CS65K Robotics

Modelling, Planning and Control

Chapter 2: Kinematics

Section 2.7

LECTURE 4: DIRECT KINEMATICS – MATRIX FOR ROTATIONS AND DISPLACEMENT

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Objectives

- The **homogeneous representation** of a vector is adopted
- Homogeneous transformations are introduced as a compact representation of position and orientation
- Composition of homogeneous transformations to derive the direct kinematics equation of an open-chain manipulator is illustrated
- **Denavit-Hartenberg** parameters are introduced
- A formula is derived to compute the transformation matrix from one link to the next one in a kinematic chain
- A computationally recursive operating procedure is illustrated

Objectives

- The direct kinematics equation is computed for a number of typical manipulator structures
- Composition of the kinematics of the arm with the kinematics of the wrist is presented
- The joint space and operational space concepts are illustrated

Homogeneous Coordinates

SECTION 1

Homogeneous Coordinates

- Homogeneous coordinates, introduced by August Ferdinand Möbius, make calculations of graphics and geometry possible in projective space. Homogeneous coordinates are a way of representing N-dimensional coordinates with N+1 numbers.
- To make 2D Homogeneous coordinates, we simply add an additional variable, w, into existing coordinates. Therefore, a point in Cartesian coordinates, (X, Y) becomes (x, y, w) in Homogeneous coordinates. And X and Y in Cartesian are re-expressed with x, y and w in Homogeneous as;

$$X = x/w$$

$$Y = y/w$$

Homogeneous Coordinates

- For instance, a point in Cartesian (1, 2) becomes (1, 2, 1) in Homogeneous. If a point, (1, 2), moves toward infinity, it becomes (∞,∞) in Cartesian coordinates. And it becomes (1, 2, 0) in Homogeneous coordinates, because of $(1/0, 2/0) \approx (\infty,\infty)$.
- Notice that we can express the point at infinity without using "∞".

Why is it called "homogeneous"?

- As mentioned before, in order to convert from Homogeneous coordinates (x, y, w) to Cartesian coordinates, we simply divide x and y by w;

$$\underset{\text{Homogeneous}}{(x, y, w)} \quad \Leftrightarrow \quad \underset{\text{Cartesian}}{\left(\frac{x}{w}, \frac{y}{w}\right)}$$

Why is it called "homogeneous"?

- Converting Homogeneous to Cartesian, we can find an important fact. Let's see the following example;

$$
\begin{array}{cccc}
\text{Homogeneous} & & \text{Cartesian} & \\
(1,2,3) & \Rightarrow & \left(\frac{1}{3}, \frac{2}{3}\right) & \\
(2,4,6) & \Rightarrow & \left(\frac{2}{6}, \frac{4}{6}\right) & = \left(\frac{1}{3}, \frac{2}{3}\right) \\
(4,8,12) & \Rightarrow & \left(\frac{4}{12}, \frac{8}{12}\right) & = \left(\frac{1}{3}, \frac{2}{3}\right) \\
\vdots & & \vdots & \\
(1a,2a,3a) & \Rightarrow & \left(\frac{1a}{3a}, \frac{2a}{3a}\right) & = \left(\frac{1}{3}, \frac{2}{3}\right)
\end{array}
$$

Why is it called "homogeneous"?

- As you can see, the points (1, 2, 3), (2, 4, 6) and (4, 8, 12) correspond to the same Euclidean point (1/3, 2/3). And any scalar product, (1a, 2a, 3a) is the same point as (1/3, 2/3) in Euclidean space. Therefore, these points are *"homogeneous"* because they represent the same point in Euclidean space (or Cartesian space). In other words, Homogeneous coordinates are scale invariant.

Homogeneous Transformation

SECTION 2

Homogeneous Transformation

p(x, y)

$$p = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$x' = S_x\, x$$
$$y' = S_y\, y$$

$$S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotate

$$x' = x\cos(\theta) - y\sin(\theta)$$
$$y' = x\sin(\theta) + y\cos(\theta)$$

$$S = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translate

$$x' = x + a$$
$$y' = y + b$$

$$S = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

Transformations

- A **transformation** is a process that manipulates a polygon or other two-dimensional object on a plane or coordinate system. Mathematical transformations describe how two-dimensional figures move around a plane or coordinate system.
- A **preimage** or inverse image is the two-dimensional shape before any transformation. The **image** is the figure after transformation.

Types of Transformations

There are five different transformations in math:

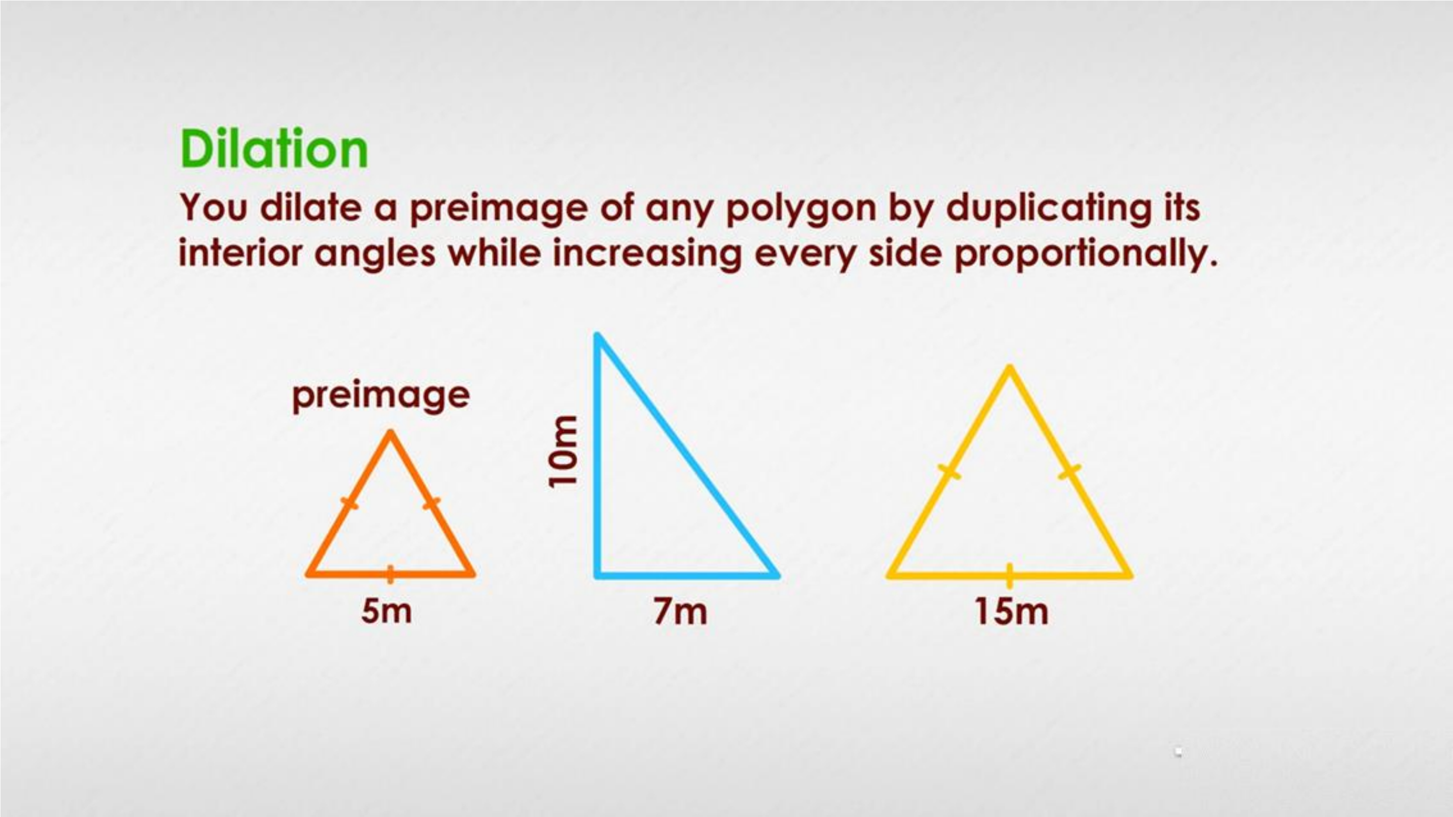
1. **Dilation** (Scaling) -- The image is a larger or smaller version of the preimage; "shrinking" or "enlarging."
2. **Reflection** -- The image is a mirrored preimage; "a flip."
3. **Rotation** -- The image is the preimage rotated around a fixed point; "a turn."
4. **Shear** -- All the points along one side of a preimage remain fixed while all other points of the preimage move parallel to that side in proportion to the distance from the given side; "a skew.,"
5. **Translation** -- The image is offset by a constant value from the preimage; "a slide."

Coordinate Change and Transformation are Inverse Operations

- A coordinate system rotate for θ is equivalent to the operation for an object to rotate for –θ.
- Therefore, the two operations are inverse operation to each other.
- Transformation:

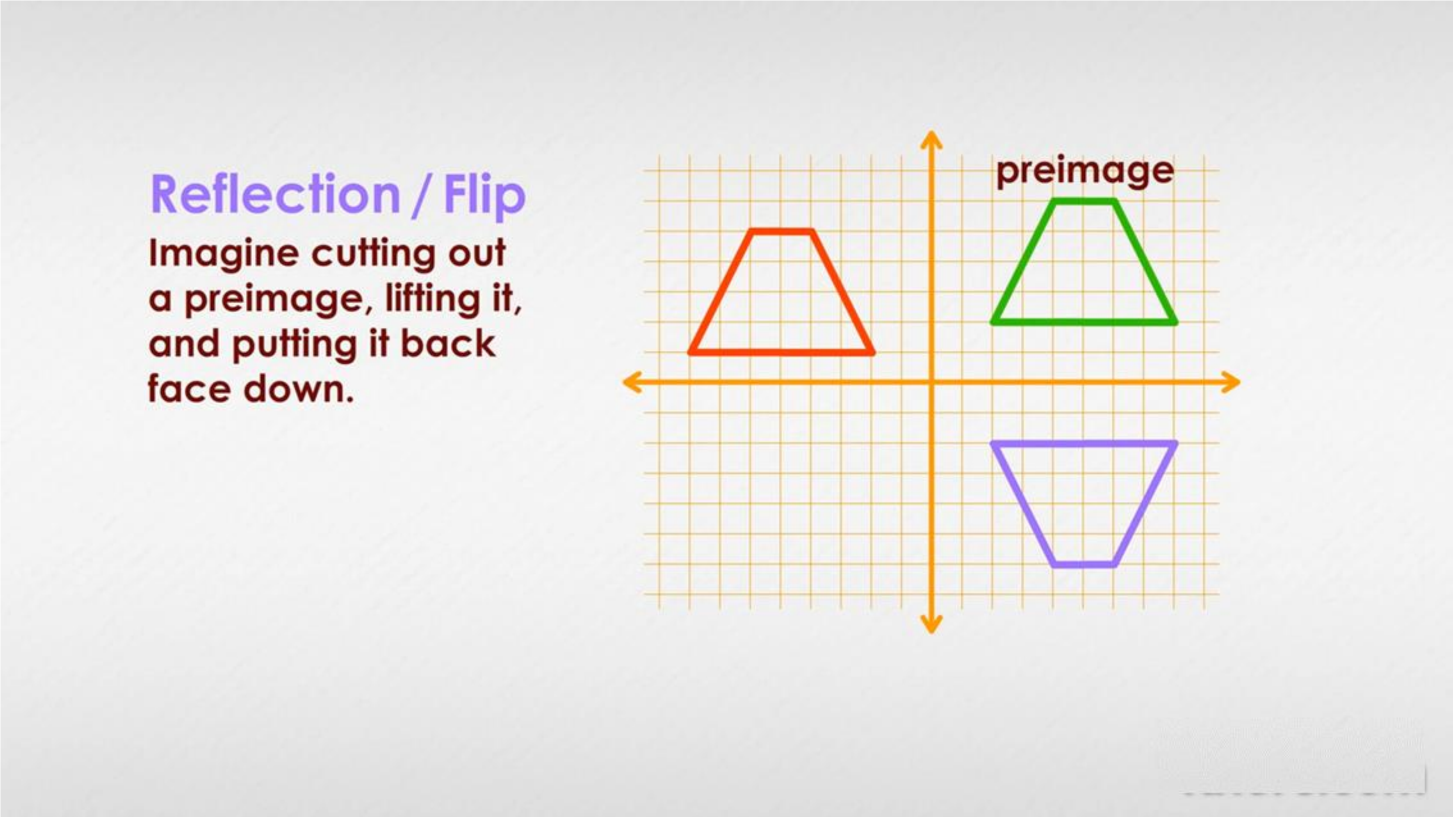
  $(x, y) \rightarrow (x', y')$

  Sometimes, I use (X, Y)



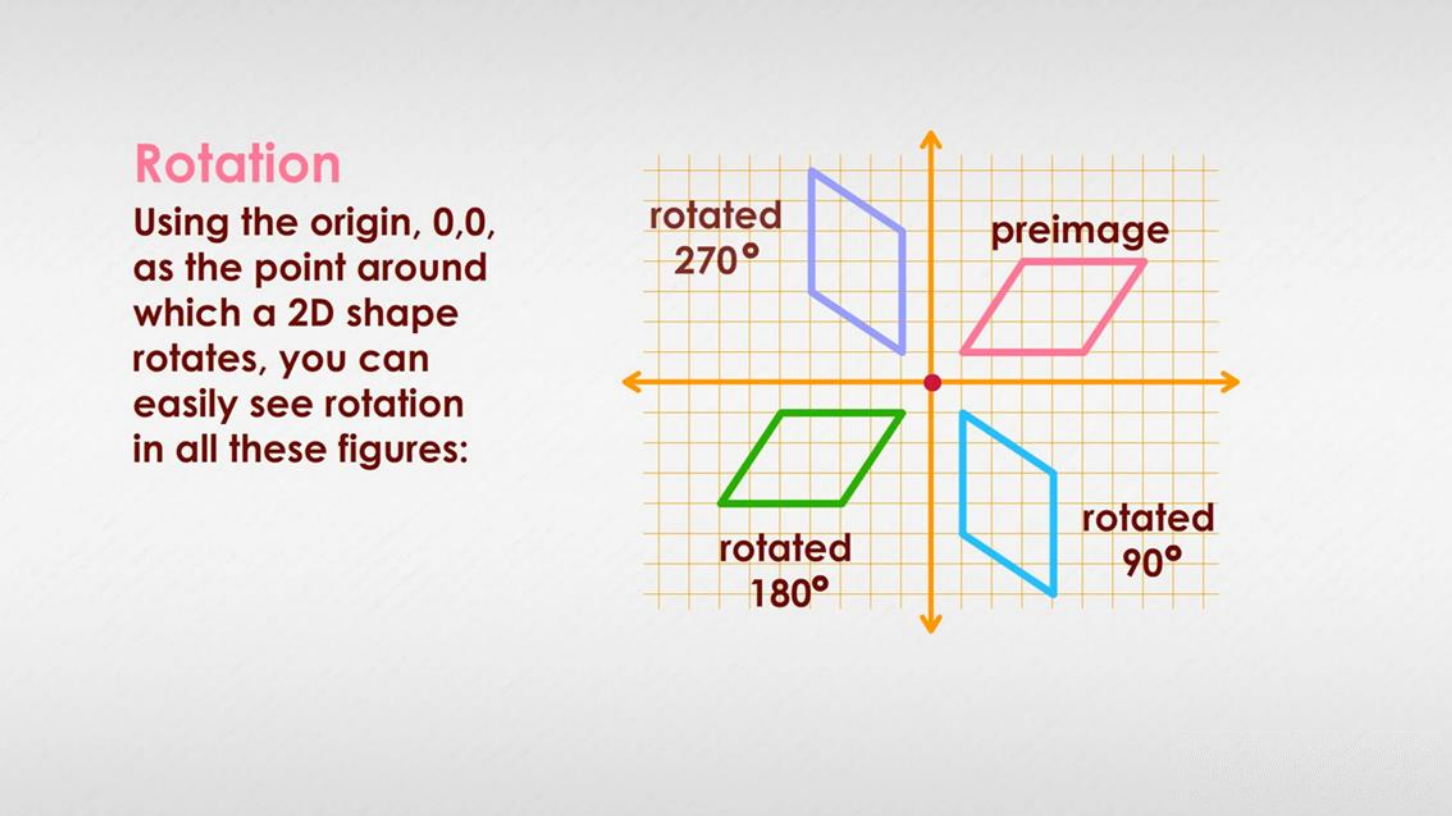
Dilation

- Dilate a preimage of any polygon is done by duplicating its interior angles while increasing every side proportionally. You can think of dilating as resizing. Which triangle image, yellow or blue, is a dilation of the orange preimage?



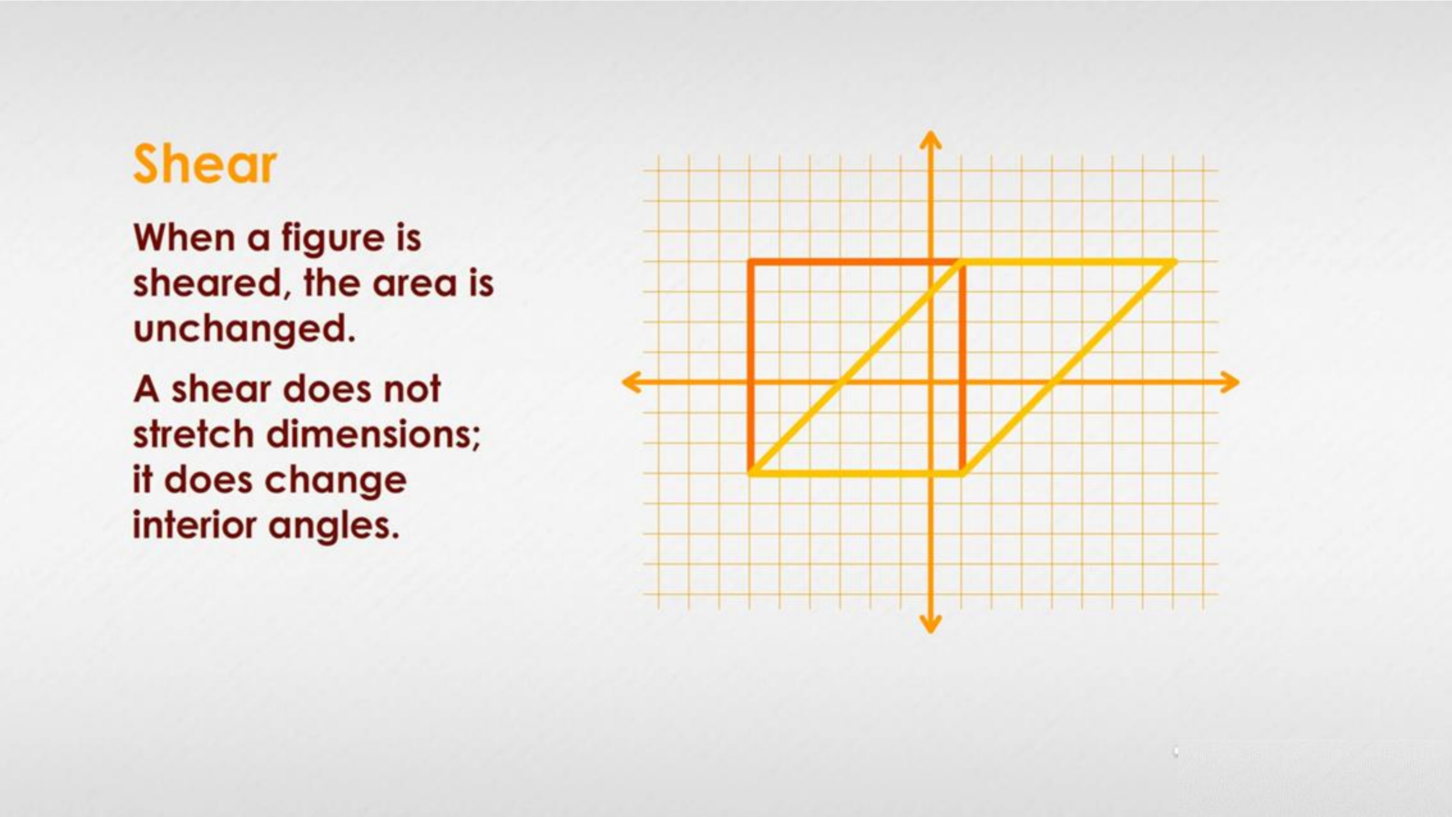
Reflection

- Imagine cutting out a preimage, lifting it, and putting it back face down. That is a reflection or a flip. A reflection image is a mirror image of the preimage. Which trapezoid image, red or purple, is a reflection of the green preimage?



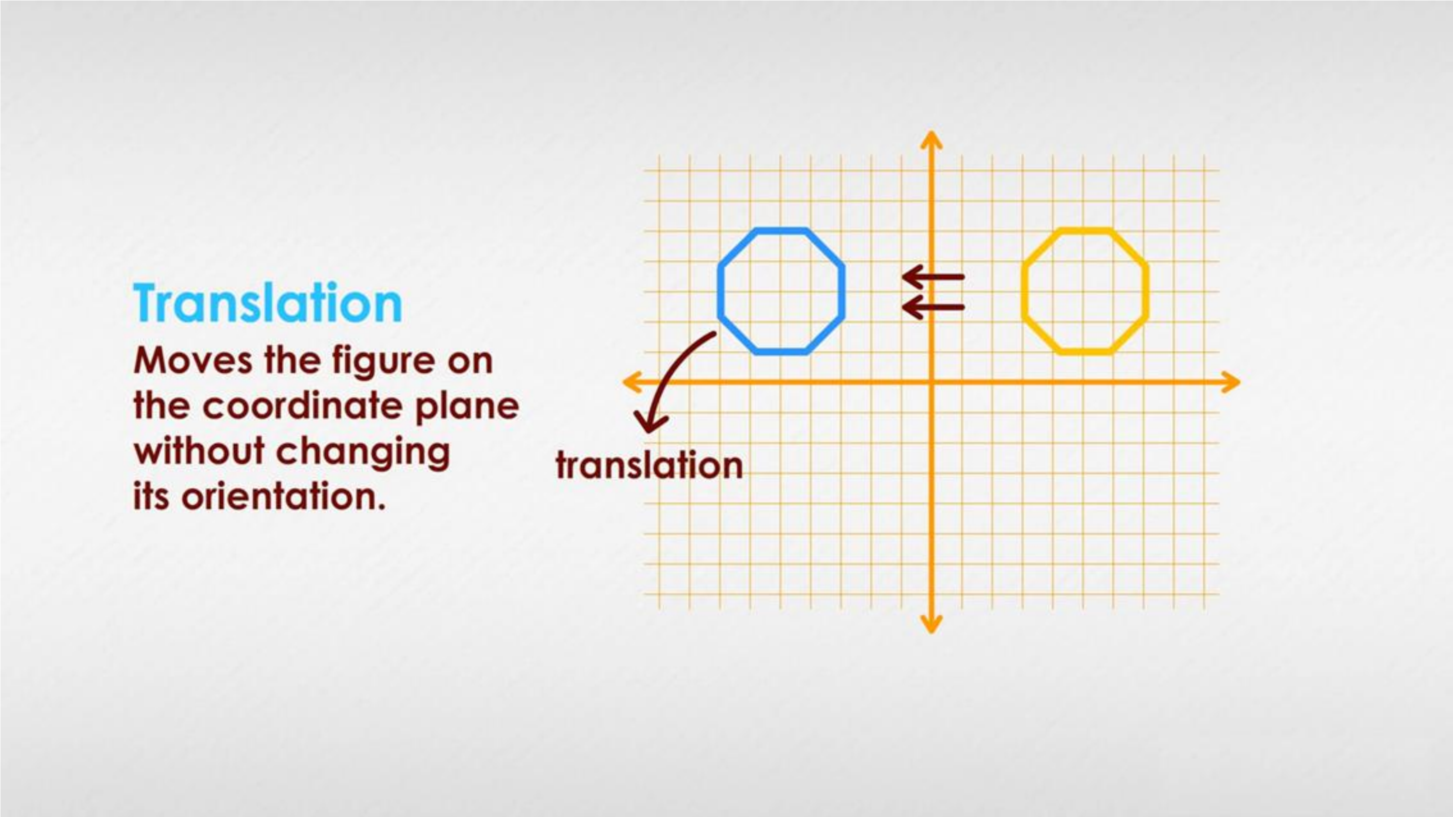
Rotation

- Using the origin, (0, 0), as the point around which a two-dimensional shape rotates, you can easily see rotation in all these figures:



Shear

- Here is a square preimage. To shear it, you "skew it," producing an image of a rhombus:



Translation

- A translation moves the figure from its original position on the coordinate plane without changing its orientation. Which octagon image below, pink or blue, is a translation of the yellow preimage?

Homogeneous Representation of a Vector

- Coordinate transformation (*translation* + *rotation*)

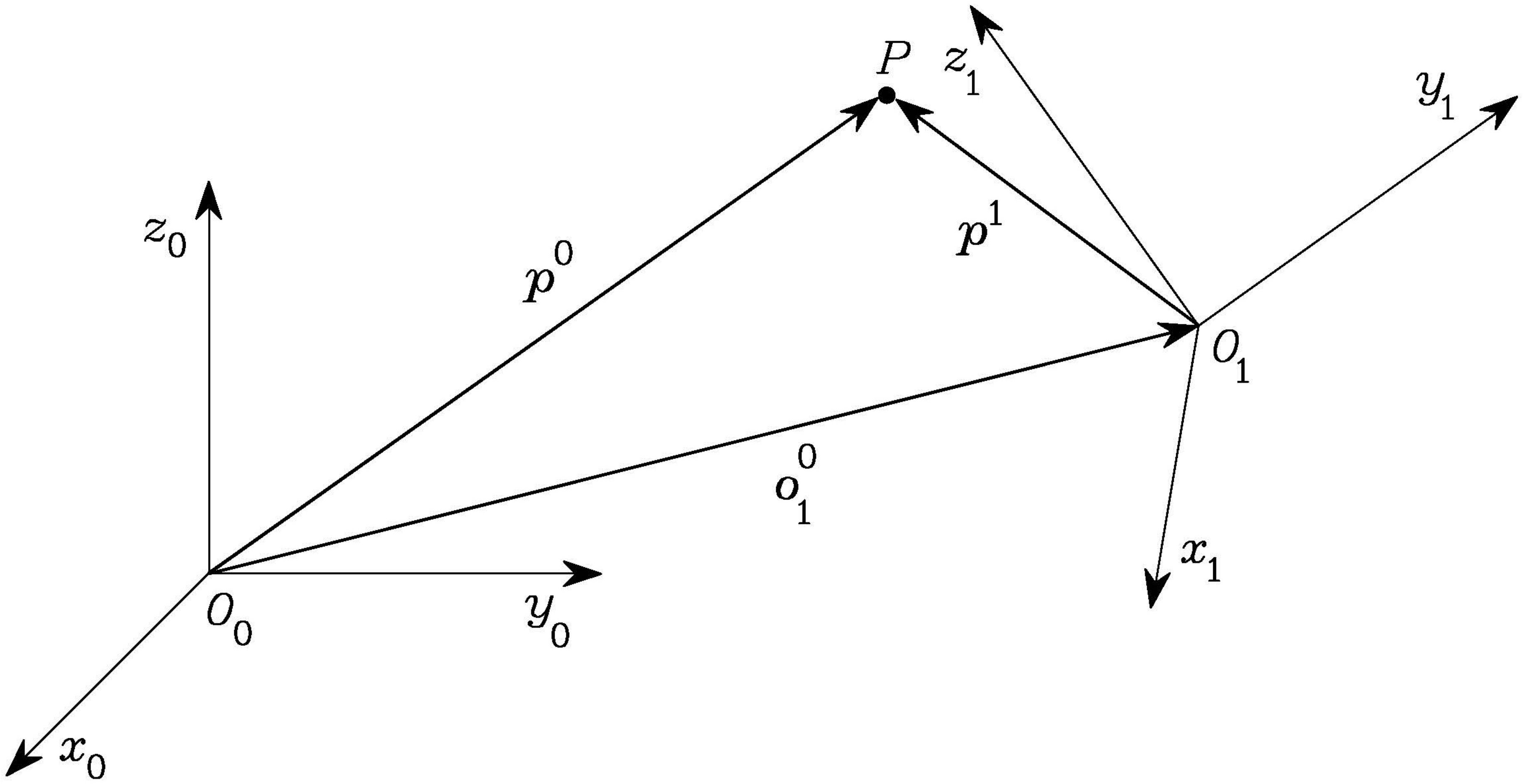
$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1$$

- Inverse transformation

$$\boldsymbol{p}^1 = -\boldsymbol{R}_0^1 \boldsymbol{o}_1^0 + \boldsymbol{R}_0^1 \boldsymbol{p}^0$$

- Homogeneous representation

$$\widetilde{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix}$$



Representation of a point in different coordinate frames

Homogeneous Transformation Matrix

$R_1^0: Rotational\ Matrix$

$o_1^0: Displacement\ Vector$

$$A_1^0 = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ 0^T & 1 \end{bmatrix}$$

Homogeneous Transformation Matrix

- Coordinate transformation

$$\widetilde{\boldsymbol{p}}^0 = \boldsymbol{A}_1^0 \widetilde{\boldsymbol{p}}^1$$

- Inverse transformation

$$\widetilde{\boldsymbol{p}}^1 = \boldsymbol{A}_0^1 \widetilde{\boldsymbol{p}}^0 = \left(\boldsymbol{A}_1^0\right)^{-1} \widetilde{\boldsymbol{p}}^0$$

with

$$\boldsymbol{A}_0^1 = \begin{bmatrix} \boldsymbol{R}_1^{0T} & -\boldsymbol{R}_1^{0T} \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_0^1 & -\boldsymbol{R}_0^1 \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

Properties

- Orthogonality does not hold

$$\boldsymbol{A}^{-1} \neq \boldsymbol{A}^T$$

- Sequence of coordinate transformations

$$\widetilde{\boldsymbol{p}}^0 = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \dots \boldsymbol{A}_n^{n-1} \widetilde{\boldsymbol{p}}^n$$

Type of Joints

SECTION 3

Manipulator

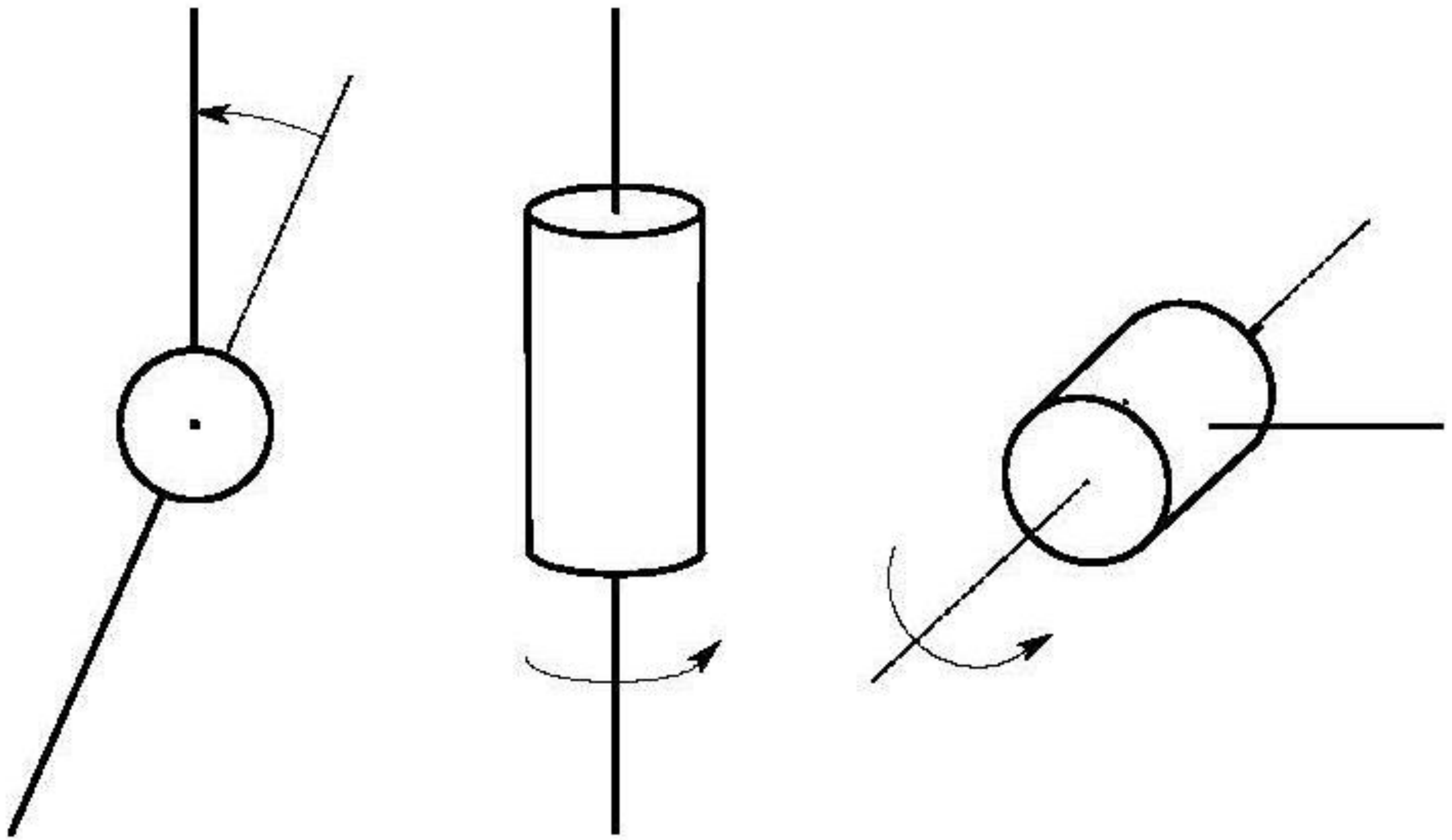
- Series of rigid bodies (*links*) connected by means of kinematic pairs or *joints*

Kinematic chain (from base to end-effector)

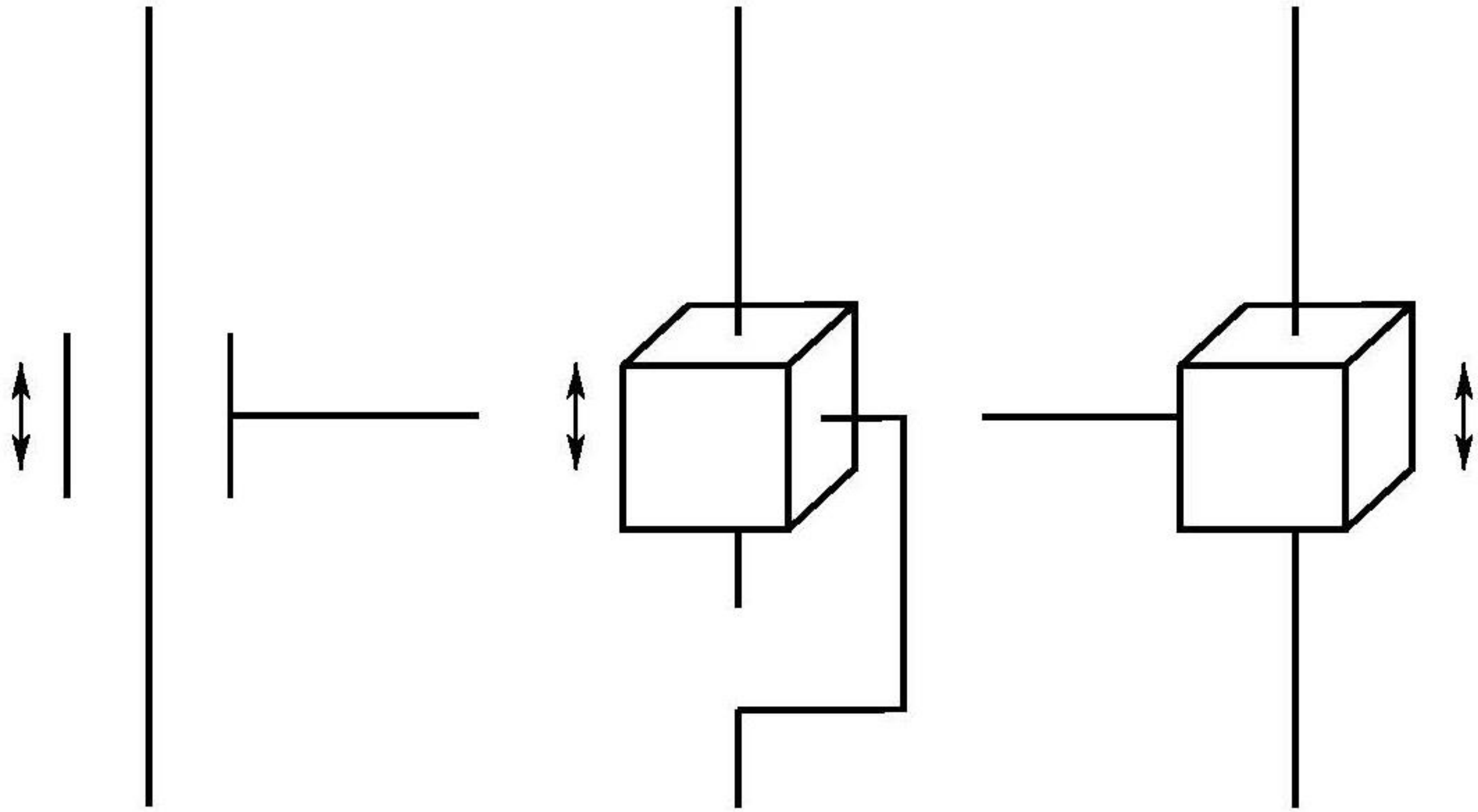
- Open (only one sequence of links connecting the two ends of the chain)
- Closed (a sequence of links forms a loop)

Degrees of freedom (DOFs) uniquely determine the manipulator's *posture*

- Each DOF is typically associated with a joint articulation and constitutes a *joint variable*



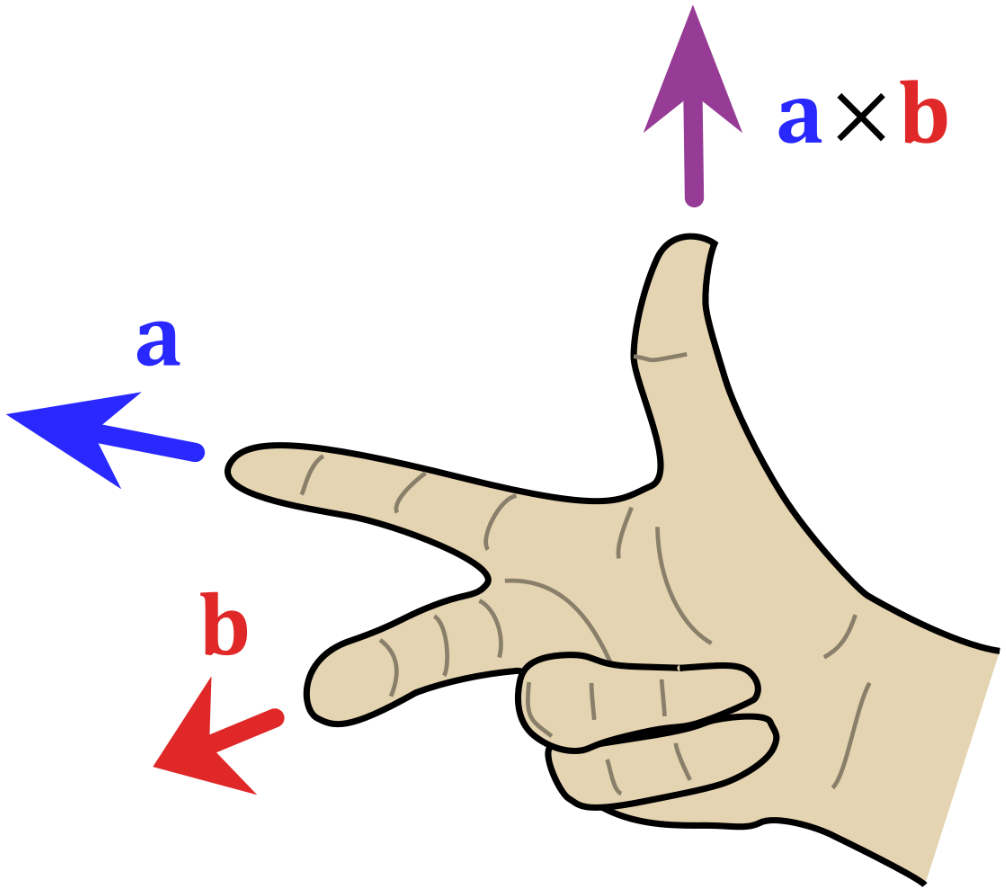
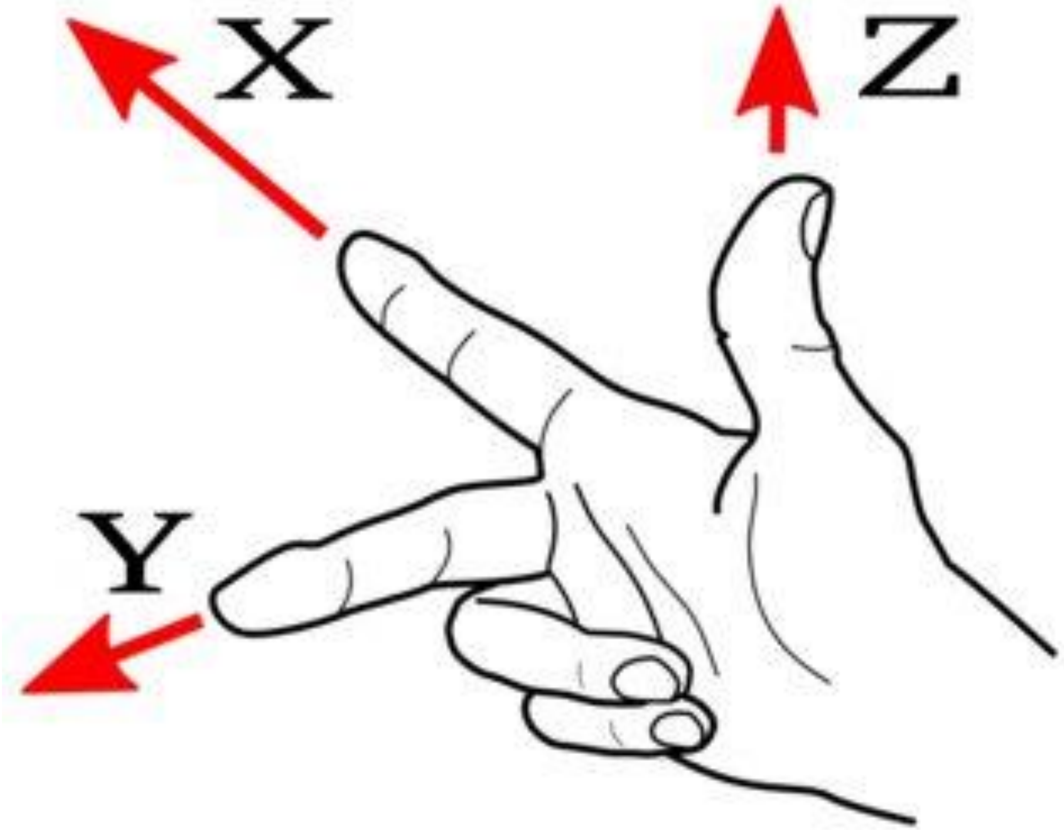
Revolute joints



Prismatic joints

Base Frame and End-effector Frame

SECTION 4


a×b
a
b

X
Z
Y

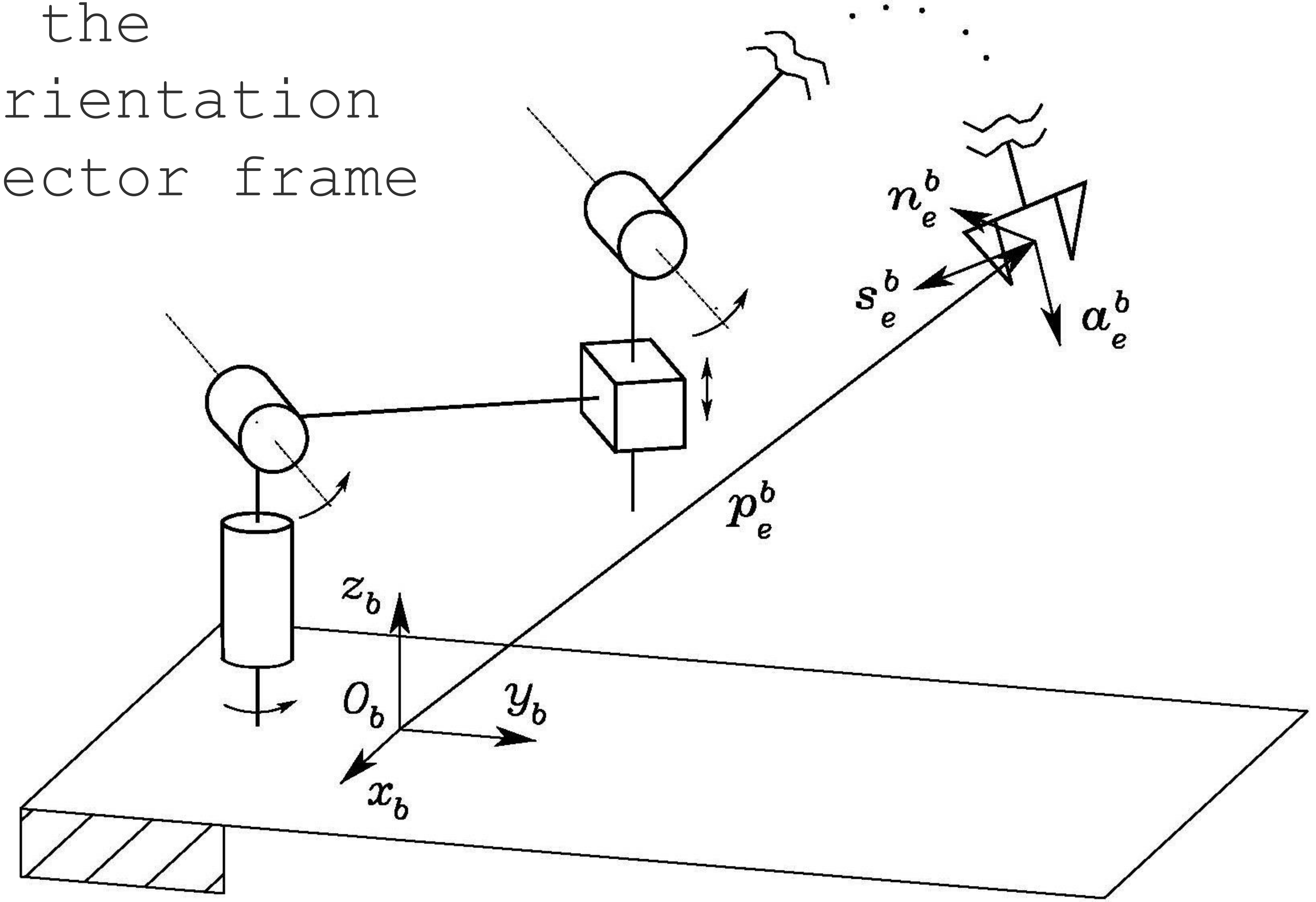
Base Frame and End-effector Frame

- Joint variables $\boldsymbol{q} = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix}^T$
- End-effector frame with respect to base frame $\boldsymbol{R}_e^b = \begin{bmatrix} \boldsymbol{n}_e^b & \boldsymbol{s}_e^b & \boldsymbol{a}_e^b \end{bmatrix}$

Direct kinematics equation

$$\boldsymbol{T}_e^b(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{n}_e^b(\boldsymbol{q}) & \boldsymbol{s}_e^b(\boldsymbol{q}) & \boldsymbol{a}_e^b(\boldsymbol{q}) & \boldsymbol{p}_e^b(\boldsymbol{q}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Description of the position and orientation of the end-effector frame



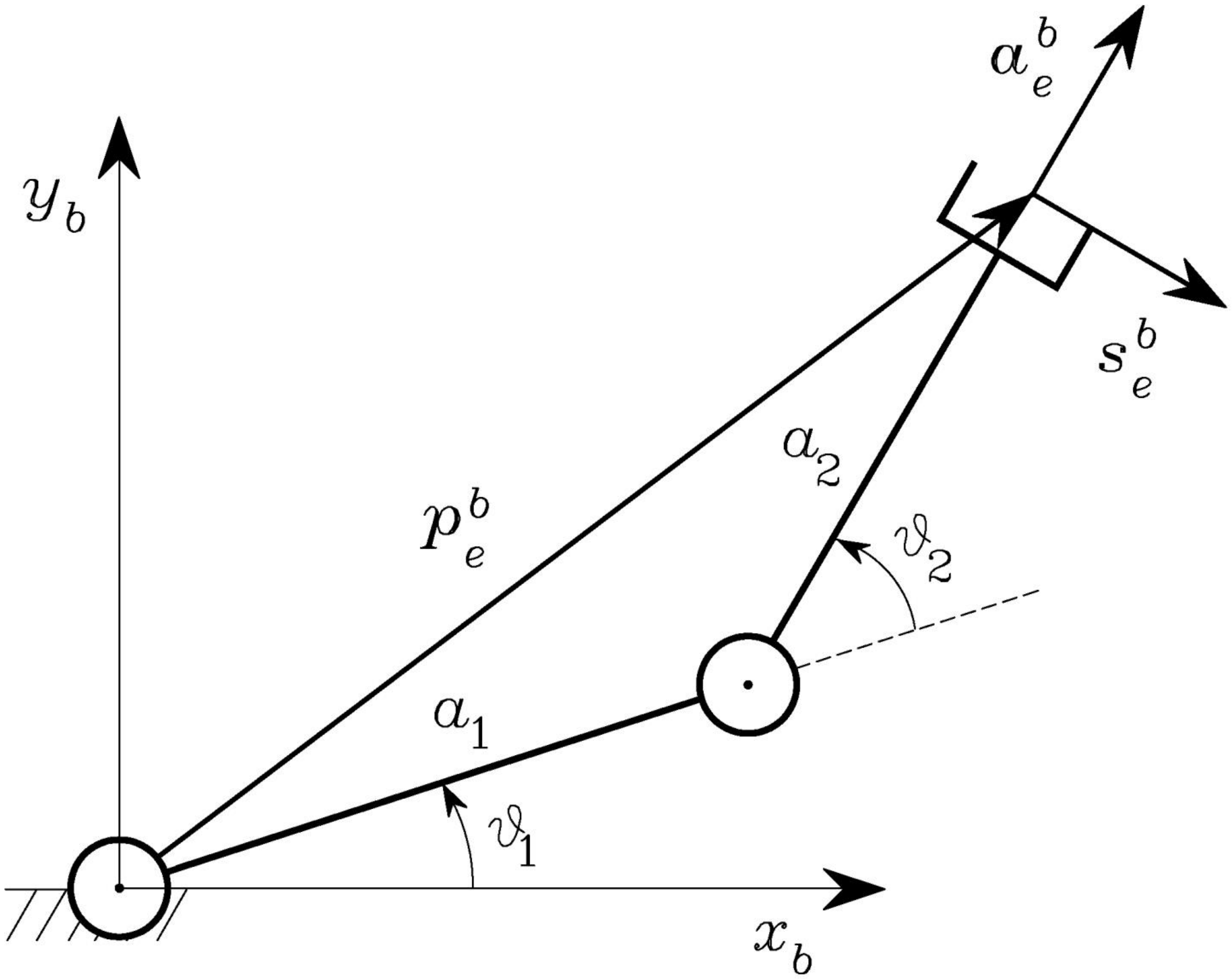
Two-link Planar Arm

SECTION 5

Two-link Planar Arm

$$T_e^b(q) = \begin{bmatrix} n_e^b & s_e^b & a_e^b & p_e^b \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & s_{12} & c_{12} & a_1 c_1 + a_2 c_{12} \\ 0 & -c_{12} & s_{12} & a_1 s_1 + a_2 s_{12} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Two-link Planar Arm

Open Chain

SECTION 6

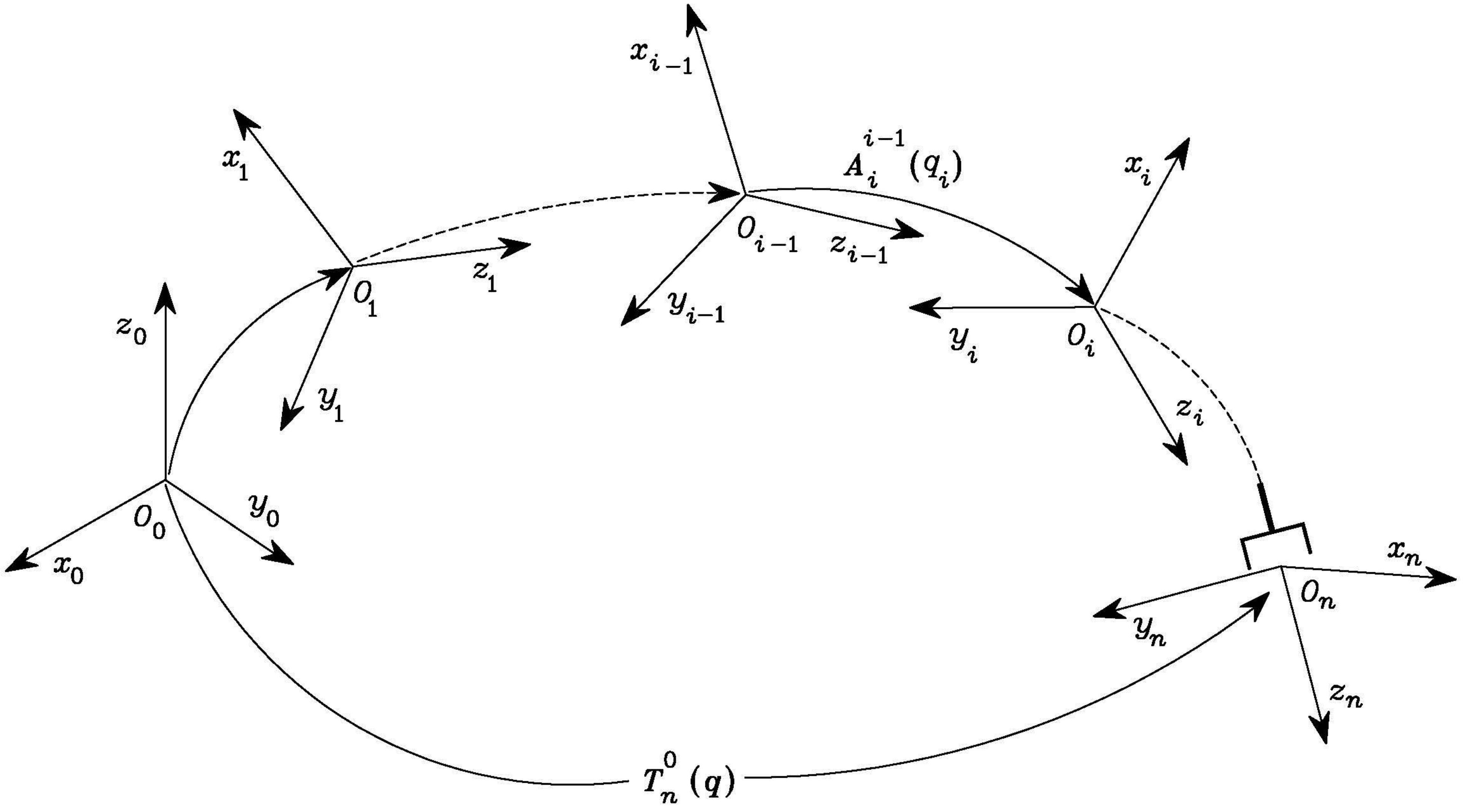
Open Chain

- Manipulator direct kinematics

$$\boldsymbol{T}_n^0(\boldsymbol{q}) = \boldsymbol{A}_1^0(q_1)\boldsymbol{A}_2^1(q_2)\dots\boldsymbol{A}_n^{n-1}(q_n)$$

- End-effector frame with respect to base frame

$$\boldsymbol{T}_e^b(\boldsymbol{q}) = \boldsymbol{T}_0^b\boldsymbol{T}_n^0(\boldsymbol{q})\boldsymbol{T}_e^n$$



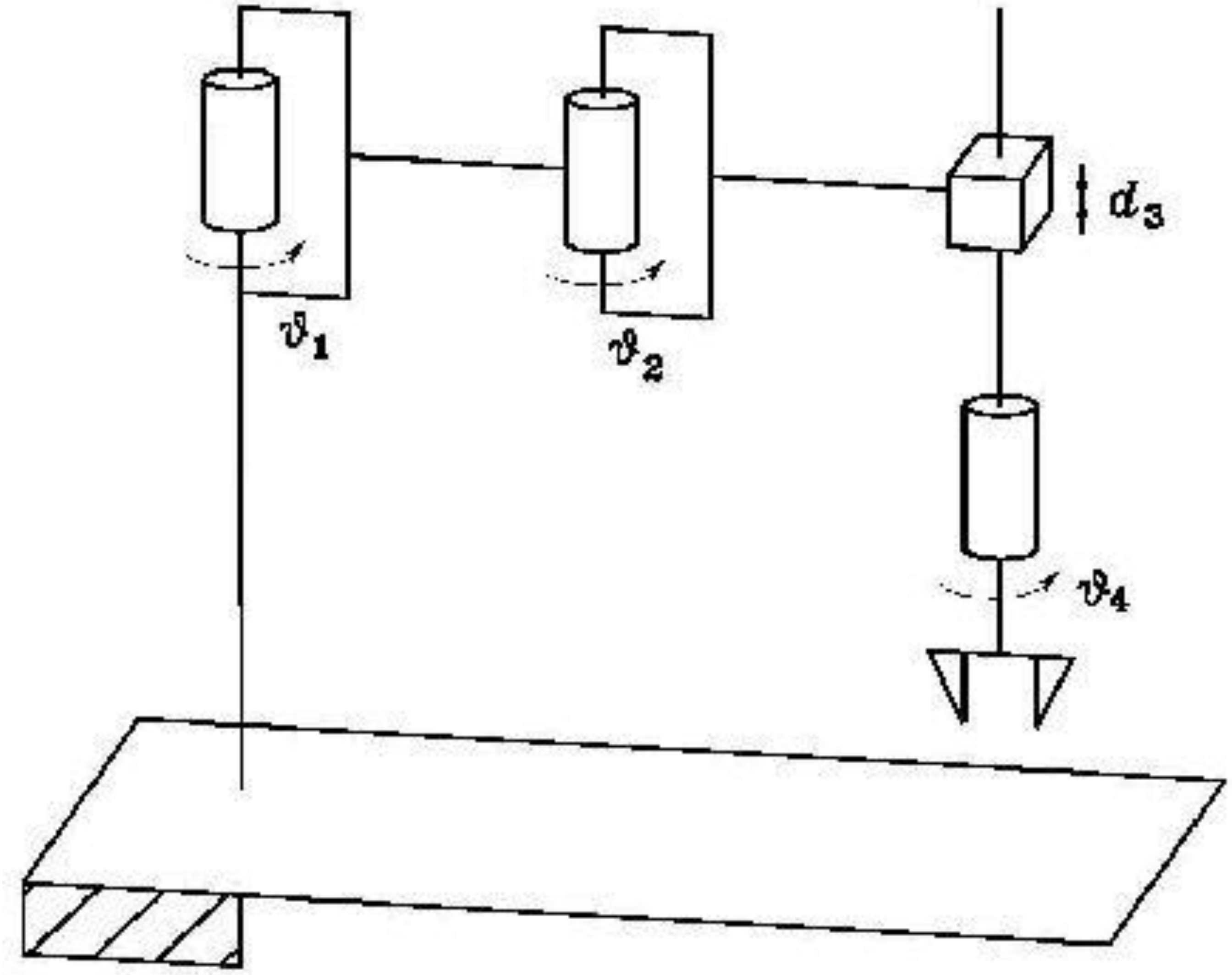
Coordinate transformations in an open kinematic chain

Summary

SECTION 7

Summary

1. By applying the rules for inverting a block-partitioned matrix, verify the expression of the homogeneous transformation matrix A_0^1
2. Find the direct kinematics equation for the SCARA manipulator in the figure.



SCARA manipulator