned} u(x, 0) &= a + b(0)e^{-x} \\ &= a \\ &= 1. \end{aligned}$$

Thus, we must have that $a = 1$ and $b \in \mathbb{R}$ for the function $u(x,t) = a + b\xi = a + b(te^{-x})$ to solve the PDE and initial value $u(x,0) = 1$.

■

Is the PDE with the initial value $f(x) = 1$ *well-posed* or *ill-posed*? Why?

Solution. From our above work, we found that any solution $u(x,t) = a + b\xi = a + bte^{-x}$ with $a = 1$ and $b \in \mathbb{R}$ solves the PDE and satisfies the initial value $u(x,0) = 1$. Consequently, there are infinitely many solutions to the PDE with initial value $f(x) = 1$.

For PDE to be well-posed, it must have a unique solution. Since the PDE with initial value $f(x) = 1$ has infinitely many solutions, it is an $\boxed{\text{ill-posed}}$ PDE. ■