TYPOS IN LOGAN'S *Applied Partial Differential Equations, 3rd ed.* SOLUTIONS

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Contents

1 Chapter 1

1.2 Chapter 1, Section 2 (1.2)

Exercise 1.2.7. Solve the initial boundary value problem

$$u_t + cu_x = \lambda u, \ \ x, t > 0$$

$$u(x, 0) = 0, \ x > 0, \ u(0, t) = g(t), \ t > 0.$$

Solution. We should find that $\phi(t) = e^{-\lambda t/c} g(-t/c)$ (notice the negative in the exponent). This follows from the fact that if $\lambda(-ct)e^{\lambda t} = g(t)$, then substituting $t = -t/c$ gives us the negative in the exponent. This gives us $u(x, t) = g(t - x/c)e^{-\lambda x/c}$, in $0 \leq x < ct$, which matches the solutions. ■

Exercise 1.2.12. Find a formula that implicitly defines the solution$u = u(x, t)$ of the initial value problem for the reaction-advection equation

$$u_t + cu_x = -\frac{\alpha u}{\beta + u}, \ \ x \in \mathbb{R}, \ t > 0,$$

$$u(x, 0) = f(x), \ \ x \in \mathbb{R}$$

Here, v, α, β are positive constants. Show from the implicit formula that you can always solve for u in terms of x and t.

Solution. $f(x)$ should be $f(x - ct)$, as we need to change back to $x - t$ coordinates. ■

1.3 Chapter 1, Section 3

Exercise 1.3.2. Let $u = u(x, t)$ satisfy the heat flow model

$$u_t = ku_{xx}, \ \ 0 < x < l, \ t > 0$$

$$u(0, t) = u(l, t) = 0, \ \ t > 0,$$

$$u(x, 0) = u_0(x), 0 \leq x \leq l.$$

Show that

$$\int_0^l u(x, t)^2 \, dx \leq \int_0 u_0(x)^2 \, dx, \ t \geq 0.$$

Hint: Let $E(t) = \int_0^l u(x, t)^2 \, dx$ and show that $E'(t) \leq 0$. What can be said about $u(x, t)$ if $u_0(x) = 0$?

Solution. We have that $E(t) \leq E(0) = \int_0^l u_0(x)^2 \, dx$. This does not affect the solution, though, as if $u_0 \equiv 0$, then $E(0) = 0$ still.

■

Exercise 1.3.6. Heat flow in a metal rod with a unit internal heat source is governed by the problem

$$u_t = ku_{xx} + 1, \ 0 < x < 1, \ t > 0,$$

$$u(0, t) = 0, \ u(1, t) = 1, \ t > 0.$$

What will be the steady-state temperature in the bar after a long time? Does it matter that no initial condition is given?

Solution. The answer should be

$$u(x) = -\frac{1}{2k}x^2 + \left(1 + \frac{1}{2k}\right) x.$$

Originally, there is a $=$ sign rather than a $+$. ■

1.5 Chapter 1, Section 5 (1.5)

Exercise 1.5.5. The total energy of the string governed by equation (1.37) with boundary conditions (1.40) is defined by

$$E(t) = \int_0^\ell \left(\frac{1}{2}\rho_0 u_t^2 + \frac{1}{2}\tau_0 u_x^2 \right) dx.$$

Show that the total energy is constant for all $t \geq 0$. Hint: Multiply (1.37) by u_t and note that $(u_t^2)_t = 2u_t u_{tt}$ and $(u_t u_x)_x = u_t u_{xx} + u_{tx} u_x$. Then show that

$$\frac{d}{dt}\int_0^\ell \rho_0 u_t^2\, dx = [2\tau_0 u_t u_x]_0^\ell - \frac{d}{dt}\int_0^\ell \tau_0 u_x^2\, dx.$$

Solution. The hint should include an extra factor of 2 (colored in red). ■

Exercise 1.5.8. At the end ($x = 0$) of a long tube ($x \geq 0$) the density of air changes according to the formula $\tilde{\rho}(0,t) = 1 - \cos 2t$ for $t \geq 0$, and $\tilde{\rho}(0,t) = 0$ for $t < 0$. Find a solution to the wave equation in the domain $x > 0, -\infty < t < \infty$, in the form of a right-traveling wave that satisfies the given boundary condition. Take $c = 1$ and plot the solution surface.

Solution. We should have that

$$\tilde{\rho}(0,t) = F(-ct) = 1 - \cos(2t).$$

This tells us that

$$F(t) = 1 - \cos(2(t - x/c)).$$

Notice that the factors of 2 are inside the cos term rather than outside (as a coefficient). ■

2 Chapter 2

2.2 Chapter 2, Section 2 (2.2)

Exercise 2.2.3. Solve the Cauchy problem

$$u_{tt} = c^2 u_{xx}, \;\; u(x,0) = \phi(x),\, u_t(x,0) = 0,$$

by differentiating the solution to the Cauchy problem

$$u_{tt} = c^2 u_{xx}, \;\; u(x,0) = 0,\, u_t(x,0) = \phi(x).$$

Solution. This solution is not available in Logan's solutions, but there is an extra "to" in the problem statement. The problem statement here is correct. ■

2.5 Chapter 2, Section 5 (2.5)

Exercise 2.5.1. Write a formula for the solution to the problem

$$u_{tt} - c^2 u_{xx} = \sin(x),\, x \in \mathbb{R},\, t > 0; \;\; u(x,0) = u_t(x,0) = 0,\, x \in \mathbb{R}.$$

Solution. The solution is

$$u(x,t) = \frac{1}{2c} \int_0^\infty \int_{x+c(t-s)}^{x+c(t+s)} \sin(s)\, ds\, dt.$$

■

Exercise 2.5.3. Using Duhamel's principle, find a formula for the solution to the initial value problem for the convection equation

$$u_t + cu_x = f(x,t),\, x \in \mathbb{R},\, t > 0; \;\; u(x,0) = 0,\, x \in \mathbb{R}.$$

Solution. The solution contains no typos, but should be split into two solutions (pertaining to Exercises 3 and 4.) ■

Exercise 2.5.4. Solve the problem

$$u_t + 2u_x = xe^{-t},\, x \in \mathbb{R},\, t > 0; \;\; u(x,0) = 0,\, x \in \mathbb{R}.$$

Solution. The given answer of

$$u(x,t) = -(x-2t)(e^{-t} - 1) - 2te^{-t} + 2(1 - e^{-t})$$

is correct, but may be more clear in its simplest form, which is the result one should get after integration by parts:

$$u(x,t) = (-x-2)e^{-t} + x - 2t + 2.$$

■

2.7 Chapter 2, Section 7 (2.7)

There are also a few minor typos in the problem statements for 2.7.9, 2.7.10, and 2.7.11. Logan writes that the x domain for the PDE is $x \in$, when it should be $x \in \mathbb{R}$.

Exercise 2.7.15. Solve the Cauchy problem for the advection-diffusion equation using Fourier transforms:

$$u_t = Du_{xx} - cu_x, \;\; x \in \mathbb{R}, t > 0; \;\; u(x,0) = \phi(x), \, x \in \mathbb{R}.$$

Solution. The solution writes (in the final two lines) that

$$\mathcal{F}^{-1}\left(e^{-i\xi ct}e^{-D\xi^2 t}\right) = \frac{1}{\sqrt{4\pi Dt}}e^{-(x+vt)^2/4Dt}.$$

This should actually be

$$\mathcal{F}^{-1}\left(e^{-i\xi ct}e^{-D\xi^2 t}\right) = \frac{1}{\sqrt{4\pi Dt}}e^{-(x-ct)^2/4Dt}.$$

Similarly, the desired result after convolution should just be

$$u(x,t) = \phi \star \frac{1}{\sqrt{4\pi Dt}}e^{-(x-ct)^2/4Dt}.$$

■

Exercise 2.7.18. Use Fourier transforms to derive d'Alembert's solution (2.14) to the Cauchy problem for the wave equations (2.12)-(2.13) when the initial velocity is zero, i.e., $g(x) \equiv 0$.

Solution. The problem erroneously states that the PDE is $u_{tt} = c^2 u_{xx} = 0$, when it should be $u_{tt} - c^2 u_{xx} = 0$. This is a very minor typo. ■

4 Chapter 4

4.7 Chapter 4, Section 7 (4.7): Bounded Domains

Example 4.28. (Homogenizing the boundary conditions) Consider the problem

$$\begin{gathered} u_t - 3u_{xx} = 0, \;\; 0 < x < 1, \, t > 0, \\ u(0,t) = 2e^{-t}, \;\; u(1,t) = 1, \\ u(x,0) = x^2, \;\; 0 < x < 1. \end{gathered}$$

Solution. We follow the procedure written in the book, taking

$$w(x,t) = u(x,t) - \left(2e^{-t} + \left(1 - 2e^{-t}\right) x\right).$$

Then w solves the problem

$$\begin{gathered} w_t - 3w_{xx} = 2e^{-t}(1-x), \;\; 0 < x < 1, \, t > 0, \\ w(0,t) = w(1,t) = 0, \, t > 0, \\ w(x,0) = x^2 + x - 2, \;\; 0 < x < 1. \end{gathered}$$

Logan writes that $w(x,0) = x^2 + x$, but notice that

$$\begin{aligned} w(x,0) &= u(x,0) - \left(2e^{-0} + \left(1 - 2e^{-0}\right) x\right) \\ &= u(x,0) - (2 - x) \\ &= x^2 + x - 2. \end{aligned}$$

■

A Completed Problems

The following list consists of problems I have done. The list is non-exhaustive, but contains problems that I believe I did not find typos in.

A.1 Chapter 1

- Section 1.1: 1-11
- Section 1.2: 1-11
- Section 1.3: 2, 3, 4, 6, 9
- Section 1.5: 3, 4, 5, 8, 9
- Section 1.7: 6
- Section 1.9: 1, 2, 5

A.2 Chapter 2

- Section 2:1, 1, 2, 6, 7
- Section 2.2: 2, 4, 6, 7
- Section 2.4: 1, 2, 3, 4
- Section 2.5: 1-5
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A.3 Chapter 3

- Section 3.1: 1
- Section 3.2: 4, 5, 6
- Section 3.3: 5, 6

A.4 Chapter 4

- Section 4.1: 1
- Section 4.2: 5, 9
- Section 4.4: 1
- Section 4.7: 7