

# A Finite Set of Generators for the Arc Algebra

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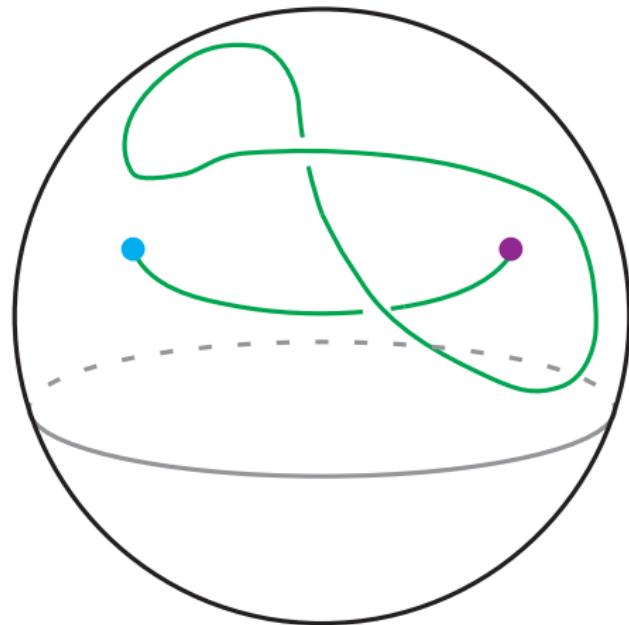
Cornell University

10 January 2015

# The Arc Algebra of a Surface

- ▶ defined in 2011 by J. Roger and T. Yang and has important connections to both quantum topology and hyperbolic geometry
- ▶ generalization of the Jones polynomial (specifically the Kauffman bracket skein algebra)
- ▶ applies to thickened surfaces with punctures

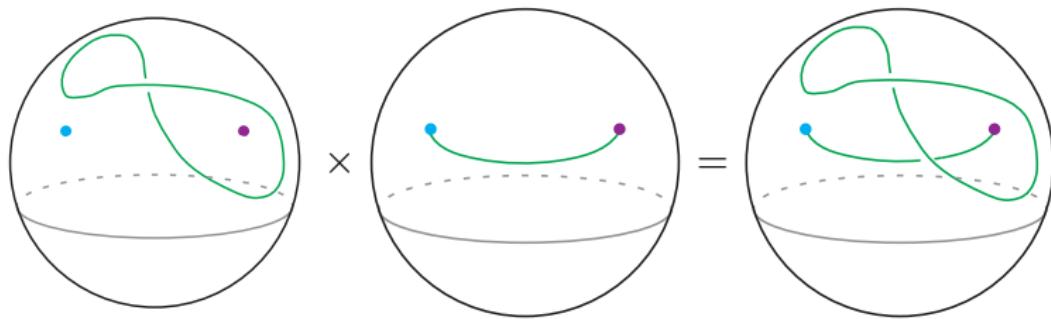
# The Arc Algebra - An Element



## The Arc Algebra - Definition

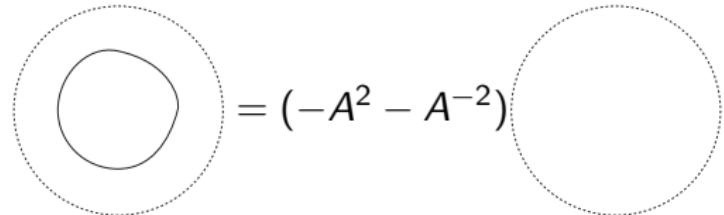
- ▶ Let  $F_{g,n}$  denote the surface of genus  $g$  with  $n$  punctures.
- ▶ Let  $R_n$  be the ring  $\mathbb{Z}[A^{\pm \frac{1}{2}}][v_1, \dots, v_n]$ .
- ▶ The arc algebra  $\mathcal{A}(F_{g,n})$  consists of formal linear combinations of framed curves (unions of knots and arcs) that lie in the thickened surface  $F_{g,n} \times [0, 1]$  subject to four relations.
- ▶ Multiplication is by stacking, induced by  $F_{g,n} \times [0, 1] = F_{g,n} \times [0, \frac{1}{2}] \cup F_{g,n} \times [\frac{1}{2}, 1]$ .

# The Arc Algebra - Multiplication

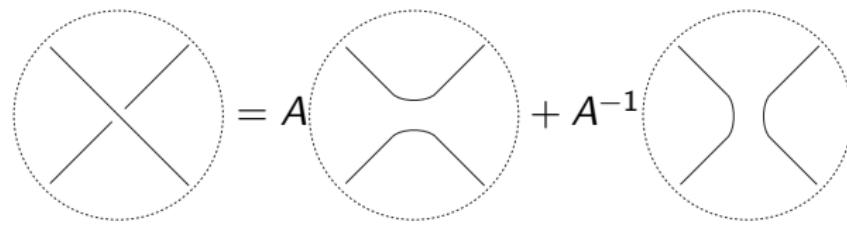


# The Arc Algebra - Kauffman Bracket Relations

Two of the four relations are the same as the skein algebra:



A diagram showing a circle with a single crossing. The left side of the equation has this circle with a solid line inside. The right side of the equation is  $= (-A^2 - A^{-2})$ . The right side consists of two separate circles, each with a dotted line inside.



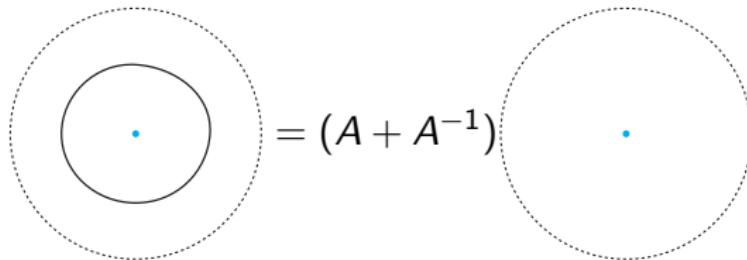
A diagram showing three circles with crossings. The left side of the equation has a circle with two crossing lines forming an 'X'. The middle term is  $= A$  followed by a circle with two crossing lines forming a 'W' shape. The right term is  $+ A^{-1}$  followed by a circle with two crossing lines forming a 'V' shape.

# The Arc Algebra - Puncture Relations

There are two more relations for punctured surfaces:

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$$= (A + A^{-1})$$

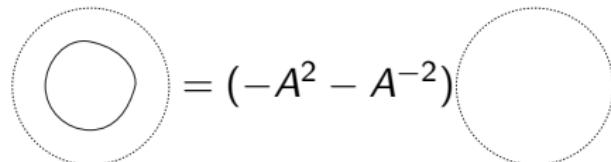
# The Arc Algebra - Puncture Relations

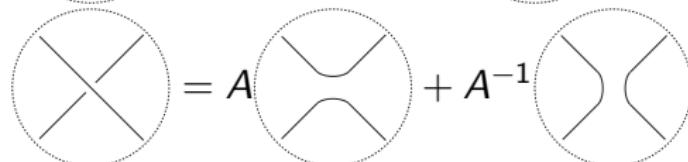
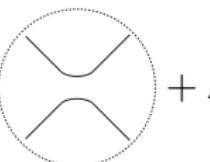
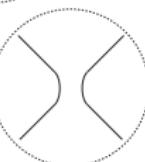
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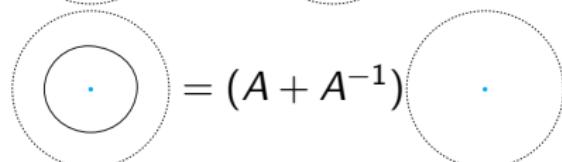
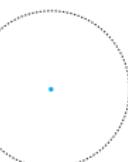
$$\begin{array}{ccc} \text{Diagram: Two concentric circles with a dot at the center of the inner circle.} & = & (A + A^{-1}) \\ & & \text{Diagram: Two concentric circles with a dot at the center of the outer circle.} \end{array}$$

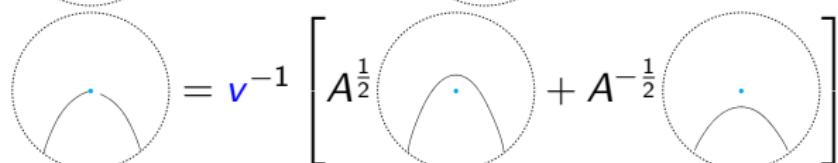
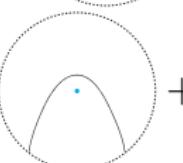
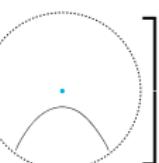
$$\begin{array}{ccc} \text{Diagram: Two concentric circles with a blue dot at the top-left point of the inner circle's boundary.} & = & v^{-1} \left[ A^{\frac{1}{2}} \right. \\ & & \text{Diagram: Two concentric circles with a blue dot at the top point of the inner circle's boundary.} \\ & & \left. + A^{-\frac{1}{2}} \right] \\ & & \text{Diagram: Two concentric circles with a blue dot at the top-right point of the inner circle's boundary.} \end{array}$$

# The Arc Algebra - All Relations


$$= (-A^2 - A^{-2})$$


$$= A$$
   $+ A^{-1}$  

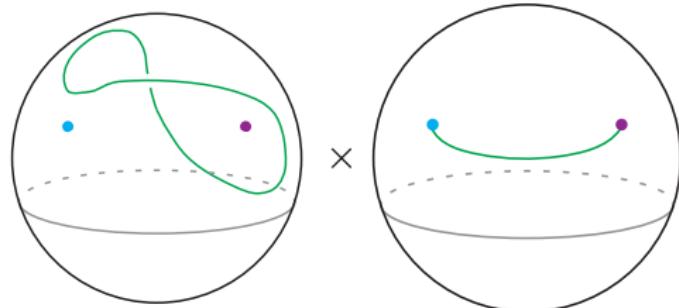

$$= (A + A^{-1})$$
 


$$= \nu^{-1} \left[ A^{\frac{1}{2}} \right.$$
   $+ A^{-\frac{1}{2}}$    $\left. \right]$

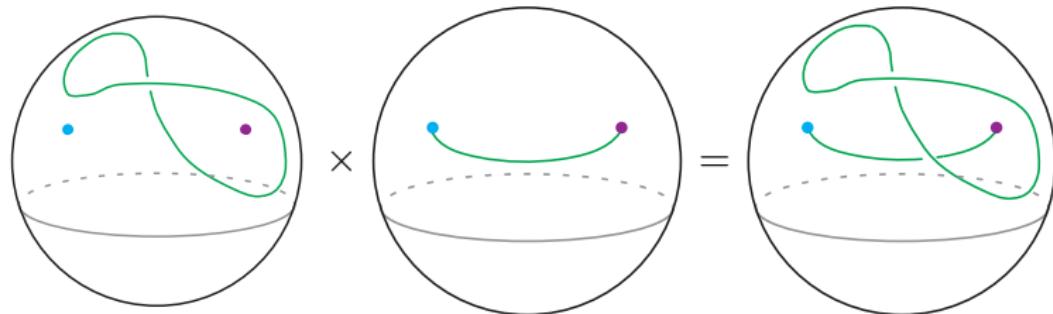
# Questions about the Arc Algebra

1. Given a surface, can we find a set of arcs and links that generate the arc algebra?
2. Given a surface, can we find a complete presentation (generators and relations) for the arc algebra?
3. Is the arc algebra always finitely generated?

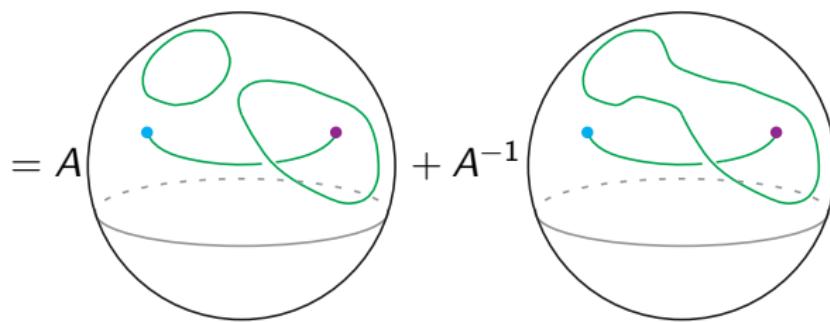
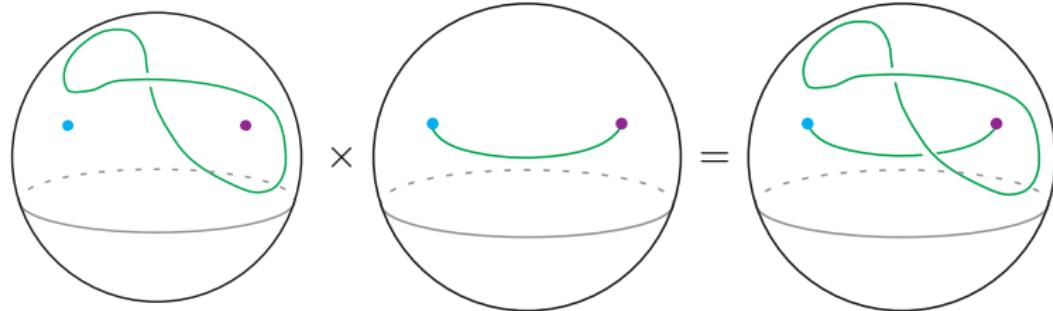
## Example on the Twice-Punctured Sphere



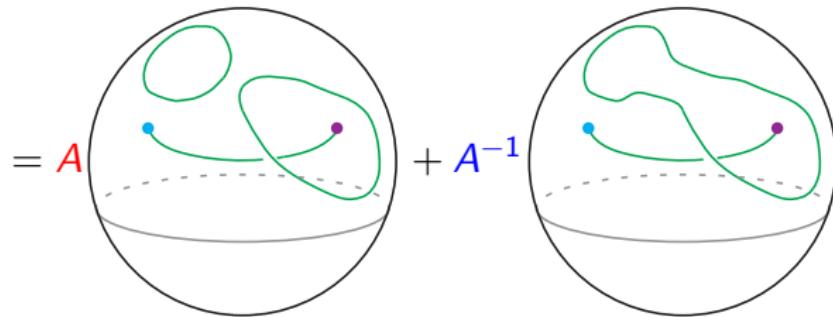
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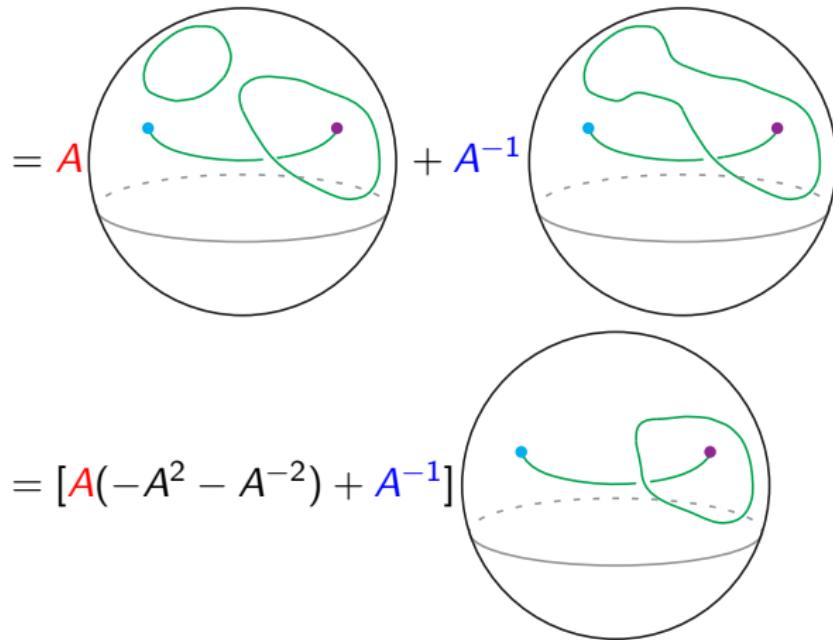
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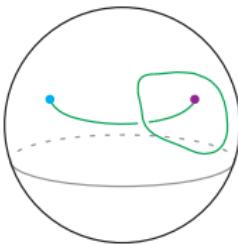


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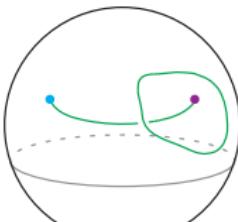
## Example on the Twice-Punctured Sphere

$$= [A(-A^2 - A^{-2}) + A^{-1}]$$

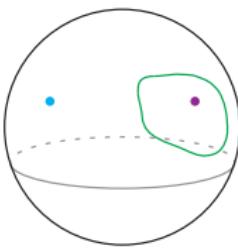
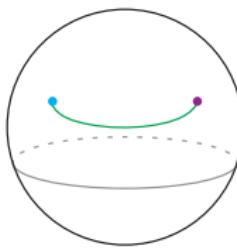


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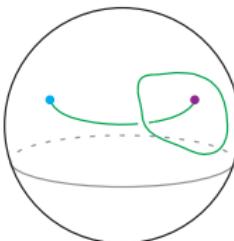


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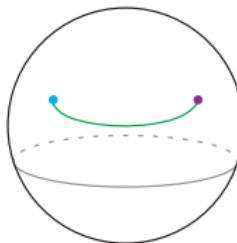
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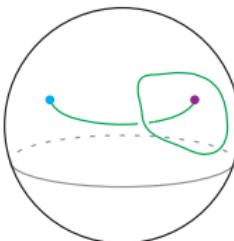


$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$

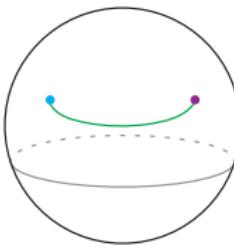


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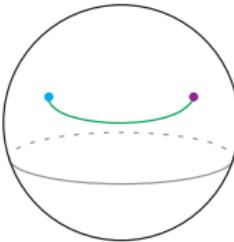
$$= [A(-A^2 - A^{-2}) + A^{-1}]$$



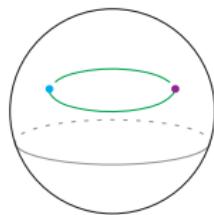
$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$



$$= (-A^4 - A^2)$$



## Another Example on the Twice-Punctured Sphere



## Another Example on the Twice-Punctured Sphere

$$\text{Diagram of a twice-punctured sphere with two green curves.} = \nu^{-1} \left[ A^{1/2} \text{Diagram of a twice-punctured sphere with one green curve and one blue dot} + A^{-1/2} \text{Diagram of a twice-punctured sphere with one green curve and one purple dot} \right]$$

## Another Example on the Twice-Punctured Sphere

$$\text{Diagram of a twice-punctured sphere with two green curves.}$$
$$= \textcolor{blue}{v}^{-1} \left[ A^{1/2} \text{Diagram with one curve and punctures} + A^{-1/2} \text{Diagram with two curves and punctures} \right] =$$
$$(\textcolor{blue}{v}\textcolor{purple}{v})^{-1} \left[ A \text{Diagram with one curve and puncture} + \text{Diagram with one curve and puncture} + \text{Diagram with two curves and punctures} + A^{-1} \text{Diagram with one curve and puncture} \right]$$

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$$\begin{aligned} & \text{Diagram of a twice-punctured sphere with two green curves connecting the punctures.} \\ & = \textcolor{blue}{v}^{-1} \left[ A^{1/2} \text{Diagram with one green curve and one puncture dot} + A^{-1/2} \text{Diagram with one green curve and one puncture dot} \right] = \\ & \quad \left( \textcolor{blue}{v} \textcolor{purple}{v} \right)^{-1} \left[ A \text{Diagram with one puncture dot and one green oval} + \text{Diagram with one puncture dot and one green oval} + \text{Diagram with one puncture dot and one green oval} + A^{-1} \text{Diagram with one puncture dot and one green oval} \right] \\ & = (\textcolor{blue}{v} \textcolor{purple}{v})^{-1} [A(A+A^{-1}) + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-1}(A+A^{-1})] \end{aligned}$$

## Another Example on the Twice-Punctured Sphere

$$\begin{aligned} & \text{Diagram of a twice-punctured sphere with two green curves intersecting at a point on the equator.} \\ & = \textcolor{blue}{v}^{-1} \left[ A^{1/2} \text{Diagram with one green curve and punctures at opposite ends of the curve} + A^{-1/2} \text{Diagram with one green curve and punctures at the endpoints} \right] = \\ & \quad \text{Diagram of a twice-punctured sphere with one green curve and one puncture.} \\ & \quad + \text{Diagram of a twice-punctured sphere with one green curve and one puncture.} \\ & \quad + \text{Diagram of a twice-punctured sphere with one green curve and one puncture.} \\ & \quad + A^{-1} \text{Diagram of a twice-punctured sphere with one green curve and one puncture.} \\ & = (\textcolor{blue}{v}\textcolor{violet}{v})^{-1} [A(A+A^{-1}) + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-1}(A+A^{-1})] \\ & = (\textcolor{blue}{v}\textcolor{violet}{v})^{-1} (-A^2 + 2 - A^{-2}) \end{aligned}$$

# Algebra of the Twice-Punctured Sphere

1. Consider any diagram on a twice-punctured sphere.

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3. If two arcs meet at a puncture, we can remove them from the puncture with a relation.
4. We can remove all unknots (around punctures and not) with a relation and put them in the coefficients.

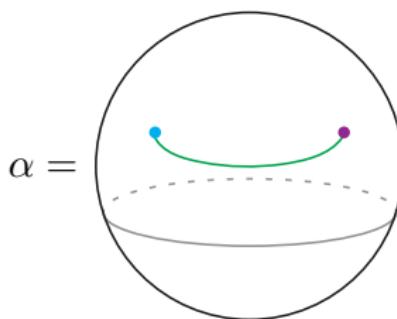
# Algebra of the Twice-Punctured Sphere

1. Consider any diagram on a twice-punctured sphere.
2. We can remove all crossings with a relation to get a sum of diagrams with coefficients.
3. If two arcs meet at a puncture, we can remove them from the puncture with a relation.
4. We can remove all unknots (around punctures and not) with a relation and put them in the coefficients.
5. What is left?

# Presentation of the Twice-Punctured Sphere

## Theorem

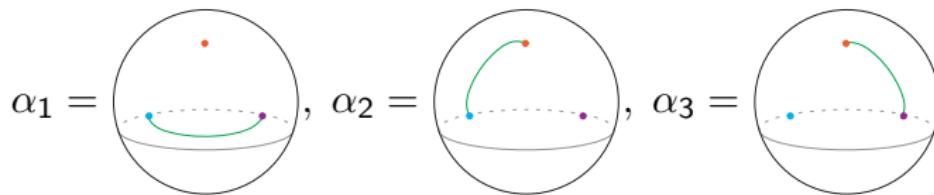
*The arc algebra of the twice-punctured sphere is generated by the unique simple arc between the two punctures,  $\alpha$ , with the relation  $\alpha^2 = -\frac{1}{v_1 v_2} (A - A^{-1})^2$ .*



# Presentation of the Thrice-Punctured Sphere

## Theorem

*The arc algebra for the thrice-punctured sphere is generated by three simple arcs*



*and has relations*

$$\alpha_i \alpha_{i+1} = \alpha_{i+1} \alpha_i = \frac{1}{v_{i+2}} (A^{\frac{1}{2}} + A^{-\frac{1}{2}}) \alpha_{i+2}$$

$$\alpha_i^2 = \frac{1}{v_{i+1} v_{i+2}} (A^{\frac{1}{2}} + A^{-\frac{1}{2}})^2$$

*where subscripts are interpreted modulo 3.*

# The Arc Algebra is Finitely Generated for all $F_{g,n}$

## Theorem

When  $n = 0$  or  $n = 1$ , the arc algebra  $\mathcal{A}(F_{g,n})$  can be generated by  $2^{2g}$  knots. For  $n > 1$ , it can be generated by a set of  $(2^{2g} - 1)(n)$  knots and  $2^{2g} \binom{n}{2}$  arcs.

## Proof.

The proof is based on Doug Bullock's corresponding result for the skein algebra. Inductively reduce any diagram so that it is expressed in terms of a finite set of generating diagrams. These generators can be counted combinatorially. □

## Acknowledgements

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## References

- [1] Doug Bullock. A finite set of generators for the Kauffman bracket skein algebra. *Math. Z.*, 231(1):91-101, 1999
- [2] Doug Bullock and Józef H. Przytycki. Multiplicative structure of the Kauffman bracket skein module quantizations. *Proc. Amer. Math. Soc.*, 128(3):923-931, 2000.
- [3] Julian Roger and Tian Yang. The skein algebra of arcs and links and the decorated Teichmüller space, arXiv:1110.2748v2, 2012.

