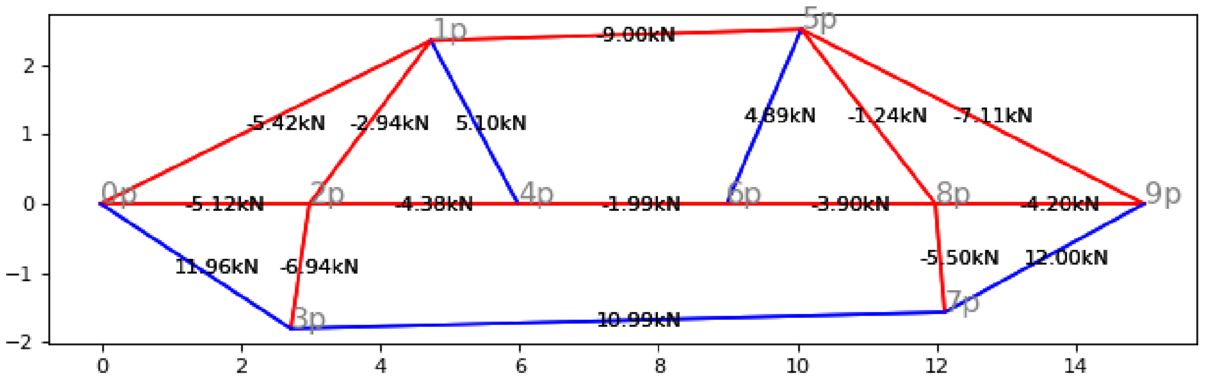
Final Iteration



We iterated some more and arrived at this final design. This design reduces the cost significantly and all members are also within their breaking limit. Joints along the x-axis are placed in the same way as the first design iteration.

The picture uses conventional coordinate axes, where right is positive x and up is positive y. **1p** is located at (4.73, 2.36), **5p** is located at (10.06, 2.52), **3p** is located at (2.72, -1.81), and **7p** is located at (12.11, -1.57). Points **0p, 2p, 4p, 6p, 8p, 9p** are evenly spaced 3 m apart, starting at (0, 0) and ending at (15, 0).

There are 10 total joints with two equations per joint for each coordinate axis. This yields 20 equations. There are also 3 more linear equations of equilibrium for the total bridge. In addition, there are 17 beams with unknown tension, and 2 unknown forces at the left anchor point and 1 unknown force at the right anchor point. Therefore, the result is a 23 by 20 matrix, representing 23 equations and 20 unknowns.

After solving the matrix, we get a reduced row echelon form variant, which we could interpret as forces for each beam. The tensile forces (in kN) are listed here (Negative tensile forces mean compression):

0p-1p: -5.42	0p-2p: -5.12	1p-2p: -2.94	1p-5p: -9.00	2p-3p: -6.93	2p-4p: -4.38
3p-0p: 11.95	4p-1p: 5.09	4p-6p: -1.98	5p-8p: -1.23	5p-9p: -7.11	6p-5p: 4.89
6p-8p: -3.90	7p-3p: 10.98	8p-7p: -5.50	8p-9p: -4.19	9p-7p: 12.00	

The fixed forces at each anchor point are (in kN, conventional coordinate axes):
fixed-0p-x: 0 fixed-0p-y: 11.25 fixed-9p-y: 11.25

Cost Analysis

We calculate the length of the beams by using the Euclidean distance of the two endpoints. The total length of the beams are 62.03 m. Since there are 10 joints, there are 10 gusset places. Therefore, the total cost is

$$10 \;*\; 5 \;+\; 62.03 \;*\; 10 \;=\; \$\,670.30.$$

Details on force calculation

Because calculating forces and solving simultaneous equations can be very tedious, we opted to put our constraints into a large matrix. Rows are equations, columns are variables, and matrix values are coefficients. In other words, we reform the problem into a linear **Ax = b** problem.

This is the **A** matrix.

$$\begin{array}{cccccccccccccccccccc|c}
\textbf{0p-1p} & \textbf{0p-2p} & \textbf{1p-2p} & \textbf{1p-5p} & \textbf{2p-3p} & \textbf{2p-4p} & \textbf{3p-0p} & \textbf{4p-1p} & \textbf{4p-6p} & \textbf{5p-8p} & \textbf{5p-9p} & \textbf{6p-5p} & \textbf{6p-8p} & \textbf{7p-3p} & \textbf{8p-7p} & \textbf{8p-9p} & \textbf{9p-7p} & \textbf{fixed-0p-x} & \textbf{fixed-0p-y} & \textbf{fixed-9p-y} & \\
0.9 & 1. & 0. & 0. & 0. & 0. & 0.83 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & \mathbf{0p}_x \\
0.45 & 0. & 0. & 0. & 0. & 0. & -0.55 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & \mathbf{0p}_y \\
-0.9 & 0. & -0.6 & 1. & 0. & 0. & 0. & 0.47 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & \mathbf{1p}_x \\
-0.45 & 0. & -0.8 & 0.03 & 0. & 0. & 0. & -0.88 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & \mathbf{1p}_y \\
0. & -1. & 0.6 & 0. & -0.15 & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & \mathbf{2p}_x \\
0. & 0. & 0.8 & 0. & -0.99 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & \mathbf{2p}_y \\
0. & 0. & 0. & 0. & 0.15 & 0. & -0.83 & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & \mathbf{3p}_x \\
0. & 0. & 0. & 0. & 0.99 & 0. & 0.55 & 0. & 0. & 0. & 0. & 0. & 0. & 0.03 & 0. & 0. & 0. & 0. & 0. & 0. & \mathbf{3p}_y \\
0. & 0. & 0. & 0. & 0. & -1. & 0. & -0.47 & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & \mathbf{4p}_x \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.88 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & \mathbf{4p}_y \\
0. & 0. & 0. & -1. & 0. & 0. & 0. & 0. & 0. & 0.61 & 0.89 & -0.39 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & \mathbf{5p}_x \\
0. & 0. & 0. & -0.03 & 0. & 0. & 0. & 0. & 0. & -0.79 & -0.46 & -0.92 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & \mathbf{5p}_y \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1. & 0. & 0. & 0.39 & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & \mathbf{6p}_x \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.92 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & \mathbf{6p}_y \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1. & -0.08 & 0. & 0.88 & 0. & 0. & 0. & \mathbf{7p}_x \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -0.03 & 1. & 0. & 0.48 & 0. & 0. & 0. & \mathbf{7p}_y \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -0.61 & 0. & 0. & -1. & 0. & 0.08 & 1. & 0. & 0. & 0. & 0. & \mathbf{8p}_x \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.79 & 0. & 0. & 0. & 0. & -1. & 0. & 0. & 0. & 0. & 0. & \mathbf{8p}_y \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -0.89 & 0. & 0. & 0. & 0. & -1. & -0.88 & 0. & 0. & 0. & \mathbf{9p}_x \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.46 & 0. & 0. & 0. & 0. & 0. & -0.48 & 0. & 0. & 1. & \mathbf{9p}_y \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & \mathbf{Total}_x \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 1. & \mathbf{Total}_y \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 15. & \mathbf{Total}_{Moment}
\end{array}$$

The **b** column matrix (shown here as the transposed for space constraint).

$$\begin{bmatrix} 0. & 2.25 & 0. & 0. & 0. & 4.5 & 0. & 0. & 0. & 4.5 & 0. & 0. & 0. & 4.5 & 0. & 0. & 0. & 4.5 & 0. & 2.25 & 0. & 22.5 & 168.36 \end{bmatrix}$$

The solution **x** column matrix (shown here as the transposed).

$$\begin{bmatrix} -5.42 & -5.12 & -2.94 & -9. & -6.94 & -4.38 & 11.96 & 5.1 & -1.99 & -1.24 & -7.11 & 4.89 & -3.9 & 10.99 & -5.5 & -4.2 & 12. & -0. & 11.25 & 11.25 \end{bmatrix}$$

For example, let’s look at the **2p** point. **2p** will be pulled left (negative x) by **0p-2p**, pulled right (positive x) by **2p-4p**, pulled up (positive y) by **1p-2p**, pulled down (negative y) by **2p-3p**, and pulled down (negative y) by the train force of 4.5 kN. All these relationships are expressed in the fifth and sixth row, corresponding to **2p**$_x$ and **2p**$_y$ respectively. In the **b** column matrix, the sum of the *x* forces must be 0, and the sum of the *y* forces must be 4.5 (because of the -4.5kN force of the train).

As another example, let's look at **3p**. **3p** would have two equations describing its equilibrium:

X-equilibrium: $0 = (3p \rightarrow 7p)cos\theta_{3p7p} - (2p \rightarrow 3p)cos\theta_{2p3p} - (0p \rightarrow 3p)cos\theta_{0p3p}$

where $cos\theta_{0p3p}$ is the angle of beam **0p-3p** to the x-axis.

Y-equilibrium: $0 = -(3p \rightarrow 7p)sin\theta_{3p7p} + (2p \rightarrow 3p)sin\theta_{2p3p} + (0p \rightarrow 3p)sin\theta_{0p3p}$

We recognize that these two are linear equations with coefficients as the sine and cosine angles of each beam.

From the geometry of the bridge,
$cos\theta_{3p7p} = 1,\ cos\theta_{2p3p} = 0.15,\ cos_{0p3p} = 0.83$ Correspondingly, the **3p**$_x$ row has these exact same coefficients.

Calculation of angles

These cosine and sine values are encoded into the matrix coefficients. For example, to calculate $sin\theta_{3p7p}$ or $cos\theta_{3p7p}$, we do this.

$$\frac{\vec{v}_{3p} - \vec{v}_{7p}}{||\vec{v}_{3p} - \vec{v}_{7p}||} = \begin{bmatrix} cos(\theta_{3p7p}) \\ sin(\theta_{3p7p}) \end{bmatrix}$$

$\vec{v}_{3p}$ is the position vector (as column vector form) to the point **3p**. Absolute value brackets denote the Euclidean distance. We perform this calculation for all beams, and use the cosine and sine values in the matrix as coefficients.

Solving for x and error estimation

Thus, the problem is to find a solution **x** to **Ax = b**, and **x** will be the forces of all our beam members. We already have **A** and **b**, as described above. The solution uses a linear least-squares algorithm based on Singular Value Decomposition to find the minimal solution of **x** to the equation. The solution is given above.

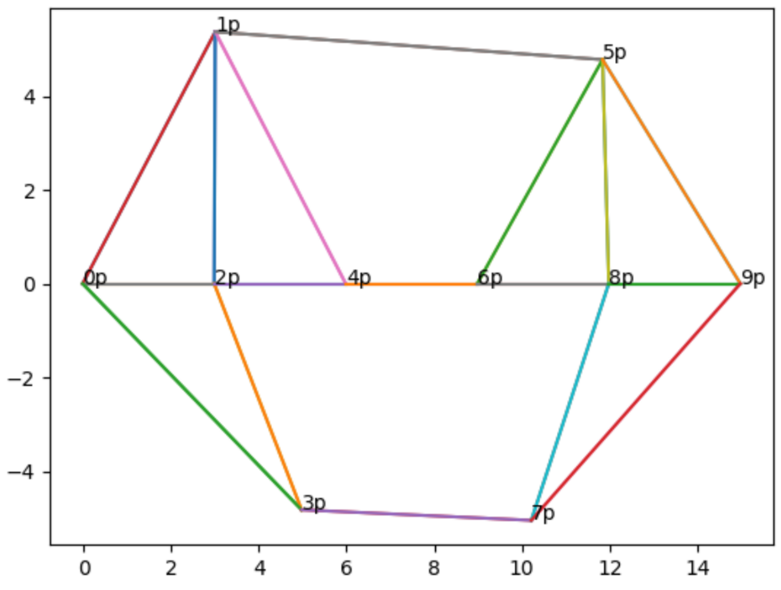
Since this is an overdetermined linear system (more equations than variables), we must verify that all equations are indeed satisfied in the computed solution. Therefore, if we take the magnitude of the error squared, we should get a meaningful estimate of how "wrong" the solution is.

$$||\mathbf{Ax} - \mathbf{b}||^2$$

This error is $7.7 * 10^{-6}$, which is sufficiently close to zero.

Methodology

The hand-drawn design was originally this:



However, even though it satisfied all constraints, the cost was very high. The truss members were very far apart and not optimized. Therefore, we used a computer program to "optimize the bridge." In other words, we move the movable joints (**1p, 3p, 5p, 7p**) around to minimize the cost.

The optimization procedure starts from this, a known, working bridge that does not break. Then, it chooses a point from **{1p, 3p, 5p, 7p}** at random to shift a random amount. After the shift, we calculate the forces again using the least squares algorithm. If the new bridge satisfies the constraints and is less costly than the old bridge, we "commit" that random change to the old bridge. Else, we discard the change and restart. After several iterations of this random search, we reach a local minimum in cost. The achieved local minimum after about 200 iterations is our final design, shown above.