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Article *in* Regular and Chaotic Dynamics · March 2015

DOI: 10.1134/S1560354715020045

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ISSN 1560-3547, Regular and Chaotic Dynamics, 2015, Vol. 20, No. 2, pp. 153–172. © Pleiades Publishing, Ltd., 2015.

Dynamics and Control of an Omniwheel Vehicle

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Received January 14, 2015; accepted March 2, 2015

Abstract—A nonholonomic model of the dynamics of an omniwheel vehicle on a plane and a sphere is considered. A derivation of equations is presented and the dynamics of a free system are investigated. An explicit motion control algorithm for the omniwheel vehicle moving along an arbitrary trajectory is obtained.

MSC2010 numbers: 70F25, 70E18, 70E55, 70E60

DOI: 10.1134/S1560354715020045

Keywords: omniwheel, roller-bearing wheel, nonholonomic constraint, dynamical system, invariant measure, integrability, controllability

INTRODUCTION

The dynamics of various vehicles (carriages) with conventional wheels are described in many publications (for more details and an extensive list of references on this topic, see, e.g., [30]). Nevertheless, even in this subject, which is far from being new, there remain a great number of open problems and problem statements for which the motion has not been completely explored. The omniwheel (also called *roller-bearing wheel* in the Russian-language literature) was invented not very long ago, and there has been little published research on the dynamics of various omniwheel vehicles so far. This concerns both free dynamics analysis and control problems [42]. We refer the reader to the review article [19] for a more detailed treatment of the kinematics of omniwheel systems.

In this paper we consider the dynamics of a free and controlled motion of the omniwheel vehicle on a plane and a sphere in the nonholonomic setting. An analogous problem is formulated in [31, 32], where a fairly particular case of the vehicle on a plane is studied; in [45, 46] equations for a more general case are obtained and the dynamics of a reduced system for a special configuration of wheels of the vehicle are analyzed. We note that in [45, 46] the equations of motion are derived using the procedure [39], which, in our opinion, involves unjustifiably cumbersome calculations. We use here equations in quasi-velocities with undetermined multipliers which correspond to nonholonomic constraints. This is a much more convenient way to derive equations of motion. In particular, after fairly simple calculations we obtain equations of motion for the omniwheel vehicle on a sphere.

The dynamics of an omniwheel vehicle on a sphere are of considerable interest from the viewpoint of dynamics and applications. This gives rise to many intriguing problems closely related to problems of controlling mobile robots of a new generation. In particular, this system is interesting as a more advanced modification of mobile devices which are propelled using various mechanisms installed inside (wheel carriages, pendulums, gyroscopes etc.; see, e.g., [1, 5, 6, 9, 10, 14, 28, 29, 38, 43, 44]) or outside a ball (the so-called Ballbots [34, 35]). The omniwheels used in such devices provide a higher maneuverability.

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In this paper we present, therefore, a preliminary analysis of the dynamics of the omniwheel vehicle on a sphere. We also note an analogy of the equations of motion obtained with those of another nonholonomic problem, which was investigated previously in connection with the presence of algebraic first integrals [17] (see also [13, Chapter 3]). A related study concerned with the dynamics and control of one of the modifications of the spherical robot is presented in [26].

We also explore the controlled motion of the omniwheel vehicle on a plane. We note that modern treatments of the controllability of systems with nonholonomic constraints are, as a rule, restricted to constructing kinematic models based on analysis of nonholonomic distributions and ignoring the dynamics of systems considered [2, 3, 28, 29, 33]. Among studies concerned with analysis of dynamical controllability we mention [31, 32], where a particular case of a three-wheel omnidirectional vehicle is discussed. In this paper we consider a more general problem of controllability of a vehicle with an arbitrary number and arrangement of omniwheels. In doing so, we use general methods of control theory (Rashevsky–Chow theorem) and present constructive control planning algorithms for motion along a given trajectory.

1. THE NONHOLONOMIC MODEL OF AN OMNIWHEEL

We recall that such a wheel has rollers fastened on the periphery (outer rim), so that only one of the rollers of the wheel is in contact with the supporting surface. Each roller rotates freely about the axis which is fixed relative to the plane of the disk, and the wheel can roll in a straight line which makes a fixed angle with the plane of the wheel. For Mecanum wheels the roller axis is fastened at an angle of 45° to the plane of the wheel, whereas for most omniwheels the roller axis lies in the plane of the wheel. The simplest nonholonomic model of an omniwheel is a flat disk for which the velocity of the contact point with the supporting surface is directed along a straight line which forms a constant angle δ with the plane of the wheel (see Fig. 1).

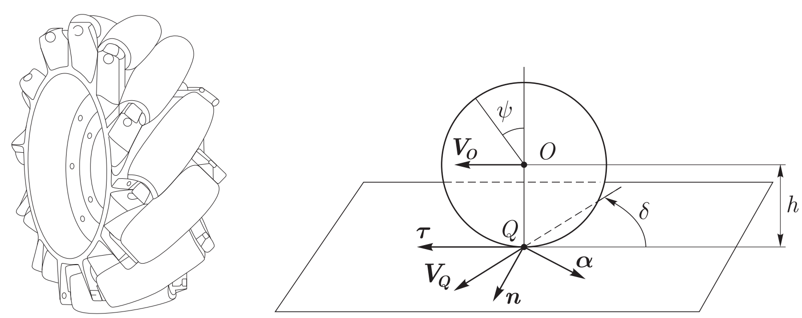


Fig. 1. The schematic and the nonholonomic model of an omniwheel.

Let $\boldsymbol{\tau}$ and $\boldsymbol{n}$ be the tangent and normal vectors to the plane of the wheel at the point of contact, such that the vector $\boldsymbol{\tau}\times\boldsymbol{n}$ is directed vertically upwards, and let $\boldsymbol{\alpha}$ be the unit vector along the axis of attachment of the rollers. Then the constraint equation is

$$(\boldsymbol{V}_Q, \boldsymbol{\alpha}) = 0, \tag{1.1}$$

where $\boldsymbol{V}_Q$ is the velocity of the point of contact. If $\boldsymbol{V}_O$ is the velocity of the center of the wheel and ψ is the angle of the wheel's rotation (measured counterclockwise when viewed from the tip of the vector $\boldsymbol{n}$), then the constraint equation can be rewritten as

$$(\boldsymbol{V}_O + h\dot{\psi}\boldsymbol{\tau}, \boldsymbol{\alpha}) = 0,$$

where h is the radius of the wheel. It is convenient to solve this equation for $\dot{\psi}$, whence

$$\dot{\psi} = -\frac{1}{sh}(\boldsymbol{V}_O, \boldsymbol{\alpha}), \quad s = (\boldsymbol{\alpha}, \boldsymbol{\tau}) = \sin\delta. \tag{1.2}$$

Remark. Since the vector $\boldsymbol{\alpha}$ is defined up to the sign, one may assume without loss of generality that $s \geqslant 0$.

2. AN OMNIWHEEL VEHICLE

We obtain the equations of motion for an omniwheel vehicle rolling on a horizontal plane and consisting of a platform (framework), to which an arbitrary number of wheels is attached in such a way that their axes are fixed relative to the platform (Fig. 2).

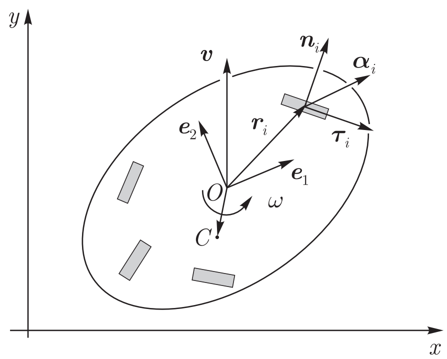


Fig. 2. Schematic of an omniwheel vehicle.

Choose a moving coordinate system $O\boldsymbol{e}_1\boldsymbol{e}_2$ rigidly attached to the platform. Let x, y, and φ denote the coordinates of the origin O and the angle of rotation of the axes $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ about the fixed coordinate system, respectively. The vectors $\boldsymbol{r}_i, \boldsymbol{\tau}_i, \boldsymbol{\alpha}_i, i = 1, \ldots, n$, specifying the position, plane and direction of the roller axis of each wheel are constant in the moving axes. Let $\boldsymbol{v} = (v_1, v_2)$ denote the velocity of the origin O of the moving coordinate system which is referred to the moving axes, and let ω be the angular velocity of the platform. Then the constraint equations (1.2) defining the direction of the velocities of the contact points can be written as

$$f_i = \dot{\psi}_i + G_i, \quad G_i = \frac{1}{s_i h_i}(\boldsymbol{v} + \omega \mathbf{J}\boldsymbol{r}_i, \boldsymbol{\alpha}_i),$$
$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

Remark. The vector $\omega \mathbf{J} r_i$ in the three-dimensional space corresponds to the vector product of the form $\boldsymbol{\omega} \times \boldsymbol{r}_i$, where $\boldsymbol{\omega} = (0, 0, \omega)$.

With no constraints imposed, the kinetic energy of the system consists of the kinetic energy of the platform

$$T_0 = \frac{1}{2} m_0 \boldsymbol{v}^2 + \frac{1}{2} I_0 \omega^2 + m_0 \omega (\mathbf{J}\boldsymbol{R}_{oc}, \boldsymbol{v}),$$

where m_0 is the mass of the framework, I_0 is its moment of inertia relative to the point O, and $\boldsymbol{R}_{oc}$ is the vector $\boldsymbol{OC}$ ($\boldsymbol{C}$ is the center of mass of the platform), and of the kinetic energy of each of the wheels

$$T_i = \frac{1}{2} m_i (\boldsymbol{v} + \omega \mathbf{J}\boldsymbol{r}_i, \boldsymbol{v} + \omega \mathbf{J}\boldsymbol{r}_i) + \frac{1}{2} I_i \dot{\psi}_i^2 + \frac{1}{2} \widetilde{I}_i \boldsymbol{\omega}^2,$$

where m_i is the mass of the i-th wheel, I_i and $\widetilde{I}_i$ are the central moments of inertia of the wheel relative to the axis and the diameter, respectively. Here and in the sequel we assume that the centers of mass of the wheels lie on their axes, and the mass distribution of the wheel possesses axial symmetry, i.e., the central tensor of inertia of the wheel has the form $\mathbf{I}_i = \mathrm{diag}(I_i, \widetilde{I}_i, \widetilde{I}_i)$. For the total kinetic energy we finally obtain

$$T = T_0 + \sum_{i=1}^{n} T_i = \frac{1}{2} m\boldsymbol{v}^2 + \frac{1}{2} I\omega^2 + m\omega(\mathbf{J}\boldsymbol{r}_c, \boldsymbol{v}) + \frac{1}{2}\sum_i I_i \dot{\psi}_i^2,$$

$$m = m_0 + \sum_i m_i \text{ is the total mass of the system,}$$

$$I = I_0 + \sum_i m_i r_i^2 + \widetilde{I}_i \text{ is the total moment of inertia relative to the point } O,$$

$$\boldsymbol{r}_c = m^{-1}(m_0 \boldsymbol{R}_{oc} + \sum_i m_i \boldsymbol{r}_i) \text{ is the center of mass of the entire system.}$$

Remark. In this approach we neglect the inertia of the rollers.

We write for this system the equations of motion in quasi-velocities with undetermined multipliers in the form [13]

$$\begin{gathered}
\left(\frac{\partial T}{\partial \boldsymbol{v}}\right)^{\boldsymbol{\cdot}} + \omega\mathbf{J}\frac{\partial T}{\partial \boldsymbol{v}} = \sum_k \lambda_k \frac{\partial f_k}{\partial \boldsymbol{v}} = \sum_k \lambda_k \frac{\partial G_k}{\partial \boldsymbol{v}},\\
\left(\frac{\partial T}{\partial \omega}\right)^{\boldsymbol{\cdot}} + \left(\mathbf{J}\boldsymbol{v}, \frac{\partial T}{\partial \boldsymbol{v}}\right) = \sum_k \lambda_k \frac{\partial f_k}{\partial \omega} = \sum_k \lambda_k \frac{\partial G_k}{\partial \omega},\\
\left(\frac{\partial T}{\partial \dot{\psi}_i}\right)^{\boldsymbol{\cdot}} = \lambda_i + M_i, \quad i = 1, \ldots, n,
\end{gathered}$$

where M_i is the moment of forces applied to the axes of the wheels.

From the last equation and the constraint equations (2.1) we find the undetermined multipliers in the form

$$\lambda_i = -I_i \dot{G}_i - M_i = -\frac{I_i}{s_i h_i}(\dot{\boldsymbol{v}} + \dot{\omega}\mathbf{J}\boldsymbol{r}_i, \boldsymbol{\alpha}_i) - M_i. \tag{2.2}$$

Substituting them into the first two equations and adding the kinematic relations governing the motion of the moving axes $O\boldsymbol{e}_1\boldsymbol{e}_2$, we obtain a complete system of equations of motion in the form

$$\begin{gathered}
(\boldsymbol{\Gamma} + m\mathbf{E})\dot{\boldsymbol{v}} + m\dot{\omega}(\mathbf{J}\boldsymbol{r}_c + \boldsymbol{R}) + m\omega\mathbf{J}(\boldsymbol{v} + \omega\mathbf{J}\boldsymbol{r}_c) = -\sum_i \mu_i \boldsymbol{\alpha}_i,\\
\hat{I}\dot{\omega} + m(\mathbf{J}\boldsymbol{r}_c + \boldsymbol{R}, \dot{\boldsymbol{v}}) + m\omega(\boldsymbol{v}, \boldsymbol{r}_c) = -\sum_i \mu_i(\mathbf{J}\boldsymbol{r}_i, \boldsymbol{\alpha}_i),\\
\dot{x} = v_1\cos\varphi - v_2\sin\varphi, \quad \dot{y} = v_1\sin\varphi + v_2\cos\varphi, \quad \dot{\varphi} = \omega,\\
\boldsymbol{\Gamma} = \sum_i \frac{I_i}{s_i^2 h_i^2}\boldsymbol{\alpha}_i \otimes \boldsymbol{\alpha}_i, \quad \boldsymbol{R} = m^{-1}\sum_i \frac{I_i}{s_i^2 h_i^2}(\mathbf{J}\boldsymbol{r}_i, \boldsymbol{\alpha}_i)\boldsymbol{\alpha}_i,\\
\hat{I} = I + \sum_i \frac{I_i}{s_i^2 h_i}(\mathbf{J}\boldsymbol{r}_i, \boldsymbol{\alpha}_i)^2, \quad \mu_i = \frac{M_i}{s_i h_i},
\end{gathered} \tag{2.3}$$

where $\mathbf{E}$ is the identity matrix.

Remark. Recall that the tensor product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as follows:

$$\boldsymbol{a} \otimes \boldsymbol{b} = \|a_i b_j\|.$$

Note that the matrix $\boldsymbol{\Gamma}$ is degenerate if and only if all vectors $\boldsymbol{\alpha}_i$ are parallel to each other, otherwise $\boldsymbol{\Gamma}$ is nondegenerate.

If the moments of forces μ_i applied to the wheel axes have the form $\mu_i = \omega\overline{\mu}_i$, where $\overline{\mu}_i$ are independent of $\boldsymbol{v}$ and ω, then Eqs. (2.3) admit the singular invariant measure [45]

$$\mu = \frac{1}{\omega}\,dv_1\,dv_2\,d\omega\,dx\,dy\,d\varphi.$$

Arbitrariness in the choice of the origin and the rotation of the moving axes can be used in order to simplify either the equations of motion or the integrals of the system (if any).

3. FREE MOTION

Consider a free motion of the omniwheel vehicle. Set $\mu_i = 0, i = 1, \ldots, n$ in Eqs. (2.3). Note that the simplest stable solutions of the free system are used in controlling the system by the method of gaits [36, 37]: the required complex motion of the system is achieved by the simplest motions of the free system, which are "conjugated" by means of controllable motions.

Choose a moving coordinate system such that the following relations hold:

$$\mathbf{J}\boldsymbol{r}_c + \boldsymbol{R} = 0, \quad \boldsymbol{\Gamma} = \mathrm{diag}(\Gamma_1, \Gamma_2). \tag{3.1}$$

The equations of motion are represented as

$$\begin{gathered}(\boldsymbol{\Gamma} + m\mathbf{E})\dot{\boldsymbol{v}} = m\omega^2\boldsymbol{r}_c - m\omega\mathbf{J}\boldsymbol{v}, \quad \hat{I}\dot{\omega} = -m\omega(\boldsymbol{v}, \boldsymbol{r}_c), \\ \dot{x} = v_1\cos\varphi - v_2\sin\varphi, \quad \dot{y} = v_1\sin\varphi + v_2\cos\varphi, \quad \dot{\varphi} = \omega.\end{gathered}$$

This system admits an energy integral and a linear integral:

$$\begin{gathered}\mathcal{E} = \frac{1}{2}\Big((\Gamma_1 + m)v_1^2 + (\Gamma_2 + m)v_2^2 + \hat{I}\omega^2\Big), \\ \mathcal{F} = \hat{I}\omega + (\Gamma_1 + m)\xi_2 v_1 - (\Gamma_2 + m)\xi_1 v_2,\end{gathered}$$

where $\boldsymbol{r}_c = (\xi_1, \xi_2)$. Since $\boldsymbol{\Gamma}$ is a sum of positive definite symmetric matrices, the eigenvalues Γ_1 and Γ_2 are nonnegative.

Remark. For this choice of moving axes the energy of the system is given by the diagonal quadratic form, whereas in the system of the center of mass it is not diagonal.

Reduced system. Consider separately a reduced system governing the evolution of the variables v_1, v_2, and ω. There are three different cases where the system can be reduced (by nondimensionalization) to the simplest form.

1) $\xi_1 \neq 0$, $\xi_2 \neq 0$.

After the substitution $v_1 = \xi_2 u_1$, $v_2 = -\xi_1 u_2$ the equations are represented as

$$\begin{gathered}a_1\dot{u}_1 = \omega(\omega - u_2), \quad a_2\dot{u}_2 = -\omega(\omega - u_1), \quad b\dot{\omega} = -\omega(u_1 - u_2), \\ a_1 = \frac{(\Gamma_1 + m)\xi_2}{m\xi_1}, \quad a_2 = \frac{(\Gamma_2 + m)\xi_1}{m\xi_2}, \quad b = \frac{\hat{I}}{m\xi_1\xi_2},\end{gathered} \tag{3.2}$$

and the integrals become

$$\mathcal{F} = a_1u_1 + a_2u_2 + b\omega, \quad \mathcal{E} = \frac{1}{2}(a_1u_1^2 + a_2u_2^2 + b\omega^2).$$

2) $\xi_1 = 0$, $\xi_2 \neq 0$.

After the substitution $v_1 = \xi_2 u_1$, $v_2 = -\xi_2 u_2$ we obtain

$$\begin{gathered}a_1\dot{u}_1 = -\omega u_2, \quad a_2\dot{u}_2 = -\omega(\omega - u_1), \quad b\dot{\omega} = \omega u_2, \\ a_1 = \frac{\Gamma_1 + m}{m}, \quad a_2 = \frac{\Gamma_2 + m}{m}, \quad b = \frac{\hat{I}}{m\xi_2^2},\end{gathered} \tag{3.3}$$

and the integrals are

$$\mathcal{F} = a_1u_1 + b\omega, \quad \mathcal{E} = \frac{1}{2}(a_1u_1^2 + a_2u_2^2 + b\omega^2).$$

3) $\xi_1 = \xi_2 = 0$.

The system has the form

$$a_1\dot{v}_1 = \omega v_2, \quad a_2\dot{v}_2 = -\omega v_1, \quad \dot{\omega} = 0,$$
$$a_1 = \frac{\Gamma_1 + m}{m}, \quad a_2 = \frac{\Gamma_2 + m}{m},$$

and the integrals are

$$\mathcal{F} = \omega, \quad \mathcal{E} = \frac{1}{2}(a_1 v_1^2 + a_2 v_2^2).$$

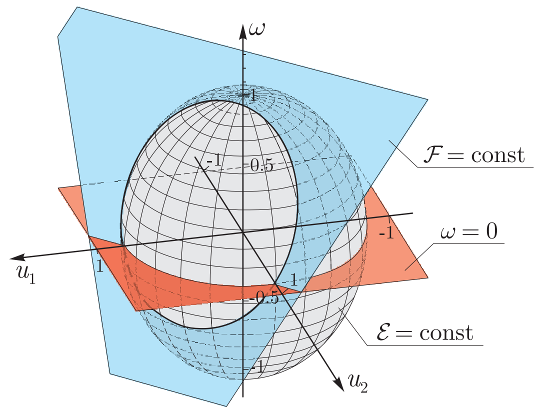


Fig. 3. Intersection of the surfaces of constant energy $\mathcal{E} = \text{const}$ and the linear integral $\mathcal{F} = \text{const}$ with $a_1 = 1.5$, $a_2 = 1.4$, $b = 1$.

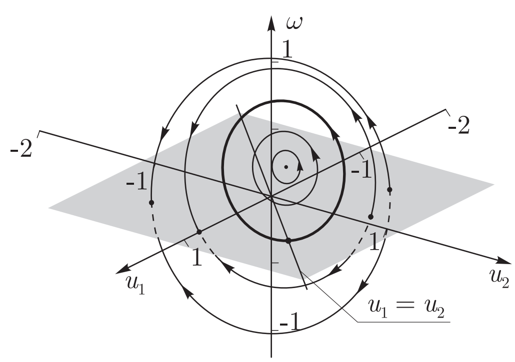


Fig. 4. A typical view of the trajectories of a reduced system in case 1 for $a_1 = 1.5$, $a_2 = 1.4$, $b = 1$.

Cases 1 and 2 are similar to each other, the trajectories of both the reduced and the complete system form a two-parameter family defined by the values of the integrals $\mathcal{F} = f$, $\mathcal{E} = h$.

Remark. The initial conditions for the variables x, y, and φ do not add any new parameters, because they can be excluded by choosing fixed axes.

In the three-dimensional space of variables u_1, u_2, and ω the trajectories are given by the intersection of the "ellipsoid of energy" $\mathcal{E} = h$ with the plane of the linear integral $\mathcal{F} = f$ (see Figs. 3–5), and according to (3.2) and (3.3), the plane $\omega = 0$ has been filled with fixed points of the reduced system. Thus, 4 types of trajectories of the reduced system can be distinguished:

type 1 — fixed points in which the plane of the linear integral is tangent to the ellipsoid of energy,

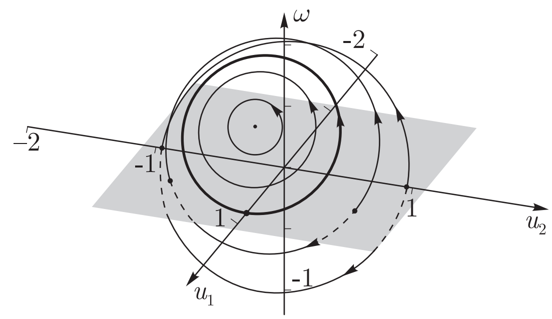


Fig. 5. A typical view of the trajectories of a reduced system in case 2 for $a_1 = 1.5$, $a_2 = 1.1$, $b = 1$.

type 2 — fixed points lying in the plane $\omega = 0$,

type 3 — asymptotic trajectories given by curves which intersect the plane $\omega = 0$ and from which the points of intersection with this plane are excluded,

type 4 — periodic trajectories defined by curves which do not intersect the plane $\omega = 0$.

Bifurcation diagram. Consider the plane of values of the first integrals $\mathbb{R}^2 = \{(f, h)\}$; the trajectories of the reduced system of one of the above-mentioned types correspond to those points of this plane which correspond to possible values of the integrals $\mathcal{E}$ and $\mathcal{F}$. The fixed points of the system of type 1 in this case are the critical trajectories of the system under consideration; on the plane (f, h) they correspond to the bifurcation curve σ_1 which is defined by the equation

$$h = \frac{f^2}{2(a_1 + a_2 + b)} \quad \text{in case 1} \qquad \text{or} \qquad h = \frac{f^2}{2(a_1 + b)} \quad \text{in case 2.}$$

This curve restricts below the region of possible values of the first integrals $\mathcal{E}$ and $\mathcal{F}$. The fixed points on the plane $\omega = 0$ (i.e., of type 2) are the critical trajectories of the system under consideration if the curve of intersection of the integrals $\mathcal{E} = \text{const}$ and $\mathcal{F} = \text{const}$ is tangent to them (see Figs. 4 and 5); they correspond to the bifurcation curve σ_2 given by the equation

$$h = \frac{f^2}{2(a_1 + a_2)} \text{ in case 1} \qquad \text{or} \qquad h = \frac{f^2}{2a_1} \text{ in case 2.}$$

This curve on the plane (f, h) separates the region of values of the integrals which correspond to periodic trajectories of type 3 from the region corresponding to the trajectories of type 4 (see Fig. 6).

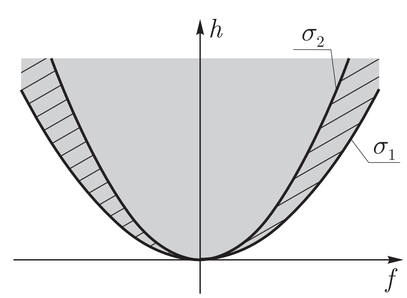


Fig. 6. Bifurcation diagram of the system in cases 1 and 2. Grey indicates the region of possible values of the first integrals. The shaded area corresponds to the periodic trajectories of the reduced system.

In the first two cases the equations of the reduced system can be easily integrated after rescaling time

$$\omega\, dt = d\varphi.$$

Case 1. The trajectory of the system (3.2) is represented as

$$\omega(\varphi) = C + A\sin(\omega_0\varphi) + B\cos(\omega_0\varphi),$$
$$u_1(\varphi) = C - \frac{Ab}{a_1+a_2}(\sin(\omega_0\varphi) + a_1\omega_0\cos(\omega_0\varphi)) - \frac{Bb}{a_1+a_2}(\cos(\omega_0\varphi) - a_2\omega_0\sin(\omega_0\varphi)),$$
$$u_2(\varphi) = C' - \frac{Ab}{a_1+a_2}(\sin(\omega_0\varphi) - a_1\omega_0\cos(\omega_0\varphi)) - \frac{Bb}{a_1+a_2}(\cos(\omega_0\varphi) + a_1\omega_0\sin(\omega_0\varphi)),$$
$$\omega_0^2 = \frac{a_1+a_2+b}{a_1a_2b}.$$

The constants of integration A, B, and C are related to the integrals $\mathcal{F}$ and $\mathcal{E}$ by

$$\mathcal{F} = (a_1+a_2+b)C, \quad 2\mathcal{E} = (a_1+a_2+b)\left(C^2 + \frac{b}{a_1+a_2}(A^2+B^2)\right).$$

The trajectory of the vehicle on the plane in this case is defined by the quadratures

$$\dot\varphi = \omega(\varphi), \quad \dot x = \xi_2u_1(\varphi)\cos\varphi + \xi_1u_2(\varphi)\sin\varphi, \quad \dot y = \xi_2u_1(\varphi)\sin\varphi - \xi_1u_2(\varphi)\cos\varphi.$$

Case 2. The trajectories of the system (3.3) are given by the relations

$$\omega(\varphi) = C + A\sin(\omega_0\varphi) + B\cos(\omega_0\varphi),$$
$$u_1(\varphi) = C - \frac{b}{a_1}(A\sin(\omega_0\varphi) + B\cos(\omega_0\varphi)),$$
$$u_2(\varphi) = \omega_0b(A\cos(\omega_0\varphi) - B\sin(\omega_0\varphi)),$$
$$\omega_0^2 = \frac{a_1+b}{a_1a_2b},$$

where the constants of integration satisfy the relations

$$\mathcal{F} = (a_1+b)C, \quad 2\mathcal{E} = (a_1+b)\left(C^2 + \frac{b}{a_1}(A^2+B^2)\right).$$

The corresponding quadratures for x, y, and φ have the form

$$\dot\varphi = \omega(\varphi), \quad \dot x = \xi_2(u_1(\varphi)\cos\varphi + u_2(\varphi)\sin\varphi), \quad \dot y = \xi_2(u_1(\varphi)\sin\varphi - u_2(\varphi)\cos\varphi).$$

Case 3 (the most simple case): the trajectories in the space of variables ω, v_1, and v_2 are ellipses formed by the intersection of the elliptic cylinder $\mathcal{E} = h = \mathrm{const}$ with the planes $\mathcal{F} = f = \mathrm{const}$. The motion along the trajectory is governed by

$$\varphi = \omega_0t + \varphi_0,$$
$$v_1 = \frac{A\sin(\kappa_1\kappa_2\varphi) + B\cos(\kappa_1\kappa_2\varphi)}{\kappa_2}, \quad v_2 = \frac{A\cos(\kappa_1\kappa_2\varphi) - B\sin(\kappa_1\kappa_2\varphi)}{\kappa_1},$$
$$\omega_0 = f, \quad A^2+B^2 = 2\kappa_1^2\kappa_2^2h, \quad \varphi_0 = \mathrm{const},$$

where $\kappa_1 = \frac{1}{\sqrt{a_1}}$, $\kappa_2 = \frac{1}{\sqrt{a_2}}$, and A and B are the constants of integration, which can be regarded as multivalued integrals of the system. Integrating the relations (2.3) for x and y, we find

$$x - x_0 = \frac{1}{2\omega_0\kappa_1\kappa_2}\left(-\frac{\kappa_1-\kappa_2}{\omega_1}(A\cos\omega_1\varphi - B\sin\omega_1\varphi) - \frac{\kappa_1+\kappa_2}{\omega_2}(A\cos\omega_2\varphi + B\sin\omega_2\varphi)\right),$$
$$y - y_0 = \frac{1}{2\omega_0\kappa_1\kappa_2}\left(-\frac{\kappa_1-\kappa_2}{\omega_1}(A\sin\omega_1\varphi + B\cos\omega_1\varphi) - \frac{\kappa_1+\kappa_2}{\omega_2}(A\sin\omega_2\varphi - B\cos\omega_2\varphi)\right),$$
$$\omega_1 = 1 + \kappa_1\kappa_2, \quad \omega_2 = 1 - \kappa_1\kappa_2.$$

As above, the quantity A^2+B^2 expressed from these equations is a single-valued periodic function in $\varphi \bmod 2\pi$; it defines the invariant surface in the configuration space (x, y, φ), which is diffeomorphic to a torus, and is given by the relation

$$\mathcal{G} = \frac{\kappa_1^2}{(1+\kappa_1^2)^2}\eta_1^2 + \frac{\kappa_2^2}{(1+\kappa_2^2)^2}\eta_2^2 = \mathrm{const},$$
$$\eta_1 = (x-x_0)\cos\varphi + (y-y_0)\sin\varphi, \quad \eta_2 = -(x-x_0)\sin\varphi + (y-y_0)\cos\varphi.$$

The projection of this surface onto the plane of variables (x, y) defines the region in which the motion of point O occurs (this region has the form of a ring).

Absolute motion. Consider a typical motion pattern of the vehicle, which corresponds to different types of trajectories of the reduced system. The motion is described by the trajectory of the origin of the moving axes $(x(t), y(t))$ and by the angle of rotation of the platform $\varphi(t)$.

1. The fixed points of type 1 correspond to the solutions of the reduced system

$$\omega = \Omega^{(0)} = \text{const}, \quad v_1 = V_1^{(0)} = \text{const}, \quad v_2 = V_2^{(0)} = \text{const}.$$

Integrating the equations for x, y, and φ, we find

$$\varphi(t) = \Omega^{(0)}t + \varphi_0,$$

$$\begin{pmatrix} x(t) - x_0 \\ y(t) - y_0 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi}{2} + \varphi(t)\right) & -\sin\left(\frac{\pi}{2} + \varphi(t)\right) \\ \sin\left(\frac{\pi}{2} + \varphi(t)\right) & \cos\left(\frac{\pi}{2} + \varphi(t)\right) \end{pmatrix} \begin{pmatrix} \frac{V_1^{(0)}}{\Omega^{(0)}} \\ \frac{V_2^{(0)}}{\Omega^{(0)}} \end{pmatrix}.$$

Thus, the vehicle uniformly rotates about the vertical axis, and the point O moves in a circle of radius $\frac{\left(V_1^{(0)}\right)^2 + \left(V_2^{(0)}\right)^2}{(\Omega^{(0)})^2}$ (the angular velocities of both motions coincide).

2. The fixed points of type 2 (i.e., on the plane $\omega = 0$) correspond to

$$\omega = 0, \quad v_1 = V_1^{(0)} = \text{const}, \quad v_2 = V_2^{(0)} = \text{const},$$

hence,

$$\varphi = \varphi_0 = \text{const}, \quad x(t) = x_0 + V_x t, \quad y(t) = y_0 + V_y t,$$
$$V_x = V_1^{(0)} \cos\varphi_0 + V_2^{(0)} \sin\varphi_0 = \text{const},$$
$$V_y = -V_1^{(0)} \sin\varphi_0 + V_2^{(0)} \cos\varphi_0 = \text{const},$$

that is, the vehicle moves in a straight line; in the case where the center of mass is located in front of the origin of coordinates defined by (3.1), these motions are stable; otherwise they are unstable.

Remark. The stability is considered with respect to variations of the initial velocities.

3. The asymptotic trajectories of type 3 correspond to motions of the vehicle where it tends to rectilinear uniform motion as $t \to \pm\infty$. During the motion the vehicle rotates through some finite angle about the vertical axis.

4. The periodic trajectories of type 4 correspond to the most complex motions of the vehicle. In this case the vehicle rotates about the vertical axis with variable angular velocity without changing the direction of rotation, and the trajectory of the origin of the moving coordinate system is a complex curve in the circular domain on the plane and consists of repetitive segments rotating about some point on the plane.

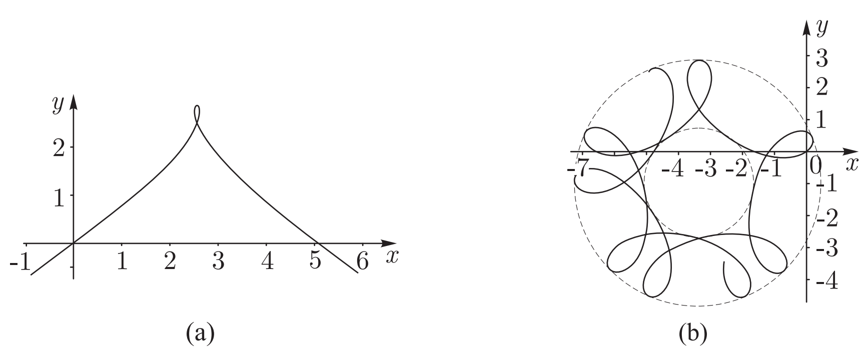


Fig. 7. A typical view of trajectories of the origin of coordinates, O, on the plane in the case of asymptotic (a) and periodic (b) motion of the reduced system ($b = 1$, $a_1 = 1.5$, $a_2 = 1.1$, $\xi_1 = 0$, $\xi_2 = 1$).

Examples. Consider three schemes of attachment of omniwheels to the platform which are found in the literature (the arrangement of wheels and the directions of the roller axes are shown in Fig. 8). In the first two vehicles, conventional omniwheels are used (the axis of the rollers lies in the plane of the wheel), in the last case Mecanum wheels are used (the axis of the rollers is attached to the rim at an angle of 45°).

In each of the cases considered we choose a coordinate system attached to the vehicle, as shown in Fig. 8. We shall assume that the coordinates of the center of mass in this system are

$$\boldsymbol{r}_c^{(0)} = (c_1, c_2).$$

For each of the schemes the wheels are assumed to be identical (the first two vehicles have conventional omniwheels, while the last vehicle has Mecanum wheels). We define the quantity

$$m_w = \frac{I_w}{s_w^2 h_w^2},$$

as the reduced mass of the wheel m_w, where I_w is the moment of inertia of the wheel relative to the axis, h_w is its radius, $s_w = 1$ for the omniwheel and $s_w = \frac{\sqrt{2}}{2}$ for the Mecanum wheel.

Using (3.1), we find the position of the origin O' of the coordinate system in which the energy of the system is diagonal, and denote the corresponding coordinates by (η_1, η_2). Calculating the position of the center of mass in the new coordinate system, we find the quantities (ξ_1, ξ_2). The corresponding expressions for these quantities are presented in Fig. 8.

4. CONTROLLED MOTION

4.1. Complete Controllability

In this section we present an analysis of the controllability of this system based on the Rashevsky – Chow theorem.

Theorem 1. *If among the vector fields $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ and among the fields composed of them by successive applications of the Lie bracket one can point out n vector fields $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n$ that are linearly independent at any point of the region $\mathcal{G}$, where $\dim \mathcal{G} = n$, then one can come from any point of the region $\mathcal{G}$ to any other point by moving a finite number of times along the trajectories of the fields $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$.*

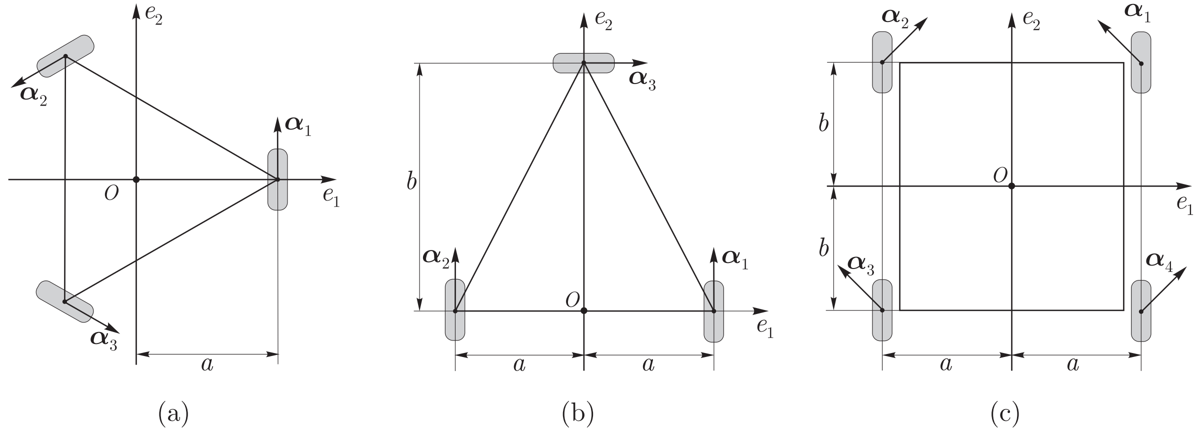
As shown in [9], if one passes to new vector fields using the linear transformation

$$\boldsymbol{X}_k = \sum_{k'=1}^{m} S_{kk'}(\boldsymbol{z}) \widetilde{\boldsymbol{X}}_{k'}, \quad k = 1, \ldots, m,$$

where $\mathbf{S}$ is an $(m \times m)$-matrix nondegenerate at each point $\boldsymbol{z} \in \mathcal{G}$, then the theorem of complete controllability remains valid also for the fields $\widetilde{\boldsymbol{X}}_1, \ldots, \widetilde{\boldsymbol{X}}_m$.

Let us write Eqs. (2.3) in a standard form used in control theory

$$\dot{\boldsymbol{z}} = \boldsymbol{X}_0(\boldsymbol{z}) + \sum_i \boldsymbol{X}_i \mu_i, \tag{4.1}$$



(a)
$$\boldsymbol{\Gamma} = \mathrm{diag}\left(\frac{3}{2}m_w, \frac{3}{2}m_w\right)$$
$$\eta_1 = \frac{2mc_1}{2m + 3m_w},\ \eta_2 = \frac{2mc_2}{2m + 3m_w}$$
$$\xi_1 = \frac{3m_w c_1}{2m + 3m_w},\ \xi_2 = \frac{3m_w c_2}{2m + 3m_w}$$

(b)
$$\boldsymbol{\Gamma} = \mathrm{diag}(m_w, 2m_w)$$
$$\eta_1 = \frac{mc_1}{m + 2m_w},\ \eta_2 = \frac{mc_2 + m_w b}{m + m_w}$$
$$\xi_1 = \frac{2m_w c_1}{m + 2m_w},\ \xi_2 = -\frac{m_w(b - c_2)}{m + m_w}$$

(c)
$$\boldsymbol{\Gamma} = \mathrm{diag}(2m_w, 2m_w)$$
$$\eta_1 = \frac{mc_1}{m + 2m_w},\ \eta_2 = \frac{mc_2}{m + 2m_w}$$
$$\xi_1 = \frac{2m_w c_1}{m + 2m_w},\ \xi_2 = \frac{2m_w c_2}{m + 2m_w}$$

Fig. 8. Possible schemes of omniwheel vehicles and the corresponding parameter values appearing in equations of motion.

where $\boldsymbol{z} = (v_1, v_2, \omega, x, y, \varphi)$,

$$\boldsymbol{X}_0 = -\mathbf{S}^{-1}\begin{pmatrix} -m\omega(v_2+\omega\xi_1) \\ m\omega(v_1-\omega\xi_2) \\ m\omega(\boldsymbol{v},\boldsymbol{r}_c) \\ v_1\cos\varphi - v_2\sin\varphi \\ v_1\sin\varphi + v_2\cos\varphi \\ \omega \end{pmatrix}, \quad \boldsymbol{X}_i = -\mathbf{S}^{-1}\begin{pmatrix} \alpha_{i_1} \\ \alpha_{i_2} \\ (\mathbf{J}\boldsymbol{r}_i, \boldsymbol{\alpha}_i) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.2}$$

$$\mathbf{S} = \begin{pmatrix} \boldsymbol{\Gamma} + m\mathbf{E} & m(\mathbf{J}\boldsymbol{r}_c + \boldsymbol{R}) & 0 & 0 & 0 \\ m(\mathbf{J}\boldsymbol{r}_c + \boldsymbol{R})^{\mathrm{T}} & \widehat{I} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here the vector field $\boldsymbol{X}_0$ corresponds to the free motion of the vehicle and $\boldsymbol{X}_i$ are the "control" vector fields. In control theory, the terms $\boldsymbol{X}_0$ in (4.1) are called *drift*. To prove complete controllability for such systems, it is necessary not only to show the completeness of the vector fields (4.2) and their commutators (according to Theorem 1), but also to prove the Poisson stability of the drift, that is, to prove that for a free system in the phase space there exists everywhere a dense set of Poisson stable points.

We first consider the question of completeness of the vector fields (4.2) and their commutators. As stated above, it suffices to consider the vector fields $\widetilde{\boldsymbol{X}}_0 = -\mathbf{S}\boldsymbol{X}_0$, $\widetilde{\boldsymbol{X}}_i = -\mathbf{S}\boldsymbol{X}_i$ and their commutators

$$\boldsymbol{Y}_i = \big[\widetilde{\boldsymbol{X}}_0, \widetilde{\boldsymbol{X}}_i\big], \quad \boldsymbol{Y}_{ij} = \big[\widetilde{\boldsymbol{X}}_i, \widetilde{\boldsymbol{X}}_j\big] = 0.$$

It is straightforward to show that if the rank of the linear span of the vector fields $\widetilde{\boldsymbol{X}}_i$ is equal to three, then the vector fields $\boldsymbol{X}_i$ and $\boldsymbol{Y}_i$ form a complete basis in the entire phase space. The above condition for independence of the vector fields $\widetilde{\boldsymbol{X}}_i$ leads to some restrictions on the structure of the system. Their physical meaning will be discussed below.

We now consider the Poisson stability of the free system. As shown in [11], all motions of the reduced system (for the variables v_1, v_2, ω) are broken down into periodic and asymptotic motions. Obviously, due to the existence of asymptotic trajectories the free system is not Poisson stable. One can only point out regions in the phase space in which the condition of Poisson stability is satisfied. These regions coincide with the regions filled with periodic trajectories of the reduced system, and in the space (v_1, v_2, ω) they form the interiors of two elliptic cones.

4.2. Explicit Control Along a Trajectory

Although the complete controllability of the system (2.3) has not been proved so far, one can present an explicit algorithm for developing a control for the motion along a predetermined trajectory.

Consider the control of the vehicle by choosing control torques μ_i, $i = 1, \ldots, n$, in such a way that the vehicle moves in a prescribed manner $\gamma(t) = (x(t), y(t), \varphi(t))$. According to Eqs. (2.3), the following proposition turns out to be valid:

Proposition 1. *If the vehicle rests on no less than three driving omniwheels for which the inequality*

$$\det\begin{pmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ (\mathbf{J}\boldsymbol{r}_1, \boldsymbol{\alpha}_1) & (\mathbf{J}\boldsymbol{r}_2, \boldsymbol{\alpha}_2) & (\mathbf{J}\boldsymbol{r}_3, \boldsymbol{\alpha}_3) \end{pmatrix} \neq 0, \tag{4.3}$$

is satisfied, then one can always choose control torques $\mu_i(t)$, $i = 1, \ldots, n$, *in such a way that the prescribed motion* $\gamma(t)$ *is realized.*

To prove the proposition, we rewrite the equations of the reduced system (for v_1, v_2, and ω) as

$$\mathbf{S}\dot{\boldsymbol{z}}^{(3)} = \widetilde{\boldsymbol{X}}_0^{(3)}(\boldsymbol{z}^{(3)}) + \mathbf{G}\boldsymbol{\mu}, \tag{4.4}$$

where $\boldsymbol{z}^{(3)} = (v_1, v_2, \omega)$, $\widetilde{\boldsymbol{X}}_0^{(3)} = \big(m\omega(v_2 + \omega\xi_1), -m\omega(v_1 - \omega\xi_2), m\omega(\boldsymbol{v}, \boldsymbol{r}_c)\big)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)$ is a vector composed of control torques, and $\mathbf{G}$ is a constant matrix depending on the design of the vehicle and having the form

$$\mathbf{G}_N = \begin{pmatrix} \alpha_{11} & \ldots & \alpha_{N1} \\ \alpha_{12} & \ldots & \alpha_{N2} \\ (\mathbf{J}\boldsymbol{r}_1, \boldsymbol{\alpha}_1) & \ldots & (\mathbf{J}\boldsymbol{r}_N, \boldsymbol{\alpha}_N) \end{pmatrix}. \tag{4.5}$$

The system (4.4) can be solved for μ_i, and thus we can obtain the control law $\mu(t)$ from the known dependencies $\boldsymbol{z}^{(3)}(t)$. For the system (4.4) to be solvable, we need to impose restrictions on the structure of the vehicle such that the rank of the matrix (4.5) is equal to three. This requires that the condition (4.3) be satisfied. □

The condition (4.3) implies, in particular, that the controllability of the system does not admit parallelism of all vectors α_i. Moreover, if the condition $\alpha_i \parallel \alpha_j \parallel \boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$ holds for two wheels, this pair of wheels can be regarded as one wheel, since the corresponding columns in the matrix (4.5) will be equal, and the corresponding control torques will appear in (4.4) only as the sums $\mu_i + \mu_j$.

We present a general algorithm for calculating the control torques for the motion along the predetermined trajectory $\boldsymbol{\gamma}(t) = (x(t), y(t), \varphi(t))$.

1. Calculate the time dependence of v_1, v_2, ω, and their derivatives from the formulae

$$v_1 = \dot{x}\cos\varphi + \dot{y}\sin\varphi, \quad v_2 = -\dot{x}\sin\varphi + \dot{y}\cos\varphi, \quad \omega = \dot{\varphi}.$$

2. Solve the system (4.4) for any of the three control torques for which the condition (4.3) is satisfied. As a result, we obtain the dependence of these three torques on time and the remaining $N-3$ torques.

3. The remaining torques can have an arbitrary time dependence. This arbitrariness can be used, in particular, for control optimization (for example, for minimization of power inputs for the motion).

Further, we present specific examples for the control of motion along the simplest trajectories for the vehicle design shown in Fig. 8c. The vectors $\boldsymbol{r}_i$, $\boldsymbol{\alpha}_i$ and the quantities s_i are expressed as follows:

$$\boldsymbol{r}_1 = -\boldsymbol{r}_3 = (a, b), \boldsymbol{r}_2 = -\boldsymbol{r}_4 = (-a, b),$$
$$\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_3 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_4 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad s_{1,2,3,4} = \frac{1}{\sqrt{2}}.$$

Figure 9 shows an experimental model of such a vehicle developed at the Laboratory of Nonlinear Analysis and Design of New Types of Vehicles of the Udmurt State University. The numerical values of its geometric and mass characteristics are: $m = 3\,581$ g, $a = 24.44$ cm, $b = 15.22$ cm, $h_{1,2,3,4} = 4.35$ cm, $I = 78\,346.51$ gcm^2, $I_i = 1\,196.07$ gcm^2, $\boldsymbol{r}_c = (0, 0)$. Below we present the results of numerical calculations for the above values of the characteristics of the vehicle. We use arbitrariness in the choice of one of the control torques to minimize the general control torque $\sum \mu_i^2$. In the case at hand this condition reduces to the equation $\mu_1 + \mu_2 - \mu_3 - \mu_4 = 0$.

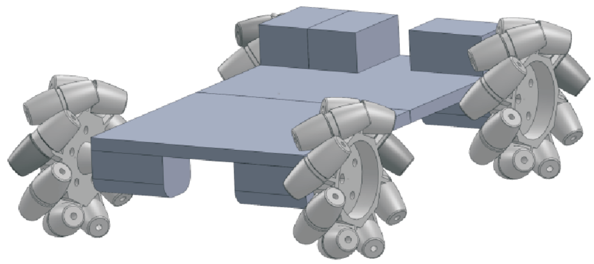


Fig. 9. 3D-model of a Mecanum wheel vehicle.

Example 1. Motion in a straight line over distance l_0 with rotation through an angle ψ_0:

$$x(t) = 0, \quad y(t) = l_0 \sin^2 \frac{\pi}{2} t, \quad \varphi(t) = \psi_0 \sin^2 \frac{\pi}{2} t, \quad t = [0, 1]. \tag{4.6}$$

The time dependences of the control torques for $\psi_0 = 2\pi$, $l_0 = 1$ m are shown in Fig. 10.

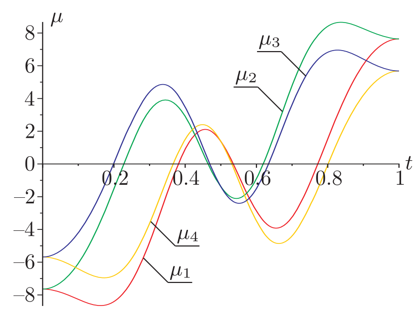


Fig. 10. Time dependence of the control torques for motion along the trajectory (4.6).

Example 2. Motion along one period of the sine curve (the orientation of the vehicle coincides with the direction of instantaneous velocity):

$$\begin{gathered} x(t) = l_0 s, \quad y(t) = R \sin(2\pi s), \\ \varphi(t) = \operatorname{arctg}\left(2\pi \frac{R}{l_0} \cos(2\pi s)\right), \quad s = \sin^2 \frac{\pi}{2} t, \quad t = [0, 1], \end{gathered} \tag{4.7}$$

where R is the amplitude of the sine curve and l_0 is the length of one period.

An example of dependence of the control torques for the motion along a sine curve for $R = 0.1$, $l_0 = 1$ is given in Fig. 11.

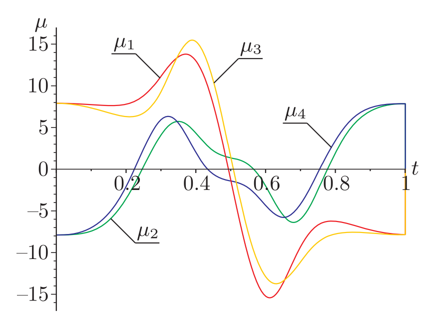


Fig. 11. Time dependence of the control torques for motion along the trajectory (4.7).

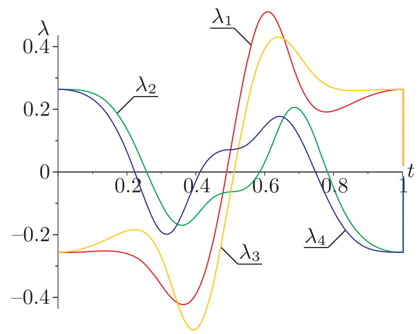


Fig. 12. Time dependence of the reactions λ_i for motion along the trajectory (4.7).

In conclusion, in Fig. 12 we present the time dependence of the reactions λ_i (2.2) for motion of the vehicle along the sine curve (example 2). As seen in the figure, the value of the reactions is an order smaller than the value of the control torques and is very irregular. It would be interesting to compare the dependencies obtained with results of calculations, for example, within the framework of the dry friction model.

5. AN OMNIWHEEL VEHICLE ON THE SURFACE OF A SPHERE

Suppose an omniwheel vehicle moves on the surface of a fixed sphere (in a gravitational field, see Fig. 13).

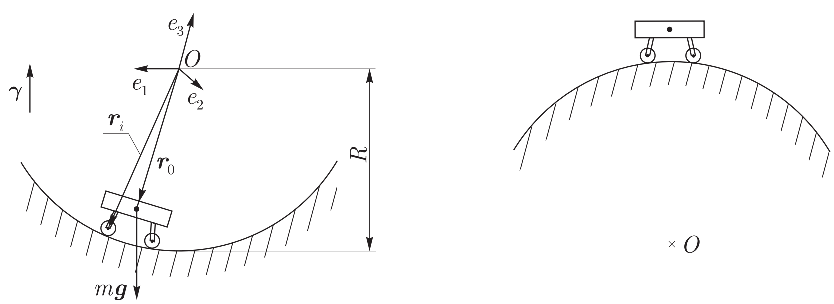


Fig. 13

In this case it is convenient to regard the vehicle as a rigid body with a fixed point, which is chosen to be the center of the sphere. We choose a moving coordinate system $Oe_1e_2e_3$ with origin at the center of the sphere and with axes coinciding with the principal axes of inertia of the system. Let $\boldsymbol{\omega}$ denote the angular velocity of the vehicle and $\boldsymbol{\omega}_i$ the total angular velocity of each wheel; if the angle of rotation of the wheel about the vehicle is ψ_i, the following relations hold:

$$\boldsymbol{\omega}_i = \boldsymbol{\omega} + \dot{\psi}_i \boldsymbol{n}_i, \quad i = 1, \ldots, n,$$

where $\boldsymbol{n}_i$ is the unit vector directed along the axis of the wheel. For the velocity of the contact point Q_i of the wheel with the sphere we have

$$\boldsymbol{V}_{Q_i} = \boldsymbol{\omega} \times \boldsymbol{r}_i + \boldsymbol{\omega}_i \times \frac{h_i}{R - h_i}\boldsymbol{r}_i, \quad i = 1, \ldots, n,$$

where R and h_i are the radii of the sphere and the wheel, respectively. Using this relation, we represent the constraint equations (1.1) as

$$f_i = \dot{\psi}_i + \frac{1}{s_i h_i}(\boldsymbol{\omega}, \boldsymbol{r}_i \times \boldsymbol{\alpha}_i) = 0, \quad s_i = \frac{(\boldsymbol{n}_i \times \boldsymbol{r}_i, \boldsymbol{\alpha}_i)}{R},$$

where $\boldsymbol{\alpha}_i$ is the unit vector along the roller axis at the point of contact. In the chosen coordinate system the vectors $\boldsymbol{\alpha}_i$, $\boldsymbol{n}_i$, and $\boldsymbol{r}_i$ are constant.

We shall assume that for each wheel the axis of rotation passes through the center of mass of the wheel and is its principal axis of inertia. Then the kinetic energy of the system can be represented as

$$\begin{aligned} T &= \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}_0\boldsymbol{\omega}) + \frac{1}{2}\sum_i \big(m_i(\boldsymbol{\omega}\times\boldsymbol{r}_i)^2 + (\boldsymbol{\omega}_i, \mathbf{I}_i\boldsymbol{\omega}_i)\big) \\ &= \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) + \sum J_i\dot{\psi}_i(\boldsymbol{\omega}, \boldsymbol{n}_i) + \frac{1}{2}\sum_i J_i\dot{\psi}_i^2, \\ \mathbf{I} &= \mathbf{I}_0 + \sum_i m_i(\boldsymbol{r}_i^2\mathbf{E} - \boldsymbol{r}_i\otimes\boldsymbol{r}_i) + \mathbf{I}_i, \quad J_i = (\boldsymbol{n}_i, \mathbf{I}_i\boldsymbol{n}_i), \end{aligned} \tag{5.1}$$

where $\mathbf{I}_0$ is the tensor of inertia of the framework relative to the center of the sphere, $\mathbf{I}_i$ is the tensor of inertia of the wheel relative to its center of mass, and $\mathbf{I}$ is the tensor of inertia of the entire vehicle relative to the center of the sphere. Since the axis of rotation of the wheel coincides with its principal axis of inertia, the relation $\mathbf{I}_i\boldsymbol{n}_i = J_i\boldsymbol{n}_i$ holds.

In this case the equations for quasi-velocities referred to the moving coordinate axes have the form

$$\begin{gathered} \left(\frac{\partial T}{\partial\boldsymbol{\omega}}\right)^{\cdot} + \boldsymbol{\omega}\times\frac{\partial T}{\partial\boldsymbol{\omega}} = \sum_i \lambda_i\frac{\partial f_i}{\partial\boldsymbol{\omega}} - mg\boldsymbol{r}_0\times\boldsymbol{\gamma}, \\ \left(\frac{\partial T}{\partial\dot{\psi}_k}\right)^{\cdot} = \sum_i \lambda_i\frac{\partial f_i}{\partial\dot{\psi}_k} + M_k = \lambda_k + M_k, \quad k = 1, \ldots, n, \end{gathered}$$

where $\boldsymbol{r}_0$ is the vector from the center of the sphere to the center of mass of the vehicle and $\boldsymbol{\gamma}$ is the unit vector of the vertical. Eliminating the undetermined multipliers λ_i using the second equation and the constraint (5.1), we find

$$\begin{gathered} \mathbf{I}\dot{\boldsymbol{\omega}} + \sum_i \left[\frac{J_i}{s_i^2h_i^2}(\dot{\boldsymbol{\omega}}, \boldsymbol{\rho}_i)\boldsymbol{\rho}_i - \frac{J_i}{s_ih_i}\Big((\dot{\boldsymbol{\omega}}, \boldsymbol{\rho}_i)\boldsymbol{n}_i + (\dot{\boldsymbol{\omega}}, \boldsymbol{n}_i)\boldsymbol{\rho}_i\Big)\right] \\ + \boldsymbol{\omega}\times\left(\mathbf{I}\boldsymbol{\omega} - \sum_i \frac{J_i}{s_ih_i}(\boldsymbol{\omega}, \boldsymbol{\rho}_i)\boldsymbol{n}_i\right) = -\sum \mu_i\boldsymbol{\rho}_i - mg\boldsymbol{r}_0\times\boldsymbol{\gamma}, \\ \boldsymbol{\rho}_i = \boldsymbol{r}_i\times\boldsymbol{\alpha}_i, \end{gathered}$$

where $\mu_i = \frac{M_i}{s_ih_i}$. Adding the kinematic relation governing the evolution of the fixed unit vector $\boldsymbol{\gamma}$ in the moving axes, we finally rewrite the equation of motion as

$$\begin{gathered} \widehat{\mathbf{I}}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\widehat{\mathbf{J}}\boldsymbol{\omega} = -\sum_i \mu_i\boldsymbol{\rho}_i - mg\boldsymbol{r}_0\times\boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times\boldsymbol{\omega}, \\ \widehat{\mathbf{I}} = \mathbf{I} + \sum_i \left(\frac{J_i}{s_i^2h_i^2}\boldsymbol{\rho}_i\otimes\boldsymbol{\rho}_i - \frac{J_i}{s_ih_i}\big(\boldsymbol{\rho}_i\otimes\boldsymbol{n}_i + \boldsymbol{n}_i\otimes\boldsymbol{\rho}_i\big)\right), \\ \widehat{\mathbf{J}} = \mathbf{I} - \sum_i \frac{J_i}{s_ih_i}\boldsymbol{n}_i\otimes\boldsymbol{\rho}_i. \end{gathered} \tag{5.2}$$

If the matrices $\widehat{\mathbf{I}}$ and $\widehat{\mathbf{J}}$ are diagonalized simultaneously (i.e., $\big[\widehat{\mathbf{I}}, \widehat{\mathbf{J}}\big] = 0$) and μ_i are independent of the angular velocity $\boldsymbol{\omega}$, then Eqs. (5.2) admit the standard invariant measure $d\boldsymbol{\omega}\,d\boldsymbol{\gamma}$.

Without the control torques ($\mu_i = 0, i = 1, \ldots, n$) Eqs. (5.2) are analogous to the Euler–Poisson equations in rigid body dynamics:

$$\widehat{\mathbf{I}}\dot{\boldsymbol{\omega}} = (\widehat{\mathbf{J}}\boldsymbol{\omega})\times\boldsymbol{\omega} - mg\boldsymbol{r}_0\times\boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times\boldsymbol{\omega}. \tag{5.3}$$

The system (5.2) can be regarded as a nonholonomic analog of the Euler – Poisson equations. It admits two additional integrals

$$F_0 = \boldsymbol{\gamma}^2 = 1, \quad E = \frac{1}{2}(\widehat{\mathbf{I}}\boldsymbol{\omega}, \boldsymbol{\omega})$$

(the integral E coincides with the kinetic energy of the system calculated with the constraints taken into account). When the gravitational field is zero ($g = 0$), one has the third additional integral

$$F_1 = (\widehat{\mathbf{I}}\boldsymbol{\omega}, \widehat{\mathbf{J}}\boldsymbol{\omega}).$$

Remark 1. In Eqs. (5.3) the matrix $\widehat{\mathbf{I}}$ is symmetric and hence can always be diagonalized by an orthogonal transformation. After such a transformation the vector $\boldsymbol{r}_0$ in (5.3) has in general three nonzero components.

We note that in nonholonomic mechanics there may exist specific restrictions on the system's parameters when an invariant measure exists (see, e.g., [16, 24, 41]). Therefore, the problem of existence (in the general case) of an invariant measure and two additional integrals in the system (5.2), which are necessary for integrability by the Euler – Jacobi theorem, remains open. At the same time it should be noted that in nonholonomic systems a hierarchy of possible types of dynamical behavior may arise, depending on combinations of tensor invariants (see, e.g., [4, 7, 16]). Moreover, the emergence of strange attractors [8, 25, 27] is possible.

We consider in more detail the case in which there is no gravitational field, $\widehat{\mathbf{J}}$ and $\widehat{\mathbf{I}}$ are diagonal matrices and, moreover, their product $\widehat{\mathbf{J}}\widehat{\mathbf{I}}$ is a positive definite matrix (in this case there exists a standard invariant measure and the integrals F_0, F_1, and E). Then, if we define the momentum $\boldsymbol{M} = (\widehat{\mathbf{J}}\widehat{\mathbf{I}})^{\frac{1}{2}}\boldsymbol{\omega}$, then the equations of motion take the form

$$\begin{gathered}\dot{\boldsymbol{M}} = \boldsymbol{M} \times \mathbf{A}\boldsymbol{M}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \mathbf{B}\boldsymbol{M}, \\ \mathbf{A} = \frac{\det \widehat{\mathbf{J}}^{\frac{1}{2}}}{\det \widehat{\mathbf{I}}^{\frac{1}{2}}}\mathbf{J}^{-1}, \quad \mathbf{B} = (\widehat{\mathbf{J}}\widehat{\mathbf{I}})^{-\frac{1}{2}}.\end{gathered} \tag{5.4}$$

To abbreviate some of our forthcoming formulae, we introduce the notation $\mathbf{A} = \mathrm{diag}(a_1, a_2, a_3)$ $\mathbf{B} = \mathrm{diag}(b_1, b_2, b_3)$. The resulting system (5.4) also describes the limiting case of the Poincaré – Zhukovskii equations (see § 3, Chapter 3 in [13]).

In [17, 18], A. V. Borisov and A. V. Tsygvintsev obtained a necessary condition for existence of an additional algebraic integral by using the Kovalevskaya – Painlevé method

$$k^2 a_{23}a_{21}a_{13} - a_{32}b_1^2 - a_{13}b_2^2 - a_{21}b_3^2 = 0, \quad a_{ij} = a_i - a_j, \tag{5.5}$$

where $k = \frac{p}{q}$, p, $q \in \mathbb{Z}$. If k is odd and positive, the additional integral F_2 is linear in γ, has degree k in $\mathbf{M}$ and can be written in the form [17, 18]

$$F_2 = (\boldsymbol{\gamma}, \boldsymbol{S}(\boldsymbol{M})),$$

where the vector $\boldsymbol{S}(\boldsymbol{M})$ can be represented as

$$\boldsymbol{S}(\boldsymbol{M}) = \mathrm{diag}(M_1, M_2, M_3)\Phi_k(\boldsymbol{M})\boldsymbol{T}, \quad \Phi_k = (\mathbf{C}_1^{-1}\mathbf{K})(\mathbf{C}_3^{-1}\mathbf{K})\ldots(\mathbf{C}_{k-2}^{-1}\mathbf{K}),$$

here $\mathbf{K}$ and $\mathbf{C}_n$ are the matrices

$$\mathbf{K} = \mathrm{diag}(M_1^2, M_2^2, M_3^2), \quad \mathbf{C}_n = \begin{pmatrix} -na_{32} & b_3 & -b_2 \\ -b_3 & -na_{13} & b_1 \\ b_2 & -b_1 & -na_{21} \end{pmatrix},$$

and the vector $\boldsymbol{T}$ is the kernel of the matrix $\mathbf{C}_k$. Under the condition $\mathbf{B} = k\mathbf{A}$, for which the condition (5.5) holds identically, the equations of motion (5.4) were integrated in terms of elliptic sigma functions in [22]. The problem of Hamiltonization of (5.4) remains unresolved.

If we set $b_1 = 0$ and $b_2 = 0$ in (5.4), then γ_3 remains constant and, moreover, we obtain the integral

$$F_3 = \gamma_1 \sin\varphi + \gamma_2 \cos\varphi, \quad \varphi = \frac{b_3}{\sqrt{a_{13}a_{32}}}\ln(\sqrt{a_{13}}M_1 + \sqrt{a_{32}}M_2).$$

We note that nonalgebraic integrals also arise in the Suslov problem with the inhomogeneous constraint [23] (whose practical realization is unknown).

Remark. If the matrices $\widehat{\mathbf{I}}$ and $\widehat{\mathbf{J}}$ are diagonalized simultaneously and, moreover, a pair of eigenvalues of each of the matrices $\widehat{\mathbf{I}}$ and $\widehat{\mathbf{J}}$ coincides, then the system (5.3) can be integrated by quadratures.

Example. Consider in more detail a vehicle on a sphere for which a wheel attachment scheme is shown in Fig. 8b. If all wheels have the same radius h, then by choosing the axes $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ according to Fig. 8b we obtain

$$\mathbf{I} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & I_{23} \\ 0 & I_{23} & I_{33} \end{pmatrix},$$

$$\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = (0, 1, 0), \quad \boldsymbol{\alpha}_3 = (1, 0, 0),$$

$$\boldsymbol{r}_1 = (a, 0, \sqrt{R_w^2 - a^2}), \quad \boldsymbol{r}_2 = (-a, 0, \sqrt{R_w^2 - a^2}), \quad \boldsymbol{r}_3 = (0, b, \sqrt{R_w^2 - b^2}),$$

$$R_w = R - h.$$

In this case, $\boldsymbol{\rho}_i = \boldsymbol{r}_i \times \boldsymbol{\alpha}_i = R_w \boldsymbol{n}_i$, $i = 1, 2, 3$, and for the matrices $\widehat{\mathbf{I}}$ and $\widehat{\mathbf{J}}$ we obtain

$$\widehat{\mathbf{I}} = \mathbf{I} + m_w R_w^2 \left(1 - \frac{2h}{R_w}\right) \begin{pmatrix} 2\left(1 - \frac{a^2}{R_w^2}\right) & 0 & 0 \\ 0 & 1 - \frac{b^2}{R_w^2} & -\frac{b}{R_w}\sqrt{1 - \frac{b^2}{R_w^2}} \\ 0 & -\frac{b}{R_w}\sqrt{1 - \frac{b^2}{R_w^2}} & \frac{2a^2 + b^2}{R_w^2} \end{pmatrix},$$

$$\widehat{\mathbf{J}} = \mathbf{I} - m_w h R_w \begin{pmatrix} 2\left(1 - \frac{a^2}{R_w^2}\right) & 0 & 0 \\ 0 & 1 - \frac{b^2}{R_w^2} & -\frac{b}{R_w}\sqrt{1 - \frac{b^2}{R_w^2}} \\ 0 & -\frac{b}{R_w}\sqrt{1 - \frac{b^2}{R_w^2}} & -\frac{2a^2 + b^2}{R_w^2} \end{pmatrix}.$$

The matrices $\widehat{\mathbf{J}}$ and $\widehat{\mathbf{I}}$ commute under the condition

$$2h = R,$$

i.e., Eqs. (5.2) admit in this case an invariant measure with constant density.

Remark. This relation coincides in form with the condition for existence of an integral in the problems of a ball rolling on a sphere [17, 18]. A topological analysis of the reduced system in the problem of a ball rolling without slipping on a sphere is presented in [15], while absolute dynamics remain unexplored (for example, analysis of the complete system for a ball on a plane is carried out in [12, 40]).

ACKNOWLEDGMENTS

The research concerned with controllability analysis was supported by the grant of the Russian Science Foundation (project no. 14-19-01303). The investigation of free dynamics was carried out within the framework of the state assignment for institutions of higher education.

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