2 AMMI Model

The AMMI model is defined as:

$$y_{ij} = \mu + g_i + e_j + \sum_{k=1}^{Q} \lambda_k \gamma_{ik} \delta_{jk} + \epsilon_{ij}, \quad \epsilon_{ij} \sim \mathcal{N}\left(0, \tau^{-1}\right)$$

2.1 Notation

- $\boldsymbol{y}$ - vector of observations;
- μ - the grand mean;
- $\boldsymbol{g}$ - vector of genotype;
- $\boldsymbol{e}$ - vector of environment;
- $\sum_{k=1}^{t} \lambda_k \gamma_{ik} \delta_{jk}$ the bilinear term.
- $\tau = 1/\sigma_E^2$ - precision parameter.
- n_g - number of genotypes
- n_e - number of environments
- $n = n_g \times n_e$ - number of observations
- $\Theta = \{\mu, \boldsymbol{g}, \boldsymbol{e}, \boldsymbol{\lambda}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\tau}\}$
- $\widetilde{f(\phi)} = E_{q(\phi)}[f(\phi)]$

2.2 AMMI Model - t=1

$$y_{ij} = \mu + g_i + e_j + \lambda \gamma_i \delta_j + \epsilon_{ij}$$

Prior

$$\begin{aligned}
p(\mu) &= \mathcal{N}(\mu_\mu; \sigma_\mu^2) \\
p(\boldsymbol{g}) &= \prod_{i=1}^{n_g} \mathcal{N}(0; \sigma_g^2) \\
p(\boldsymbol{e}) &= \prod_{j=1}^{n_e} \mathcal{N}(0; \sigma_e^2) \\
p(\lambda) &= \mathcal{N}^+(0; \sigma_\lambda^2) \\
p(\boldsymbol{\gamma}) &= \mathcal{N}^+(0; 1) \prod_{i=2}^{n_g} \mathcal{N}(0; 1) \\
p(\boldsymbol{\delta}) &= \prod_{j=1}^{n_e} \mathcal{N}(0; 1) \\
p(\tau) &= \mathcal{G}(a; b)
\end{aligned}$$

Let us get the conditional distributions.

$$
\begin{aligned}
p(\boldsymbol{y}|\boldsymbol{\Theta}) &= \prod_{i=1}^{n_g}\prod_{j=1}^{n_e}\mathcal{N}(\mu+g_i+e_j+\lambda\gamma_i\delta_j;\tau^{-1}) \\
&= \prod_{i=1}^{n_g}\prod_{j=1}^{n_e}\frac{\tau^{-1}}{\sqrt{2\pi}}\exp\left\{-\frac{\tau}{2}\left(y-(\mu+g_i+e_j+\lambda\gamma_i\delta_j)^2\right)\right\} \\
\log p(\boldsymbol{y}|\boldsymbol{\Theta}) &\propto \sum_{i=1}^{n_g}\sum_{j=1}^{n_e}\left\{-\frac{\log\tau}{2}-\frac{\tau}{2}\left(y-(\mu+g_i+e_j+\lambda\gamma_i\delta_j)^2\right)\right\} \\
&\propto \sum_{i=1}^{n_g}\sum_{j=1}^{n_e}\left\{-\frac{\log\tau}{2}-\frac{\tau}{2}\left(y^2-2\mu y_{ij}-2y_{ij}g_i-2y_{ij}e_j-2y_{ij}\lambda\gamma_i\delta_j+\mu^2+2\mu g_i+2\mu e_j+2\mu\lambda\gamma_i\delta_j+\right.\right. \\
&\quad \left.\left. g_i^2+2g_ie_j+2g_i\lambda\gamma_i\delta_j+e_j^2+2e_j\lambda\gamma_i\delta_j+\lambda^2\gamma_i^2\delta_j^2\right)\right\}.
\end{aligned}
$$

Variational distributions.

1. **Variational distribution of μ.**

$$
\begin{aligned}
q(\mu) &\propto \exp\{E_{-\mu}[\log p(\boldsymbol{y}|\boldsymbol{\Theta})+\log p(\mu)]\} \\
\exp q(\mu) &\propto E_{-\mu}\left\{\sum_{i=1}^{n_g}\sum_{j=1}^{n_e}-\frac{\tau}{2}\left[-2y_{ij}\mu+\mu^2+2\mu g_i+2\mu e_j+2\mu\sum_{k=1}^{t}\lambda_k\gamma_{ik}\delta_{jk}\right]-\frac{1}{2\sigma_\mu^2}(\mu^2-2\mu\mu_\mu)\right\} \\
&\propto \mu^2\left[-\frac{1}{2}\left(n\tilde{\tau}+\frac{1}{\sigma_\mu^2}\right)\right]+\mu\left[\tilde{\tau}\left(\sum_{i=1}^{n_g}\sum_{j=1}^{n_e}y_{ij}-n_e\sum_{i=1}^{n_g}\tilde{g}_i-n_g\sum_{j=1}^{n_e}\tilde{e}_j-\sum_{i=1}^{n_g}\sum_{j=1}^{n_e}\sum_{k=1}^{t}\tilde{\lambda}_k\tilde{\gamma}_{ik}\tilde{\delta}_{jk}\right)+\frac{\mu_\mu}{\sigma_\mu^2}\right] \\
q(\mu) &\sim \mathcal{N}(\mu_{q(\mu)},\Sigma_{q(\mu)}^{-1}); \\
\mu_{q(\mu)} &= \Sigma_{q(\mu)}^{-1}\left[\tilde{\tau}\left(\sum_{i=1}^{n_g}\sum_{j=1}^{n_e}y_{ij}-n_e\sum_{i=1}^{n_g}\tilde{g}_i-n_g\sum_{j=1}^{n_e}\tilde{e}_j-\sum_{i=1}^{n_g}\sum_{j=1}^{n_e}\sum_{k=1}^{t}\tilde{\lambda}_k\tilde{\gamma}_{ik}\tilde{\delta}_{jk}\right)+\frac{\mu_\mu}{\sigma_\mu^2}\right] \\
\Sigma_{q(\mu)} &= n\tilde{\tau}+\frac{1}{\sigma_\mu^2}
\end{aligned}
$$

2. **Variational distribution of $\boldsymbol{g}$.**

$$\begin{aligned}
q(\boldsymbol{g}) &\propto \exp\{E_{-\boldsymbol{g}}[\log p(\boldsymbol{y}|\boldsymbol{\Theta}) + \log p(\boldsymbol{g})]\} \\
\exp q(\boldsymbol{g}) &\propto E_{-g_i}\left\{\sum_{i=1}^{n_g}\sum_{j=1}^{n_e} -\frac{\tau}{2}\left[-2y_{ij}g_i + 2\mu g_i + g_i^2 + 2g_ie_j + 2g_i\sum_{k=1}^{t}\lambda_k\gamma_{ik}\delta_{jk}\right] - \sum_{i=1}^{n_g}\frac{1}{2\sigma_g^2}g_i^2\right\} \\
&\propto \sum_{i=1}^{n_g}\left\{g_i^2\left[-\frac{1}{2}\left(n\tilde{\tau} + \frac{1}{\sigma_g^2}\right)\right] + g_i\left[\tilde{\tau}\left(\sum_{j=1}^{n_e}y_{ij} - n_e\tilde{\mu} - \sum_{j=1}^{n_e}\tilde{e_j} - \sum_{j=1}^{n_e}\sum_{k=1}^{t}\tilde{\lambda}_k\tilde{\gamma}_{ik}\tilde{\delta}_{jk}\right)\right]\right\} \\
q(\boldsymbol{g}) &\sim \prod_{i=1}^{n_g}\mathcal{N}(\mu_{q(g_i)}, \Sigma^{-1}_{q(g_i)}); \\
\mu_{q(g_i)} &= \Sigma^{-1}_{q(g_i)}\left[\tilde{\tau}\left(\sum_{j=1}^{n_e}y_{ij} - n_e\tilde{\mu} - \sum_{j=1}^{n_e}\tilde{e_j} - \sum_{j=1}^{n_e}\sum_{k=1}^{t}\tilde{\lambda}_k\tilde{\gamma}_{ik}\tilde{\delta}_{jk}\right)\right] \\
\Sigma_{q(g_i)} &= n_e\tilde{\tau} + \frac{1}{\sigma_g^2}
\end{aligned}$$

3. **Variational distribution of $\boldsymbol{e}$.**

$$\begin{aligned}
q(\boldsymbol{e}) &\propto \exp\{E_{-\boldsymbol{e}}[\log p(\boldsymbol{y}|\boldsymbol{\Theta}) + \log p(\boldsymbol{e})]\} \\
\exp q(\boldsymbol{e}) &\propto E_{-e_j}\left\{\sum_{i=1}^{n_g}\sum_{j=1}^{n_e} -\frac{\tau}{2}\left[-2y_{ij}e_j + 2\mu e_j + e_j^2 + 2g_ie_j + 2e_j\sum_{k=1}^{t}\lambda_k\gamma_{ik}\delta_{jk}\right] - \sum_{i=1}^{n_g}\frac{1}{2\sigma_e^2}e_j^2\right\} \\
&\propto \sum_{j=1}^{n_e}\left\{e_j^2\left[-\frac{1}{2}\left(n\tilde{\tau} + \frac{1}{\sigma_e^2}\right)\right] + e_j\left[\tilde{\tau}\left(\sum_{i=1}^{n_g}y_{ij} - n_g\tilde{\mu} - \sum_{i=1}^{n_g}\tilde{g}_i - \sum_{i=1}^{n_g}\sum_{k=1}^{t}\tilde{\lambda}_k\tilde{\gamma}_{ik}\tilde{\delta}_{jk}\right)\right]\right\} \\
q(\boldsymbol{g}) &\sim \prod_{j=1}^{n_e}\mathcal{N}(\mu_{q(e_j)}, \Sigma^{-1}_{q(e_j)}); \\
\mu_{q(e_j)} &= \Sigma^{-1}_{q(e_j)}\left[\tilde{\tau}\left(\sum_{i=1}^{n_g}y_{ij} - n_g\tilde{\mu} - \sum_{i=1}^{n_g}\tilde{g}_i - \sum_{i=1}^{n_g}\sum_{k=1}^{t}\tilde{\lambda}_k\tilde{\gamma}_{ik}\tilde{\delta}_{jk}\right)\right] \\
\Sigma_{q(e_j)} &= n\tilde{\tau} + \frac{1}{\sigma_e^2}
\end{aligned}$$

4. **Variational distribution of λ_1.**

$$
\begin{aligned}
q(\lambda_1) &\propto \exp\{E_{-\lambda_1}[\log p(\boldsymbol{y}|\boldsymbol{\Theta}) + \log p(\lambda_1)]\} \\
\exp q(\lambda_1) &\propto E_{-\lambda_1}\left\{\sum_{i=1}^{n_g}\sum_{j=1}^{n_e} -\frac{\tau}{2}\left[-2y_{ij}\lambda_1\gamma_i\delta_j + 2\mu\lambda\gamma_i\delta_j + 2g_i\lambda 1\gamma_i\delta_j + 2e_j\lambda 1\gamma_i\delta_j + (\lambda 1\gamma_i\delta_j)^2\right] - \right. \\
&\quad \left. \frac{\lambda_1^2}{2\sigma_\lambda^2(1-\Phi(0))}\right\} \\
q(\lambda_1) &\sim \mathcal{TN}(\mu_{q(\lambda_1)}, \Sigma_{q(\lambda_1)}^{-1}); \\
\mu_{q(\lambda_1)} &= \Sigma_{q(\lambda_1)}^{-1}\left[\tilde{\tau}\sum_{i=1}^{n_g}\sum_{j=1}^{n_e}\gamma_i\delta_j(-2y_{ij} + 2\mu + 2g_i + 2e_j)\right] \\
\Sigma_{q(\lambda_1)} &= \tilde{\tau}\sum_{i=1}^{n_g}\sum_{j=1}^{n_e}(\widetilde{\gamma^2}\widetilde{\delta^2}) + \frac{1}{\sigma_\lambda^2(1-\Phi(0)}
\end{aligned}
$$

5. **Variational distribution of $\boldsymbol{\gamma}$.**

6. **Variational distribution of $\boldsymbol{\delta}$.**

7. **Variational distribution of τ.**