CS229 Fall 2017
Problem Set #1: Supervised Learning

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Logistic regression

(a) Given that

$$J(\theta) = \frac{1}{m}\sum_{i=1}^{m} log(1 + e^{-y^{(i)}\theta^T x^{(i)}})$$

we can get

$$\frac{\partial J(\theta)}{\partial \theta_i} = \frac{1}{m}\sum_{k=1}^{m} \frac{-y^{(k)}x_i^{(k)}}{1 + e^{y^{(k)}\theta^T x^{(k)}}}$$

then

$$\frac{\partial J(\theta)}{\partial \theta_i \partial \theta_j} = \frac{1}{m}\sum_{k=1}^{m} \frac{x_i^{(k)}x_j^{(k)}e^{y^{(k)}\theta^T x^{(k)}}}{(1 + e^{y^{(k)}\theta^T x^{(k)}})^2}$$

which is H_{ij}
so

$$\begin{aligned} Z^T H Z &= \sum_{i=1}^{n}\sum_{j=1}^{n} z_i H_{ij} z_j \\ &= \frac{1}{m}\sum_{k=1}^{m} \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} z_i x_i^{(k)} x_j^{(k)} z_j e^{y^{(k)}\theta^T x^{(k)}}}{(1 + e^{y^{(k)}\theta^T x^{(k)}})^2} \end{aligned}$$

known that

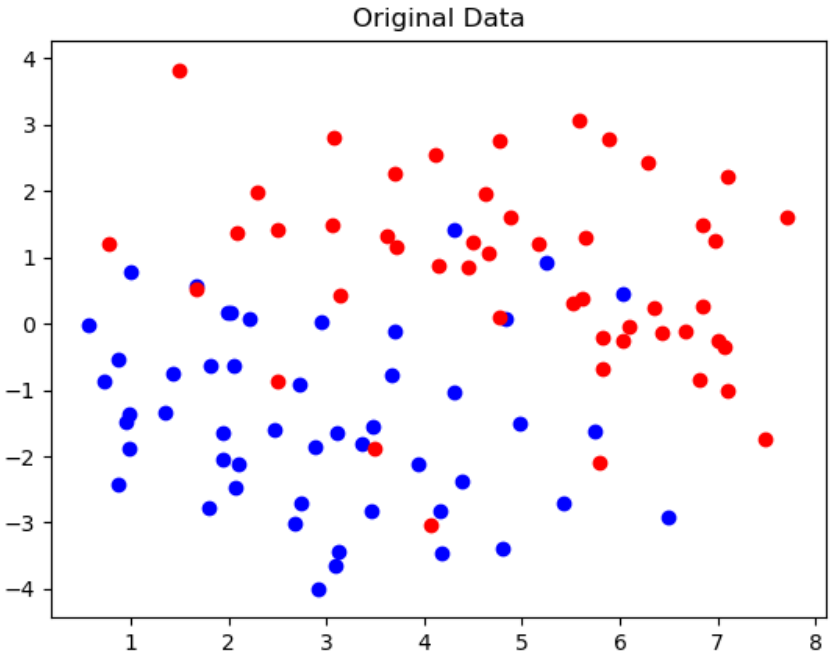
$$\sum_{i=1}^{n}\sum_{j=1}^{n} z_i x_i^{(k)} x_j^{(k)} z_j = (X^T Z)^2 \geq 0$$

$$\frac{e^{y^{(k)}\theta^T x^{(k)}}}{(1 + e^{y^{(k)}\theta^T x^{(k)}})^2} > 0$$

we can easily get

$$Z^T H Z \geq 0$$

(b) Firstly, we plot the original data



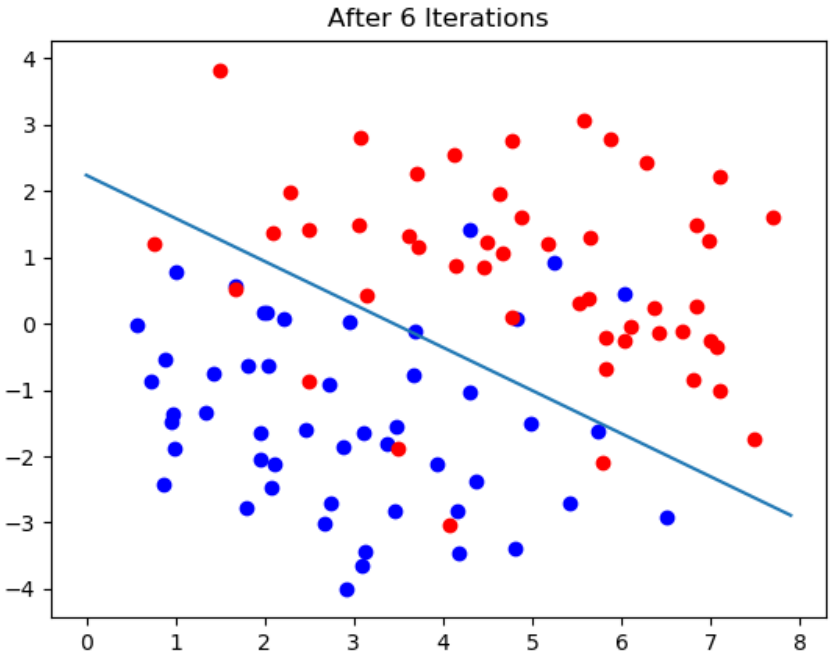
To implement Newton's Method, we calculate the value of

$$\frac{1}{m}\sum_{k=1}^{m}\frac{-y^{(k)}x_i^{(k)}}{1+e^{y^{(k)}\theta^T x^{(k)}}}$$

and **Hessian**
then using the **update rule**

$$\theta := \theta - H^{-1}\bigtriangledown_\theta l(\theta)$$

(c) Through 6 iterations, we finally get the result



and the θ is [-2.6205116, 0.76037154, 1.17194674].

Poisson regression and the exponential family

(a) Firstly, we get

$$\begin{aligned} p(y;\lambda) &= \frac{e^{-\lambda}\lambda^y}{y!} \\ &= \frac{1}{y!}exp(log\lambda^y - loge^{\lambda}) \\ &= \frac{1}{y!}exp(log\lambda y - \lambda) \end{aligned}$$

It's easy to know that

$$\begin{aligned} log\lambda = \eta, \lambda = e^{\eta} \\ T(y) = y \\ a(\eta) = -e^{\eta} \\ b(y) = \frac{1}{y!} \end{aligned}$$

and Poisson distribution is in the exponential family.

(b) Known that if $Y \sim P(\lambda)$, then $E[Y] = \lambda$,

$$\begin{aligned} h_\theta(x) &= E[y|x;\theta] \\ &= \lambda \\ &= e^{\eta} = e^{\theta^T x} \end{aligned}$$

so the response function for the family is

$$g(\eta) = e^{\eta}$$

.

(c) Firstly, we can get likelihood function $L(\theta)$

$$\begin{aligned} L(\theta) &= \Pi_{k=1}^{m} P(y^{(k)}|x^{(k)};\lambda) \\ &= \Pi_{k=1}^{m} \frac{e^{-\lambda}\lambda^{y^{(k)}}}{y^{(k)}!} \end{aligned}$$

Then, the log-likelihood function $l(\theta)$

$$\begin{aligned} l(\theta) &= \sum_{k=1}^{m}(-logy^{(k)}!) + (-\lambda) + y^{(k)}log\lambda \\ &= \sum_{k=1}^{m}(-logy^{(k)}!) + (-e^{\theta^T x^{(k)}}) + y^{(k)}\theta^T x^{(k)} \end{aligned}$$

Then, we can get the derivative of the log-likelihood with respect to θ_i

$$\frac{\partial l(\theta)}{\partial \theta_i} = \sum_{k=1}^{m} (y^{(k)} - e^{\theta^T x^{(k)}}) x_i^{(k)}$$

So, the stochastic gradient ascent learning rule is

$$\theta_i = \theta_i + \alpha (y^{(k)} - e^{\theta^T x^{(k)}}) x_i^{(k)}$$

(d) Firstly, we can know that

$$p(y|x;\theta) = b(y) exp(\eta^T y - a(\eta))$$

then, for a simple training data (x, y) in stochastic gradient ascent,

$$\frac{\partial p(y|x;\theta)}{\partial \theta_i} = x_i (y - \frac{\partial a(\eta)}{\partial \eta})$$

known that,

$$\int_y p(y|x;\eta) = 1$$

we can imply derivation in both sides, and get

$$\int_y p(y|x;\eta)(y - \frac{\partial a(\eta)}{\partial \eta}) = 0$$

finally, we get

$$\frac{\partial a(\eta)}{\partial \eta} = \int_y p(y|x;\eta) y = h_\theta(x)$$

so the gradient we get is

$$\frac{\partial p(y|x;\theta)}{\partial \theta_i} = x_i (y - h_\theta(x))$$

the update rule is

$$\theta_i = \theta_i - \alpha (h_\theta(x) - y) x_i$$

Gaussian discriminant analysis

(a) Firstly, we have

$$p(y=1|x;\phi,\Sigma,\mu_1,\mu_-1) = \frac{p(x|y=1)p(y=1)}{p(x|y=1)p(y=1)+p(x|y=-1)p(y=-1)} = \frac{1}{1+\frac{p(x|y=-1)p(y=-1)}{p(x|y=1)p(y=1)}}$$

apply such calculattion, we can get

$$p(y=1|x;\phi,\Sigma,\mu_1,\mu_-1) = \frac{1}{1+exp(-y(ln\frac{\phi}{1-\phi}+\frac{1}{2}((x-\mu_{-1})^T\Sigma^{-1}(x-\mu_{-1})-(x-\mu_1)^T\Sigma^{-1}(x-\mu_1))} = \frac{1}{1+exp(-y(ln\frac{\phi}{1-\phi}+(\mu_1-\mu_{-1})\Sigma^{-1}x+\frac{1}{2}(\mu_{-1}\Sigma^{-1}\mu_{-1}-\mu_1\Sigma^{-1}\mu_1)))}$$

It's the same when $y=-1$. They can be written in the form of a logistic function where

$$\theta = (\mu_1-\mu_{-1})\Sigma^{-1}$$

$$\theta_0 = ln\frac{\phi}{1-\phi}+1/2(\mu_{-1}\Sigma^{-1}\mu_{-1}-\mu_1\Sigma^{-1}\mu_1)$$

(b) Let m_1 be the number of samples with label $y=1$, and m_{-1} be the number of samples with label $y=-1$. The log-likelihood function is

$$l(\theta,\mu_{-1},\mu_1,\Sigma) = mlog(\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}})+m_1log\phi+m_{-1}log(1-\phi)+\sum_{i=1}^{m}(-\frac{1}{2})(x^{(i)}-\mu_{y^{(i)}})^T\Sigma^{-1}(x^{(i)}-\mu_{y^{(i)}})$$

so we can derive the partial derivate of each parameter,

$$\frac{\partial l}{\partial\phi} = \frac{m_1-m\phi}{\phi(1-\phi)}$$

$$\frac{\partial l}{\partial\mu_1} = \frac{\sum_{i=1}^{m_1}(x^{(i)}-\mu_1)}{\Sigma}$$

$$\frac{\partial l}{\partial\mu_{-1}} = \frac{\sum_{j=1}^{m_1}(x^{(j)}-\mu_{-1})}{\Sigma}$$

$$\frac{\partial l}{\partial\Sigma} = \frac{-\frac{1}{2}(m\Sigma-\sum_{i=1}^{m}(x^{(i)}-\mu_{y^{(i)}})^T(x^{(i)}-\mu_{y^{(i)}}))}{\Sigma^2}$$

let these partial derivates to be zero, we can get the maximum likelihood estimates the same as problem statement.

(c) As shown in (b)

Linear invariance of optimization algorithms

(a)