First Term Test F.5A M2

4-11-2019

1a) $LHS = \frac{\sin 4x}{1-\cos 4x}$

$= \frac{\sin 2(2x)}{1-\cos 2(2x)}$

$= \frac{2\sin(2x)\cos(2x)}{1-(1-2\sin^2(2x))}$

$= \frac{\cancel{2\sin(2x)}\cos(2x)}{\cancel{2}\sin^{\cancel{2}}(2x)}$

$= \frac{\cos(2x)}{\sin(2x)}$

$= \cot 2x$ //

$RHS = \cot 2x$

$\therefore \frac{\sin 4x}{1-\cos 4x} = \cot 2x$

1b) $\int \frac{\sin 4x}{\sin 2x(1-\cos 4x)} dx$

$= \int \left[\frac{1}{\sin 2x} \cdot \cot 2x\right] dx$

$= \int \operatorname{cosec} 2x \cot 2x \, dx$

$= \frac{1}{2}\int \operatorname{cosec} 2x \cot 2x \, d(2x)$

$= \frac{1}{2}[-\operatorname{cosec} 2x] + C$ //

2) $\displaystyle\int \frac{x^3}{\sqrt{1+9x^2}}\,dx$

let $x = \frac{1}{3}\tan\theta$

$dx = \frac{1}{3}\sec^2\theta\,d\theta$

$$= \int \frac{\frac{1}{27}\tan^3\theta}{\sqrt{1+\tan^2\theta}}\left(\frac{1}{3}\sec^2\theta\right)d\theta$$

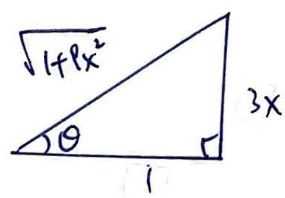
$x = \frac{1}{3}\tan\theta$

$\tan^{-1}(3x) = \theta$

$$= \frac{1}{81}\int \tan^3\theta\sec\theta\,d\theta$$

$$= \frac{1}{81}\int \tan^{\cancel{3}}\theta\,\cancel{\sec\theta}\,\frac{d(\sec\theta)}{\cancel{\sec\theta\tan\theta}}$$

$$= \frac{1}{81}\int \tan^2\theta\,d(\sec\theta)$$



$$= \frac{1}{81}\int (\sec^2\theta - 1)\,d(\sec\theta)$$

$$= \frac{1}{81}\left(\frac{\sec^3\theta}{3}\right) - \frac{1}{81}(\sec\theta) + C$$

$$= \frac{\sec^3\theta}{243} - \frac{\sec\theta}{81} + C$$

$$= \cancel{(1+9x^2)^{\frac{3}{2}}}$$

$$= \frac{(1+9x^2)^{\frac{3}{2}}}{243} - \frac{(1+9x^2)}{81} + C$$

3) $\int x^2(\ln x)^2\,dx$

$= \int \cancel{x^2}(\ln x)^2 \frac{d(x^3)}{3\cancel{x^2}}$

$= \frac{1}{3}\int (\ln x)^2\, d(x^3)$

$= \frac{1}{3}\left[x^3(\ln x)^2\right] - \frac{1}{3}\int x^3\, d(\ln x)^2$

$= \frac{1}{3}\left[x^3(\ln x)^2\right] - \frac{1}{3}\int \cancel{x^3}(2\ln x)\left(\frac{1}{\cancel{x}}\right) dx$

$= \frac{1}{3}\left[x^3(\ln x)^2\right] - \frac{2}{3}\int x^2 \ln x\, dx$

$= \frac{1}{3}\left[x^3(\ln x)^2\right] - \frac{2}{3}\int \cancel{x^2}\ln x \frac{d(x^3)}{3\cancel{x^2}}$

$= \frac{1}{3}\left[x^3(\ln x)^2\right] - \frac{2}{9}\int \ln x\, d(x^3)$

$= \frac{1}{3}\left[x^3(\ln x)^2\right] - \frac{2}{9}\left[x^3\ln x\right] + \frac{2}{9}\int x^3\, d(\ln x)$

$= \frac{1}{3}\left[x^3(\ln x)^2\right] - \frac{2}{9}\left[x^3\ln x\right] + \frac{2}{9}\int x^2\, dx$

$= \frac{1}{3}\left[x^3(\ln x)^2\right] - \frac{2}{9}\left[x^3\ln x\right] + \frac{2}{9}\left[\frac{x^3}{3}\right] + C$

$= \frac{1}{3}\left[x^3(\ln x)^2\right] - \frac{2}{9}\left[x^3\ln x\right] + \frac{2x^3}{27} + C$

$= \frac{1}{3}\left[x^3(\ln x)^2\right] - \frac{2}{9}\left[x^3\ln x\right] + \frac{2}{27}x^3 + C_4$

4) $\int_{\ln 3}^{\ln 8} e^x\sqrt{1+e^x}\, dx$

$= \int_{\ln 3}^{\ln 8} \cancel{e^x}\sqrt{1+e^x}\, \frac{d(e^x+1)}{\cancel{e^x}}$

$= \int_{\ln 3}^{\ln 8} \sqrt{1+e^x}\, d(e^x+1)$

$= \int_{\ln 3}^{\ln 8} (1+e^x)^{\frac{1}{2}}\, d(e^x+1)$

$= \left[\frac{(1+e^x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{\ln 3}^{\ln 8}$

$= \left[\frac{2(1+e^x)^{\frac{3}{2}}}{3}\right]_{\ln 3}^{\ln 8}$

$= \frac{2}{3}\left[(1+e^{\ln 8})^{\frac{3}{2}}\right] - \frac{2}{3}\left[(1+e^{\ln 3})^{\frac{3}{2}}\right]$

$= 18 - \frac{16}{3}$

$= \frac{38}{3}$ //

5) $\int_0^4 \frac{x+2}{\sqrt{2x+1}}\, dx$

$= \frac{1}{2}\int_1^9 \frac{\frac{u-1}{2}+2}{\sqrt{u}}\, du$

let $u = 2x+1$

$\frac{du}{dx} = 2$

$\frac{du}{2} = dx$

when $x=4, u=9$

when $x=0, u=1$

$u = 2x+1$

$u-1 = 2x$

$x = \frac{u-1}{2}$

$$= \frac{1}{2}\int_1^9 \left(\frac{u-1}{2}+2\right)u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2}\int_1^9 \left(\frac{u+3}{2}\right)u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4}\int_1^9 (u+3)\,u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4}\int_1^9 \left(\frac{u}{\sqrt{u}}+\frac{3}{\sqrt{u}}\right)du$$

$$= \frac{1}{4}\int_1^9 u^{\frac{1}{2}}\,du + \frac{1}{4}\int_1^9 \frac{3}{\sqrt{u}}\,du$$

$$= \frac{1}{4}\left[\frac{2u^{\frac{3}{2}}}{3}\right]_1^9 + \frac{1}{4}\left[6\sqrt{u}\right]_1^9$$

$$= \frac{1}{4}\left[\frac{2(9)^{\frac{3}{2}}}{3}\right] - \frac{1}{4}\left(\frac{2(1)^{\frac{3}{2}}}{3}\right) + \frac{1}{4}\left[6\sqrt{9}\right] - \frac{1}{4}\left[6\sqrt{1}\right]$$

$$= \frac{9}{2} - \frac{\cancel{2}^{\,1}}{\cancel{12}\,6} + \frac{9}{2} - \frac{3}{2}$$

$$= \frac{9}{2} - \frac{1}{6} + \frac{9}{2} - \frac{3}{2}$$

$$= \frac{22}{3}$$ //

6) $\int_0^{\pi} e^x \sin 3x \, dx$

$$= \int_0^{\pi} \cancel{e^x} \sin 3x \frac{d(e^x)}{\cancel{e^x}}$$

$$= \int_0^{\pi} \sin 3x \, d(e^x)$$

$$= \left[e^x \sin 3x\right]_0^{\pi} - \int_0^{\pi} e^x \, d(\sin 3x)$$

$$= (0) - (0) - \int_0^{\pi} e^x (3\cos 3x) \, dx$$

$$= -3\int_0^{\pi} e^x \cos 3x \, dx$$

$$= -3\int_0^{\pi} \cancel{e^x} \cos 3x \frac{d(e^x)}{\cancel{e^x}}$$

$$= -3\int_0^{\pi} \cos 3x \, d(e^x)$$

$$= -3\left[e^x \cos 3x\right]_0^{\pi} + 3\int_0^{\pi} e^x (-3\sin 3x) \, dx$$

$$= -3\left[-e^{\pi} - 1\right] - 9\int_0^{\pi} e^x \sin 3x \, dx$$

$$\therefore \int_0^{\pi} e^x \sin 3x \, dx = -3\left[-e^{\pi} - 1\right] - 9\int_0^{\pi} e^x \sin 3x \, dx$$

$$10\int_0^{\pi} e^x \sin 3x \, dx = 3e^{\pi} + 3$$

$$\int_0^{\pi} e^x \sin 3x \, dx = \frac{3e^{\pi} + 3}{10}$$

7a) $RHS = \int_0^a f(a-x)\,dx$ let $u = a-x$

$\frac{du}{dx} = -1$

$dx = -du$

$= \int_a^0 f(u)(-du)$

$= -\int_a^0 f(u)\,du$ when $x=a$, $u=0$

when $x=0$, $u=a$

$= \int_0^a f(u)\,du$

$LHS = \int_0^a f(x)\,dx$

$\because$ x and u are dummy variables.

$\therefore \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

7b) let $f(x) = \ln(1+\tan x)$

$\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left[1+\tan\left(\frac{\pi}{4}-x\right)\right] dx$

$\uparrow$

LHS:

$$= \int_0^{\frac{\pi}{4}} \ln\left\{1+\frac{\tan(\frac{\pi}{4})-\tan x}{1+\tan(\frac{\pi}{4})\tan x}\right\} dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left\{\frac{1+\tan x}{1+\tan x}+\frac{1-\tan x}{1+\tan x}\right\} dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left[\frac{2}{1+\tan x}\right] dx \quad = RHS$$

$\therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\frac{2}{1+\tan x}\,dx$

7c) $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

$\because \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\frac{2}{1+\tan x}\,dx$ (proved)

$\therefore 2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx + \int_0^{\frac{\pi}{4}} \ln\frac{2}{1+\tan x}\,dx$

$= \int_0^{\frac{\pi}{4}} \ln\left(\cancel{(1+\tan x)}\frac{2}{\cancel{(1+\tan x)}}\right)dx$

$= \int_0^{\frac{\pi}{4}} \ln(2)\,dx$

黄老師做法!

$= \left[x\ln 2\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4}\ln 2$

$\therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{\left(\frac{\pi}{4}\ln 2\right)}{2}$

$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{\pi}{8}\ln 2$

OR

$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

By part(b),

$= \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

$= \int_0^{\frac{\pi}{4}} \ln\frac{2}{1+\tan x}\,dx$

$= \int_0^{\frac{\pi}{4}} \ln 2\,dx - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

$\therefore \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln 2\,dx - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

$2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \left[x\ln 2\right]_0^{\frac{\pi}{4}}$

$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{\pi}{8}\ln 2$

鍾老師做法!

PP

8a) $\int x \sin x \, dx$

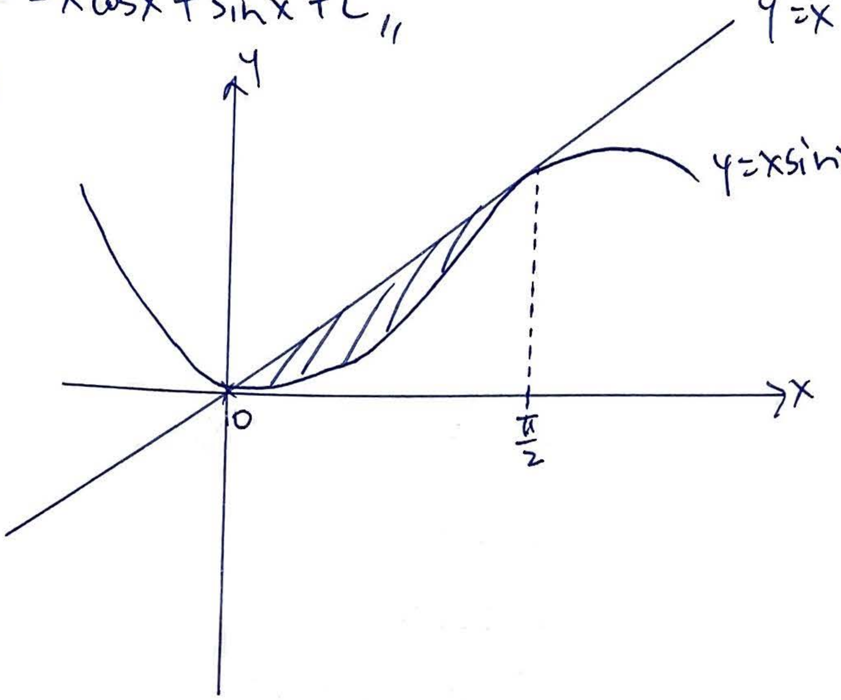
$= \int x \cancel{\sin x} \frac{d(\cos x)}{-\cancel{\sin x}}$

$= -\int x \, d(\cos x)$

$= -[x \cos x] + \int \cos x \, dx$

$= -x \cos x + \sin x + C$ //

8b)



$\begin{cases} y = x \\ y = x \sin x \end{cases}$

$x = x \sin x$

$x = \frac{\pi}{2}$

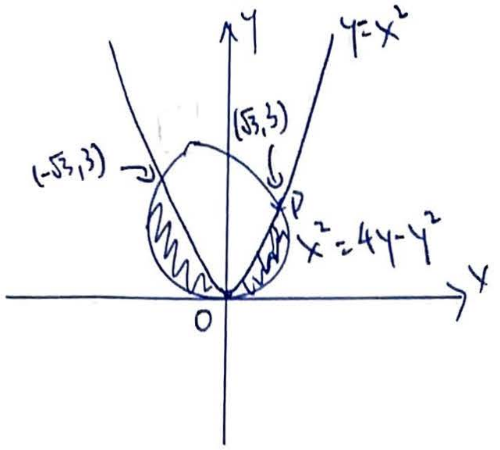
Area $= \int_0^{\frac{\pi}{2}} (x - x\sin x) \, dx$

$= \int_0^{\frac{\pi}{2}} x \, dx - \int_0^{\frac{\pi}{2}} x \sin x \, dx$

$= \left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} - \left[-x\cos x + \sin x\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi^2}{8} - \left[(+1) - (0)\right] \qquad = \frac{\pi^2}{8} - 1$ //

9a)



$\begin{cases} y=x^2 \\ x^2=4y-y^2 \end{cases}$

$y=4y-y^2$

$0=-y^2+4y-y$

$\therefore y=0$ or $y=3$

Put $y=3$ into $y=x^2$

$3=x^2$

$\pm\sqrt{3}=x$

Point P is in quad I,

$\therefore$ the coordinates of the point P is $P(\sqrt{3},3)$

Area: $2\int_0^3 \left(\sqrt{4y-y^2}-\sqrt{y}\right)dy$

$=2\int_0^3 \left\{\sqrt{4-(y-2)^2}-y^{\frac{1}{2}}\right\}dy$

$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \sqrt{4-4\sin^2\theta}\,(2\cos\theta)\,d\theta-2\left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^3$

$=8\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \cos^2\theta\,d\theta-\frac{4}{3}\left[(3^{\frac{3}{2}})-0\right]$

$=8\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{1+\cos 2\theta}{2}\,d\theta-4\sqrt{3}$

$4y-y^2$

$=-y^2+4y$

$=-[y^2-4y]$

$=-[(y)^2-2(y)(2)+(2)^2-(2)^2]$

$=-[(y-2)^2-4]$

$=-(y-2)^2+4$

let $(y-2)=2\sin\theta$

$\frac{dy}{d\theta}=2\cos\theta$

when $y=3$, $\theta=\frac{\pi}{6}$

when $y=0$, $\theta=-\frac{\pi}{2}$

$=4\left[\theta+\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}-4\sqrt{3}$

$=4\left[\left(\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right)-\left(-\frac{\pi}{2}\right)\right]-4\sqrt{3}$

$=\frac{8\pi}{3}-3\sqrt{3}$ //

-End-