Homework 8

Partial Differential Equations, Spring 2023 David Yang

HW 9 Problems

Chapter 4.1 Problem 5

Consider heat flow in a rod of length l where the heat is lost across the lateral boundary is given by Newton's law of cooling. The model is

$$\begin{gathered} u_t = ku_{xx} - hu, \ \ 0 < x < l, \\ u = 0 \text{ at } x = 0,\ x = l, \ \ \text{for all } t > 0, \\ u = f(x) \text{ at } t = 0,\ 0 \leq x \leq l, \end{gathered}$$

where $h > 0$ is the heat loss coefficient.

a) **Find the equilibrium temperature.**

Solution. The equilibrium temperate u has the property that $u_t = 0$. Thus, our PDE becomes

$$ku_{xx} - hu = 0.$$

Equivalently, after dividing both sides by k, we have that

$$u_{xx} - \frac{h}{k}u = 0.$$

Solving this ODE using the characteristic polynomial (which is $r^2 - \frac{h}{k} = 0$), we find that the general solution is

$$u(x,t) = ae^{-\sqrt{\frac{h}{k}}x} + be^{\sqrt{\frac{h}{k}}x}.$$

Our bondary conditions tell us that $u(0,t) = u(l,t) = 0$. To satisfy $u(0,t) = 0$, we must have

$$\begin{aligned} u(0,t) &= ae^{-\sqrt{\frac{h}{k}}0} + be^{\sqrt{\frac{h}{k}}0} \\ &= a + b = 0. \end{aligned}$$

On the other hand, to satisfy $u(l,t) = 0$, we must have

$$u(l,t) = ae^{-\sqrt{\frac{h}{k}}l} + be^{\sqrt{\frac{h}{k}}l} = 0.$$

Multiplying our first equation by $e^{-\sqrt{\frac{h}{k}}l}$ and subtracting it from the second equation, we have that

$$b\left(e^{\sqrt{\frac{h}{k}}l} - e^{\sqrt{-\frac{h}{k}}l}\right) = 0,$$

which can only be satisfied if $b = 0$. Similarly, since $a + b = 0$, we also know that $a = 0$.

Thus, our equilibrium temperature is $u(x,t) = \boxed{0}$.

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b) **Solve the problem.**

Solution. We will use the separation of variables method to solve the problem. Consider a separated solution of the form

$$u(x,t) = y(x)g(t).$$

For this solution, we have that

$$u_t = y(x)g'(t) \text{ and } u_{xx} = y''(x)g(t).$$

Substituting these partials into our original PDE $u_t = ku_{xx} - hu$, we get that

$$y(x)g'(t) = ky''(x)g(t) - hy(x)g(t).$$

Since neither $y(x)$ nor $g(t)$ are 0, we can separate this equation further by dividing both sides by $y(x)g(t)$. This gives us

$$\frac{g'(t)}{g(t)} = \frac{ky''(x) - hy(x)}{y(x)}.$$

For this equation to hold true for all values of $x \in (0, l)$ and $t > 0$ is for them to evaluate to the same constant, and so we have

$$\frac{g'(t)}{g(t)} = \frac{ky''(x) - hy(x)}{y(x)} = C$$

for some constant C.

Rewriting the above equation as two separate ODEs, we have that

$$\begin{cases} g'(t) = Cg(t) \\ y''(x) = \frac{C+h}{k}y(x) \end{cases}$$

We will begin by solving for $y(x)$. Note that, just like in our in-class example, nontrivial solutions $y(x)$ that satisfy the boundary conditions $y(0) = y(l) = 0$ will only occur when $C + h < 0$. Consequently, we can solve the ODE

$$y''(x) - \frac{(C+h)}{k}y(x) = 0$$

when $\frac{C+h}{k} < 0$ to get the general form for $y(x)$:

$$y(x) = a\sin\left(\sqrt{\frac{-(C+h)}{k}}x\right) + b\cos\left(\sqrt{\frac{-(C+h)}{k}}x\right).$$

By the boundary conditions, we know that $y(0) = 0$ and $y(l) = 0$. To satisfy $y(0) = 0$, we must have that

$$y(0) = a\sin(0) + b\cos(0) = 0,$$

so $b = 0$. On the other hand, for $y(l) = 0$, we must have that

$$\begin{aligned} y(l) &= a\sin\left(\sqrt{\frac{-(C+h)}{k}}l\right) + 0\cos\left(\sqrt{\frac{-(C+h)}{k}}l\right) \\ &= a\sin\left(\sqrt{\frac{-(C+h)}{k}}l\right) = 0. \end{aligned}$$

Since the sine function is 0 at integer multiples of π, we know that

$$\sqrt{\frac{-(C+h)}{k}}l = \pi n$$

for some integer n. Equivalently, we have that

$$\begin{aligned}\sqrt{\frac{-(C+h)}{k}} &= \frac{\pi n}{l} \\ \frac{-(C+h)}{k} &= \left(\frac{\pi n}{l}\right)^2\end{aligned}$$

and so solving for C gives us

$$C = -k\left(\frac{\pi n}{l}\right)^2 - h.$$

Recall that our solution for $y(x)$ is

$$y(x) = a\sin\left(\sqrt{\frac{-(C+h)}{k}}x\right)$$

with $C = -k\left(\frac{\pi n}{l}\right)^2 - h$ for positive integers n.

Substituting our expression for C, we find that our general solution for y is

$$y_n(x) = a_n \sin\left(\frac{\pi n}{l}x\right) \text{ for positive integers } n.$$

We can now solve our other ODE for $g(t)$. We had that $g'(t) = Cg(t)$. Solving and using the general value for C, we have that $g(t) = e^{Ct}$ so

$$g_n(t) = e^{\left(-k\left(\frac{\pi n}{l}\right)^2 - h\right)t}$$

Thus, our product solutions that satisfy the original PDE and its boundary conditions are

$$u_n(x,t) = y_n(t)g_n(t) = a_n e^{\left(-k\left(\frac{\pi n}{l}\right)^2 - h\right)t}\sin\left(\frac{\pi n}{l}x\right).$$

Using superposition, our solution $u(x,t)$ is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{\left(-k\left(\frac{\pi n}{l}\right)^2 - h\right)t}\sin\left(\frac{\pi n}{l}x\right)}$$

where a_n are the Fourier coefficients defined on page 148: $a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{\pi n}{l}x\right)$.

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Chapter 3.2 Problem 3(a)

Let $f(x) = 0$ **for** $0 < x < 1$ **and** $f(x) = 1$ **for** $1 < x < 3$**.**

a) **Find the first** 4 **nonzero terms of the Fourier cosine series of** f**.**

Solution. By definition, the Fourier cosine series of f is

$$\frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{3}\right),$$

where

$$b_0 = \frac{2}{3}\int_0^3 f(x)\,dx = \frac{2}{3}\int_1^3 1\,dx = \frac{4}{3}$$

and

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 f(x)\cos\left(\frac{n\pi x}{3}\right)\,dx \\ &= \frac{2}{3}\int_1^3 \cos\left(\frac{n\pi x}{3}\right)\,dx \end{aligned}$$

for positive integers n. Simplifying further, we get that

$$\begin{aligned} b_n &= \frac{2}{3}\left[\frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\right]_1^3 \\ &= \frac{2}{n\pi}\left(\sin(n\pi) - \sin\left(\frac{n\pi}{3}\right)\right) \end{aligned}$$

Note that $\sin(n\pi) = 0$ for all integer n. Using this fact, we can simplify our general term to

$$b_n = \frac{2}{n\pi}\left(-\sin\left(\frac{n\pi}{3}\right)\right)$$

and plugging in a few values of n to determine the first nonzero coefficients, we find that

$$b_0 = \frac{4}{3}$$

$$\begin{aligned} b_1 &= \frac{2}{1\pi}\left(-\sin\left(\frac{\pi}{3}\right)\right) = -\frac{\sqrt{3}}{\pi} \\ b_2 &= \frac{2}{2\pi}\left(-\sin\left(\frac{2\pi}{3}\right)\right) = -\frac{\sqrt{3}}{2\pi} \\ b_3 &= \frac{2}{3\pi}\left(-\sin\left(\frac{3\pi}{3}\right)\right) = 0 \\ b_4 &= \frac{2}{4\pi}\left(-\sin\left(\frac{4\pi}{3}\right)\right) = \frac{\sqrt{3}}{4\pi}. \end{aligned}$$

Thus, since the Fourier cosine series of f is

$$\frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{3}\right),$$

we find that the first 4 nonzero terms of the Fourier cosine series of f are

$$\frac{b_0}{2} = \boxed{\frac{2}{3}}$$

$$b_1 \cos\left(\frac{1\pi x}{3}\right) = \boxed{-\frac{\sqrt{3}}{\pi}\cos\left(\frac{\pi x}{3}\right)}$$

$$b_2 \cos\left(\frac{2\pi x}{3}\right) = \boxed{-\frac{\sqrt{3}}{2\pi}\cos\left(\frac{2\pi x}{3}\right)}$$

$$b_4 \cos\left(\frac{4\pi x}{3}\right) = \boxed{\frac{\sqrt{3}}{4\pi}\cos\left(\frac{4\pi x}{3}\right)}$$

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Chapter 3.2 Problem 6

If c_n are Fourier coefficients of f and f_n is an orthonormal set, show that

$$\left(\sum_{n=1}^{N} c_n f_n,\, f - \sum_{n=1}^{N} c_n f_n\right) = 0.$$

Also answer for this problem: Why does this formula makes sense? In your (very brief) answer, you can relate this formula to a result you may have learned in linear algebra if you studied orthogonal projection and orthogonal decomposition.

Solution. By the linearity property of inner products, we know that

$$\begin{aligned}&\left(\sum_{n=1}^{N} c_n f_n,\, f - \sum_{n=1}^{N} c_n f_n\right)\\ &= \left(\sum_{n=1}^{N} c_n f_n,\, f\right) - \left(\sum_{n=1}^{N} c_n f_n, \sum_{n=1}^{N} c_n f_n\right).\end{aligned}$$

Notice that by definition, $f = \sum_{n=1}^{\infty} c_n f_n$, so

$$\begin{aligned} f &= \sum_{n=1}^{\infty} c_n f_n \\ &= \sum_{n=1}^{N} c_n f_n + \sum_{n=N+1}^{\infty} c_n f_n. \end{aligned}$$

Substituting this back into our inner product expression above, we get

$$\begin{aligned}&\left(\sum_{n=1}^{N} c_n f_n,\, f\right) - \left(\sum_{n=1}^{N} c_n f_n, \sum_{n=1}^{N} c_n f_n\right)\\ &= \left(\sum_{n=1}^{N} c_n f_n, \sum_{n=1}^{N} c_n f_n + \sum_{n=N+1}^{\infty} c_n f_n\right) - \left(\sum_{n=1}^{N} c_n f_n, \sum_{n=1}^{N} c_n f_n\right).\end{aligned}$$

Using the linearity property once again on the first term, we find that this expression is simply

$$= \left(\sum_{n=1}^{N} c_n f_n, \sum_{n=1}^{N} c_n f_n\right) + \left(\sum_{n=1}^{N} c_n f_n, \sum_{n=N+1}^{\infty} c_n f_n\right) - \left(\sum_{n=1}^{N} c_n f_n, \sum_{n=1}^{N} c_n f_n\right).$$

Grouping our terms and simplifying, we get that

$$\begin{aligned}&\left(\sum_{n=1}^{N} c_n f_n, \sum_{n=1}^{N} c_n f_n\right) + \left(\sum_{n=1}^{N} c_n f_n, \sum_{n=N+1}^{\infty} c_n f_n\right) - \left(\sum_{n=1}^{N} c_n f_n, \sum_{n=1}^{N} c_n f_n\right)\\ &= \left(\left(\sum_{n=1}^{N} c_n f_n, \sum_{n=1}^{N} c_n f_n\right) - \left(\sum_{n=1}^{N} c_n f_n, \sum_{n=1}^{N} c_n f_n\right)\right) + \left(\sum_{n=1}^{N} c_n f_n, \sum_{n=N+1}^{\infty} c_n f_n\right).\end{aligned}$$

$$= \left(\sum_{n=1}^{N} c_n f_n, \sum_{n=N+1}^{\infty} c_n f_n \right).$$

However, since $\{f_n\}$ is an orthonormal set, $(f_i, f_j) = 0$ when $i \neq j$, and so this term simplifies to 0.

Thus, we have that

$$\left(\sum_{n=1}^{N} c_n f_n, f - \sum_{n=1}^{N} c_n f_n \right) = 0$$

as desired.

Note: This formula makes sense as we apply the fact that f_n is an orthonormal set. If we subtract off the first N terms in the Fourier series, we are left with a sum that is "orthogonal" to our original sum, so we should find that the inner product is 0. I think we may have briefly covered orthogonal projections in linear algebra; perhaps we could imagine this as an orthogonal projection of an orthogonal subspace onto another one (which should be 0). I am not too sure about this interpretation. ■