Intersection of Two Lines

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1 INTERSECTION OF TWO LINES

1.1 Different Forms of a Line

Vertical Line Form

A line in **vertical line form** is defined as

$$\boxed{x = x_1} \tag{1}$$

Note that a vertical line has an undefined slope and passes through the point $(x_1, 0)$.

Slope-Intercept Form

A line in **slope-intercept form** is defined as

$$\boxed{y = mx + b} \tag{2}$$

where m is the slope of the line and b is the y-intercept, i.e.

$$y(b) = 0$$

Point-Slope Form

A line in **point-slope form** is defined as

$$\boxed{y - y_1 = m(x - x_1)} \tag{3}$$

where m is the slope of the line and (x_1, y_1) is a point on the line. Since (x_1, y_1) is a point on the line, we know (from the slope-intercept form of Eq. (2)) that

$$y_1 = mx_1 + b$$

where b is the y-intercept. To find b, we can simply rearrange the above equation.

$$\boxed{b = y_1 - mx_1} \tag{4}$$

Two Point Form

Two points can completely specify a line that passes through both of them. Therefore, a line in **two point form** is defined as a set of two ordered pairs:

$$\boxed{\{(x_1, y_1), (x_2, y_2)\}} \tag{5}$$

The slope of the line passing through the two points can be calculated as

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}} \tag{6}$$

Now that we have the slope, m, we can also find the y-intercept, b, using point-slope form, selecting (x_1, y_1) as our point. From Eq. (4),

$$b = y_1 - mx_1$$

Substituting Eq. (6) into the equation above,

$$\boxed{b = y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_1} \tag{7}$$

1.2 Converting to Point-Slope Form

Algorithm 2 (to find the intersection of two lines) in the next section will depend on the two lines both being in point-slope form. Therefore, in this section, we develop Algorithm 1 to put a line in any form into point-slope form. To do this, we make use of all the equations from Section 1.1, and we also note that each form of a line requires a different number of variables to define it:

1. vertical line form: defined by 1 variable (x_1)
2. slope-intercept form: defined by 2 variables (m and b)
3. point-slope form: defined by 3 variables (m, x_1, and y_1)
4. two point form: defined by 4 variables (x_1, y_1, x_2, and y_2)

Algorithm 1: `get_point_slope`

Converts a line of any form to point-slope form.

Given:

- $\boldsymbol{\ell} \in \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3,$ or $\mathbb{R}^4$ - vector defining the line

Procedure:

1. Input given in vertical line form ($\boldsymbol{\ell} = \begin{bmatrix} x_1 \end{bmatrix}$).

   if $\boldsymbol{\ell} \in \mathbb{R}$
   $x_1 = \boldsymbol{\ell}$
   $y_1 = 0$
   $m = \texttt{NaN}$
   end

2. Input given in slope-intercept form ($\boldsymbol{\ell} = \begin{bmatrix} m & b \end{bmatrix}$).

   if $\boldsymbol{\ell} \in \mathbb{R}^2$
   $x_1 = 0$
   $y_1 = \ell_2$
   $m = \ell_1$
   end

3. Input given in point-slope form ($\boldsymbol{\ell} = \begin{bmatrix} x_1 & y_1 & m \end{bmatrix}$).

   if $\boldsymbol{\ell} \in \mathbb{R}^3$
   $x_1 = \ell_1$
   $y_1 = \ell_2$
   $m = \ell_3$
   end

4. Input given in two point form ($\boldsymbol{\ell} = \begin{bmatrix} x_1 & y_1 & x_2 & y_2 \end{bmatrix}$).

   if $\boldsymbol{\ell} \in \mathbb{R}^4$

$m = \dfrac{\ell_4 - \ell_2}{\ell_3 - \ell_1}$
if $|m| = \infty$
  $x_1 = \ell_1$
  $y_1 = 0$
  $m = \texttt{NaN}$
else
  $x_1 = \ell_1$
  $y_1 = \ell_2$
end
end

Return:

- $x_1 \in \mathbb{R}$ - x-coordinate of point on line
- $y_1 \in \mathbb{R}$ - y-coordinate of point on line
- $m \in \mathbb{R}$ - slope of line

Note:

- If the line is vertical, the the function returns $x_1 = x_1$, $y_1 = 0$, and $m = \texttt{NaN}$ since a vertical line passes through the point $(x_1, 0)$ and has an undefined slope.
- In step 4, to define if a line is vertical, we check for an infinite slope because of how many programming languages perform division by 0. However, since rigorously the slope of the line is undefined, we use `NaN` to define the slope of a vertical line.

1.3 Intersection of Two Lines

Consider the following two lines given in point-slope form:

$$\text{Line 1: } y - y_1 = m_1 (x - x_1)$$
$$\text{Line 2: } y - y_2 = m_2 (x - x_2)$$

Our goal is to find the intersection of these two lines, $(x_{\text{int}}, y_{\text{int}})$. However, there are *seven* cases we must consider, which can be categorized into three groups:

1. Single intersection point.
   (a) Line 1 nonvertical + line 2 nonvertical + not parallel (Section 1.3.1).
   (b) Line 1 vertical + line 2 nonvertical (Section 1.3.2).
   (c) Line 1 nonvertical + line 2 vertical (Section 1.3.3).
2. Infinite number of intersection points (collinear lines).
   (a) Vertical + collinear (Section 1.3.4).
   (b) Nonvertical + collinear (Section 1.3.5).
3. Intersection at infinity (parallel lines).
   (a) Vertical + parallel (Section 1.3.6).
   (b) Nonvertical + parallel (Section 1.3.7).

1.3.1 Line 1 Nonvertical + Line 2 Nonvertical + Not Parallel

This is the most common case, where the two lines are not parallel, not collinear, and neither is vertical.

If the two lines intersect at $(x_{\text{int}}, y_{\text{int}})$, then

$$y_{\text{int}} - y_1 = m_1 (x_{\text{int}} - x_1) \tag{8}$$

$$y_{\text{int}} - y_2 = m_2 (x_{\text{int}} - x_2) \tag{9}$$

Solving Eqs. (8) and (9) for y_{int},

$$y_{\text{int}} = y_1 + m_1 (x_{\text{int}} - x_1) \tag{10}$$

$$y_{\text{int}} = y_2 + m_2 (x_{\text{int}} - x_2) \tag{11}$$

Equating Eqs. (10) and (11),

$$y_1 + m_1 (x_{\text{int}} - x_1) = y_2 + m_2 (x_{\text{int}} - x_2)$$

Solving for x_{int},

$$\boxed{x_{\text{int}} = \frac{(m_1 x_1 - m_2 x_2) - (y_1 - y_2)}{m_1 - m_2}} \tag{12}$$

To obtain y_{int}, we can use either line. We choose to use line 1.

$$\boxed{y_{\text{int}} = y_1 + m_1 (x_{\text{int}} - x_1)} \tag{13}$$

1.3.2 Line 1 Vertical + Line 2 Nonvertical

If line 1 is vertical, then its equation is given by

$$x = x_1 \tag{14}$$

Since line 2 is not vertical but line 1 is, the intersection *must* occur at

$$\boxed{x_{\text{int}} = x_1} \tag{15}$$

The point-slope form of line 2 is

$$y - y_2 = m_2(x - x_2)$$

At the intersection point,

$$y_{\text{int}} - y_2 = m_2(x_{\text{int}} - x_2) \tag{16}$$

Solving for y_{int},

$$\boxed{y_{\text{int}} = y_2 + m_2(x_{\text{int}} - x_2)} \tag{17}$$

1.3.3 Line 1 Nonvertical + Line 2 Vertical

If line 2 is vertical, then its equation is given by

$$x = x_2 \tag{18}$$

Since line 1 is not vertical but line 2 is, the intersection *must* occur at

$$\boxed{x_{\text{int}} = x_2} \tag{19}$$

The point-slope form of line 1 is

$$y - y_1 = m_1(x - x_1)$$

At the intersection point,

$$y_{\text{int}} - y_1 = m_1(x_{\text{int}} - x_1) \tag{20}$$

Solving for y_{int},

$$\boxed{y_{\text{int}} = y_1 + m_1(x_{\text{int}} - x_1)} \tag{21}$$

1.3.4 Vertical + Collinear

If both lines are nonlinear, then the point-slope forms given by Algorithm 1 will have $m_1 = m_2 = \texttt{NaN}$ and $x_1 = x_2$. While collinear lines don't have a single intersection point, in this case we can specify

$$\boxed{x_{\text{int}} = x_1 = x_2} \tag{22}$$

For y_{int}, we can just say

$$\boxed{y_{\text{int}} = \texttt{NaN}} \tag{23}$$

1.3.5 Nonvertical + Collinear

If the two lines are nonvertical and collinear, we know that $m_1 = m_2$. Consider putting the two lines in slope-intercept form:

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

If both lines are the same (i.e. they are collinear), then the y-intercept, b should be the same for both (i.e. $b_1 = b_2 = b$). Substituting $(x, y) = (x_1, y_1)$ into the top equation and $(x, y) = (x_2, y_2)$ into the bottom equation,

$$y_1 = m_1 x_1 + b$$
$$y_2 = m_2 x_2 + b$$

Solving both equations for b,

$$b = y_1 - m_1 x_1$$
$$b = y_2 - m_2 x_2$$

Thus, if $(y_1 - m_1 x_1)$ is the same as $(y_2 - m_2 x_2)$, and if $m_1 = m_2$, then the two lines are nonvertical and collinear. In this case, there is no intersection, so the intersection point is given by

$$\boxed{x_{\text{int}} = \texttt{NaN}} \tag{24}$$

$$\boxed{y_{\text{int}} = \texttt{NaN}} \tag{25}$$

1.3.6 Vertical + Parallel

If both lines are vertical, then we know $m_1 = \texttt{NaN}$ and $m_2 = \texttt{NaN}$, and that the equations of the lines are given by $x = x_1$ and $x = x_2$. If the two lines are also parallel, then $x_1 \neq x_2$. Since parallel lines intersect at infinity,

$$\boxed{x_{\text{int}} = \infty} \tag{26}$$

$$\boxed{y_{\text{int}} = \infty} \tag{27}$$

1.3.7 Nonvertical + Parallel

If the two lines are nonvertical and parallel, we know that $m_1 = m_2$. Consider putting the two lines in slope-intercept form:

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

If the two lines are parallel, then the y-intercept, b should be different for both. Substituting $(x, y) = (x_1, y_1)$ into the top equation and $(x, y) = (x_2, y_2)$ into the bottom equation,

$$y_1 = m_1 x_1 + b_1$$
$$y_2 = m_2 x_2 + b_2$$

Solving both equations for b,

$$b_1 = y_1 - m_1 x_1$$
$$b_2 = y_2 - m_2 x_2$$

Thus, if $(y_1 - m_1 x_1)$ is different than $(y_2 - m_2 x_2)$, and if $m_1 = m_2$, then the two lines are nonvertical and parallel. Since parallel lines intersect at infinity,

$$\boxed{x_{\text{int}} = \infty} \tag{28}$$

$$\boxed{y_{\text{int}} = \infty} \tag{29}$$

1.4 Algorithm

Algorithm 2: `line_intersection`

Finds the intersection of two lines.

Given:

- $\boldsymbol{\ell}_1 \in \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$, or $\mathbb{R}^4$ - vector defining the line 1
- $\boldsymbol{\ell}_2 \in \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$, or $\mathbb{R}^4$ - vector defining the line 2

Procedure:

1. Convert both lines to point-slope form (Algorithm 1).

$$[x_1, y_1, m_1] = \texttt{get\_point\_slope}(\boldsymbol{\ell}_1)$$
$$[x_2, y_2, m_2] = \texttt{get\_point\_slope}(\boldsymbol{\ell}_2)$$

2. Case #1: line 1 nonvertical + line 2 nonvertical + not parallel.

if $(m_1 \neq \texttt{NaN})$ **and** $(m_2 \neq \texttt{NaN})$ **and** $(m_1 \neq m_2)$

$$x_{\text{int}} = \frac{(m_1 x_1 - m_2 x_2) - (y_1 - y_2)}{m_1 - m_2}$$
$$y_{\text{int}} = y_1 + m_1 (x_{\text{int}} - x_1)$$

3. Case #2: line 1 vertical + line 2 nonvertical.

else if $(m_1 = \texttt{NaN})$ **and** $(m_2 \neq \texttt{NaN})$

$$x_{\text{int}} = x_1$$
$$y_{\text{int}} = y_2 + m_2 (x_{\text{int}} - x_2)$$

4. Case #3: line 1 nonvertical + line 2 vertical.

   else if $(m_1 \neq \texttt{NaN})$ **and** $(m_2 = \texttt{NaN})$

   $x_{\text{int}} = x_2$

   $y_{\text{int}} = y_1 + m_1 (x_{\text{int}} - x_1)$

5. Case #4: vertical + collinear lines.

   else if $(m_1 = \texttt{NaN})$ **and** $(m_2 = \texttt{NaN})$ **and** $(x_1 = x_2)$

   $x_{\text{int}} = x_1$

   $y_{\text{int}} = \texttt{NaN}$

   Warn the user that the two lines are collinear.

6. Case #5: nonvertical + collinear lines.

   else if $(m_1 = m_2)$ **and** $((y_1 - m_1 x_1) = (y_2 - m_2 x_2))$

   $x_{\text{int}} = \texttt{NaN}$

   $y_{\text{int}} = \texttt{NaN}$

   Warn the user that the two lines are collinear.

7. Case #6: vertical + parallel lines.

   else if $(m_1 = \texttt{NaN})$ **and** $(m_2 = \texttt{NaN})$ **and** $(x_1 \neq x_2)$

   $x_{\text{int}} = \infty$

   $y_{\text{int}} = \infty$

   Warn the user that the two lines are parallel.

8. Case #7: nonvertical + parallel lines.

   else if $(m_1 = m_2)$ **and** $((y_1 - m_1 x_1) \neq (y_2 - m_2 x_2))$

   $x_{\text{int}} = \infty$

   $y_{\text{int}} = \infty$

   Warn the user that the two lines are parallel.

   end

Return:

- $x_{\text{int}} \in \mathbb{R}$ - x-coordinate of line intersection
- $y_{\text{int}} \in \mathbb{R}$ - y-coordinate of line intersection