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A wheeled mobile robot is here modelled as a planar rigid body that rides on an arbitrary number of wheels. The relationship between the rigid body motion of the robot and the steering and drive rates of wheels is developed. In particular, conditions are obtained that guarantee that rolling without skidding or sliding can occur. Explicit differential equations are derived to describe the rigid body motions that arise in such ideal rolling trajectories. The simplest wheel configuration that permits access of arbitrary rigid-body motions is determined. Then the question of slippage due to misalignment of the wheels is investigated by minimization of a nonsmooth convex dissipation functional that is derived from Coulomb's Law of friction. It is shown that this minimization principle is equivalent to the construction of quasi-static motions. Examples are presented to illustrate the models.

1. Introduction

In this paper, we analyze the kinematics of a *wheeled mobile robot*, or *WMR*. Such robots ride on a system of wheels and axles, some of which may be steerable or driven. There are many wheel and axle configurations that have been used for WMRs (Whitaker 1962; Lewis and Bejczy 1973; Smith and Coles 1973; Hollis 1977; Everett 1979; Giralt, Sobek and Chatila 1979; Moravec 1980; Iijima, Kanayama and Yuma 1981a; Iijima, Kanayama and Yuma 1981b; Balmer, Jr. 1982; Carlisle 1983; Helmers 1983b; Helmers 1983a; Ichikawa, Ozaki and Sadakane 1983; Moravec 1983; Johnson 1984; Nilsson 1984; Podnar, Dowling and Blackwell 1984; Rogers 1984; Helmers 1985; Holland 1985; Marrs 1985; Wallace et al. 1985; Wilson 1985; Moravec 1986). For other related work, see Kanayama and Miyake (1985), Fortune and Wilfong (1988), Wilfong (1988). The ultimate objective of our investigations is the complete description of the kinematics and inverse kinematics of such robots during low-speed maneuvering.

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Much of the research cited above is described in a recent study of WMRs (Muir and Neuman 1987). This work develops a formalism that is used first to model the kinematics of each wheel, and second to amalgamate the information about individual wheels to describe the kinematics of the WMR regarded as a whole. A condition is developed that determines whether, given the configuration of the wheels in the WMR, ideal rolling is possible. If not, a least-squares fit to rolling is obtained. Some of the same questions are considered here. However, our approach is somewhat different, and, we believe, complementary. One major difference is that our analysis of cases in which ideal rolling fails is based on physical models of friction.

Attention is restricted to the problem of maneuvering a WMR on a horizontal plane. Precise and explicit connections between the steering and drive rates of the various wheels and the position and orientation of the robot are obtained. The inverse kinematic problem of determining the steering and drive rates that produce a prescribed robot trajectory is also resolved. We determine the simplest configuration of steerable and driven wheels that allows the robot to be maneuvered in arbitrary planar motions. Slippage resulting from wheel configurations that are incompatible with ideal rolling is also considered. Our description of slippage is based on the use of Coulomb's Law to model friction. It is shown that a quasi-static analysis, in which forces and moments are balanced, is equivalent to the minimization of a non-smooth, convex *dissipation* functional. This dissipation, or *friction*, functional is the weighted sum of the euclidian norms of certain vectors. For illustration, a complete analysis of the simplest application is presented.

The qualification of low speed arises in this work because our model of rolling does not consider inertial forces and accelerations. It will become apparent that this rolling model retains its validity until the inertial forces arising from accelerations are so large as to saturate the available frictional forces between the wheels and the surface. Thus the low-speed model may in fact be valid for relatively fast motions, provided either that the turns are not too tight or that the friction between the wheels and the surface is sufficiently large.

In Section 2, notation and definitions are introduced, and some basic results are presented. In the first instance, all wheels are assumed to roll; i. e., there is no slipping or sliding. This requirement places compatibility conditions on the motions of the wheels. These conditions and their consequences are discussed in Section 3, which is a reformulation and extension of results obtained in Alexander and Maddocks (1988). In this prior work, attention was focused on intrinsic properties of rigid-body trajectories, such as curvatures and centers of rotation. In the present development, more emphasis is given to formulations that would allow efficient numerical treatment. The conclusion of Section 3 is that the specification of steering and drive rates of two wheels are necessary and sufficient to specify arbitrary planar motions. The four rates under our control (two steering, two driving) must satisfy one (transcendental) compatibility condition. The explicit connections between the steering and drive rates and the rigid-body motion are described in Section 4, where a resolution of the inverse-kinematic problem is also given. A practical design is suggested in Section 5, and, in Section 6, the kinematics of three existing WMRs are compared with our proposed design.

Our kinematic development is then extended to consider failure of the rolling model due to slippage of the wheels. There are two distinct circumstances in which slippage will occur. The first has already been mentioned: the rolling model can fail to be a good approximation because of large inertial forces that saturate the available friction. This mode of failure of the rolling model is associated with high-speed maneuvering. The second mode of failure is that the steering and driving controls of the wheels are not compatible with ideal rolling. This lack of compatibility can arise at any speed. The first circumstance, which we call *skidding*, is not considered in this paper except to note that the friction functional described in Section 7 provides a technique to predict the onset of skidding. Quasi-static evolution equations for low speed maneuvering involving the second mode of failure, which we call *slippage* (sometimes called *scrubbing*), are developed in Section 7 and illustrated in Section 8.

2. Notation and Definitions, Elementary Results

A WMR is modelled as a planar rigid *robot body* that moves over a horizontal reference plane on wheels that are connected to the body by axles. For our purposes, the only role of the body of the WMR is to carry a moving coordinate system. Thus the *body coordinates* of a vector $\mathbf{x}$ are denoted ${}^{\mathrm{B}}\mathbf{x} = (x_1, x_2)$, while the underlying two-dimensional *space coordinates* are denoted ${}^{\mathrm{R}}\mathbf{x}$. The ith *axle*, $i = 1, \ldots, m$, is attached to the body at the *axle* or *constraint point* $\mathbf{x}_i$ with body coordinates ${}^{\mathrm{B}}\mathbf{x}_i$ and space coordinates ${}^{\mathrm{R}}\mathbf{x}_i$. The body coordinates are known constants; the space coordinates are unknown functions of time. The axle is supported by a single *simple wheel*, which is idealized as a disc, without thickness, of radius R_i that lies in a vertical plane through the axle point. A wheel can rotate in its vertical plane about its center point (which is attached to the axle). If it is *driven*, it will rotate at a prescribed angular speed. Otherwise it is *passive*, and its rotation rate is determined by the kinematics of the WMR. It may also be possible for the axle to rotate in the WMR about the vertical through the axle point. If the wheel (or axle) is *steered*, this rotation rate is prescribed. Otherwise the wheel is *unsteered*. A *fixed* wheel is one for which the axle cannot rotate. Technically, a fixed wheel is a special type of steered wheel, but it is useful to maintain a distinction.

There are a number of other types of wheels that are adopted in robot design, of which a simple wheel [called a *conventional wheel* in §3 of Muir and Neuman (1987)] is but the most straightforward. Our kinematic analysis can be expanded to include many of these more complicated wheels. Some remarks along this line are included in Sections §5 and 6, where practical wheel configurations for WMRs are considered.

Regarded as a rigid body in three-dimensional space, a simple wheel has a three-dimensional vector-valued angular velocity $\boldsymbol{\omega}_i$. Since the wheel remains in a vertical plane

1.. Schematic of vectors associated with wheel i. $\mathbf{a}_i$: axle vector along axle; $\mathbf{v}_i$: velocity of axle point $\mathbf{x}_i$; $\boldsymbol{\omega}_i$: angular velocity of wheel; $\mathbf{k}$: unit vertical vector. 2. Schematic of coordinate systems. ${}^{\mathrm{B}}\mathbf{x}_i$: body coordinates of axle point i; ${}^{\mathrm{R}}\mathbf{x}_i$: coordinates of axle point i with respect to frame fixed in space; ${}^{\mathrm{B}}\theta_i$: angle of axle vector $\mathbf{a}_i$ with respect to body frame; ${}^{\mathrm{R}}\theta_i$: angle of axle vector $\mathbf{a}_i$ with respect to fixed frame.

at all times, the 2-dimensional horizontal component of $\boldsymbol{\omega}_i$, denoted $\mathbf{a}_i$, is perpendicular to the plane of the wheel. The vector $\mathbf{a}_i$ is called the *axle vector.* The magnitude ρ_i of the axle vector is the *rotation rate* of the wheel. The axle vector $\mathbf{a}_i$ makes a *steering angle* ${}^{\mathrm{B}}\theta_i$ from a reference direction fixed in the body of the WMR, and an angle ${}^{\mathrm{R}}\theta_i$ from a reference direction fixed in space. The angles ${}^{\mathrm{B}}\theta_i$ and ${}^{\mathrm{R}}\theta_i$ are well-defined whenever $\rho_i \neq 0$. The conventions adopted here assume ${}^{\mathrm{B}}\theta_i$ to be interpreted modulo 2π and ρ_i to be nonnegative. In practical implementations, it may be more convenient to interpret ${}^{\mathrm{B}}\theta_i$ modulo π and to allow ρ_i to be negative (i. e., the wheel may rotate backward). We remark that the definitions of ${}^{\mathrm{B}}\theta_i$ and ${}^{\mathrm{R}}\theta_i$ differ by $\pi/2$ from those given in Alexander and Maddocks (1988). The scalar *pivot* or *steering speed* is defined to be ${}^{\mathrm{B}}\omega_i = {}^{\mathrm{B}}\theta_i'(t)$. The vertical component of the wheel angular velocity is ${}^{\mathrm{R}}\theta_i'(t) = {}^{\mathrm{B}}\omega_i + \Omega$, where Ω is the angular velocity of the robot body in the ambient space.

It is apparent that the motion of a simple wheel relative to the body of the WMR is mathematically characterized by the two scalars ρ_i and ${}^{\mathrm{B}}\theta_i$, or by ρ_i, ${}^{\mathrm{B}}\omega_i$, and an initial value ${}^{\mathrm{B}}\theta_i(0)$. Moreover, the steering angle and rotation rate are the physically relevant kinematic parameters. It should be stressed that the two scalars ρ_i and ${}^{\mathrm{B}}\theta_i$ uniquely characterize the components ${}^{\mathrm{B}}\mathbf{a}_i$ of the axle vector $\mathbf{a}_i$ *in the robot body*. The first major goal of this paper is to investigate how the assumption of ideal rolling determines the motion of the robot body in the ambient space in terms of the ${}^{\mathrm{B}}\mathbf{a}_i$.

The position (or configuration) of the robot body is determined by any two of the axle-point vectors ${}^{\mathrm{R}}\mathbf{x}_i(t)$. Alternatively the body configuration is totally specified given one vector ${}^{\mathrm{R}}\mathbf{x}_i(t)$ and the *body orientation angle* Θ between the reference directions fixed in the body and in space. The angular velocity of the body is $\Omega = \Theta'(t)$. It is apparent that

$$ {}^{\mathrm{R}}\theta_i(t) = \Theta(t) + {}^{\mathrm{B}}\theta_i(t), \tag{2.1} $$

and, by differentiation,

$$ {}^{\mathrm{R}}\theta_i'(t) = \Omega(t) + {}^{\mathrm{B}}\omega_i(t). \tag{2.2} $$

The fact that the robot is a rigid body is expressed by the *rigidity conditions*

$$ \frac{d}{dt}\left|{}^{\mathrm{R}}\mathbf{x}_i(t) - {}^{\mathrm{R}}\mathbf{x}_j(t)\right| = \frac{d}{dt}\left|{}^{\mathrm{B}}\mathbf{x}_i(t) - {}^{\mathrm{B}}\mathbf{x}_j(t)\right| = 0, \qquad i, j = 1, \ldots m. \tag{2.3} $$

If the *axle-point velocity* $\mathbf{v}_i(t) = {}^{\mathrm{R}}\mathbf{v}_i(t) = {}^{\mathrm{R}}\mathbf{x}_i'(t)$ is introduced, and the differentiation is performed, (2.3) can be expressed as

$$ \big(\mathbf{x}_i(t) - \mathbf{x}_j(t)\big) \cdot \big(\mathbf{v}_i(t) - \mathbf{v}_j(t)\big) = 0, \qquad i, j = 1, \ldots m. \tag{2.4} $$

This condition can be reformulated in another useful way. Let α_{ij} be the signed angle between the reference direction in the body and the line joining $\mathbf{x}_i$ to $\mathbf{x}_j$. By choosing basis vectors parallel and perpendicular to $\mathbf{x}_i - \mathbf{x}_j$ and considering components in those directions, (2.4) can be rewritten:

$$ |\mathbf{v}_i| \sin({}^{\mathrm{B}}\theta_i - \alpha_{ij}) - |\mathbf{v}_j| \sin({}^{\mathrm{B}}\theta_j - \alpha_{ij}) = 0. \tag{2.5} $$

One of the most elementary, yet elegant, results in kinematics (Chasles' theorem or Descartes' principle of instantaneous motion) asserts that the conditions in (2.4) imply that at each instant the motion of a planar rigid body coincides with either (1) a pure rotation about some point (the *instantaneous center of rotation* or *ICR*) or (2) a pure translation.

3. Rolling

The condition of rolling of the ith wheel relates the axle vector $\mathbf{a}_i$, which characterizes the wheel motion relative to the body, to the axle-point velocity $\mathbf{v}_i$, which partially describes the motion of the robot body in space. Explicitly, the *rolling conditions* are:

1. the directed angle from the axle-point velocity $\mathbf{v}_i$ to the axle vector $\mathbf{a}_i$ is $\pi/2$,
2. $|\mathbf{v}_i| = 2\pi R_i\,|\mathbf{a}_i| = 2\pi R_i \rho_i$.

In mathematical terms, $\mathbf{v}_i = 2\pi R_i \mathbf{a}_i \times \mathbf{k}$. Here, and throughout, $\mathbf{k}$ will denote the unit vertical vector, and any two-dimensional vectors, such as $\mathbf{a}_i$ and $\mathbf{v}_i$, are considered to be naturally embedded in three-dimensional space. Consequently the vector product is well-defined. Physically, $\mathbf{v}_i$ lies in the plane of the wheel, and its magnitude is determined by the rotation rate of the wheel.

The condition of rolling combines naturally with the notion of instantaneous center of rotation. If the instantaneous motion is a translation, then the axle vector of any rolling wheel must be orthogonal to the translation and be of given magnitude. More interestingly, if the body is rotating with instantaneous velocity $\Omega \neq 0$, and the ith wheel is rolling, then the ICR must be located at the point

$$\mathbf{x}_i + 2\pi R_i \Omega^{-1} \mathbf{a}_i. \tag{3.1}$$

Here we simplify notation by assuming, with no loss of generality, that $R_i = R_j = 1/2\pi$. This assumption merely represents a scaling of each axle vector $\mathbf{a}_i$. It then follows from (3.1) that any two rolling wheels, say the ith and the jth, must satisfy the *rolling compatibility conditions*:

$$\Omega(\mathbf{x}_i - \mathbf{x}_j) = (\mathbf{a}_j - \mathbf{a}_i). \tag{3.2}$$

This expression remains valid in the translational case $\Omega = 0$. It is apparent from (3.2) that once the body components of the axle vectors corresponding to two rolling wheels are given, then the angular velocity Ω, and therefore the axle vector of any other rolling wheel, is determined. On the other hand, even the first two axle vectors are not independent, for the vector equation (3.2) must be solvable for the scalar Ω. Note that the condition for (3.2) to be solvable is nothing other than (2.4), which, taken with the rolling conditions, states that $\mathbf{a}_j - \mathbf{a}_i$ must be parallel to $\mathbf{x}_i - \mathbf{x}_j$. Provided this solvability condition is satisfied, we find that

$$\Omega = -\frac{(\mathbf{a}_i - \mathbf{a}_j) \cdot (\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^2}. \tag{3.3}$$

4. Rigid-Body Kinematics

In this section, we explicitly find the differential equations that determine the location of the robot body—in the reference frame—in terms of the axle-point locations and axle-vectors in the body frame, which are assumed to be known, and the initial location. Thus ${}^{\mathrm{B}}\mathbf{x}_i$, ${}^{\mathrm{B}}\mathbf{x}_j$, ${}^{\mathrm{B}}\mathbf{a}_i$, and ${}^{\mathrm{B}}\mathbf{a}_j$ are four known sets of coordinates that satisfy rolling compatibility condition (3.2). The variables that will be adopted are the space coordinates of one axle point ${}^{\mathrm{R}}\mathbf{x}_i$ and the body orientation Θ. There are four parameters associated with the two axle vectors, namely two steering angles and two rotation rates. These four scalars must satisfy the solvability condition (2.5). Although only three of the four scalars are independent, there is no convenient way to explicitly eliminate one of them.

The differential equations governing the motion are

$$
{}^{\mathrm{R}}\mathbf{x}_i' = {}^{\mathrm{R}}\mathbf{v}_i, \tag{4.1a}
$$

$$
\Theta' = \Omega. \tag{4.1b}
$$

However, with the assumption of rolling, the reference coordinates $\mathbf{v}_i$ can be expressed in terms of ρ_i, ${}^{\mathrm{B}}\theta_i$, and Θ, and (3.3) provides an expression for Ω. Thus (4.1) can be recast as a system of three first-order equations ((4.2a) is a vector equation written in terms of two coordinates):

$$
{}^{\mathrm{R}}\mathbf{x}_i' = \rho_i\big(\sin({}^{\mathrm{B}}\theta_i + \Theta), -\cos({}^{\mathrm{B}}\theta_i + \Theta)\big), \tag{4.2a}
$$

$$
\Theta' = -\, x_{ij}^{-1}\big[\rho_i \cos({}^{\mathrm{B}}\theta_i - \alpha_{ij}) - \rho_j \cos({}^{\mathrm{B}}\theta_j - \alpha_{ij})\big], \tag{4.2b}
$$

where $x_{ij} = \left|{}^{\mathrm{B}}\mathbf{x}_i - {}^{\mathrm{B}}\mathbf{x}_j\right|$. Here the reference direction in space is taken to lie along the positive ${}^{\mathrm{R}}x_1$-axis. Equations (4.2), taken with a set of initial conditions, determine the location and orientation of the robot, given a set of specified compatible parameters. The equation (4.2a), with i replaced by j, can be solved for ${}^{\mathrm{R}}\mathbf{x}_j$, and provided that (2.5) is satisfied, $\left|{}^{\mathrm{R}}\mathbf{x}_i(t) - {}^{\mathrm{R}}\mathbf{x}_j(t)\right|$ is automatically constant. Alternatively, ${}^{\mathrm{R}}\mathbf{x}_j$ can be constructed from the formula

$$
{}^{\mathrm{R}}\mathbf{x}_j = {}^{\mathrm{R}}\mathbf{x}_i + x_{ij}\big(\cos(\alpha_{ij} + \Theta), \sin(\alpha_{ij} + \Theta)\big). \tag{4.3}
$$

The above analysis resolves the inverse kinematic problem *en passant*. For if ${}^{\mathrm{R}}\mathbf{x}_i$ and Θ are prescribed as functions of time, then (4.2a) can be regarded as two coupled transcendental equations that are uniquely solvable for the parameters $\rho_i \geq 0$, and ${}^{\mathrm{B}}\theta_i$ modulo 2π. Moreover, with given ${}^{\mathrm{R}}\mathbf{x}_i$ and Θ, (4.3) provides an expression for ${}^{\mathrm{R}}\mathbf{x}_j$. Consequently, (4.2a) provides a unique $\rho_j \geq 0$, and ${}^{\mathrm{B}}\theta_j$, modulo 2π. Similarly, if ${}^{\mathrm{R}}\mathbf{x}_i$ and ${}^{\mathrm{R}}\mathbf{x}_j$ satisfying (2.2) are prescribed as functions of time, then (4.3) provides Θ, and (4.2a) and (4.3) can be solved for the parameters as before. Any set of four parameters obtained in this way are automatically compatible.

More specifically, suppose that the position of a point $\mathbf{w}$ of the WMR and Θ are prescribed as functions of time. There is no loss of generality in assuming that $\mathbf{w}$ is actually an axle point ${}^{\mathrm{R}}\mathbf{x}_i$, because a "virtual" wheel can be imagined. From (4.2a), we obtain

$$\rho_i = |{}^{\mathrm{R}}\mathbf{x}_i'|, \tag{4.4a}$$

$$^{\mathrm{B}}\theta_i = \beta_i - \Theta, \tag{4.4b}$$

where β_i is the known angle between ${}^{\mathrm{R}}\mathbf{x}_i'$ and the ${}^{\mathrm{R}}x_2$-axis of spatial coordinates. To obtain the parameters for the other wheels, differentiate (4.3) to obtain

$${}^{\mathrm{R}}\mathbf{x}_j' = {}^{\mathrm{R}}\mathbf{x}_i' + x_{ij}\Theta'\big(-\sin(\alpha_{ij}+\Theta), \cos(\alpha_{ij}+\Theta)\big),$$

and use (4.2a) and (4.3) to obtain

$$\rho_j = \sqrt{\rho_i^2 + x_{ij}^2(\Theta')^2 - 2\rho_i x_{ij}\Theta' \cos({}^{\mathrm{B}}\theta_i \alpha_{ij})}.$$

Thus the inverse kinematic equations are

$$\rho_j = \sqrt{|{}^{\mathrm{R}}\mathbf{x}_i'|^2 + x_{ij}^2(\Theta')^2 - 2|{}^{\mathrm{R}}\mathbf{x}_i'|x_{ij}\Theta'\cos(\beta_i - \alpha_{ij} - \Theta)} \tag{4.5a}$$

$${}^{\mathrm{B}}\theta_j = \tan^{-1}\left(\frac{|{}^{\mathrm{R}}\mathbf{x}_i'|\sin\beta_i - x_{ij}\Theta'\sin(\alpha_{ij}+\Theta)}{|{}^{\mathrm{R}}\mathbf{x}_i'|\cos\beta_i - x_{ij}\Theta'\cos(\alpha_{ij}+\Theta)}\right) - \Theta. \tag{4.5b}$$

Formulae (4.5) give closed-form expressions for the parameters ρ_j, ${}^{\mathrm{B}}\theta_j$ of an arbitrary wheel in terms of known quantities; there are no differential equations involved.

One of the simplest practical problems in the kinematics of a WMR is to move the robot body from one configuration, as represented by ${}^{\mathrm{R}}\mathbf{x}_i(0)$ and $\Theta(0)$ say, to a final configuration ${}^{\mathrm{R}}\mathbf{x}_i(T)$ and $\Theta(T)$. The above analysis demonstrates that to each rigid-body trajectory linking the initial and final configurations, or equivalently to each pair of functions ${}^{\mathrm{R}}\mathbf{x}_i(t)$ and $\Theta(t)$ with correct initial and final values, there corresponds a unique set of parameters. Consequently, many algorithms for the generation of controllable parameters can be developed — one for each rigid-body trajectory. For example, the mathematically simplest choice involves functions ${}^{\mathrm{R}}\mathbf{x}_i(t)$ and $\Theta(t)$ that are linear in time. In practice, other, non-kinematic, considerations enter into the question of which generating algorithm is, in some sense, best. For example, an optimal-control study could start from this analysis and would further determine which set of feasible control parameters actually minimized some functional, perhaps total work.*

The analysis of this section takes no account of constraints on the values that parameters may assume. For example, most practical designs will involve a maximum steering lock, and accordingly the parameter ${}^{\mathrm{B}}\theta_i$ must lie in some predetermined range. The development presented here certainly allows the admissibility of a given trajectory and its associated parameters to be checked *a posteriori*, but we do not attempt the much harder problem of the *a priori* characterization of trajectories that are attainable with a given restricted set of parameters.

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* Possibly the total work of the designer.

5. Practical Designs

The analysis of the previous sections has demonstrated that in order to be able to maneuver a WMR in arbitrary planar rigid body motions, it is both necessary (in the considered class of designs) and sufficient that there be two driven and steered simple wheels. At first sight it appears that any two-wheeled robot must immediately fall, because it makes only two-point contact with the supporting plane. This argument is invalid, because the center of gravity of the robot body may lie below the axle attachment points. See, for example, Helmers (1983a), which discusses a robot with two non-simple wheels that are mounted in non-vertical planes. Nevertheless, the body of a robot with only two wheels will be susceptible to pendulum-like oscillations that may be excited during maneuvers. Consequently, we consider robots with wheel configurations that provide an intrinsically more stable, three-point contact with the supporting plane.

When the third support point is provided by a third simple wheel and rolling occurs, the analysis of Section 3 applies to completely determine the third axle vector in terms of the first two axle vectors. Consequently, of the six controllable parameters associated with three simple wheels, only three are independent. If there is an error in any one of these parameters, slippage must occur. Accordingly, it is apparent that there is considerable, essentially redundant, work required to calculate compatible wheel parameters. There are three plausible approaches to the removal of this redundancy.

First, the *mechanics* of the maneuvering robot may guarantee that some of the *kinematic* compatibility conditions are automatically satisfied when some of the wheels are not driven or steered. For example, if the third simple wheel is steered correctly and no control of its rotation rate is made, it will automatically rotate at a compatible speed in response to the frictional forces exerted by the surface. In contrast, there are no torques that would automatically orient the third wheel and allow it to be unsteered. Consequently, all redundancies cannot be eliminated merely by ignoring certain control parameters.

Second, compatibility problems are reduced if some mobility of the robot body is sacrificed by restriction of the range of some of the controls. For example in a three-wheeled robot, all compatibility conditions are removed if two steering angles ${}^{\mathrm{B}}\theta_i$ are held constant and the associated drive rates are left passive. Having two fixed wheels limits possible rolling motions to the unacceptably small set of circular arcs with prescribed radius, *unless* the fixed axle vectors are parallel to each other *and* to the line between the two axle points. This last configuration is equivalent to having two simple wheels mounted on the same axle with independent rotation rates (such as the rear wheels of an automobile). This special three-wheel configuration with one steerable, two fixed, one driven, and two passive, wheels is a viable design that was adopted for the WMR Neptune (Podnar, Dowling and Blackwell 1984). Its kinematics are further discussed in Section 6.

Third, if restricted classes of planar rigid body motions are not acceptable, then a mechanically stable robot, with minimal control compatibility conditions, can be achieved by having two steered, driven wheels and a passive *castor*. For our purposes, a castor is a second (small) planar rigid body that has mounted in it a fixed simple wheel and that is attached to the robot body at a free pivot point. The addition of a castor to two simple wheels at an appropriate point certainly makes the robot body mechanically stable. The four kinematic parameters associated with the two simple wheels must satisfy one compatibility condition, but the castor can be left entirely passive with no rolling condition being violated. This last fact is physically obvious and is supported by practical experience. It can be derived from our mathematical analysis but will not be considered here.

6. Examples of Rolling

As an application of our theory, we consider the basic kinematic problem of determining the steering and drive rates that move a WMR from one prescribed rigid body configuration to another. Consider first a completely mobile robot, that is, a robot that can traverse arbitrary rigid-body motions with all wheels undergoing ideal rolling. Two such designs are the CMU Rover (Moravec 1983), which has three driven, steered simple wheels and the design suggested in the previous section, namely a WMR with two driven, steered simple wheels, and a supporting, passive castor. Because the WMR is assumed to be completely mobile, there is no question of which trajectories linking the given starting and ending configurations are actually accessible. We shall therefore pick a simple trajectory and determine the parameters that move one axle point $\mathbf{x}_i$ uniformly along the ${}^{\mathrm{R}}x_2$-axis from (0,0) to (0,h) in unit time, while simultaneously rotating the robot uniformly from $\Theta(0) = \Theta^0$ to $\Theta(1) = \Theta^1$. We set

$$ {}^{\mathrm{R}}\mathbf{x}_i(t) = (0,0) + t(0,h), \tag{6.1a} $$

$$ \Theta(t) = \Theta^0(1-t) + t\Theta^1. \tag{6.1b} $$

From (4.4b), ${}^{\mathrm{B}}\theta_i = \beta_i - \Theta = -\Theta^0(1-t) - t\Theta^1$. Moreover, from (4.4a), $\rho_i = |{}^{\mathrm{R}}\mathbf{x}_i'| = h$. The parameters at another simple wheel mounted at the axle point ${}^{\mathrm{R}}\mathbf{x}_j$ are obtained from equations (4.5), which reduce to

$$ \rho_j = \sqrt{h^2 + x_{ij}^2(\Theta^1-\Theta^0)^2 - 2hx_{ij}(\Theta^1-\Theta^0)\cos\big(\alpha_{ij}+\Theta(t)\big)}, \tag{6.2a} $$

$$ {}^{\mathrm{B}}\theta_j = -\tan^{-1}\left(\frac{x_{ij}(\Theta^1-\Theta^0)\sin\big(\alpha_{ij}+\Theta(t)\big)}{h - x_{ij}(\Theta^1-\Theta^0)\cos\big(\alpha_{ij}+\Theta(t)\big)}\right) - \Theta(t), \tag{6.2b} $$

and $\Theta(t)$ is given by (4.1b). Here x_{ij}, defined in Section 5, and α_{ij}, defined in Section 2, are constants determined by the geometry of the WMR. If any of the simple wheels of the robot are not driven exactly as is specified by (6.2), slippage of the wheels must occur, and the analysis of Sections 7 and 8 will be required.

Another practical WMR wheel configuration is kinematically equivalent to a child's tricycle. The kinematics of such vehicles are considered in Alexander and Maddocks (1988). The WMR Newt (Hollis 1977) is one tricycle-like WMR. It has two parallel driven wheels that are fixed in direction, along with a passive castor for stability. Another existing robot, Neptune (Podnar, Dowling and Blackwell 1984), has two passive parallel wheels fixed in direction and a driven, steered wheel. Because tricycle-like WMRs have a fixed axle, they are not completely mobile, and therefore arbitrary rigid-body trajectories cannot be specified. For example, such a WMR cannot be moved parallel to the fixed axle. However, the path of any given point not on the fixed axle can be prescribed arbitrarily (Alexander and Maddocks 1988). In particular, a WMR with the kinematics of a tricycle can be maneuvered with ideal rolling in any rigid-body trajectory for which the center of curvature always lies on the extension of the axle vector. Consequently the robot can be maneuvered with ideal rolling from one body configuration to another in a motion comprised of piecewise smooth arcs of pure rotations, and the prior analysis can be used to construct the appropriate steering and drive rates.

The Stanford Cart (Moravec 1980; Moravec 1983) is another WMR that is essentially kinematically equivalent to a tricycle. It has two parallel driven wheels fixed in direction and a pair of steerable wheels configured as a small automobile. For our purposes the only difference between the Stanford cart and a tricycle is that there are more wheels, and therefore more redundant controllable parameters that must be accurately activated to obviate slippage. For WMRs with four or more wheels, there is almost certain to be some slippage due to incompatibility of activated wheels.

7. Slippage

Sections 2–5 provide an analysis that uses a kinematic model of ideal rolling of simple wheels to predict the motion of a WMR in ambient space when the axle-vector body coordinates ${}^{\mathrm{B}}\mathbf{a}_i$ are given as functions of time. The analysis does not involve a force balance and does not explicitly solve Newton's laws of motion. Instead, it is implicitly assumed that the undetermined static frictional forces, which act between the wheels and the supporting plane, are sufficiently large to balance the inertial forces that arise from accelerations. Naturally, the validity of this assumption depends on the relative sizes of the inertial and static frictional forces. Whenever the inertial forces dominate, the rolling model ceases to be valid. The subsequent motion involves a phenomenon we call skidding, and any analysis would have to be based on the full system of Newton's laws. On the other hand, the rolling model also fails, even if the inertial forces are negligible, whenever the wheel controls are incompatible [i. e., whenever rolling conditions (3.2) are not satisfied]. The subsequent motion involves the phenomenon we call *slippage*. The purpose of this section is to present a quasi-static model that can predict motions that include slippage due to wheel incompatibility. The model described is quasi-static because, in distinction from the preceding analysis, the inertial forces are assumed to be sufficiently small as to be totally negligible.

In the previously described model of rolling, we obtained closed-form expressions for the axle-point velocities $\mathbf{v}_i$ in terms of the controls, as represented by the axle vectors $\mathbf{a}_i$, and consequently obtained differential equations that describe the axle-point trajectories. Here we pursue a less ambitious goal; namely, to find an algorithm that determines the current axle-point velocities $\mathbf{v}_i$ given the current controls $\mathbf{a}_i$. We shall also consider the problem in which one or more drive controls are left passive. In this case, the direction of the axle vector is known, but its magnitude is undetermined.

Our first step is the construction of a *friction functional* that represents the power dissipated by Coulomb frictional forces during any planar rigid body motion of the WMR. The friction functional depends on the axle-point coordinates ${}^{\mathrm{B}}\mathbf{x}_i$, the axle-vector coordinates ${}^{\mathrm{B}}\mathbf{a}_i$, and on the planar rigid-body velocity of the robot, which is regarded as the unknown. Of course, a planar rigid-body velocity is a three-vector describing the planar linear velocity of some reference point and the angular velocity of the body. We will take the reference point to be the origin of the body coordinates ${}^{\mathrm{B}}\mathbf{x}$, and $\mathbf{y}$ will denote the unique three-vector, with vertical component $\Omega = \mathbf{y} \cdot \mathbf{k}$, for which $\mathbf{y} \times \mathbf{k}$ is the linear velocity of the reference point. With this definition of $\mathbf{y}$, any axle-point velocity $\mathbf{v}_i$ can be written

$$\mathbf{v}_i = (\mathbf{y} - \Omega \mathbf{x}_i) \times \mathbf{k}. \tag{7.1}$$

Here, as before, planar vectors such as $\mathbf{x}_i$ are viewed as being embedded in three-dimensional space, so that the vector product is well defined. We shall show that the friction functional is a convex function of $\mathbf{y}$. Consequently the minimization problem is well posed. It is, however, mathematically delicate, because the friction functional is neither strictly convex nor everywhere differentiable. Thus a minimizer need not be unique, and the standard conditions applying at a minimizer(namely, the vanishing of the derivative) must be interpreted in the sense of the subdifferential from convex analysis.

We obtain an evolutionary system for the WMR from the *principle of least dissipation*, which states that the actual motion of the WMR is the one that minimizes the friction functional at all times. It is then shown that motions that satisfy minimal dissipation are quasi-static in the sense that the forces and moments that arise are in equilibrium. Accordingly, the principle of least dissipation leads to approximate solutions of Newton's laws of motion, provided that inertial terms are negligible.

Consider a simple wheel that is undergoing a combination of rolling and slippage. Define the *rolling velocity* $\mathbf{r}_i$ by $\mathbf{r}_i = \mathbf{a}_i \times \mathbf{k}$ where, as before, the $\mathbf{a}_i$ are the axle vectors (scaled with a factor $2\pi R_i$). In the case of ideal rolling, $\mathbf{r}_i$ coincides with the velocity $\mathbf{v}_i$ of the ith axle point, but in the case of combined rolling and sliding, there can be a discrepancy. We define the *slippage velocity* $\mathbf{s}_i$ to be this discrepancy. Explicitly, for all i,

$$\mathbf{s}_i = \mathbf{v}_i - \mathbf{r}_i = \{\mathbf{y} - \Omega \mathbf{x}_i - \mathbf{a}_i\} \times \mathbf{k}. \tag{7.2}$$

It should also be remarked that the wheel rotation rate ρ_i satisfies

$$\rho_i = |\mathbf{r}_i| = |\mathbf{a}_i|. \tag{7.3}$$

According to Coulomb's law, the sliding friction $\mathbf{f}_i$ between a particle and a surface is given by the expression

$$\mathbf{f}_i = -\frac{\alpha_i \mathbf{s}_i}{|\mathbf{s}_i|}, \tag{7.4}$$

where $\mathbf{s}_i \neq \mathbf{0}$ is the relative velocity of the particle and the surface, and $\alpha_i \geq 0$ is the product of the normal load and the coefficient of sliding friction. If the coefficients of sliding and static friction are assumed equal, (7.4) also describes Coulomb's law of static friction, provided that in the limit $\mathbf{s}_i \to \mathbf{0}$, $\mathbf{s}_i/|\mathbf{s}_i|$ is interpreted as an arbitrary vector with magnitude less than or equal to one.

The total instantaneous power P being dissipated by sliding friction is the sum of the scalar products

$$P = -\sum_i \mathbf{f}_i \cdot \mathbf{s}_i, \tag{7.5}$$

or, by (7.4),

$$P = \sum_i \alpha_i |\mathbf{s}_i|. \tag{7.6}$$

Of course, the construction of (7.6) applies to the contact points between the wheels of the WMR and the supporting plane, with the relative velocity being the slippage velocity $\mathbf{s}_i$. Consequently, (7.2) can be used to rewrite (7.6) as

$$P = P(\mathbf{y}; {}^{\mathrm{B}}\mathbf{x}_i, {}^{\mathrm{B}}\mathbf{a}_i, \alpha_i) = \sum_i \alpha_i |(\mathbf{A}_i \mathbf{y} - {}^{\mathrm{R}}\mathbf{a}_i) \times \mathbf{k}|. \tag{7.7}$$

Here A_i denotes the 2×3 matrix defined by

$$\mathbf{A}_i = \begin{pmatrix} 1 & 0 & -{}^{\mathrm{B}}x_1^i \\ 0 & 1 & -{}^{\mathrm{B}}x_2^i \end{pmatrix}.$$

We call (7.7) the *dissipation* or *friction functional*. The variable $\mathbf{y}$ is unknown, but ${}^{\mathrm{B}}\mathbf{x}_i$, ${}^{\mathrm{B}}\mathbf{a}_i(t)$ and α_i are viewed as known parameters. Because $(\mathbf{A}_i\mathbf{y} - {}^{\mathrm{R}}\mathbf{a}_i)$ lies in the horizontal plane and $\mathbf{k}$ is the vertical unit vector, the dissipation functional can be rewritten as:

$$P = P(\mathbf{y}; {}^{\mathrm{B}}\mathbf{x}_i, {}^{\mathrm{B}}\mathbf{a}_i, \alpha_i) = \sum_i \alpha_i |\mathbf{A}_i \mathbf{y} - {}^{\mathrm{R}}\mathbf{a}_i|. \tag{7.8}$$

It is straightforward to show that the dissipation functional is convex in $\mathbf{y}$. In particular, by the triangle inequality, we have that for $0 \leq \tau \leq 1$,

$$\sum_i \alpha_i \big|\mathbf{A}_i\big(\tau\mathbf{y}_1 + (1-\tau)\mathbf{y}_2\big) - {}^{\mathrm{B}}\mathbf{a}_i\big| \leq \tau \sum_i \alpha_i |\mathbf{A}_i\mathbf{y}_1 - {}^{\mathrm{B}}\mathbf{a}_i| + (1-\tau) \sum_i \alpha_i |\mathbf{A}_i\mathbf{y}_2 - {}^{\mathrm{B}}\mathbf{a}_i|. \tag{7.9}$$

Consequently the first variation of the dissipation functional provides both necessary and sufficient conditions to determine minima.

Provided that no term $|\mathbf{A}_i\mathbf{y} - {}^{\mathrm{B}}\mathbf{a}_i|$ vanishes, the dissipation function is a differentiable function of $\mathbf{y}$, and the first-order conditions are obtained by differentiation with respect to $\mathbf{y}$:

$$\sum_i \alpha_i \mathbf{A}_i^{\mathrm{T}} \frac{\mathbf{A}_i\mathbf{y} - {}^{\mathrm{B}}\mathbf{a}_i}{|\mathbf{A}_i\mathbf{y} - {}^{\mathrm{B}}\mathbf{a}_i|} = \mathbf{0}. \tag{7.10}$$

Moreover, the theory of convex functionals—see for example Rockafellar (1970)—can be applied to treat the singular cases. In particular, provided that any singular terms

$$\frac{\mathbf{A}_i\mathbf{y} - {}^{\mathrm{B}}\mathbf{a}_i}{|\mathbf{A}_i\mathbf{y} - {}^{\mathrm{B}}\mathbf{a}_i|} \tag{7.11}$$

are interpreted as arbitrary vectors $\mathbf{z}_i$ with $|\mathbf{z}_i| \leq 1$, the left side of (7.10) always represents the subdifferential of the dissipation functional. Consequently, any minimizing vector $\mathbf{y}$ must satisfy equation (7.10) either in the classical sense or, when there are singular terms, in a generalized sense involving vectors $\mathbf{z}_i$.

Equations (7.10) can be expanded to:

$$\sum_i \alpha_i \frac{\mathbf{A}_i\mathbf{y} - {}^{\mathrm{B}}\mathbf{a}_i}{|\mathbf{A}_i\mathbf{y} - {}^{\mathrm{B}}\mathbf{a}_i|} = \mathbf{0}, \tag{7.12}$$

and

$$\sum_i \alpha_i {}^{\mathrm{B}}\mathbf{x}_i \cdot \frac{\mathbf{A}_i\mathbf{y} - {}^{\mathrm{B}}\mathbf{a}_i}{|\mathbf{A}_i\mathbf{y} - {}^{\mathrm{B}}\mathbf{a}_i|} = \mathbf{0}. \tag{7.13}$$

The definition of $\mathbf{A}_i$, taken with equations (7.2) and (7.4), implies that the cross-product with $\mathbf{k}$ of equations (7.12) and (7.13) can be written in the form

$$\sum_i \mathbf{f}_i = \mathbf{0}, \tag{7.14}$$

and

$$\sum_i (\mathbf{x}_i \times \mathbf{f}_i) \cdot \mathbf{k} = \mathbf{0} \qquad (7.15).$$

Consequently, minimization of the friction functional with respect to the velocity $\mathbf{y}$ generates *quasi-static* motions, in which the frictional forces and moments acting on the robot are in equilibrium. This conclusion is valid whether (7.10) represents a classic derivative or a subdifferential. The non-smooth case arises precisely in the absence of slippage at one or more wheels, and the associated indeterminacy in the subdifferential is exactly matched by the indeterminacy in Coulomb's law of static friction.

It should be observed that the dissipation functional is always nonnegative and is zero only if each term in the sum (7.7) vanishes. Provided that there are at least two wheels, this requirement leads to an overdetermined system of equations involving the three-vector $\mathbf{y}$. The solvability conditions for this overdetermined system are precisely the rolling compatibility conditions (3.2) derived previously. Accordingly the principle of minimal dissipation predicts ideal rolling of all wheels whenever ideal rolling is possible.

We have shown that the principle of minimal dissipation provides a quasi-static dynamic system that models motions involving a combination of rolling and slipping. The principle characterizes instantaneous rigid-body motions from minimization over the variable $\mathbf{y}$ while the (known) parameters ${}^{\mathrm{B}}\mathbf{x}_i$, ${}^{\mathrm{B}}\mathbf{a}_i$, and α_i are held fixed. A minimizing vector $\mathbf{y}$ determines the linear velocity (in body coordinates) of the origin of the body-coordinate system and the angular velocity of the WMR. For most parameter values the minimal $\mathbf{y}$ will be unique, but the example of the next section shows that nonuniqueness is possible.

In the case of a passive wheel the interpretation is slightly different. Then the axle vector ${}^{\mathrm{B}}\mathbf{a}_i$ is of the form $\rho_i{}^{\mathrm{B}}\mathbf{u}_i$, where ${}^{\mathrm{B}}\mathbf{u}_i$ is a given unit vector, but the rotation rate ρ_i is *a priori* unknown. Any unknown rotation rates are treated as additional variables over which the slippage functional must be minimized, and further first-order conditions are obtained. More precisely, partial differentiation with respect to each passive rotation rate ρ_i implies

$$(\mathbf{A}_i\mathbf{y} - \rho_i{}^{\mathrm{B}}\mathbf{u}_i) \cdot {}^{\mathrm{B}}\mathbf{u}_i = 0. \tag{7.16}$$

However this equation is equivalent to the geometric condition that the slippage velocity is orthogonal to the rolling velocity, or $\mathbf{s}_i \cdot \mathbf{r}_i = 0$. Hence there is a minimization principle at each passive wheel: for given $\mathbf{v}_i$ and ${}^{\mathrm{B}}\mathbf{u}_i$, the rotation rate ρ_i is such that the slippage vector $\mathbf{s}_i$ has minimal length. Alternatively, equation (7.16) characterizes the rotation rate for which the frictional force is parallel to the axle vector. Consequently, the frictional force exerts no torque about the axle vector, which is in accord with quasi-static motion.

8. Examples Involving Slippage

Before a general analysis of the kinematics of WMRs with slippage can be attempted, the dissipation functional must be better understood. We present, as an illustrative example, an analysis of the case of two wheels.

As was noted in Section 5, when a WMR has two wheels both of which are driven and steered, then there is one constraint (3.2) on the parameters that must be satisfied in order for ideal rolling to be possible; if this constraint is violated, slippage must occur. We shall analyze the motion, including slippage, for such two-heeled robots. The remarks concerning castors made in Section 5 apply with equal force here and imply that castors have no effect on the kinematics of a WMR. Thus it is apparent that the subsequent analysis also applies to robots with two simple wheels and passive stabilizing castors.

Denote the wheels by $i = 1, 2$. Recall that the $\alpha_i \geq 0$ are the product of the normal load on the ith wheel and the coefficient of friction. There are two cases to consider: (1) the symmetric case $\alpha_1 = \alpha_2$ and (2) the asymmetric case $\alpha_1 \neq \alpha_2$.

In the case of two wheels with asymmetric weights, a minimal dissipation solution must have one wheel undergoing ideal rolling. This follows from the following reasoning. In the smooth case (7.12) reduces to the weighted sum of two unit vectors, and accordingly has a solution if and only if the two unit vectors are anti-parallel and $\alpha_1 = \alpha_2$. Therefore, when $\alpha_1 \neq \alpha_2$, the minimum must occur at a nondifferentiable point of the dissipation functional. In this case the simplest approach is to consider the functional (7.8) directly. Suppose that $\alpha_1 < \alpha_2$. It is then apparent that (7.8) is minimized when it is the second wheel that undergoes ideal rolling. The ideal rolling condition

$$\mathbf{A}_2\mathbf{y} = \mathbf{a}_2, \tag{8.1}$$

provides an underdetermined set of equations for $\mathbf{y}$ that reduce to the condition

$$\mathbf{y} - \Omega\mathbf{k} = \Omega\mathbf{x}_2 + \mathbf{a}_2,$$

where the scalar angular velocity Ω is arbitrary. Then the dissipation functional (7.8) reduces to

$$\alpha_1|\Omega(\mathbf{x}_1 - \mathbf{x}_2) + (\mathbf{a}_1 - \mathbf{a}_2)|, \tag{8.2}$$

which is minimized when

$$\Omega = -\frac{(\mathbf{a}_1 - \mathbf{a}_2)\cdot(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^2}. \tag{8.3}$$

Remarkably, the angular velocity Ω in (8.3) that is predicted by the slippage model coincides with the angular velocity in (3.3) that was predicted by the ideal rolling model. Notice that the second wheel undergoes ideal rolling in such a manner that the slippage of the first wheel is minimized. Furthermore, the minimal dissipation is strictly positive unless $\mathbf{a}_1 - \mathbf{a}_2$ is parallel to $\mathbf{x}_1 - \mathbf{x}_2$. In this last case the compatibility condition for ideal rolling of both wheels is satisfied, $P = 0$, and no slippage occurs.

Consider now the symmetric case. Here there is no difference between the smooth and non-smooth cases. Equation (7.12) always implies that the horizontal component of $\mathbf{y}$ lies on the straight line segment between $\Omega^{\mathrm{R}}\mathbf{x}_1 + {}^{\mathrm{R}}\mathbf{a}_1$ and $\Omega^{\mathrm{R}}\mathbf{x}_2 + {}^{\mathrm{R}}\mathbf{a}_2$. That is, for $0 \leq \gamma \leq 1$,

$$\mathbf{y} - \Omega\mathbf{k} = (1-\gamma)(\Omega^{\mathrm{R}}\mathbf{x}_1 + {}^{\mathrm{R}}\mathbf{a}_1) + \gamma(\Omega^{\mathrm{R}}\mathbf{x}_2 + {}^{\mathrm{R}}\mathbf{a}_2). \tag{8.4}$$

Then (7.13) further implies that

$$\Omega = -\frac{(\mathbf{a}_1 - \mathbf{a}_2)\cdot(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^2}, \tag{8.5}$$

as before. Notice that although the angular velocity Ω of the WMR is uniquely determined, condition (8.4) does *not* provide a unique solution for the horizontal component of $\mathbf{y}$. Consequently the WMR has several possible linear velocities, all corresponding to quasi-static motion. In practice, it might be expected that the load α_i on one wheel or the other will dominate at any particular time, and the non-uniqueness will be resolved so that the dominant wheel will experience ideal rolling. Small perturbations could cause this dominance to jump intermittently from one wheel to the other, and the motion of the WMR will vary discontinuously.*

9. Discussion

This paper concerns the forward and inverse kinematics of a WMR with simple wheels that is maneuvering over a horizontal plane. The investigation has two main parts. In the first instance the wheels are assumed to be undergoing ideal rolling with static contact between the wheels and supporting plane. We describe the explicit relations between the motion of the wheels in the robot body—as characterized by the steering and driving rates—and the motion of the robot body in the plane, and *vice versa*. The analysis allows us to ascertain the mobility of any given robot design. That is, for a given wheel configuration, we can determine those rigid-body trajectories that are accessible with ideal rolling of all wheels, and we can construct the steering and drive rates that access the trajectory.

The central result in our kinematic investigations is the ideal rolling compatibility conditions, which imply that only three steering and drive rates are independent. The second part of the paper concerns motions that ensue when these rolling compatibility conditions fail. Our model for combined rolling and slipping is formulated and analyzed as a minimization principle involving a friction functional that measures the dissipation due to Coulomb friction. It is shown that this principle of minimal dissipation is equivalent to the characterization of quasi-static motions for which frictional forces and torques are in equilibrium. It should be stressed that the unknowns in our analysis are linear and angular velocities rather than forces and torques. This remark may be of practical significance, because the servo-control of velocities is typically easier than that of forces. We also mention that the theory for numerical minimization of non-smooth convex functionals of a class encompassing the dissipation functional is well developed (Overton 1983).

The dissipation functional constructed here has connections with Rayleigh's dissipation function (Lord Rayleigh 1896; Whittaker 1961, p. 230; Pars 1965, §10.11). The classic form of Rayleigh's dissipation function arises in the Lagrangian description of a finite-dimensional dynamic system that has dissipative forces that are linearly dependent on the speeds and directly opposed to the velocities. For such systems the Rayleigh dissipation function has a smooth dependence on the coordinates and is quadratic in the speeds. Of course, Coulomb's model of sliding friction leads to dissipative forces that

* J. C. Alexander has experienced this effect in a very badly aligned automobile. As a WMR, an automobile can be modelled with two steerable wheels and one driven wheel (the rear axle). The effect is the same as in the text. Usually one wheel will dominate; however, on a wet road, the coefficients of friction can vary so that the other wheel will dominate for a second or two causing the automobile to suddenly lurch sideways. The author remembers the sensation well.

are directly opposed to the velocity but are independent of speed. Consequently we are led to the more complicated non-smooth dissipation functional described above. The principle of minimal dissipation is also closely related to ideas introduced by Peshkin and Sanderson (1986) to describe quasi-static sliding of systems of particles, but rolling was not considered there. We also mention the work of Moreau (1979), who exploits methods of convex analysis, and of Goyal and Ruina (1988). Both of these works construct models for the forces that arise at wheels in terms of anisotropic friction laws. However, the thrust of their research is considerably different from our development, which is focused on a simultaneous analysis of several wheels with the unknown being the rigid-body velocity of the robot.

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