Name: ____________ ( ) Class: ______ Date: ________

Time: 30 minutes Marks: ________/34

Chapter Test 11

1. Let $A = \begin{pmatrix} 2 & 3 & -4 \\ -1 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ -1 & 4 \\ 3 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 7 \\ 4 & -3 \end{pmatrix}$. Find $C^2 - AB$. (4 marks)

2. **(a)** Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$. (3 marks)

(b) Hence, evaluate $\begin{vmatrix} 4 & 8 & 16 \\ 1 & 9 & 81 \\ 1 & 5 & 25 \end{vmatrix}$. (4 marks)

3. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}$.

(a) Show that $A^3 - 4A - I = 0$, where I is the 3×3 identity matrix. (4 marks)

(b) Using the result of **(a)**, find A^{-1}. (4 marks)

4. Let $A = \begin{pmatrix} 1 & 0 \\ -8 & 3 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$.

(a) Find P^{-1}. (2 marks)

(b) Let $D = P^{-1}AP$.

(i) Find D. (2 marks)

(ii) Prove that both D and A are non-singular. (2 marks)

(c) Find D^n and A^n, where n is a positive integer. (5 marks)

5. Let $A = \begin{pmatrix} 3 & 1 & -4 \\ 1 & 0 & 1 \\ -5 & -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 & 1 \\ -5 & -20 & -7 \\ -1 & -2 & -1 \end{pmatrix}$.

(a) Is AB a scalar matrix? Explain your answer. (2 marks)

(b) If $\det(A) = 2$, find $\det(B)$ without expansion. (2 marks)

~~~End~~~
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Chapter Test 11 (M2)

1. $A = \begin{pmatrix} 2 & 3 & -4 \\ -1 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ -1 & 4 \\ 3 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 7 \\ 4 & -3 \end{pmatrix}$

$C^2 - AB$

$= \begin{pmatrix} 2 & 7 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 2 & 7 \\ 4 & -3 \end{pmatrix} - \begin{pmatrix} 2 & 3 & -4 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ -1 & 4 \\ 3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 32 & -7 \\ -4 & 37 \end{pmatrix} - \begin{pmatrix} -5 & 12 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 37 & -19 \\ -5 & 37 \end{pmatrix}$ //

2a) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ a-b & b & c-b \\ a^2-b^2 & b^2 & c^2-b^2 \end{vmatrix}$

$= (a-b)(c-b) \begin{vmatrix} 0 & 1 & 0 \\ 1 & b & 1 \\ a+b & b^2 & c+b \end{vmatrix}$

$= -(a-b)(c-b) \begin{vmatrix} 1 & 1 \\ a+b & a+b \end{vmatrix}$

$= (b-a)(c-b)[(c+b)-(a+b)]$

$= (b-a)(c-b)(c-a)$ //

2b) $\begin{vmatrix} 4 & 8 & 16 \\ 1 & 9 & 81 \\ 1 & 5 & 25 \end{vmatrix} = 4\begin{vmatrix} 1 & 2 & 4 \\ 1 & 9 & 81 \\ 1 & 5 & 25 \end{vmatrix}$

$= 4\begin{vmatrix} 1 & 1 & 1 \\ 2 & 9 & 5 \\ 4 & 81 & 25 \end{vmatrix}$

$= 4(9-2)(5-9)(5-2)$

$= -336_{//}$

3a) $A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}$

$= A^3 - 4A - I$

$= \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix} -$

$4\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix} - 4\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 5 & -4 & 4 \\ -4 & 1 & 0 \\ 8 & 0 & -3 \end{pmatrix} - \begin{pmatrix} 4 & -4 & 4 \\ -4 & 0 & 0 \\ 8 & 0 & -4 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O_{/}$

3b) A^{-1}

$$A^3-4A-I=0 \longrightarrow A^3-4A=I$$

$$A(A^2-4I)=I=(A^2-4I)A$$

$\therefore A^{-1}$ Exists

$$A^{-1}=A^2-4I$$

$$=\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}-4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}-\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & -1 & 0 \\ -1 & -3 & -1 \\ 0 & -2 & -1 \end{pmatrix}_{//}$$

4a) $A=\begin{pmatrix} 1 & 0 \\ -8 & 3 \end{pmatrix}$, $P=\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$

P^{-1}

$$\det(P)=\begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix}=1\neq 0$$

$\therefore P^{-1}$ Exists

$$\therefore P^{-1} = \frac{1}{det(P)} \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}_{//}$$

4bi) $D = P^{-1}AP$

$$D = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -8 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ -12 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}_{//}$$

4bii) $det(D) = \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3 \neq 0$

$\therefore D$ is non-singular

$det(P^{-1}AP) = det(D)$

$det(P^{-1}) \cdot det(A) \cdot det(P) = 3$

$det(P^{-1}) \cdot det(P) \cdot det(A) = 3$

$det(P^{-1}P) \cdot det(A) = 3$

$det(I) \cdot det(A) = 3$

$det(A) = 3 \neq 0$

$\therefore A$ is non-singular

4c) $D^1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3^1 \end{pmatrix}$

$D^2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3^2 \end{pmatrix}$

$D^3 = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 27 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3^3 \end{pmatrix}$

$D^4 = \begin{pmatrix} 1 & 0 \\ 0 & 27 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 81 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3^4 \end{pmatrix}$

$\therefore D^n = \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix}$

$(P^{-1}AP)^n = D^n$

$P^{-1}A\underbrace{P \cdot P^{-1}}_{I}A\underbrace{P \cdot P^{-1}}_{I}AP \cdots \underset{P^{-1}AP}{\wedge} P^{-1}AP = D^n$

$P \times P^{-1}A^n P \times P^{-1} = P \times D^n \times P^{-1}$

$A^n = PD^nP^{-1}$

$A^n = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$

$A^n = \begin{pmatrix} 1 & 0 \\ 4 & 3^n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$

$A^n = \begin{pmatrix} 1 & 0 \\ 4-4(3^n) & 3^n \end{pmatrix}$ //

5a) $A = \begin{pmatrix} 3 & 1 & -4 \\ 1 & 0 & 1 \\ -5 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 & 1 \\ -5 & -20 & -7 \\ -1 & -2 & -1 \end{pmatrix}$

AB

$= \begin{pmatrix} 3 & 1 & -4 \\ 1 & 0 & 1 \\ -5 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ -5 & -20 & -7 \\ -1 & -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2I$

$\therefore$ AB is a scalar matrix, because the whose elements on the main diagonal are the same.

5b) $\det(A) = 2$

$\det(AB) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$

$\det(AB) = 8$

$\det(A) \cdot \det(B) = 8$

$2 \cdot \det(B) = 8$

$\det(B) = 4$ //