# Homework 6

Partial Differential Equations, Spring 2023 David Yang

HW 6 Problems

1. **Find a solution to the diffusion PDE**

$$u_t - u_{xx} = 0 \text{ for } x \in \mathbb{R},\, t > 0$$

**with initial value**

$$u(x,0) = e^{-x^2/4} \text{ for } x \in \mathbb{R}.$$

*Solution.* We will use the solution formula for the diffusion equation on an unbounded domain. Recall that the solution to the diffusion PDE $u_t - ku_{xx} = 0$ on the unbounded domain with initial value $u(x,0) = \phi(x)$ is is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-(x-y)^2/(4kt)}\, dy.$$

Applying the solution formula to this problem (where $k = 1$ and $\phi(x) = e^{-x^2/4}$), we have that the solution to the given diffusion PDE on the unbounded domain is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-y^2/4} e^{-(x-y)^2/(4t)}\, dy.$$

Simplifying the integrand by combining the exponential terms, we get that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-ty^2/(4t)} e^{-(x-y)^2/(4t)}\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2 - ty^2}{4t}}\, dy \end{aligned}$$

Expanding the numerator and then completing the square, we get that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2 - ty^2}{4t}}\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x^2 - 2xy + (t+1)y^2)}{4t}}\, dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(t+1)\left(y - \frac{x}{t+1}\right)^2 + x^2 \cdot \frac{t}{t+1}}{4t}}\, dy. \end{aligned}$$

Factoring out a $e^{-\frac{x^2 \cdot \frac{t}{t+1}}{4t}} = e^{-\frac{x^2}{4(t+1)}}$ term from the integrand, we get that

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4(t+1)}} \int_{-\infty}^{\infty} e^{-\frac{(t+1)\left(y - \frac{x}{t+1}\right)^2}{4t}}\, dy$$

We will now make a substitution to transform the integrand into $e^{-r^2}$: let

$$r = \frac{\left(y - \frac{x}{t+1}\right)\sqrt{t+1}}{\sqrt{4t}}.$$

Then we also have that

$$dr = \frac{\sqrt{t+1}}{\sqrt{4t}}\, dy.$$

Making the substitution for $r$ in our solution $u(x,t)$, we find that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4(t+1)}} \int_{y=-\infty}^{y=\infty} e^{-\frac{(t+1)\left(y-\frac{x}{t+1}\right)^2}{4t}}\, dy \\ &= \frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}} \int_{r=-\infty}^{r=\infty} e^{-r^2}\, dr. \end{aligned}$$

But we also know that $\int_{-\infty}^{\infty} e^{-r^2}\, dr = \sqrt{\pi}$, so plugging this back into our solution, we find that

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}} \int_{r=-\infty}^{r=\infty} e^{-r^2}\, dr \\ &= \left(\frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}}\right) \cdot \sqrt{\pi} \\ &= \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}}. \end{aligned}$$

Thus, our solution to the given diffusion PDE with the given initial value is

$$\boxed{u(x,t) = \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}}.}$$

■

2. **Show that your solution to # 1 satisfies the property that, for all $t > 0$,**

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u(x,0)\,dx.$$

**In other words, $\int_{-\infty}^{\infty} u(x,t)\,dx$ is a *conserved quantity* (constant with respect to $t$).**

*Solution.* To show that this property holds, we will show that the left and right hand sides simplify to the same expression. We will begin by working with the left-hand side.

Plugging in our solution for $u(x,t)$ from the first problem into the left-hand side of the equation, we have that

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}}\,dx.$$

Let us make the substitution $y = \frac{x}{\sqrt{4(t+1)}}$. We also have that $dy = \frac{1}{\sqrt{4(t+1)}}\,dx$. Substituting this into our integral, we get that

$$\begin{aligned}
\int_{-\infty}^{\infty} u(x,t)\,dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{4(t+1)}}\,dx \\
&= 2\int_{-\infty}^{\infty} \frac{1}{\sqrt{4(t+1)}} e^{-\frac{x^2}{4(t+1)}}\,dx \\
&= 2\int_{y=-\infty}^{y=\infty} e^{-y^2}\,dy.
\end{aligned}$$

We will now show that the right-hand side $\int_{-\infty}^{\infty} u(x,0)\,dx$ can be rewritten in the same form. Plugging in the initial condition $u(x,0) = e^{-x^2/4}$ into this integral, we get

$$\int_{-\infty}^{\infty} u(x,0)\,dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{4}}\,dx.$$

Let us now make the substitution $y = \frac{x}{2}$. This also gives us $dy = \frac{1}{2}\,dx$. Substituting this back into our integral, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} u(x,0)\,dx &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{4}}\,dx \\
&= 2\int_{-\infty}^{\infty} \frac{1}{2} e^{-\frac{x^2}{4}}\,dx \\
&= 2\int_{y=-\infty}^{y=\infty} e^{-y^2}\,dy.
\end{aligned}$$

Thus, we see that

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u(x,0)\,dx = 2\int_{-\infty}^{\infty} e^{-y^2}\,dy = 2\sqrt{\pi}.$$

and so we find that $\int_{-\infty}^{\infty} u(x,t)\,dx$ is a *conserved quantity* (constant with respect to $t$), with $\int_{-\infty}^{\infty} u(x,t)\,dx = 2\sqrt{\pi}$ for all $t \geq 0$, as desired. ■

3. (a) **If $u$ solves the diffusion equation on the infinite domain ($x \in \mathbb{R}$), with bounded initial value $u(x,0) = \phi(x)$ that has the property that**

$$\lim_{x \to -\infty} \phi(x) = a \textbf{ and } \lim_{x \to \infty} \phi(x) = b \quad \textbf{(a, b constants)}.$$

**What is the value of $\lim_{t \to \infty} u(x,t)$?**

*Solution.* $u$ is a solution to the diffusion equation on the infinite domain with bounded initial value $u(x,0) = \phi(x)$. Consequently, the solution (in the Poisson Integral representation form) is

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} \phi(x - r\sqrt{4kt})\, dr.$$

Splitting the integral up into two separate integrals, we get that

$$u(x,t) = \frac{1}{\sqrt{\pi}} \left[ \int_{-\infty}^{0} e^{-r^2} \phi(x - r\sqrt{4kt})\, dr + \int_{0}^{\infty} e^{-r^2} \phi(x - r\sqrt{4kt})\, dr \right]$$

Thus,

$$\lim_{t \to \infty} u(x,t) = \lim_{t \to \infty} \frac{1}{\sqrt{\pi}} \left[ \int_{-\infty}^{0} e^{-r^2} \phi(x - r\sqrt{4kt})\, dr + \int_{0}^{\infty} e^{-r^2} \phi(x - r\sqrt{4kt})\, dr \right].$$

Per the given hint, we can pass the limit with respect to $t$ inside the integral, so we get that

$$\lim_{t \to \infty} u(x,t) = \frac{1}{\sqrt{\pi}} \left[ \int_{-\infty}^{0} \lim_{t \to \infty} e^{-r^2} \phi(x - r\sqrt{4kt})\, dr + \int_{0}^{\infty} \lim_{t \to \infty} e^{-r^2} \phi(x - r\sqrt{4kt})\, dr \right].$$

We can evaluate each of the limits separately: note that when $r < 0$ (corresponding to the first integral), we have that

$$\begin{aligned} \lim_{t \to \infty} e^{-r^2} \phi(x - r\sqrt{4kt})\, dr &= e^{-r^2} \lim_{t \to \infty} \phi(x - r\sqrt{4kt}) \\ &= e^{-r^2} \lim_{y \to \infty} \phi(y) \\ &= e^{-r^2} b. \end{aligned}$$

Similarly, we have that when $r > 0$ (corresponding to the second integral),

$$\begin{aligned} \lim_{t \to \infty} e^{-r^2} \phi(x - r\sqrt{4kt})\, dr &= e^{-r^2} \lim_{t \to \infty} \phi(x - r\sqrt{4kt}) \\ &= e^{-r^2} \lim_{y \to -\infty} \phi(y) \\ &= e^{-r^2} a. \end{aligned}$$

Plugging these results back into our equation, we find that

$$\begin{aligned} \lim_{t \to \infty} u(x,t) &= \frac{1}{\sqrt{\pi}} \left[ \int_{-\infty}^{0} \lim_{t \to \infty} e^{-r^2} \phi(x - r\sqrt{4kt})\, dr + \int_{0}^{\infty} \lim_{t \to \infty} e^{-r^2} \phi(x - r\sqrt{4kt})\, dr \right] \\ &= \frac{1}{\sqrt{\pi}} \left[ \int_{-\infty}^{0} e^{-r^2} b\, dr + \int_{0}^{\infty} e^{-r^2} a\, dr \right] \end{aligned}$$

Factoring out the constants $a$ and $b$ and using the fact that

$$\int_{-\infty}^{0} e^{-r^2}\,dr = \int_{0}^{\infty} e^{-r^2}\,dr = \frac{\sqrt{\pi}}{2},$$

we have that

$$\begin{aligned}
\lim_{t\to\infty} u(x,t) &= \frac{1}{\sqrt{\pi}}\left[\int_{-\infty}^{0} e^{-r^2} b\,dr + \int_{0}^{\infty} e^{-r^2} a\,dr\right] \\
&= \frac{1}{\sqrt{\pi}}\left[b\int_{-\infty}^{0} e^{-r^2}\,dr + a\int_{0}^{\infty} e^{-r^2}\,dr\right] \\
&= \frac{1}{\sqrt{\pi}}\left[b\frac{\sqrt{\pi}}{2} + a\frac{\sqrt{\pi}}{2}\right] \\
&= \frac{1}{2}(a+b).
\end{aligned}$$

Thus, we have that

$$\boxed{\lim_{t\to\infty} u(x,t) = \frac{1}{2}(a+b)}.$$

■

(b) **Review Eq 2.5 on page 82 of Logan, which is a solution for the PDE**

$$w_t = kw_{xx} \text{ for } x \in \mathbb{R},\ t > 0$$

$$w(x,0) = 0 \text{ for } x < 0\,;\ w(x,0) = 1 \text{ for } x > 0.$$

**What is** $\lim_{t\to\infty} w(x,t)$ **and does this agree with your result in 3(a)?**

*Solution.* Eq 2.5 on page 82 of Logan gives us a solution to the given PDE:

$$w(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_{0}^{x/(\sqrt{4kt})} e^{-r^2}\,dr.$$

Taking the limit of $w(x,t)$ as $t$ approaches $\infty$, we find that

$$\lim_{t\to\infty} w(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\lim_{t\to\infty}\int_{0}^{x/(\sqrt{4kt})} e^{-r^2}\,dr.$$

Since $\lim_{t\to\infty}\frac{x}{\sqrt{4kt}} = 0$, we have that

$$\begin{aligned}
\lim_{t\to\infty} w(x,t) &= \frac{1}{2} + \frac{1}{\sqrt{\pi}}\lim_{t\to\infty}\int_{0}^{x/(\sqrt{4kt})} e^{-r^2}\,dr \\
&= \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_{0}^{0} e^{-r^2}\,dr \\
&= \frac{1}{2} + \frac{1}{\sqrt{\pi}}\cdot 0 \\
&= \frac{1}{2}.
\end{aligned}$$

Since $\lim_{x\to-\infty} w(x,0) = 0$ and $\lim_{x\to\infty} w(x,0) = 1$, and

$$\lim_{t\to\infty} w(x,t) = \frac{1}{2} = \frac{1}{2}(0+1),$$

we see that our answer does match up with our result from 3(a), as desired. ■