17. HKALE 2007 AL Pure Mathematics Paper 2 Q10

(a) Using integration by substitution, find $\int \sqrt{a^2 - x^2}\, dx$, where a is a positive constant. (3 marks)

(b) Let a and b be positive constants. Consider the curve E: $y = \frac{b}{a}\sqrt{a^2 - x^2}$, where $0 < x < a$. Prove that the straight line $y = mx + c$ is a tangent to E if and only if $m < 0$ and $c = \sqrt{a^2 m^2 + b^2}$. (4 marks)

$\Delta = 0$

(c) Consider the curve E_1: $y = \sqrt{27 - 3x^2}$, where $0 < x < 3$, and the curve E_2: $y = \sqrt{9 - \frac{x^2}{3}}$, where $0 < x < 3\sqrt{3}$.

(i) Find the point of intersection of E_1 and E_2.

(ii) Let L be the common tangent to E_1 and E_2.

(1) Find the equation of L.

(2) Find the area of the region bounded by E_1, E_2 and L. (8 marks)

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(a) $\int \sqrt{a^2-x^2}\,dx$

let $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$$= \int \sqrt{a^2-(a^2\sin^2\theta)}\; a\cos\theta\, d\theta$$

$$= \int \sqrt{a^2(1-\sin^2\theta)}\; a\cos\theta\, d\theta$$

$$= \int a^2\sqrt{1-\sin^2\theta}\; \cos\theta\, d\theta$$

$$= a^2\int \cos^2\theta\, d\theta$$

$$= a^2\int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{a^2}{2}\int 1\, d\theta + \frac{a^2}{2}\int \cos 2\theta\, d\theta$$

$$= \frac{a^2}{2}[\theta] + \frac{a^2}{4}\int \cos 2\theta\, d(2\theta)$$

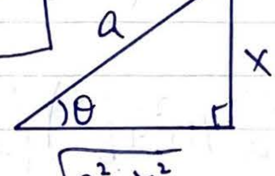
$$= \frac{a^2}{2}[\theta] + \frac{a^2}{4}[\sin 2\theta] + C$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{\cancel{a^2}}{\cancel{4}}\left[\frac{\cancel{2}x\sqrt{a^2-x^2}}{\cancel{a^2}}\right] + C$$

$x = a\sin\theta$

$\theta = \sin^{-1}\left(\frac{x}{a}\right)$

$$\boxed{= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2} + C}$$



$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin\theta = \frac{x}{a}$

$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$

$$\sin 2\theta = 2\left(\frac{x}{a}\right)\left(\frac{\sqrt{a^2-x^2}}{a}\right)$$

$$= \frac{2x\sqrt{a^2-x^2}}{a^2}$$

(b) $E: y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \le x \le a$

$$\begin{cases} y = \frac{b}{a}\sqrt{a^2 - x^2} & \text{—— (1)} \\ y = mx + c & \text{—— (2)} \end{cases}$$

substituting (2) into (1)

$$mx + c = \frac{b}{a}\sqrt{a^2 - x^2}$$

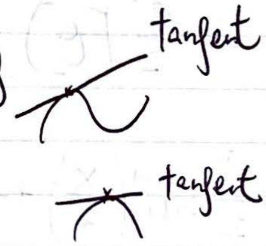
$$(mx+c)^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$m^2x^2 + 2mcx + c^2 = b^2 - \frac{b^2}{a^2}x^2$$

$$a^2m^2x^2 + 2a^2mcx + a^2c^2 = a^2b^2 - b^2x^2$$

$$\underbrace{(a^2m^2 + b^2)}_{a}x^2 + \underbrace{2a^2mc}_{b}x + \underbrace{(a^2c^2 - a^2b^2)}_{c} = 0$$

eg



$\Delta = 0$

$b^2 - 4ac = 0$

$$(2a^2mc)^2 - 4(a^2m^2 + b^2)(a^2c^2 - a^2b^2) = 0 \qquad \Delta = 0$$

$$4a^4m^2c^2 - 4(a^2m^2 + b^2)(a^2c^2 - a^2b^2) = 0$$

$$a^4m^2c^2 - (a^2m^2 + b^2)(a^2c^2 - a^2b^2) = 0$$

$$a^4m^2c^2-[a^4m^2c^2-a^4m^2b^2+a^2b^2c^2-a^2b^4]=0$$

$$\cancel{a^4m^2c^2}-\cancel{a^4m^2c^2}+a^4m^2b^2-a^2b^2c^2+a^2b^4=0$$

$$a^4m^2b^2-a^2b^2c^2+a^2b^4=0$$

$$-a^2b^2c^2=-a^2b^4-a^4m^2b^2$$

$$\frac{-a^2b^2c^2}{-a^2b^2}=\frac{-a^2b^4}{-a^2b^2}-\frac{a^4m^2b^2}{-a^2b^2}$$

$$c^2=b^2+a^2m^2$$

$$c=\pm\sqrt{a^2m^2+b^2}$$

$c=+\sqrt{a^2m^2+b^2}$ or $c=-\sqrt{a^2m^2+b^2}$

$\downarrow$ (rejected)

$m<0$

(c) $E_1: y=\sqrt{27-3x^2}$ $0<x<3$

$E_2: y=\sqrt{9-\frac{x^2}{3}}$ $0<x<3\sqrt{3}$

$E: y=\frac{b}{a}\sqrt{a^2-x^2}$

$E_1 : y = \sqrt{27 - 3x^2}$ $\qquad \sqrt{3^3 - 3x^2}$

$\sqrt{3^1 \cdot 3^2 - 3x^2}$

$= \sqrt{3(3^2 - x^2)}$

b

$= \dfrac{3\sqrt{3}}{3} \sqrt{3^2 - x^2}$

a

$$C = \sqrt{a^2 m^2 + b^2} = \sqrt{(3)^2 m^2 + (3\sqrt{3})^2}$$

$$= \sqrt{9m^2 + 27}$$

$E_2 : y = \sqrt{9 - \dfrac{x^2}{3}} = \sqrt{\dfrac{1}{3}(27 - x^2)}$

$b \rightarrow$

$= \dfrac{3}{3\sqrt{3}} \sqrt{(3\sqrt{3})^2 - x^2}$

$a \rightarrow$

$$C = \sqrt{a^2 m^2 + b^2} = \sqrt{(3\sqrt{3})^2 m^2 + (3)^2}$$

$$= \sqrt{27m^2 + 9}$$

(ci) $\begin{cases} E_1 : y = \sqrt{27 - 3x^2} & \text{——— (1)} \\ E_2 : y = \sqrt{9 - \dfrac{x^2}{3}} & \text{——— (2)} \end{cases}$

Substituting (1) into (2),

$$\sqrt{27-3x^2} = \sqrt{9-\frac{x^2}{3}}$$

$$\left(\sqrt{27-3x^2}\right)^2 = \left(\sqrt{9-\frac{x^2}{3}}\right)^2$$

$$27-3x^2 = 9-\frac{x^2}{3}$$

$$18 = \frac{8x^2}{3}$$

$$x^2 = \frac{27}{4}$$

$$x = \pm\sqrt{\frac{27}{4}}$$

$x = +\sqrt{\frac{27}{4}}$ or $x = -\sqrt{\frac{27}{4}}$ (rejected)

$$x = \frac{3\sqrt{3}}{2}$$

$$\Downarrow$$

$$y = \sqrt{27-3\left(\frac{3\sqrt{3}}{2}\right)^2}$$

$$y = \sqrt{27-3\left(\frac{27}{4}\right)}$$

$$y = \frac{3\sqrt{3}}{2}$$

$\therefore$ the intersection point of E_1 and E_2 : $\left(\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}\right)$

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(c(ii)) if L is the common tangent to E_1 and E_2

$\therefore$ c of E_1 equal to c of E_2

$$\begin{cases} c = \sqrt{9m^2+27} & \text{——— (1)} \\ c = \sqrt{27m^2+9} & \text{——— (2)} \end{cases}$$

substituting (1) into (2),

$$\sqrt{9m^2+27} = \sqrt{27m^2+9}$$

$$\left(\sqrt{9m^2+27}\right)^2 = \left(\sqrt{27m^2+9}\right)^2$$

$$9m^2+27 = 27m^2+9$$

$$18 = 18m^2$$

$$m^2 = 1$$

$$m = \pm 1 \qquad (m<0)$$

$m = +1$ (rejected) or $m = -1$

$$m = -1$$

put $m=-1$ into c,

$$c = \sqrt{27(-1)^2+9}$$
$$= 6$$

$\therefore$ the equation of L (common tangent):

$$y = mx + c$$

$$y = -x + 6$$

(cii 2) For the curve of $E_1 : y = \sqrt{27-3x^2}$

$$E_1 : y = \sqrt{27-3x^2}$$

$$y^2 = (27-3x^2)$$

$$3x^2 + y^2 = 27$$

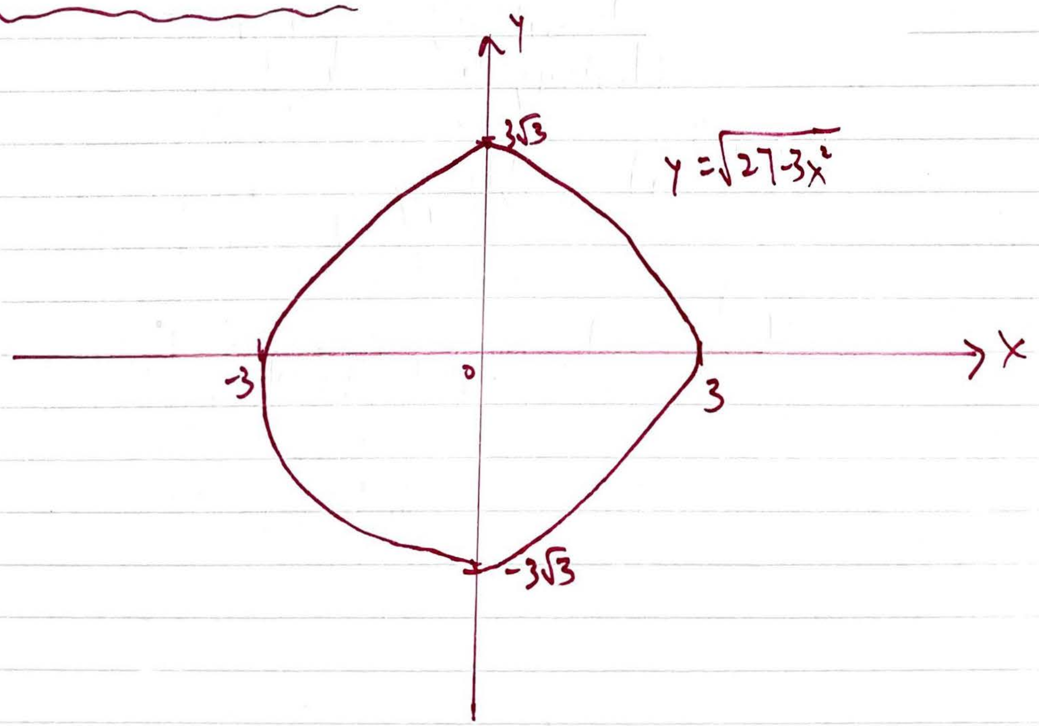
$$3x^2 + y^2 = (3\sqrt{3})^2$$

$$\frac{3x^2}{(3\sqrt{3})^2} + \frac{y^2}{(3\sqrt{3})^2} = \frac{\cancel{(3\sqrt{3})^2}}{\cancel{(3\sqrt{3})^2}}$$

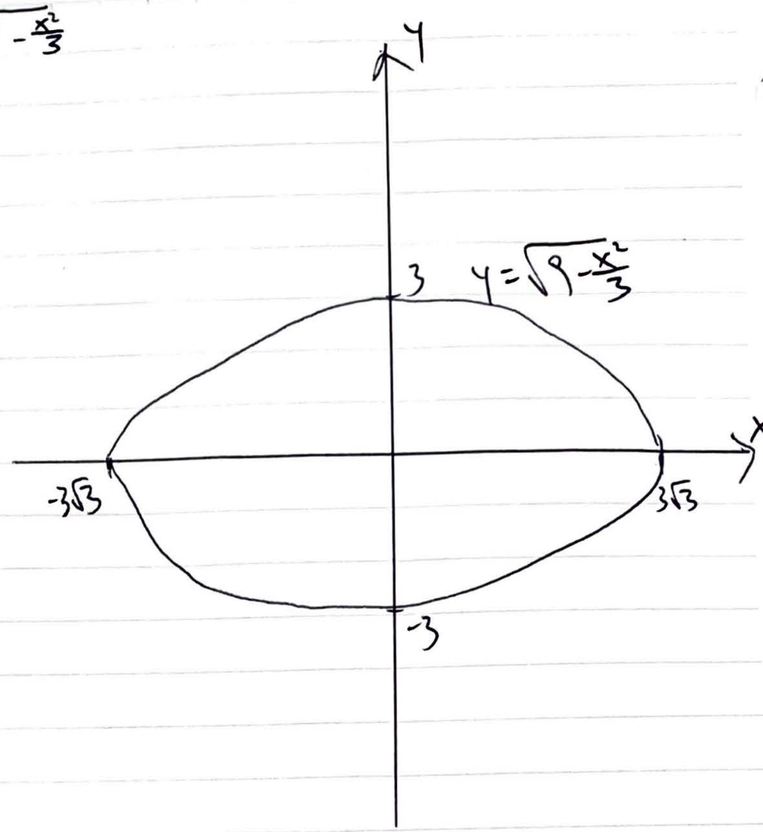
$$\frac{x^2}{3^2} + \frac{y^2}{(3\sqrt{3})^2} = 1$$

ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{3^2} + \frac{y^2}{(3\sqrt{3})^2} = 1$$



For the curve of $E_2 : y=\sqrt{9-\frac{x^2}{3}}$



$$E_2 : y = \sqrt{9-\frac{x^2}{3}}$$

$$y^2 = 9-\frac{x^2}{3}$$

$$\frac{x^2}{3}+y^2 = 9$$

$$\frac{x^2}{3} \div 9 + \frac{y^2}{9} = \frac{9}{9}$$

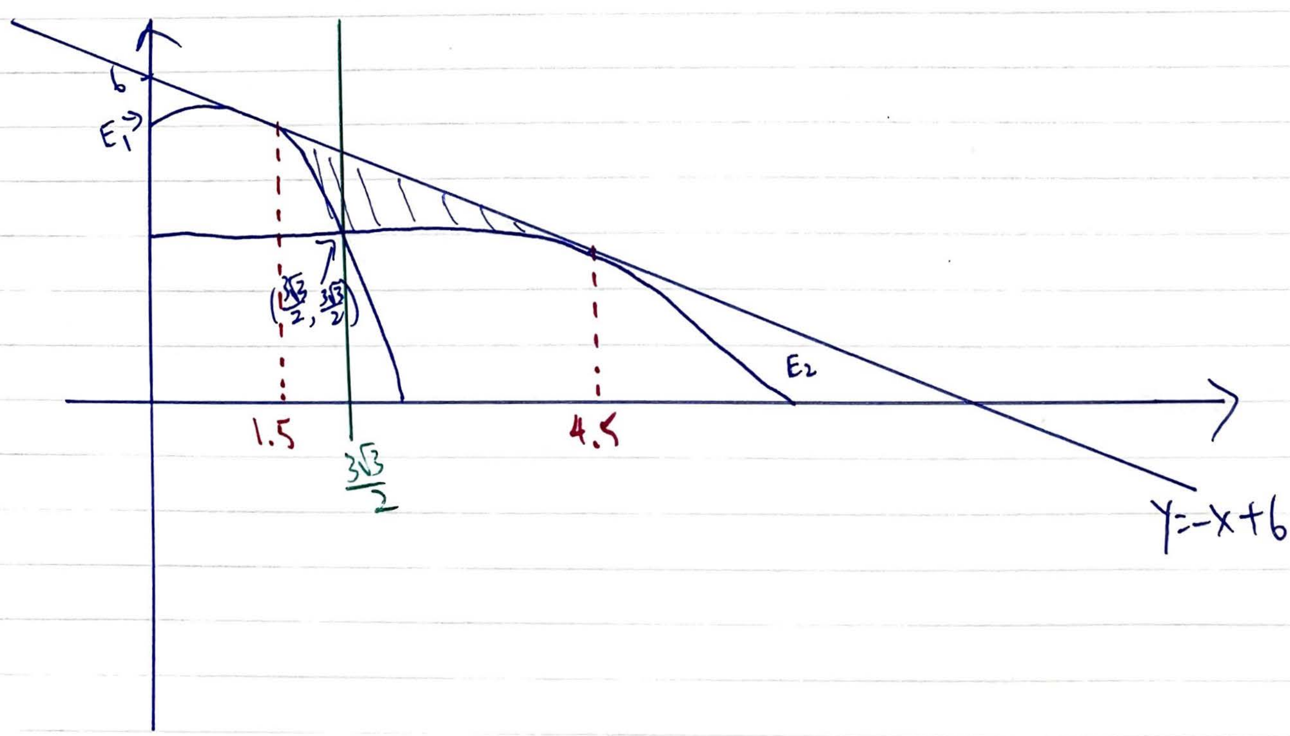
$$\frac{x^2}{27} + \frac{y^2}{9} = 1$$

$$\frac{x^2}{(3\sqrt{3})^2} + \frac{y^2}{3^2} = 1$$

$E_1 : \frac{x^2}{3^2} + \frac{y^2}{(3\sqrt{3})^2} = 1$ $\quad y=\sqrt{27-3x^2}$ $\quad 0<x<3$ ← quad I

$E_2 : \frac{x^2}{(3\sqrt{3})^2} + \frac{y^2}{3^2} = 1$ $\quad y=\sqrt{9-\frac{x^2}{3}}$ $\quad 0<x<3\sqrt{3}$ ← quad I

$L : y = -x+6$



$$\begin{cases} y = -x+6 & \text{— (1)} \\ y = \sqrt{27-3x^2} & \text{— (2)} \end{cases}$$

Substituting (1) into (2),

$$-x+6 = \sqrt{27-3x^2}$$

$$(-x+6)^2 = 27-3x^2$$

$$(6-x)^2 = 27-3x^2$$

$$36-2(6)(x)+x^2 = 27-3x^2$$

$$36-12x+x^2 = 27-3x^2$$

$$4x^2-12x+9 = 0$$

$$\therefore x = 1.5 \text{ (repeated)}$$

$$\begin{cases} y = -x+6 & \text{— (4)} \\ y = \sqrt{9-\frac{x^2}{3}} & \text{— (5)} \end{cases}$$

Substituting (4) into (5),

$$-x+6 = \sqrt{9-\frac{x^2}{3}}$$

$$36-12x+x^2 = 9-\frac{x^2}{3}$$

$$108-36x+3x^2 = 27-x^2$$

$$4x^2-36x+81 = 0$$

$$\therefore x = 4.5 \text{ (repeated)}$$

$$\text{Area}: \left[\int_{1.5}^{\frac{3\sqrt{3}}{2}} (-x+6) - \sqrt{27-3x^2}\,dx\right] + \left[\int_{\frac{3\sqrt{3}}{2}}^{4.5} (-x+6) - \sqrt{9-\frac{x^2}{3}}\,dx\right]$$

$$= \left[\int_{1.5}^{\frac{3\sqrt{3}}{2}} (-x+6)\,dx - \int_{1.5}^{\frac{3\sqrt{3}}{2}} \sqrt{3}\,\underbrace{\sqrt{3^2-x^2}}_{}\,dx\right] + \quad \text{(using part (a))}$$

Part (a)

$$\int \sqrt{a^2-x^2}\,dx$$

$$\vdots$$

$$\left[\int_{\frac{3\sqrt{3}}{2}}^{4.5} (-x+6)\,dx - \int_{\frac{3\sqrt{3}}{2}}^{4.5} \frac{1}{\sqrt{3}}\underbrace{\sqrt{(3\sqrt{3})^2-x^2}}_{\text{using part (a)}}\,dx\right]$$

$$= \left[\left(\frac{-x^2}{2}+6x\right)_{1.5}^{\frac{3\sqrt{3}}{2}} - \sqrt{3}\left[\frac{3^2}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{3^2-x^2}}{2}\right]_{1.5}^{\frac{3\sqrt{3}}{2}}\right] -$$

$$\left[\left(\frac{-x^2}{2}+6x\right)_{\frac{3\sqrt{3}}{2}}^{4.5} - \frac{1}{\sqrt{3}}\left(\frac{(3\sqrt{3})^2}{2}\sin^{-1}\left(\frac{x}{3\sqrt{3}}\right) + \frac{x\sqrt{(3\sqrt{3})^2-x^2}}{2}\right]_{\frac{3\sqrt{3}}{2}}^{4.5}\right]$$

$$= \left[6\cdot\frac{3\sqrt{3}}{2} - \frac{\left(\frac{3\sqrt{3}}{2}\right)^2}{2}\right] - \left[6(1.5) - \frac{(1.5)^2}{2}\right] - \sqrt{3}\left[\frac{3^2}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2}\right]_{1.5}^{\frac{3\sqrt{3}}{2}}$$

$$+ \cancel{\left[6\left(\frac{3\sqrt{3}}{2}\right)\right.} + \left[6(4.5) - \frac{(4.5)^2}{2}\right] - \left[6\left(\frac{3\sqrt{3}}{2}\right) - \frac{\left(\frac{3\sqrt{3}}{2}\right)^2}{2}\right]$$

$$- \frac{1}{\sqrt{3}}\left[\frac{(3\sqrt{3})^2}{2}\sin^{-1}\left(\frac{x}{3\sqrt{3}}\right) + \frac{x\sqrt{27-x^2}}{2}\right]_{\frac{3\sqrt{3}}{2}}^{4.5}$$

$$= 9 - \frac{3\sqrt{3}}{2}\pi$$

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