CS65K Robotics

Modelling, Planning and Control

Chapter 2: Kinematics

Section 2.12

LECTURE 6: INVERSE KINEMATICS

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Objectives

- What is Inverse Kinematics
- Cast Study - Three-link Planar Arm
  - The difficulties of solving the inverse kinematics problem are evidenced
  - Computation of closed-form solutions requires either algebraic or geometric intuition
  - The two solutions of the three-link planar arm are found

Objectives

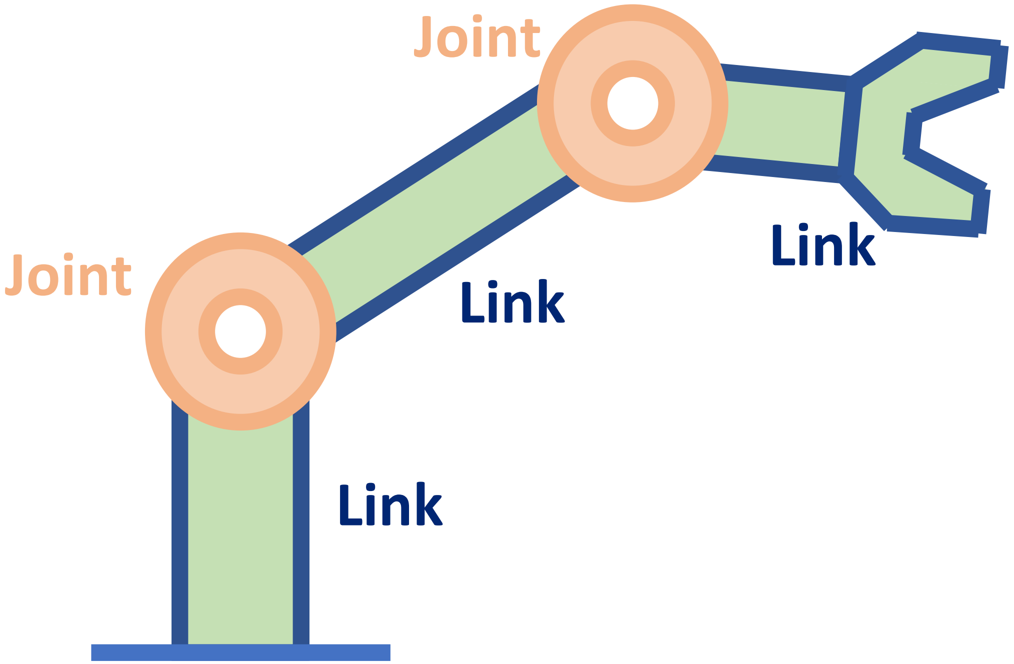
- Cast Study - Manipulators with Spherical Wrist
  - The inverse kinematics problem is simplified for manipulators having a spherical wrist
  - The solution of the anthropomorphic arm is computed and the four admissible postures corresponding to a given wrist position are found
  - The solution of the spherical wrist is computed

Forward Kinematics

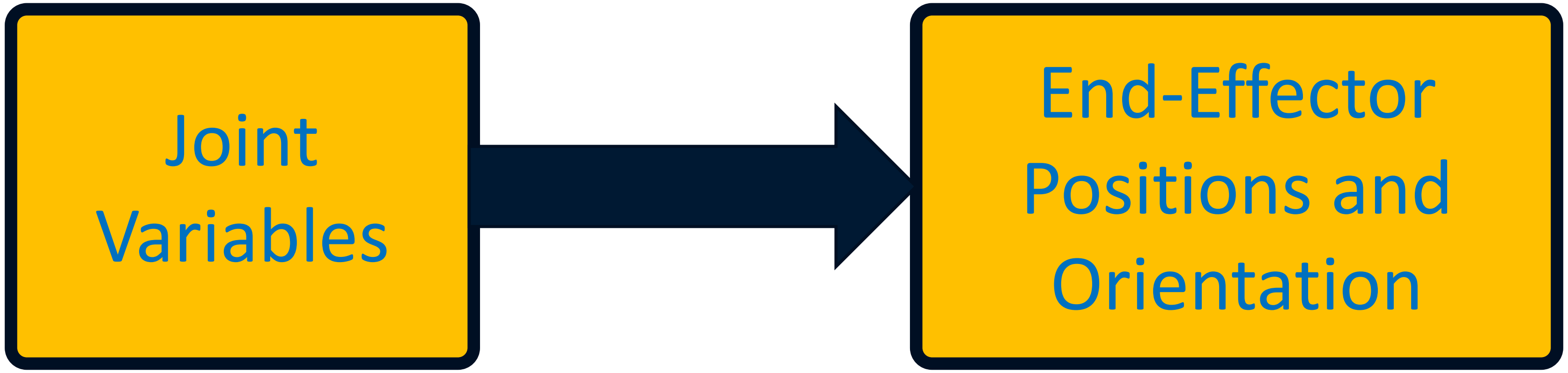
SECTION 1

Forward Kinematics

- Animator specifies joint variables: $\theta_1, \theta_2, a_1, a_2, a_3, a_4$
- Computer finds the positions of end-effector: $[x, y, z]$



Forward Kinematics

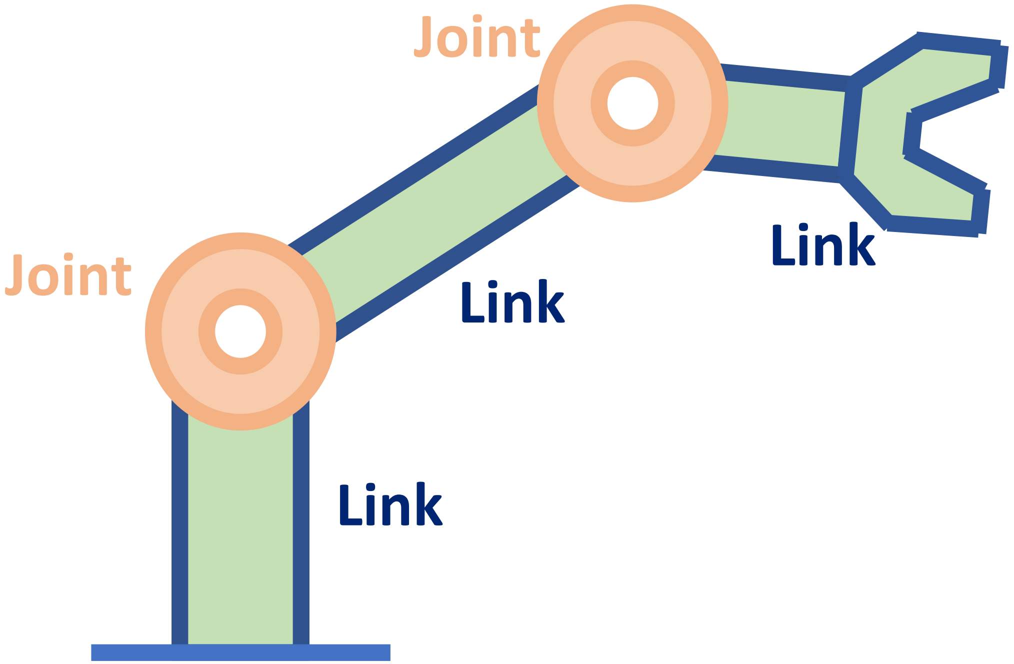


Inverse Kinematics

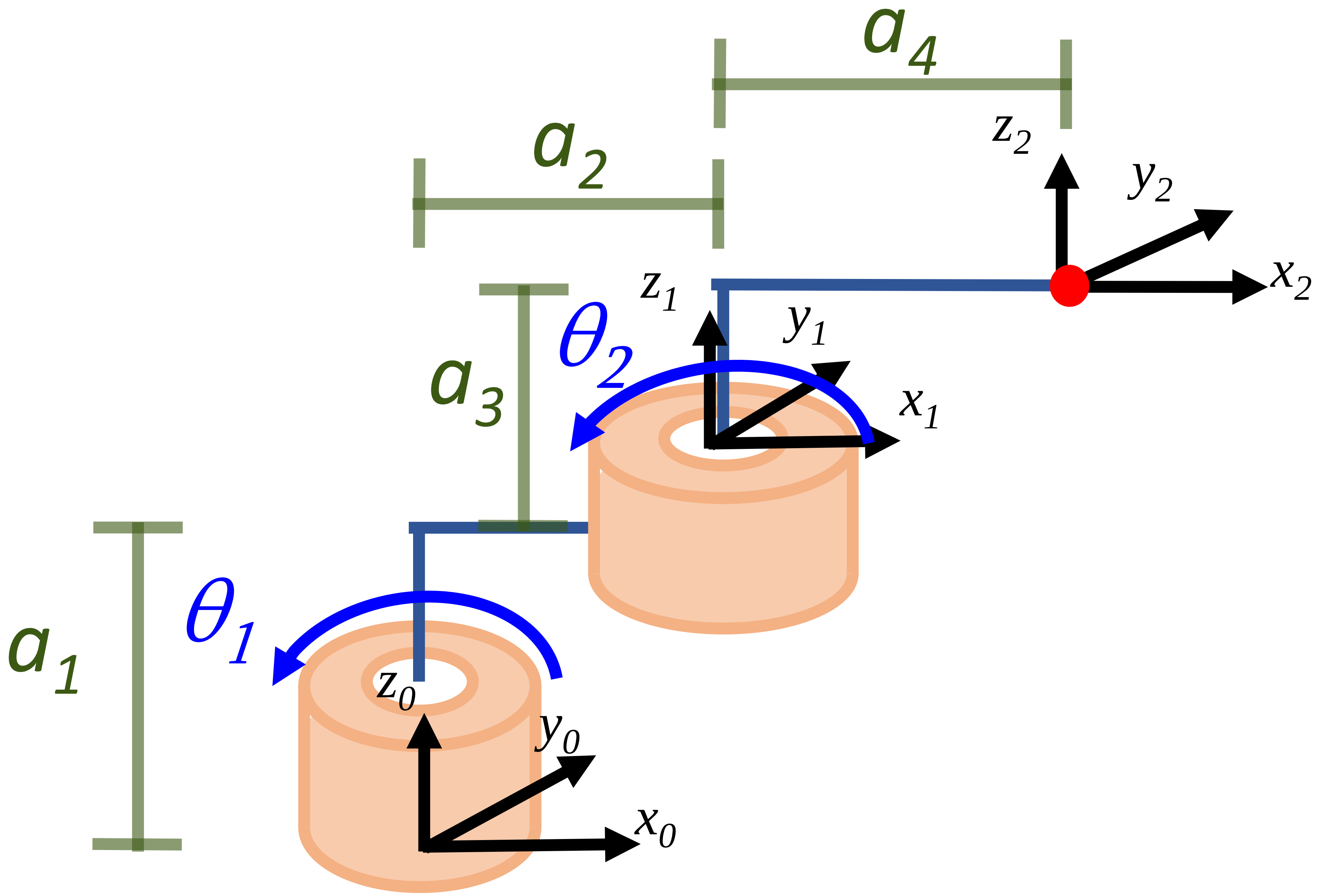
SECTION 2

Inverse Kinematics

- Animator specifies the positions of end-effector: [x, y, z]
- Computer finds the joint variables: θ_1, θ_2. *Note:* a_1, a_2, a_3, a_4 *are fixed*



Inverse Kinematics



a_4
a_2
z_2
y_2
x_2
z_1
y_1
θ_2
a_3
x_1
a_1
θ_1
z_0
y_0
x_0

Inverse Kinematics

$$H_2^0 = H_1^0 \, H_2^1$$

$$H_2^0 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & a_2\cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) & 0 & a_2\sin(\theta_1) \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & a_4\cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & a_4\sin(\theta_2) \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$p = H_2^0 \, p'$$

Inverse Kinematics (New Notation)

$$H_2^0 = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & a_2\,C\theta_1 \\ S\theta_1 & C\theta_1 & 0 & a_2\,S\theta_1 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & a_4\,C\theta_2 \\ S\theta_2 & C\theta_2 & 0 & a_4\,S\theta_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C(\theta_1+\theta_2) & -S(\theta_1+\theta_2) & 0 & a_4C(\theta_1+\theta_2) + a_2\,C\theta_1 \\ S(\theta_1+\theta_2) & C(\theta_1+\theta_2) & 0 & a_4S(\theta_1+\theta_2) + a_2\,S\theta_1 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$p = H_2^0\,p'$$

Given Original Point of Frame 2

$$p = H_2^0 \, p'$$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} C(\theta_1+\theta_2) & -S(\theta_1+\theta_2) & 0 & a_4 C(\theta_1+\theta_2) + a_2\, C\theta_1 \\ S(\theta_1+\theta_2) & C(\theta_1+\theta_2) & 0 & a_4 S(\theta_1+\theta_2) + a_2\, S\theta_1 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x = a_4 C(\theta_1+\theta_2) + a_2\, C\theta_1$$
$$y = a_4 S(\theta_1+\theta_2) + a_2\, S\theta_1$$

Non-linear Equation

Given x, y, to find θ_1 and θ_2

$$x = a_4 C(\theta_1 + \theta_2) + a_2\, C\theta_1$$
$$y = a_4 S(\theta_1 + \theta_2) + a_2\, S\theta_1$$

- It is a non-linear system of equation. Only numerical solution can be easily found. Symbolic reduction is almost impossible.

Cast Study - Three-link Planar Arm

SECTION 3

Inverse Kinematics Problem

Inverse kinematics

$$\boldsymbol{T} \Rightarrow \boldsymbol{q}$$

$$\boldsymbol{x} \Rightarrow \boldsymbol{q}$$

Complexity

- Possibility to find closed-form solutions (nonlinear equations to solve)
- Existence of multiple solutions
- Existence of infinite solutions (kinematically redundant manipulator)
- No admissible solutions, in view of the manipulator kinematic structure

Solutions

Computation of closed-form solutions

- Algebraic intuition
- Geometric intuition

No closed-form solutions

- Numerical solution techniques

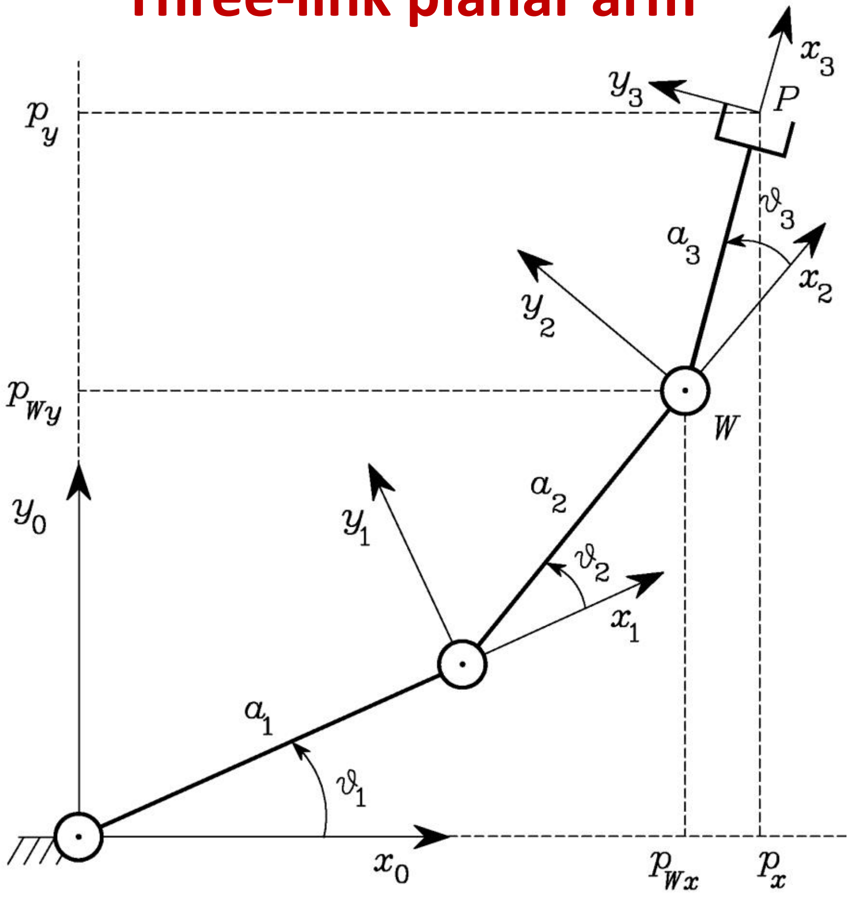
Solution of Three-link Planar Arm

$$\phi = \vartheta_1 + \vartheta_2 + \vartheta_3$$

$$p_{W_x} = p_x - a_3 c_\phi = a_1 c_1 + a_2 c_{12}$$

$$p_{W_y} = p_y - a_3 s_\phi = a_1 s_1 + a_2 s_{12}$$

Three-link planar arm



Algebraic solution

Squaring and summing …

$$
\begin{aligned}
c_2 &= \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1a_2} \qquad \Longrightarrow \vartheta_2 = \mathrm{Atan2}(s_2, c_2) \\
s_2 &= \pm\sqrt{1 - c_2^2} \\
s_1 &= \frac{(a_1 + a_2c_2)p_{Wy} - a_2s_2p_{Wx}}{p_{Wx}^2 + p_{Wy}^2} \qquad \Longrightarrow \vartheta_1 = \mathrm{Atan1}(s_1, c_1) \\
c_1 &= (a_1 + a_2c_2)p_{Wx} + a_2s_2p_{Wy}
\end{aligned}
$$

$$\vartheta_3 = \phi - \vartheta_1 - \vartheta_2$$

Geometric solution

- Application of cosine theorem to the triangle formed by links a_1, a_2 and the segment connecting points W and O

$$p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\pi - \vartheta_2) \qquad \sqrt{p_{Wx}^2 + p_{Wy}^2} \le a_1 + a_2$$

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1a_2} \quad \Longrightarrow \quad \vartheta_2 = \pm\cos^{-1}(c_2)$$

$$\vartheta_2 \in (-\pi, 0) : \text{elbow-up posture}$$

$$\vartheta_2 \in (0, \pi) : \text{elbow-down posture}$$

$$\alpha = \mathrm{Atan2}(p_{Wy}, p_{Wx})$$

$$c_\beta\sqrt{p_{Wx}^2 + p_{Wy}^2} = a_1 + a_2c_2 \quad \Longrightarrow \quad \beta = \cos^{-1}\left(\frac{p_{Wx}^2 + p_{Wy}^2 + a_1^2 - a_2^2}{2a_1\sqrt{p_{Wx}^2 + p_{Wy}^2}}\right)$$

$$\vartheta_1 = \alpha \pm \beta$$

Cast Study - Manipulators with Spherical Wrist

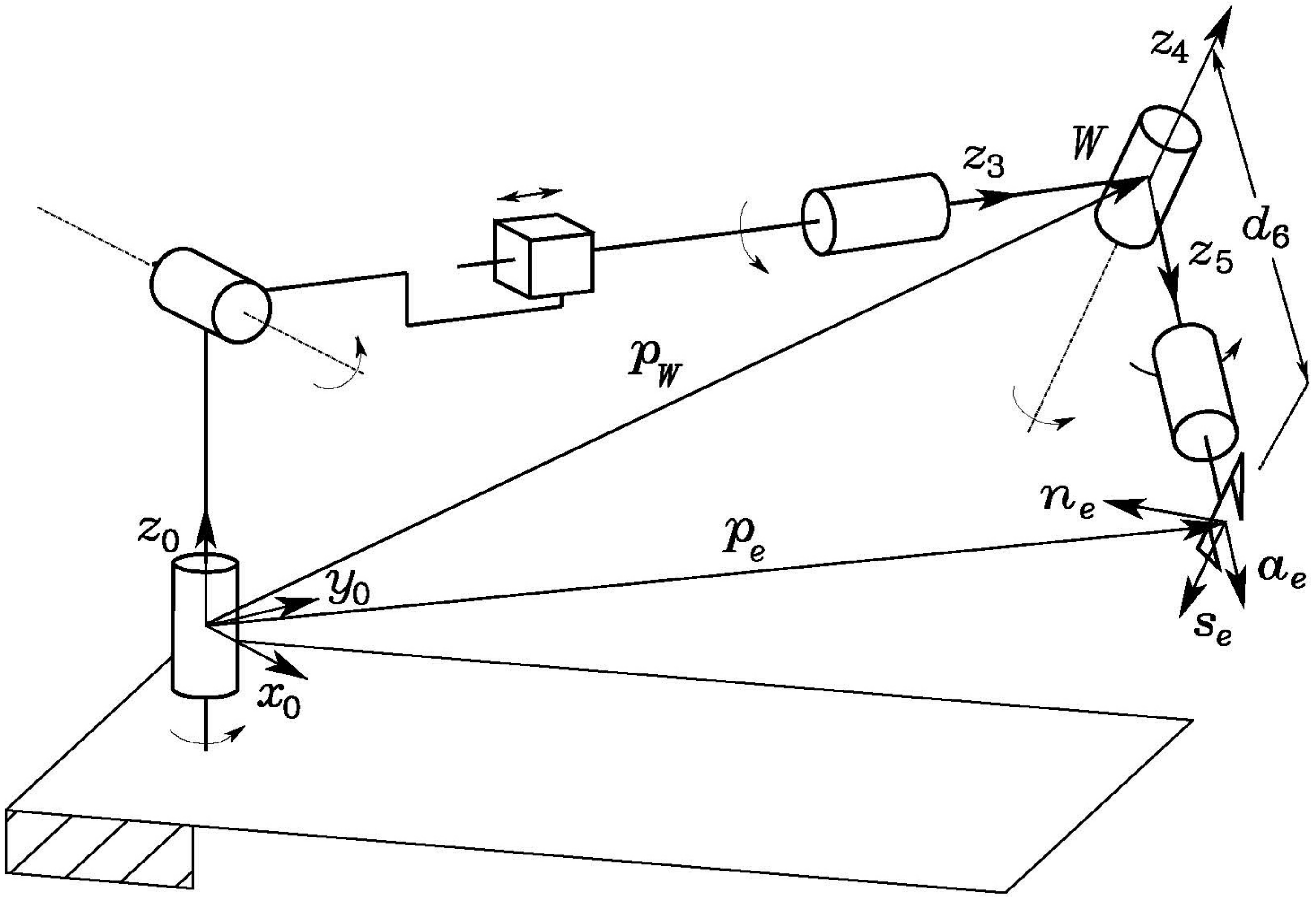
SECTION 4

Kinematic Decoupling

$$\boldsymbol{p}_W = \boldsymbol{p}_e - d_6\boldsymbol{a}_e$$

Solution decoupling

- Compute wrist position $\boldsymbol{p}_W(q_1, q_2, q_3)$
- Solve inverse kinematics for (q_1, q_2, q_3)
- Compute $\boldsymbol{R}_3^0(q_1, q_2, q_3)$
- Compute $\boldsymbol{R}_6^3(\vartheta_4, \vartheta_5, \vartheta_6) = \boldsymbol{R}_3^{0T}\boldsymbol{R}_e$
- Solve inverse kinematics for orientation $(\vartheta_4, \vartheta_5, \vartheta_6)$



Manipulator with spherical wrist

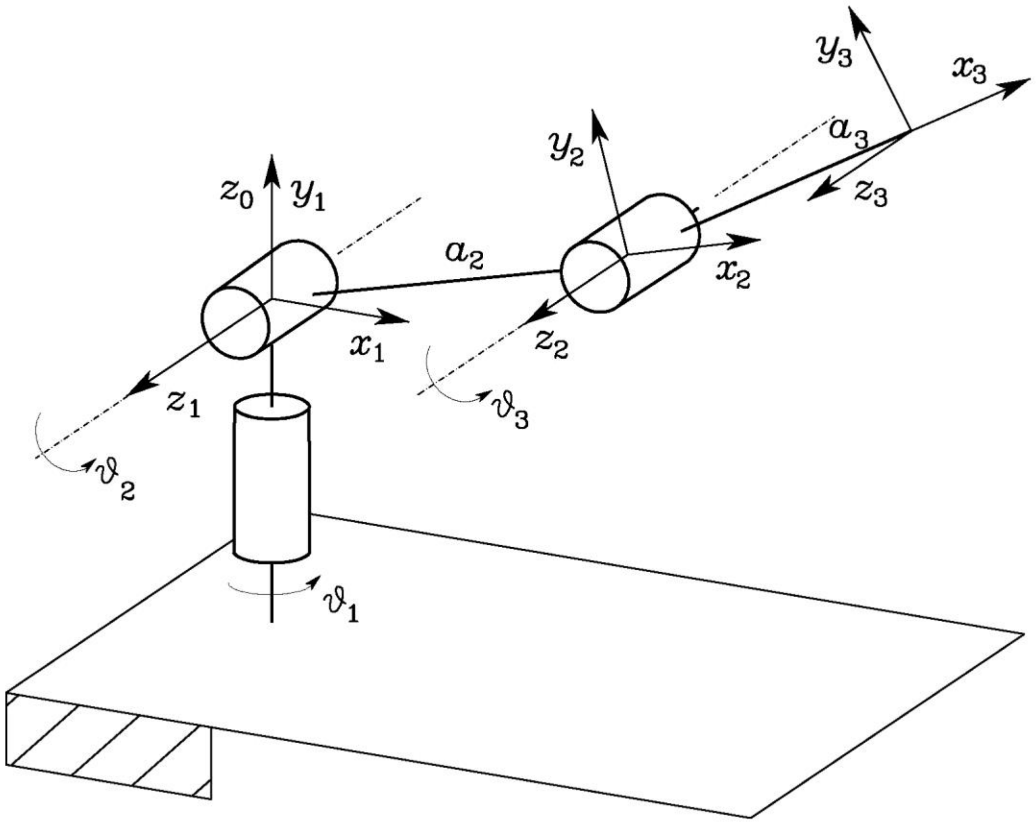
Solution of Anthropomorphic Arm

$$p_{W_x} = c_1(a_2c_2 + a_3c_{23})$$

$$p_{W_y} = s_1(a_2c_2 + a_3c_{23})$$

$$p_{W_z} = a_2s_2 + a_3s_{23}$$

Anthropomorphic arm



Solution of Anthropomorphic Arm II

- Squaring and summing:

$$
\begin{aligned}
p_{Wx}^2 &+ p_{Wy}^2 + p_{Wz}^2 = a_2^2 + a_3^2 + 2a_2a_3c_3 \\
c_3 &= \frac{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 - a_2^2 - a_3^2}{2a_2a_3} \qquad \Longrightarrow \vartheta_3 = \mathrm{Atan2}(s_3, c_3) \\
s_3 &= \pm\sqrt{1 - c_3^2} \qquad \vartheta_{3,\mathrm{I}} \in [-\pi, \pi] \quad \vartheta_{3,\mathrm{II}} = -\vartheta_{3,\mathrm{I}} \\
p_{Wx}^2 &+ p_{Wy}^2 = (a_2c_2 + a_3c_{23})^2 \\
c_2 &= \frac{\pm\sqrt{p_{Wx}^2 + p_{Wy}^2}(a_2 + a_3c_3) + p_{Wz}a_3s_3}{a_2^2 + a_3^2 + 2a_2a_3c_3} \qquad \Longrightarrow \vartheta_2 = \mathrm{Atan2}(s_2, c_2) \\
s_2 &= \frac{p_{Wz}(a_2 + a_3c_3) \mp \sqrt{p_{Wx}^2 + p_{Wy}^2}\,a_3s_3}{a_2^2 + a_3^2 + 2a_2a_3c_3}
\end{aligned}
$$

Computation of Second Angle

- Squaring and summing …

$$p_{Wx}^2 + p_{Wy}^2 = (a_2 c_2 + a_3 c_{23})^2 \Longrightarrow a_2 c_2 + a_3 c_{23} = \pm\sqrt{p_{Wx}^2 + p_{Wy}^2}$$

$$c_2 = \frac{\pm\sqrt{p_{Wx}^2 + p_{Wy}^2}(a_2 + a_3 c_3) + p_{Wz} a_3 s_3}{a_2^2 + a_3^2 + 2a_2 a_3 c_3} \Longrightarrow \vartheta_2 = \mathrm{Atan2}(s_2, c_2)$$

$$s_2 = \frac{p_{Wz}(a_2 + a_3 c_3) \mp \sqrt{p_{Wx}^2 + p_{Wy}^2}\, a_3 s_3}{a_2^2 + a_3^2 + 2a_2 a_3 c_3}$$

Admissible Solutions

$$\vartheta_{2,\mathrm{I}} = \operatorname{Atan2}\Big((a_2 + a_3c_3)p_{Wz} - a_3s_3^+\sqrt{p_{Wx}^2 + p_{Wy}^2},\; (a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^+p_{Wz}\Big) \qquad s_3^+ = \sqrt{1-c_3^2}$$

$$\vartheta_{2,\mathrm{II}} = \operatorname{Atan2}\Big((a_2 + a_3c_3)p_{Wz} + a_3s_3^+\sqrt{p_{Wx}^2 + p_{Wy}^2},\; -(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^+p_{Wz}\Big)$$

$$\vartheta_{2,\mathrm{III}} = \operatorname{Atan2}\Big((a_2 + a_3c_3)p_{Wz} - a_3s_3^-\sqrt{p_{Wx}^2 + p_{Wy}^2},\; (a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^-p_{Wz}\Big) \qquad s_3^- = -\sqrt{1-c_3^2}$$

$$\vartheta_{2,\mathrm{IV}} = \operatorname{Atan2}\Big((a_2 + a_3c_3)p_{Wz} + a_3s_3^-\sqrt{p_{Wx}^2 + p_{Wy}^2},\; -(a_2 + a_3c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3s_3^-p_{Wz}\Big)$$

Computation of First Angle

$$
\begin{aligned}
p_{Wx} &= \pm c_1 \sqrt{p_{Wx}^2 + p_{Wy}^2} \\
p_{Wy} &= \pm s_1 \sqrt{p_{Wx}^2 + p_{Wy}^2} \\
&\Downarrow \\
\vartheta_{1,\mathrm{I}} &= \mathrm{Atan2}(p_{Wy}, p_{Wx})
\end{aligned}
$$

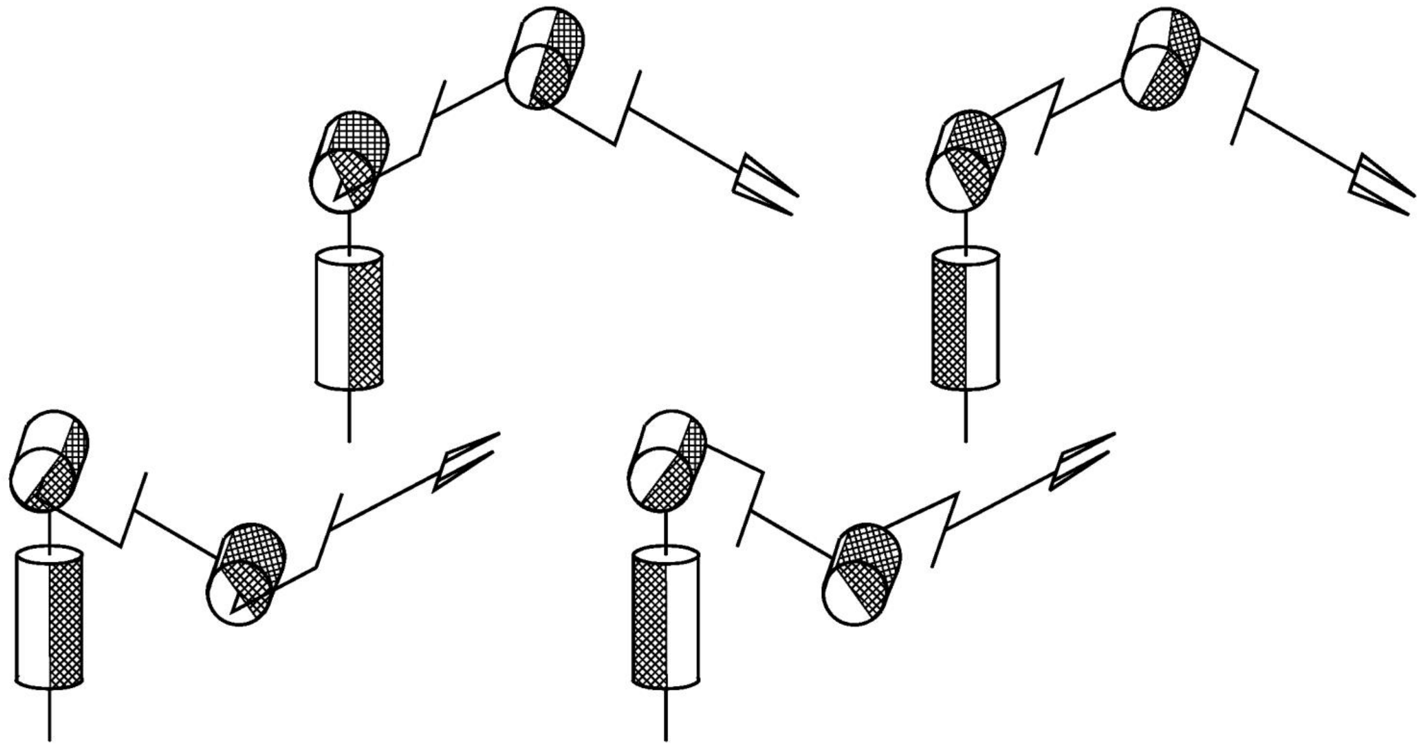
$$
\vartheta_{1,\mathrm{II}} = \mathrm{Atan2}(-p_{Wy}, -p_{Wx}) \Longrightarrow \vartheta_{1,\mathrm{II}} = \begin{cases} \mathrm{Atan2}(p_{Wy}, p_{Wx}) - \pi & p_{Wy} \geq 0 \\ \mathrm{Atan2}(p_{Wy}, p_{Wx}) + \pi & p_{Wy} < 0 \end{cases}
$$

Four solutions:

$$
(\vartheta_{1,\mathrm{I}}, \vartheta_{2,\mathrm{I}}, \vartheta_{3,\mathrm{I}}) \quad (\vartheta_{1,\mathrm{I}}, \vartheta_{2,\mathrm{III}}, \vartheta_{3,\mathrm{II}}) \quad (\vartheta_{1,\mathrm{II}}, \vartheta_{2,\mathrm{II}}, \vartheta_{3,\mathrm{I}}) \quad (\vartheta_{1,\mathrm{II}}, \vartheta_{2,\mathrm{IV}}, \vartheta_{3,\mathrm{II}})
$$

It is possible to find the solutions only if at least $p_{Wx} \neq 0$ or $p_{Wy} \neq 0$

Computation of First Angle II



The four configurations of an anthropomorphic arm compatible with a given wrist position

Solution of Spherical Wrist

- $(\vartheta_4, \vartheta_5, \vartheta_6)$ constitute a set of Euler angles ZYZ with respect to Frame 3

$$\boldsymbol{R}_6^3 = \begin{bmatrix} n_x^3 & s_x^3 & a_x^3 \\ n_y^3 & s_y^3 & a_y^3 \\ n_z^3 & s_z^3 & a_z^3 \end{bmatrix}$$

$$\vartheta_4 = \mathrm{Atan2}(a_y^3, a_x^3)$$

$$\vartheta_5 = \mathrm{Atan2}\left(\sqrt{(a_x^3)^2 + (a_y^3)^2}, a_z^3\right) \qquad \vartheta_5 \in (0, \pi)$$

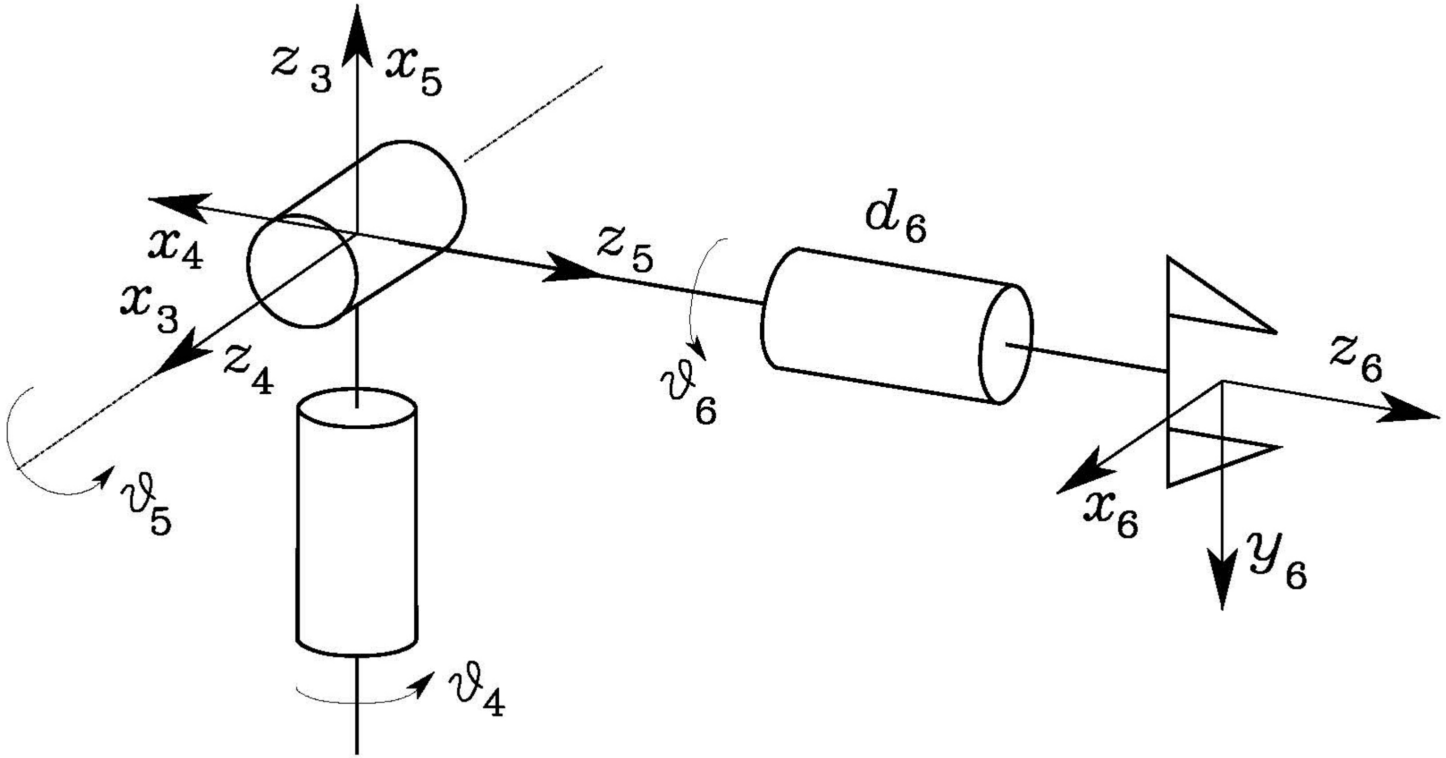
$$\vartheta_6 = \mathrm{Atan2}(s_z^3, -n_z^3)$$

$$\vartheta_4 = \mathrm{Atan2}(-a_y^3, -a_x^3)$$

$$\vartheta_5 = \mathrm{Atan2}\left(-\sqrt{(a_x^3)^2 + (a_y^3)^2}, a_z^3\right) \qquad \vartheta_5 \in (-\pi, 0)$$

$$\vartheta_6 = \mathrm{Atan2}(-s_z^3, n_z^3)$$

Solution of Spherical Wrist II

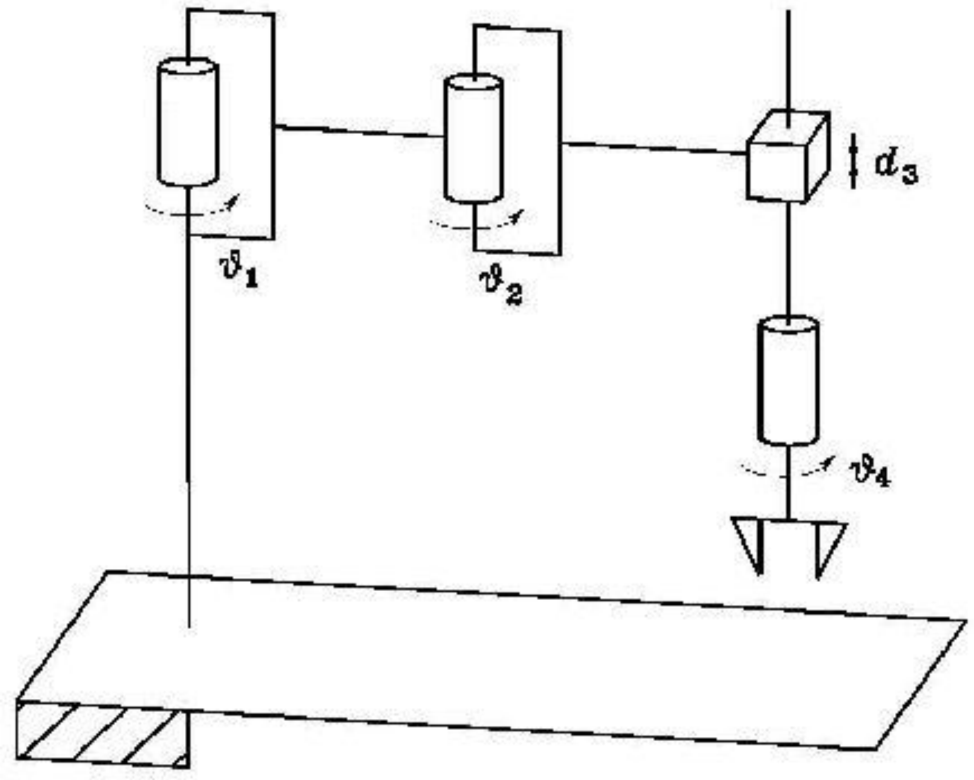


Spherical wrist

Further Insights

1. With reference to the inverse kinematics of the anthropomorphic arm, discuss the number of solutions in the singular cases of θ_2 and θ_4

2. Solve the inverse kinematics for the SCARA manipulator in the figure.



SCARA manipulator

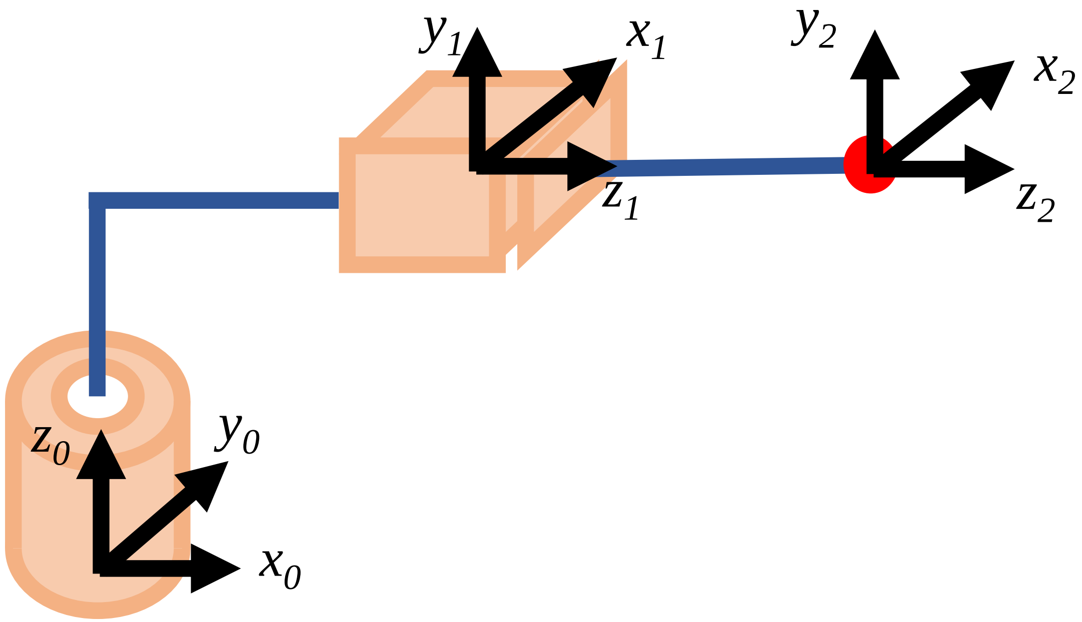
Graphical Method

SECTION 5

Graphical Method

- Inverse kinematics is the problem in which we know a position we want the end-effector to go to, and we need to find the values of the joint variables that move the end-effector to that position.
- In this section, we learn the 'graphical approach' to inverse kinematics, see some examples, and use the inverse kinematics equations to manipulate a robot arm of the similar model.

Example 1: Cylindrical Manipulator (2 DOF)



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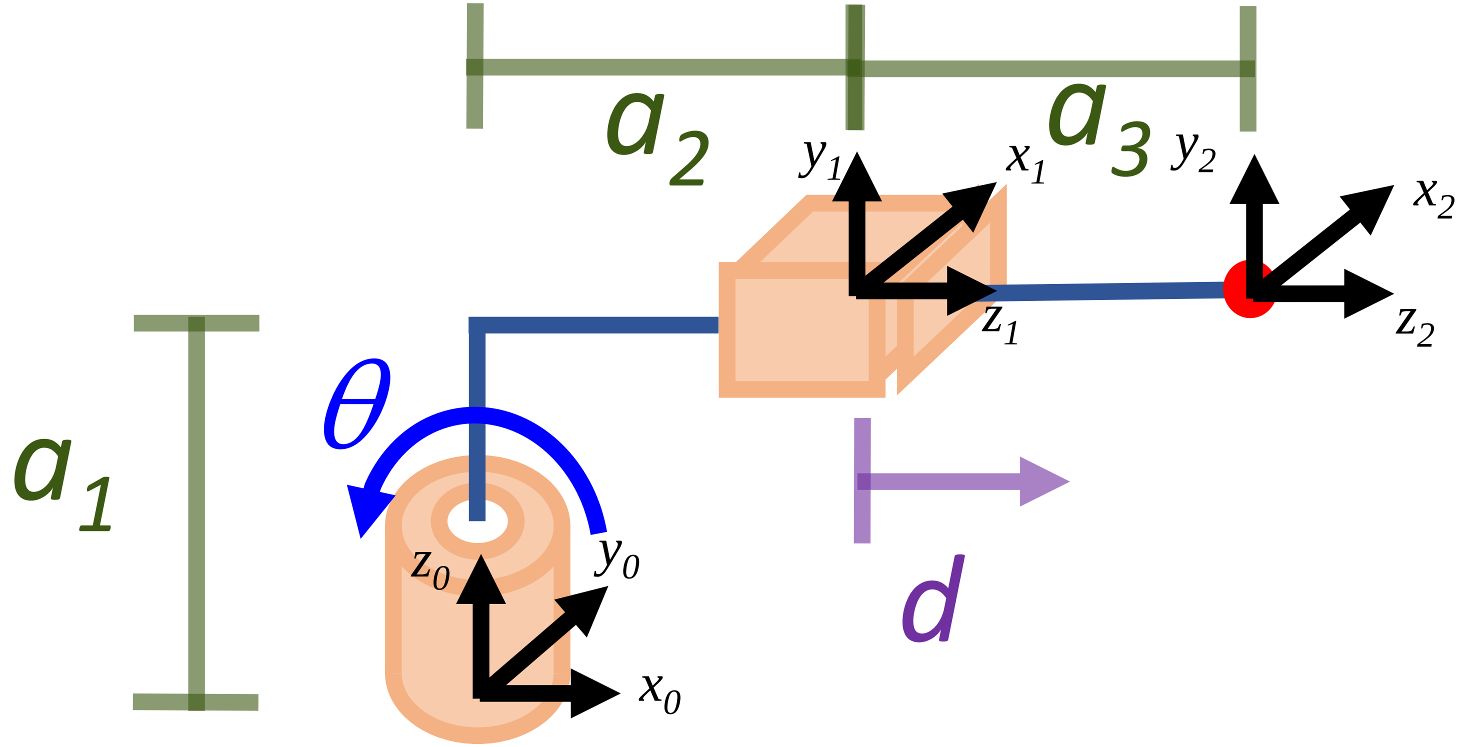
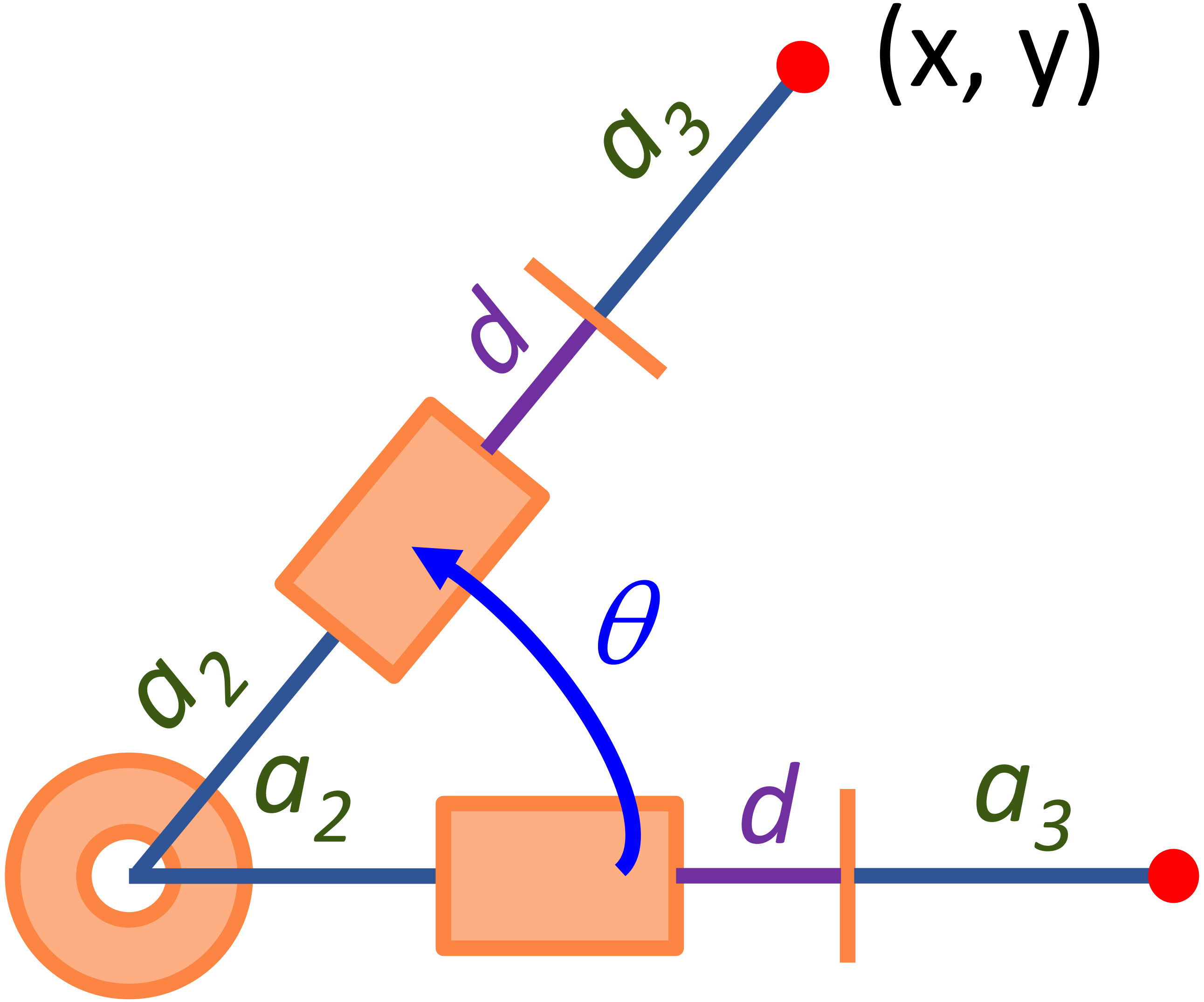




Given x, y
Solve (d, θ)

(1) $r = a_2 + a_3$

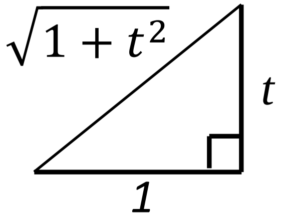
(2) $x = (r + d)\cos(\theta)$
(3) $y = (r + d)\sin(\theta)$

(4) $\frac{y}{x} = \tan(\theta) = t$

$\theta = \tan^{-1}(t)$

$\sin(\theta) = \frac{t}{\sqrt{1+t^2}} = \frac{y}{r+d}$

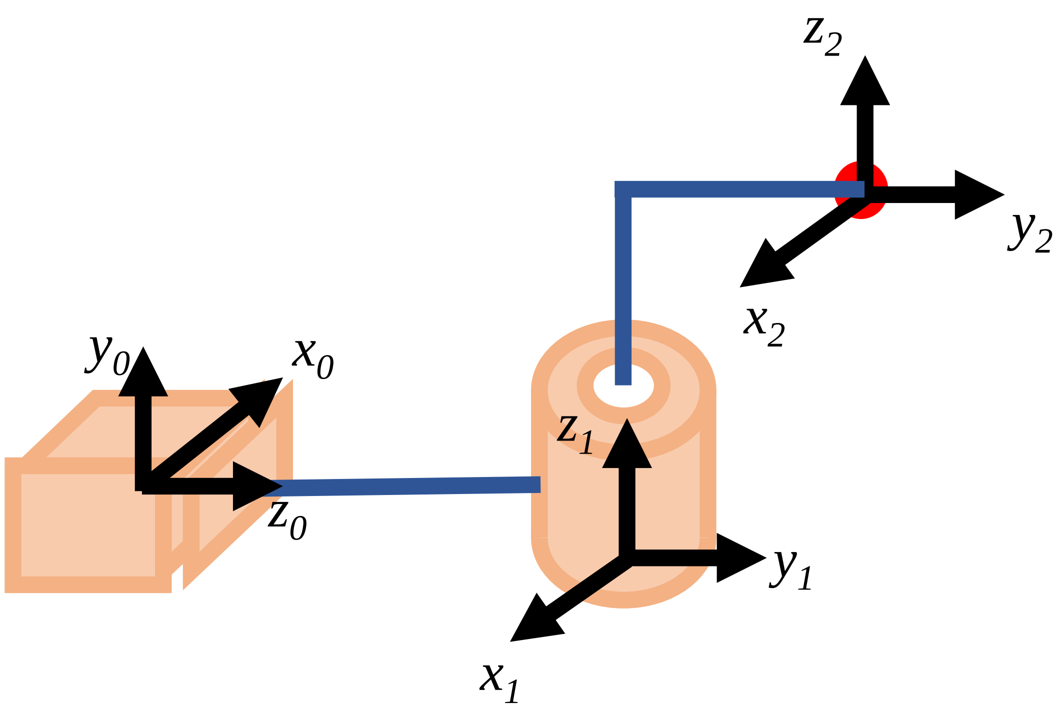
$\frac{\sqrt{1+t^2}}{t} = \frac{r+d}{y}$



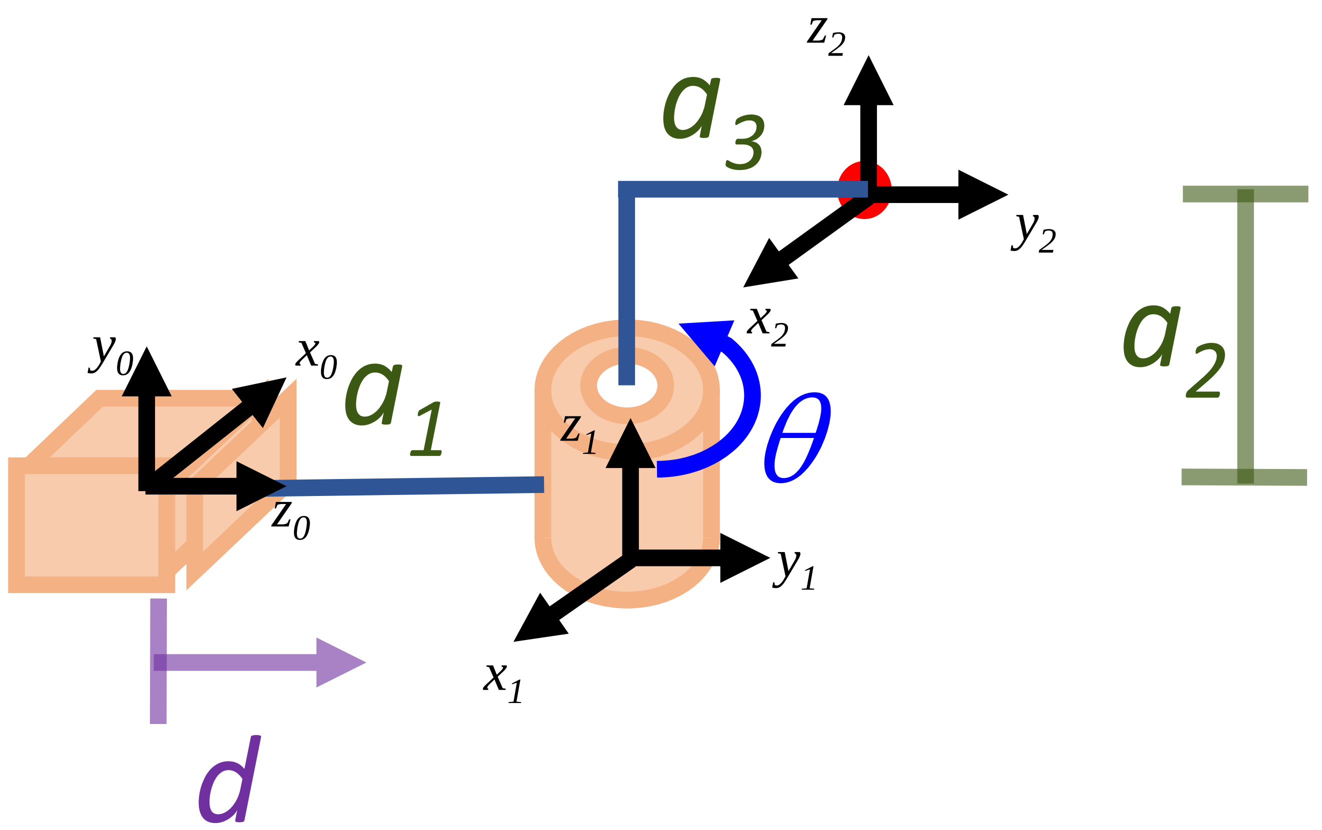
$d = \frac{y\sqrt{1+t^2}}{t} - r = \frac{y\sqrt{x^2+y^2}}{\frac{y}{x}x} - r$

$= \sqrt{x^2+y^2} - (a_2 + a_3)$

Example 2: Manipulator (2 DOF)

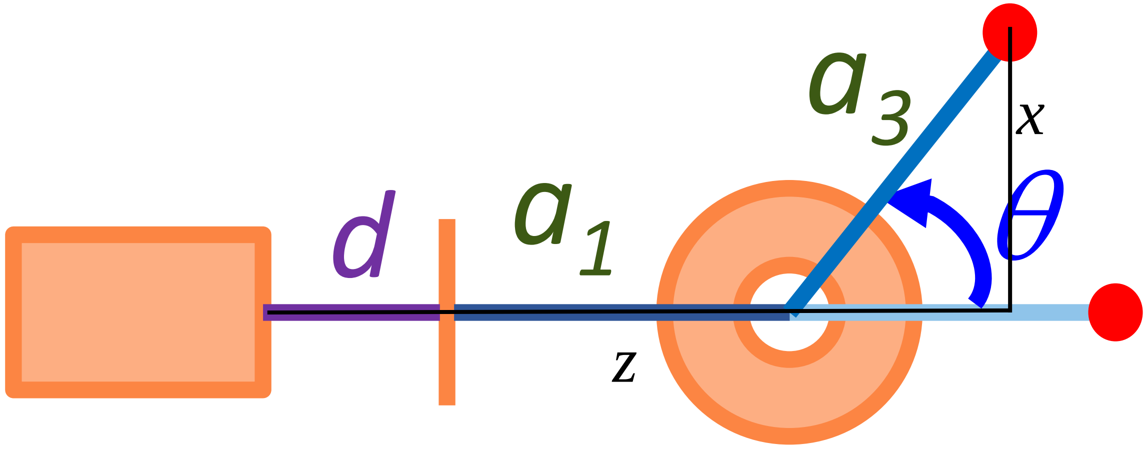


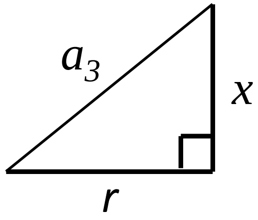
Example 2: Manipulator (2 DOF)



Given (x, z), y fixed

To find (d, θ)

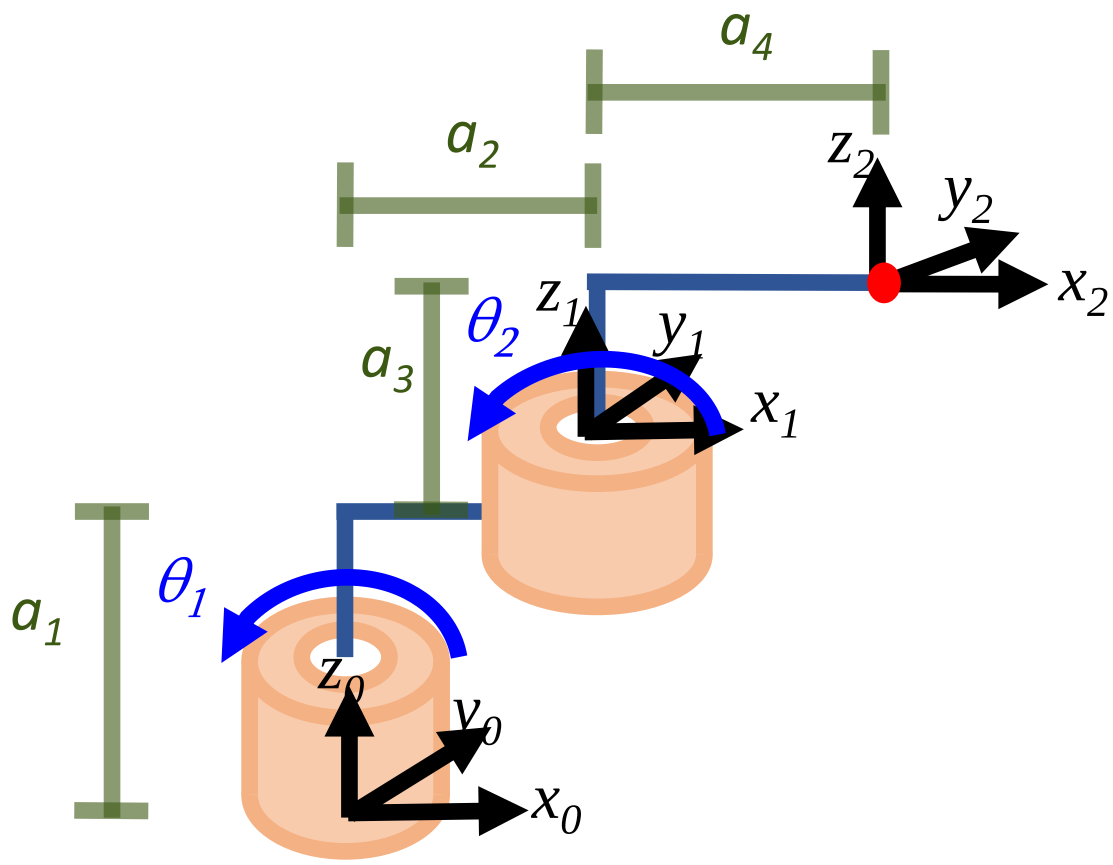


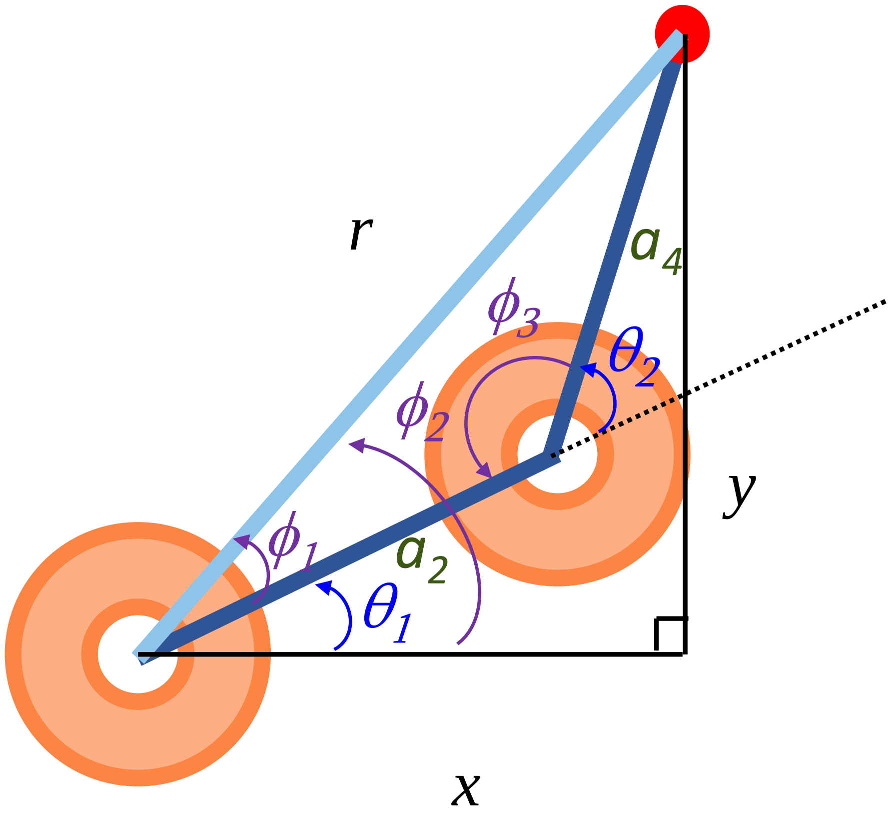


$$r = \sqrt{a_3^2 - x^2}$$

$$d = z - a_1 - \sqrt{a_3^2 - x^2}$$

Example 3: Manipulator (2 DOF)





Given (x, y)
To find (θ_1, θ_2)

$$\theta_1 = \phi_2 - \phi_1 \qquad (1)$$

$$\theta_2 = 180 - \phi_3 \qquad (2)$$

$$\phi_1 = cos^{-1}\left(\frac{a_2^2 + r^2 - a_4^2}{2a_2 r}\right) \qquad (3)$$

$$\phi_2 = tan^{-1}\left(\frac{y}{x}\right) \qquad (4)$$

$$\phi_3 = cos^{-1}\left(\frac{a_2^2 + a_4^2 - r^2}{2a_2 a_4}\right) \qquad (5)$$