Final Project Draft

Partial Differential Equations, Spring 2023

David Yang and Martina Kampel

1 Introduction

In this paper, we discuss a partial differential equation that models the evolution in time of an age-structured population. Our equation is useful for age-structured models, demographic models where the population at time t has an age distribution superimposed on it. In other words, at a given time t, the ages of the individuals in a population are also considered. Age-structured models are related to general physiologically-structured models, where any other variable (such as size or weight) can replace the age variable. For example, the following paper uses a physiologically structured population model to model a "size-structured consumer population feeding on a non-structured prey population", studying the qualitative and quantitative population dynamics of a planktivorous fish population. Consequently, the ideas we discuss in this paper, though focused on age, can also be extended to other physiological structures.

Anderson Gray McKendrick (A.G. McKendrick) was a Scottish physician and epidemiologist. In 1926, McKendrick published a paper titled *Applications of mathematics to medical problems*. He aimed to study the transfer of disease caused by interactions between people and apply mathematical modeling to epidemiology; his paper introduced a version of the McKendrick-von Forester equation. [**mck1925**] McKendrick's paper went relatively unseen; independently, in 1959, biophysics Heinz von Foerster discovered the same equation when studying cell divisions. [**keyfitz˙keyfitz**] The equation itself was later named in recognition of their independent work, as what we now know as the **McKendrick–von Forester equation**.

The **McKendrick–von Forester equation** is a form of the advection equation that incorporates the mortality rate at some age to study the population density at a given age and time.

In our project, we will derive the general age-structured model in Logan's *Applied Partial Differential Equations* [**logan**]:

$$u_t = -u_a - m(a)u, \;\; a > 0,\, t > 0$$

$$u(0,t) = \int_0^\infty b(a,t)u(a,t)\,da, \;\; t > 0$$

$$u(a,0) = f(a), \;\; a \geq 0$$

where the newly introduced terms $b(a,t)$ and $f(a)$ represent the average reproduction rate and the population of female organisms, respectively.

We will then discuss the stable age structure to see what happens over a long time. Finally, we plan on presenting the renewal equation (a simplified version of the age-structured model above where $b = b(a)$ and $m =$ constant), which we will solve using the method of characteristics.

In conclusion, we will discuss various extensions of the McKendrick-von Forester equation and its importance in helping scientists understand population models.

2 Mathematical Content

2.1 Context

Consider a population of female organisms with age structure at time $t = 0$ given by $f(a)$. Equivalently, $f(a)da$ is the number of females between age a and age $a + da$ in the population.

Though age should technically be finite, let the domain for a be $[0, \infty)$. The goal of the McKendrick-von Forester equation is to model the age structure $u = u(a, t)$ for the population for any time $t > 0$. By definition, $u(a, t)da$ represents the number of females at time t between ages a and $a+da$. The total female population at time t can consequently be represented as

$$N(t) = \int_0^\infty u(a, t)da.$$

Note that the quantity $u(0, t)$, the number of newborns at time t, is not known; this quantity depends on the reproduction rate of females and the mortality rate. We define $m(a)$ to be the *per capita mortality rate*, which will be given to us in any initial statement of the problem, and $b(a, t)$ to be the *fecundity rate* (also known as the *maternity function*), the average number of offspring per female at time t.

2.2 Formula

The general form of the **McKendrick–von Forester equation**, which models the population dynamics described above, is

$$u_t = -u_a - m(a)u.$$

This equation is a form of the advection equation where $m(a)$ represents a function that takes in age a and returns the per capita mortality rate. This is also known as the *force of mortality.* We use $u = u(a, t)$ to represent the density of a population of age a and time t, for nonnegative a and t. Note that a and t are measured in the same units.

We have initial condition

$$u(a, 0) = f(a), \quad a \geq 0.$$

where, as stated before, $f(a)$ represents the initial population of female organisms of age a at time $t = 0$. We also have boundary condition

$$u(0, t) = \int_0^\infty b(a, t)u(a, t)\, da, \quad t > 0.$$

Here, we see that u is part of our boundary condition. This is known as a *nonlocal boundary condition*, where the unknown solution is a part of the condition. As stated previously, $u(a, t)$ represents the number of females at time t at age a and $b(a, t)$ is the average reproduction rate of females of age a at time t. Summing over all possible ages, from 0 to infinity, will give us $u(0, t) = B(t)$ where $B(t)$ represents the total amount of offspring produced by females of all ages at time t.

Furthermore, notice that in particular, the McKendrick–von Forester equation is the advection equation with speed one and sink term given by the mortality rate; notice that the flux is $\phi = u$, or the density of the population at that age.

This means we can use a certain age, a, and a time, t, and then compute the density of the population of that particular age at that time.

We have now derived the general age-structured model.

$$u_t = -u_a - m(a)u, \;\; a > 0, \, t > 0$$

$$u(0,t) = \int_0^\infty b(a,t)u(a,t)\,da, \;\; t > 0$$

$$u(a,0) = f(a), \;\; a \geq 0$$