CS65K Robotics

Modelling, Planning and Control

Chapter 11: Mobile Robots

LECTURE 16: MOBILE ROBOTS

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Objectives

- The fundamental problems of mobile robotics are introduced and different control architectures are discussed
- Nonholonomic constraints are described and the integrability conditions are presented
- Pfaffian constraints are exploited to derive the kinematic model of a nonholonomic mobile robot
- The model of a unicycle is derived along with its properties
- The model of a bicycle is derived along with its properties

Objectives

- The dynamic model of a mobile robot is derived adopting the Lagrange formulation
- The dynamic model of a unicycle is computed

Mobile Robots

SECTION 1

Mobile Robots

- Fundamental problems of mobile robotics
  - Localization (where are we?)
  - Path and trajectory planning (how do we reach the goal?)
  - Motion control (how do we move?)
- To be able to solve these three problems simultaneously in environments which are:
  - uncertain
  - unstructured
  - dynamic

    $\Downarrow$
- Some degree of autonomy is needed
- From now on, wheeled mobile robots are considered

Control Architectures

Deliberative architecture

Perception

- *Proprioceptive*: robot position, orientation, velocity
- *Exteroceptive*: relative position of obstacles, other robots, people

Wide range of available sensors

- Proprioceptive: encoders, inertial navigation systems (INS), global positioning systems (GPS)
- Exteroceptive: range finders, cameras, tactile sensors, proximity sensors

Control Architectures

Other architectures

Reactive architecture

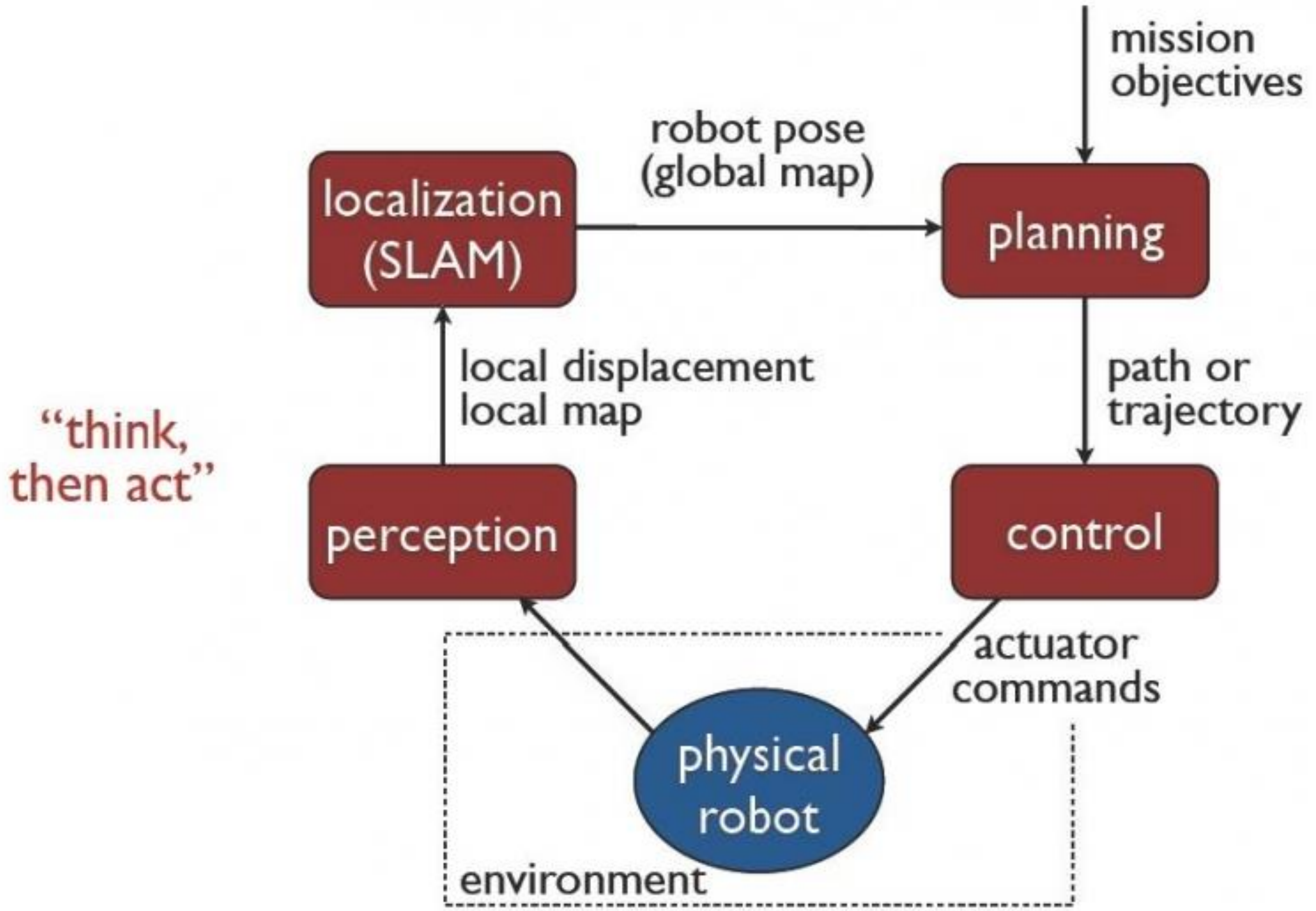
- Do not think, (re-)act

Hybrid architecture

- Think and act simultaneously

Behavioral architecture

- Think about modality of action (behavior)



Deliberative architecture

Configuration Space and Constraints

SECTION 2

Configuration space

- Generalized coordinates

$$\boldsymbol{q} \in \mathcal{C}$$

- Locally $\boldsymbol{q} \in \mathbb{R}^n$: the configuration space is a *manifold*

Constraints

Bilateral (expressed as equalities)
Scleronomic (not depending on time)
Holonomic (integrable)

$$h_i(\boldsymbol{q}) = 0 \qquad i = 1, \ldots, k < n$$

- $n - k$ accessible configurations
- k generalized coordinates as a function of the remaining $n - k$ (implicit function theorem)

$$\Downarrow$$

$$\frac{dh_i(\boldsymbol{q})}{dt} = \frac{\partial h_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = 0 \qquad i = 1, \ldots, k$$

Holonomic vs Nonholonomic Constraints

Kinematic constraints (positions + velocities)

$$a_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$$

- constrain the instantaneous admissible motion of the mechanical system by reducing the set of generalized velocities that can be attained at each configuration

Linear in the velocities (*Pfaffian form*)

$$\boldsymbol{a}_i^T(\boldsymbol{q})\dot{\boldsymbol{q}} = 0 \quad i = 1, \ldots, k < n \qquad \Longrightarrow \qquad \boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}$$

- if all integrable then *holonomic*, else *nonholonomic*

Holonomic constraint

$$a^T(q)\dot{q} = 0 \quad \implies \quad h(q) = c$$

- motion confined to level surface of h of dimension $n-1$

Nonholonomic constraint

- velocities constrained to null space of $a^T(q)$ of dimension $n-1$

In general n-dimensional mechanical system subject to k nonholonomic constraints

- can access its whole configuration space C
- at any configuration its velocities must belong to $(n-k)$-dimensional space

Case Study

SECTION 3

Disk rolling on a plane (without slipping)

CASE STUDY 1

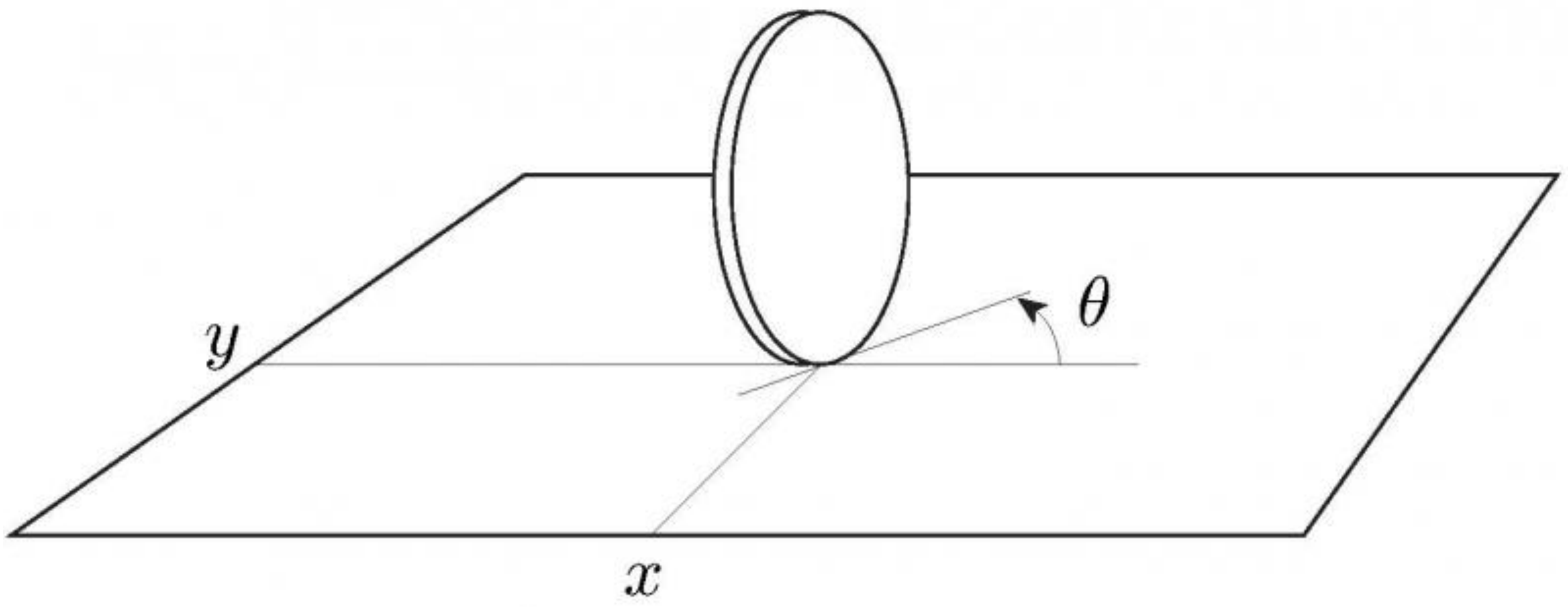
Disk rolling on a plane (without slipping)

Configuration vector $\boldsymbol{q} = [\ x \quad y \quad \theta\ ]^T$

Pure rolling constraint in Pfaffian form

$$\dot{x} \sin\theta - \dot{y} \cos\theta = [\ \sin\theta \quad -\cos\theta \quad 0\ ]\, \dot{\boldsymbol{q}} = 0$$

- Velocity of contact point has zero component in the direction orthogonal to sagittal plane
- Angular velocity of disk around the vertical axis is unconstrained



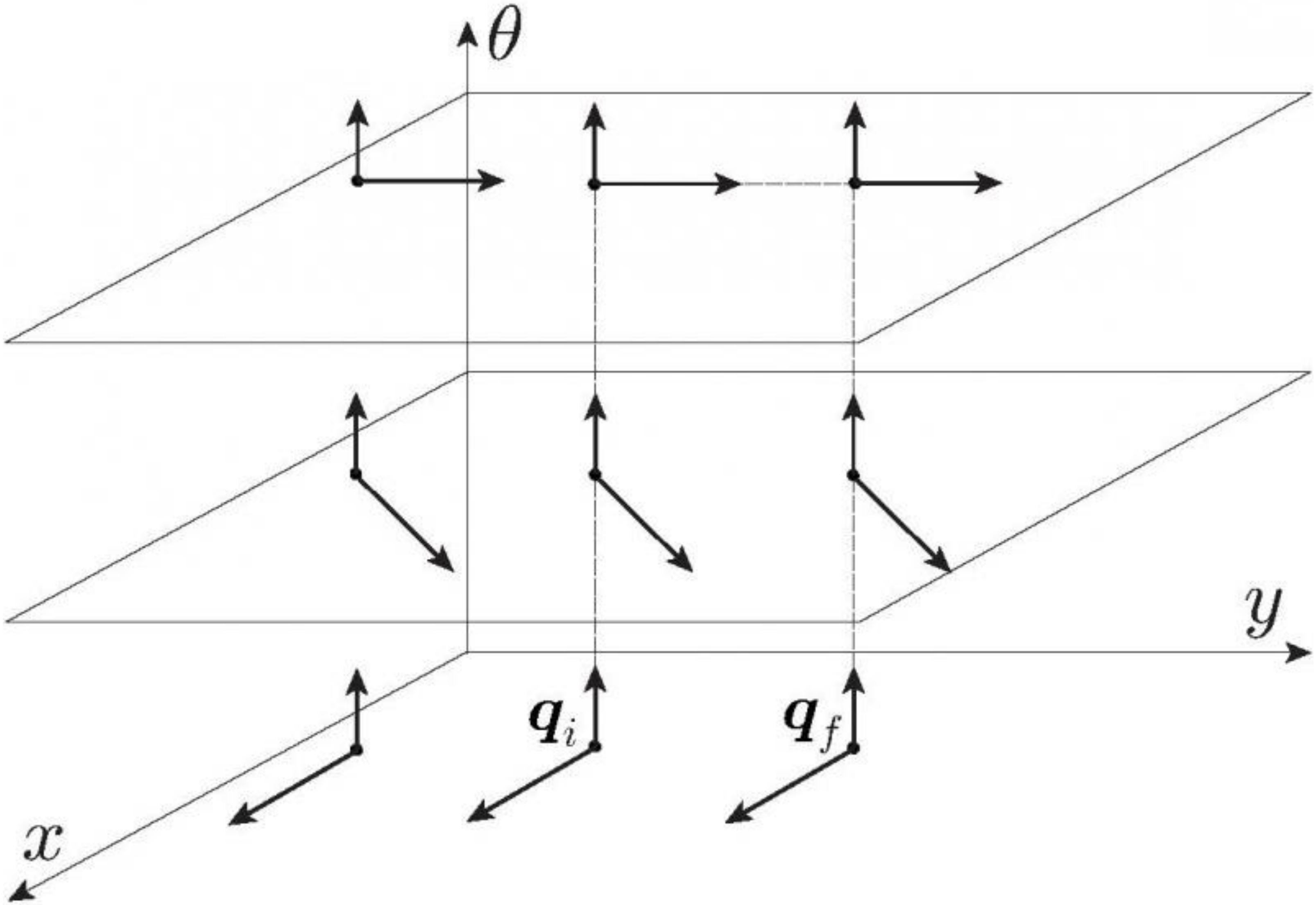
Generalized coordinates for a disk rolling on a plane

Example 2

CASE STUDY 2

Example 2

- There is no loss of accessibility
- The disk can go from any initial configuration $\boldsymbol{q}_i = [\begin{array}{ccc} x_i & y_i & \theta_i \end{array}]^T$ to any final configuration $\boldsymbol{q}_f = [\begin{array}{ccc} x_f & y_f & \theta_f \end{array}]^T$ without violating the constraint, by the sequence of three manoeuvres
    - Rotate the disk around its vertical axis so as to reach the orientation θ_v for which the sagittal axis goes through the final contact point (x_f, y_f)
    - Roll the disk on the plane at a constant orientation θ_v until the contact point reaches its final position (x_f, y_f)
    - Rotate again the disk around its vertical axis to change the orientation from θ_v to θ_f



A local representation of the configuration space for the rolling disk with an example manoeuvre that transfers from the initial to the final configuration (dashed line)

Integrability Conditions

In the presence of Pfaffian kinematic constraints, integrability conditions can be used to decide whether the system is holonomic or nonholonomic

- Single Pfaffian constraint $\boldsymbol{a}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \sum_{j=1}^{n} a_j(\boldsymbol{q})\dot{q}_j = 0$
- Integrability: $\exists h(\boldsymbol{q}), \gamma(\boldsymbol{q}) \neq 0 : \gamma(\boldsymbol{q}) a_j(\boldsymbol{q}) = \frac{\partial h(\boldsymbol{q})}{\partial q_j} \quad j = 1, \ldots, n$
- Conversely: if $\exists$ an integrating factor $\gamma(\boldsymbol{q}) \neq 0$ such that $\gamma(\boldsymbol{q})\boldsymbol{a}(\boldsymbol{q})$ is the gradient of a scalar function $h(\boldsymbol{q})$, then the constraint is integrable

Integrability Conditions

- An equivalent condition not containing the unknown function $h(\boldsymbol{q})$ is obtained by using Schwarz theorem on the symmetry of second derivatives

It is sufficient to observe that $\frac{\partial(\gamma a_k)}{\partial q_j} = \frac{\partial(\gamma a_j)}{\partial q_k} \qquad j, k = 1, \ldots, n, \quad j \neq k$

- A different approach is needed in case of multiple Pfaffian constraints

$$\frac{\partial \gamma a_j}{\partial q_k} = \frac{\partial^2 h}{\partial q_k \partial q_j} = \frac{\partial^2 h}{\partial q_j \partial q_k} = \frac{\partial \gamma a_k}{\partial q_j}$$

Disk Rolling on a Plane

- Pure rolling constraint $\dot{x}\sin\theta - \dot{y}\cos\theta = \begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix} \dot{\mathbf{q}} = 0$

- Applying integrability condition

$$\sin\theta \frac{\partial\gamma}{\partial y} = -\cos\theta \frac{\partial\gamma}{\partial x}$$
$$\cos\theta \frac{\partial\gamma}{\partial\theta} = \gamma \sin\theta$$
$$\sin\theta \frac{\partial\gamma}{\partial\theta} = -\gamma \cos\theta$$

Disk Rolling on a Plane

- Squaring and adding the last two equations gives $\partial\gamma/\partial\theta = \pm\gamma$

$$\Longrightarrow \quad \begin{aligned} \gamma\cos\theta &= \gamma\sin\theta \\ \gamma\sin\theta &= -\gamma\cos\theta \end{aligned}$$

whose only solution is $\gamma = 0 \implies$ Nonholonomic constraint!

Kinematic Model for Mobile Robots

SECTION 4

System of Pfaffian constraints

$$A^T(q)\dot{q} = 0$$

- The admissible velocities are those belonging to $\mathcal{N}(A^T(q))$ which has dimension $n-k$
  - Basis $\{g_1(q), \ldots, g_{n-k}(q)\}$
- Solutions of nonlinear dynamic system

$$\dot{q} = \sum_{j=1}^{m} g_j(q)u_j = G(q)u \qquad m = n-k$$

System of Pfaffian constraints

This equation can be regarded as that of a nonlinear first-order dynamic system with state $\boldsymbol{q} \in \mathbb{R}^n$ and control input $\boldsymbol{u} \in \mathbb{R}^m$

- The solutions of the system are the trajectories $\boldsymbol{q}(t)$ executed by the constrained mechanical system
- Infinite possible choices exist for $\boldsymbol{G}(\boldsymbol{q})$, while $\boldsymbol{u}$ might have a different meaning from control input

Kinematic model $\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{u}$

- Driftless system $\boldsymbol{u} = 0 \Longrightarrow \dot{\boldsymbol{q}} = 0$

Kinematic Model II

- It can be shown that nonholonomy of kinematic constraint is equivalent to controllability of kinematic model (nonlinear system)
  - If the system is controllable, given two arbitrary configurations $\boldsymbol{q}_i, \boldsymbol{q}_f \in \mathcal{C}$, there exists a choice of $\boldsymbol{u}(t)$ that steers the system from $\boldsymbol{q}_i$ to $\boldsymbol{q}_f$, i.e. there exists a trajectory $\boldsymbol{q}(t)$ joining the two configurations and satisfying the kinematic constraints
  - If the system is not controllable, the kinematic constraints reduce the set of accessible configurations in $\mathcal{C}$, depending on the dimension ν of the accessible subspace

  $m < \nu < n \Longrightarrow$ constraints are only partially integrable (nonholonomic system)

  $m = \nu \Longrightarrow$ constraints are completely integrable (holonomic system)

Kinematic Model II

- Nonholonomy of Pfaffian constraints can be established by analyzing controllability of the associated kinematic model
    - *Accessibility rank condition* (controllability)

$$\dim \Delta_{\mathcal{A}}(\boldsymbol{q}) = n$$

$\Delta_{\mathcal{A}}$: accessibility distribution associated with system, involutive closure of $\Delta = \mathrm{span}\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_m\}$

Unicycle

SECTION 5

Unicycle

Vehicle with single orientable wheel

- Generalized coordinates: $\boldsymbol{q} = [\ x \quad y \quad \theta\ ]^T$
- Pure rolling constraint for the wheel

$$\dot{x}\sin\theta - \dot{y}\cos\theta = [\ \sin\theta \quad -\cos\theta \quad 0\ ]\,\dot{\boldsymbol{q}} = 0$$

velocity of contact point is zero in the direction orthogonal to the sagittal axis of the vehicle (*zero motion line*)

$$\boldsymbol{G}(\boldsymbol{q}) = [\ \boldsymbol{g}_1(\boldsymbol{q}) \quad \boldsymbol{g}_2(\boldsymbol{q})\ ] = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}$$

Unicycle II

Kinematic model

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

- *Driving velocity* v : angular speed of the wheel around its horizontal axis multiplied by the wheel radius
- *Steering velocity* ω : wheel angular speed around the vertical axis

Unicycle II

- Lie Bracket $$[\boldsymbol{g}_1, \boldsymbol{g}_2](\boldsymbol{q}) = \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix}$$

always linearly independent from $\boldsymbol{g}_1(\boldsymbol{q}), \boldsymbol{g}_2(\boldsymbol{q})$

$$\Downarrow$$

$$\dim \Delta_{\mathcal{A}} = \dim \Delta_2 = \dim \operatorname{span}\{\boldsymbol{g}_1, \boldsymbol{g}_2, [\boldsymbol{g}_1, \boldsymbol{g}_2]\} = 3$$

- The unicycle is controllable with degree of nonholonomy $\kappa = 2$

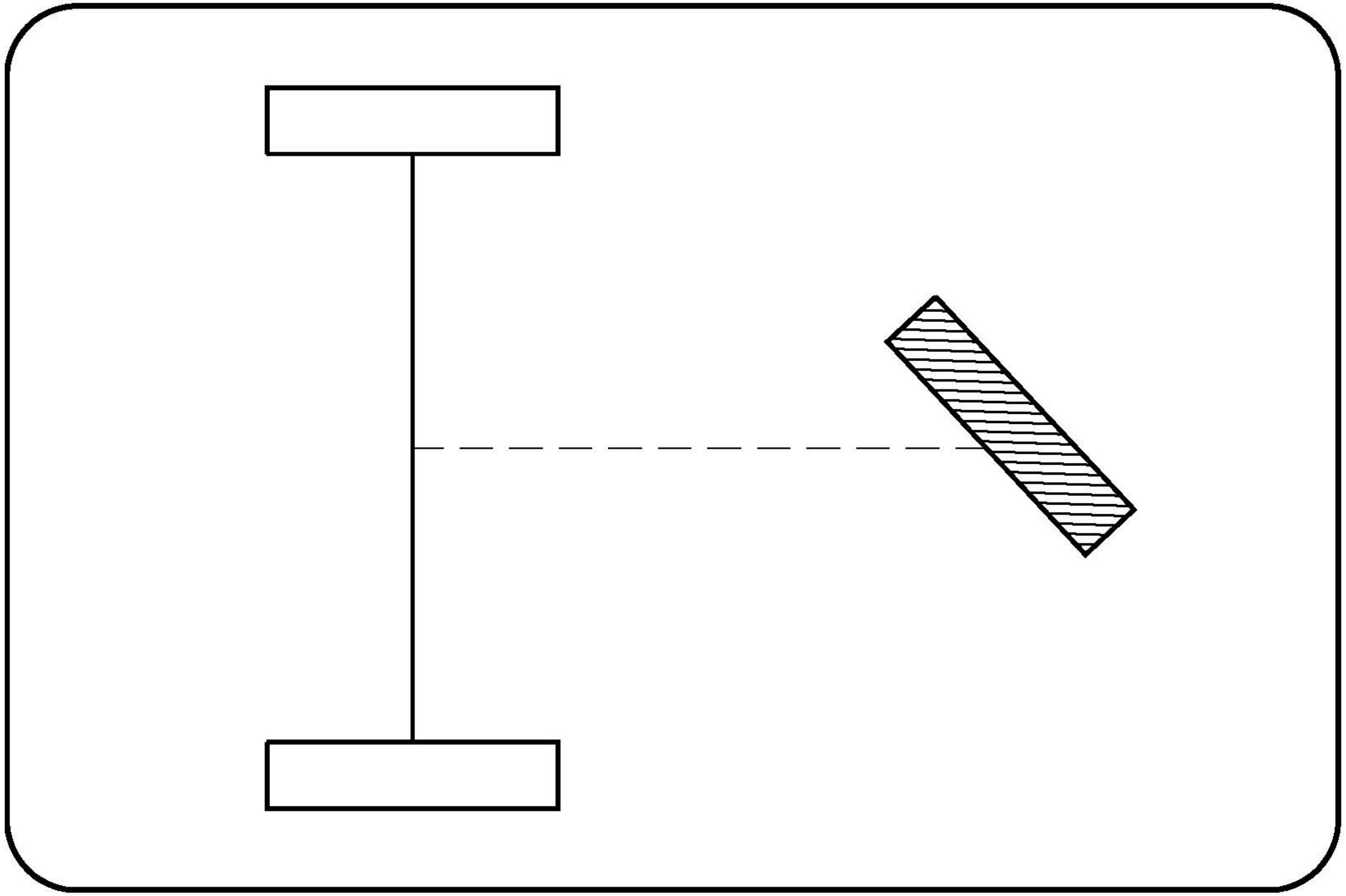
Kinematically Equivalent Vehicles

No real robot has single wheel for obvious problems of mechanical stability

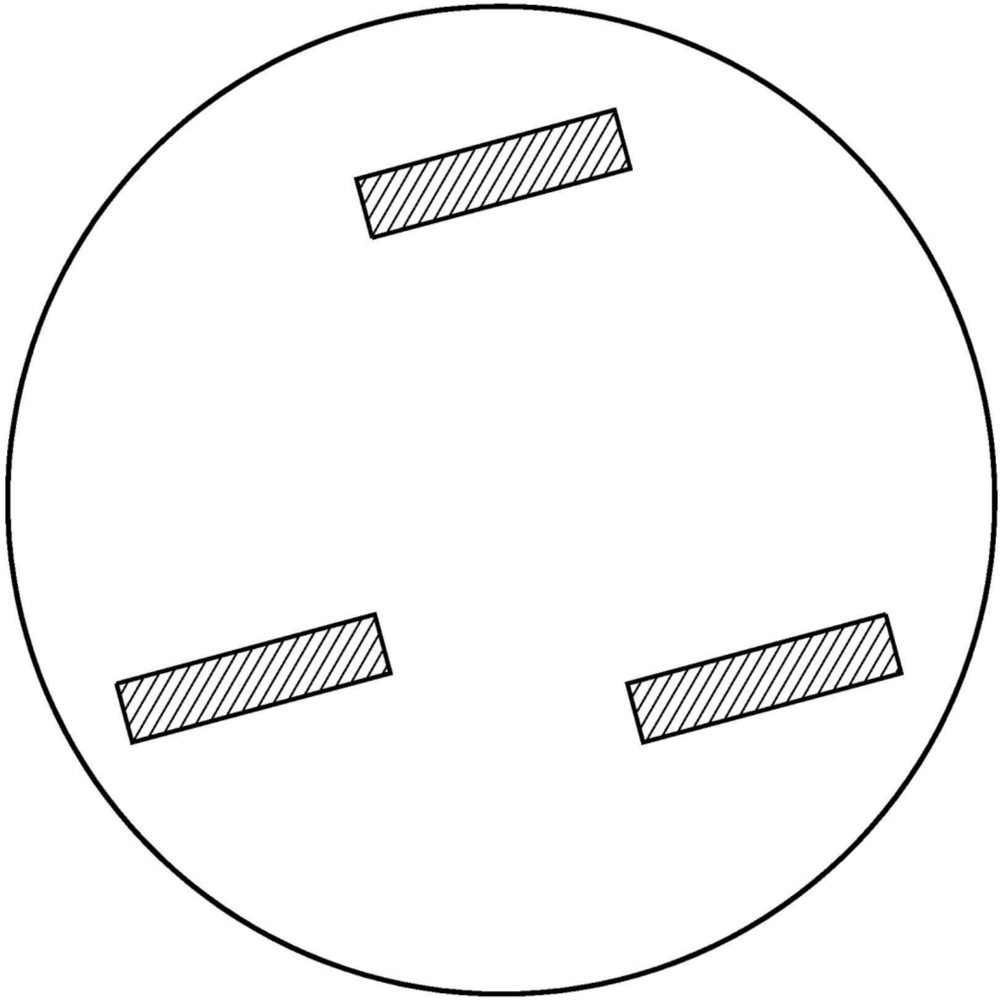
- *Differential drive* vehicle

$$v = \frac{r\left(\omega_R + \omega_L\right)}{2} \qquad \omega = \frac{r\left(\omega_R - \omega_L\right)}{d}$$

  - r : radius of the wheels
  - d : distance between their centres
  - ω_L : angular speed of left wheel
  - ω_R : angular speed of right wheel
- *Synchro drive* vehicle
  - v, ω : common to the three orientable wheels
  - (x, y) : any point of the robot (centroid)
  - θ : common orientation of the wheels



A differential-drive mobile robot



A synchro-drive mobile robot

Bicycle

SECTION 6

Vehicle having an orientable wheel and a fixed wheel

- Generalized coordinates: $\boldsymbol{q} = [\ x \quad y \quad \theta \quad \phi\ ]^T$
  - (x, y) : Cartesian coordinates of contact point between rear wheel and ground
  - θ : orientation of vehicle with respect to x axis
  - ϕ : steering angle of front wheel with respect to vehicle
- Two pure rolling constraints, one for each wheel

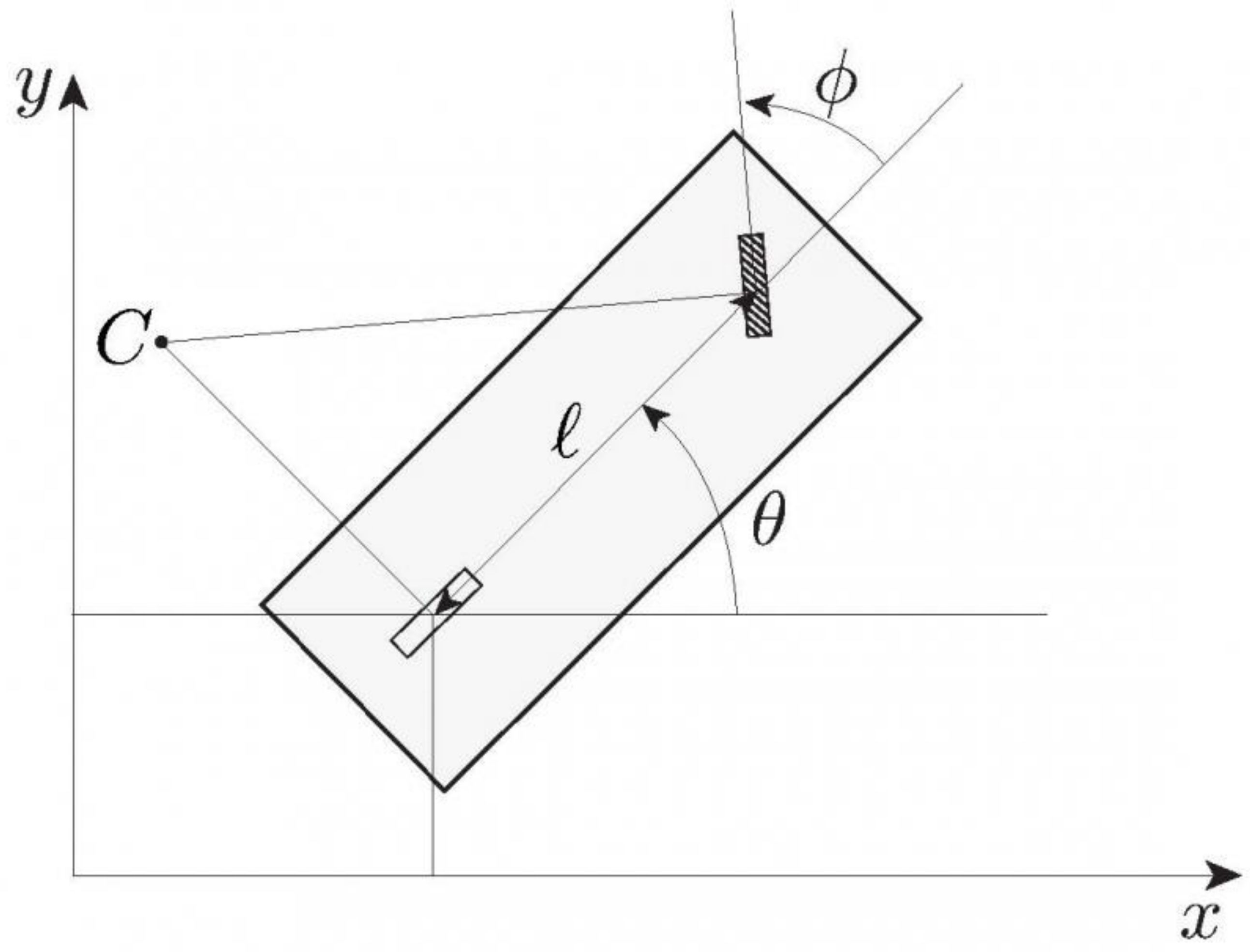
$$\dot{x}_f \sin(\theta + \phi) - \dot{y}_f \cos(\theta + \phi) = 0$$

$$\dot{x} \sin\theta - \dot{y} \cos\theta = 0$$

- Cartesian position of centre of front wheel

$$x_f = x + \ell \cos\theta$$

$$y_f = y + \ell \sin\theta$$



Generalized coordinates and instantaneous centre of rotation for a bicycle

Bicycle II

Pfaffian constraints $A^T(q) = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -\ell\cos\phi & 0 \end{bmatrix}$

- Constant rank $k = 2 \Longrightarrow$ admissible velocities as a linear combination of basis of

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi/\ell \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2 \qquad \mathcal{N}(A^T(q)) \quad (n-k=2)$$

- Orientable front wheel $\Longrightarrow u_2 = \omega$ (*steering* velocity)

Front-wheel Drive

$u_1 = v$: *driving* velocity

- Kinematic model

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \cos\phi \\ \sin\theta \cos\phi \\ \sin\phi/\ell \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega = \boldsymbol{g}_1(\boldsymbol{q})v + \boldsymbol{g}_2(\boldsymbol{q})\omega$$

$$\boldsymbol{g}_3(\boldsymbol{q}) = [\boldsymbol{g}_1, \boldsymbol{g}_2](\boldsymbol{q}) = \begin{bmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ -\cos\phi/\ell \\ 0 \end{bmatrix} \qquad \boldsymbol{g}_4(\boldsymbol{q}) = [\boldsymbol{g}_1, \boldsymbol{g}_3](\boldsymbol{q}) = \begin{bmatrix} -\sin\theta/\ell \\ \cos\theta/\ell \\ 0 \\ 0 \end{bmatrix}$$

both linearly independent from $\boldsymbol{g}_1, \boldsymbol{g}_2$

$$\Longrightarrow \dim \Delta_4 = \dim \Delta_3 = \dim \operatorname{span}\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4\} = 4$$

The front-wheel drive bicycle is controllable with degree of nonholonomy $\kappa = 3$

Rear-wheel Drive

$u_1 = v/\cos\phi$ (v *driving* velocity of rear wheel)

- Kinematic model ($\phi \neq \pm\pi/2$)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/\ell \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega \qquad g_3(q) = [g_1, g_2](q) = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\ell\cos^2\phi} \\ 0 \end{bmatrix} \qquad g_4(q) = [g_1, g_3](q) = \begin{bmatrix} -\frac{\sin\theta}{\ell\cos^2\phi} \\ \frac{\cos\theta}{\ell\cos^2\phi} \\ 0 \\ 0 \end{bmatrix}$$

linearly independent from $g_1(q), g_2(q)$

$\Downarrow$

The rear-wheel drive bicycle is controllable with degree of $\kappa = 3$ nonholonomy

Kinematically Equivalent Vehicles (Mechanically Balanced)

SECTION 7

Tricycle

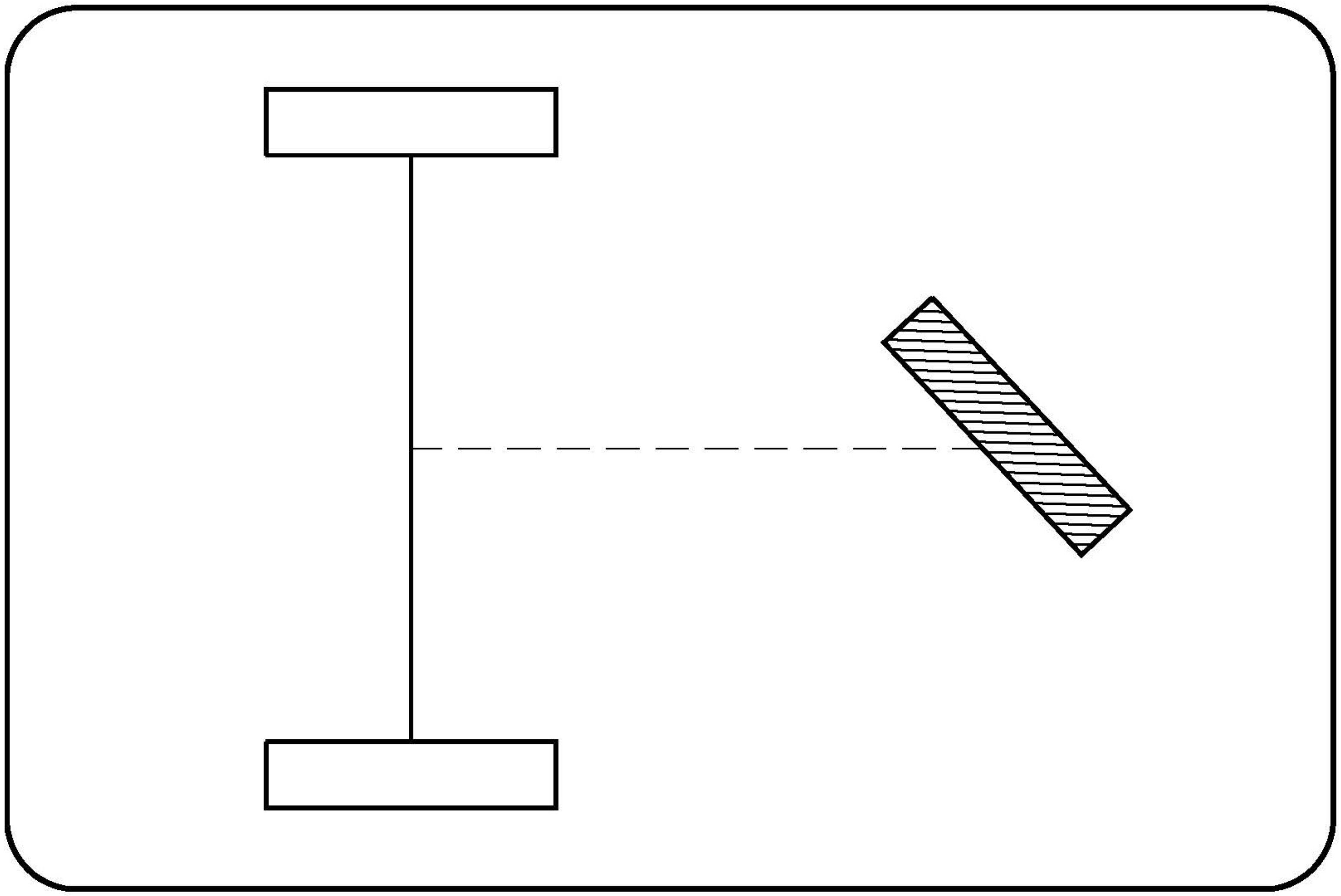
Car-like

- Same kinematic models (front-wheel or rear-wheel drive)

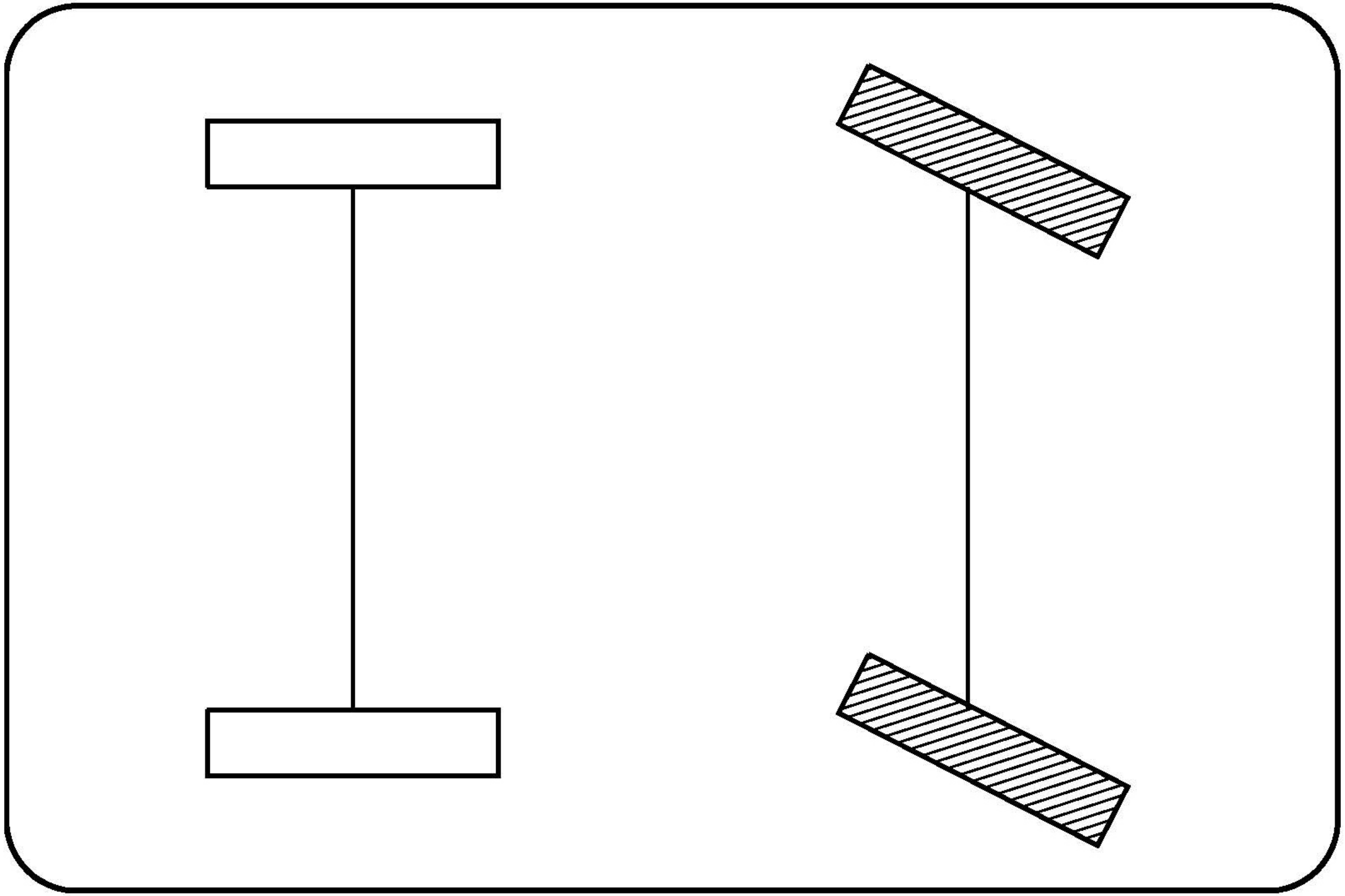
  (x, y): Cartesian coordinates of midpoint of rear wheel axle

  θ : orientation of vehicle

  ϕ: steering angle



A tricycle mobile robot



A car-like mobile robot

Lagrange Formulation

SECTION 8

Lagrange Formulation

Similar to robot manipulators, accounting for nonholonomy constraints on generalized velocities $\Longrightarrow$ No full linearization by state feedback!

n-dimensional mechanical system subject to $k < n$ Pfaffian constraints

- Lagrangian

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} - \mathcal{U}(\boldsymbol{q})$$

$$\Downarrow$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\right)^T = \boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau} + \boldsymbol{A}(\boldsymbol{q})\boldsymbol{\lambda}$$

Lagrange Formulation

$\boldsymbol{S}(\boldsymbol{q})$ is an $(n \times m)$ matrix mapping the $m = n - k$ external inputs $\boldsymbol{\tau}$ to generalized forces performing work on $\boldsymbol{q}$

- Dynamic model of constrained mechanical system

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau} + \boldsymbol{A}(\boldsymbol{q})\boldsymbol{\lambda}$$

$$\boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}$$

$$\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{q}}\left(\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)\right)^T + \left(\frac{\partial \mathcal{U}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T$$

Lagrange Formulation II

$$G(q) : A^T(q)G(q) = 0$$

$$\dot{q} = G(q)v = \sum_{i=1}^{m} g_i(q)\, v_i$$

$$G^T(q)\left(B(q)\ddot{q} + n(q, \dot{q})\right) = G^T(q)S(q)\tau$$

- *Reduced* dynamic model

Lagrange Formulation II

- Differentiating kinematic model and inserting into reduced dynamic model yields $\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{v}} + \boldsymbol{m}(\boldsymbol{q}, \boldsymbol{v}) = \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau}$

$$
\begin{aligned}
\boldsymbol{M}(\boldsymbol{q}) &= \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{B}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q}) \\
\boldsymbol{m}(\boldsymbol{q}, \boldsymbol{v}) &= \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{G}}(\boldsymbol{q})\boldsymbol{v} + \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{n}(\boldsymbol{q}, \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v}) \\
\dot{\boldsymbol{G}}(\boldsymbol{q})\boldsymbol{v} &= \sum_{i=1}^{m}\left(v_i \frac{\partial \boldsymbol{g}_i}{\partial \boldsymbol{q}}(\boldsymbol{q})\right)\boldsymbol{G}(\boldsymbol{q})\boldsymbol{v}
\end{aligned}
$$

- *State-space reduced* model

$$
\begin{aligned}
\dot{\boldsymbol{q}} &= \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v} \\
\dot{\boldsymbol{v}} &= \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{m}(\boldsymbol{q}, \boldsymbol{v}) + \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau}
\end{aligned}
$$

Second-order Kinematic Model of Constrained Mechanical System

SECTION 9

Second-order Kinematic Model

Partial feedback linearization

- Assumption $\det\left(G^T(q)S(q)\right) \neq 0$
- Linearizing control $$\tau = \left(G^T(q)S(q)\right)^{-1}(M(q)a + m(q,v))$$

$$\Downarrow$$

$$\dot{q} = G(q)v$$
$$\dot{v} = a$$

measurement of v required (not directly accessible)

$$v = G^\dagger(q)\dot{q} = \left(G^T(q)G(q)\right)^{-1} G^T(q)\dot{q}$$

Second-order kinematic model of constrained mechanical system

$$\dot{x} = f(x) + G(x)u = \begin{bmatrix} G(q)v \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} u$$

Unicycle Dynamic Model

SECTION 10

Unicycle Dynamic Model

m : mass

I : moment of inertia around the vertical axis through its centre

τ_1 : driving force

τ_2 : steering torque

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} + \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \lambda$$

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

$$\begin{aligned} \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \boldsymbol{0} \\ \boldsymbol{G}(\boldsymbol{q}) &= \boldsymbol{S}(\boldsymbol{q}) \\ \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q}) &= \boldsymbol{I} \\ \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{B}\dot{\boldsymbol{G}}(\boldsymbol{q}) &= \boldsymbol{0} \end{aligned}$$

- State-space reduced model

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v} \\ \dot{\boldsymbol{v}} &= \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{\tau} \end{aligned}$$

Unicycle Second-order Kinematic Model

- Linearizing control

$$\boldsymbol{\tau} = \boldsymbol{M}\boldsymbol{u} = \begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \boldsymbol{u}$$

$$\Downarrow$$

$$\dot{\xi} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$