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Free and Controlled Motions of an Omniwheel Vehicle

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Abstract—The dynamics of a vehicle whose three omniwheels are symmetrically arranged is considered in the case when the vehicle moves on a horizontal plane. Two wheels are parallel to each other, whereas the third one is perpendicular to them; the centers of the wheels are located at the vertices of an isosceles triangle. A phase portrait is constructed under the assumption that there are no external actions (except for gravity). The stability conditions for uniform rectilinear motions are compared with the Chaplygin sleigh model. The stability and bifurcation of steady motions are discussed in the case of constant control.

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1. THE EQUATIONS OF MOTION OF AN OMNIWHEEL VEHICLE

An omniwheel is a disk with rollers mounted around its periphery; these rollers are able to freely rotate about the tangent to the rim of the disk at the mount point of each roller [1]. The simplest mechanical model of an omniwheel is a disk such that the velocity of the contact point between the disk and the supporting surface is perpendicular to its plane.

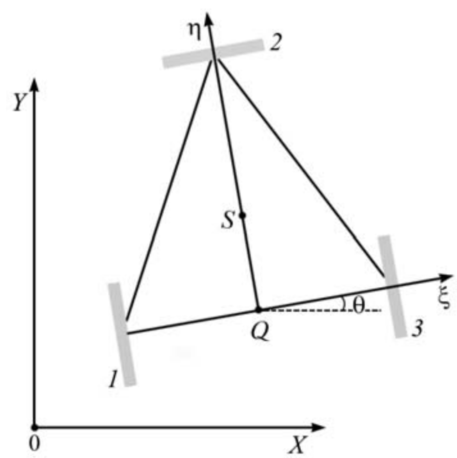

Fig. 1

In this paper we consider a vehicle moving along a rough horizontal plane. The following three omniwheels are mounted on the vehicle: wheels *1* and *3* are parallel to each other, whereas wheel *2* is perpendicular to them; the centers of the wheels are located at the vertices of an isosceles triangle (Fig. 1). During motion, the planes of the wheels are vertical and maintain their constant position relative to the vehicle body. The above system is a nonholonomic mechanical system with three degrees of freedom.

Let $OXYZ$ be a fixed coordinate system and $Q\xi\eta\zeta$ be a coordinate system rigidly attached to the vehicle body. The plane OXY coincides with the supporting plane; the OZ axis is vertical. The point Q is the midpoint of the segment connecting the centers of the parallel wheels; the $Q\xi$ axis coincides with their axis of rotation; the $Q\eta$ axis passes through the center-of-mass point S of the system (this point is situated on the axis of geometric symmetry). Let the distance between Q and S be equal to D.

Let θ be the angle between the OX and $Q\xi$ axes. Let (x, y) be the coordinates of the point S on the plane OXY. The center-of-mass velocity can be represented as the following linear combination of the unit vectors of the moving coordinate system: $\mathbf{v}_s = R\nu_1\mathbf{e}_\xi + R\nu_2\mathbf{e}_\eta$ (here R is the radius of the wheels). Let χ_i be the rotation angle of the ith wheel relative to the vehicle ($i = 1, 2, 3$). Let $d_1, d_2, d_3 = d_1$ be the distance between the point Q and the centers of the wheels. The contact point velocity is perpendicular to the plane of the wheel; therefore we have

$$\dot{\chi}_1 = \nu_2 + \delta_1\dot{\theta}, \qquad \dot{\chi}_2 = \nu_1 + (\delta_2 - \Delta)\,\dot{\theta}, \qquad \dot{\chi}_3 = -\nu_2 + \delta_1\dot{\theta}, \tag{1}$$

where the lengths Δ and δ_i are dimensionless: $\Delta = R^{-1}D$ and $\delta_i = R^{-1}d_i$. These equations express the three differential constraints imposed on the system. The sum of the first and third equations in (1) can be integrated in order to relate χ_1, χ_3, and θ to each other. The remaining two constraints are nonholonomic.

Let us consider the motion of the vehicle by inertia; in other words, we assume that the system is subjected to the ideal constraints and to the gravity force only. In order to derive the required equations of motion, we use the compact form proposed in [2, 3]. To accomplish this, we write the kinetic energy of the system as

$$2T = M(\dot{x}^2 + \dot{y}^2) + I_S\dot{\theta}^2 + J(\dot{\chi}_1^2 + \dot{\chi}_2^2 + \dot{\chi}_3),$$

where M is the total mass of the system, I_S is the total moment of inertia with respect to the vertical axis passing through the point S, and J is the inertia moment of each wheel with respect to its axis of rotation. Now we introduce the dimensionless parameters Λ and λ as follows: $I_S = \Lambda^2 MR^2$ and $J = \lambda^2 MR^2$. Let $\nu_3 = \Lambda\dot{\theta}$ be velocity. The pseudovelocities can be represented in the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} R\cos\theta & -R\sin\theta & 0 \\ R\sin\theta & R\cos\theta & 0 \\ 0 & 0 & 1/\Lambda \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \tag{2}$$

Thus, here we consider the case when the differential constraints (1) are imposed on the mechanical system whose position is described by the variables $\{q_i\} = \{x, y, \theta, \chi_1, \chi_2, \chi_3\}$ with Lagrangian $L = T$ (the potential energy of gravity does not change, since the vehicle moves on a horizontal plane). The differential constraints (1) can be rewritten in matrix form as

$$\begin{pmatrix} \dot{\chi}_1 \\ \dot{\chi}_2 \\ \dot{\chi}_3 \end{pmatrix} = \Xi \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \quad \Xi = \begin{pmatrix} 0 & 1 & \sigma \\ -1 & 0 & \rho \\ 0 & -1 & \sigma \end{pmatrix}, \quad \sigma = \Lambda^{-1}\delta_1, \quad \rho = \Lambda^{-1}(\delta_2 - \Delta).$$

Let P_1, P_2, and P_3 be the coefficients at the pseudovelocities ν_1, ν_2, and ν_3 after substituting (1) and (2) into the sum $\sum p_i \dot{q}_i$:

$$\sum P_\alpha \nu_\alpha = \left(p_x\dot{x} + p_y\dot{y} + p_\theta\dot{\theta} + p_1\dot{\chi}_1 + p_2\dot{\chi}_2 + p_3\dot{\chi}_3\right)\big|_{(1),(2)}.$$

These coefficients have the form

$$P_1 = R(p_x\cos\theta + p_y\sin\theta) - p_2, \quad P_2 = R(-p_x\sin\theta + p_y\cos\theta) + p_1 - p_3,$$
$$P_3 = \Lambda^{-1}p_\theta + p_1\sigma + p_2\rho + p_3\sigma,$$

where p_i are the formal canonical momenta, i.e., the variables that allow one to calculate the Poisson bracket $\{F, G\} = \sum\left(\dfrac{\partial F}{\partial q_i}\dfrac{\partial G}{\partial p_i} - \dfrac{\partial F}{\partial p_i}\dfrac{\partial G}{\partial q_i}\right)$.

The required equations of motion can be represented in the form

$$\frac{d}{dt}\frac{\partial L^*}{\partial \nu_\alpha} + \{P_\alpha, L^*\} = \sum \nu_\beta \{P_\alpha, P_\beta\}^*,$$

where $L^* = T\big|_{(1),(2)} = \dfrac{1}{2}\nu^{\mathrm{T}}\mathbf{A}\nu$ is the Lagrangian after the substitution of the above constraints, $\nu^{\mathrm{T}} = (\nu_1, \nu_2, \nu_3)$ is the three-dimensional vector, and the matrix $\mathbf{A}$ is equal to $\mathbf{E} + \lambda^2\Xi^{\mathrm{T}}\Xi$ (here E is the identity matrix). Note that $\{P_\alpha, L^*\} = 0$. Finding $\{P_\alpha, P_\beta\}$, we come to the relations

$$\{P_1, P_2\} = 0, \quad \{P_1, P_3\} = \frac{\partial P_1}{\partial \theta}\frac{\partial P_3}{\partial p_\theta} = R\Lambda^{-1}(-p_x\sin\theta + p_y\cos\theta),$$
$$\{P_2, P_3\} = \frac{\partial P_2}{\partial \theta}\frac{\partial P_3}{\partial p_\theta} = R\Lambda^{-1}(-p_x\cos\theta - p_y\sin\theta).$$

Taking into account (1) and (2) and the well-known expressions $p_x = \dfrac{\partial T}{\partial \dot{x}}$ and $p_y = \dfrac{\partial T}{\partial \dot{y}}$, we obtain

$$p_x^* = M\dot{x} = MR(\cos\theta\nu_1 - \sin\theta\nu_2), \quad p_y^* = M\dot{y} = MR(\sin\theta\nu_1 + \cos\theta\nu_2).$$

Hence, $\{P_1, P_3\}^* = MR^2\Lambda^{-1}\nu_2$ and $\{P_2, P_3\}^* = -MR^2\Lambda^{-1}\nu_1$. The dynamic equations take the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial \nu_1} = \frac{MR^2}{\Lambda}\nu_2\nu_3, \quad \frac{d}{dt}\frac{\partial T^*}{\partial \nu_2} = -\frac{MR^2}{\Lambda}\nu_1\nu_3, \quad \frac{d}{dt}\frac{\partial T^*}{\partial \nu_3} = 0.$$

Transforming the left-hand sides of these equations, we get

$$\mathbf{A}\dot{\nu} = \Lambda^{-1}\mathbf{a}, \quad \boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)^{\mathrm{T}}, \quad \mathbf{a} = (\nu_2\nu_3, -\nu_1\nu_3, 0)^{\mathrm{T}}, \tag{3}$$

$$\mathbf{A} = \begin{pmatrix} 1+\lambda^2 & 0 & -\rho\lambda^2 \\ 0 & 1+2\lambda^2 & 0 \\ -\rho\lambda^2 & 0 & 1+(2\sigma^2+\rho^2)\lambda^2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 & -\kappa A_3 \\ 0 & A_2 & 0 \\ -\kappa A_3 & 0 & A_3 \end{pmatrix}.$$

The system admits the energy integral $T^* \equiv \text{const}$, the linear integral $\dfrac{\partial T^*}{\partial \nu_3} \equiv \text{const}$, and the invariant measure $\mu = \dfrac{d\nu_1 \wedge d\nu_2 \wedge d\nu_3}{|\nu_3|}$. In the space of the pseudovelocities ν_1, ν_2, and ν_3, the trajectories of the system lie on the section of the ellipsoid $MR^2\nu^{\mathrm{T}}\mathbf{A}\nu = 2h$ by the plane $\dfrac{\partial T^*}{\partial \nu_3} = MR^2(A_3\nu_3 - \kappa A_3\nu_1) = MR^2k$ (note that this ellipsoid is a level of the energy integral).

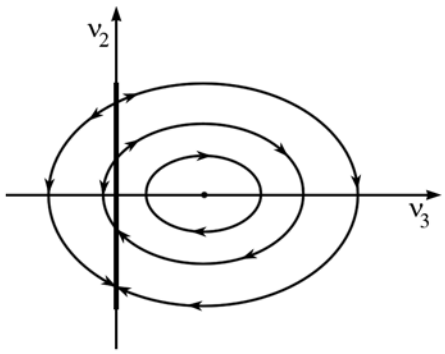

Fig. 2

Let us assume that $\rho > 0$, i.e., the center of mass is situated between the axis of the parallel wheels *1* and *3* and the wheel *2*. Let k be a level of the linear integral. Then, the projections of the phase curves onto the plane $\pi = \{\nu_1 = 0\}$ (Fig. 2) are described by the differential equations

$$A_2\dot{\nu}_2 = -\frac{1}{\Lambda}\left(\frac{A_3\nu_3 - k}{\kappa A_3}\right)\nu_3, \qquad \left(\kappa A_3 + \frac{A_1}{\kappa}\right)\dot{\nu}_3 = \frac{1}{\Lambda}\,\nu_2\nu_3.$$

In the plane π, the trajectories of the system belong to the concentric ellipses whose center lies on the line $\nu_2 = 0$; this center corresponds to the steady rotation of the vehicle about the vertical axis through the center of mass. An invariant domain of motions with a periodic behavior of phase velocities is situated near the center of the ellipse. The existence of this domain and the existence of the invariant measure allow us to study the motion of the vehicle with small torque in the wheel axes by the Anosov averaging method.

The singular line $\nu_3 = 0$ corresponds to the uniform rectilinear motion of the vehicle (during motion, the angle θ is constant, whereas the symmetry axis of the vehicle is not necessarily parallel to the direction of motion). Determining the direction of motion of the representative point along the phase trajectories, we obtain that the uniform rectilinear motions $\nu_1 = \text{const}$, $\nu_2 = \text{const}$, and $\nu_3 = 0$ are stable for $\nu_2 < 0$ and are not stable for $\nu_2 > 0$. Physically, this means that the rectilinear motion is stable if and only if, during motion, the center of mass is situated behind the axis of the parallel wheels.

If the omniwheels of a vehicle are situated at the vertices of a regular triangle and are oriented perpendicularly to the bisectors of the corresponding angles [4], then the free rectilinear motion is globally asymptotically stable.

The above stability condition $\nu_2 < 0$ is in direct opposition to the stability condition for the free rectilinear motion of the classical nonholonomic model known as the Chaplygin sleigh; note that the two-wheeled vehicle model proposed by E.A. Devyanin [5] can be reduced to this model in the case of dynamic symmetry when control and friction are absent. Below we discuss some stability conditions for the rectilinear motions of a family of mechanical systems; it should be noted that the vehicle under study and the Chaplygin sleigh belong to this family (the latter is possible if a certain parameter tends to infinity).

2. COMPARISON OF THE STABILITY CONDITIONS FOR THE VEHICLE UNDER STUDY AND FOR THE DEVYANIN TWO-WHEELED VEHICLE

The two-wheeled vehicle model (the Chaplygin sleigh model) can be constructed from the above dynamical system. First, we impose the following constraint: the projection $v_{Q\xi}$ of the velocity of the point Q onto the $Q\xi$ axis is equal to zero (in terms of the Chaplygin sleigh model, this means that the sleigh edge is situated at the point Q and is directed along the $Q\eta$ axis). Second, we remove wheel *2*. The above constraint is of the form $v_{Q\xi} = \nu_1 + \Delta\Lambda^{-1}\nu_3 = 0$.

In order to relate these two systems and to compare their behavior, we consider the following class of systems: in our omniwheel model, we introduce the viscous friction proportional to the velocity projection $v_{Q\xi}$ with a coefficient c. It is clear that we come to the original omniwheel model if $c = 0$. In addition, we expect that the behavior of a system from this class would be close to the behavior of the Chaplygin sleigh as $c \to \infty$.

This class of systems is described by (3) if, in the right-hand side, we introduce the generalized forces corresponding to viscous friction; these forces can be represented in the form $Q_i = \partial\Phi/\partial\nu_i$, where $\Phi = -\frac{c}{2}(v_{Q\xi})^2 = -\frac{c}{2}\big(\nu_1 + \Delta\Lambda^{-1}\nu_3\big)^2$ is the Rayleigh dissipative function. Thus, we come to the equations

$$\begin{aligned} &A_1\dot\nu_1 - \kappa A_3\dot\nu_3 = \Lambda^{-1}\nu_2\nu_3 - c(\nu_1 + \Delta\Lambda^{-1}\nu_3),\\ &A_2\dot\nu_2 = -\Lambda^{-1}\nu_1\nu_3,\\ &-\kappa A_3\dot\nu_1 + A_3\dot\nu_3 = -c\Delta\Lambda^{-1}(\nu_1 + \Delta\Lambda^{-1}\nu_3). \end{aligned} \tag{4}$$

For any value of c, the system admits the steady rectilinear motions $\nu_1 = 0$, $\nu_2 = v$, and $\nu_3 = 0$ (the stability conditions for these motions are different for the two systems under consideration). Linearizing Eqs. (4) in a neighborhood $\nu_1 = 0 + \delta\nu_1$, $\nu_2 = v + \delta\nu_2$, and $\nu_3 = 0 + \delta\nu_3$ of this steady motion, we come to the equation $\mathbf{A}\delta\dot\nu = \mathbf{B}\delta\nu$, where

$$\mathbf{B} = \begin{pmatrix} -c & 0 & v - c\Delta\Lambda^{-1}\\ 0 & 0 & 0\\ -c\Delta\Lambda^{-1} & 0 & -c\Delta^2\Lambda^{-2} \end{pmatrix}.$$

The eigenvalues μ of this linearized system are the solutions to the cubic equation

$$\mu\left(\big[A_1A_3 - \kappa^2A_3^2\big]\mu^2 + \big[c(\Delta^2\Lambda^{-2}A_1 + A_3 + 2\Delta\Lambda^{-1}\kappa A_3) - \kappa A_3 v\big]\mu + c\Delta\Lambda^{-1}v\right) = 0.$$

Note that the coefficient $A_1A_3 - \kappa^2A_3^2$ is positive, since this coefficient is a diagonal minor of the positive definite matrix $\mathbf{A}$ of the kinetic energy. In the case of our omniwheel vehicle ($c = 0$), the linearized system has the two zero eigenvalues that correspond to the two-dimensional manifold of the equilibrium states $\nu_3 = 0$. The stability condition has the form $-\kappa A_3 v > 0$; this means that the motions $\nu_2 = v < 0$ are stable.

If $c > 0$, we come to the following stability conditions:

$$v < c\,\frac{\Delta^2\Lambda^{-2}A_1 + A_3 + 2\Delta\Lambda^{-1}\kappa A_3}{\kappa A_3}, \qquad c\Delta\Lambda^{-1}v > 0. \tag{5}$$

Thus, the motions $\nu_2 = v < 0$ become unstable if we introduce even small friction: one of the two zero roots of the characteristic equation becomes a small positive number. The fast unstable motions $\nu_2 = v > 0$ remain unstable, whereas the slow unstable motions whose velocity v is small in magnitude become stable if the first inequality in (5) is valid. As $c \to \infty$, the stability conditions (5) become the above stability condition $v > 0$ for the Chaplygin sleigh.

3. STABILITY OF STEADY MOTIONS UNDER CONSTANT CONTROL

Now we consider the controlled motion of our vehicle. Let the control torques $M_i = bU_i - c\dot\chi_i$ be applied to the wheel axes from the direction of the vehicle body (here U_i are the control voltages). Such a description of direct current motor drives is generally accepted when considering the dynamics of controlled vehicles. In the case of the new time $\tau = \frac{c}{MR^2}\,t$, the equations of motion take the form

$$\mathbf{A}\dot\nu = \Lambda^{-1}\mathbf{a} + \mathbf{u} - \Xi^{\mathrm{T}}\Xi\nu, \qquad \mathbf{u}^{\mathrm{T}} = (u_1, u_2, u_3) = MR^2bc^{-1}(U_1, U_2, U_3)\Xi. \tag{6}$$

Let us study the steady motions of this system under constant control: $\nu_1(t) \equiv p = \text{const}$, $\nu_2(t) \equiv r = \text{const}$, and $\nu_3(t) \equiv \Lambda\omega = \text{const}$.

The most important modes of motion are the uniform rectilinear motions of the vehicle. Such motions are possible when the condition $u_3 = -\rho u_1$ is imposed on the control parameters. In addition to the rectilinear motions $p = u_1$, $r = u_2/2$, and $\omega = 0$, in this case there exist the following steady rotary motions: on the supporting plane, the mass center moves along a circle and the vehicle uniformly rotates about its center of mass: $\Lambda(\rho^2 + 2\sigma^2)\omega^2 + \rho u_1\omega + (4\Lambda\sigma^2 - u_2\rho) = 0$, $p = u_1 + \Lambda\rho^{-1}(\rho^2 + 2\sigma^2)\omega$, and $r = 2\Lambda\rho^{-1}\sigma^2$.

The rectilinear motions are asymptotically stable for $4\Lambda\sigma^2 - \rho u_2 > 0$. This condition means that the magnitude of the center-of-mass velocity projection onto the $Q\eta$ axis is less than a positive number: $\nu_2 \equiv r = u_2/2 < 2\Lambda\sigma^2/\rho$.

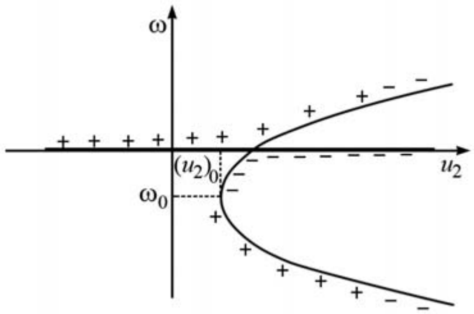

Fig. 3

Let us study the steady rotary motions. This family of motions corresponds to the parabola in the bifurcation diagram (u_2, ω) represented in Fig. 3, where the asymptotically stable motions are marked by "+", whereas the unstable motions are marked by "−". The branches of this parabola are directed to the right; its vertex is situated at the point $\omega_0 = -\dfrac{\rho u_1}{2(\rho^2 + 2\sigma^2)}$, $(u_2)_0 = \dfrac{4\Lambda\sigma^2}{\rho} - \dfrac{\rho u_1^2}{2\Lambda(\rho^2 + 2\sigma^2)}$. Linearizing the dynamic equations (6) in a neighborhood of these steady motions, we come to the following characteristic equation:

$$M_3\mu^3 + M_2\mu^2 + M_1\mu + M_0 = 0, \quad M_3 > 0, \quad M_2 > 0,$$
$$M_1 > 0, \quad M_0 = 2(\rho^2 + 2\sigma^2)\omega^2 + \rho\Lambda^{-1}u_1\omega.$$

From here it follows that the real root of this characteristic equation changes its sign for $\omega = 0$ and $\omega = \omega_0$; hence, the nature of stability also changes. Our analysis of the expression $\mathcal{R} = M_1M_2 - M_0M_3$ shows that there exist inertial and geometric parameters such that $\mathcal{R} < 0$ in a domain $(-\infty, \omega_-) \cup (\omega_+, +\infty)$ and that $\mathcal{R} \geq 0$ in the complement of this domain; note that $(\omega_-, \omega_+) \supset (\omega_0, 0)$.

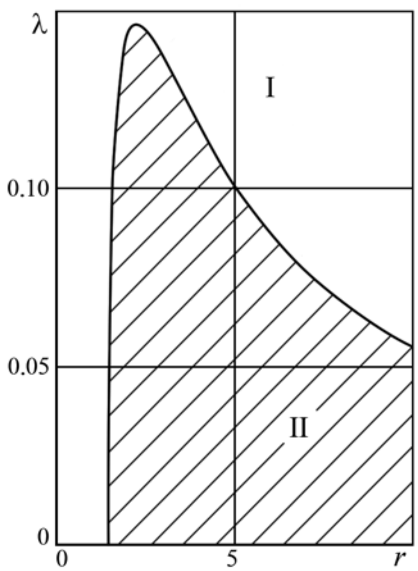

Fig. 4

Let us consider the bifurcation nature at the points $\omega = \omega_\pm$ for $u_3 = -\rho u_1 = 0$. In this case the critical points $\omega_\pm$ satisfy the equation

$$\mathcal{R} = f(\lambda, \rho, \sigma)\,\omega_\pm^2 + g(\lambda, \rho, \sigma). \tag{7}$$

This equation can be solved for the same mass-inertia characteristics (λ, ρ, σ) for which $f(\lambda, \rho, \sigma) < 0$ (the function f is a third-degree polynomial of λ^2). Our analysis of this function shows that the maximum dimensionless inertia moment of the wheel for which there exists a solution to Eq. (7) is less than 0.15.

In order to clarify the bifurcation nature for $\omega = \omega_\pm$, it is necessary to study the nonlinear equations of a disturbed motion. Analyzing the first Lyapunov coefficient L_1 [6], we come to the following conclusions: its sign does not depend on Λ and L_1 is positive in the entire domain $f(\lambda, \rho, \sigma) < 0$.

For $u_3 = -\rho u_1 = 0$, thus, in the space of the parameters (λ, ρ, σ) it is possible to construct two domains where the system behaves qualitatively different. A typical section of the parameter space by the plane $\sigma/\rho = \text{const}$ $(r = \sqrt{\sigma^2 + \rho^2})$ is represented in Fig. 4. In domain I, the steady rotary motions are always stable (i.e., there exist no critical points $\omega_\pm$); in domain II, the unstable limit cycles tend to the steady rotary motions with increasing u_2.

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