CS65K Robotics

Modelling, Planning and Control

Chapter 2: Kinematics

LESSON 4: DIRECT KINEMATICS

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Objectives

- The **homogeneous representation** of a vector is adopted
- Homogeneous transformations are introduced as a compact representation of position and orientation
- Composition of homogeneous transformations to derive the direct kinematics equation of an open-chain manipulator is illustrated
- **Denavit-Hartenberg** parameters are introduced
- A formula is derived to compute the transformation matrix from one link to the next one in a kinematic chain
- A computationally recursive operating procedure is illustrated

Objectives

- The direct kinematics equation is computed for a number of typical manipulator structures
- Composition of the kinematics of the arm with the kinematics of the wrist is presented
- The joint space and operational space concepts are illustrated

Homogeneous Coordinates

SECTION 1

Homogeneous Coordinates

- Homogeneous coordinates, introduced by August Ferdinand Möbius, make calculations of graphics and geometry possible in projective space. Homogeneous coordinates are a way of representing N-dimensional coordinates with N+1 numbers.
- To make 2D Homogeneous coordinates, we simply add an additional variable, w, into existing coordinates. Therefore, a point in Cartesian coordinates, (X, Y) becomes (x, y, w) in Homogeneous coordinates. And X and Y in Cartesian are re-expressed with x, y and w in Homogeneous as;

$$X = x/w$$

$$Y = y/w$$

Homogeneous Coordinates

- For instance, a point in Cartesian (1, 2) becomes (1, 2, 1) in Homogeneous. If a point, (1, 2), moves toward infinity, it becomes (∞,∞) in Cartesian coordinates. And it becomes (1, 2, 0) in Homogeneous coordinates, because of $(1/0, 2/0) \approx (\infty,\infty)$.
- Notice that we can express the point at infinity without using "∞".

Why is it called "homogeneous"?

- As mentioned before, in order to convert from Homogeneous coordinates (x, y, w) to Cartesian coordinates, we simply divide x and y by w;

$$\underset{\text{Homogeneous}}{(x, y, w)} \quad \Leftrightarrow \quad \underset{\text{Cartesian}}{\left(\frac{x}{w}, \frac{y}{w}\right)}$$

Why is it called "homogeneous"?

- Converting Homogeneous to Cartesian, we can find an important fact. Let's see the following example;

$$
\begin{array}{ccccc}
\text{Homogeneous} & & \text{Cartesian} & & \\
(1,2,3) & \Rightarrow & \left(\frac{1}{3},\frac{2}{3}\right) & & \\
(2,4,6) & \Rightarrow & \left(\frac{2}{6},\frac{4}{6}\right) & = & \left(\frac{1}{3},\frac{2}{3}\right) \\
(4,8,12) & \Rightarrow & \left(\frac{4}{12},\frac{8}{12}\right) & = & \left(\frac{1}{3},\frac{2}{3}\right) \\
\vdots & & \vdots & & \\
(1a,2a,3a) & \Rightarrow & \left(\frac{1a}{3a},\frac{2a}{3a}\right) & = & \left(\frac{1}{3},\frac{2}{3}\right)
\end{array}
$$

Why is it called "homogeneous"?

- As you can see, the points (1, 2, 3), (2, 4, 6) and (4, 8, 12) correspond to the same Euclidean point (1/3, 2/3). And any scalar product, (1a, 2a, 3a) is the same point as (1/3, 2/3) in Euclidean space. Therefore, these points are *"homogeneous"* because they represent the same point in Euclidean space (or Cartesian space). In other words, Homogeneous coordinates are scale invariant.

Homogeneous Transformation

SECTION 2

Homogeneous Transformation

p(x, y)

$$p = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$x' = S_x\, x$$
$$y' = S_y\, y$$

$$S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotate

$$x' = x\cos(\theta) - y\sin(\theta)$$
$$y' = x\sin(\theta) + y\cos(\theta)$$

$$S = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Tranlate

$$x' = x + a$$
$$y' = y + b$$

$$S = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

Transformations

- A **transformation** is a process that manipulates a polygon or other two-dimensional object on a plane or coordinate system. Mathematical transformations describe how two-dimensional figures move around a plane or coordinate system.
- A **preimage** or inverse image is the two-dimensional shape before any transformation. The **image** is the figure after transformation.

Types of Transformations

There are five different transformations in math:

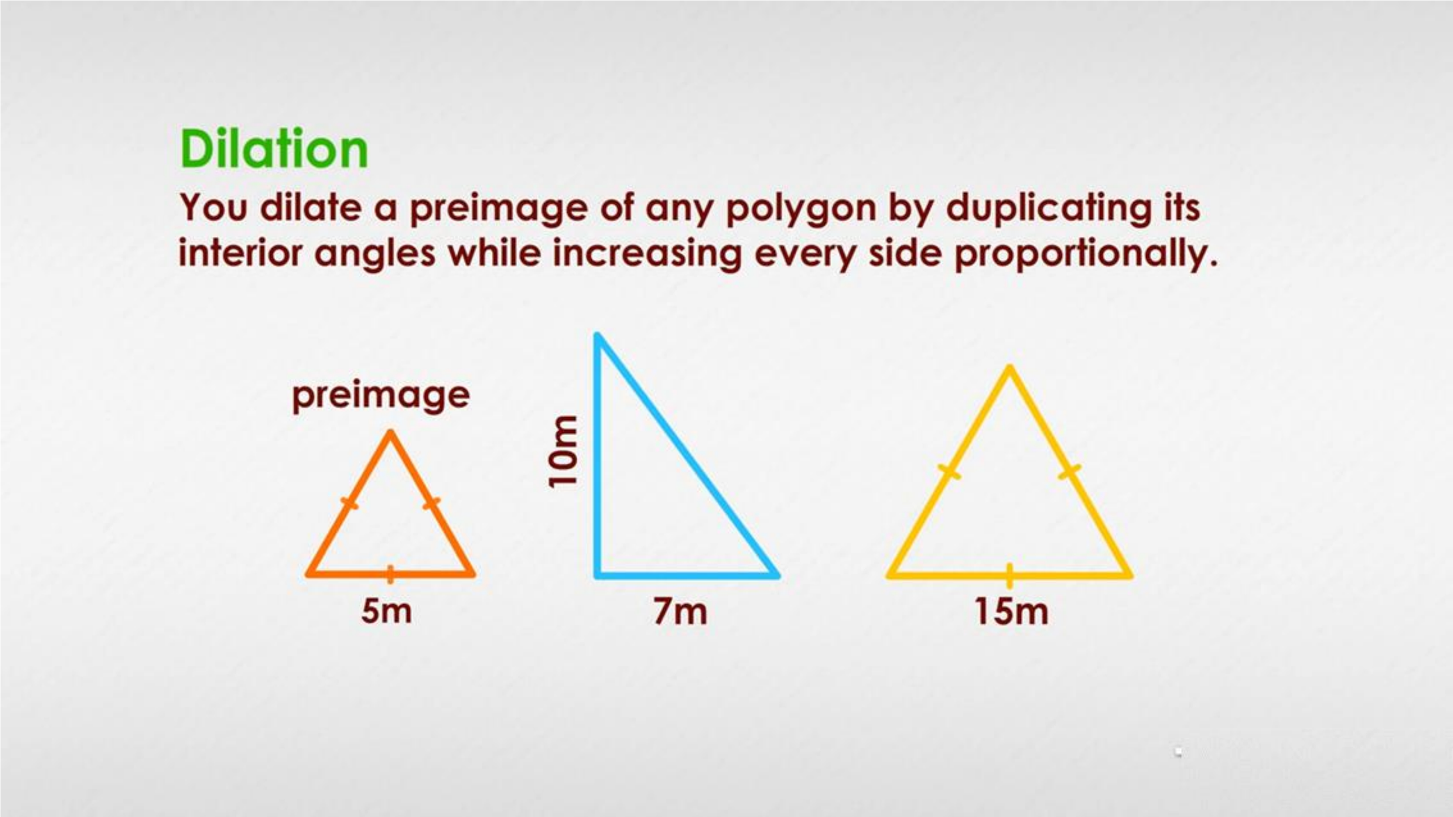
1. **Dilation** (Scaling) -- The image is a larger or smaller version of the preimage; "shrinking" or "enlarging."
2. **Reflection** -- The image is a mirrored preimage; "a flip."
3. **Rotation** -- The image is the preimage rotated around a fixed point; "a turn."
4. **Shear** -- All the points along one side of a preimage remain fixed while all other points of the preimage move parallel to that side in proportion to the distance from the given side; "a skew.,"
5. **Translation** -- The image is offset by a constant value from the preimage; "a slide."

Coordinate Change and Transformation are Inverse Operations

- A coordinate system rotate for θ is equivalent to the operation for an object to rotate for –θ.
- Therefore, the two operations are inverse operation to each other.
- Transformation:

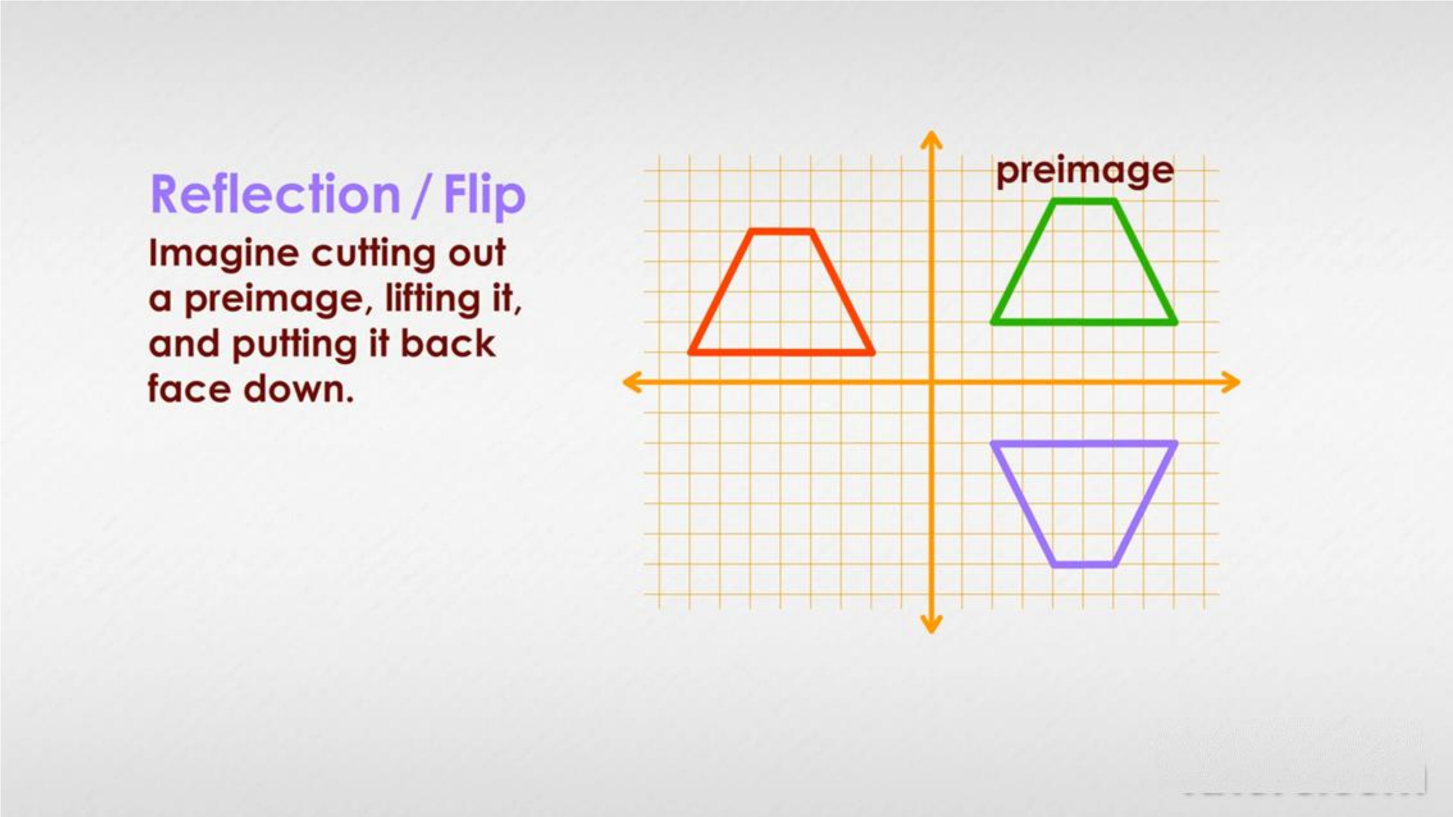
  $(x, y) \rightarrow (x', y')$

  Sometimes, I use (X, Y)



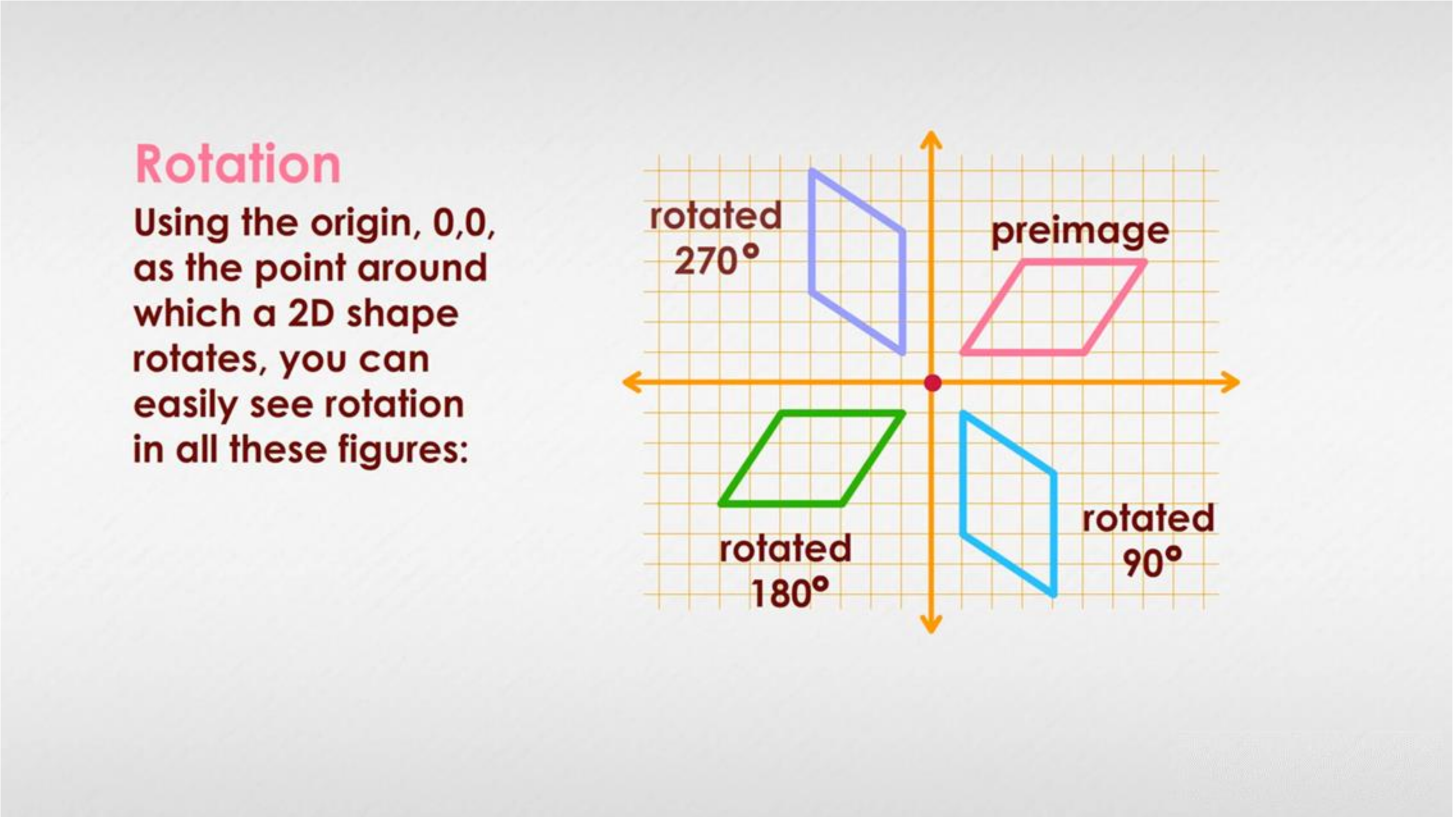
Dilation

- Dilate a preimage of any polygon is done by duplicating its interior angles while increasing every side proportionally. You can think of dilating as resizing. Which triangle image, yellow or blue, is a dilation of the orange preimage?



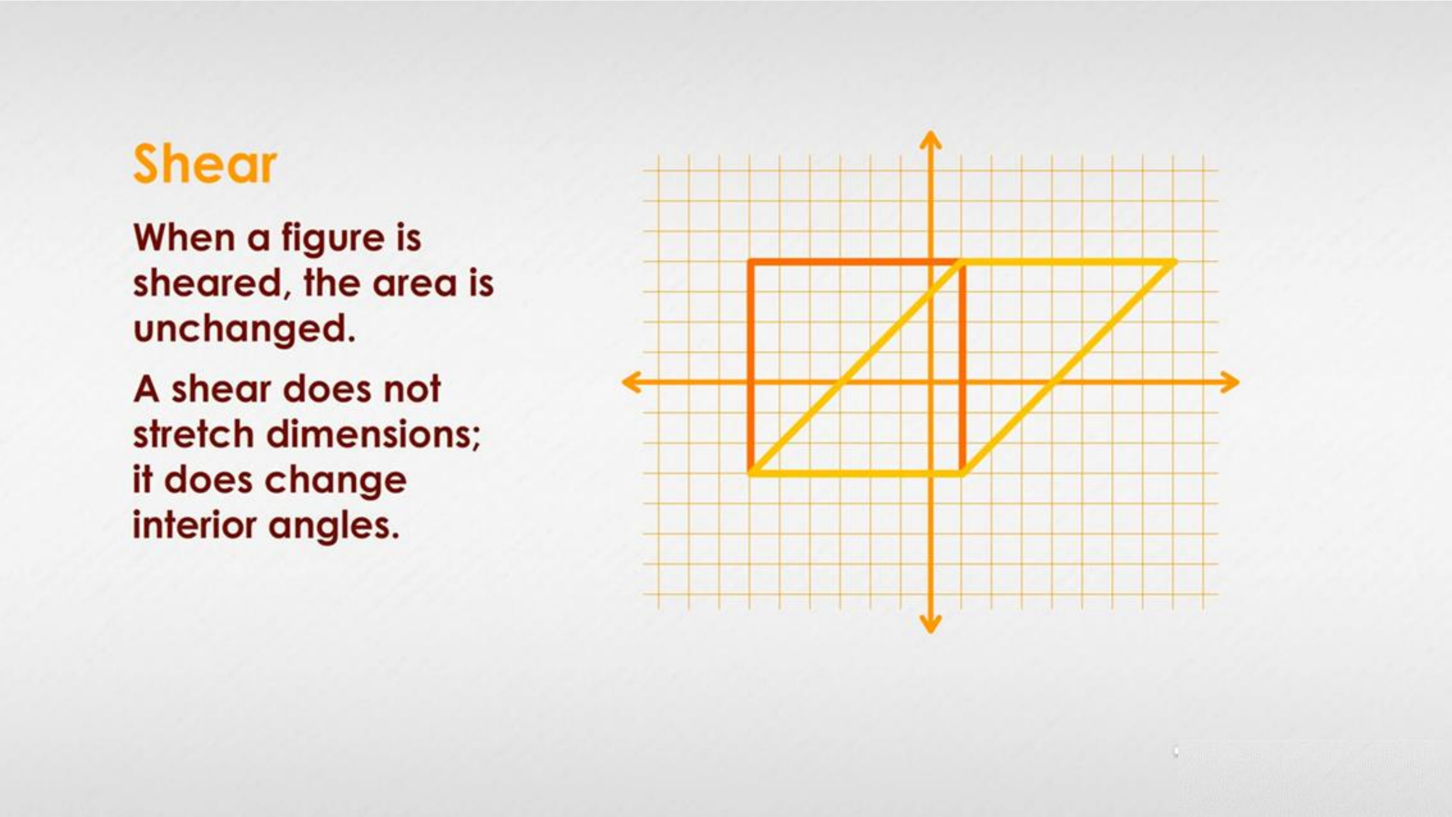
Reflection

- Imagine cutting out a preimage, lifting it, and putting it back face down. That is a reflection or a flip. A reflection image is a mirror image of the preimage. Which trapezoid image, red or purple, is a reflection of the green preimage?



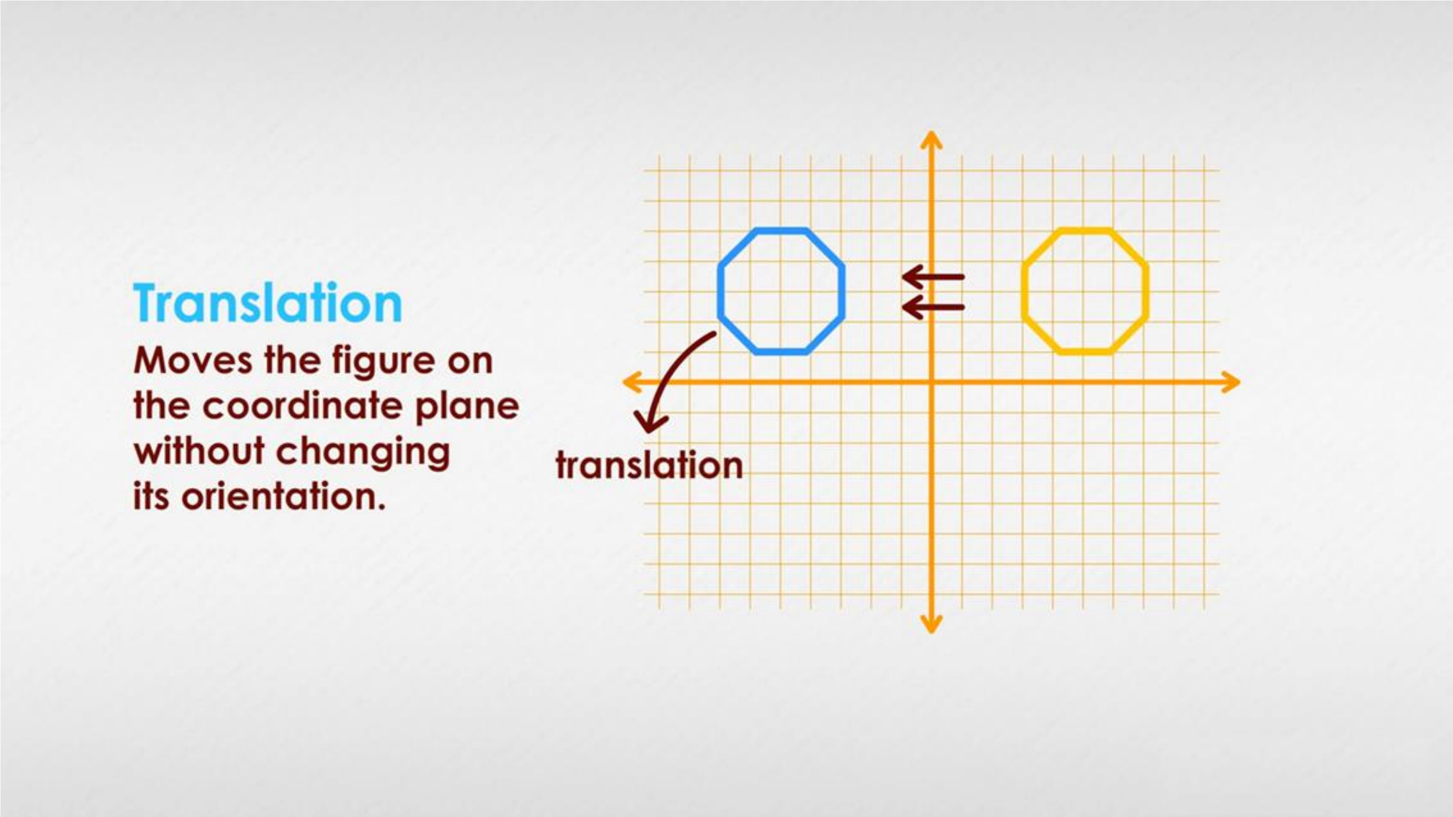
Rotation

- Using the origin, (0, 0), as the point around which a two-dimensional shape rotates, you can easily see rotation in all these figures:



Shear

- Here is a square preimage. To shear it, you "skew it," producing an image of a rhombus:



Translation

- A translation moves the figure from its original position on the coordinate plane without changing its orientation. Which octagon image below, pink or blue, is a translation of the yellow preimage?

Homogeneous Representation of a Vector

- Coordinate transformation (*translation* + *rotation*)

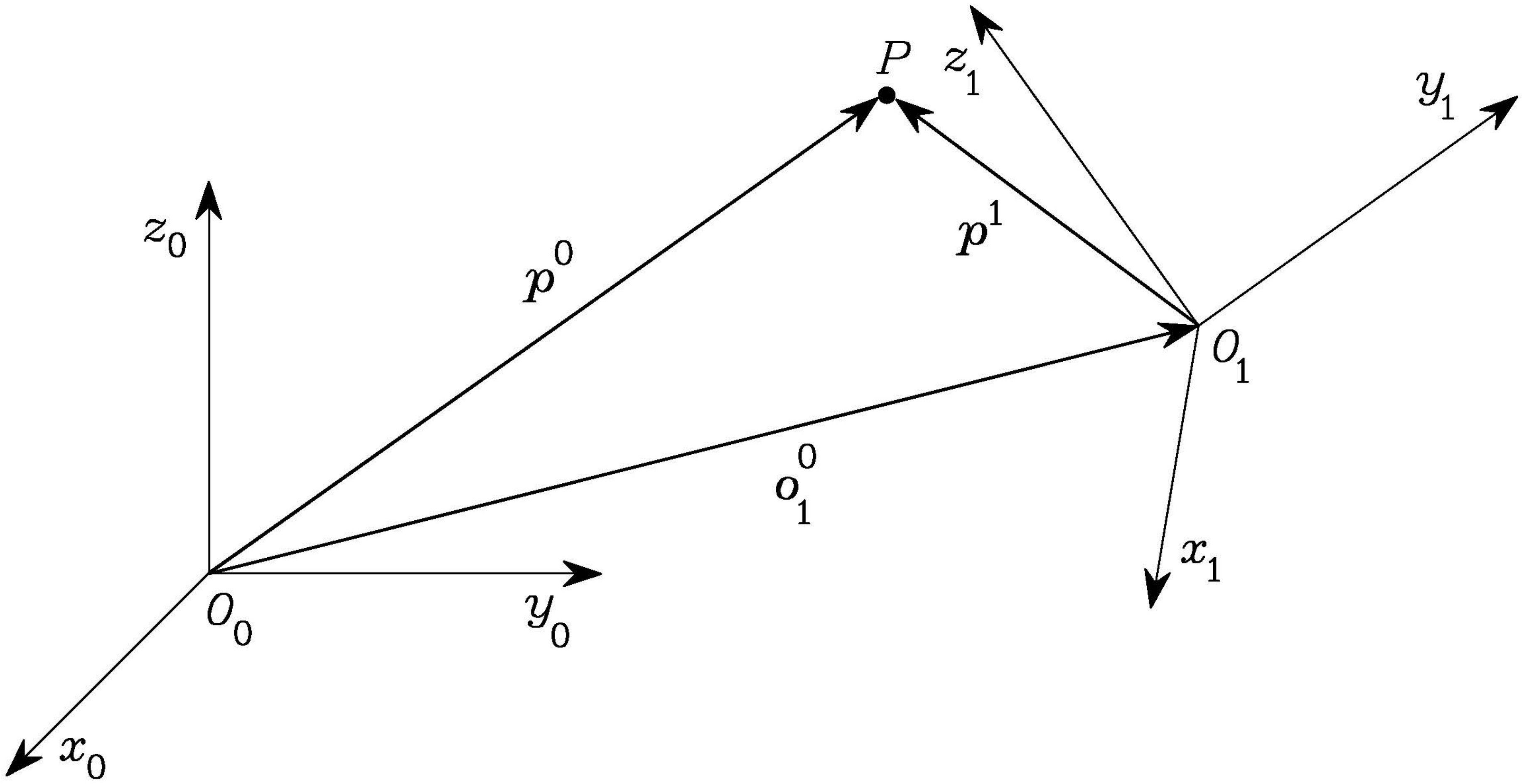
$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1$$

- Inverse transformation

$$\boldsymbol{p}^1 = -\boldsymbol{R}_0^1 \boldsymbol{o}_1^0 + \boldsymbol{R}_0^1 \boldsymbol{p}^0$$

- Homogeneous representation

$$\widetilde{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix}$$



Representation of a point in different coordinate frames

Homogeneous Transformation Matrix

$R_1^0: Rotational\ Matrix$

$o_1^0: Displacement\ Vector$

$$A_1^0 = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ 0^T & 1 \end{bmatrix}$$

Homogeneous Transformation Matrix

- Coordinate transformation

$$\tilde{\boldsymbol{p}}^0 = \boldsymbol{A}_1^0 \tilde{\boldsymbol{p}}^1$$

- Inverse transformation

$$\tilde{\boldsymbol{p}}^1 = \boldsymbol{A}_0^1 \tilde{\boldsymbol{p}}^0 = \left(\boldsymbol{A}_1^0\right)^{-1} \tilde{\boldsymbol{p}}^0$$

with

$$\boldsymbol{A}_0^1 = \begin{bmatrix} \boldsymbol{R}_1^{0T} & -\boldsymbol{R}_1^{0T} \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_0^1 & -\boldsymbol{R}_0^1 \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

Properties

- Orthogonality does not hold

$$\boldsymbol{A}^{-1} \neq \boldsymbol{A}^T$$

- Sequence of coordinate transformations

$$\tilde{\boldsymbol{p}}^0 = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \ldots \boldsymbol{A}_n^{n-1} \tilde{\boldsymbol{p}}^n$$

Type of Joints

SECTION 3

Manipulator

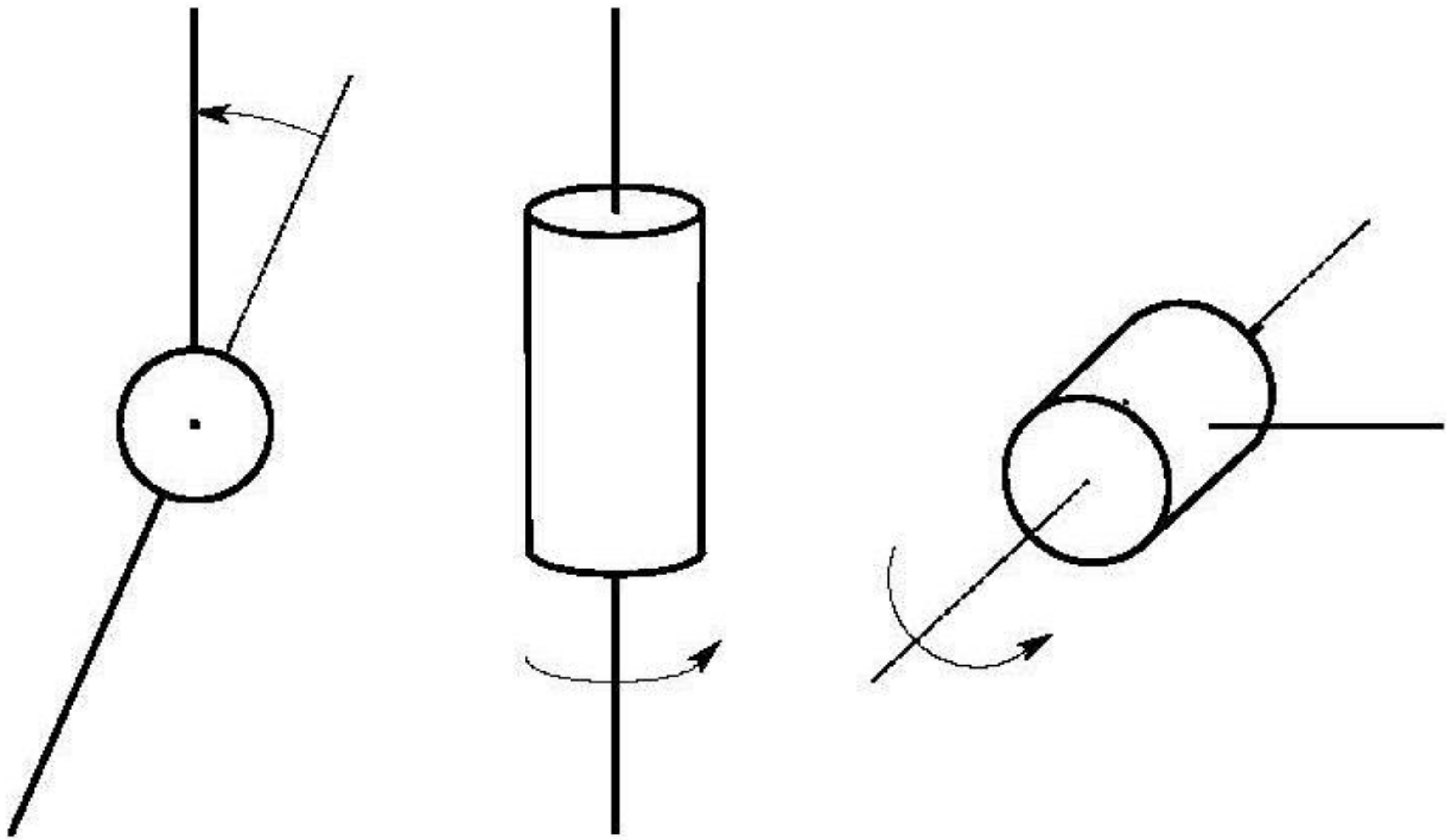
- Series of rigid bodies (*links*) connected by means of kinematic pairs or *joints*

Kinematic chain (from base to end-effector)

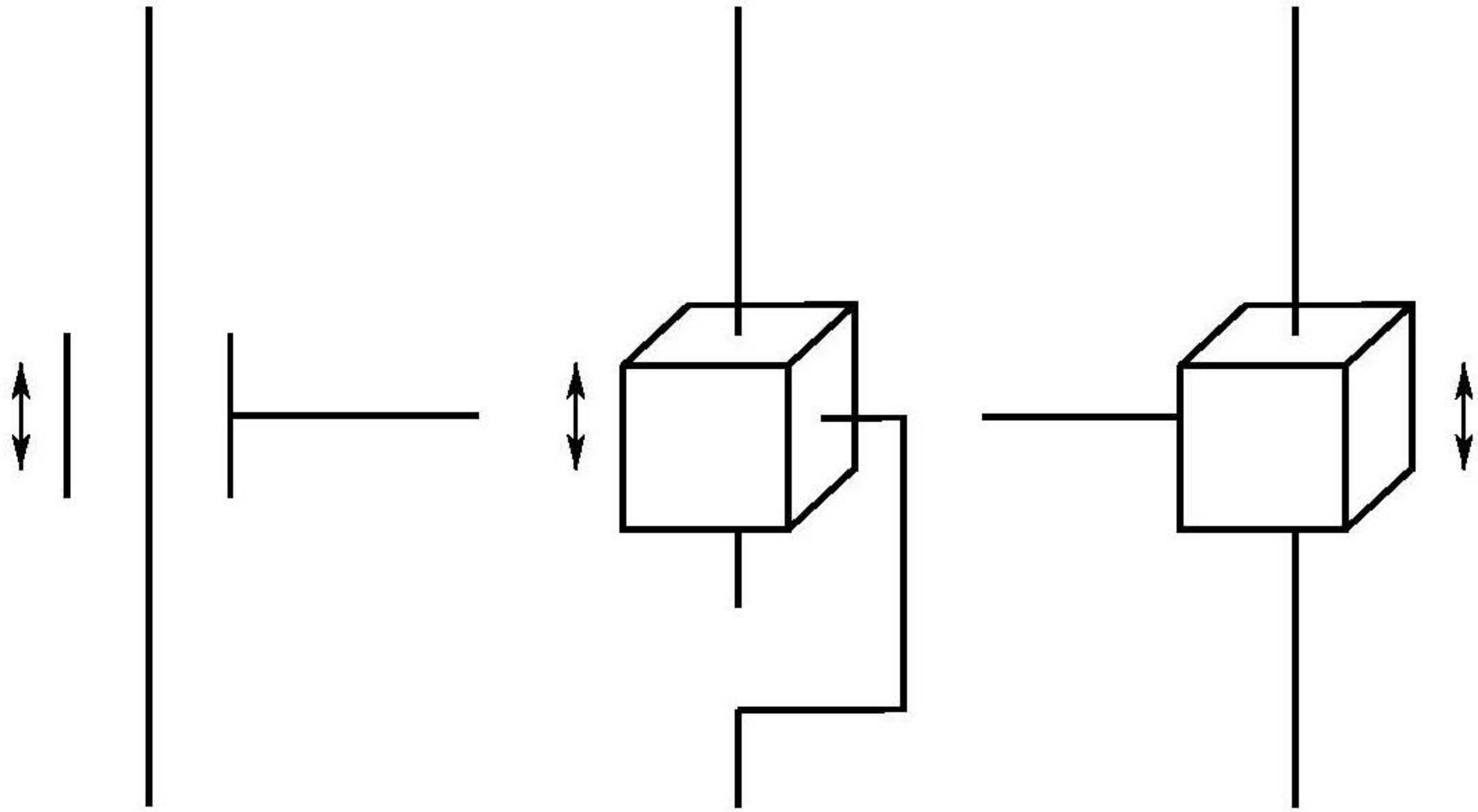
- Open (only one sequence of links connecting the two ends of the chain)
- Closed (a sequence of links forms a loop)

Degrees of freedom (DOFs) uniquely determine the manipulator's *posture*

- Each DOF is typically associated with a joint articulation and constitutes a *joint variable*



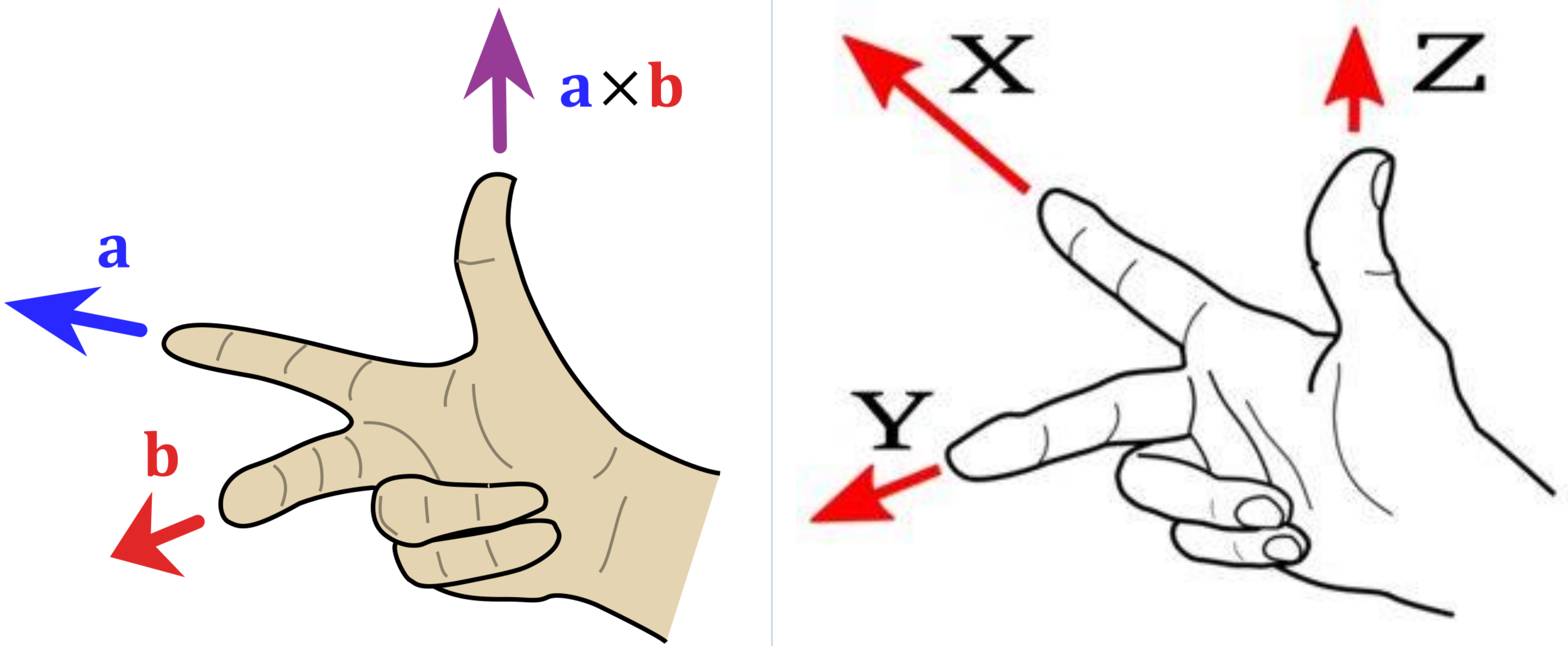
Revolute joints



Prismatic joints

Base Frame and End-effector Frame

SECTION 4



a×b
a
b
X
Z
Y

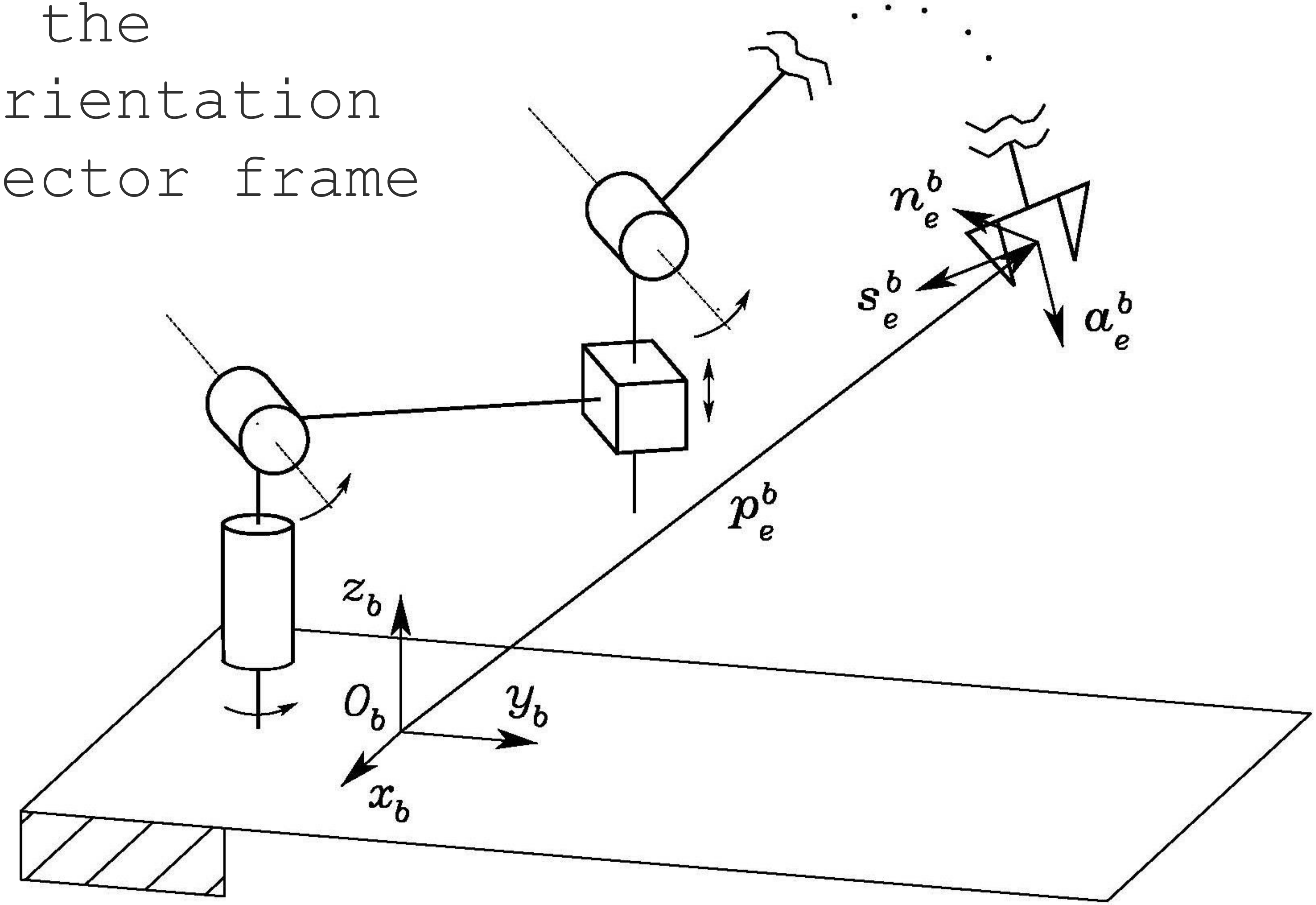
Base Frame and End-effector Frame

- Joint variables $\boldsymbol{q} = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix}^T$
- End-effector frame with respect to base frame $\boldsymbol{R}_e^b = \begin{bmatrix} \boldsymbol{n}_e^b & \boldsymbol{s}_e^b & \boldsymbol{a}_e^b \end{bmatrix}$

Direct kinematics equation

$$\boldsymbol{T}_e^b(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{n}_e^b(\boldsymbol{q}) & \boldsymbol{s}_e^b(\boldsymbol{q}) & \boldsymbol{a}_e^b(\boldsymbol{q}) & \boldsymbol{p}_e^b(\boldsymbol{q}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Description of the position and orientation of the end-effector frame



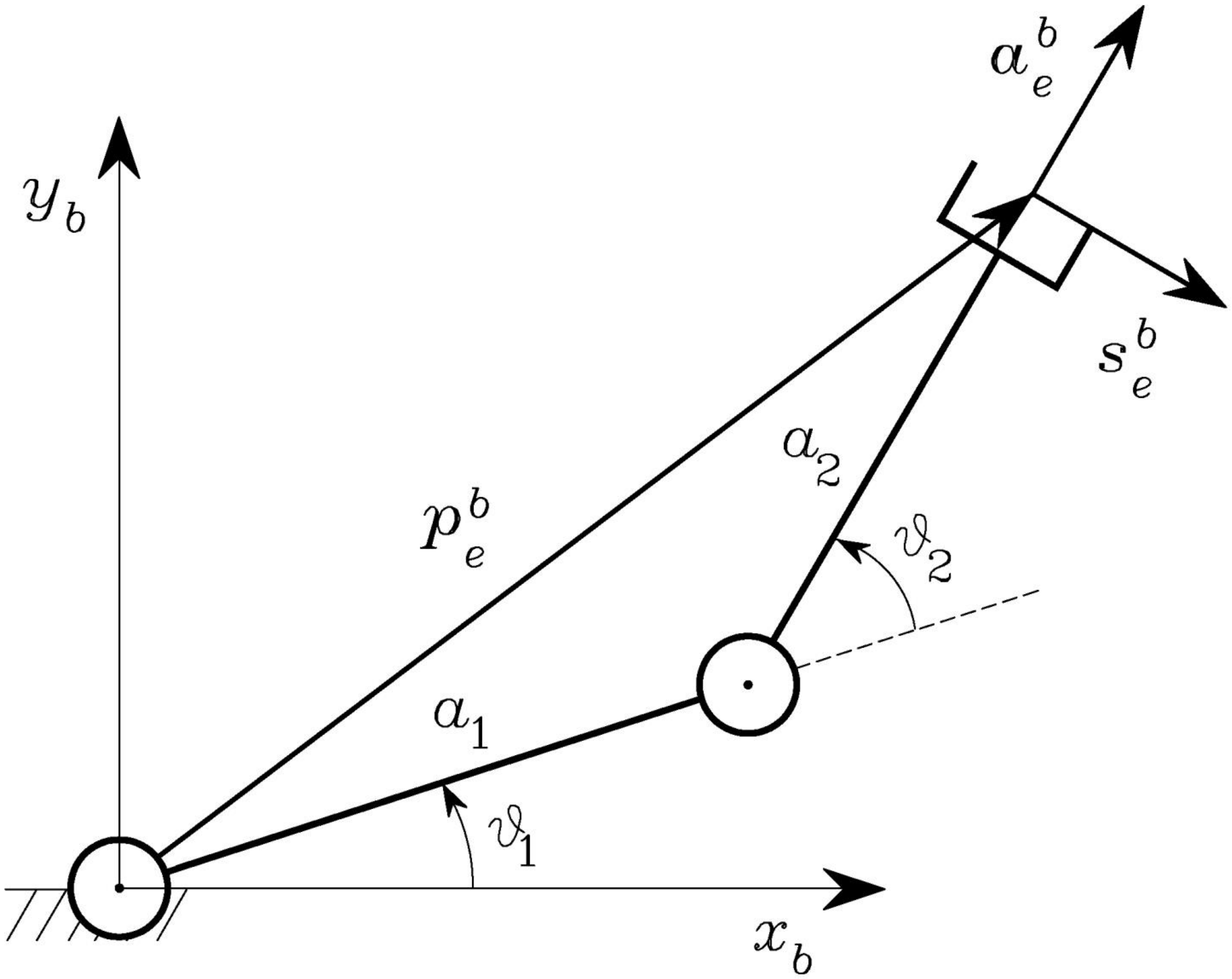
Two-link Planar Arm

SECTION 5

Two-link Planar Arm

$$\boldsymbol{T}_e^b(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{n}_e^b & \boldsymbol{s}_e^b & \boldsymbol{a}_e^b & \boldsymbol{p}_e^b \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & s_{12} & c_{12} & a_1c_1 + a_2c_{12} \\ 0 & -c_{12} & s_{12} & a_1s_1 + a_2s_{12} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Two-link Planar Arm

Open Chain

SECTION 6

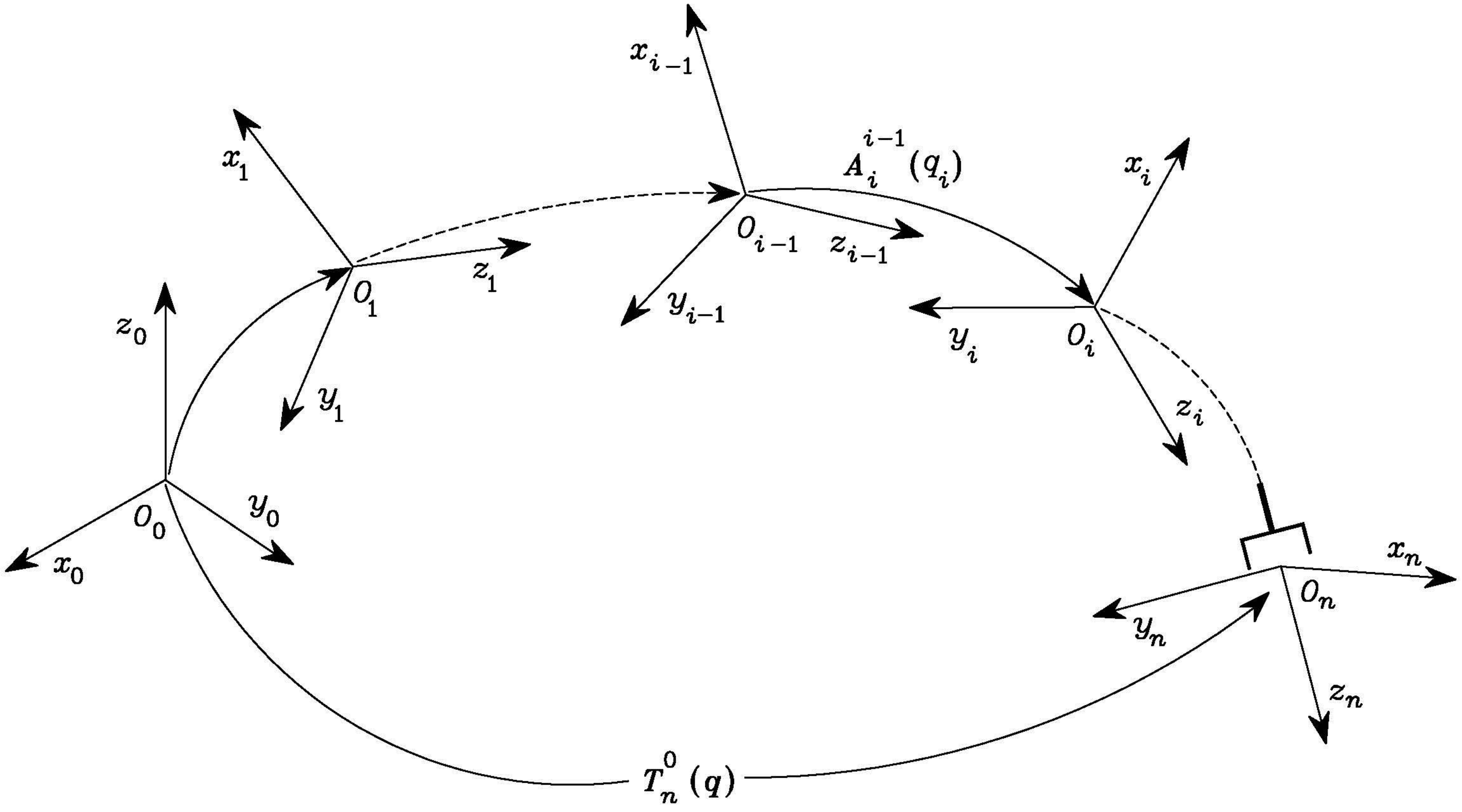
Open Chain

- Manipulator direct kinematics

$$\boldsymbol{T}_n^0(\boldsymbol{q}) = \boldsymbol{A}_1^0(q_1)\boldsymbol{A}_2^1(q_2)\dots\boldsymbol{A}_n^{n-1}(q_n)$$

- End-effector frame with respect to base frame

$$\boldsymbol{T}_e^b(\boldsymbol{q}) = \boldsymbol{T}_0^b\boldsymbol{T}_n^0(\boldsymbol{q})\boldsymbol{T}_e^n$$



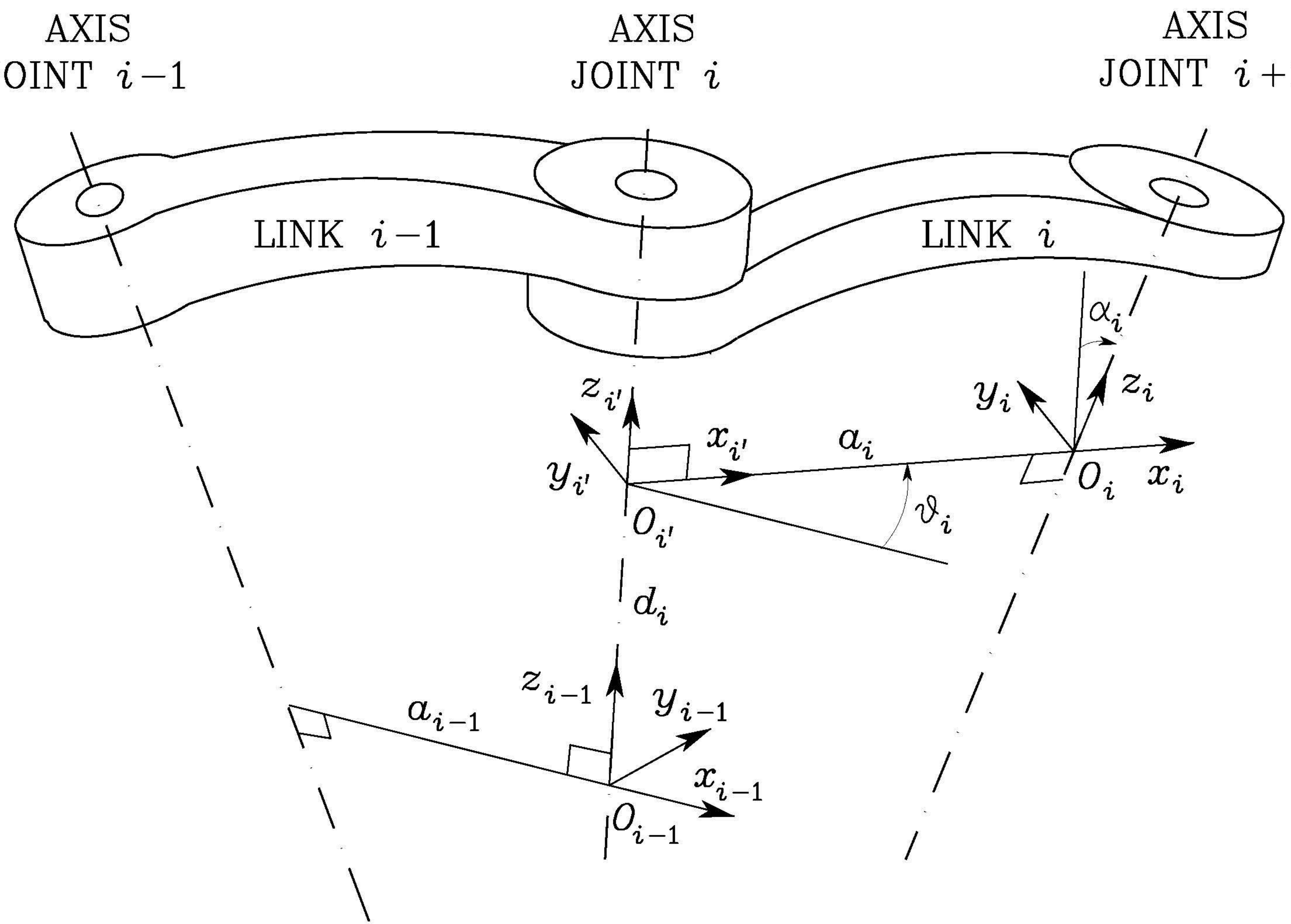
Coordinate transformations in an open kinematic chain

Denavit-Hartenberg Convention

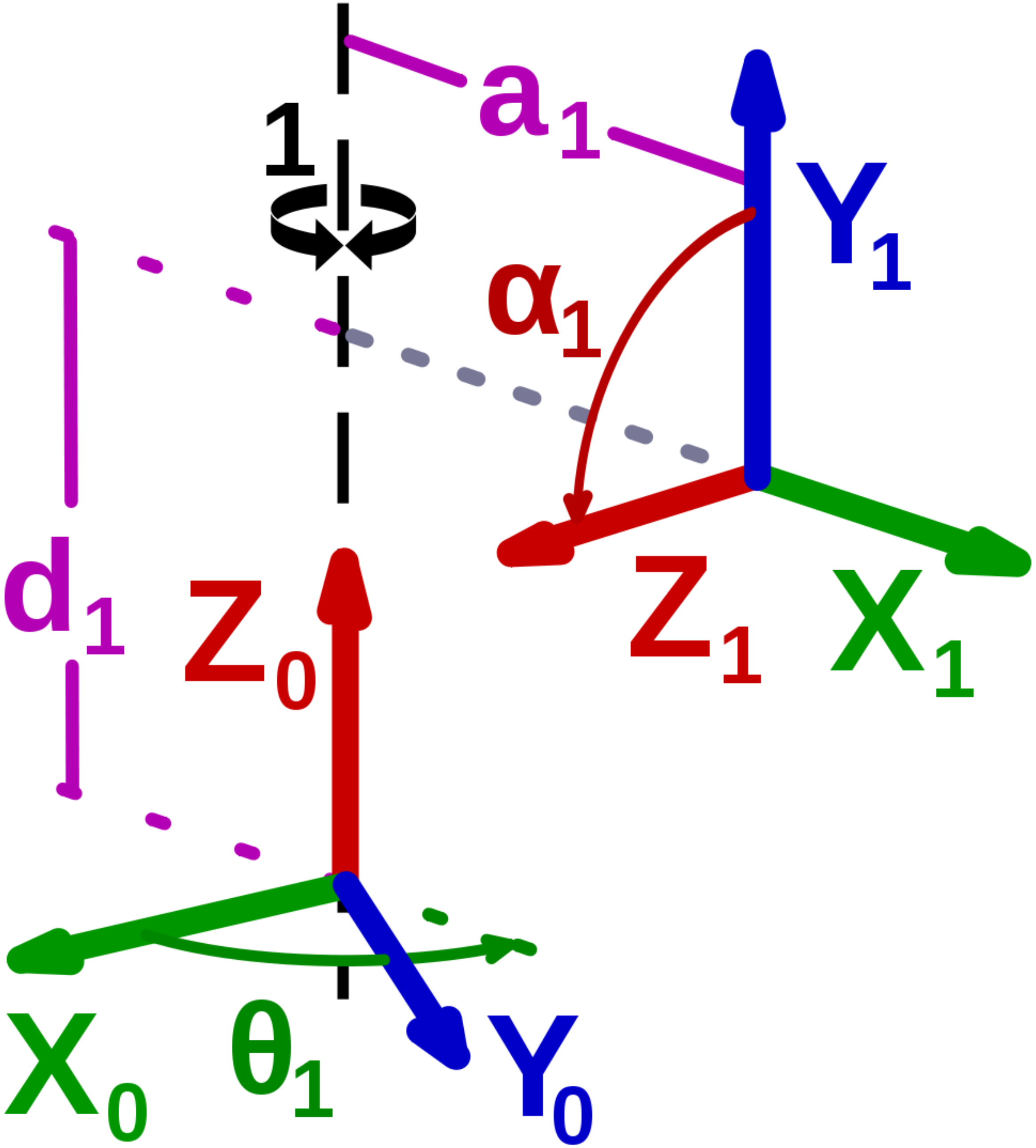
SECTION 7

Denavit-Hartenberg Convention

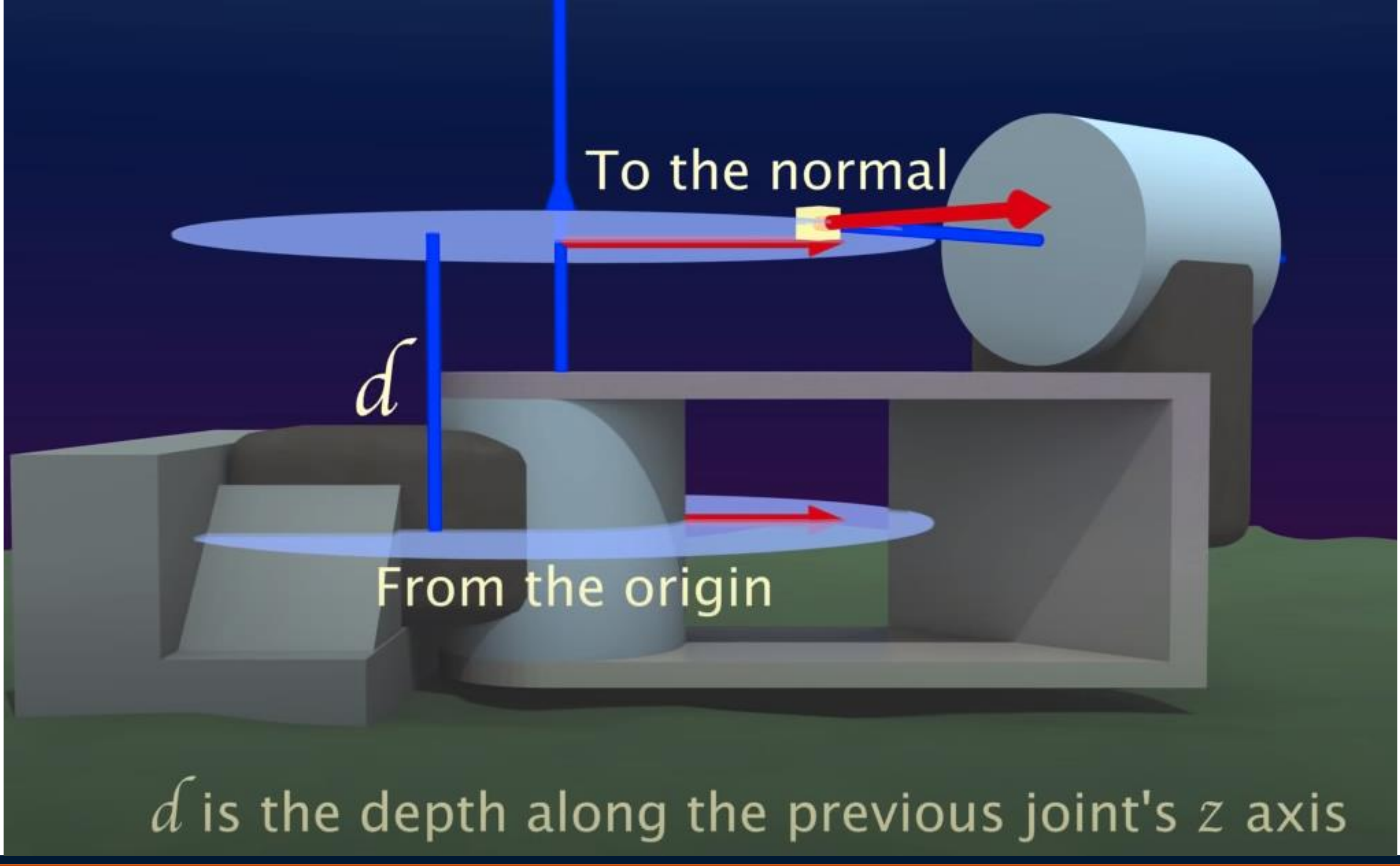
- Choose axis z_i along the axis of Joint $i+1$
- Locate the origin O_i at the intersection of axis z_i with the common normal to axes z_{i-1} and z_i. Also, locate $O_{i'}$ at the intersection of the common normal with axis z_{i-1}
- Choose axis x_i along the common normal to axes z_{i-1} and z_i with direction from Joint i to Joint $i+1$
- Choose axis y_i so as to complete a right-handed frame

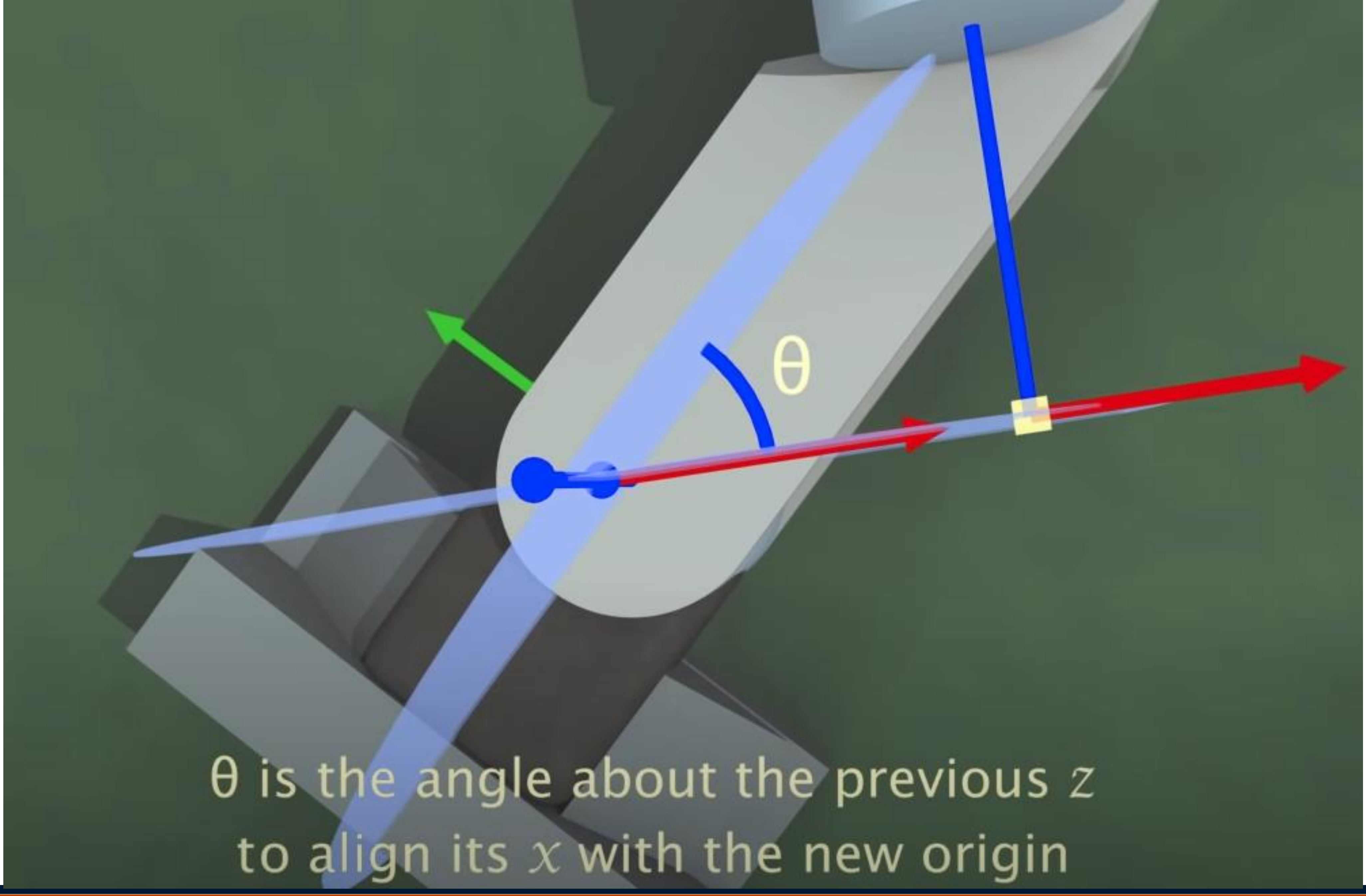


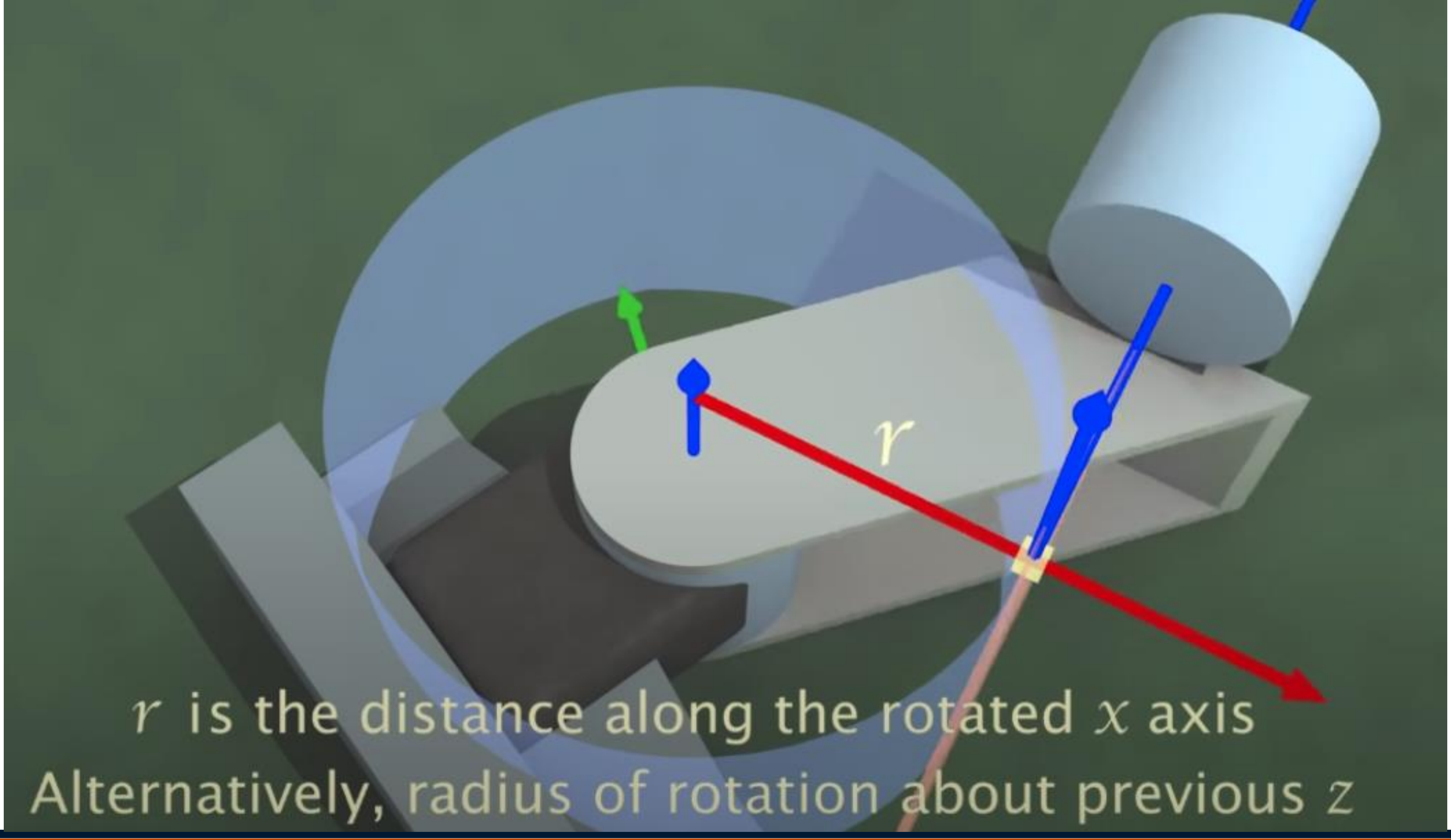
Denavit-Hartenberg kinematic parameters

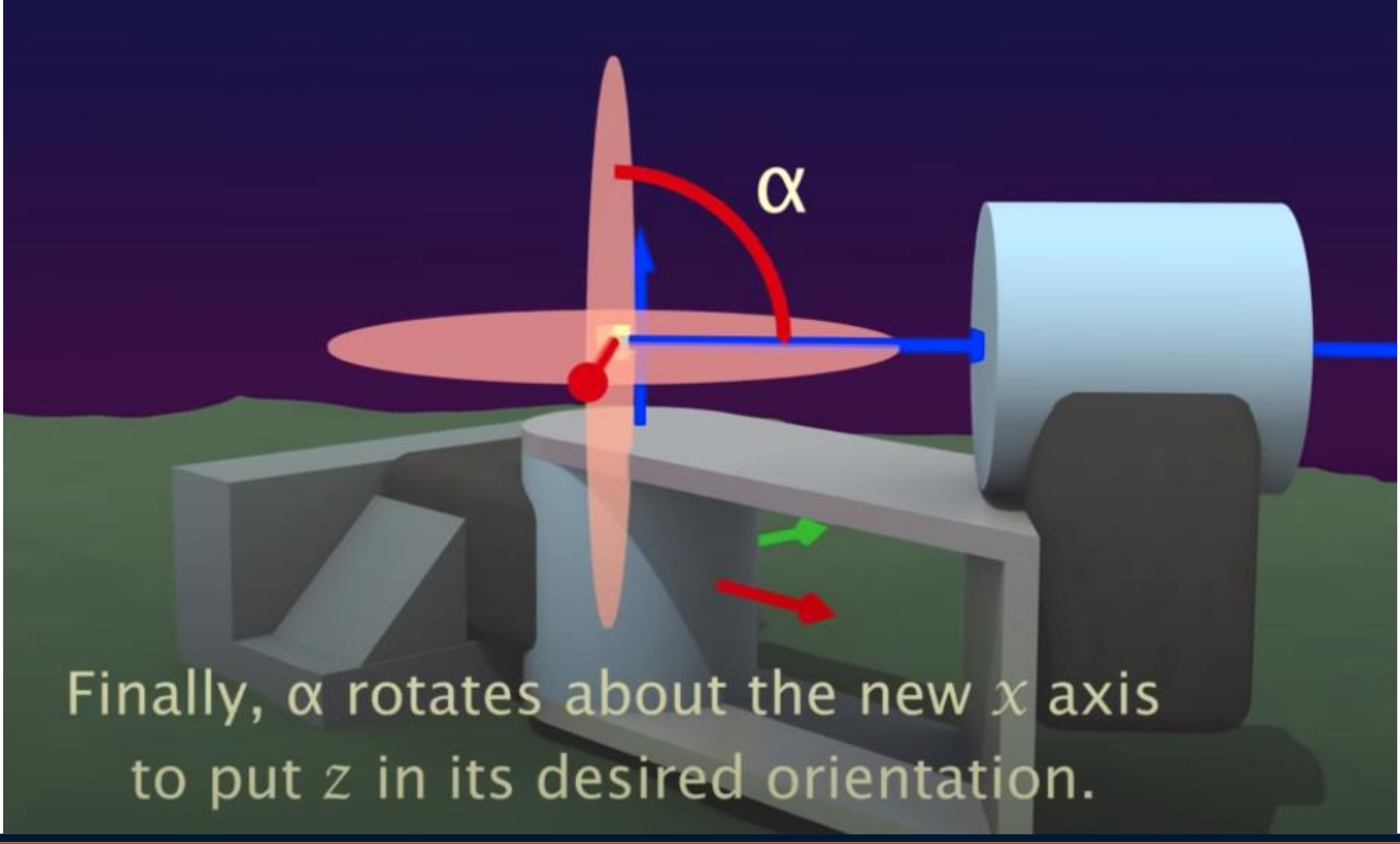


Denavit-Hartenberg Parameters


To the normal
d
From the origin
d is the depth along the previous joint's z axis


θ
θ is the angle about the previous z
to align its x with the new origin


r
r is the distance along the rotated x axis
Alternatively, radius of rotation about previous z


α
Finally, α rotates about the new x axis
to put z in its desired orientation.

Denavit-Hartenberg Convention II

Nonunique definition of the link frame in the following cases

- For Frame 0, only the direction of axis z_0 is specified; then O_0 and x_0 can be arbitrarily chosen
- For Frame n, since there is no Joint $n+1$, z_n is not uniquely defined while x_n has to be normal to axis z_{n-1}. Typically, Joint n is revolute, and thus z_n is to be aligned with the direction of z_{n-1}
- When two consecutive axes are parallel, the common normal between them is not uniquely defined
- When two consecutive axes intersect, the direction of x_i is arbitrary
- When Joint i is prismatic, the direction of z_{i-1} is arbitrary

Denavit-Hartenberg Parameters

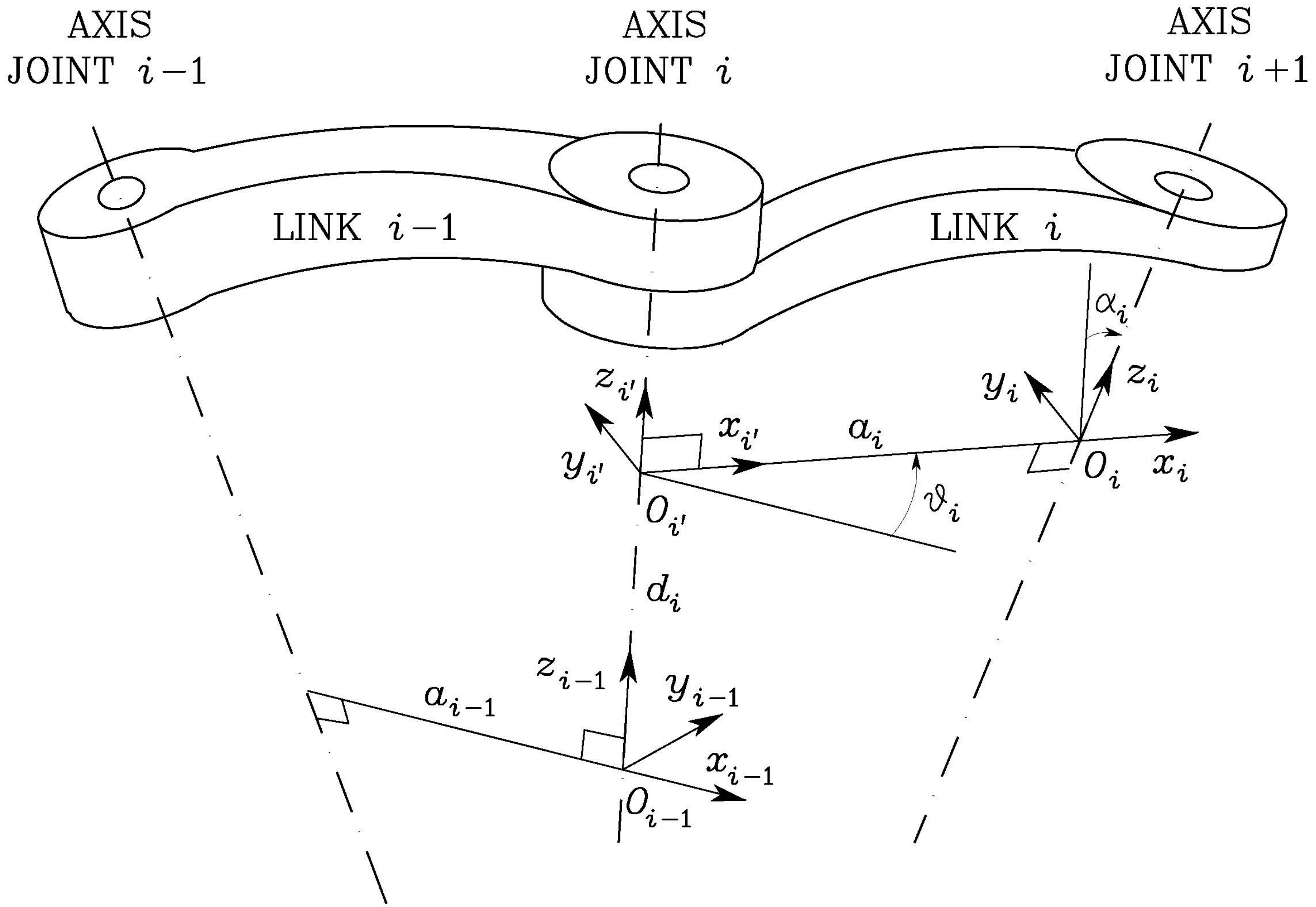
a_i: distance between O_i and $O_{i'}$

d_i: coordinate of $O_{i'}$ along z_{i-1}

α_i: angle between axes z_{i-1} and z_i about axis x_i to be taken positive when rotation is made counter-clockwise

ϑ_i: angle between axes x_{i-1} and x_i about axis z_{i-1} to be taken positive when rotation is made counter-clockwise

- a_i and α_i are always constant
- If Joint i is *revolute* the variable is ϑ_i
- If Joint i is *prismatic* the variable is d_i



Denavit-Hartenberg parameters

Coordinate Transformation

- Transformation from Frame i - 1 to Frame i'

$$A_{i'}^{i-1} = \begin{bmatrix} \cos\vartheta_i & -\sin\vartheta_i & 0 & 0 \\ \sin\vartheta_i & \cos\vartheta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Transformation from Frame i' to Frame i - 1

$$A_{i}^{i'} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Coordinate Transformation II

- The resulting coordinate transformation is obtained by post-multiplication of the single transformations

$$\boldsymbol{A}_i^{i-1}(q_i) = \boldsymbol{A}_{i'}^{i-1}\boldsymbol{A}_i^{i'} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i}c\alpha_i & s_{\vartheta_i}s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i}c_{\alpha_i} & -c_{\vartheta_i}s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating Procedure

1. Find and number consecutively the joint axes; set the directions of axes $z_0, \dots, z_n$

2. Choose Frame 0 by locating the origin on axis z_0; axes x_0 and y_0 are chosen so as to obtain aright-handed frame. If feasible, it is worth choosing Frame 0 to coincide with the base frame

Execute steps from 3. to 5. for $i = 1, \dots, n-1$

3. Locate the origin O_i at the intersection of z_i with the common normal to axes z_{i-1} and z_i. If axes z_{i-1} and z_i are parallel and Joint i is revolute, then locate O_i so that $d_i = 0$; if Joint i is prismatic, locate O_i at a reference position for the joint range (mechanical limit)

4. Choose axis x_i along the common normal to axes z_{i-1} and z_i with direction from Joint i to Joint $i+1$

5. Choose axis y_i so as to obtain a right-handed frame

Operating Procedure II

6. Choose Frame n; if Joint n is revolute, then align z_n with z_{n-1}, otherwise, if Joint n is prismatic, then choose z_n arbitrarily. Axis x_n is set according to step 4.

7. For $i = 1, \ldots, n$, form the table of parameters $a_i, d_i, \alpha_i, \vartheta_i$

8. On the basis of the parameters in 7. compute the homogeneous transformation matrices $\boldsymbol{A}_i^{i-1}(q_i)$ for $i = 1, \ldots, n$

9. Compute the homogeneous transformation $\boldsymbol{T}_n^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \ldots \boldsymbol{A}_n^{n-1}$ that yields the position and orientation of Frame n with respect to Frame 0

10. Given $\boldsymbol{T}_0^b$ and $\boldsymbol{T}_e^n$, compute the direct kinematics function as $\boldsymbol{T}_e^b(\boldsymbol{q}) = \boldsymbol{T}_0^b \boldsymbol{T}_n^0(\boldsymbol{q}) \boldsymbol{T}_e^n$ that yields the position and orientation of the end-effector frame with respect to the base frame.

Kinematics of Typical Manipulator Structures

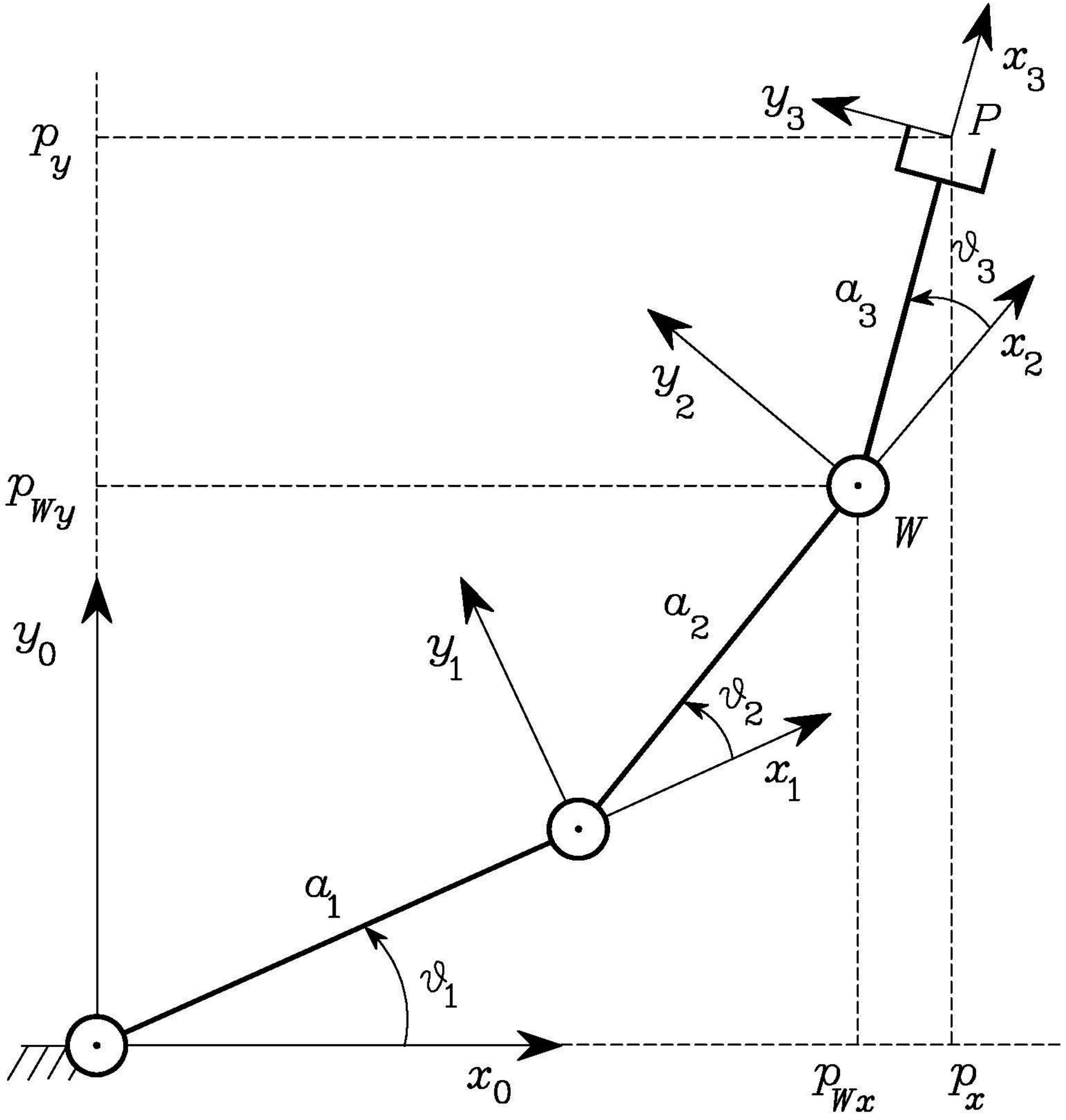
SECTION 8

Three-link Planar Arm

DH parameters

Link	a_i	α_i	d_i	ϑ_i
1	a_1	0	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

$$\boldsymbol{A}_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Three-link planar arm with frame assignment

Three-link Planar Arm II

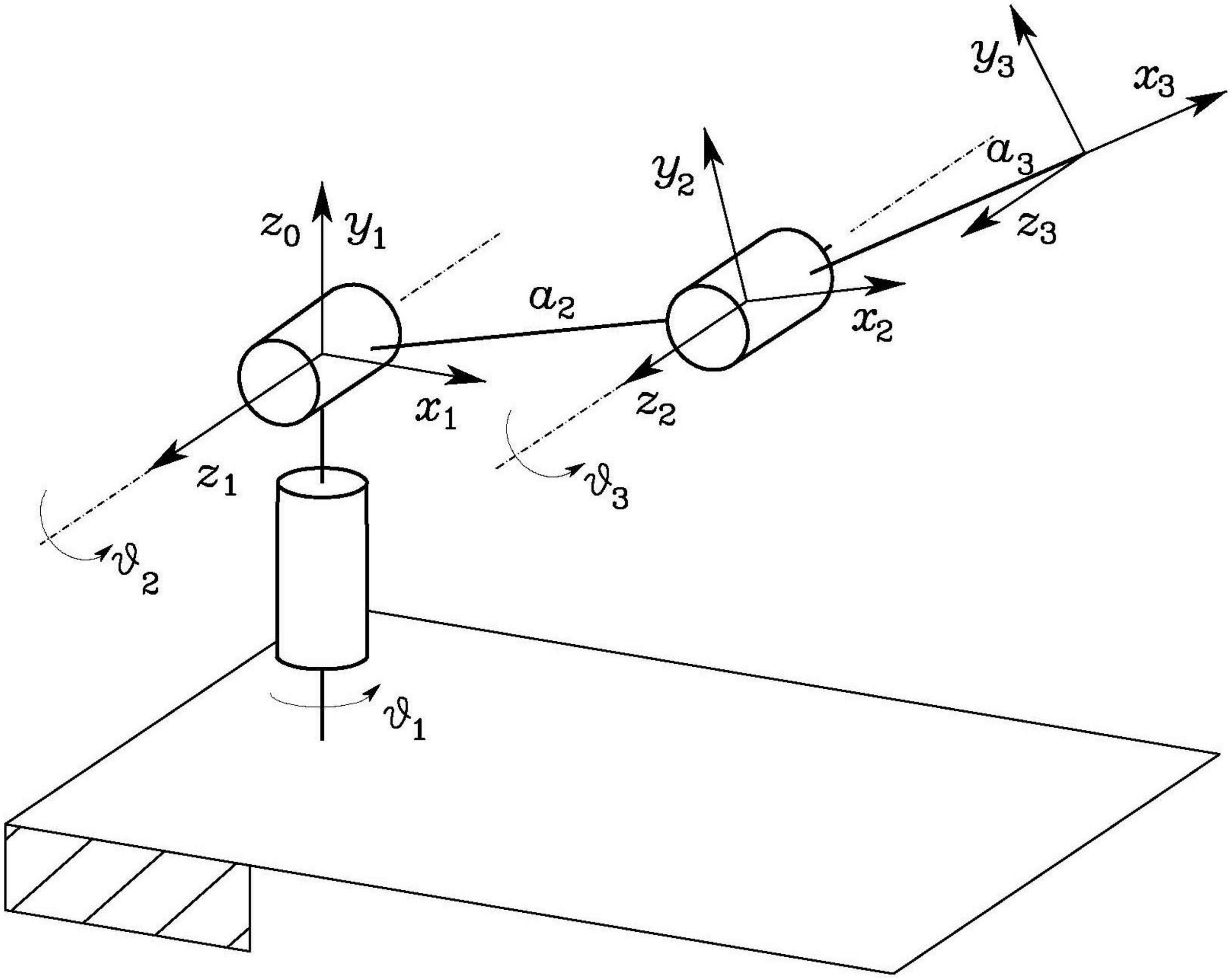
$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0\boldsymbol{A}_2^1\boldsymbol{A}_3^2 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} + a_3c_{123} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Anthropomorphic Arm

DH parameters

Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

$$\boldsymbol{A}_1^0(\vartheta_1) = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{A}_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad i = 2, 3$$



Anthropomorphic arm with frame assignment

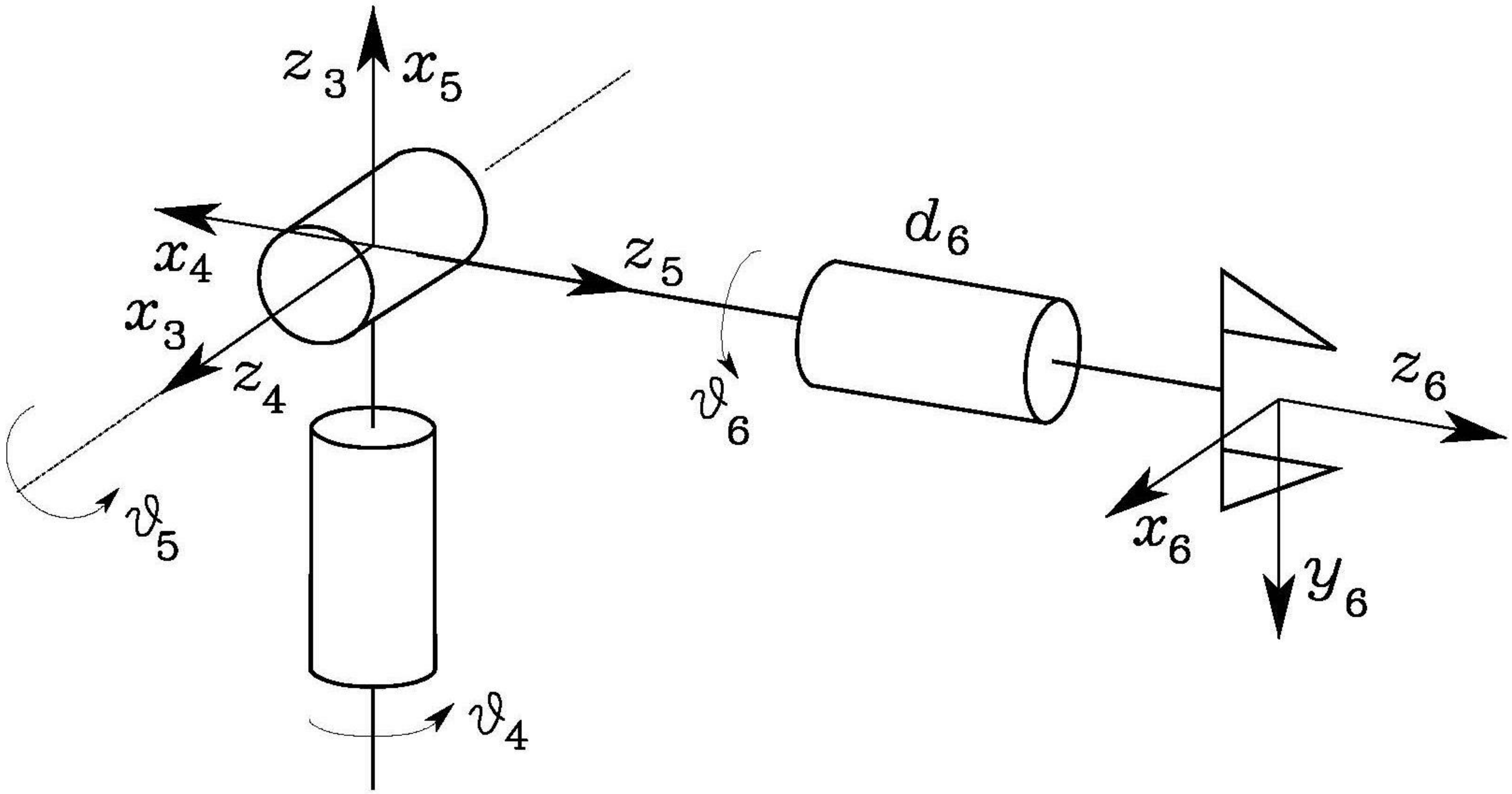
Anthropomorphic Arm II

$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{T}_1^0 \boldsymbol{T}_2^1 \boldsymbol{T}_3^2 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & c_1(a_2 c_2 + a_3 c_{23}) \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & s_1(a_2 c_2 + a_3 c_{23}) \\ s_{23} & c_{23} & 0 & a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Spherical Wrist

DH parameters

Link	a_i	α_i	d_i	ϑ_i
4	0	$-\pi/2$	0	ϑ_4
5	0	$\pi/2$	0	ϑ_5
6	0	0	d_6	ϑ_6



Spherical wrist with frame assignment

Spherical Wrist II

$$A_4^3(\vartheta_4) = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5^4(\vartheta_5) = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

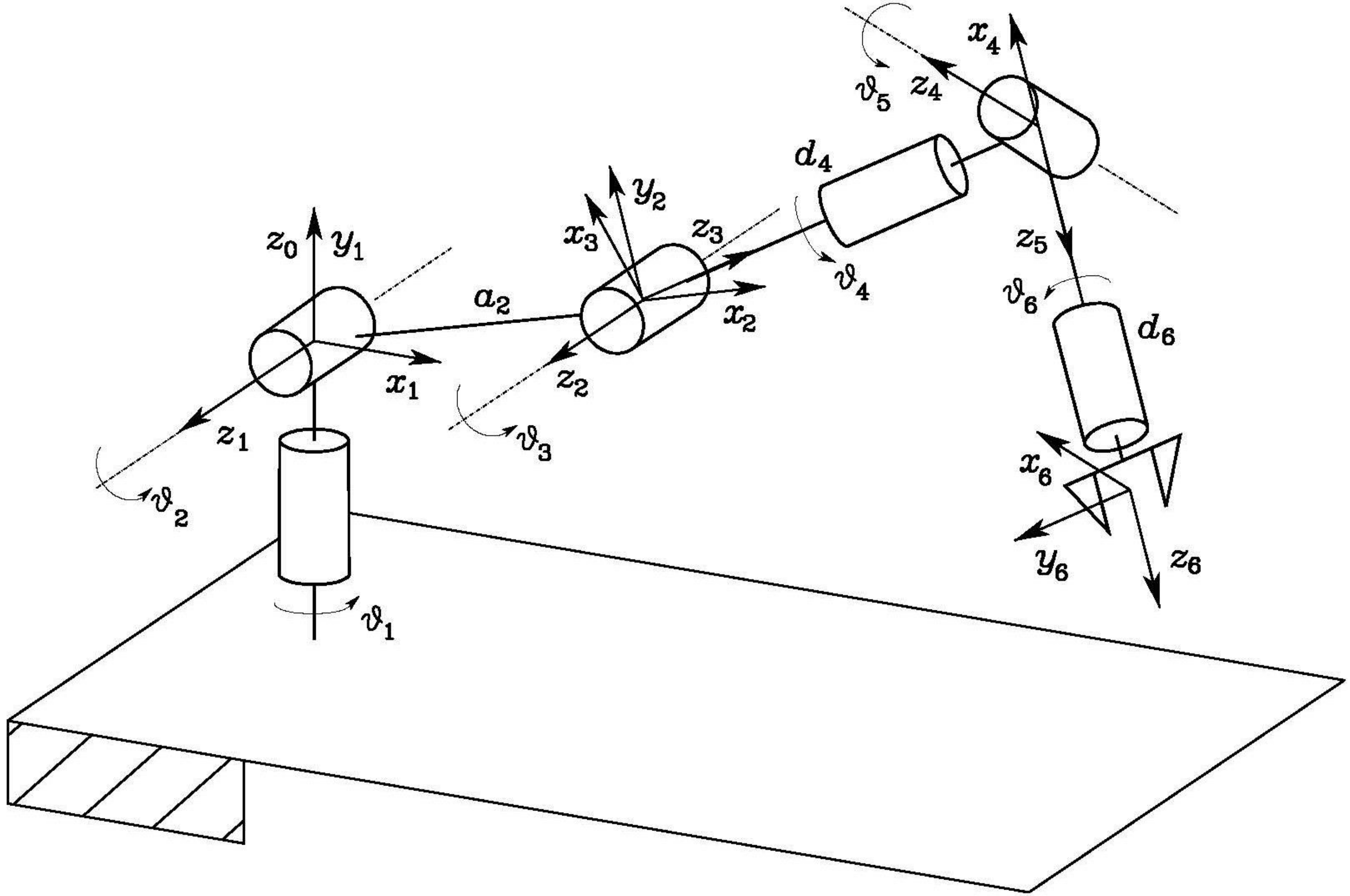
$$A_6^5(\vartheta_6) = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_6^3(q) = A_4^3 A_5^4 A_6^5 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Anthropomorphic Arm with Spherical Wrist

DH parameters

Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	a_2	0	0	ϑ_2
3	0	$\pi/2$	0	ϑ_3
4	0	$-\pi/2$	d_4	ϑ_4
5	0	$\pi/2$	0	ϑ_5
6	0	0	d_6	ϑ_6



Anthropomorphic arm with spherical wrist with frame assignment

Anthropomorphic Arm with Spherical Wrist II

$$A_3^2(\vartheta_3) = \begin{bmatrix} c_3 & 0 & s_3 & 0 \\ s_3 & 0 & -c_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_4^3(\vartheta_4) = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Anthropomorphic Arm with Spherical Wrist III

$$\boldsymbol{n}_6^0 = \begin{bmatrix} c_1\big(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\big) + s_1(s_4c_5c_6 + c_4s_6) \\ s_1\big(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\big) - c_1(s_4c_5c_6 + c_4s_6) \\ s_{23}(c_4c_5c_6 - s_4s_6) + c_{23}s_5c_6 \end{bmatrix}$$

$$\boldsymbol{s}_6^0 = \begin{bmatrix} c_1\big(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6\big) + s_1(-s_4c_5s_6 + c_4c_6) \\ s_1\big(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6\big) - c_1(-s_4c_5s_6 + c_4c_6) \\ -s_{23}(c_4c_5s_6 + s_4c_6) - c_{23}s_5s_6 \end{bmatrix}$$

$$\boldsymbol{a}_6^0 = \begin{bmatrix} c_1(c_{23}c_4s_5 + s_{23}c_5) + s_1s_4s_5 \\ s_1(c_{23}c_4s_5 + s_{23}c_5) - c_1s_4s_5 \\ s_{23}c_4s_5 - c_{23}c_5 \end{bmatrix}$$

$$\boldsymbol{p}_6^0 = \begin{bmatrix} a_2c_1c_2 + d_4c_1s_{23} + d_6\big(c_1(c_{23}c_4s_5 + s_{23}c_5) + s_1s_4s_5\big) \\ a_2s_1c_2 + d_4s_1s_{23} + d_6\big(s_1(c_{23}c_4s_5 + s_{23}c_5) - c_1s_4s_5\big) \\ a_2s_2 - d_4c_{23} + d_6(s_{23}c_4s_5 - c_{23}c_5) \end{bmatrix}$$

Joint Space and Operational Space

Joint space

$$\boldsymbol{q} = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix}^T$$

- $q_i = \vartheta_i$ (revolute joint)
- $q_i = d_i$ (prismatic joint)

Operational space

$$\boldsymbol{x}_e = \begin{bmatrix} \boldsymbol{p}_e \\ \boldsymbol{\phi}_e \end{bmatrix} \qquad \boldsymbol{x}_e \in \mathrm{R}^m, m \le n$$

- $\boldsymbol{p}_e$ (position)
- $\boldsymbol{\phi}_e$ (orientation)

Direct kinematics equation

$$\boldsymbol{x}_e = \boldsymbol{k}(\boldsymbol{q})$$

$m < n$: Kinematically *redundant* manipulator

Joint Space and Operational Space

Examples

- Three-link planar arm

$$\boldsymbol{x}_e = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \boldsymbol{k}(\boldsymbol{q}) = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ \vartheta_1 + \vartheta_2 + \vartheta_3 \end{bmatrix}$$

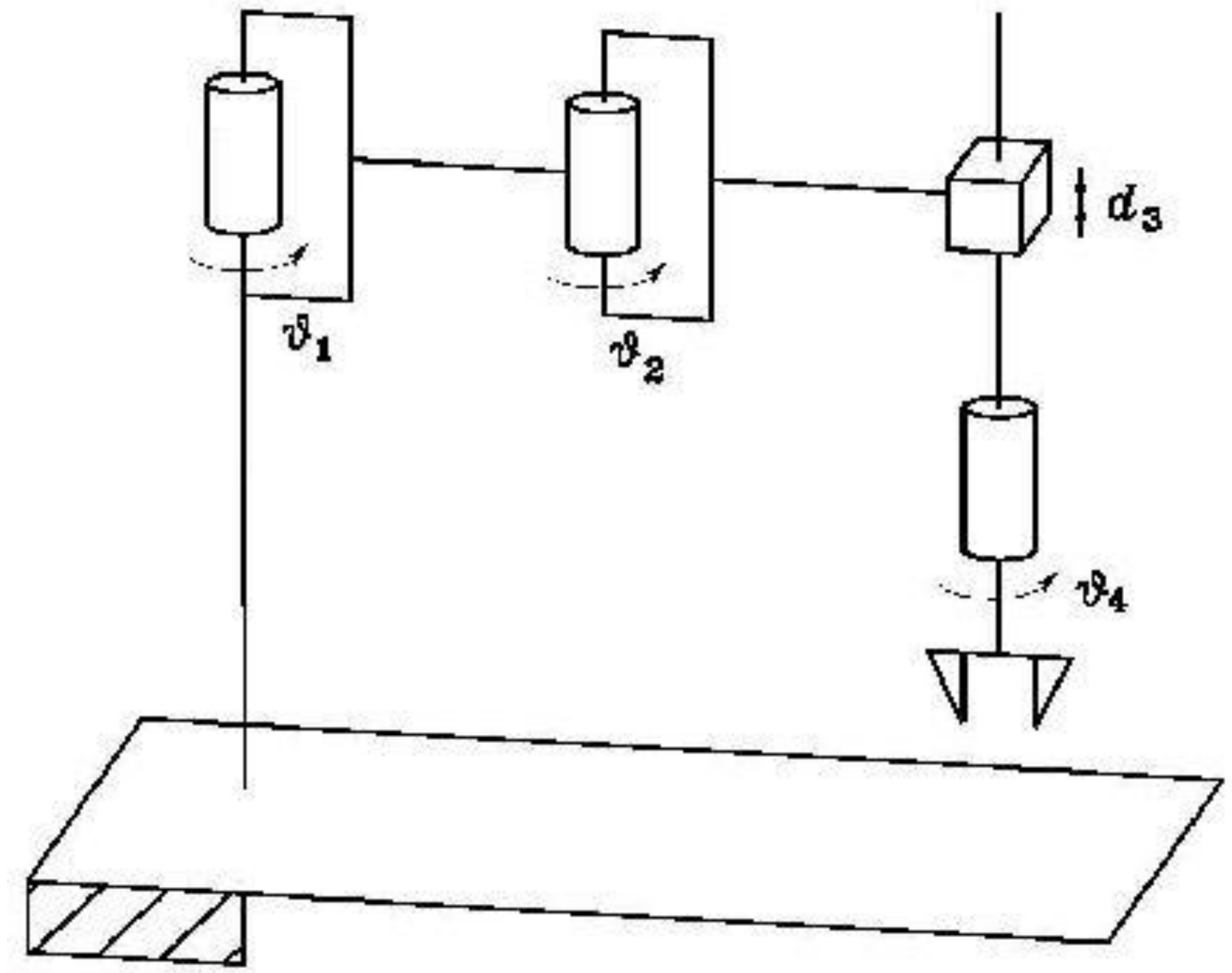
- In the most general case of a six-dimensional operational space ($m = 6$), the computation of the three components of the function $\phi_e(\boldsymbol{q})$ cannot be performed in closed form but goes through the computation of the elements of the rotation matrix $\boldsymbol{R}_e(\boldsymbol{q}) = [\boldsymbol{n}_e(\boldsymbol{q}) \quad \boldsymbol{s}_e(\boldsymbol{q}) \quad \boldsymbol{a}_e(\boldsymbol{q})]$ via inverse formulae

Summary

SECTION 9

Summary

1. By applying the rules for inverting a block-partitioned matrix, verify the expression of the homogeneous transformation matrix A_0^1
2. Find the direct kinematics equation for the SCARA manipulator in the figure.



SCARA manipulator