Audio Engineering Society
Convention Paper
Presented at the 115th Convention
2003 October 10–13 New York, NY, USA

Discrete-Time Shelf Filter Design for Analog Modeling

David P. Berners[1], Jonathan S. Abel[1]

[1] *Universal Audio, Inc., Santa Cruz, CA 95060, USA*

Correspondence should be addressed to David P. Berners (dpberner@uaudio.com <mailto:dpberner@uaudio.com>)

ABSTRACT
A method for the design of discrete-time second-order shelf filters is developed which allows the response of an analog shelf filter to be approximated in the digital domain. For filters whose features approach the Nyquist limit, the proposed method provides a closer approximation to the analog response than direct application of the bilinear transform. Three types of resonant shelf filter are discussed, and design examples are presented.

1. INTRODUCTION

Bilinear transforms can be used to map the entire continuous-time frequency axis onto the unit circle in the z-plane [1]. Analog prototype filters can be carried into the discrete-time domain using bilinear tranforms while preserving filter order and stability. Through the use of the bilinear warping constant, a single frequency in the continuous-time domain can be positioned anywhere on the unit circle in the z-plane. All other frequencies will be displaced by various amounts, with the amount of warping becoming more severe near the Nyquist limit. Because of the increased warping at higher frequencies, the bilinear transform is a poor choice for discrete-time modeling of filters with features near the Nyquist limit. Oversampling can be used to move the Nyquist Frequency much higher than the features of the filter, but this leads to increased computational cost, processing latency, and artifacts due to the upsampling process.

In [2], a design method is developed for parametric filters which does not suffer from the warping due to the bilinear transform. This approach can be readily extended to the design of shelf filters.

The second-order shelf filter typically makes use of two complex-conjugate poles and two complex-conjugate zeros to accomplish shelving. If the poles lie at a higher frequency than the zeros, the shelf is either a low-cut shelf or a high-boost shelf; otherwise, it is a low-boost or high-cut shelf. The poles and zeros can have independently varying amounts of resonance. Often, one set of features (either the poles or zeros) will be either critically damped or slightly underdamped, and the other set of features will have an adjustable amount of resonance.

In the design of the discrete-time filter, five features of the transfer function can be fixed, as afforded by the five independent coefficients available for use in the biquad. Salient features of the shelf include: low- and high-frequency asymptotic gains, resonant frequencies of the poles and zeros, and response magnitudes at the resonant frequencies. Depending upon the locations and amounts of damping of the shelf resonances, each of these six features may become more or less important. In this paper, criteria are given for the selection of features to be fixed in the filter design, and methods for determining filter coefficients are presented based on the desired features. Design examples are given.

2. ANALOG SHELF-FILTER DESIGN

Standard shelf-filter design leads to the transfer function

$$H(s) = \frac{\frac{\gamma}{\omega_s^2}s^2 + \frac{2\sqrt{\gamma}}{\omega_s}s + 1}{\frac{1}{\omega_s^2}s^2 + \frac{2}{\omega_s}s + 1} \tag{1}$$

for a high-shelf filter at frequency ω_s with asymptotic high-frequency gain γ. For a high-shelf cut-filter, the transfer function becomes

$$H(s) = \frac{\frac{1}{\omega_s^2}s^2 + \frac{2}{\omega_s}s + 1}{\frac{\gamma}{\omega_s^2}s^2 + \frac{2\sqrt{\gamma}}{\omega_s}s + 1} \tag{2}$$

where $1/\gamma$ becomes the high-frequency asymptotic gain. The shelf transfer function can be generalized to

$$H(s) = \frac{\frac{\gamma}{\omega_s^2}s^2 + \frac{\sqrt{\gamma}}{Q_z\omega_s}s + 1}{\frac{1}{\omega_s^2}s^2 + \frac{1}{Q_p\omega_s}s + 1} \tag{3}$$

for a *resonant* shelf filter, shown in Fig. 1.

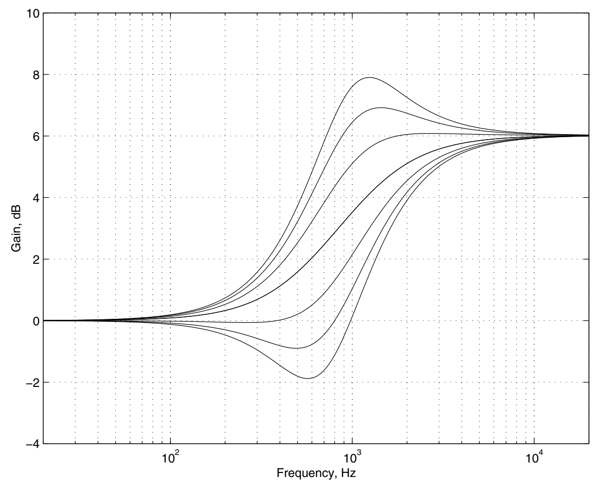

Fig. 1: *Resonant Analog Shelf Filter*

3. FILTER DISCRETIZATION

In carrying the shelf filter from the continuous to the dicrete-time domain, we must determine which features of the transfer function we would like to preserve. Let us further examine the high-shelf filter for this purpose.

The analog filter can be fully described by specification of the following features:

- Asymptotic Gain at DC
- Asymptotic Gain at High Frequency
- Q Associated With Poles
- Q Associated With Zeros
- Shelving Frequency

where, for a high-shelf filter, the *shelving frequency* is defined as the frequency of the poles for a boost filter or the frequency of the zeros for a cut filter.

In the discrete-time domain, there is a different set of features that may be more relevant for describing the transfer function:

- Frequency Locations of Resonant Peaks
- Gains at Resonant Peaks
- Asymptotic Gain at DC
- Gain at Nyquist Limit

For shelf filters with two resonant peaks, this system is overdetermined ; a second-order filter has only five degrees of freedom. For accurate modeling of such filters in the discrete-time domain, higher order filters must be used. However, for filters having only one resonant feature, this list leads to only four constraints. Discrete filter design can thus be carried out with the addition of a fifth constraint. We choose filter gain at the frequency of the non-resonant feature to complete the set of constraints.

3.1. Discrete-Time Filter Design Example

Consider a high-shelf filter H_0 with a DC gain of unity, a high-frequency gain of $\gamma_0 = 2$, a shelving frequency $f_0 \doteq \omega_0/(2\pi)$ of 8kHz, a pole resonance of $Q_{p0} = \sqrt{2}$, and a zero resonance of $Q_{z0} = \sqrt{2}/2$. We will now design a discrete-time model of this filter to be run at a sampling rate of $f_s = 44.1$kHz. We have

$$H_0(s) = \frac{\frac{\gamma_0}{\omega_0^2}s^2 + \frac{\sqrt{\gamma_0}}{Q_{z0}\omega_0}s + 1}{\frac{1}{\omega_0^2}s^2 + \frac{1}{Q_{p0}\omega_0}s + 1}. \tag{4}$$

Our method will be to design a *pre-warped* analog filter H_1 which, upon application of the bilinear transform, will result in a discrete-time filter meeting our specifications:

$$H_1(s) = \frac{\frac{\gamma_1}{\omega_1^2}s^2 + \frac{\sqrt{\gamma_1}}{Q_{z1}\omega_1}s + 1}{\frac{1}{\omega_1^2}s^2 + \frac{1}{Q_{p1}\omega_1}s + 1}. \tag{5}$$

We will use the bilinear transform

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right). \tag{6}$$

We will plan on using a bilinear warping constant

$$T_d = \left(\frac{2}{\omega_0}\right)\tan\left(\frac{\omega_0}{2f_s}\right). \tag{7}$$

Since the resonant frequency of the shelf filter is ω_0, this choice of T_d will prevent the resonant frequency from being displaced by the bilinear transform. Thus, we can choose $\omega_1 = \omega_0$.

The DC gain of the shelf filter will be preserved across the bilinear transform, so we need do nothing in order to preserve this feature. However, in order to preserve the filter gain at the Nyquist limit, we need to alter the transfer function in the analog domain.

The gain of the discrete-time shelf filter at the Nyquist limit will be equal to the asymptotic high-frequency gain of the analog prototype. Therefore we need to choose

$$\gamma_1 = |H_0(f_s/2)|, \tag{8}$$

or

$$\gamma_1 = \left[\frac{\left(1 - \omega_\eta^2\frac{\gamma_0}{\omega_0^2}\right)^2 + \left(\omega_\eta^2\frac{\gamma_0}{Q_{z0}^2\omega_0^2}\right)}{\left(1 - \frac{\omega_\eta^2}{\omega_0^2}\right)^2 + \left(\frac{\omega_\eta^2}{Q_{p0}^2\omega_0^2}\right)}\right]^{1/2}, \tag{9}$$

where $\omega_\eta = 2\pi f_s/2$.

At this point, all parameters have been determined except Q_{p1} and Q_{z1}. These two parameters will be determined iteratively, in such a way that the resulting filter has the desired gains at ω_1 and $\omega_1/\sqrt{\gamma_1}$, the characteristic frequencies of the filter poles and zeros.

In order to carry out the iteration, we first need to obtain the desired gains $|H_{1Desired}(\omega_1)|$ and $|H_{1Desired}(\omega_1/\sqrt{\gamma_1})|$. At ω_1, this is straightforward. Since we have chosen T_d in such a way that ω_1 is not displaced by the bilinear transform, we can simply set $|H_{1Desired}(\omega_1)| = |H_0(\omega_1)|$. However, in order to determine the desired gain for H_1 at $\omega_1/\sqrt{\gamma_1}$, we need to evaluate H_0 at the frequency *to which* $\omega_1/\sqrt{\gamma_1}$ *will be warped* by the bilinear transform. This frequency is readily found as

$$\hat{\omega} = 2\arctan\left(\frac{T_d\omega_1}{2\sqrt{\gamma}}\right)f_s, \tag{10}$$

leading to $|H_{1Desired}\left(\omega_1/\sqrt{\gamma_1}\right)| = |H_0(\hat{\omega})|$.

We now are able to set up an iterative adjustment to Q_{p1} and Q_{z1}. Q_{p1} will be adjusted based upon $|H_1(\omega_1)|$, while Q_{z1} will be adjusted according to $|H_1\left(\omega_1/\sqrt{\gamma_1}\right)|$. Frequencies other than ω_1 and $\omega_1/\sqrt{\gamma_1}$ could be used as the basis for determining Q_{p1} and Q_{z1}. However, since ω_1 and $\omega_1/\sqrt{\gamma_1}$ are the natural frequencies of the poles and zeros of the filter, this choice provides the most robust and quickly converging iteration: a change in Q_{p1} will affect the gain of H_1 at ω_1 more than at any other frequency, and a change in Q_{z1} will affect the gain of H_1 at $\omega_1/\sqrt{\gamma_1}$ more than at any other frequency.

The iteration works as follows: Beginning with $Q_{p1} = Q_{p0}$ and $Q_{z1} = Q_{z0}$,

1. $Q_{p1} = Q_{p1} \cdot |H_{1Desired}(\omega_1)/H_1(\omega_1)|$
2. $Q_{z1} = Q_{z1} \cdot |H_1\left(\omega_1/\sqrt{\gamma_1}\right)/H_{1Desired}\left(\omega_1/\sqrt{\gamma_1}\right)|$
3. repeat

After a handful of iterations, Q_{p1} and Q_{z1} will have converged to stable values. We have used a very conservative nine iterations to good effect for a wide variety of shelf filters.

The simple update expressions within the iteration result from the fact that, at the natural frequencies, the zeroth and second order terms of a resonator combine to zero, leaving the entire gain of the resonator proportional to the linear term:

$$\left|\frac{\gamma}{\omega_0^2}s^2 + \frac{\sqrt{\gamma}}{Q\omega_0}s + 1\right|\Bigg|_{\omega=\frac{\omega_0}{\sqrt{\gamma}}} = \frac{1}{Q}. \qquad (11)$$

Figure 2 shows the responses of the filters H_0 and H_1. The original filter H_0 appears as a solid line, the pre-warped filter H_1 dashed. Various points on the plots show how the effects of the bilinear transform are accounted for by the pre-warped filter. The high-frequency asymptote of filter H_1 will be warped to point A on H_0, the Nyquist limit. Point B shows the natural frequency of the resonant poles. This point is kept fixed through our choice of bilinear warping constant T_d. Finally, point C on H_1 will be warped to point D on H_0.

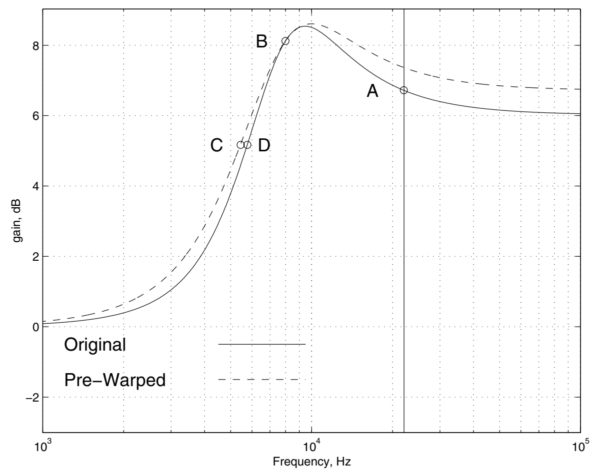

Fig. 2: *Original and Pre-Warped Analog Filters*

A discrete-time filter $H(z)$ can be obtained by applying the bilinear transform to $H_1(s)$ with warping constant $T_d = \left(\frac{2}{\omega_0}\right)\tan\left(\frac{\omega_0}{2f_s}\right)$.

Comparing the discrete-time filter to the analog filter yields the following results:

- responses match at DC
- responses match at Nyquist limit
- responses match at characteristic frequency of poles
- responses match at characteristic frequency of zeros of the discrete-time filter
- characteristic frequencies of poles match for the two filters

This exhausts the five degrees of freedom afforded by the biquad structure.

Figure 3 shows transfer functions for the prototype analog filter, the resulting discrete-time filter, and a second discrete-time filter obtained by direct application of the bilinear transform to the prototype filter.

Figure 4 shows a similar set of plots, this time for a filter with the same γ_0 and ω_0, but with $Q_{z0} = \sqrt{2}$ and $Q_{p0} = \sqrt{2}/2$.

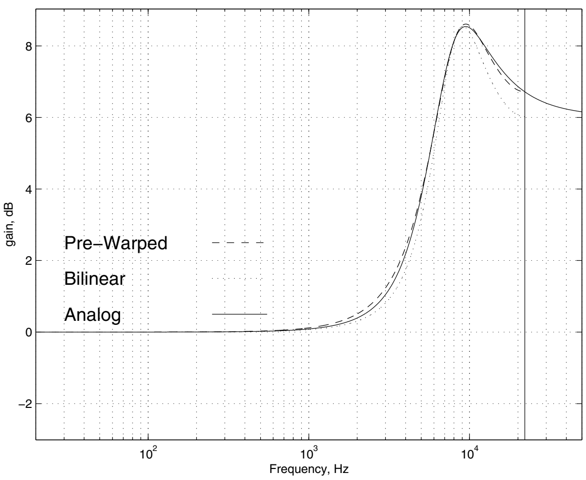

Fig. 3: *Filter Design Example 1*

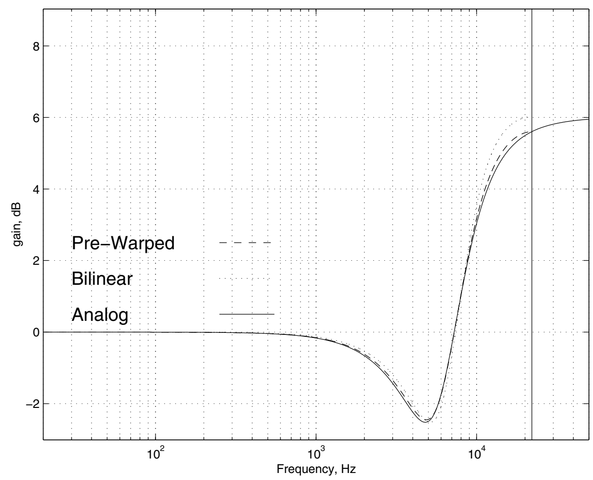

Fig. 4: *Filter Design Example 2*

The design method for this type of filter is similar. Here, the bilinear warping constant used will be

$$T_d = \left(\frac{2}{\omega_0/\sqrt{\gamma_0}}\right)\tan\left(\frac{\omega_0/\sqrt{\gamma_0}}{2f_s}\right) \qquad (12)$$

because the biggest resonance of ther filter is now at $\omega_0/\sqrt{\gamma_0}$. In order to preserve this resonance position when adjusting the gain at the Nyquist limit, for this filter ω_0 and $\sqrt{\gamma_0}$ must be adjusted proportionally, rather than simply adjusting γ_0 as before. All other design steps remain unchanged.

4. CONSLUSIONS

A simple iterative method for discretization of second-order shelf filters has been proposed which is inexpensive and order-preserving. For shelf filters having one resonant transition, the natural frequency of resonance and gain at resonance, as well as the DC gain, gain at the Nyquist limit, and gain at the location of the filter's other natural frequency, are preserved.

5. REFERENCES

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