CS65K Robotics

Modelling, Planning and Control

Chapter 2: Kinematics

LESSON 3: REPRESENTATION OF ORIENTATION

DR. ERIC CHOU

IEEE SENIOR MEMBER

Cartesian Coordination

- The rotation matrix is introduced as the fundamental tool to describe the orientation of a rigid body
- The three equivalent geometrical meanings of a rotation matrix are explained
- The rule for composition of elementary rotations is presented

Kinematics

SECTION 1

Kinematics

- The branch of mechanics concerned with the motion of objects without reference to the forces which cause the motion.
- Relationship between the joint positions and the end-effector position and orientation

Intro to Kinematics

$d = X_F - X_0$ $v_F = v_0 + at$

$$\boxed{X_F = X_0 + V_0 t + \frac{1}{2}at^2}$$

$d = vt$ $v_F^2 = v_0^2 + 2ad$

Representations of orientation

- Rotation matrix
- Euler angles
- Four-parameter representations

Direct kinematics

- Homogeneous transoformations
- Denavit-Hartenberg convention
- Kinematics of typical manipulator structures

Inverse kinematics

- Solution of three-link planar arm
- Solution of anthropomorphic arm
- Solution of spherical wrist

Degree of Freedom

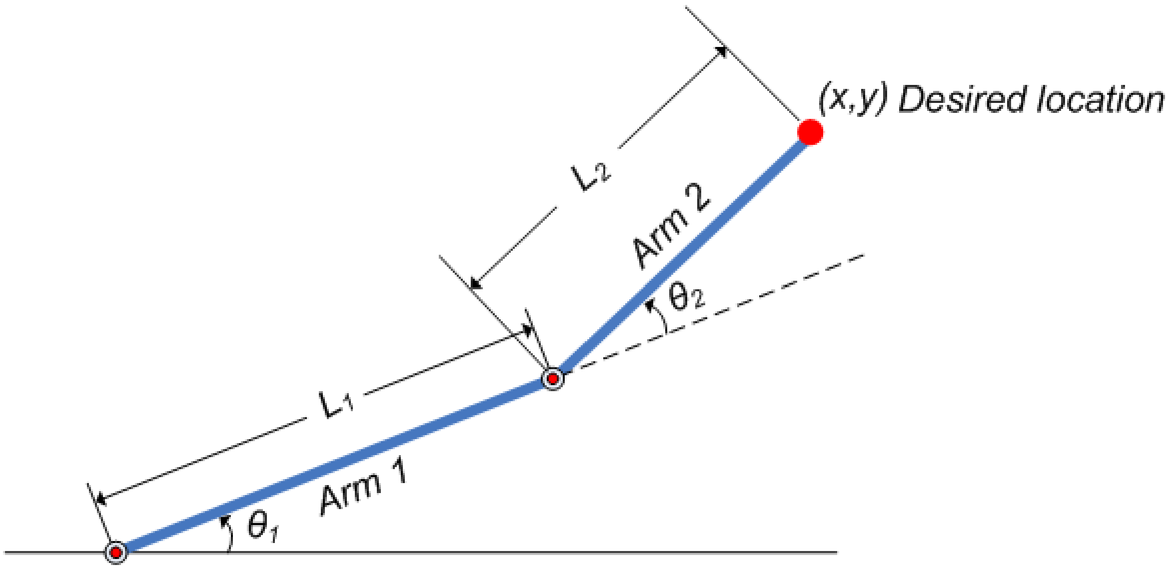
SECTION 2

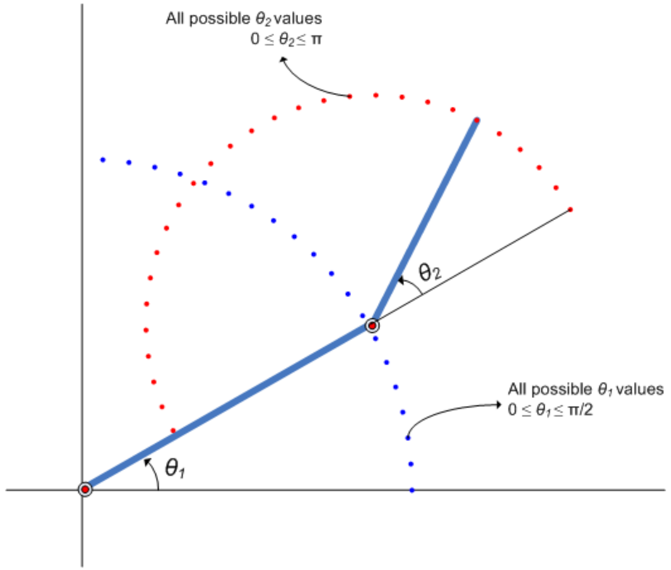
Soft body

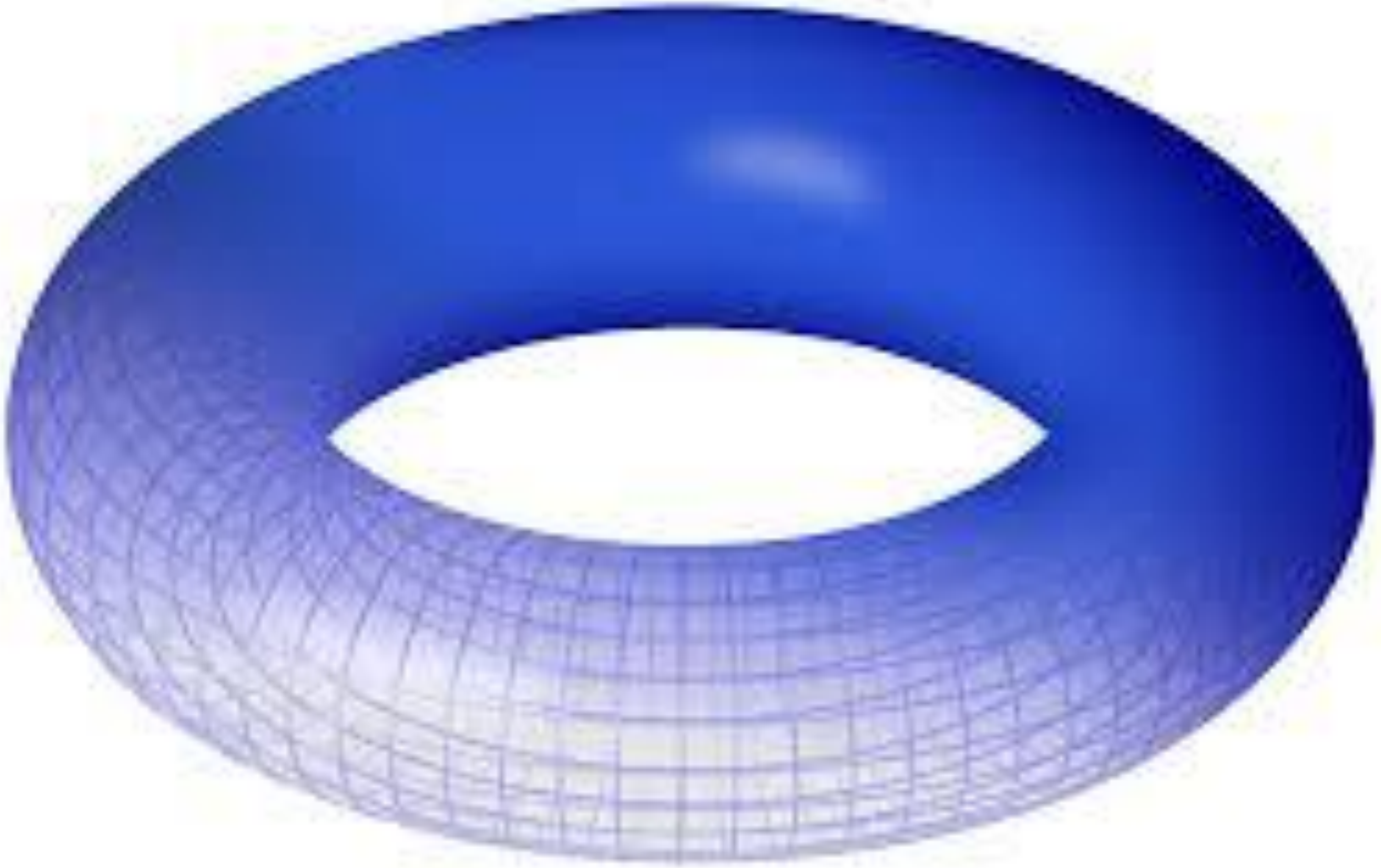
Rigid body

Terminology

- **Configuration:** A specification of the position of all points of a robot
- **C-Space:** The space of all configurations
- **DOF:** The dimension of the C-Space. (Degree of Freedom)

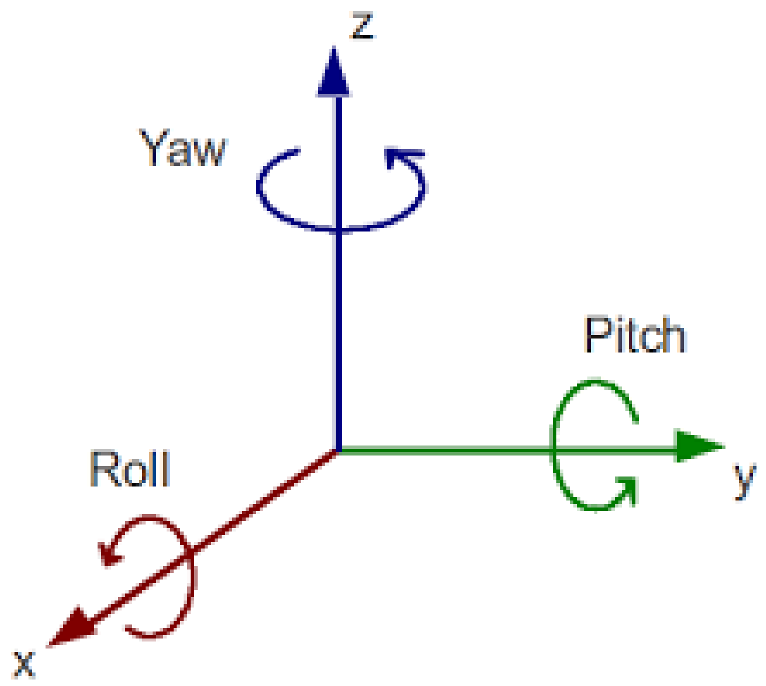






Degree of Freedom

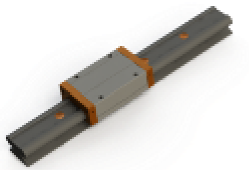
$DOF = \sum(freedoms\ of\ points) + \#\ of\ independent\ constraints$



Axis: n

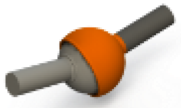
Angle: C_2^n

•**Prismatic** — Allows translation along a single standard axis (x, y, or z). Joint blocks can contain up to three prismatic joint primitives, one for each translational DoF. Prismatic primitives are labelled P*, where the asterisk denotes the axis of motion, e.g., Px, Py, or Pz.



•**Revolute** — Allows rotation about a single standard axis (x, y, or z). Joint blocks can contain up to three revolute joint primitives, one for each rotational DoF. Revolute primitives are labelled R*, where the asterisk denotes the axis of motion, e.g., Rx, Ry, or Rz.

•**Spherical** — Allows rotation about any 3-D axis, [*x*, *y*, *z*]. Joint blocks contain no more than one spherical primitive, and never in combination with revolute primitives. Spherical primitives are labelled S.



- **Lead Screw Primitive** — Allows coupled rotation and translation on a standard axis (e.g., z). This primitive converts between rotation at one end and translation at the other. Joint blocks contain no more than one lead screw primitive. Lead screw primitives are labeled LS*, where the asterisk denotes the axis of motion.

- **Constant Velocity Joint** — Allows rotation at constant velocity between intersecting though arbitrarily aligned shafts. Joint blocks contain no more than one constant velocity primitive. Constant velocity primitives are labelled CV.

Joint Block	Constraint I	Entities I	Constraint II	Entities II	Offsets	Notes
Cartesian Joint					$\theta_I = \theta_{II} = 0°$	
Cylindrical Joint					$d_I = 0$	
Planar Joint					$d_I = 0$	
Prismatic Joint					$d_I = d_{II} = 0$	1
					$d_I = 0,\ \theta_{II} = 90°$	
					$d_I, d_{II} \geq 0$	2
Rectangular Joint					$d_I, \theta_{II} \geq 0$	
					$d_I \geq 0, \theta_{II} = 0°$	
Revolute Joint						3
					$d_I = 0, d_{II} \geq 0$	
Spherical Joint					$d_I = 0$	

Simscape™ Multibody™ software can successfully import an Autodesk Inventor® assembly model with any joint (all are supported) and the following constraints:

- Angle offset
- Mate/flush
- Insert

Simscape Multibody software can successfully import CAD assemblies whose constraints join these constraint entities:

- Circle/arc
- Cone
- Cylinder
- Line
- Plane
- Point

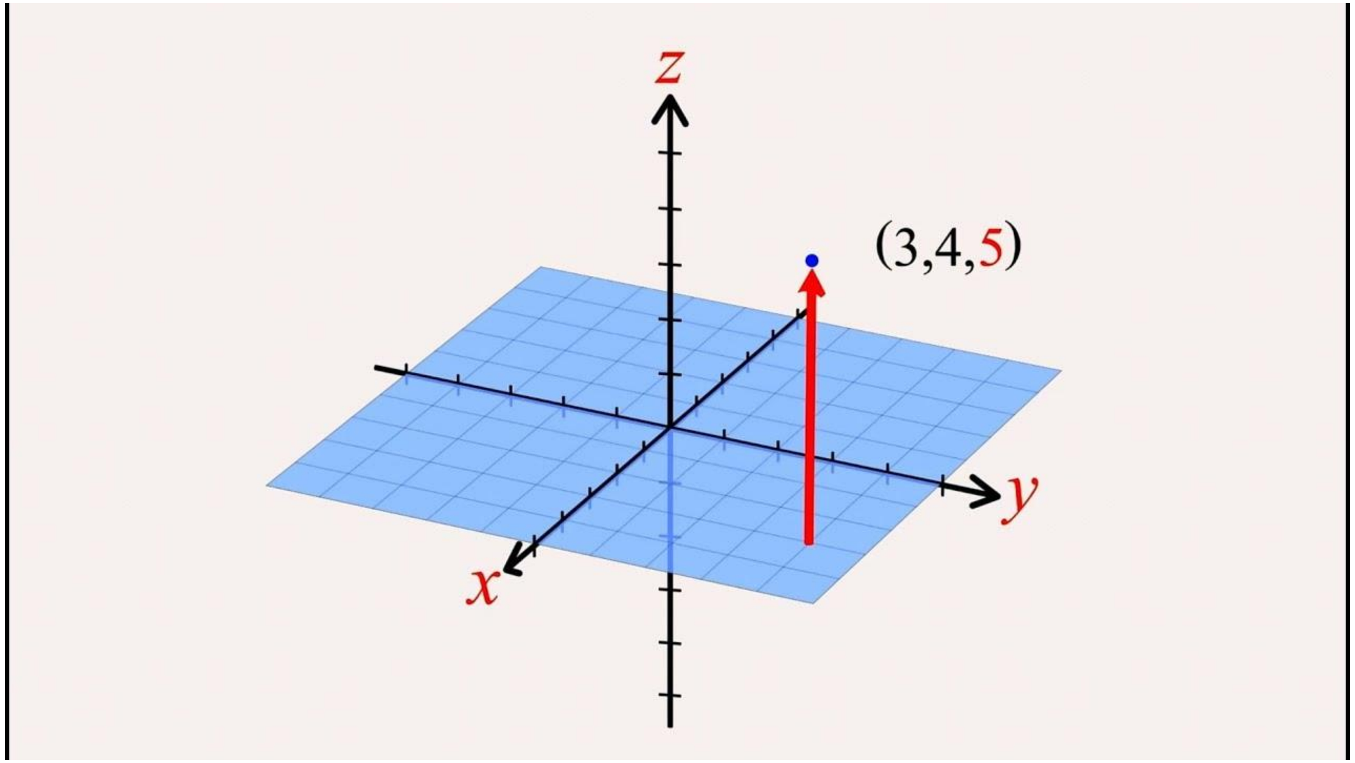
Joint Block	Joint Primitives							Joint DoFs
6-DOF Joint	Px	Py	Pz	Rx	Ry	Rz	S	3 T + 3 R
Cartesian Joint	Px	Py	Pz	Rx	Ry	Rz	S	3 T + 0 R
Cylindrical Joint	Px	Py	Pz	Rx	Ry	Rz	S	1 T + 1 R
Planar Joint	Px	Py	Pz	Rx	Ry	Rz	S	2 T + 1 R
Prismatic Joint	Px	Py	Pz	Rx	Ry	Rz	S	1 T + 0 R
Rectangular Joint	Px	Py	Pz	Rx	Ry	Rz	S	2 T + 0 R
Revolute Joint	Px	Py	Pz	Rx	Ry	Rz	S	0 T + 1 R
Spherical Joint	Px	Py	Pz	Rx	Ry	Rz	S	0 T + 3 R
Telescoping Joint	Px	Py	Pz	Rx	Ry	Rz	S	1 T + 3 R
Universal Joint	Px	Py	Pz	Rx	Ry	Rz	S	0 T + 2 R
Weld Joint	Px	Py	Pz	Rx	Ry	Rz	S	0 T + 0 R

Pose of a Rigid Body

SECTION 3

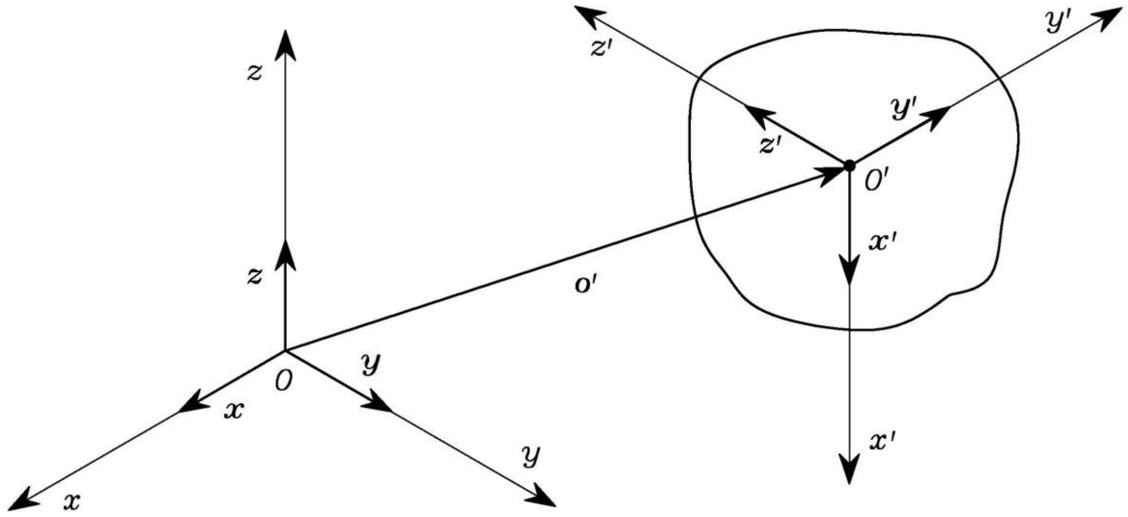
Position

$$o' = \begin{bmatrix} o'_x \\ o'_y \\ o'_z \end{bmatrix}$$



Orientation

$$\boldsymbol{x}' = x'_x \boldsymbol{x} + x'_y \boldsymbol{y} + x'_z \boldsymbol{z} \qquad \boldsymbol{y}' = y'_x \boldsymbol{x} + y'_y \boldsymbol{y} + y'_z \boldsymbol{z} \qquad \boldsymbol{z}' = z'_x \boldsymbol{x} + z'_y \boldsymbol{y} + z'_z \boldsymbol{z}$$



Position and orientation of a rigid body

Rotation

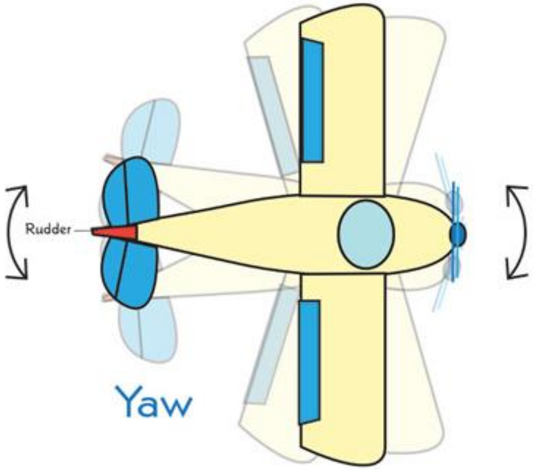
Rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}$$

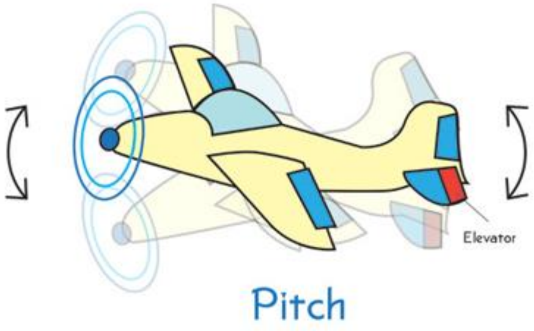
Orthogonality Property

$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I} \quad \Longrightarrow \quad \boldsymbol{R}^T = \boldsymbol{R}^{-1}$$

Use the ailerons to control
Roll

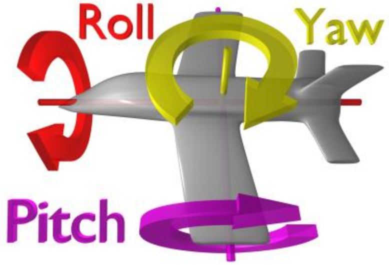


Use the rudder to control
Yaw



Use the elevators to control
Pitch

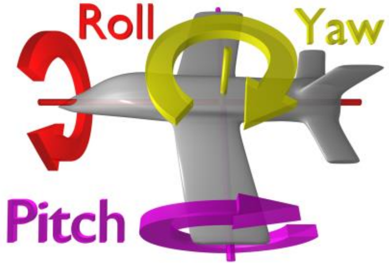
Elementary Rotations (Yaw)



- rotation of α about z

$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

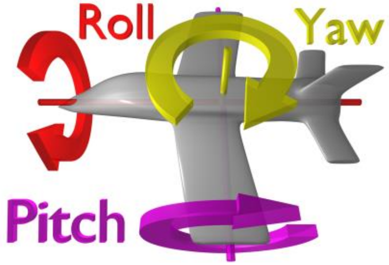
Elementary Rotations (Pitch)



- rotation of β about y

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

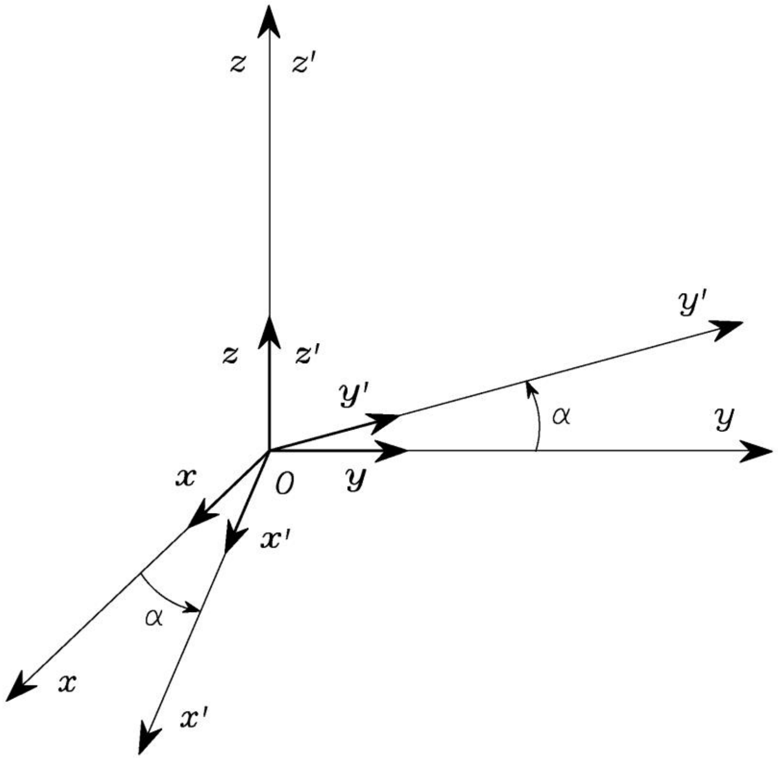
Elementary Rotations (Roll)



- rotation of γ about x

$$\boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

Elementary Rotations II



Rotation of frame by an angle about a coordinate axis

Representation of a Vector

SECTION 4

Representation of a Vector

$$\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad \boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{p} &= p'_x \boldsymbol{x}' + p'_y \boldsymbol{y}' + p'_z \boldsymbol{z}' = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} \boldsymbol{p}' \\ &= \boldsymbol{R}\boldsymbol{p}' \\ \boldsymbol{p}' &= \boldsymbol{R}^T \boldsymbol{p} \end{aligned}$$

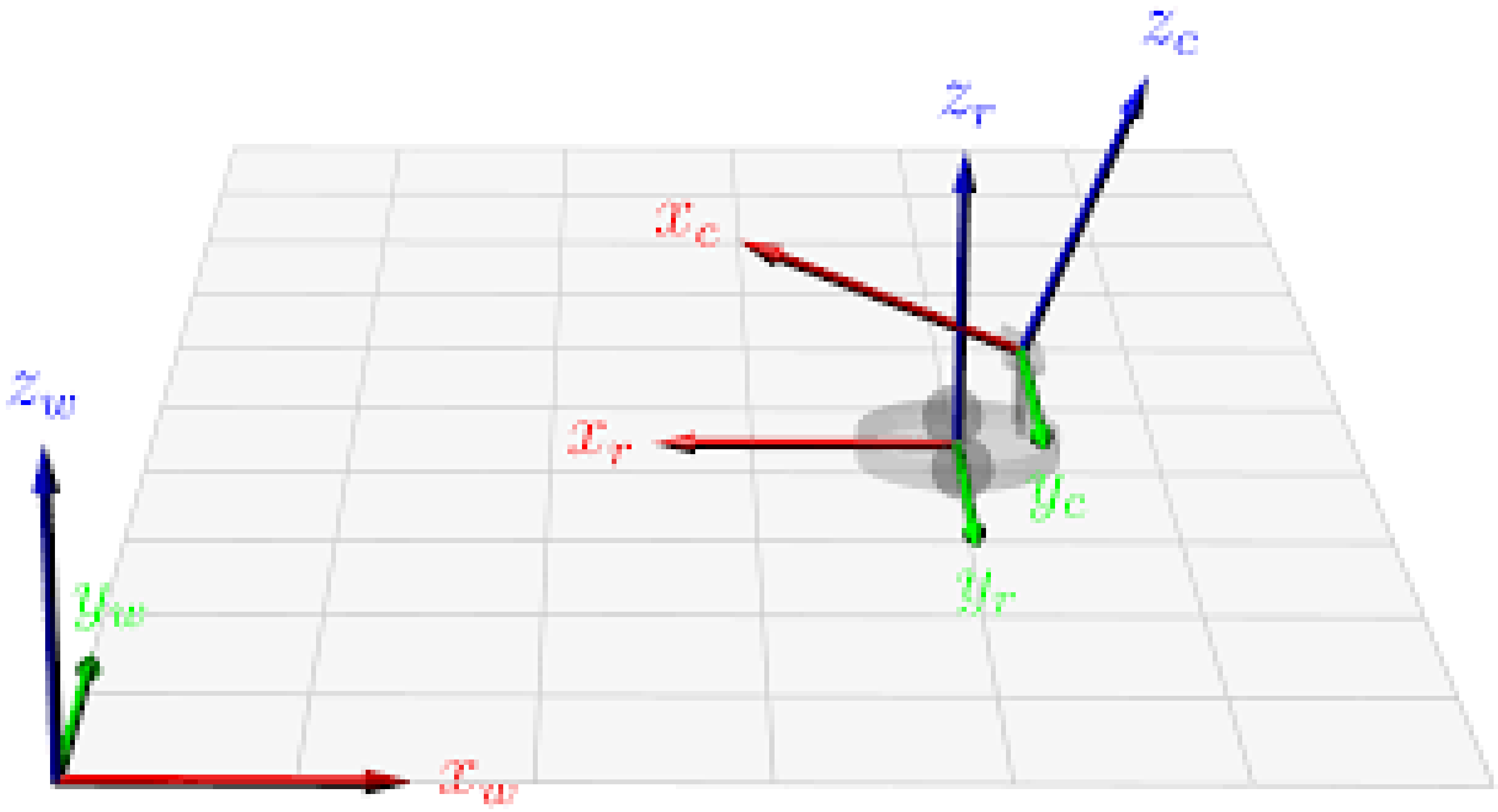
Transformation Matrix

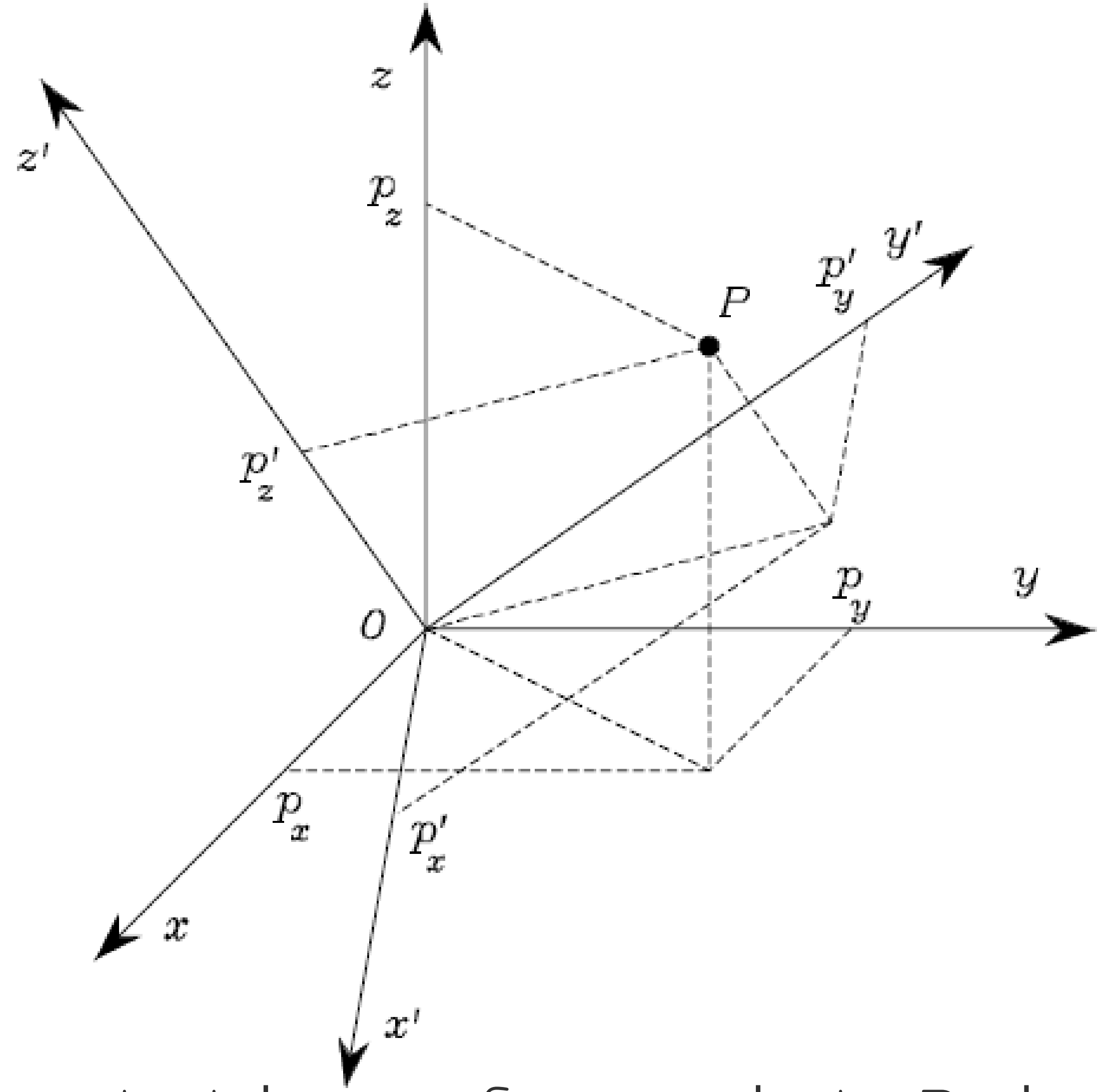
$$\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}'$$

The rotation matrix ***R*** represents the *transformation matrix* of the vector coordinates in frame *O–xyz* into the coordinates of the same vector in frame *O–x'y'z'*. In view of the orthogonality property, the inverse transformation is simply given by

$$p' = R^T p.$$

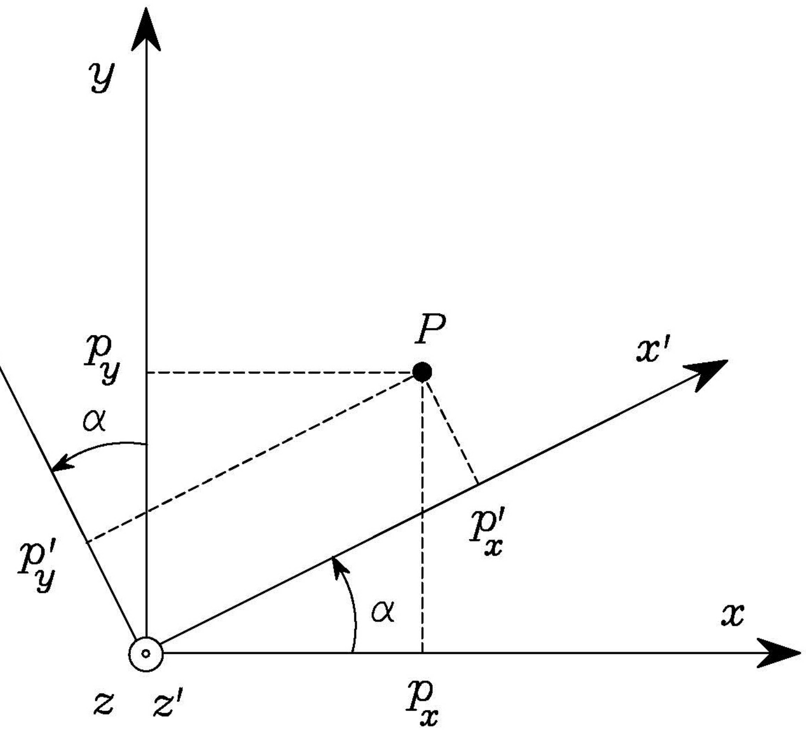
Coordinate Frames





Representation of a point P in two different coordinate frames

Rotated Frame (2D)



Representation of a point P in rotated frames

Rotated Frame (2D)

Example 2.1

Consider two frames with common origin mutually rotated by an angle *α* about the axis *z*. Let ***p*** and ***p'*** be the vectors of the coordinates of a point *P*, expressed in the frames *O–xyz* and *O–x'y'z'*, respectively. On the basis of simple geometry, the relationship between the coordinates of *P* in the two frames is

$$p_x = p'_x \cos \alpha - p'_y \sin \alpha$$

$$p_y = p'_x \sin \alpha + p'_y \cos \alpha$$

$$p_z = p'_z$$

Therefore, the matrix (2.6) represents not only the orientation of a frame with respect to another frame, but it also describes the transformation of a vector from a frame to another frame with the same origin.

Rotation of a Vector

$$\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}'$$

$$\boldsymbol{p}^T\boldsymbol{p} = \boldsymbol{p}'^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{p}'$$

Rotation of a vector of an angle about a coordinate axis

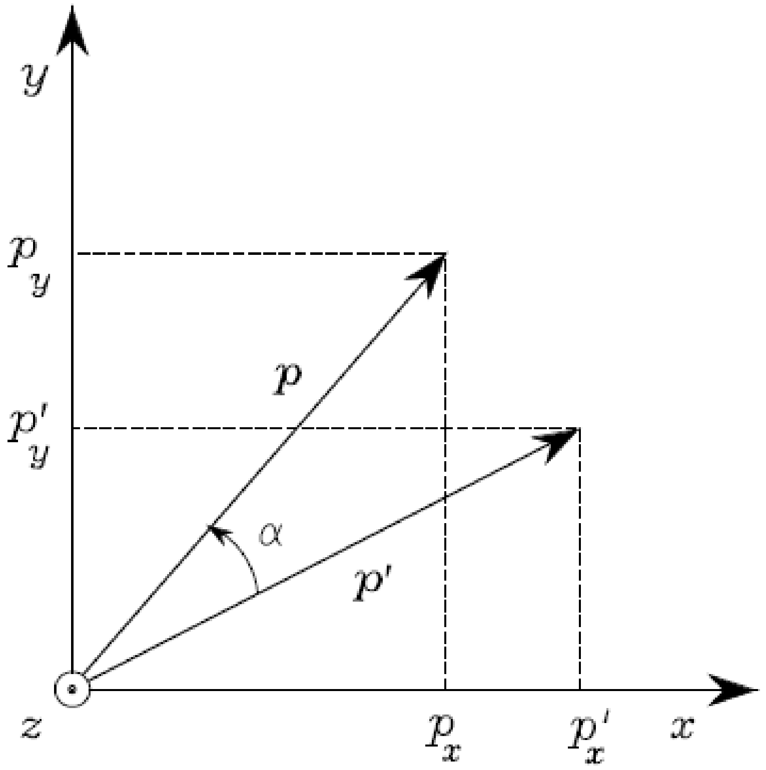
$$p_x = p'_x \cos\alpha - p'_y \sin\alpha$$

$$p_y = p'_x \sin\alpha + p'_y \cos\alpha$$

$$p_z = p'_z$$

$$\boldsymbol{p} = \boldsymbol{R}_z(\alpha)\boldsymbol{p}'$$

Rotated of a Vector (2D)



Rotation of a vector

Rotated of a Vector (2D)

Example 2.2

Consider the vector p which is obtained by rotating a vector p’ in the plane xy by an angle α about axis z of the reference frame. Let (p'_x, p'_y, p'_z) be the coordinates of the vector p’. The vector p has components

$$p_x = p'_x \cos \alpha - p'_y \sin \alpha$$

$$p_y = p'_x \sin \alpha + p'_y \cos \alpha$$

$$p_z = p'_z$$

It is easy to recognize that p can be expressed as

$$p = R(\alpha)p'$$

where Rz(α) is the same rotation matrix as in Matrix for Rotation Yaw

Summary

In sum, a rotation matrix attains three *equivalent geometrical meanings*:

- It describes the mutual orientation between two coordinate frames; its column vectors are the direction cosines of the axes of the rotated frame with respect to the original frame.
- It represents the coordinate transformation between the coordinates of a point expressed in two different frames (with common origin).
- It is the operator that allows the rotation of a vector in the same coordinate frame.

Composition of Rotation Matrices

SECTION 5

Composition of Rotation Matrices

- Given three rotated frames 0, 1, 2

$$\boldsymbol{p}^1 = \boldsymbol{R}_2^1 \boldsymbol{p}^2$$

$$\boldsymbol{p}^0 = \boldsymbol{R}_1^0 \boldsymbol{p}^1$$

$$\boldsymbol{p}^0 = \boldsymbol{R}_2^0 \boldsymbol{p}^2$$

$$\boldsymbol{R}_i^j = (\boldsymbol{R}_j^i)^{-1} = (\boldsymbol{R}_j^i)^T$$

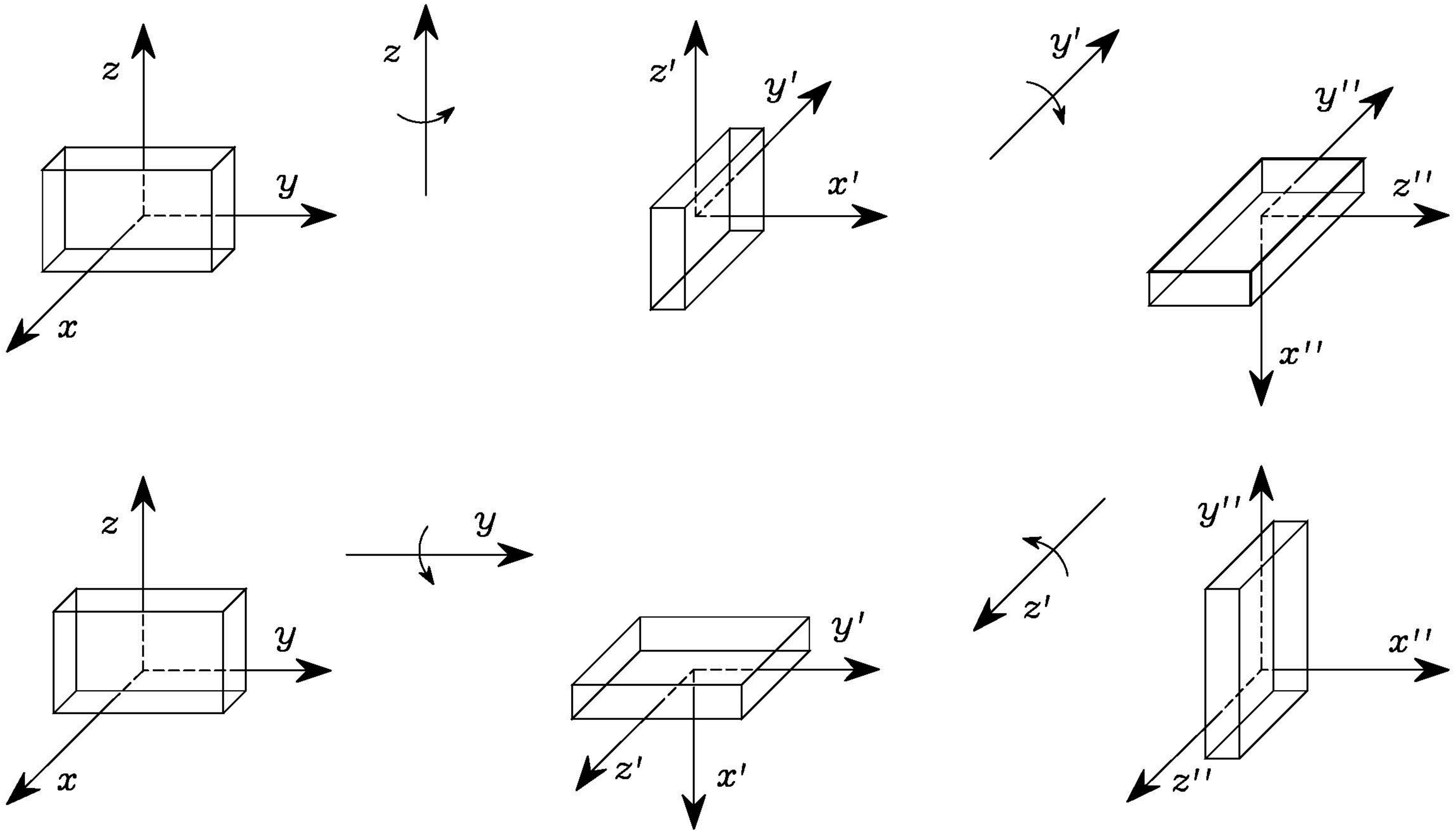
Composition of Rotation Matrices

- Rotation with respect to current frame

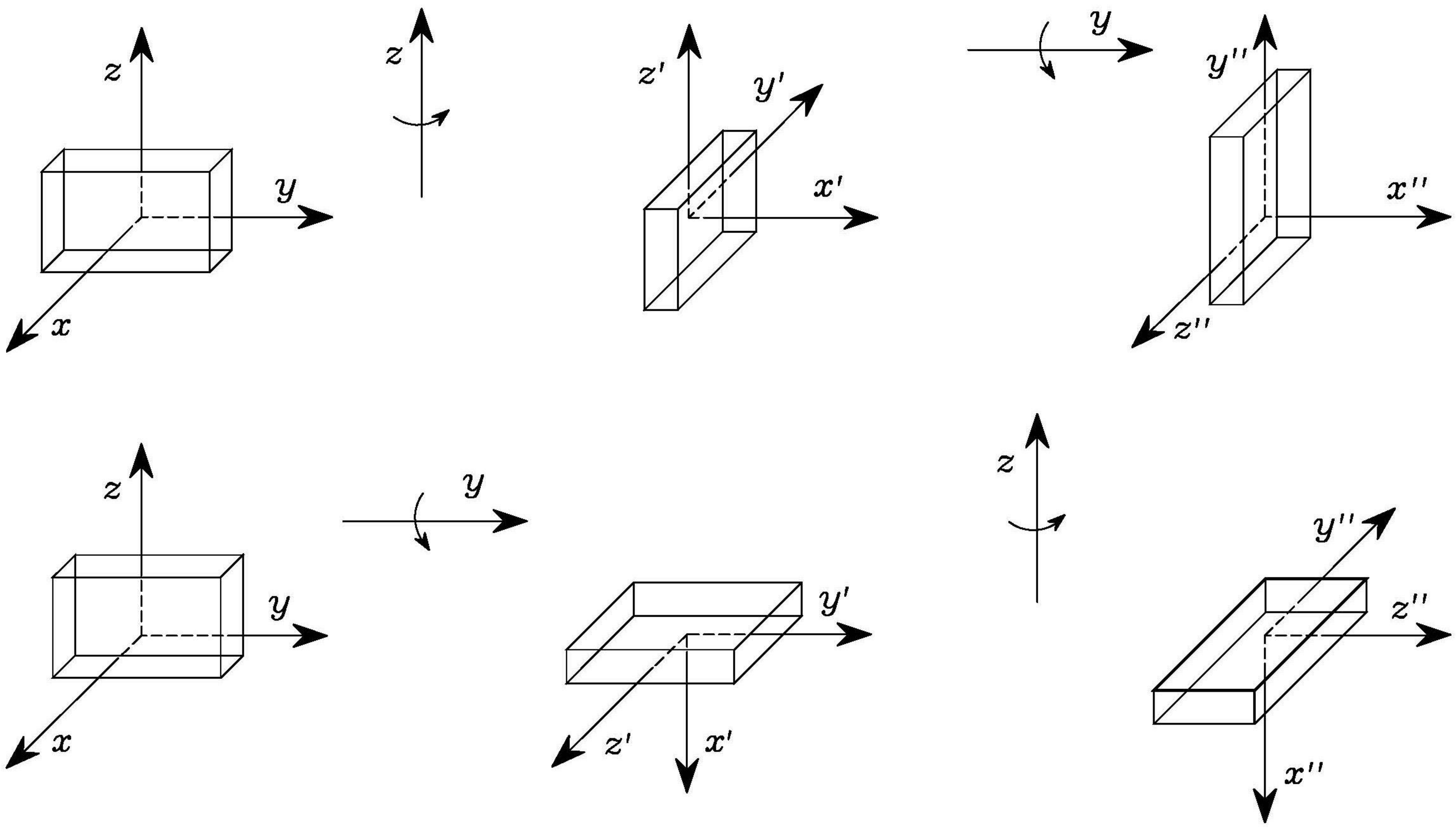
$$\boldsymbol{R}_2^0 = \boldsymbol{R}_1^0 \boldsymbol{R}_2^1$$

- Rotation with respect to fixed frame

$$\boldsymbol{R}_2^0 = \boldsymbol{R}_2^1 \boldsymbol{R}_1^0$$



Successive rotations of an object
about axes of current frame



Successive rotations of an object about axes of fixed frame

Euler Angle

SECTION 6

Minimal Representations

- **Euler angles** are introduced as a minimal representation of orientation
- **RPY angles** are presented as an alternative set of Euler angles
- Both direct and inverse formulae are derived for the two sets

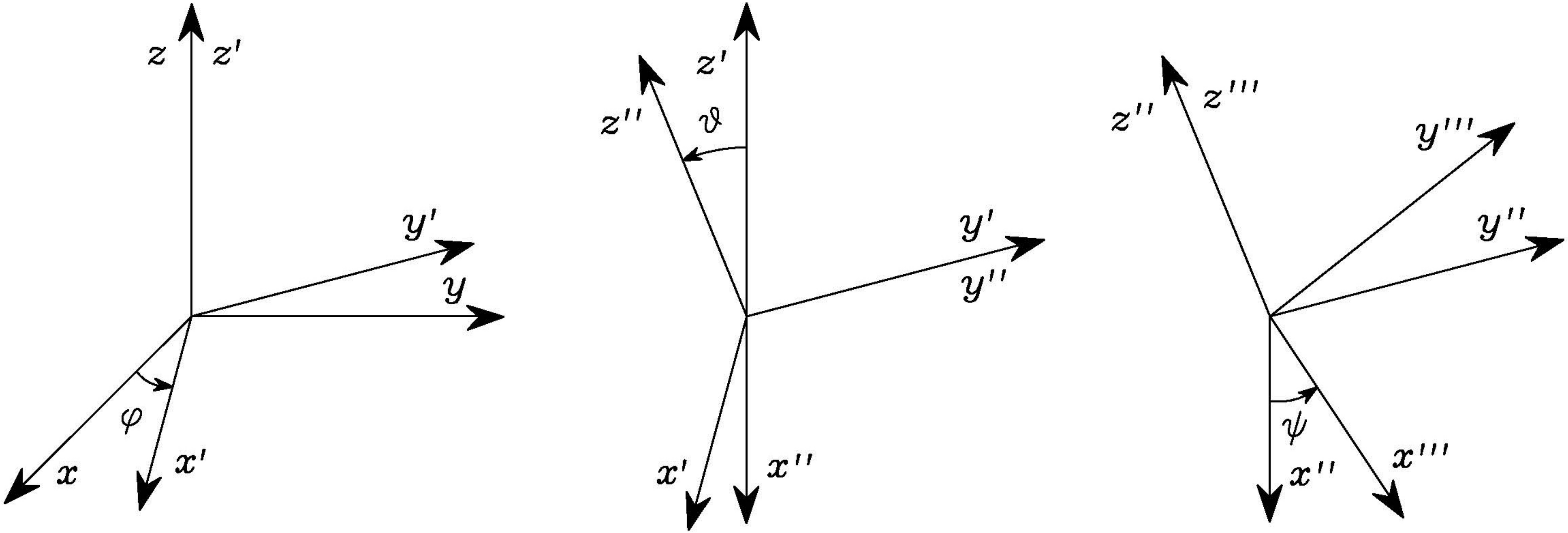
Euler Angles

Rotation matrix

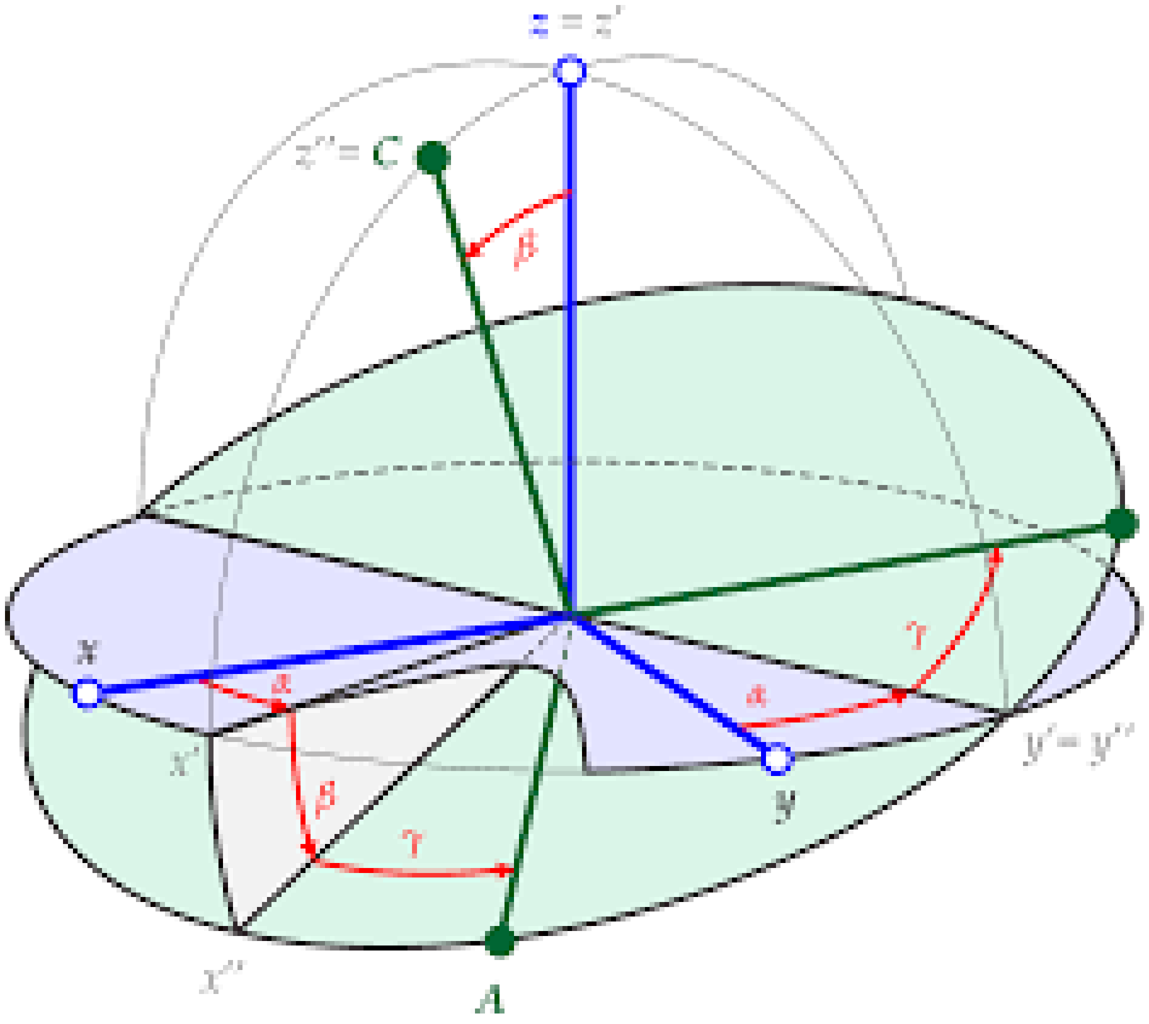
- 9 parameters with 6 constraints

Minimal representation of orientation

- 3 independent parameters



Representation of Euler angles ZYZ



Euler Angles

Euler Angles

ZYZ Angles

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)$$

$$= \begin{bmatrix} \cos\varphi\cos\vartheta\cos\psi - \sin\varphi\sin\psi & -\cos\varphi\cos\vartheta\sin\psi - \sin\varphi\cos\psi & \cos\varphi\sin\vartheta \\ \sin\varphi\cos\vartheta\cos\psi + \cos\varphi\sin\psi & -\sin\varphi\cos\vartheta\sin\psi + \cos\varphi\cos\psi & \sin\varphi\sin\vartheta \\ -\sin\vartheta\cos\psi & \sin\vartheta\sin\psi & \cos\vartheta \end{bmatrix}$$

ZYZ Angles

The rotation described by *ZYZ angles* is obtained as composition of the following elementary rotations:

- Rotate the reference frame by the angle ϕ about axis z; this rotation is described by the matrix $\boldsymbol{R}_z(\phi)$ which is formally defined in (2.6).
- Rotate the current frame by the angle θ about axis y'; this rotation is described by the matrix $\boldsymbol{R}_{y'}(\theta)$ which is formally defined in (2.7).
- Rotate the current frame by the angle ψ about axis z''; this rotation is described by the matrix $\boldsymbol{R}_{z''}(\psi)$ which is again formally defined in (2.6).

Inverse Problem

Given

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

the three ZYZ angles are for

$$\vartheta \in (0, \pi)$$

$$\varphi = \mathrm{Atan2}(r_{23}, r_{13})$$

$$\vartheta = \mathrm{Atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$

$$\psi = \mathrm{Atan2}(r_{32}, -r_{31})$$

Else for

$$\vartheta \in (-\pi, 0)$$

$$\varphi = \mathrm{Atan2}(-r_{23}, -r_{13})$$

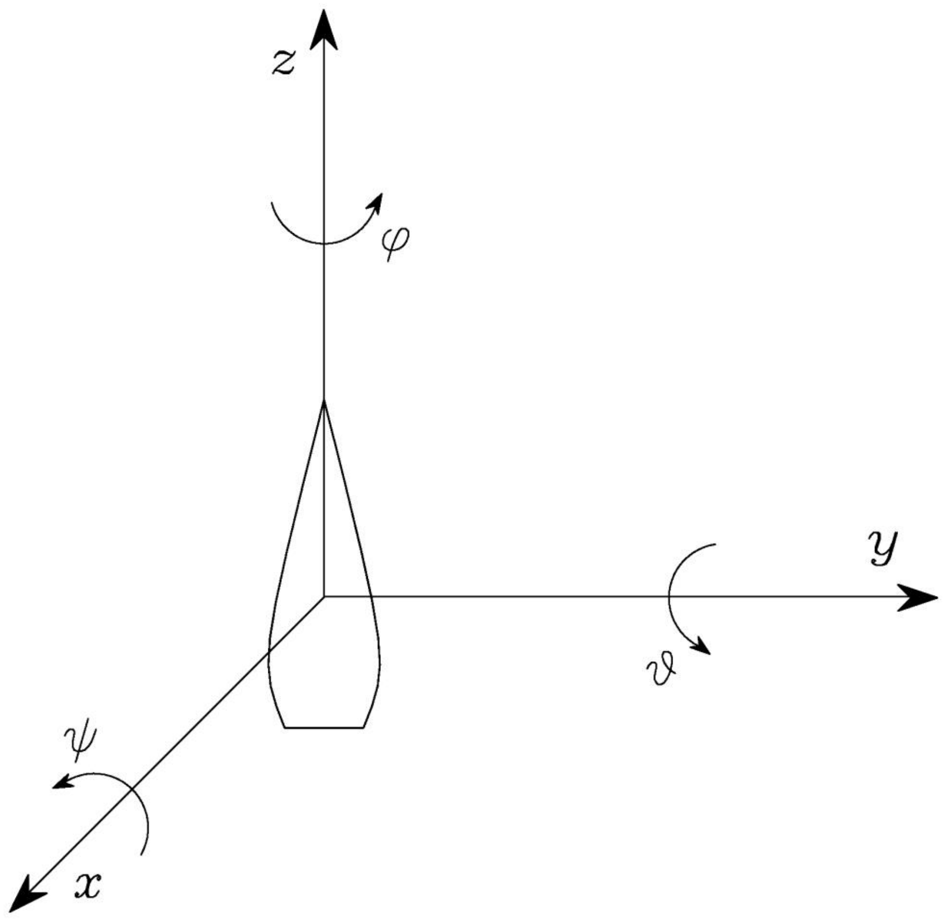
$$\vartheta = \mathrm{Atan2}\left(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$

$$\psi = \mathrm{Atan2}(-r_{32}, r_{31})$$

RPY Angles

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_{\mathbf{z}}(\varphi)\boldsymbol{R}_{\mathbf{y}}(\vartheta)\boldsymbol{R}_{\mathbf{x}}(\psi)$$

$$= \begin{bmatrix} \cos\varphi\cos\vartheta & \cos\varphi\sin\vartheta\sin\psi - \sin\varphi\cos\psi & \cos\varphi\sin\vartheta\cos\psi + \sin\varphi\sin\psi \\ \sin\varphi\cos\vartheta & \sin\varphi\sin\vartheta\sin\psi + \cos\varphi\cos\psi & \sin\varphi\sin\vartheta\cos\psi - \cos\varphi\sin\psi \\ -\sin\vartheta & \cos\vartheta\sin\psi & \cos\vartheta\cos\psi \end{bmatrix}$$



Representation of Roll-Pitch-Yaw angles

Inverse Problem

Given

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

the three RPY angles are for

$$\vartheta \in (-\pi/2, \pi/2)$$

$$\varphi = \mathrm{Atan2}(r_{21}, r_{11})$$

$$\vartheta = \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}\right)$$

$$\psi = \mathrm{Atan2}(r_{32}, r_{33})$$

Inverse Problem

else for $\vartheta \in (\pi/2, 3\pi/2)$

$$\varphi = \text{Atan2}(-r_{21}, -r_{11})$$

$$\vartheta = \text{Atan2}\left(-r_{31}, -\sqrt{r_{32}^2 + r_{33}^2}\right)$$

$$\psi = \text{Atan2}(-r_{32}, -r_{33})$$

Four-parameter Representations

SECTION 7

Four-parameter Representations

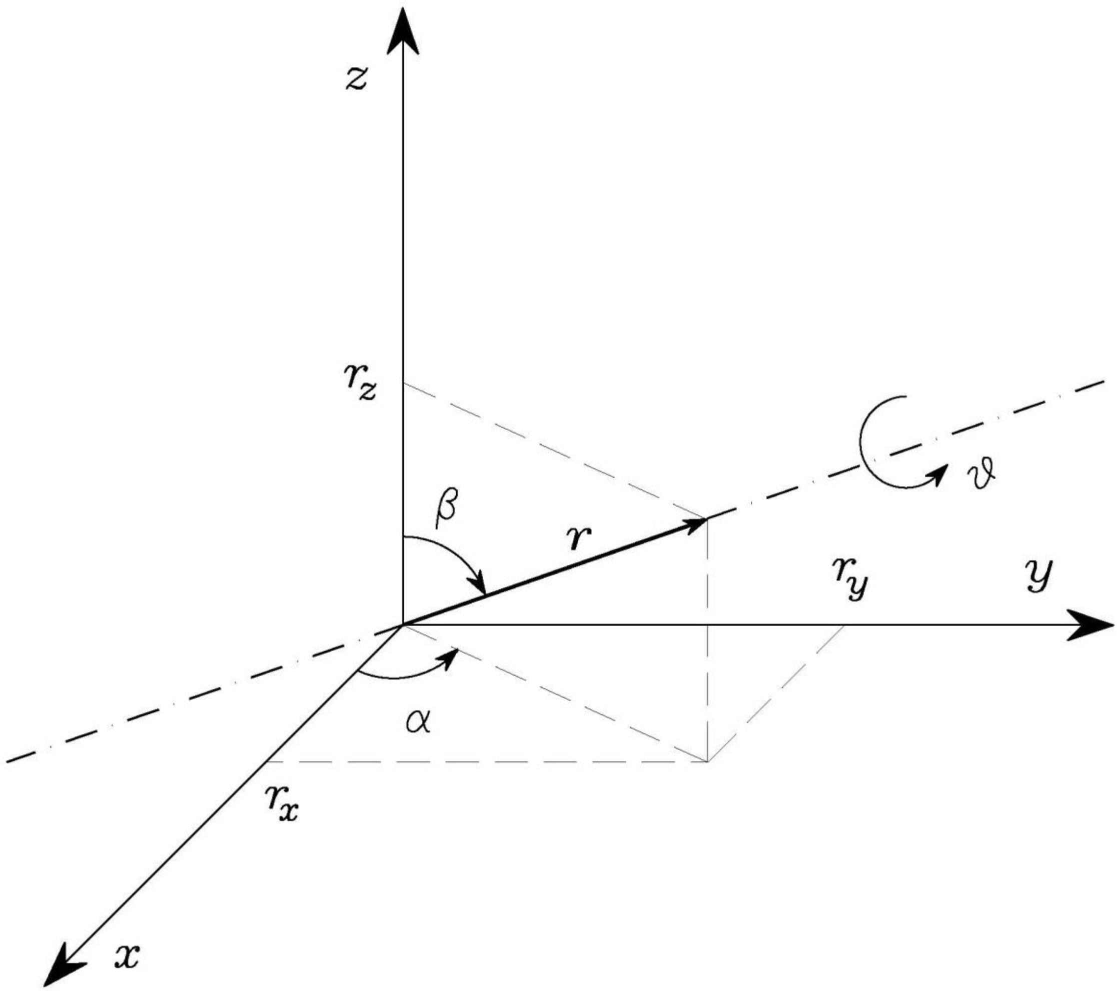
- The angle and axis description is introduced as a four-parameter representation of orientation
- The unit quaternion is adopted as an alternative four-parameter representation or orientation
- Both direct and inverse formulae are derived for the two representations

Angle and Axis

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\vartheta)\boldsymbol{R}_y(-\beta)\boldsymbol{R}_z(-\alpha)$$

$$\sin\alpha = \frac{r_y}{\sqrt{r_x^2 + r_y^2}} \qquad \cos\alpha = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\sin\beta = \sqrt{r_x^2 + r_y^2} \qquad \cos\beta = r_z$$



Rotation of an angle about an axis

Angle and Axis II

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \begin{bmatrix} r_x^2(1-\cos\vartheta)+\cos\vartheta & r_x r_y(1-\cos\vartheta)-r_z\sin\vartheta & r_x r_z(1-\cos\vartheta)+r_y\sin\vartheta \\ r_x r_y(1-\cos\vartheta)+r_z\sin\vartheta & r_y^2(1-\cos\vartheta)+\cos\vartheta & r_y r_z(1-\cos\vartheta)-r_x\sin\vartheta \\ r_x r_z(1-\cos\vartheta)-r_y\sin\vartheta & r_y r_z(1-\cos\vartheta)+r_x\sin\vartheta & r_z^2(1-\cos\vartheta)+\cos\vartheta \end{bmatrix}$$

$$\boldsymbol{R}(-\vartheta, -\boldsymbol{r}) = \boldsymbol{R}(\vartheta, \boldsymbol{r})$$

Inverse Problem

Given

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

the angle and axis are for $\sin\vartheta \neq 0$ $\quad r_x^2 + r_y^2 + r_z^2 = 1$

$$\vartheta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\boldsymbol{r} = \frac{1}{2\sin\vartheta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Unit Quaternion

Four-parameter representation

$$Q = \{\eta, \epsilon\}$$

$$\eta = \cos\frac{\vartheta}{2}$$

$$\epsilon = \sin\frac{\vartheta}{2}\boldsymbol{r}$$

with $\eta^2 + \epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2 = 1$

$(\vartheta, \boldsymbol{r})$ and $(-\vartheta, -\boldsymbol{r})$ give the same quaternion

$$\boldsymbol{R}(\eta, \epsilon) = \begin{bmatrix} 2(\eta^2 + \epsilon_x^2) - 1 & 2(\epsilon_x\epsilon_y - \eta\epsilon_z) & 2(\epsilon_x\epsilon_z + \eta\epsilon_y) \\ 2(\epsilon_x\epsilon_y + \eta\epsilon_z) & 2(\eta^2 + \epsilon_y^2) - 1 & 2(\epsilon_y\epsilon_z - \eta\epsilon_x) \\ 2(\epsilon_x\epsilon_z - \eta\epsilon_y) & 2(\epsilon_y\epsilon_z + \eta\epsilon_x) & 2(\eta^2 + \epsilon_z^2) - 1 \end{bmatrix}$$

Inverse Problem

Given

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\eta = \frac{1}{2}\sqrt{r_{11} + r_{22} + r_{33} + 1}$$

the quaternion is for $\eta \geq 0$

$$\epsilon = \frac{1}{2}\begin{bmatrix} \mathrm{sgn}(r_{32} - r_{23})\sqrt{r_{11} - r_{22} - r_{33} + 1} \\ \mathrm{sgn}(r_{13} - r_{31})\sqrt{r_{22} - r_{33} - r_{11} + 1} \\ \mathrm{sgn}(r_{21} - r_{12})\sqrt{r_{33} - r_{11} - r_{22} + 1} \end{bmatrix}$$

Inverse Problem II

Quaternion extracted from $\boldsymbol{R}^{-1} = \boldsymbol{R}^T$

$$Q^{-1} = \{\eta, -\epsilon\}$$

Quaternion product

$$Q_1 * Q_2 = \{\eta_1\eta_2 - \epsilon_1^T\epsilon_2, \eta_1\epsilon_2 + \eta_2\epsilon_1 + \epsilon_1 \times \epsilon_2\}$$

Further Insights

1. Find the rotation matrix corresponding to the set of Euler angles ZXZ.
2. Discuss the inverse solution for the Euler angles ZYZ in the case $\sin(\vartheta)$.
3. Discuss the inverse solution for the Roll-Pitch-Yaw angles in the case $\cos(\vartheta)$. Verify the formula for the rotation matrix corresponding to the rotation by an angle about an arbitrary axis.
4. Verify the inverse formulae for the angle and the unit vector of the axis corresponding to a rotation matrix. Find inverse formulae in the case of $\sin(\vartheta)$.
5. Verify the formula for the rotation matrix corresponding to the unit quaternion.
6. Verify the inverse formulae for the unit quaternion corresponding to a rotation matrix.