Music 421A
Winter 2011-2012
Homework #1
Introduction to Windows and Spectral Analysis
31 points
Due in one week (Thursday, January 19, 2012, at 5:00pm)

Theory Problems

1. (5 pts) Give the expression of a rectangular window which, when convolved with itself, gives the length-M triangular window defined as (M can be assumed to be odd)

$$w_\Lambda(n) = \begin{cases} 1 - \frac{|n|}{(M-1)/2}, & -\frac{(M-1)}{2} \le n \le \frac{M-1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

Note that, under this definition, the triangular window includes zeros on each end of the window (*i.e.*, there are only $M-2$ nonzero samples in the window). See the Bartlett window section[1] of the text for further discussion. Windows are normally defined to include only non-zero values, so that $M-1$ in the above definition would normally be $M+1$.

Solution:

(5 pts) We have the triangular window defined as

$$w_\Lambda(n) = \begin{cases} 1 - \frac{|n|}{(M-1)/2} & , \quad -\frac{(M-1)}{2} \le n \le \frac{M-1}{2} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

For a general discrete time convolution of a sequence $x(n)$ and $y(n)$, the length of non-zero samples in the resulting signal is N_x+N_y-1 where N_x and N_y are the lengths of the sequences x and y respectively. Our length M triangular window includes two zero endpoints, so the nonzero portion is length $M-2$. A length M_R rectangular window convolved with itself gives $2M_R-1$ nonzero points. Thus, we need $M-2 = 2M_R-1$, or $M_R = (M-1)/2$.

To obtain the amplitude, we can realize that the peak of the triangle happens at $n=0$ in the convolution, which is the point where the rectangular windows fully overlap. Since we know also the length $M_R = (M-1)/2$, this leads to the simple equation $(M-1)/2 \times A^2 = 1$, where A is the amplitude of the rectangular window. Thus,

$$w_R(n) = \begin{cases} \frac{1}{\sqrt{(M-1)/2}} & , \quad -\frac{(M-3)}{4} \le n \le \frac{(M-3)}{4} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Naturally, M should be odd such that $(M-1)/2$ is also odd. Otherwise, we would have non-integer indices for the rectangular window. However, the formula and the convolution still work if we allow for non-integer indices. The expression makes sense only for $M \ge 3$ even though a length-1 window is valid (both triangle and rectangle).

[1] http://ccrma.stanford.edu/~jos/sasp/Matlab_Bartlett_Window.html

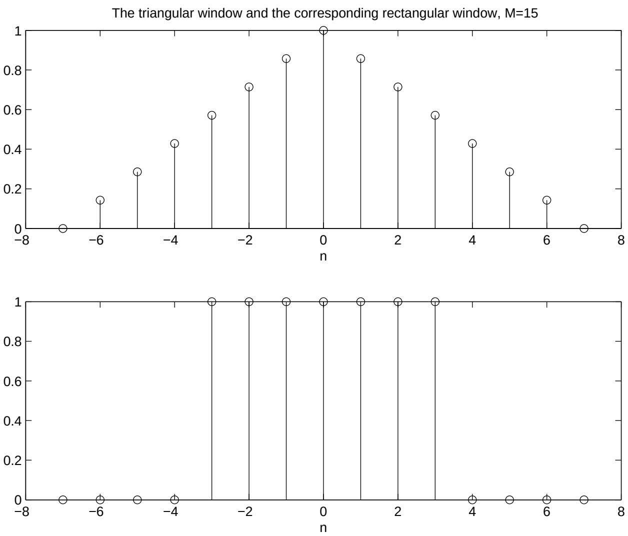


Figure 1: Triangular window and corresponding rectangular window.

scoring:
(-2 pts) wrong length
(-2 pts) wrong amplitude
(-1 pt) slop on signs or $\geq$

2. (10 points)(1 pt per item part or subpart) Consider the window transform of the length-M triangular window. Find the following in terms of M. [Hint: remember that convolution in the time domain corresponds to pointwise multiplication in the frequency-domain, and vice versa.]

   (a) Main-lobe width in normalized radian frequency (radians per sample) [Normalized radian frequency ωT ranges from $-\pi$ to π).]

   (b) Main-lobe width in Hertz (Hz) when the sampling rate is $f_s = 44100$ (Frequency f in Hz ranges from $-f_s/2$ to $f_s/2$.)

   (c) Side-lobe width in normalized radian frequency

   (d) Sidelobe width in Hz for $f_s = 44100$ Hz

   (e) First sidelobe level, in dB AND in linear amplitude relative to peak = 1. [Hint: You may assume M is large.]

   (f) Roll-off rate, in dB per octave. [Hint: You may assume M is large.]

   (g) The minimum frequency difference for two sinusoids to be resolvable by the window. Give your answer in normalized radian frequency (radians per sample), radian frequency (radians per second), and Hz (cycles per second), always assuming $f_s = 44100$.

Solution: It helps to think of the length M triangular window as a convolution of two rectangular windows of the same length $(M-1)/2$. Using the convolution theorem

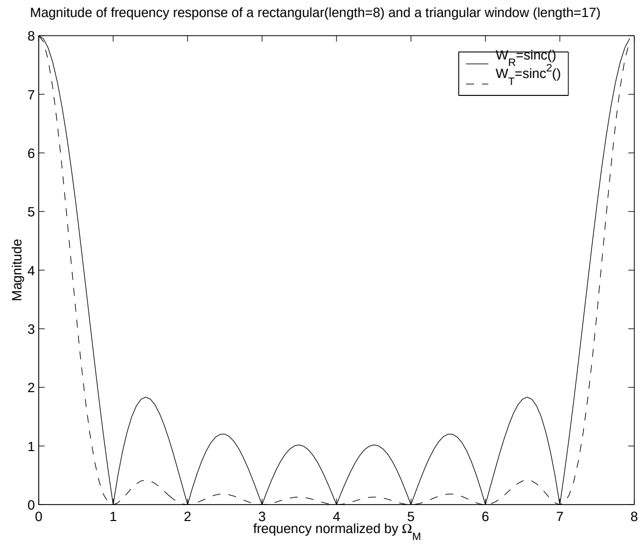


Figure 2: Triangular window transform and corresponding rectangular window transform

$$w(n) * w(n) \longleftrightarrow W(\omega) \cdot W(\omega) = W^2(\omega)$$

we then have

(a) (1 pt) See SASP page 58. The mainlobe width will be the same for the triangle window transform and the rectangle window transform of half the length. In reference to the transform of a rectangular window of length $M' = (M-1)/2$, mainlobe width $= 2\Omega_{M'} = 2\frac{2\pi}{M'} = 2\frac{4\pi}{M-1}$

(b) (1 pt) Now convert frequencies: divide by π and multiply by $f_s/2$ to get $\frac{4f_s}{M-1} = 176400/(M-1)$

(c) (1 pt) Sidelobe width $= \frac{2\pi}{M'} = 2 \cdot \frac{2\pi}{(M-1)}$

(d) (1 pt) sidelobe width $= \frac{f_s}{M'} = \frac{2f_s}{(M-1)}$

(e) (1 pt each of 2 parts) For large enough M, the first sidelobe level = -26dB, which makes sense when we consider the first sidelobe level for a rectangle window is -13dB and we have the magnitude squared here via frequency domain multiplication.

The plots of various values of M are shown in Figure 2 to have sidelobe level independent of M when M is large.

To convert to linear amplitude, we consider $-26dB = 20\log_{10}(\text{amplitude}/1)$ which yields amplitude = .0501.

(f) (1 pt) The triangular window has two finite and uniquely defined derivatives, as a result of being a convolution of a rectangular window with itself. Therefore, the roll-off rate is twice that of the rectangular i.e., -12dB/octave (or -40dB/decade).

(g) (1 pt for each of 3 parts) We want about 2 periods of the difference frequency inside the rectangular window used to make the triangular window. Because the rectangular

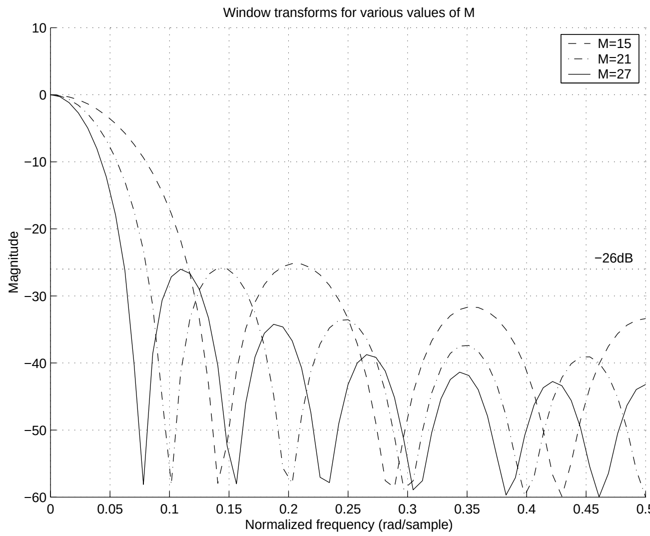


Figure 3: Various lengths of triangular window showing first sidelobe level at -26dB for large M

window is half as long, it is equivalent to say we need 4 difference period cycles in the triangular window. Or, thinking in the frequency domain, to resolve two peaks easily, we want them to be separated by at least one mainlobe width i.e., $4 \times 2\pi/(M-1)$ rad/sample (normalized radians), $4 \times 2\pi f_s/(M-1)$ rad/sec (non-normalized radians), $4 \times f_s/(M-1)$ Hz corresponding to the frequency resolution of $4/(M-1)$ Hz.

An example of rectangular window of width 8 and the corresponding triangular window of width 17 is shown in Figure 2.

Note:

(a) For large M, $M-1 \approx M$ which we might have used when thinking in continuous time where a convolution of a function with itself gives a result twice the original width.

(b) For a general discrete time convolution of a sequence $x(n)$ and $y(n)$, the length of non-zero samples in the resulting signal is $N_x + N_y - 1$ where N_x and N_y are the lengths of the sequences x and y respectively. However, in our case, the triangular window is defined to include two zeros at both ends. The resulting triangular window length then has an additional two samples.

3. (7 points) What is the length of the triangular window which can resolve two sinusoids whose frequencies are ω_1 and ω_2 respectively?

   (a) (5 pts) Give your answer in seconds, in terms of ω_1 and ω_2.

   (b) (2 pts) Give your answer in samples, assuming $\omega_1 = 2\pi 440$ rad/s, and $\omega_2 = 2\pi 490$ rad/s, and the sampling rate is 44100 Hz (CD quality).

Solution:

The period corresponding to the frequency difference of $|\omega_1 - \omega_2|$ is $T_d = 2\pi/|\omega_1 - \omega_2|$. For a **rectangular** window, we want two "difference frequency" cycles. To make a triangle window, you use a convolution of two rectangle windows, each of which is (a half sample less than) half as long as the resultant triangle window. Thus, we want two cycles in a "half-length" rectangle window, or four difference frequency cycles in the required triangle window. Thus the window length required (in seconds) is

$$\boxed{T_w \geq 4 \cdot \frac{2\pi}{|\omega_1 - \omega_2|}}$$

If we want an answer in number of samples, $M \geq (T_w f_s) = 8\pi\, f_s/|\omega_1 - \omega_2|$ where f_s is the sampling rate. At $f_s = 44100$, we have $M \geq 1108354/|\omega_1 - \omega_2|$.

Lab Assignments

Lab assignments are due 24 hours after the theory problems (typically Friday at 5:00pm):

1. (4 pts) Using MATLAB, plot the magnitude and phase response of the triangular window given in problem 1, using the below code. Note: you can copy and paste this code from this PDF.

```
M=15;
Nfft = 1024;
n=-(M-1)/2:(M-1)/2;
wtrian = 1-abs(n)*2/(M-1);
nzeros = (Nfft-length(wtrian)-1)/2;
wtrian = fftshift([zeros(1,nzeros+1) wtrian zeros(1,nzeros)]);
Wtr=fftshift(fft(wtrian));
figure;
title('Magnitude and phase response of triangular window-M=15');
subplot(211);
plot([0:2*pi/Nfft:2*pi*(1-1/Nfft)]-pi,20*log10(abs(Wtr)/max(abs(Wtr))));
xlabel('Normalized frequency');
ylabel('Magnitude (dB)');
axis([-pi pi ylim]);grid;
subplot(212);
plot([0:M/Nfft:M*(1-1/Nfft)]-M/2,angle(Wtr));
xlabel('Normalized frequency');
ylabel('Phase');axis([-pi pi ylim]);
grid;
```

Once you have printed out the plots using the code, triple label the frequency axis by hand in the following units. You should label the minimum frequency, the maximum frequency, and the frequencies corresponding to the first zero crossing of the window transform magnitude.

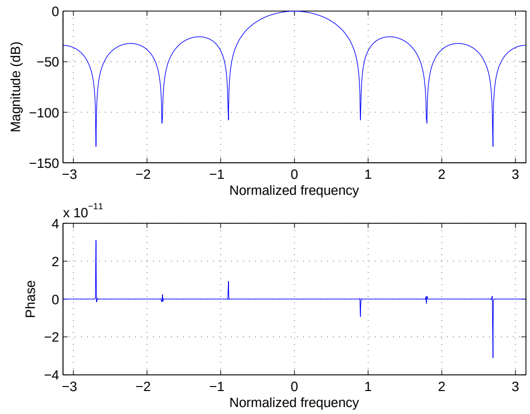


Figure 4: Frequency response of a triangular window

(a) radians per sample
(b) radians per second
(c) Hz (cycles per second)

Note: Don't worry about the units MATLAB automatically gives you (which are the numeric indices of the frequency samples). We will address that when we get to zero padding.

Solution:

(4 pts) The frequency response is shown in Figure 1. Note that the phase is always zero as a result of the transform being real and positive for all values of ω. The extremely small glitches are the result of the finite-precision.

Your axes should be labeled as follows:

units	minimum / maximum frequency	first zero crossing
rad/sample	$\pm\pi$	$4\pi/(M-1)$
rad/second	$\pm\pi f_s$	$4\pi f_s/(M-1)$
Hz	$\pm f_s/2$	$2f_s/(M-1)$

2. (5 pts) Using MATLAB, execute the following code:

```
N = 1024; % window length
w = (.42 - .5*cos(2*pi*(0:N-1)/(N-1)) + .08*cos(4*pi*(0:N-1)/(N-1)))';
% Blackman window
```

A Blackman window is now stored in w. Then write matlab code to do the following:

(a) Create a 1000 Hz sinusoid (cosine) of the same duration N samples, assuming a sampling rate of 8192 Hz. Give it amplitude of 0.6 and a phase 0.25*2*π radians.

(b) Window the sinusoid with the Blackman window. (Implement via Time Domain multiplication.)

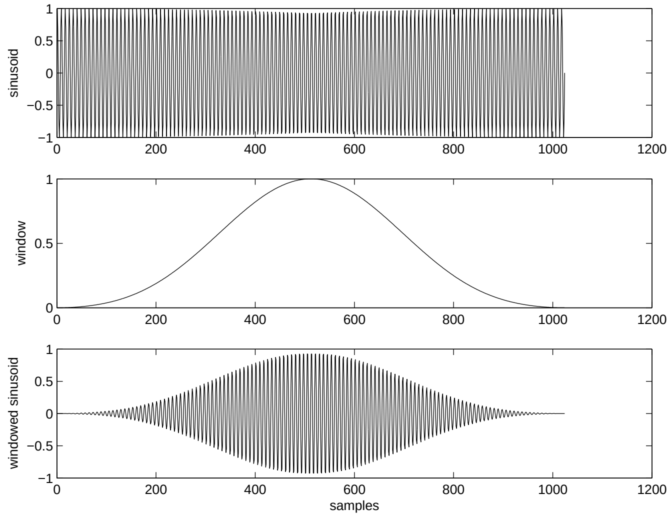


Figure 5: window, sinusoid, and windowed sinusoid

(c) In separate subplots, plot the window, the sinusoid, and the windowed sinusoid.

(d) Plot the log-mag-spectrum, normalizing the peak to 0 dB. Use the fft() function in matlab with zero padding (set a larger number than N as an argument of the matlab function, e.g. fft(x,4*N)).

(e) Find the worst-case side-lobe level graphically.

(f) (Bonus) Change the window function to

```
w = boxcar(N);
```

which is a convenient way of making an N sample rectangle window. Do the log-mag-spectrum plot again for a 1024 Hz sinusoid and also a 1023 Hz sinusoid. These two new plots look very different from each other. Why? We will cover this soon...

Solution: (1 pt for each of 5 parts, 2 point bonus) The following code lines achieve the desired result, though working variations are acceptable.

(a) (1 pt)

```
x = 0.6*cos(2*pi/8192*1000*(0:N-1)+0.25*2*pi);
```

(b) (1 pt)

```
y = w'.*x;
```

(c) (1 pt)

```
subplot(311); plot(x); ylabel('sinusoid');
subplot(312); plot(w); ylabel('window');
subplot(313); plot(y); ylabel('windowed sinusoid'); xlabel('samples');
```

See Figure 5 for a plot.

(d) (1 pt)

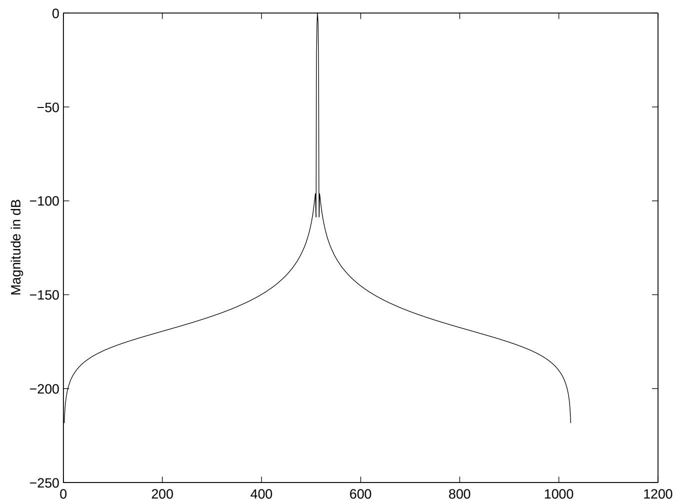


Figure 6: log-mag spectrum

```
plot(fftshift(20*log10(abs(fft(y))./max(abs(fft(y))))));
ylabel('Magnitude in dB');
```

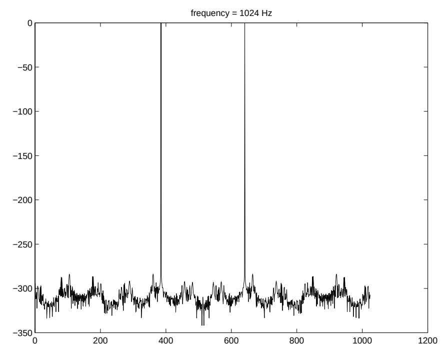
See Figure 6 for a plot. Frequency axes need not be labeled, but magnitude must be labeled in dB as required by the problem.

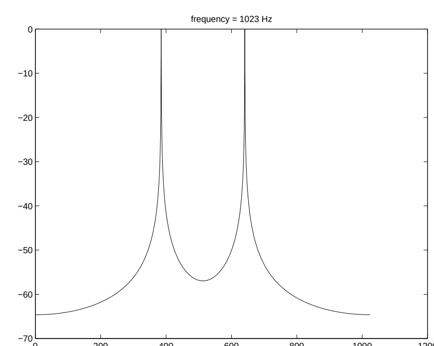
(e) (1 pt) Graphically, the worst case side lobe level appears to be -66.8dB or 63.6dB in this case. Recalling that the DFT is just a sampled DTFT, however, the actual side lobe level is about -58.1dB (revealed by zero padding to be covered in the future). Acceptable answers are between -58 and -67 dB.

(f) (2 pt BONUS)

```
x = 0.6*cos(2*pi/8192*1024*(0:N-1)+0.25*2*pi);
%(then other code, then plot as in part d);
x = 0.6*cos(2*pi/8192*1023*(0:N-1)+0.25*2*pi);
%(then other code, plotting as in part d);
```

Plots show very different results due to the sampled nature of the DFT. When the frequency is 1024, the frequency domain samples hit the "low" points, i.e. the zero crossings of the DFT, because an integer number of cycles fits into the time domain window. When the frequency is 1023, this no longer occurs.


frequency = 1024 Hz
0
−50
−100
−150
−200
−250
−300
−350
0
200
400
600
800
1000
1200


frequency = 1023 Hz
0
−10
−20
−30
−40
−50
−60
−70
0
200
400
600
800
1000
1200