Chapter 2

Linear DAE

(external perturbation independent of $x(t)$)

2.1 Linear Differential Algebraic Equations

In this chapter linear implicit differential equations or Differential Algebraic Equations (DAEs) of the form

$$E\dot{x}(t) = Ax(t) + f(t) \tag{2.1}$$

are studied. In the simplest case, $E, A \in \mathbb{R}^{n\times n}$ are square constant matrices and $f(t) : \mathbb{R} \to \mathbb{R}^n$ is some ~~inhomogeneity~~. Such equations occur for example by linearization of autonomous non-linear problems with respect to constant (or critical) solutions, where $f(t)$ plays the role of a perturbation. Other cases are modelling of linear electrical circuits, simple mechanical systems or, in general, linear systems with additional algebraic linear s constraints. The ~~First~~ question that we need to answer~~, it~~ regards existence and uniqueness of ~~the~~ s solution $x(t) : \mathbb{R} \to \mathbb{R}^n$. Of course the only interesting case is when E is singular (not invertible). Otherwise it is sufficient to invert E and the DAEs can be written as a Ordinary Differential Equations (ODEs) as follows,

$$\dot{x}(t) = E^{-1}Ax(t) + E^{-1}f(t). \tag{2.2}$$

For eq. (2.2) it is sufficient to assume $f(t)$ Lebesgue measurable ~~locally Lipschitz~~ and existence and uniqueness are ensured, see for example [1]. Conversely when matrix E is singular, it is not a priori clear if the solution exists or not. A singular matrix E means that ODEs are mixed with linear algebraic constraints. In order to clarify the existence of a solution for eq. (2.1) we need the following definition.

Definition 2.1.1. *~~The~~ M ~~matrix~~ matrix pair (E, A) is said to be a* regular pencil *if there exist a scalar $\lambda_0 \in \mathbb{R}$, such that*

$$\det(A - \lambda_0 E) \neq 0. \tag{2.3}$$

The key idea is that if ~~the~~ pair (E, A) is a regular pencil, then under an appropriate change of coordinates it is possible to simplify the problem. The idea is made more precise by the following theorem.

Theorem 2.1.1 (Weirstrass canonical form). *~~The~~ M ~~matrix~~ matrix pair $E, A \in \mathbb{R}^{n\times n}$ is a regular pencil if, and only if, there exist invertible matrices P and Q such that*

$$(PEQ, PAQ) = \left(\begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}\right), \tag{2.4}$$

where N is an ~~an~~ upper triangular nilpotent ~~matrix,~~ matrix; J is a matrix in Jordan canonical form and I is the identity matrix of suitable ~~opportune~~ dimensions.

Just for the convenience of the reader we recall the definition of nilpotent matrix.

Definition 2.1.2. *A square matrix N is said to be nilpotent if* there *exists ~~a number~~* an integer *$\nu \in \mathbb{N}$ such that $N^\nu = 0$.*

Remark 2.1.1. *The smallest ν that satisfies $N^\nu = 0$ is usually called* degree *of N.*

Remark 2.1.2. *~~All the~~* Any *upper triangular matrix with zeros along the main diagonal ~~are~~* is *nilpotent.*

The decoupling form in eq. (2.4) is called *Weierstrass canonical form*, for further details see [2]. Moreover the two matrices have the following structure;

$$N = \begin{bmatrix} N_1 & 0 & \dots & 0 \\ 0 & N_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & N_{n_1} \end{bmatrix}, \tag{2.5a}$$

$$J = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{n_2}. \end{bmatrix}. \tag{2.5b}$$

Here N_i with $i = 1, \dots, n_1$ and J_i with $i = 1, \dots, n_2$ are Jordan blocks. The blocks N_i are associated to zero eigenvalues of matrix E, therefore they appear as follows

the corresponding

$$N_i = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}, \quad i = 1, \dots, n_1, \tag{2.6}$$

perhaps would be best stated in terms of eigenvectors?

the and dimension of each block N_i is equal to the difference between the algebraic and the geometric multiplicity of ~~each~~ a eigenvalue. Conversely blocks J_i with $i = 1, \dots, n_2$ are classic al Jordan blocks with the following structure,

$$J_k = \begin{bmatrix} \lambda_k & 1 & & & \\ & \lambda_k & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_k \end{bmatrix}, \quad k = 1, \dots, n_2. \tag{2.7}$$

We now sketch a proof for the Weierstrass canonical form. The proof will be constructive and based on a sequential multiplication of non-singular matrices.

We only show sufficiency (WHAT ABOUT NECESSITY?)

Proof. Let us consider the regular pencil pair (E, A), surely we can write $(E, A + \lambda_0 E - \lambda_0 E)$. Now let us define the matrix $T_0 = (A - \lambda_0 E)$, which is non singular by assumption. Therefore the quantity $T_0^{-1}(E, A + \lambda_0 E - \lambda_0 E) = \left(T_0^{-1}E, I + \lambda_0 T_0^{-1} E\right)$. Now let us call T_1 the non-singular transformation matrix such that $T_1^{-1} T_0^{-1} E T_1$ is in Jordan canonical form. Then the pair $T_1^{-1}\left(T_0^{-1}E, I + \lambda_0 T_0^{-1}E\right) T_1$ has the following structure,

$$\left(\begin{bmatrix} \tilde{N} & 0 \\ 0 & \tilde{J} \end{bmatrix}, \begin{bmatrix} I + \lambda_0 \tilde{N} & 0 \\ 0 & I + \lambda_0 \tilde{J} \end{bmatrix} \right). \tag{2.8}$$

Here the block $\tilde{N}$ is associated to the zero eigenvalues of the singular matrix $T_0^{-1}E$. Therefore $\tilde{N}$ is upper triangular with zero elements on the main diagonal. Thanks to this ~~the~~ property, matrix $(I+\lambda_0\tilde{N})$ instead is invertible. Therefore we can consider the non singular transformation matrix T_2 defined as follows,

$$T_2 = \begin{bmatrix} (I+\lambda_0\tilde{N}) & 0 \\ 0 & \tilde{J} \end{bmatrix}. \tag{2.9}$$

left-multiplying

Now ~~multiplying left side of~~ eq. (2.8) by T_2^{-1} we obtain the following structure,

$$\left(\begin{bmatrix} (I+\lambda_0\tilde{N})^{-1}\tilde{N} & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & \tilde{J}^{-1}(I+\lambda_0\tilde{J}) \end{bmatrix} \right). \tag{2.10}$$

Finally it is sufficient to consider ~~the~~ matrices T_{31} and T_{32} such that $T_{31}^{-1}(I+\lambda_0\tilde{N})^{-1}\tilde{N}T_{31}$ and $T_{32}^{-1}\tilde{J}^{-1}(I+\lambda_0\tilde{J})T_{32}$ are in Jordan canonical form and build ~~the~~ matrix T_3 as follows,

$$T_3 = \begin{bmatrix} T_{31} & 0 \\ 0 & T_{32} \end{bmatrix}. \tag{2.11}$$

Thus the final change of coordinates yields,

You need to prove that $(I+\lambda_0\tilde{N})^{-1}\tilde{N}$ is nilpotent!

$$T_3^{-1}\left(\begin{bmatrix} (I+\lambda_0\tilde{N})^{-1}\tilde{N} & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & \tilde{J}^{-1}(I+\lambda_0\tilde{J}) \end{bmatrix} \right) T_3 = \left(\begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \right), \tag{2.12}$$

and this completes the proof. Just to summarize, we notice that the overall transformation matrices (P,Q) are given by $P=(T_0T_1T_2T_3)^{-1}$ and $Q=T_1T_3$. □

The Weirstrass canonical form is quite useful since we can re-write system ~~in eq.~~ (2.1) using the transformation matrices (P,Q) just introduced, thus

$$PEQQ^{-1}\dot{x}(t) = PAQQ^{-1}x(t) + Pf(t). \tag{2.13}$$

Now considering the following change of coordinates,

$$z(t) = Q^{-1}x(t), \tag{2.14a}$$
$$g(t) = Pf(t), \tag{2.14b}$$

we obtain the following equivalent DAEs

$$\begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}. \tag{2.15}$$

Here we partitioned the $z(t)$ and $g(t)$ vectors according to the dimensions of respectively J and N. Obviously $z(t) = \mathrm{col}(z_1(t), z_2(t))$ and $g(t) = \mathrm{col}(g_1(t), g_2(t))$. It is also interesting to notice that the two sets of equations corresponding to $z_1(t)$ and $z_2(t)$ are decoupled

$$N\dot{z}_1 = z_1 + g_1(t), \tag{2.16a}$$
$$\dot{z}_2 = Jz_2 + g_2(t). \tag{2.16b}$$

It is straightforward to see that the equation associated to $z_2(t)$ is just an ODE. Then the unique solution, is ~~provided~~ given by

$$z_2(t) = e^{Jt}z_{20} + \int_0^t e^{J(t-\tau)}g_2(\tau)\,\mathrm{d}\tau, \quad \forall t \geq 0, \tag{2.17}$$

from the initial condition

with $z_{20} = z_2(t=0)$ ~~initial condition for z_2~~, (for further details see for example [3]). ~~The~~ Eq. (2.16a) is instead again a DAEs , but since ~~the~~ matrix N is nilpotent we can ~~carry out a lot~~ manipulate it. It is convenient to re-write eq. (2.16a) as

$$z_1(t) = N\dot{z}_1(t) - g_1(t). \tag{2.18}$$

Now substituting eq. (2.18) inside itself and differentiating yields,

$$
\begin{aligned}
z_1(t) &= N\frac{\mathrm{d}}{\mathrm{d}t}\left[N\dot{z}_1(t) - g_1(t)\right] - g_1(t) = N^2\frac{\mathrm{d}^2 z_1(t)}{\mathrm{d}t^2} - N\frac{\mathrm{d}g_1(t)}{\mathrm{d}t} - g_1(t) \\
&= N^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[N\dot{z}_1(t) - g_1(t)\right] - N\frac{\mathrm{d}g_1(t)}{\mathrm{d}t} - g_1(t) = N^3\frac{\mathrm{d}^3 z_1(t)}{\mathrm{d}t^3} - N^2\frac{\mathrm{d}^2 g_1(t)}{\mathrm{d}t^2} - \dots \\
&\vdots \\
&= N^\nu\frac{\mathrm{d}^\nu z_1(t)}{\mathrm{d}t^\nu} - N^{\nu-1}\frac{\mathrm{d}^{\nu-1} g_1(t)}{\mathrm{d}t^{\nu-1}} - \dots - N\frac{\mathrm{d}g_1(t)}{\mathrm{d}t} - g_1(t).
\end{aligned}
$$

establishing differential properties of solutions.

~~Therefore we have~~ solution ~~$z_1(t)$ in recurrence form~~. The procedure stops when we reach the degree ν of matrix N, indeed $N^\nu = 0$ and ~~the solution can be expressed as a~~ we may find a unique solution expressed as a finite summation of ~~different terms as follows~~ increasing derivatives of g as follows:

$$z_1(t) = -\sum_{k=0}^{\nu-1} N^k \frac{\mathrm{d}^k g_1(t)}{\mathrm{d}t^k}. \tag{2.20}$$

Remark 2.1.3. *Sometimes index ν is also called* Kronecker index. *Notice also that ν is the number of times that we differentiated function $g_1(t)$ in order to obtain the solution $z_1(t)$.*

Finally we are ready to express the unique solution ~~for~~ to the original problem in eq. (2.1). It is sufficient to invert the change of coordinates in eq. (2.14a) and we obtain

$$x(t) = Qz(t) = Q\begin{bmatrix} -\sum_{k=0}^{\nu-1} N^k \frac{\mathrm{d}^k g_1(t)}{\mathrm{d}t^k} \\ e^{Jt} z_{20} + \int_0^t e^{J(t-\tau)} g_2(\tau)\,\mathrm{d}\tau \end{bmatrix} \tag{2.21}$$

Remark 2.1.4. *Notice that the initial condition $z_{10} = z_1(t=0)$ cannot be chosen freely, but has to satisfy the following,*

$$z_{10} = -\sum_{k=0}^{\nu-1} N^k \frac{\mathrm{d}^k g_1(0)}{\mathrm{d}t^k}. \tag{2.22}$$

*Therefore also the initial condition $x_0 = x(t=0)$ for eq. (2.1) cannot be chosen completely freely. This intuitively express*es *the fact that also the initial condition has to be consistent with constrains expressed by E.*

It would be nice to show that if this does not hold, then constraint (2.1) is violated at $t=0$.

Summarizing what we discussed we have the following (where we only proved the "if" part)

Theorem 2.1.2. *Problem in eq. (2.1) with initial condition $x_0 = x(t=0)$ ~~have~~ has a* unique *solution if, and only if the following conditions holds:*

- *function $f(t)$ is ~~locally Lipschitz,~~* Lebesgue measurable
- *the pair (E, A) is a regular pencil,*
- *the initial condition x_0 satisfies $Q^{-1}x_0 = \begin{bmatrix} -\sum_{k=0}^{\nu-1} N^k \frac{\mathrm{d}^k g_1(0)}{\mathrm{d}t^k} \\ * \end{bmatrix}$.*

Moreover, the corresponding solution is given by (2.21).