Homework 1

Partial Differential Equations, Spring 2023 — David Yang

Logan Chapter 1.1, Problem 6

Verify that

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

is a solution to the wave equation $u_{tt} = c^2 u_{xx}$, where c is a constant and g is a given continuously differentiable function.

Solution. We will calculate the partial derivatives u_{tt} and u_{xx} separately. We will first calculate u_t and then u_{tt}. Note that

$$u_t = \frac{d}{dt}u(x,t) = \frac{d}{dt}\left(\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds\right).$$

By Leibniz's rule, we know that

$$\frac{d}{dt}\left(\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds\right) = \frac{1}{2c}\left[g(x+ct)\frac{d}{dt}(x+ct) - g(x-ct)\frac{d}{dt}(x-ct)\right].$$

Simplifying, we find that

$$u_t = \frac{1}{2c}\left[g(x+ct)\cdot c + g(x-ct)\cdot c\right] = \frac{g(x+ct)+g(x-ct)}{2}.$$

We can now take the partial of u_t with respect to t to find u_{tt}. We get that

$$u_{tt} = \frac{g'(x+ct)\frac{d}{dt}(x+ct) + g'(x-ct)\frac{d}{dt}(x-ct)}{2} = c\frac{g'(x+ct)-g'(x-ct)}{2}.$$

We will follow a similar procedure to calculate u_x and u_{xx}. We find that

$$u_x = \frac{d}{dx}u(x,t) = \frac{d}{dx}\left(\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds\right).$$

Applying Leibniz's Rule, we get that

$$\frac{d}{dx}\left(\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds\right) = \frac{1}{2c}\left[g(x+ct)\frac{d}{dx}(x+ct) - g(x-ct)\frac{d}{dx}(x-ct)\right].$$

Simplifying, we get that

$$u_x = \frac{g(x+ct)-g(x-ct)}{2c}.$$

Taking the partial of u_x with respect to x to determine u_{xx}, we find that

$$u_{xx} = \frac{g'(x+ct)\frac{d}{dx}(x+ct) - g'(x-ct)\frac{d}{dx}(x-ct)}{2c} = \frac{g'(x+ct)-g'(x-ct)}{2c}.$$

We see that

$$u_{tt} = c\frac{g'(x+ct) - g'(x-ct)}{2} = c^2\left(\frac{g'(x+ct) - g'(x-ct)}{2c}\right) = c^2u_{xx}$$

and so $u_{tt} = c^2u_{xx}$, as desired. $\blacksquare$