CS65K Robotics

Modelling, Planning and Control

Chapter 3: Differential Kinematics and Statics

Section 3.1-3.4

LECTURE 7: DIFFERENTIAL KINEMATICS AND JACOBIAN

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Objectives

- The differential kinematics and statics concepts are introduced
- The derivative of a rotation matrix is expressed in terms of the end-effector angular velocity
- The velocity composition rule is characterized
- The contributions of joint velocities to the link linear and angular velocities are computed
- Formulae for prismatic and revolute joints are derived
- Formulae to compute the columns of the Jacobian are derived
- The Jacobian is computed for typical manipulator structures

Differential Kinematics

SECTION 1

Differential Kinematics

Relationship between the joint velocities and the end-effector linear and angular velocities Jacobian

- Derivative of a rotation matrix
- Jacobian computation
- Jacobian of typical manipulation structures

Differential Kinematics

Differential Kinematics

- Kinematic singularities
- Analysis of redundancy
- Analytical Jacobian

Differential Kinematics

Inverse Kinematics Algorithms

- Jacobian (pseudo-)inverse
- Jacobian transpose
- Orientation error

Statics

SECTION 2

statics

Relationship between the generalized forces applied to the end-effector and the generalized forces applied to the joints Statics

- Kineto-statics duality
- Velocity and force transformation
- Manipulability ellipsoids

Geometric Jacobian

SECTION 3

Geometric Jacobian

$$\boldsymbol{T}_e(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{R}_e(\boldsymbol{q}) & \boldsymbol{p}_e(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

Differential Kinematics Equation

$$\boldsymbol{v}_e = \begin{bmatrix} \dot{\boldsymbol{p}}_e \\ \omega_e \end{bmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{bmatrix} \boldsymbol{J}_P(\boldsymbol{q}) \\ \boldsymbol{J}_O(\boldsymbol{q}) \end{bmatrix} \dot{\boldsymbol{q}}$$

$$\dot{\boldsymbol{p}}_e = \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\omega_e = \boldsymbol{J}_O(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Derivative of a Rotation Matrix

$$\boldsymbol{R}(t)\boldsymbol{R}^T(t) = \boldsymbol{I}$$

- Differentiating …

$$\dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t) + \boldsymbol{R}(t)\dot{\boldsymbol{R}}^T(t) = \boldsymbol{O}$$

- Skew-symmetric operator

$$\boldsymbol{S}(t) = \dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t)$$

$$\boldsymbol{S}(t) + \boldsymbol{S}^T(t) = \boldsymbol{O}$$

- Angular velocity

$$\dot{\boldsymbol{R}} = \boldsymbol{S}(\omega)\boldsymbol{R}$$

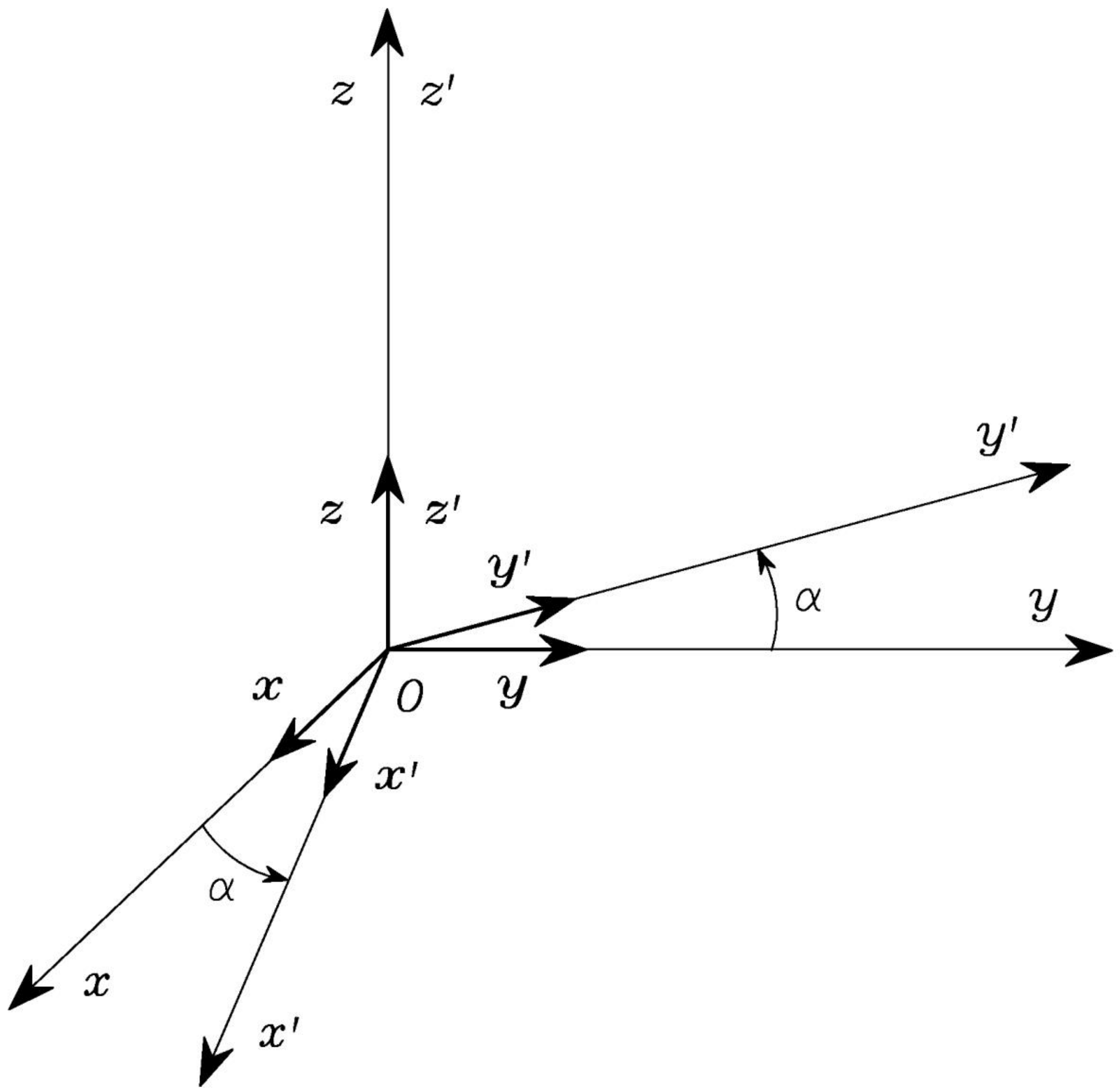
$$\boldsymbol{S} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Example

$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Differentiating …

$$\boldsymbol{S}(t) = \begin{bmatrix} -\dot{\alpha}\sin\alpha & -\dot{\alpha}\cos\alpha & 0 \\ \dot{\alpha}\cos\alpha & -\dot{\alpha}\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\alpha} & 0 \\ \dot{\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boldsymbol{S}(\omega(t))$$



Elementary rotation about coordinate axis

Velocity Composition

SECTION 4

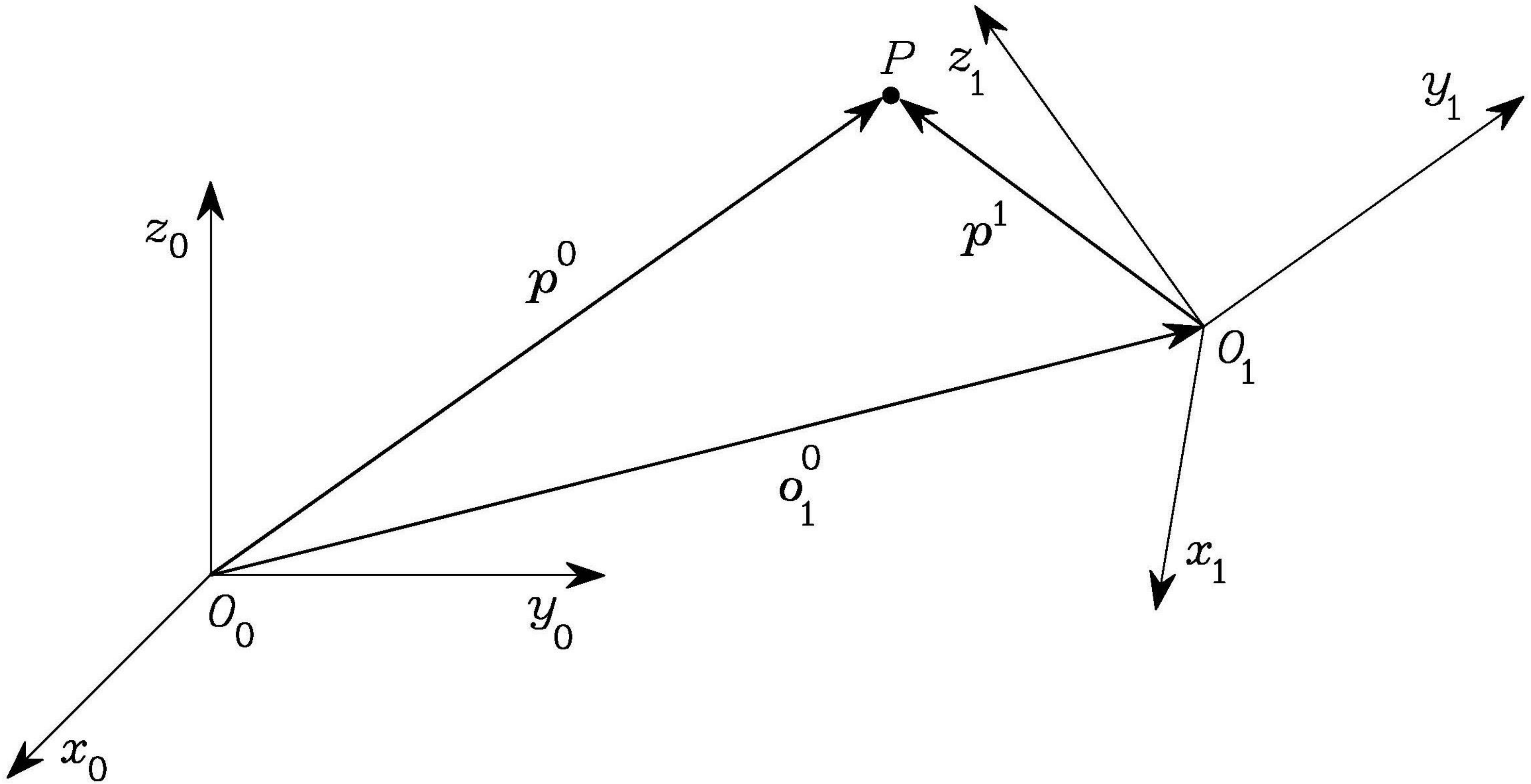
Velocity Composition

$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1$$

- Differentiating …

$$\begin{aligned} \dot{\boldsymbol{p}}^0 &= \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \dot{\boldsymbol{R}}_1^0 \boldsymbol{p}^1 = \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \boldsymbol{S}(\omega_1^0) \boldsymbol{R}_1^0 \boldsymbol{p}^1 \\ &= \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \omega_1^0 \times \boldsymbol{r}_1^0 \qquad \boldsymbol{r}_1^0 = \boldsymbol{R}_1^0 \boldsymbol{p}^1 \end{aligned}$$

- If $\boldsymbol{P}^1$ is *fixed* in Frame 1, then $\dot{\boldsymbol{p}}^0 = \dot{\boldsymbol{o}}_1^0 + \omega_1^0 \times \boldsymbol{r}_1^0$



Characterization of generic link of a manipulator

Linear and Angular Velocities

SECTION 5

Linear and Angular Velocities

Linear Velocity

$$\boldsymbol{p}_i = \boldsymbol{p}_{i-1} + \boldsymbol{R}_{i-1}\boldsymbol{r}_{i-1,i}^{i-1}$$

- Differentiating …

$$\begin{aligned}\dot{\boldsymbol{p}}_i &= \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{R}_{i-1}\dot{\boldsymbol{r}}_{i-1,i}^{i-1} + \omega_{\mathbf{i-1}} \times \boldsymbol{R}_{\mathbf{i-1}}\boldsymbol{r}_{\mathbf{i-1,i}}^{\mathbf{i-1}} \\ &= \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{v}_{i-1,i} + \omega_{i-1} \times \boldsymbol{r}_{i-1,i} \qquad \boldsymbol{v}_{i-1,i} = \boldsymbol{R}_{i-1}\dot{\boldsymbol{r}}_{i-1,i}^{i-1}\end{aligned}$$

Angular Velocity

Angular Velocity

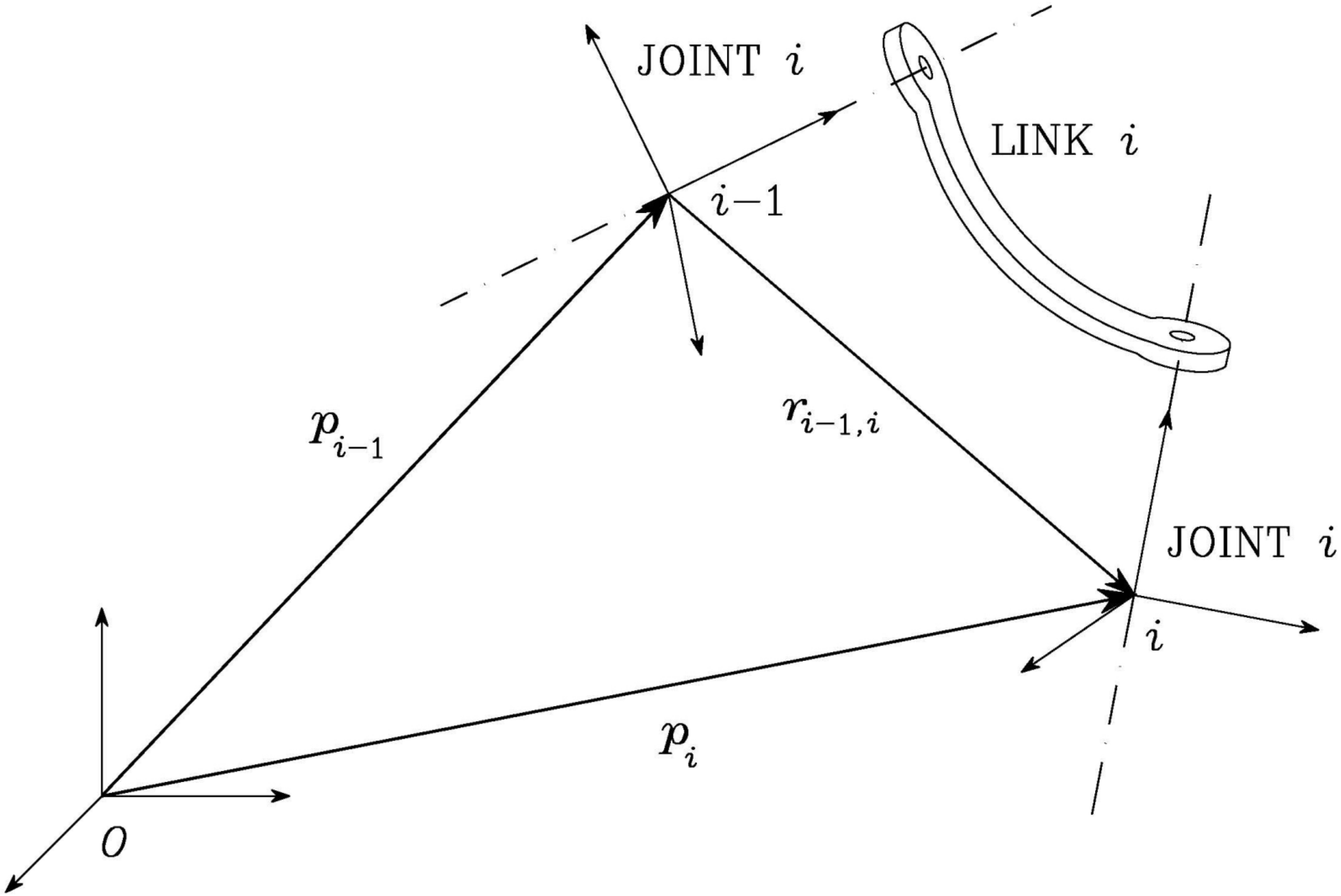
$$\boldsymbol{R}_i = \boldsymbol{R}_{i-1}\boldsymbol{R}_i^{i-1}$$

- Differentiating …

$$\boldsymbol{S}(\omega_i)\boldsymbol{R}_i = \boldsymbol{S}(\omega_{i-1})\boldsymbol{R}_i + \boldsymbol{R}_{i-1}\boldsymbol{S}(\omega_{i-1,i}^{i-1})\boldsymbol{R}_i^{i-1} \quad = \quad \boldsymbol{S}(\omega_{i-1})\boldsymbol{R}_i + \boldsymbol{S}(\boldsymbol{R}_{i-1}\omega_{i-1,i}^{i-1})\boldsymbol{R}_i$$

$$\boldsymbol{R}_{i-1}\boldsymbol{S}(\omega_{i-1,i}^{i-1})\boldsymbol{R}_i^{i-1} \quad = \quad \boldsymbol{S}(\boldsymbol{R}_{i-1}\omega_{i-1,i}^{i-1})\boldsymbol{R}_i$$

$$\omega_i = \omega_{i-1} + \boldsymbol{R}_{i-1}\omega_{i-1,i}^{i-1} = \omega_{i-1} + \omega_{i-1,i}$$



Representation of vectors needed for the computation of the velocity contribution of a revolute joint to the end-effector linear velocity

Joint Velocities

SECTION 6

Joint Velocities

Link velocities

$$\omega_i = \omega_{i-1} + \omega_{i-1,i}$$

$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{v}_{i-1,i} + \omega_{i-1} \times \boldsymbol{r}_{i-1,i}$$

Prismatic joint velocity

$$\omega_i = \omega_{i-1} \qquad \omega_{i-1,i} = \mathbf{0}$$

$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \dot{d}_i \boldsymbol{z}_{i-1} + \omega_i \times \boldsymbol{r}_{i-1,i} \qquad \boldsymbol{v}_{i-1,i} = \dot{d}_i z_{i-1}$$

Revolute joint velocity

$$\omega_i = \omega_{i-1} + \dot{\vartheta}_i \boldsymbol{z}_{i-1} \qquad \omega_{i-1,i} = \dot{\vartheta}_i \boldsymbol{z}_{i-1}$$

$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \omega_i \times \boldsymbol{r}_{i-1,i} \qquad \boldsymbol{v}_{i-1,i} = \omega_{i-1,i} \times \boldsymbol{r}_{i-1,i}$$

Linear Velocity

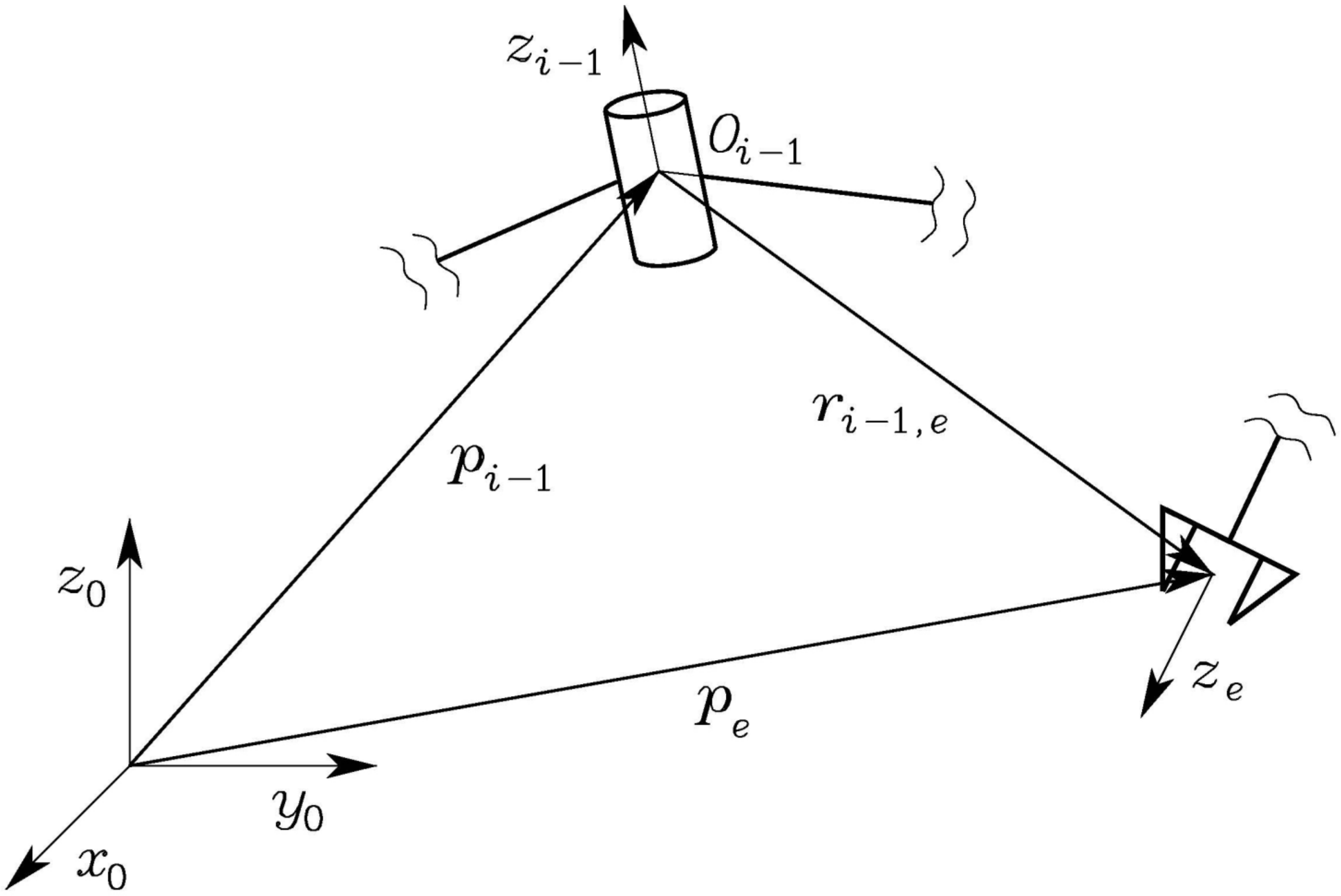
$$\dot{\boldsymbol{p}}_e = \sum_{i=1}^{n} \frac{\partial \boldsymbol{p}_e}{\partial q_i} \dot{q}_i = \sum_{i=1}^{n} \jmath_{P_i} \dot{q}_i$$

- Prismatic joint

$$\dot{q}_i \jmath_{P_i} = \dot{d}_i \boldsymbol{z}_{i-1} \qquad \Longrightarrow \qquad \jmath_{P_i} = \boldsymbol{z}_{i-1}$$

- Revolute joint

$$\dot{q}_i \jmath_{P_i} = \dot{\vartheta}_i \boldsymbol{z}_{i-1} \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1}) \qquad \Longrightarrow \qquad \jmath_{P_i} = \boldsymbol{z}_{i-1} \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1})$$



Representation of vectors needed for the computation of the velocity contribution of a revolute joint to the end-effector linear velocity

Angular Velocity

$$\omega_e = \omega_n = \sum_{i=1}^{n} \omega_{i-1,i} = \sum_{i=1}^{n} \jmath_{Oi}\dot{q}_i$$

- Prismatic joint

$$\dot{q}_i \jmath_{Oi} = \mathbf{0} \qquad \Longrightarrow \qquad \jmath_{Oi} = \mathbf{0}$$

- Revolute joint

$$\dot{q}_i \jmath_{Oi} = \dot{\vartheta}_i \boldsymbol{z}_{i-1} \qquad \Longrightarrow \qquad \jmath_{Oi} = \boldsymbol{z}_{i-1}$$

Jacobian Columns

SECTION 7

Jacobian Column

- Prismatic Joint

$$\begin{bmatrix} j_{Pi} \\ j_{Oi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_{i-1} \\ \boldsymbol{0} \end{bmatrix}$$

- Revolute Joint

$$\begin{bmatrix} \boldsymbol{z}_{i-1} \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1}) \\ \boldsymbol{z}_{i-1} \end{bmatrix}$$

$$\boldsymbol{z}_{i-1} = \boldsymbol{R}_1^0(q_1)\dots\boldsymbol{R}_{i-1}^{i-2}(q_{i-1})\boldsymbol{z}_0$$

$$\widetilde{\boldsymbol{p}}_e = \boldsymbol{A}_1^0(q_1)\dots\boldsymbol{A}_n^{n-1}(q_n)\widetilde{\boldsymbol{p}}_0$$

$$\widetilde{\boldsymbol{p}}_{i-1} = \boldsymbol{A}_1^0(q_1)\dots\boldsymbol{A}_{i-1}^{i-2}(q_{i-1})\widetilde{\boldsymbol{p}}_0$$

Expression in a Different Frame

- Jacobian depends on frame in which end-effector velocity is expressed
- Representation in different frame

$$\begin{bmatrix} \dot{\boldsymbol{p}}_e^u \\ \boldsymbol{\omega}_e^u \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}^u & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R}^u \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}}_e \\ \boldsymbol{\omega}_e \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}^u & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R}^u \end{bmatrix} \boldsymbol{J}\,\dot{\boldsymbol{q}}$$

$$\boldsymbol{J}^u = \begin{bmatrix} \boldsymbol{R}^u & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R}^u \end{bmatrix} \boldsymbol{J}$$

Case Studies

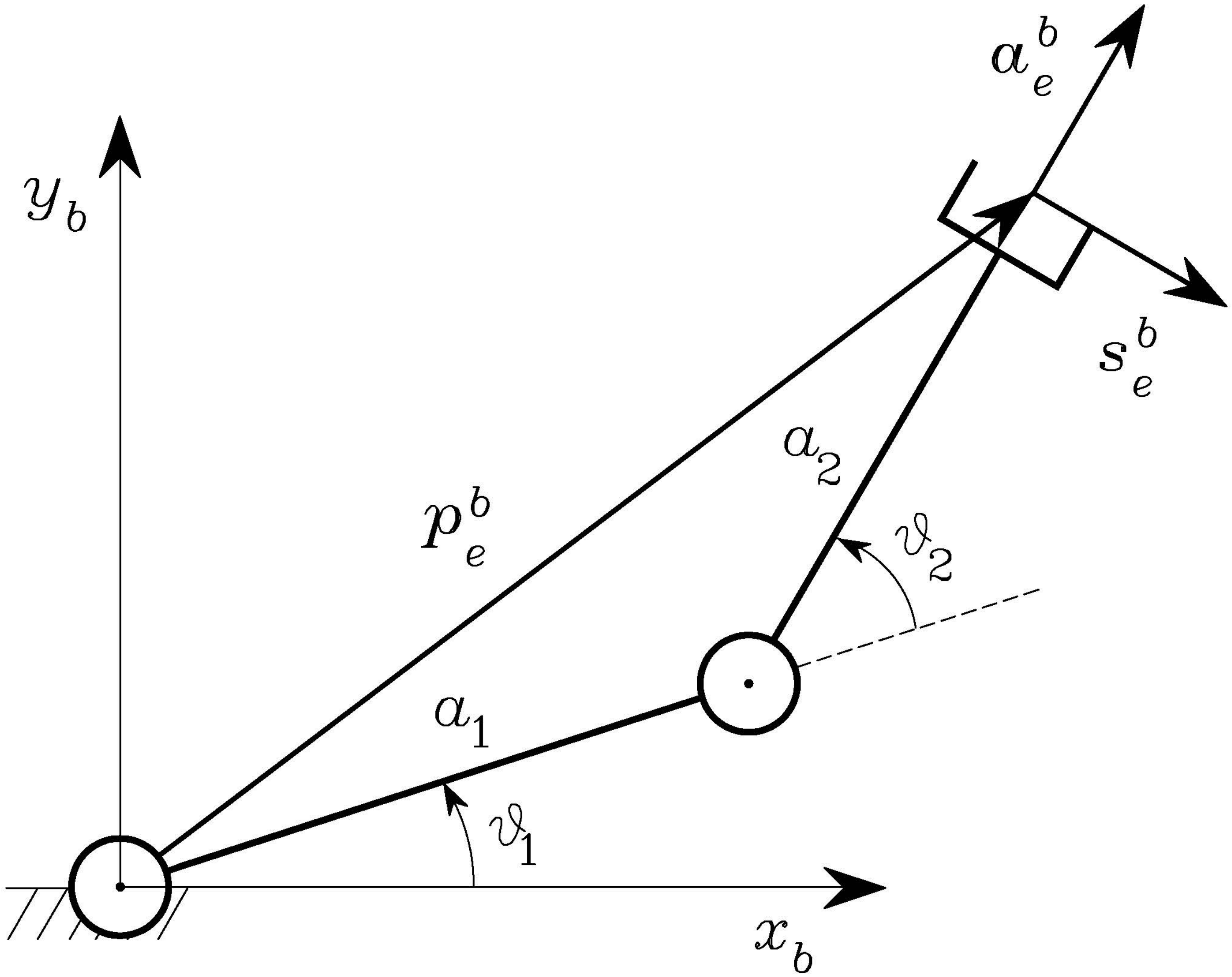
SECTION 8

Three-link Planar Arm

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p}_3 - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_3 - \boldsymbol{p}_1) & \boldsymbol{z}_2 \times (\boldsymbol{p}_3 - \boldsymbol{p}_2) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{bmatrix}$$

$$\boldsymbol{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} \quad \boldsymbol{p} = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 \end{bmatrix}$$

$$\boldsymbol{z}_0 = \boldsymbol{z}_1 = \boldsymbol{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Three-link planar arm

Three-link Planar Arm

$$\boldsymbol{J} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

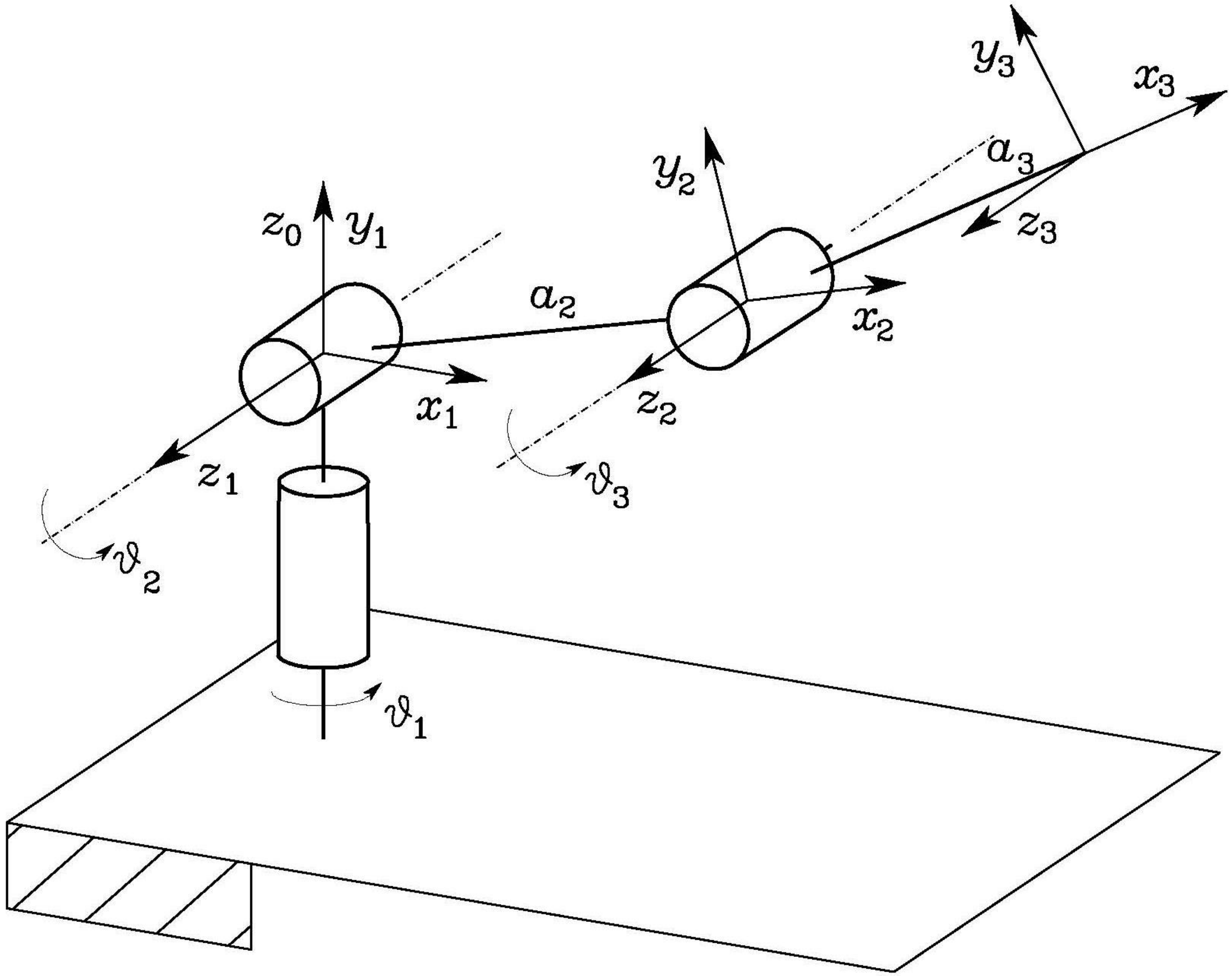
$$\boldsymbol{J}_P = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \end{bmatrix} \qquad (m = 2)$$

Anthropomorphic Arm

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p}_3 - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_3 - \boldsymbol{p}_1) & \boldsymbol{z}_2 \times (\boldsymbol{p}_3 - \boldsymbol{p}_2) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{bmatrix}$$

$$\boldsymbol{p}_0 = \boldsymbol{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_2 = \begin{bmatrix} a_2 c_1 c_2 \\ a_2 s_1 c_2 \\ a_2 s_2 \end{bmatrix} \quad \boldsymbol{p}_3 = \begin{bmatrix} c_1 (a_2 c_2 + a_3 c_{23}) \\ s_1 (a_2 c_2 + a_3 c_{23}) \\ a_2 s_2 + a_3 s_{23} \end{bmatrix}$$

$$\boldsymbol{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \boldsymbol{z}_1 = \boldsymbol{z}_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$



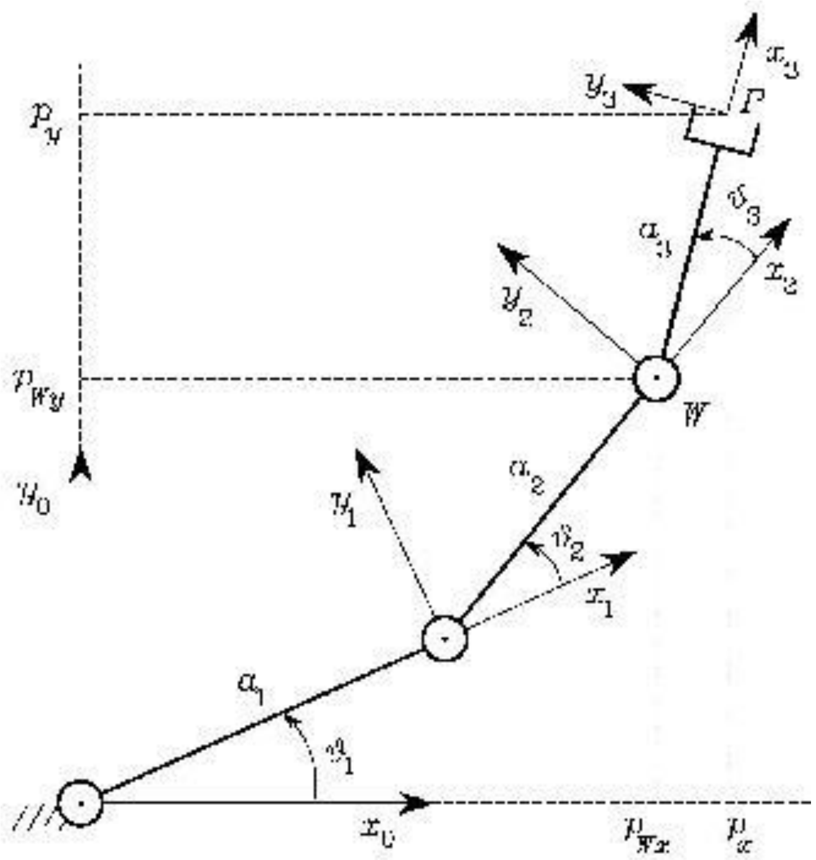
Anthropomorphic arm

Anthropomorphic Arm

$$\boldsymbol{J} = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{J}_P = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{bmatrix} \qquad (m = 3)$$

Further Insights



SCARA Manipulator

Jacobian Matrix Operations

SECTION 9

Jacobian Matrix

- If $A = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, B = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$
- Then the cross product

$$A \times B = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ -(a_x b_z - a_z b_x) \\ a_x b_y - a_y b_x \end{bmatrix}$$

Jacobian Matrix

- The Denavit-Hartenberg matrix T

$$A_i = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & r_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -s\alpha_i c\theta_i & r_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The transformation matrix T

$$T_i^0 = A_1 A_2 \ldots\ldots A_i$$

Jacobian Matrix (n= n DOF)

$$J = [J_1 \quad J_2 \quad \cdots \quad J_n]$$

where if join (i) is revolute

$$J_i = \begin{bmatrix} Z_{i-1} \times (O_n - O_{i-1}) \\ Z_{i-1} \end{bmatrix}$$

- And if joint (i) is prismatic

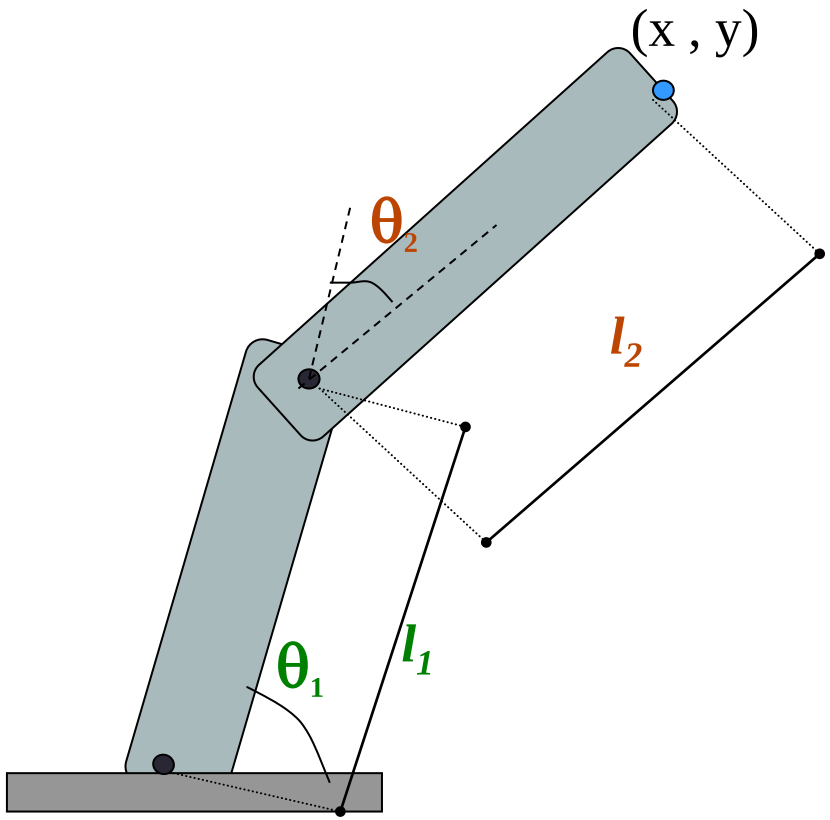
$$J_i = \begin{bmatrix} Z_{i-1} \\ 0 \end{bmatrix}$$

- Where Z_i is the first three elements in the 3rd column of the T_i^0 matrix, and O_i is the first three elements in the 4th column of the T_i^0 matrix

Jacobian Matrix

2-DOF planar robot arm, Given *l1*, *l2* , *Find: Jacobian*

- *Here, n=2,*



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable

$$A_i = \begin{bmatrix} c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -s\alpha_i c\theta_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z_0 = Z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T_1^0 = A_1.$$

$$T_2^0 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where $(\theta_1 + \theta_2)$ denoted by θ_{12} , r_i by a_i and $\cos(\theta_1 + \theta_2)$ by c_{12}

$$O_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, O_1 = \begin{bmatrix} a_1 \cos\theta_1 \\ a_1 \sin\theta_1 \\ 0 \end{bmatrix}, O_2 = \begin{bmatrix} a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

Jacobian Matrix

2-DOF planar robot arm

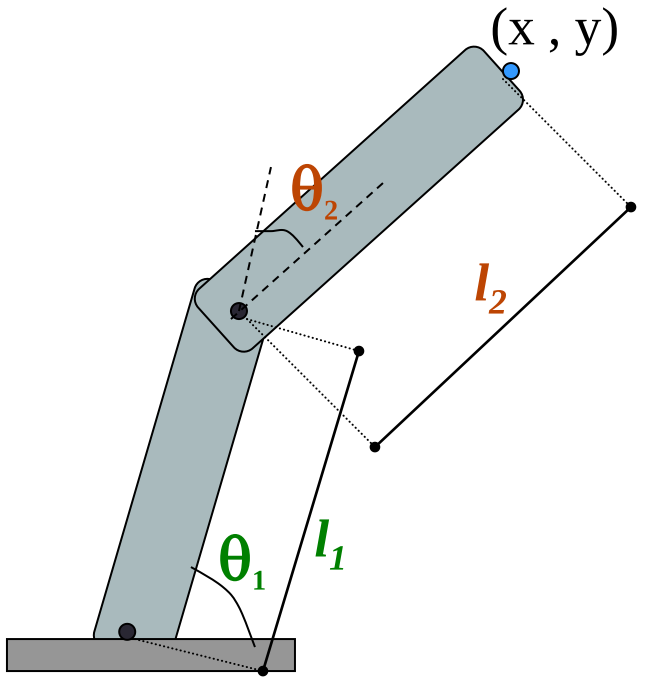
Given *l1*, *l2* , ***Find: Jacobian***

- *Here, n=2*

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable

$$J_1 = \begin{bmatrix} z_0 \times (o_2 - o_0) \\ z_0 \end{bmatrix}, J_2 = \begin{bmatrix} z_1 \times (o_2 - o_1) \\ z_1 \end{bmatrix}$$



Jacobian Matrix

$$J_1 = \begin{bmatrix} z_0 \times (o_2 - o_0) \\ z_0 \end{bmatrix} \qquad Z_0 \times (o_2 - o_0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

Jacobian Matrix

$$J_2 = \begin{bmatrix} z_1 \times (o_2 - o_1) \\ z_1 \end{bmatrix}$$

$$Z_1 \times (o_2 - o_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 \cos(\theta_1 + \theta_2) \\ a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ a_2 \cos(\theta_1 + \theta_2) & a_2 \sin(\theta_1 + \theta_2) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_2 \sin(\theta_1 + \theta_2) \\ a_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

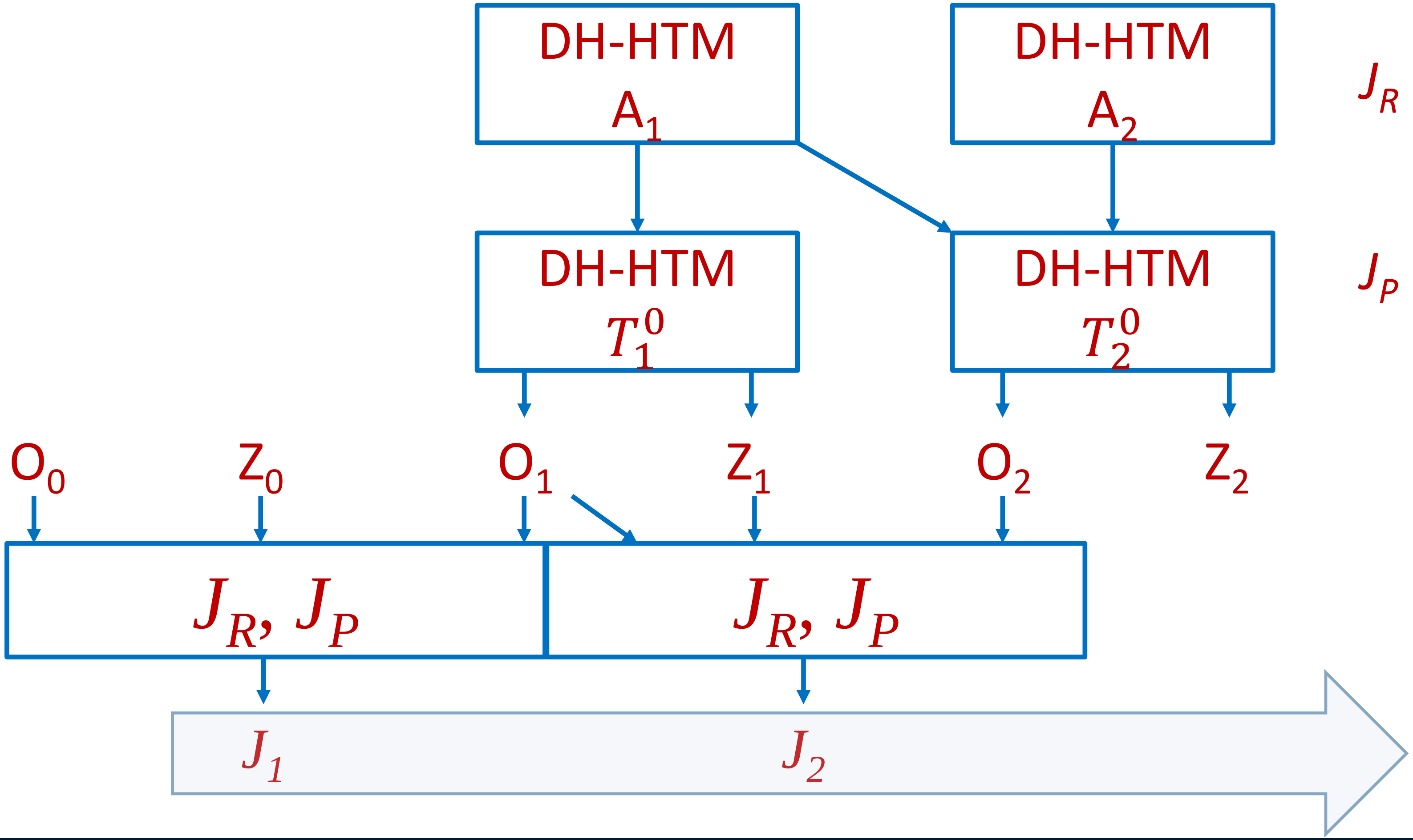
Jacobian Matrix

$$J_1 = \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad J_2 = \begin{bmatrix} -a_2 \sin(\theta_1 + \theta_2) \\ a_2 \cos(\theta_1 + \theta_2) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The required Jacobian matrix **J**

$$J = \begin{bmatrix} J_1 & J_2 \end{bmatrix}$$

Generation of Jacobian Matrix



$$J_R = \begin{bmatrix} Z_{i-1} \times (O_n - O_{i-1}) \\ Z_{i-1} \end{bmatrix}$$

$$J_P = \begin{bmatrix} Z_{i-1} \\ 0 \end{bmatrix}$$

$$J = [J_1 \quad J_2]$$