Homework 6

Partial Differential Equations, Spring 2023 David Yang

HW 6 Problems

1. **Find a solution to the diffusion PDE**

$$u_t - u_{xx} = 0 \text{ for } x \in \mathbb{R},\, t > 0$$

with initial value

$$u(x,0) = e^{-x^2/4} \text{ for } x \in \mathbb{R}.$$

Solution. We will use the solution formula for the diffusion equation on an unbounded domain. Recall that the solution to the diffusion PDE $u_t - ku_{xx} = 0$ on the unbounded domain with initial value $u(x,0) = \phi(x)$ is is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-(x-y)^2/4kt}\, dy.$$

Applying the solution formula to this problem (where $k = 1$ and $\phi(x) = e^{-x^2/4}$), we have that the solution to the given diffusion PDE on the unbounded domain is ■

2. **Show that your solution to # 1 satisfies the property that, for all $t > 0$,**

$$\int_{-\infty}^{\infty} u(x,t)\, dx = \int_{-\infty}^{\infty} u(x,0)\, dx.$$

In other words, $\int_{-\infty}^{\infty} u(x,t)\, dx$ is a *conserved quantity* (constant with respect to t).

3. (a) **If u solves the diffusion equation on the infinite domain ($x \in \mathbb{R}$), with bounded initial value $u(x,0) = \phi(x)$ that has the property that**

$$\lim_{x \to -\infty} \phi(x) = a \textbf{ and } \lim_{x \to \infty} \phi(x) = b \quad \textbf{(a, b constants)}.$$

What is the value of $\lim_{t \to \infty} u(x,t)$?

(b) **Review Eq 2.5 on page 82 of Logan, which is a solution for the PDE**

$$w_t = kw_{xx} \text{ for } x \in \mathbb{R},\, t > 0$$

$$w(x,0) = 0\text{for } x < 0\,;\, w(x,0) = 1\text{for } x > 0.$$

What is $\lim_{t \to \infty} w(x,t)$ and does this agree with your result in 3(a)?