

# Generators of the Arc Algebra

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# The Arc Algebra

Background:

- ▶ generalization of the Kauffman bracket skein algebra
- ▶ defined in 2011 by J. Roger and T. Yang and has important connections to both quantum topology and hyperbolic geometry
- ▶ applies to thickened surfaces with punctures

# The Arc Algebra

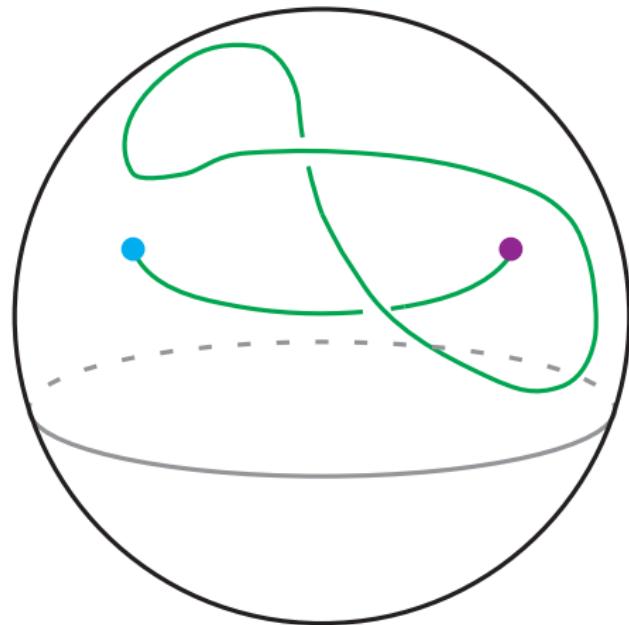
Background:

- ▶ generalization of the Kauffman bracket skein algebra
- ▶ defined in 2011 by J. Roger and T. Yang and has important connections to both quantum topology and hyperbolic geometry
- ▶ applies to thickened surfaces with punctures

Results:

- ▶ full presentations for small examples
- ▶ a finite set of generators for any surface with punctures

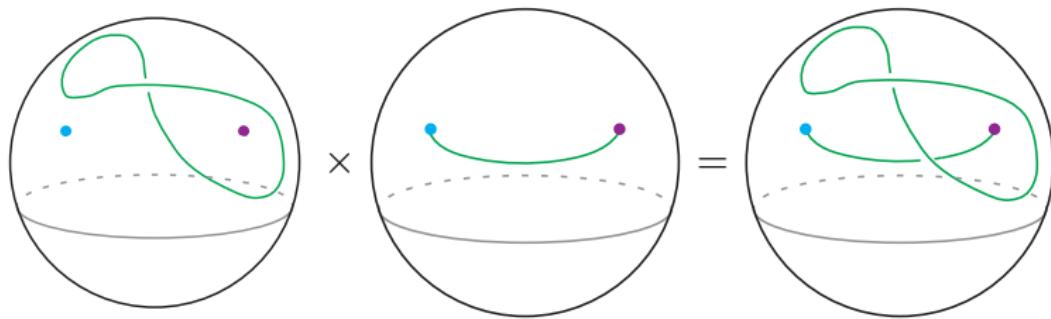
# The Arc Algebra - An Element



## The Arc Algebra - Definition

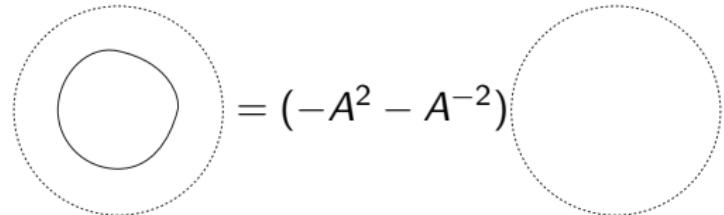
- ▶ Let  $F_{g,n}$  denote the surface of genus  $g$  with  $n$  punctures.
- ▶ Let  $R_n$  be the ring  $\mathbb{Z}[A^{\pm \frac{1}{2}}][v_1, \dots, v_n]$ .
- ▶ The arc algebra  $\mathcal{A}(F_{g,n})$  consists of formal linear combinations of framed curves (unions of knots and arcs) that lie in the thickened surface  $F_{g,n} \times [0, 1]$  subject to four relations.
- ▶ Multiplication is by stacking, induced by  $F_{g,n} \times [0, 1] = F_{g,n} \times [0, \frac{1}{2}] \cup F_{g,n} \times [\frac{1}{2}, 1]$ .

# The Arc Algebra - Multiplication

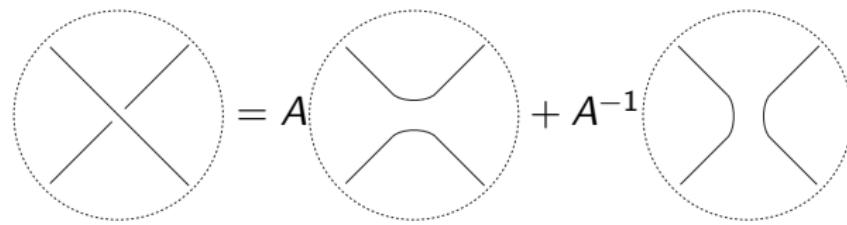


# The Arc Algebra - Kauffman Bracket Relations

Two of the four relations are the same as the skein algebra:



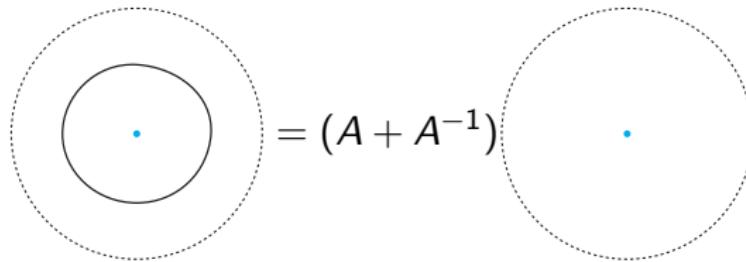
A diagram showing a circle with a single crossing. The left side of the equation has this circle with a dotted boundary. The right side of the equation is  $= (-A^2 - A^{-2})$ , followed by another circle with a dotted boundary.

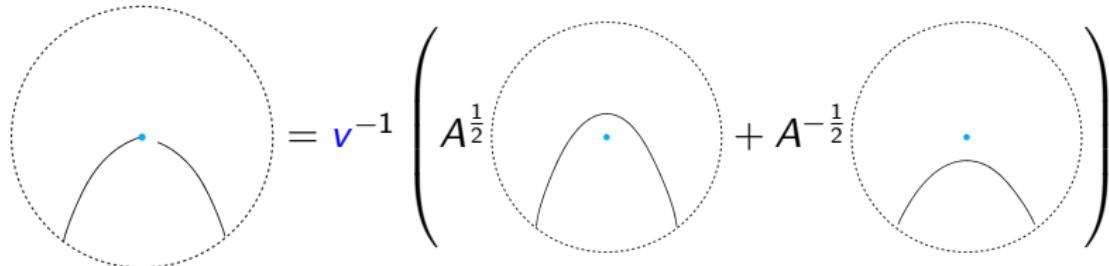


A diagram showing three circles with crossings. The left side of the equation has a circle with a single crossing (an 'X'). The middle term is  $= A$  followed by a circle with two crossings (a trefoil knot). The right term is  $+ A^{-1}$  followed by another circle with two crossings.

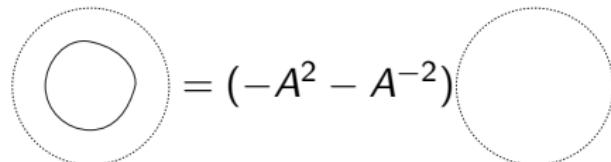
# The Arc Algebra - Puncture Relations

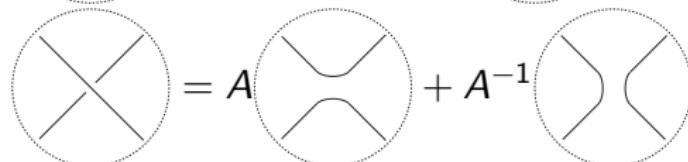
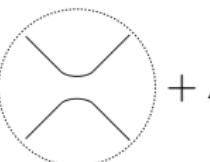
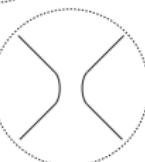
There are two more relations for punctured surfaces:

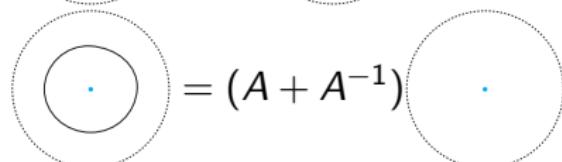
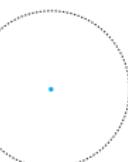

$$= (A + A^{-1})$$

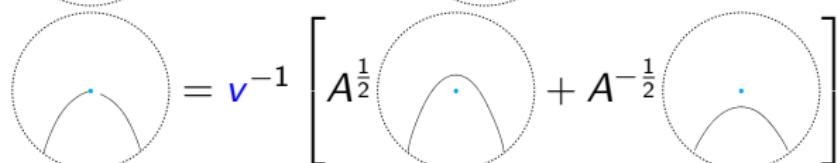
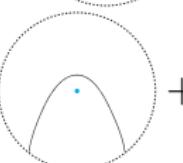
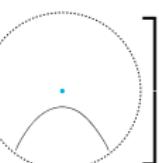

$$= v^{-1} \left( A^{\frac{1}{2}} \text{ (left diagram)} + A^{-\frac{1}{2}} \text{ (right diagram)} \right)$$

# The Arc Algebra - All Relations


$$= (-A^2 - A^{-2})$$


$$= A$$
   $+ A^{-1}$  

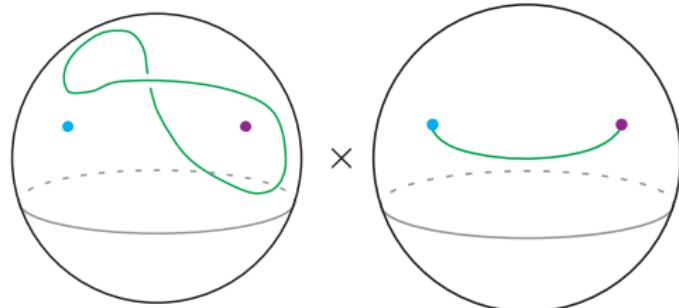

$$= (A + A^{-1})$$
 


$$= \nu^{-1} \left[ A^{\frac{1}{2}} \right.$$
   $+ A^{-\frac{1}{2}}$    $\left. \right]$

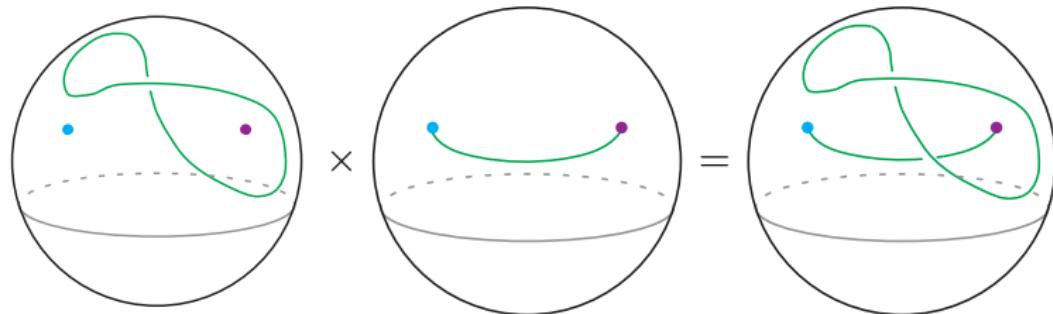
# Questions about the Arc Algebra

1. Given a surface, can we find a set of arcs and links that generate the arc algebra?
2. Given a surface, can we find a complete presentation (generators and relations) for the arc algebra?
3. Is the arc algebra always finitely generated?

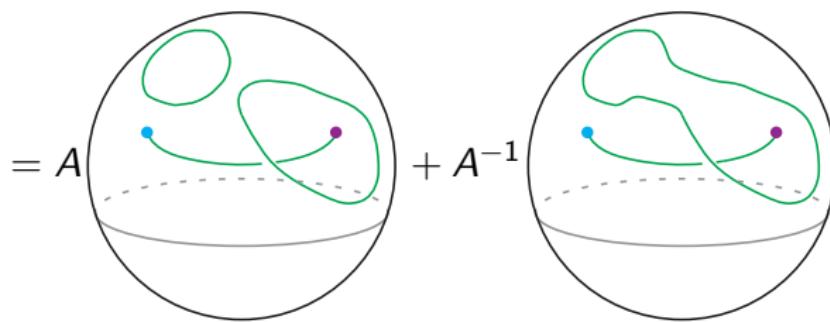
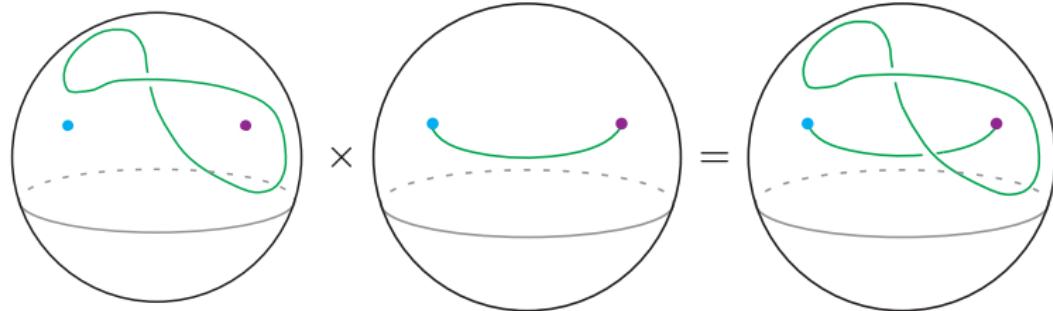
## Example on the Twice-Punctured Sphere



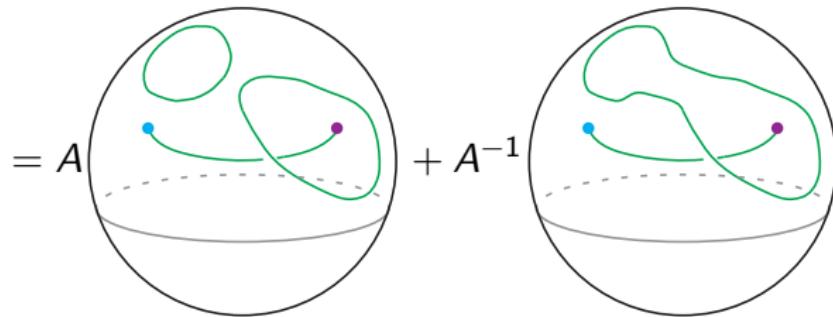
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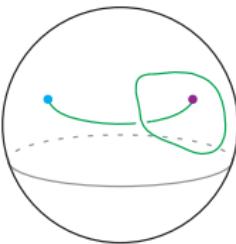


## Example on the Twice-Punctured Sphere

$$\begin{aligned} &= A \left( \text{Diagram 1: A sphere with two punctures and two green curves connecting them.} \right) + A^{-1} \left( \text{Diagram 2: A sphere with two punctures and two green curves connecting them, with one curve being more complex.} \right) \\ &= [A(-A^2 - A^{-2}) + A^{-1}] \left( \text{Diagram 3: A sphere with two punctures and two green curves connecting them, where one curve is a simple loop around the other.} \right) \end{aligned}$$

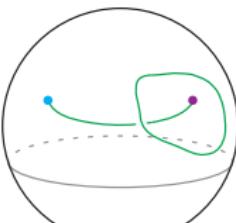
## Example on the Twice-Punctured Sphere

$$= [A(-A^2 - A^{-2}) + A^{-1}]$$

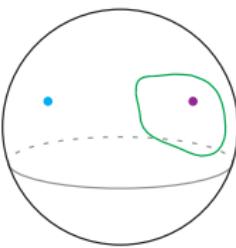
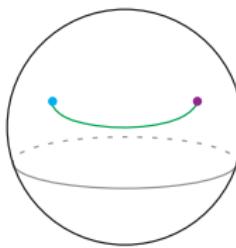


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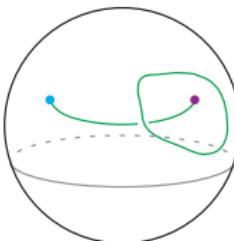


$$= [A(-A^2 - A^{-2}) + A^{-1}]$$

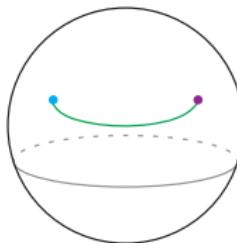
 $\times$ 

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$$= [A(-A^2 - A^{-2}) + A^{-1}]$$

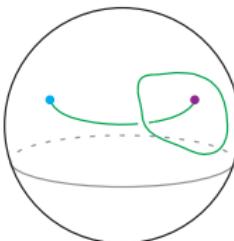


$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$

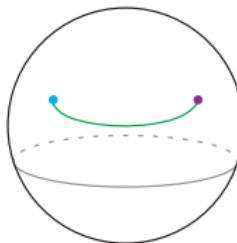


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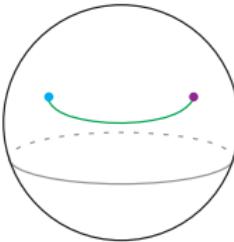
$$= [A(-A^2 - A^{-2}) + A^{-1}]$$



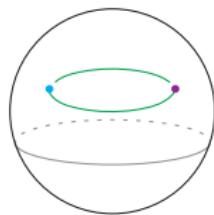
$$= [A(-A^2 - A^{-2}) + A^{-1}][A + A^{-1}]$$



$$= (-A^4 - A^2)$$



## Another Example on the Twice-Punctured Sphere



## Another Example on the Twice-Punctured Sphere

$$\text{Diagram of a twice-punctured sphere with two green curves.} = \nu^{-1} \left[ A^{1/2} \text{Diagram of a twice-punctured sphere with one green curve and one blue dot} + A^{-1/2} \text{Diagram of a twice-punctured sphere with one green curve and one purple dot} \right]$$

## Another Example on the Twice-Punctured Sphere

$$\text{Diagram of a twice-punctured sphere with two green curves.}$$
$$= \textcolor{blue}{v}^{-1} \left[ A^{1/2} \text{Diagram with one curve and punctures} + A^{-1/2} \text{Diagram with two curves and punctures} \right] =$$
$$(\textcolor{blue}{v}\textcolor{purple}{v})^{-1} \left[ A \text{Diagram with one curve and puncture} + \text{Diagram with one curve and puncture} + \text{Diagram with two curves and punctures} + A^{-1} \text{Diagram with one curve and puncture} \right]$$

## Another Example on the Twice-Punctured Sphere

$$\begin{aligned} & \text{Diagram of a twice-punctured sphere with two green curves connecting the punctures.} \\ & = \textcolor{blue}{v}^{-1} \left[ A^{1/2} \text{Diagram with one green curve and one blue dot at the top} + A^{-1/2} \text{Diagram with one green curve and one blue dot at the bottom} \right] = \\ & \quad \left( \textcolor{blue}{v} \textcolor{purple}{v} \right)^{-1} \left[ A \text{Diagram with one green curve and one blue dot at the left} + \text{Diagram with one green curve and one blue dot at the center} + \text{Diagram with one green curve and one blue dot at the right} + A^{-1} \text{Diagram with one green curve and one blue dot at the far right} \right] \\ & = (\textcolor{blue}{v} \textcolor{purple}{v})^{-1} [A(A+A^{-1}) + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-1}(A+A^{-1})] \end{aligned}$$

## Another Example on the Twice-Punctured Sphere

$$\begin{aligned} & \text{Diagram of a twice-punctured sphere with two green curves intersecting at a point on the equator.} \\ & = \textcolor{blue}{v}^{-1} \left[ A^{1/2} \text{Diagram with one green curve and punctures at opposite ends} + A^{-1/2} \text{Diagram with one green curve and punctures at the top and bottom} \right] = \\ & \quad \left( \textcolor{blue}{v} \textcolor{violet}{v} \right)^{-1} \left[ A \text{Diagram with one puncture and small green circle} + \text{Diagram with one green curve and punctures at opposite ends} + \text{Diagram with one green curve and punctures at the top and bottom} + A^{-1} \text{Diagram with one puncture and large green circle} \right] \\ & = (\textcolor{blue}{v} \textcolor{violet}{v})^{-1} [A(A+A^{-1}) + (-A^2 - A^{-2}) + (-A^2 - A^{-2}) + A^{-1}(A+A^{-1})] \\ & = (\textcolor{blue}{v} \textcolor{violet}{v})^{-1} (-A^2 + 2 - A^{-2}) \end{aligned}$$

# Algebra of the Twice-Punctured Sphere

1. Consider any diagram on a twice-punctured sphere.

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1. Consider any diagram on a twice-punctured sphere.
2. We can remove all crossings with a relation to get a sum of diagrams with coefficients.
3. If two arcs meet at a puncture, we can remove them from the puncture with a relation.
4. We can remove all unknots (around punctures and not) with a relation and put them in the coefficients.

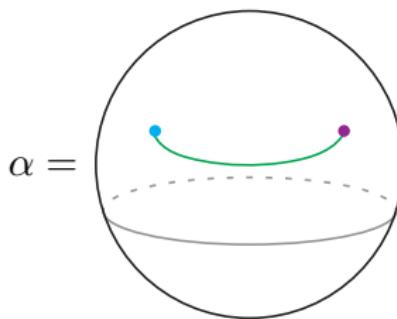
# Algebra of the Twice-Punctured Sphere

1. Consider any diagram on a twice-punctured sphere.
2. We can remove all crossings with a relation to get a sum of diagrams with coefficients.
3. If two arcs meet at a puncture, we can remove them from the puncture with a relation.
4. We can remove all unknots (around punctures and not) with a relation and put them in the coefficients.
5. What is left?

# Presentation of the Twice-Punctured Sphere

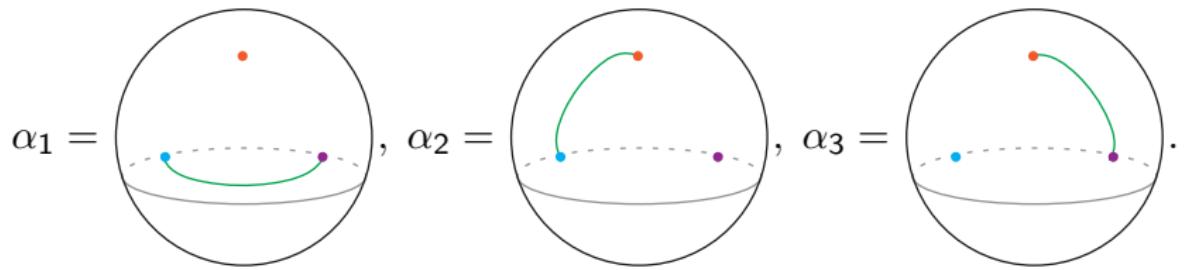
## Theorem

*The arc algebra of the twice-punctured sphere is generated by the unique simple arc between the two punctures,  $\alpha$ , with the relation  $\alpha^2 = -\frac{1}{v_1 v_2} (A - A^{-1})^2$ .*



# Algebra of the Thrice-Punctured Sphere

By the same process as in the twice-punctured sphere, a generating set is:



# Algebra of the Thrice-Punctured Sphere

As before, squaring a generator results in unknots which can be removed.

$$\begin{aligned}\alpha_i^2 &= \text{Diagram of a thrice-punctured sphere with points } i, i+1, i+2 \text{ and a central circle.} \\ &= v_{i+1}^{-1} v_{i+2}^{-1} \left( A \text{Diagram with points } i, i+1, i+2 \text{ and a central circle.} + \text{Diagram with points } i, i+1, i+2 \text{ and a central circle.} + \text{Diagram with points } i, i+1, i+2 \text{ and a central circle.} + A^{-1} \text{Diagram with points } i, i+1, i+2 \text{ and a central circle.} \right) \\ &= v_{i+1}^{-1} v_{i+2}^{-1} (A(A + A^{-1}) + (-A^2 - A^{-2}) + (A + A^{-1}) \\ &\quad + A^{-1}(A + A^{-1})) \\ &= v_{i+1}^{-1} v_{i+2}^{-1} (A^{\frac{1}{2}} + A^{-\frac{1}{2}})^2 \\ &= v_{i+1}^{-1} v_{i+2}^{-1} \delta^2.\end{aligned}$$

# Algebra of the Thrice-Punctured Sphere

Also,

$$\begin{aligned}\alpha_i \alpha_{i+1} &= \text{Diagram } 1 = v_{i+2}^{-1} \left( A^{\frac{1}{2}} \text{Diagram } 2 + A^{-\frac{1}{2}} \text{Diagram } 3 \right) \\ &= v_{i+2}^{-1} (A^{\frac{1}{2}} + A^{-\frac{1}{2}}) \alpha_{i+2} \\ &= v_{i+2}^{-1} \delta \alpha_{i+2}\end{aligned}$$

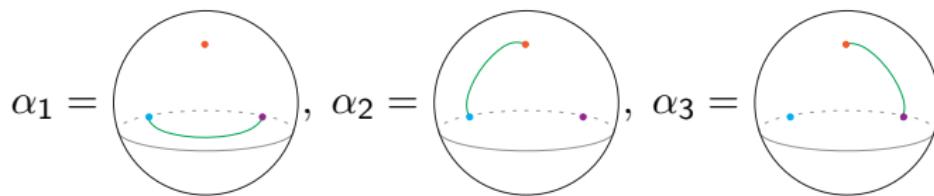

and similarly

$$\begin{aligned}\alpha_{i+1} \alpha_i &= \text{Diagram } 4 = v_{i+2}^{-1} \left( A^{\frac{1}{2}} \text{Diagram } 5 + A^{-\frac{1}{2}} \text{Diagram } 6 \right) \\ &= v_{i+2}^{-1} (A^{\frac{1}{2}} + A^{-\frac{1}{2}}) \alpha_{i+2} \\ &= v_{i+2}^{-1} \delta \alpha_{i+2}.\end{aligned}$$


# Presentation of the Thrice-Punctured Sphere

## Theorem

*The arc algebra for the thrice-punctured sphere is generated by three simple arcs*



*and has relations*

$$\alpha_i \alpha_{i+1} = \alpha_{i+1} \alpha_i = \frac{1}{v_{i+2}} (A^{\frac{1}{2}} + A^{-\frac{1}{2}}) \alpha_{i+2}$$

$$\alpha_i^2 = \frac{1}{v_{i+1} v_{i+2}} (A^{\frac{1}{2}} + A^{-\frac{1}{2}})^2$$

*where subscripts are interpreted modulo 3.*

# Algebra of the Thrice-Punctured Sphere in Matrices

$$\rho(\alpha_1) = \begin{bmatrix} 0 & v_2^{-1}v_3^{-1}\delta^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2^{-1}\delta \\ 0 & 0 & v_3^{-1}\delta & 0 \end{bmatrix},$$

$$\rho(\alpha_2) = \begin{bmatrix} 0 & 0 & v_1^{-1}v_3^{-1}\delta^2 & 0 \\ 0 & 0 & 0 & v_1^{-1}\delta \\ 1 & 0 & 0 & 0 \\ 0 & v_3^{-1}\delta & 0 & 0 \end{bmatrix},$$

$$\rho(\alpha_3) = \begin{bmatrix} 0 & 0 & 0 & v_1^{-1}v_2^{-1}\delta^2 \\ 0 & 0 & v_1^{-1}\delta & 0 \\ 0 & v_2^{-1}\delta & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

# The Arc Algebra is Finitely Generated for all $F_{g,n}$

## Theorem

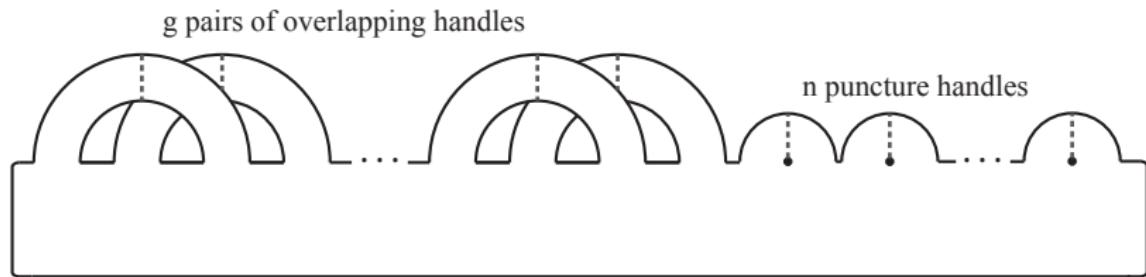
When  $n = 0$  or  $n = 1$ , the arc algebra  $\mathcal{A}(F_{g,n})$  can be generated by  $2^{2g} - 1$  knots. For  $n > 1$ , it can be generated by a set of  $(2^{2g} - 1)(n)$  knots and  $2^{2g} \binom{n}{2}$  arcs.

## Proof.

The proof is based on Doug Bullock's corresponding result for the skein algebra. Inductively reduce any diagram so that it is expressed in terms of a finite set of generating diagrams with minimal complexity. These generators can be counted combinatorially. □

## Proof Sketch - Setting Up

Remove a small disk from  $F_{g,n}$  to make  $F_{g,n}^*$ .



Generators of  $\mathcal{A}(F_{g,n}^*)$  will generate  $\mathcal{A}(F_{g,n})$ .

## Proof Sketch - Complexity 1 of 3

Reduce to curves passing through each handle (genus or puncture) at most once. There are several cases. Example:

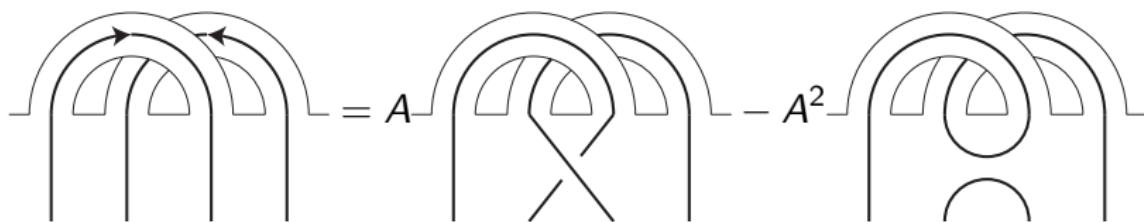
$$\text{Diagram A} = -A^2 \text{Diagram B} + A \text{Diagram C}$$

$$\text{Diagram D} = -A^2 \left( A \text{Diagram E} + A^{-1} \text{Diagram F} \right) + A \text{Diagram G} + A \text{Diagram H}$$

Each of the diagrams with crossings can be further simplified by a straightforward application of the skein relation.

## Proof Sketch - Complexity 2 of 3

Reduce to curves passing through pairs of genus handles in only the "good" way. There are two cases. Example:



## Proof Sketch - Complexity 3 of 3

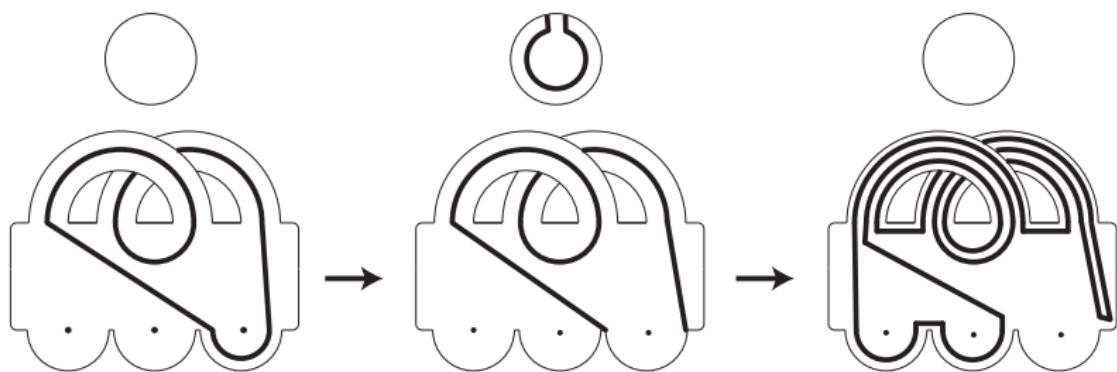
Reduce to curves that pass through puncture handles a total of 0 (if arc) or 1 (if knot) times. There are many cases. Example:



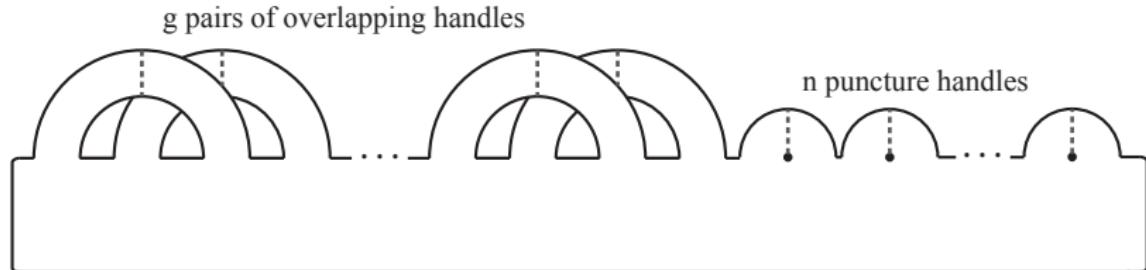
$$= v_i v_j A^{-1} \left( \text{Diagram with both handles having one self-intersection} \right) - A^{-1} \left( \text{Diagram where handle } i \text{ has one self-intersection and handle } j \text{ is a simple loop} \right)$$
$$- A^{-1} \left( \text{Diagram where handle } i \text{ is a simple loop and handle } j \text{ has one self-intersection} \right) - A^{-2} \left( \text{Diagram where both handles have one self-intersection} \right)$$

## Proof Sketch - Finishing

In addition, we can force the curves to avoid one of the puncture handles using the fact that we are generating  $\mathcal{A}(F_{g,n})$  and not  $\mathcal{A}(F_{g,n}^*)$ .



## Proof Sketch - Counting Generators



A generator can

- ▶ pass through each genus handle at most once
- ▶ pass through a pair of genus handles in only the good way
- ▶ pass through puncture handles a total of 1 time if knot
- ▶ pass through puncture handles a total of 0 times if arc
- ▶ start and end at any distinct pair of punctures if arc

these choices uniquely determine the generator.

# The Arc Algebra is Finitely Generated for all $F_{g,n}$

## Theorem

When  $n = 0$  or  $n = 1$ , the arc algebra  $\mathcal{A}(F_{g,n})$  can be generated by  $2^{2g} - 1$  knots. For  $n > 1$ , it can be generated by a set of  $(2^{2g} - 1)(n)$  knots and  $2^{2g} \binom{n}{2}$  arcs.

## Acknowledgements

Thanks to Helen Wong, Stephen Kennedy, Martin Bobb, and the Carleton College Mathematics Department. This project was partially supported by NSF Grant DMS-1105692.

## References

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- [3] Julian Roger and Tian Yang. The skein algebra of arcs and links and the decorated Teichmüller space. *J. Differential Geom.*, 96(1):95-140, 2014.

