Homework 5

Partial Differential Equations, Spring 2023 David Yang

Logan Chapter 1.7, Problem 6

Consider the PDE

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0 \quad \textbf{for } x \in \mathbb{R} \textbf{ and } t > 0$$

with initial conditions

$$u(x,0) = x^2 \quad \textbf{and} \quad u_t(x,0) = e^x \textbf{ for } x \in \mathbb{R}$$

a) **Calculate the discriminant and classify the PDE as hyperbolic, parabolic, or elliptic.**

Solution. This PDE is a second-order differential equation of the form $Au_{xx} + Bu_{xt} + Cu_{tt} = 0$, with $A = 1$, $B = -3$, and $C = -4$.

The discriminant of the PDE, then, is

$$D = B^2 - 4AC = (-3)^2 - 4(1)(-4) = \boxed{25} > 0.$$

Since this PDE has a positive discriminant, the PDE is $\boxed{\text{hyperbolic}}$. ■

b) **Solve the PDE.**

Solution. We will make a change of variables of the form

$$\begin{cases} \xi = ax + bt \\ \tau = cx + dt \end{cases},$$

as done in Logan, page 75. Since the PDE is hyperbolic, we can follow page 75 once again. We choose $a = c = 1$, and get that

$$b = \frac{-B + \sqrt{D}}{2C} = \frac{-(-3) + \sqrt{25}}{2(-4)} = -1$$

$$d = \frac{-B - \sqrt{D}}{2C} = \frac{-(-3) - \sqrt{25}}{2(-4)} = \frac{1}{4}.$$

Thus, our change of coordinates are

$$\begin{cases} \xi = x - t \\ \tau = x + \frac{1}{4}t \end{cases}.$$

■

Since our original PDE had no lower order terms, this change of coordinates transforms our PDE into the canonical form

$$U_{\xi\tau} = 0.$$

We can now simply integrate twice to solve for U. First, integrating both sides with respect to τ, we get that

$$U_\xi = \tilde{f}(\xi)$$

where $\tilde{f}(\xi)$ is a function of ξ. Integrating again, but this time with respect to ξ, we get that

$$U = f(\xi) + g(\tau),$$

where $f(\xi) = \int \tilde{f}$ is simply another function of ξ.

Our change of coordinates was $\xi = x - t$ and $\tau = x + \frac{1}{4}t$. Reverting our solution back to $x - t$ coordinates, we have that

$$u(x,t) = f(x-t) + g\left(x + \frac{1}{4}t\right).$$

We can now use our initial conditions to determine the functions f and g. First, we are given that $u(x,0) = x^2$, so we know that

$$u(x,0) = f(x-0) + g\left(x + \frac{1}{4}\cdot 0\right) = f(x) + g(x) = x^2$$

Similarly, taking the partial derivative of $u(x,t)$ with respect to t, we get that

$$u_t(x,t) = -f'(x-t) + \frac{1}{4}g'\left(x + \frac{1}{4}t\right)$$

Thus, since $u_t(x,0) = e^x$, we have that

$$u_t(x,0) = -f'(x-0) + \frac{1}{4}g'\left(x + \frac{1}{4}\cdot 0\right) = -f'(x) + \frac{1}{4}g'(x) = e^x.$$

Combining the information from the initial conditions, we have the system of equations

$$\begin{cases} f(x) + g(x) = x^2 \\ -f'(x) + \frac{1}{4}g'(x) = e^x \end{cases}$$

Taking the derivative of the first equation, we get the new system

$$\begin{cases} f'(x) + g'(x) = 2x \\ -f'(x) + \frac{1}{4}g'(x) = e^x \end{cases}$$

We can now solve for $f'(x)$ and $g'(x)$. To solve for $g'(x)$, we add the two equations and multiply both sides by $\frac{4}{5}$ to get

$$g'(x) = \frac{4}{5}\left(2x + e^x\right) = \frac{8x}{5} + \frac{4}{5}e^x.$$

We can then get $f'(x)$ from the first equation:

$$f'(x) = \frac{2x}{5} - \frac{4}{5}e^x.$$

We can now integrate these two equations to get $f(x)$ and $g(x)$. We find that

$$f(x) = \frac{x^2}{5} - \frac{4}{5}e^x \quad \text{and} \quad g(x) = \frac{4x^2}{5} + \frac{4}{5}e^x.$$

From our work before, we have that

$$u(x,t) = f(x-t) + g\left(x + \frac{1}{4}t\right).$$

Substituting the equations we found for $f(x)$ and $g(x)$, we get that

$$\begin{aligned} u(x,t) &= f(x-t) + g\left(x + \frac{1}{4}t\right) \\ &= \left(\frac{(x-t)^2}{5} - \frac{4}{5}e^{x-t}\right) + \left(\frac{4(x+\frac{1}{4}t)^2}{5} + \frac{4}{5}e^{x+\frac{1}{4}t}\right) \end{aligned}$$

and so our solution that satisfies the original PDE and its initial conditions is

$$\boxed{u(x,t) = \left(\frac{(x-t)^2}{5} - \frac{4}{5}e^{x-t}\right) + \left(\frac{4(x+\frac{1}{4}t)^2}{5} + \frac{4}{5}e^{x+\frac{1}{4}t}\right)}.$$