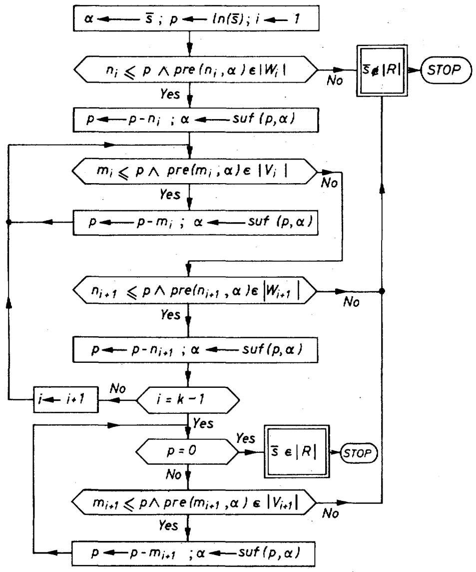

Fig. 9. Recognition algorithm of contours.

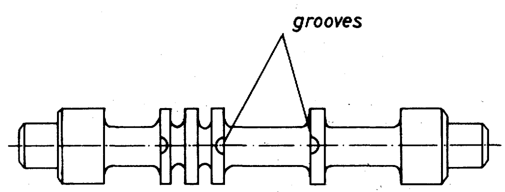

Fig. 10. Machine element with extractions (grooves).

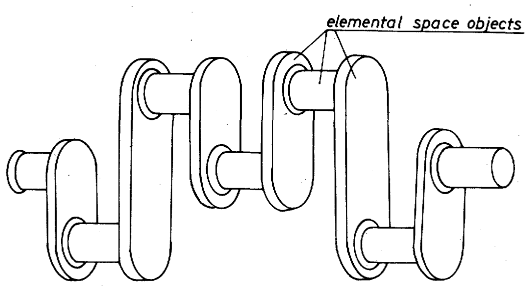

Fig. 11. Machine element composed of elemental space objects.

element from which some solids have been extracted. The description of this element leads to a description of its rotary part and extractions. The methods for the description of those extractions have been extensively studied by the authors. An example of an element with extractions is shown in Fig. 10.

Let us look at other kinds of nonrotary elements. Notice that space objects (elemental) which are made to contact each other successively (see Fig. 11) create a class of those nonrotary elements which may be described by the presented formalism extended in some way. This extension leads to a conversion of the segments previously defined into the elemental space objects. In that case we deal with the connection surfaces between the objects in contact instead of points as in the established formalism. Consequently, the parameters associated with every object (see Definition 2) in a given element specify contact surface.

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New Concepts for Three-Dimensional Shape Analysis

K. JAYARAM UDUPA AND I. S. N. MURTHY

***Abstract*—A new approach to machine representation and analysis of three-dimensional objects is presented. The representation, based on the notion of "skeleton" of an object leads to a scheme for comparing two given object views for shape relations. The objects are composed of long, thin, rectangular prisms joined at their ends. The input picture to the program is the digitized line drawing portraying the three-dimensional object. To compare two object views, two characteristic vertices called "cardinal point" and "end-cardinal point," occurring consistently at the bends and open ends of the object are detected. The skeletons are then obtained as a connected path passing through these points. The shape relationships between the objects are then obtained from the matching characteristics of their skeletons. The method explores the possibility of a more detailed and finer analysis leading to detection of features like symmetry, asymmetry and other shape properties of an object.**

***Index Terms*—Cardinal points, machine perception, pattern recognition, picture processing, shape analysis skeletal and mirror matching, skeleton of an object.**

I. INTRODUCTION

Descriptive approach to pattern recognition has been quite popular in the analysis of two-dimensional patterns [1]–[4]. It is readily realized that the description and analysis of three-dimensional objects from a straightforward extension of these concepts to be equivalent two-dimensional line drawing is quite complex. However attempts have been made [5]–[8] with significant success for machine recognition of three-dimensional objects through an analysis of their two-dimensional line drawings. These techniques concentrate on a set of vertices in the given object and their implications are exploited to obtain a structural model for the object.

It is felt that a more general approach to three-dimensional pattern description should involve the notion of a "spine-like-skeleton" of the object or the concept of learning as in [8]. The

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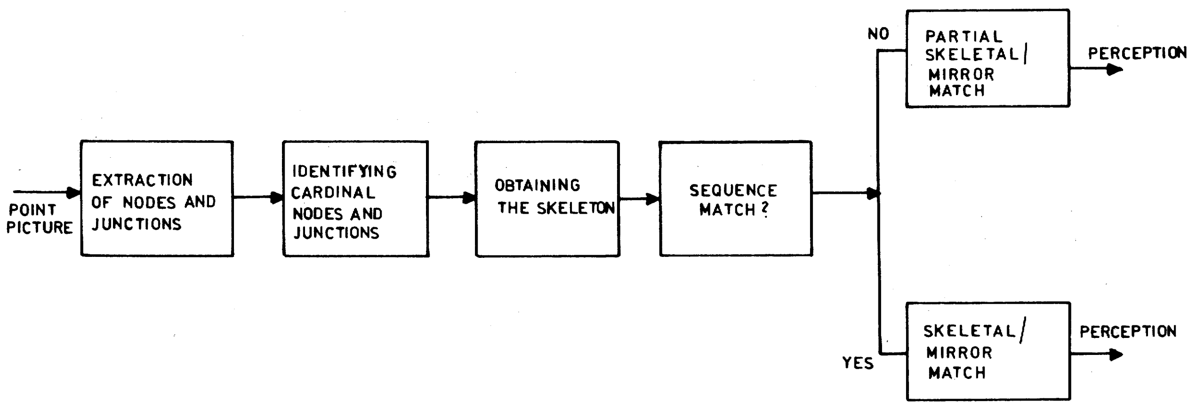


Fig. 1. Block diagram of the scheme.

problem of defining the "skeleton" of an object unambiguously (as is done for two-dimensional patterns in [2]) is quite challenging.

In the present work a scheme is presented for comparing two given views of objects for shape relationships between them. The scheme makes use of the representational model developed in [9] and is based on two concepts of skeleton and cardinal point for three-dimensional objects. This has similarities with the configurational model discussed in [10] for two-dimensional objects. Cardinal points are nodes or junctions that characterize certain invariant properties consistently occurring at bends and open ends of the object. The skeleton is an ordered and connected track passing through cardinal points and can be thought of as a spine of the given object. It is shown that the matching properties of the skeletons of two given object views reveal how the objects portrayed by the two views are related to each other. The tasks considered here are somewhat similar to those considered in [6], wherein a syntax-directed program determines whether two given objects are identical, mirror images of each other or are structurally different. The present method, because of its general approach, can in addition determine if there exist partially identical or partially mirror image matches. Also, by comparing the skeletons of the two halves of an object symmetry, if any, can be detected. The block diagram representation of the scheme is shown in Fig. 1. The object in Fig. 2(a) whose skeleton is shown in Fig. 2(b) is considered for purposes of illustration.

II. Shape Analysis

Input Format

The point picture obtained by digitizing the line drawing portraying a three-dimensional object is the input to the program. For testing purposes this has been done manually.

Perception through Matching the Skeletons

To solve the perceptual problems posed here the machine has to determine if the two line drawings portraying two views of three-dimensional bodies are as follows:

1) different views of objects of same shape;
2) views of objects which are mirror images of each other;
3) two views of structurally different objects; and
4) different views of objects of partially identical shape (two objects are of partially identical shape if a part of one object is identical in shape to some part of the other).

For this purpose the skeletons of the two views are obtained as explained in [9]. Let sequences S_1 and S_2 represent the skeletons of the two views.

$$S_1 = a_1^1 \cdots a_1^1 b_1^2 \cdots b_1^2 \cdots m_1^i \cdots m_1^i \cdots k_1^l \cdots k_1^l$$
$$S_2 = a_2^1 \cdots a_2^1 b_2^2 \cdots b_2^2 \cdots m_2^r \cdots m_2^r \cdots k_2^s \cdots k_2^s$$

where a_1^1, $b_1^2 \cdots k_1^l$ and a_2^1, $b_2^2 \cdots k_2^s$ are slope-code numbers 1, $\cdots$,6 of the line segments in the skeleton (slope-code numbers are codes representing a coded axis system with reference to which the given view is a basic view. A basic view of an object bounded by planes $P_1, \cdots, P_N$ is a view in which any two mutually perpendicular planes P_i,P_j are parallel to two of the three planes XY,YZ,ZX). Thus, for example $(a_1^1 a_1^1 \cdots a_1^1)$ represents a line segment in the skeleton between two cardinal pairs and is labeled as a section. A section, hence, is a string of same slope-code number. The number of elements in a section indicates the length of the corresponding line segment in the line drawing. Keeping the above objectives in mind the following definitions will be made before we proceed further.

Two given views are said to sequence match if and only if the number of sections in the two sequences representing the two skeletons of the given views is same, i.e., $l = s$ if the two sequences S_1 and S_2 given above are to sequence match.

Two given views are said to skeletal match if

a) the two views sequence match;

b) there exist mappings $f_i: s_1^i \rightarrow s_2^i$, $i = 1, \cdots s$ (s_1^i and s_2^i are ith sections of the sequences S_1 and S_2, respectively) such that $f_1 = f_2 = f_s$ or, there exist identity mappings $I_j: s_1^j \rightarrow s_2^j$, $j = k,l,p, \cdots 1 \leqslant (k,l,p \cdots) \leqslant s$ such that $(f_1,f_2, \cdots ,f_{k-1} I_k, f_{k+1}, \cdots ,f_{l-1}, I_l, f_{l+1}, \cdots ,f_s)$ is the set of all mappings from S_1 to S_2, $f_1 = f_2 = \cdots = f_{k-1} = f_{k+1} = \cdots f_{l-1} = f_{l+1} = \cdots = f_s$ and $I_k = \cdots = I_l = \cdots = \cdots I_p = \cdots$. The functions $f_1,f_2, \cdots$ are as defined above.

The mappings defined above are obtained as follows. Write down the slope-code numbers in the order 1 2 3 4 5 6. Strike out the digit n and its "conjugate" $(n + 3)$ modulo 6 where n is the slope-code number appearing in both the sections s_1^j and s_2^j for some $j = k,p,l \cdots, \cdots$ (hence s_1^j and s_2^j are identity maps of each other). Now the mapping $f_i: s_1^i \rightarrow s_2^i$ where say $s_1^i = (aa \cdots a)$ and $s_2^i = (bb \cdots)$, $i \neq j, i = 1, \cdots ,s$ is obtained as the following relation: f_i associates, with a, an uncancelled digit to the immediate right of a, etc. If there are no identity mappings the first condition of b) above may be directly applied. For illustration let it be required to check the skeletal match between S_1 = 1111 2222 6666 and S_2 = 3333 2222 1111. Noting that $s_1^2 \rightarrow s_2^2$ is an identity map I_2, in the slope-code scale cancel out digits 2 and 5: 1 $\cancel{2}$ 3 4 $\cancel{5}$ 6. Now $f_1: s_1^1 \rightarrow s_2^1$ maps every slope-code number 1 to a slope-code number 3. Thus, f_1 may be identified as (with reference to the slope-code scale) associating an uncancelled slope-code number with another uncancelled slope-code number to the immediate right of it. It may be noticed that $f_3: s_1^3 \rightarrow s_2^3$ is also defined by the same relation if it is to map 6 to 1 (notice that the slope-code scale

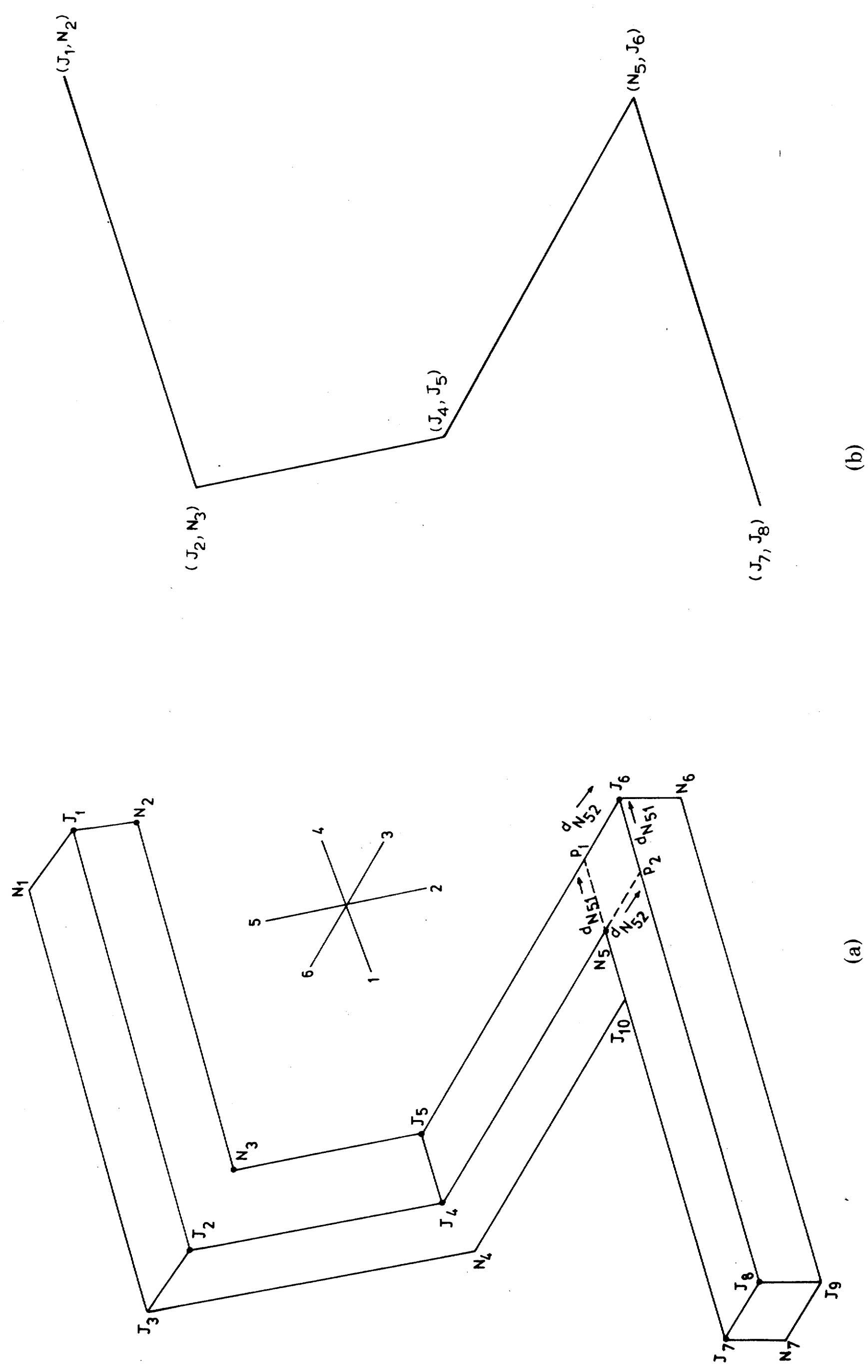


Fig. 2. (a) Object view for the running example. (b) Skeleton of the object in Fig. 2(a).

is cyclic and hence to the immediate right of 6 is 1). Thus, in addition to S_1 and S_2 sequence matching they skeletal match also since $f_1 = f_3$.

Now the task administered to the machine is executed in the following steps.

Step 1—Identical Shape:

If S_1 and S_2 do not sequence match then the two possibilities are that the objects they represent are structurally different or of partially identical shape. These two cases are investigated later.

If S_1 and S_2 sequence match but do not skeletal match,[1] then add 3 modulo 6 to every element (conjugate each element) of the sequence S_2 and reverse the sequence[2] (tail to head and vice versa) to obtain the modified sequence S_2'. If S_2' and S_1 skeletal match, they represent objects of identical shape. If they do not, then the possibilities are that they represent structurally different, mirror image, or partially identical objects.

Step 2—Mirror Image:

We shall now investigate the possibility of one of the objects represented by S_1 or S_2 being a mirror image of the other. Obtain the nine modified sequences $S_2(1), S_2(2), \cdots, S_2(9)$ from S_2 using the mappings $f_1 = f_2 = \cdots = f_s = F_1, f_1 = f_2 = \cdots = f_s = F_2, \cdots, \cdots, f_1 = f_2 = f_s = F_9$, respectively given in Table I. However, all those sections of S_2 which have slope-code numbers 1 or 4 as their elements are identically mapped on to corresponding sections in $S_2(1), S_2(2), S_2(3)$, sections in S_2 with 2 or 5 as their elements are mapped identically on to corresponding sections in $S_2(4), S_2(5), S_2(6)$ and sections in S_2 with 3 or 6 as their elements are mapped identically on to corresponding sections in $S_2(7), S_2(8), S_2(9)$. As can be verified these nine modified sequences correspond to the nine basic views of the object represented by S_2. Now, we say the views represented by S_1 and S_2 mirror match if there exist mappings $f_i: s_1^i \to s_{2k}^i$ (where s_1^i and s_{2k}^i are the ith sections of S_1 and $S_2(k)$) such that $f_1 = f_2 = \cdots = f_s$ hold, or there exist conjugate mappings $f_j^*: s_1^j \to s_{2k}^j, j = p,q,r \cdots, 1 \leq (p,q,r \cdots) \leq s$, such that $(f_1 f_2 \cdots f_{p-1} f_p^*, f_{p+1}, \cdots f_{q-1}, f_q^*, f_{q+1}, \cdots, \cdots, f_s)$ is the set of all mappings from S_1 to $S_2(k)$, $f_1 = f_2 = \cdots = f_{p-1} = f_{p+1} = \cdots = f_{q-1} = f_{q+1} = \cdots = f_s$ and $f_p^* = \cdots = f_q^* = \cdots = f_r^* = \cdots$. The mappings $f_1, f_2, \cdots$ are as defined above.

The relations defining f_i's are obtained as described for skeletal match. f_j^*'s are identified by the following relation. If $s_1^j = (aa \cdots a)$ and $s_{2k}^j = (bbb \cdots b)$ then the mapping $s_1^j \to s_{2k}^j$ is f_j^* if $b = (a + 3)$ modulo 6.

If S_1 does not mirror match any one of the sequences $S_2(1) \cdots, S_2(9)$ then obtain $S_2'(1), \cdots, S_2'(9)$ from S_2' as described earlier. Examine if S_1 mirror matches any one of $S_2'(1), \cdots, S_2'(9)$. If it does not one can conclude that the objects represented by S_1 and S_2 are not mirror images of each other. The two remaining possibilities are that they may be of partially identical shapes or may be structurally completely different objects.

Step 3—Partial Identity:

We say that two views represented by sequences S_1 and S_2 partially skeletal match if there exist mappings $f_k: s_1^{i+k-1} \to s_2^{j+k-1}$, $k = 1, 2, \cdots m_1$, i and j could be any numbers between $1, \cdots, l$ and $1, \cdots, s$, respectively (note that S_1 and S_2 need not sequence match) such that $f_1 = f_2 = \cdots = f_{m1}$; or there exist identity mappings $I_p: s_1^{i+p-1} \to s_2^{j+p-1}$, $p = t,u,v \cdots, 1 \leq (t,u,v, \cdots) \leq m_1$, such that $(f_1, f_2 \cdots f_{t-1}, I_t, f_{t+1}, \cdots, f_{u-1}, I_u, f_{u+1}, \cdots, \cdots f_{m1})$ is the set of all mappings $s_1^{i+k-1} \to s_2^{j+k-1}, k = 1, \cdots, m_1, f_1 = f_2 = \cdots = f_{t-1} = f_{t+1} = \cdots = f_{u-1} = f_{u+1} = \cdots = f_{m1}$ and $I_t = \cdots = I_u = \cdots = I_v = \cdots$. The mappings $f_1, f_2, \cdots$ are as defined above.

As before, if S_1 and S_2 do not partially skeletal match, obtain S_2 and examine if S_1 and S_2' do so. If they do not for any i and j then the objects they represent are structurally different.

[1] One of the reasons for this nonmatching may be that the assumed orientation of S_2 on S_1 is not correct.

[2] This will set right the orientation of S_2 on S_1 should there be a wrong assumption about the initial orientation of S_2 on S_1.

TABLE I
Mappings for Testing Mirror Match

Mapping	Relation
F_1	6 $\cancel{1}$ 2 3 $\cancel{4}$ 5 6 $\cancel{1}$
F_2	6 $\cancel{1}$ 2 3 $\cancel{4}$ 5 6 $\cancel{1}$
F_3	6 $\cancel{1}$ 2 3 $\cancel{4}$ 5 6 $\cancel{1}$
F_4	6 1 $\cancel{2}$ 3 4 $\cancel{5}$ 6 1
F_5	6 1 $\cancel{2}$ 3 4 $\cancel{5}$ 6 1
F_6	6 1 $\cancel{2}$ 3 4 $\cancel{5}$ 6 1
F_7	$\cancel{6}$ 1 2 $\cancel{3}$ 4 5 $\cancel{6}$ 1
F_8	$\cancel{6}$ 1 2 $\cancel{3}$ 4 5 $\cancel{6}$ 1
F_9	$\cancel{6}$ 1 2 $\cancel{3}$ 4 5 $\cancel{6}$ 1

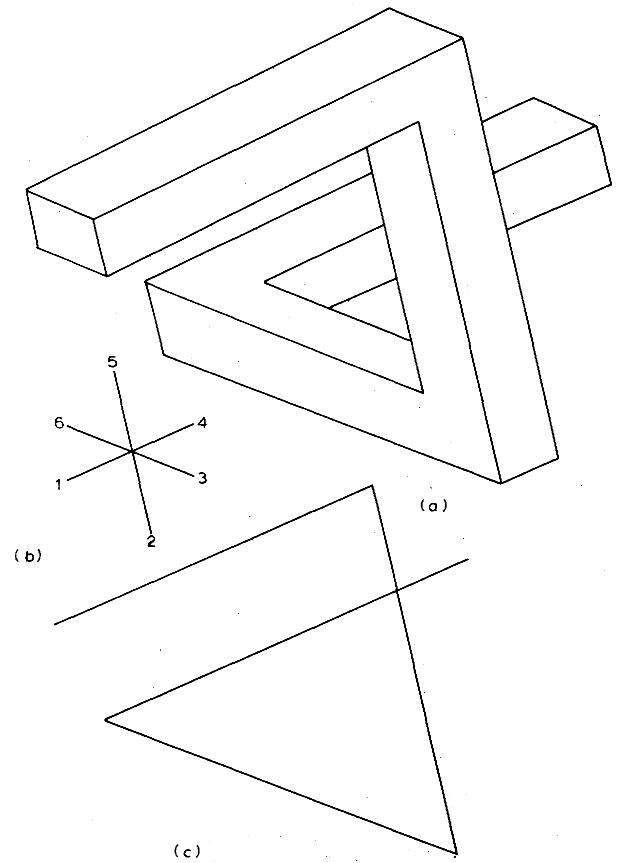
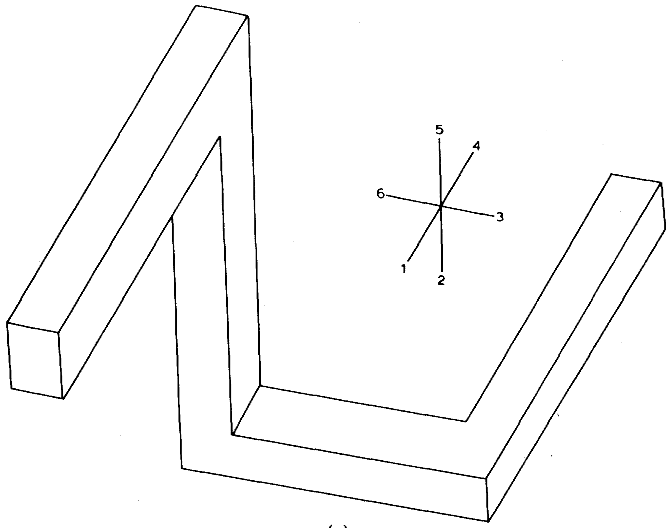
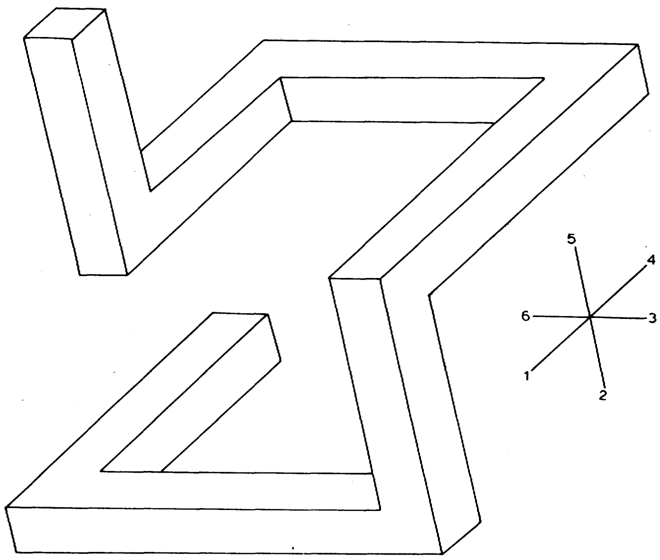


Fig. 3. (a) Another view of the object in Fig. 2(a). (b) Skeleton of Fig. 3(a). (c) Slope-code diagram for Fig. 3(a).

In a similar manner one can define partial mirror match and identify portion(s) of S_1 that mirror match some portion(s) of S_2. In general it is possible that a subsequence in S_1 skeletal or mirror matches a subsequence in S_2 and at the same time mirror or skeletal matches another subsequence also in S_2. In such cases one concludes that although complete skeletal or mirror match does not exist, the objects represented by S_1 and S_2 resemble each other to some extent. It is also possible to find out in a given object whether there are repetitions of some portions having a particular shape or its mirror image. This can be done by dividing the sequence S representing the object into subsequences and examining for skeletal and mirror matches amongst themselves. This immediately suggests that it is possible to identify symmetry in a given object by dividing S into two subsequences and examining if they mirror match.



(a)



(b)

Fig. 4. (a) Mirror image of the object in Fig. 2(a). (b) Object having partial matches with that in Fig. 2(a).

TABLE II
Procedure for Skeletal Matching of Figs. 2(a) and 3(a)

S_1: 4444 .... 6666 .... 5555 4444
S_2: 4444 .... 2222 .... 6666 4444

Mapping	Relation	Conclusion
I_1	$6\ \cancel{1}\ 2\ 3\ \cancel{4}\ 5\ 6\ 1$	$I_1 = I_4$
f_2	$6\ \cancel{1}\ 2\ 3\ \cancel{4}\ 5\ 6\ 1$	$f_2 = f_3$
f_3	$6\ \cancel{1}\ 2\ 3\ \cancel{4}\ 5\ 6\ 1$	S_1 and S_2 Skeletal match
I_4	$6\ \cancel{1}\ 2\ 3\ \cancel{4}\ 5\ 6\ 1$	

Some examples appear in Figs. 3 and 4 to illustrate some of the ideas given above. Table II explains the process of matching of the object of Fig. 2(a) with that of Fig. 3(a). The skeleton of the object view of Fig. 3(a) is shown in Fig. 3(b) and the slope-code diagram in Fig. 3(c). From Table II it is clear that the objects in addition to sequence matching skeletal match also and hence are of same shape. Fig. 4(a) shows another view which is a mirror image of that in Fig. 2(a) and their matching process is summed up in Table III. Fig. 4(b) shows an object whose shape is partially identical to that of the object in Fig. 2(a) and Table IV explains how this is identified. Fig. 5 shows a symmetric object and the symmetry is reflected in the mirror match of the two subsequences of S as shown in Table V.

III. Conclusion and Comments

A method for machine representation and analysis of three-dimensional objects based on the concept of coding the skeleton has been presented. It is shown that the cardinal points are a clue

TABLE III
Mirror Matching Figs. 2(a) and 4(a)

S_1: 44⋯ 66⋯ 55⋯ 44⋯ S_2: 44⋯ 22⋯ 33⋯ 44⋯	Conclusions
$S_2(1)$: 44⋯ 33⋯ 55⋯ 44⋯ $S_2(2)$: 44⋯ 55⋯ 66⋯ 44⋯ $S_2(3)$: 44⋯ 66⋯ 22⋯ 44⋯ $S_2(4)$: 66⋯ 22⋯ 44⋯ 66⋯ $S_2(5)$: 11⋯ 22⋯ 66⋯ 11⋯ $S_2(6)$: 33⋯ 22⋯ 11⋯ 33⋯ $S_2(7)$: 55⋯ 44⋯ 33⋯ 55⋯ $S_2(8)$: 11⋯ 55⋯ 33⋯ 11⋯ $S_2(9)$: 22⋯ 11⋯ 33⋯ 22⋯	
$S_1 \rightarrow S_2(3)$: $f_1 = f_2 = f_4 = I$, $f_3 = f_3^*$	S_1 mirror matches S_2

TABLE IV
Partial Matching Figs. 2(a) and 4(b)

S_1: 44⋯ 66⋯ 55⋯ 44⋯ S_2: 22⋯ 44⋯ 33⋯ 11⋯ 22⋯ 66⋯ 44⋯	Conclusions
$S_1 \supset S_1^{2,4}$: 66⋯ 55⋯ 44⋯ $S_2 \supset S_2^{1,3}$: 22⋯ 44⋯ 33⋯ $(S_2^{1,3})'$: 66⋯ 11⋯ , 55⋯	$f_1 \neq f_2 = f_3$ $f_1 = I$, $f_2 = f_3$ $S_1^{2,4}$ and $S_2^{1,3}$ Skeletal match
$S_1 \supset S_1^{2,4}$: 66⋯ 55⋯ 44⋯ $S_2 \supset S_2^{4,6}$: 11⋯ 22⋯ 66⋯ $S_1^{2,4} \rightarrow S_2^{4,6}(3)$: $f_1 = f_3 = I$, $f_2 = f_2^*$ where $S_2^{4,6}(3)$: 66⋯ 22⋯ 44⋯	$S_1^{2,4}$ mirror matches $S_2^{4,6}$ S_1 and S_2 partially skeletal and mirror match

TABLE V
Symmetry Detection by Mirror Match

S: 44 ⋯ 55 ⋯ 66 ⋯ 44 ⋯ 33 ⋯ 22 ⋯ 44 ⋯	Conclusion
$S \supset S^{1,4}$: 44⋯55⋯66⋯44⋯ $S \supset S^{4,7}$: 44⋯33⋯22⋯44⋯ $S^{1,4} \rightarrow S^{4,7}(1)$: $f_1 = f_2 = f_4 = I$, $f_3 = f_4^*$, where, $S^{4,7}(1)$: 44..55..33..44⋯	Subsequence $S^{1,4}$ having four sections mirror matches $S^{4,7}$ having four sections; object is symmetric

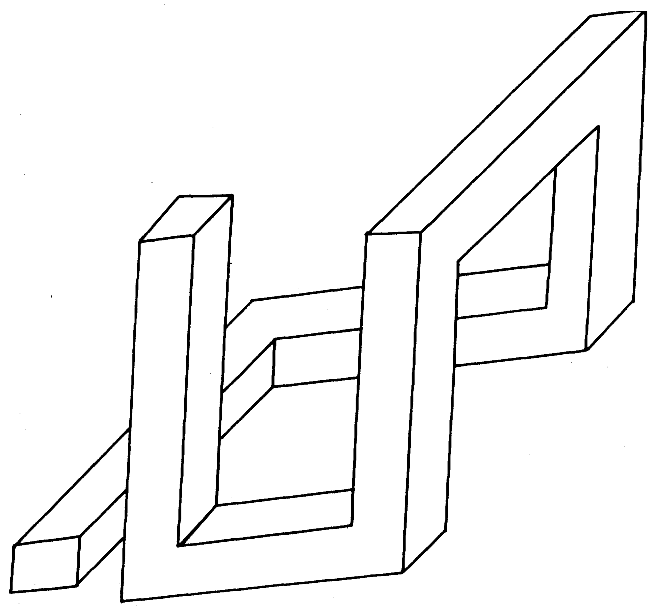


Fig. 6. Partially obscured view of an object.

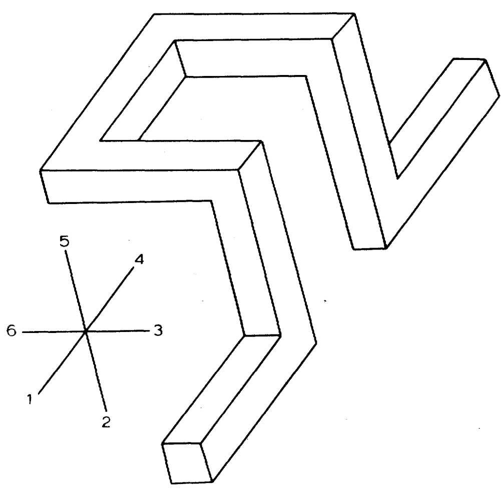


Fig. 5. Symmetric object.

to the existence of bends and open ends and determine the path of the skeleton through them. The central task considered in this correspondence is to compare two given views of objects for shape relationships. It has been shown that the matching properties of the skeletons of the two given object views reveal how the shapes of the two objects are related to each other. The present method is effective in analyzing object views with degeneracy of order one, (but not two and three [9]), and also partially obscured object views (Fig. 6).

Correlation studies on the sequences representing the skeletons of two given views may reveal finer shape relationships between the objects. It appears that the theory can be extended directly to include objects consisting of $N > 3$ intersecting planes by using a coded axis system with $n > 6$ axes dividing the three-dimensional space into equal solid angles.

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An Algorithm for Testing 2-Asummability of Boolean Functions

ANIL K. SARJE AND NRIPENDRA N. BISWAS

***Abstract*—Simple algorithms have been developed to generate pairs of minterms forming a given 2-sum and thereby to test 2-asummability of switching functions. The 2-asummability testing procedure can be easily implemented on the computer. Since 2-asummability is a necessary and sufficient condition for a switching function of upto eight variables to be linearly separable (LS), it can be used for testing LS switching functions of upto eight variables.**

***Index Terms*—Algorithm, asummability, 2-asummability, linear separability, 2-summability, S-sequence, threshold functions.**

I. Introduction

It is known [6] that 2-asummability is a necessary and sufficient condition for the linear separability of switching functions of upto 8 variables. Several methods [1], [3], [4], [8], [9] have been devised for checking the 2-asummability of switching functions. Hopcroft and Mattson [4] construct a decimal number for each minterm, which helps in the testing of 2-asummability.

Bargainer and Coates [1] have presented techniques to obtain decimal numbers for the vertices, that can be used for checking k-summability of a switching function. Using the same technique, Sureshchander [9] has formulated algorithms for testing asummability. Some techniques to reduce the amount of computation have been discussed in Ghosh *et al.* [3], where the information about index numbers of vertices has been utilized.

The 2-summable pairs of vertices have been listed by Sinha Roy [8] in the form of a slide rule which can be used to check 2-asummability of switching functions. The number of operations required however becomes very large as the number of variables, n, increases. Furthermore, the 2-summable pairs are obtained by taking all possible combinations of vertices and then grouping together those giving equal 2-sums. Since the number of possible 2-sums increases very rapidly with n, storing the lists of 2-summable pairs is not a good solution when computation is being done by machine.

In this correspondence, we shall derive an algorithm for testing 2-asummability which is simple and completely programmable.

II. Preliminaries

The two terms, minterm and vertex, have their usual meaning [2], [5] and have been used interchangeably. A minterm is denoted by a binary or by a decimal number.

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The decimal equivalent of a base-2 number, $X = x_1 x_2 \cdots x_n$, will be denoted by dX, i.e.,

$$dX = \sum_{i=1}^{n} 2^{n-i} x_i. \tag{1}$$

Thus, whereas X may denote a vertex or a minterm which can be represented in the binary or the decimal form, dX is purely a number that is obtained by converting the binary form of X into decimal and can participate in arithmetic operations.

Definition 1: The componentwise vector sum of the binary representations of a pair of minterms or vertices is called its *2-sum.* For instance, the 2-sum of the minterms 5 and 12 is

$$\begin{array}{c r} & 5 = 0\ \ 1\ \ 0\ \ 1 \\ ② & \underline{12 = 1\ \ 1\ \ 0\ \ 0} \\ & S = 1\ \ 2\ \ 0\ \ 1 \end{array}$$

We shall use the symbol ② to denote the componentwise vector sum of two minterms[1]. Since a binary number contains 0's and 1's only, a 2-sum can contain the three digits: 0(0 + 0), 1(1 + 0, 0 + 1) and 2(1 + 1).

Definition 2: A 2-sum containing only 1's will be called a *type-α 2-sum.* A 2-sum containing at least one 0 but no 2 will be called a *type-β 2-sum.* A 2-sum containing at least one 2 will be called a *type-γ 2-sum.*

A 2-sum obtained from a type-γ 2-sum S_γ by replacing all 2's by 0's will be called the type-β 2-sum for $S\gamma$.

Thus, for example, 1101 is a type-β 2-sum. 1201 is a type-γ 2-sum and 1001 is the type-β 2-sum for 1201. Obviously, for a given n, there is only one possible type-α 2-sum.

Definition 3: An *S-sequence* is a sequence obtained by arranging in ascending order the decimal designations of the entire collection of minterms contained in the pairs forming the same 2-sum S. For instance, (5,12) and (4,13) are the only pairs forming the 2-sum 1201. {5,12,4,13} is the entire collection of the minterms contained in these pairs. Arranging these in ascending order we obtain the 1201-sequence as 4, 5, 12, 13.

Since there are ${}^nC_k = \binom{n}{k}$ different ways of arranging k 1's in n locations, it can be easily verified that there are in all

$$N(n) = \sum_{k=2}^{n} \binom{n}{k} 2^{n-k} \qquad (n \geq 2) \tag{2}$$

S-sequences of n variables, with at least two 1's, out of which one is of type α,

$$\sum_{k=2}^{n-1} \binom{n}{k}$$

of type β and the remaining of type γ.

Gosh *et al.* [3] have shown that if a pair of minterms (X_1, X_2) has the same 2-sum as another pair (Y_1, Y_2), then the decimal sums of the two pairs are also equal. The converse of this theorem is not true, thus the decimal sums of different pairs may be equal even if their 2-sums are not equal. The S-sequence contains all the minterms which participate in the 2-sum equal to S. A useful property of an S-sequence is that the correct pairs giving the 2-sums S can be picked up without actually computing the 2-sums.

Theorem 1: All the 2-sum pairs forming the 2-sum S are obtained from the S-sequence by pairing up minterms equidistant from the center.

Proof: The proof follows from the fact that the minterms in S-sequence are arranged in increasing order of magnitude, and that minterms with equal 2-sums have equal decimal sums.

For example, the 1111-sequence is as follows:

$$0\quad 1\quad 2\quad 3\quad 4\quad \cdots\quad \cdots\quad \cdots\quad 14\quad 15.$$

[1] Symbol "②" was suggested by our colleague C.V.S. Rao.