Defining Numbers

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Abstract

This paper is intended to define some common operators and functions for constructing numbers. We are not going to pass through some details of how a real number is completable and such a number can be represented at a point on a continuous straight line. We are focusing on extending the dimension of numbers, that is exactly four normed division algebras, reals, complexes, quaternions, octonions. And also even if a higher dimensional numbers such as sedenions, pathions, chingons, routons, voudons… can also compatible with the algorithms.

Algebraic Properties

Reals

- Comparative
- Multiplication Commutativity
- Multiplication Associativity
- Power Associativity

Complexes

- Multiplication Commutativity
- Multiplication Associativity
- Power Associativity

Quaternions

- Multiplication Associativity
- Power Associativity

Octonions

- Power Associativity

Sedenions, Pathions, Chingons, Routons, Voudons…

- Power Associativity
- Existing Zero Divisors

Convention

Symbols n, u and v represent numbers.

— 1 operand n

— 2 operands u, v

Sometimes we want to explicitly indicate the components of a number when an expression includes both scalars and vectors below.

— scalar s

— vector v

The subscripts are just helper functions on knowing the traits of numbers.

— the scalar function $(s + v)_s = s$

— the vector function $(s + v)_v = v$

Symbols x, a and b represent complexes.

— 1 operand x

— 2 operands a, b

Sometimes we want to explicitly indicate the components of a complex when the expression includes the imaginary unit i.

— real a

— imaginary b

Otherwise any exceptional cases will be noted at that place.

Introduction

It is worth noting that complexes can have commutative operations on multiplying terms but quaternions will come out a side-effect. It is obvious that summing up terms will never cause any of multi-valued outcomes since the whole concepts when we construct a number with a scalar and a vector. That is

$$n = s + v$$

A structure of numbers which consists of 2 parts without orders. Essentially, numbers should obey the property of addition operations. The rule we make an extension $(\hat{})$ below.

$$a \textit{ is replaced by } s$$

$$bi \textit{ is replaced by } v$$

$$b \textit{ is replaced by } |v|$$

$$i \textit{ is replaced by } \operatorname{sgn} v$$

Our strategy is to reinterpret the multiples into a sum of a sequence of terms. How can we do so? By using the property of logarithm, suppose a, b and c are non-zero numbers, such that

$$\log_c ab = \log_c a + \log_c b \,(\#0)$$

Here that is no matter what the base number c is, we can have terms without multiples. The benefits with it is now you can get rid of a restriction on preventing when defining somewhere functions occur multi-valued.

Other things should be noticed is the power associative problems. We will get single result when we have a wealth of same value multiplying by one another, but sometimes somehow the situation come up with silently. For instance, you may see something like

$$n \operatorname{sgn} n_v = n_s \operatorname{sgn} n_v + |n_v| \operatorname{sgn}^2 n_v$$

in the expression. It seems like which works secretive. The factors are parallel satisfying a power associative rule.

$$n \to \sum_{i \in \mathbb{Z}} x_i \operatorname{sgn}^i n_v$$

These are the only exception to multiply remaining multi-valued form as you should keep check with the unit vector sgn of numbers and distribute it term by term $(*)$, and adding up are not affected by any such rule.

On the other hand, when we extend from complex world to quaternion world we have to check for the negative imaginary unit version. For instance, you will see something like

$$\exp n = \begin{cases} (\exp n_s)(\cos|n_v| + \operatorname{sgn} n_v \sin|n_v|), & n_v \neq 0 \\ \exp n_s\,, & n_v = 0 \end{cases} \quad (0.11)$$

$$e^{a+bi} = e^a(\cos b + i \sin b)(0.12)$$

As we know the vector component of a complex can have 2 outcomes.

$$\operatorname{sgn} bi = \begin{cases} i, & b > 0 \\ -i, & b < 0 \end{cases} \quad (\#1)$$

Extending (0.12), you should come up with the negative imaginary unit version. Suppose b is a positive real number,

$$e^{a-bi} = e^a(\cos b - i \sin b)(\#2)$$

Comparing with (0.11), when $n = a - bi$ we have

$$\operatorname{sgn} n_v = -i$$

$$\sin|n_v| = \sin b$$

$$\cos|n_v| = \cos b$$

$$\exp n_s = e^a$$

Which also match up the rule we make an extension (^), and so we call that isomorphic (&), and could try to substitute back into the original equations confirming whether it is.

And one more thing is the argument problem. For instance, you will see something like

$$\ln n = \begin{cases} \ln|n| + \operatorname{sgn} n_v \arg n\,, & n_v \neq 0 \\ \ln(-n_s) + (2k+1)\pi i, & n_s < 0 \textit{ and } n_v = 0 \textit{ where } k \in Z \\ \ln n_s + 2k\pi i, & n_s > 0 \textit{ and } n_v = 0 \textit{ where } k \in Z \end{cases} \quad (0.13)$$

$$\ln x = \ln|x| + i \arg^c x\,, \qquad a \neq 0 \quad (0.14)$$

The case of 2nd line is to fix the sgn and arg functions cannot detect negative values. A real function may map some elements of the set of reals outside its set, since the general interpretation has multiple imaginary basis vectors $(+)$. The reference angles in complex plane carrying a sign denote whether that above the x-axis.

$$\theta^c = \arctan\frac{b}{a} \quad \to \quad \arg^c x = \begin{cases} \theta^c, & \textit{In Quadrant I} \\ \pi + \theta^c, & \textit{In Quadrant II, III} \\ 2\pi + \theta^c, & \textit{In Quadrant IV} \end{cases} \quad (\#3)$$

Then which of quadrants would adjust the angle by shifting with π.

The reference angles in quaternion world carrying a sign denote whether that scalar positive.

$$\theta = \arccos\frac{n_s}{|n|} \quad \rightarrow \quad \arg n = \theta$$

Now we just consider the imaginary portion of the natural logarithms in quadrant II, III and IV.

$$\begin{cases} \theta = \pi + \theta^c \\ \theta = \pi - \theta^c \\ \theta = -\theta^c \end{cases} \rightarrow \begin{cases} i(\pi + \theta^c) = i\theta \\ i(\pi + \theta^c) = i(2\pi - \theta) \\ i(2\pi + \theta^c) = i(2\pi - \theta) \end{cases} \rightarrow \begin{cases} i(\pi + \theta^c) = i\theta \\ i(\pi + \theta^c) = -i\theta \\ i(2\pi + \theta^c) = -i\theta \end{cases}$$

With (#1), we prove that both isomorphic satisfying the rule (&).

$$i \arg^c x = \operatorname{sgn} bi \arg x$$

Operators

We define additions and subtractions

$$u \pm v = (u_s \pm v_s) + (u_v \pm v_v)$$

We define multiplications with Cayley-Dickson Construction. Given that n is positive integers. If M is a 2^n dimensional number, u_L, u_R, v_L and v_R are a 2^{n-1} dimensional number. Therefore

$$M = (u_L, u_R)(v_L, v_R) = (u_L v_L - v_R^* u_R, v_R u_L + u_R v_L^*)$$

Also defining the relation of multiples with vectors

$$uv = u_s v_s + u_s v_v + v_s u_v + u_v \times v_v - u_v \cdot v_v \Leftrightarrow u_v v_v = u_v \times v_v - u_v \cdot v_v$$

If a number is not the four normed division algebras, that exists zero divisors, we will still define the inverses (0.9) cause we are in spirit to find out an algorithm no matter how non-reliable is can also be calculated.

Fundamentals with Multiples

We define the conjugates,

$$\operatorname{conjg}(s + v) = s - v (0.1)$$

We define the dot products,

$$\operatorname{dot}(u, v) = u_s v_s + u_v \cdot v_v \Longrightarrow \operatorname{dot}(u, v) = \frac{v \operatorname{conjg} u + u \operatorname{conjg} v}{2} (0.2)$$

We define the outer products,

$$\operatorname{outer}(u, v) = \begin{vmatrix} u_s & v_s \\ u_v & v_v \end{vmatrix} + u_v \times v_v \Longrightarrow \operatorname{outer}(u, v) = \frac{v \operatorname{conjg} u - u \operatorname{conjg} v}{2} (0.3)$$

We define the even products,

$$\operatorname{even}(u, v) = u_s v_s + u_s v_v + v_s u_v - u_v \cdot v_v \Longrightarrow \operatorname{even}(u, v) = \frac{uv + vu}{2} (0.4)$$

We define the cross products,

$$\operatorname{cross}(u, v) = u_v \times v_v \Longrightarrow \operatorname{cross}(u, v) = \frac{uv - vu}{2} (0.5)$$

We define the absolutes,

$$|n| = \sqrt{\operatorname{dot}(n, n)} (0.6)$$

We define the arguments,

$$\arg n = \arccos \frac{n_s}{|n|} + 2\pi k, \qquad n \neq 0 \textit{ and } k \in \mathbb{Z} \quad (0.7)$$

We define the signs,

$$\operatorname{sgn} n = \frac{n}{|n|}, \qquad n \neq 0 \quad (0.8)$$

We define the inverses,

$$n^{-1} = \frac{\operatorname{conjg} n}{\operatorname{dot}(n, n)}, \qquad n \neq 0 \quad (0.9)$$

Consider v is a vector, such that

$$v^{-1} = -\frac{v}{\operatorname{dot}(v, v)} \quad \textit{whereas} \quad (\operatorname{sgn} v)^{-1} = -\operatorname{sgn} v$$

We define the exponents, extending the polar form with complexes to a general interpretation with Euler's formula

$$e^{ix} = \cos x + i \sin x \ (0.10)$$

$$\exp n = \begin{cases} (\exp n_s)(\cos|n_v| + \operatorname{sgn} n_v \sin|n_v|), & n_v \neq 0 \\ \exp n_s\,, & n_v = 0 \end{cases} \quad (0.11)$$

$$e^{a+bi} = e^a(\cos b + i \sin b)(0.12)$$

[Note: *The* $\operatorname{sgn} n_v$ *acting with the rest factors from* (0.11) *is isomorphic extended by* $\pm i$ *satisfying the rule* (&)]

Consider v is a vector, it is obvious that

$$e^{\pi i} = -1 \quad \textit{while} \quad e^{\pi \operatorname{sgn} v} = -1$$

We define the natural logarithms,

$$\ln n = \begin{cases} \ln|n| + \operatorname{sgn} n_v \arg n\,, & n_v \neq 0 \\ \ln(-n_s) + (2k+1)\pi i, & n_s < 0 \textit{ and } n_v = 0 \textit{ where } k \in Z \\ \ln n_s + 2k\pi i, & n_s > 0 \textit{ and } n_v = 0 \textit{ where } k \in Z \end{cases} \quad (0.13)$$

$$\ln x = \ln|x| + i \arg^c x\,, \qquad a \neq 0 \quad (0.14)$$

[Note: *The* $\operatorname{sgn} n_v$ *acting with the rest factors from* (0.13) *is isomorphic extended by* $\pm i$ *satisfying the rule* (&)]

[Note: *The case of 2nd line in* (0.13) *is to specify the default imaginary basis i for extending isomorphically* (+)]

Exponentials

We define the powers,

$$a^b = e^{b \ln a}(1.1)$$

$$u^v = e^{e^{\ln v + \ln(\ln u)}}(1.2)$$

For k is a real, defining a reference value

$$n^d = \begin{cases} |n|^d(\cos(d \arg n) + \operatorname{sgn} n_v \sin(d \arg n)), & n_v \neq 0 \\ (-n_s)^d(\cos(2k+1)\pi d + i \sin(2k+1)\pi d), & n_s < 0 \textit{ and } n_v = 0 \textit{ where } k \in Z \\ n_s^d(\cos 2k\pi d + i \sin 2k\pi d), & n_s \geq 0 \textit{ and } n_v = 0 \textit{ where } k \in Z \end{cases} \quad (1.3)$$

[Note: *The* $\operatorname{sgn} n_v$ *acting with the rest factors from* (1.3) *is isomorphic extended by* $\pm i$ *satisfying the rule* (&)]

[Note: *The case of 2nd line in* (1.3) *is to specify the default imaginary basis i for extending isomorphically* (+)]

$$x^k = \begin{cases} |x|^k(\cos(k \arg^c x) + i \sin(k \arg^c x)), & x \neq 0 \\ 0, & x = 0 \; and \; k \neq 0 \end{cases} \quad (1.4)$$

We define the roots,

$$\sqrt[b]{a} = a^{\frac{1}{b}}, \qquad b \neq 0 \quad (1.5)$$

$$\sqrt[v]{u} = u^{\frac{1}{v}}, \qquad v \neq 0 \quad (1.6)$$

For k is a real, defining a reference value

$$\sqrt[k]{v} = v^{\frac{1}{k}}, \qquad k \neq 0 \quad (1.7)$$

$$\sqrt[k]{x} = x^{\frac{1}{k}}, \qquad k \neq 0 \quad (1.8)$$

When k is a positive integer, (1.7) is just equivalent to assuming

$$n^k = N \quad where\; n \; and \; N \; are \; numbers$$

Which is

$$|n|^k(\cos(k \arg n) + \operatorname{sgn} n_v \sin(k \arg n)) = |N|(\cos(\arg N) + \operatorname{sgn} N_v \sin(\arg N))$$

Where

$$\operatorname{sgn} n_v = \operatorname{sgn} N_v$$

$$|n|^k = |N|$$

$$k \arg n = \arg N$$

When you solve with these equations, you will observe the corresponding possibility solutions should have k since the periodicity of trigonometrics are 2π on rotating a certain times will meet the first point. You can take a reference with De Moivre's Theorem if interested in (1.8).

Consider when $k = 2$,

$$n^{\frac{1}{2}} = |N|^{\frac{1}{2}}\left[\cos\frac{\arg N}{2} + \operatorname{sgn} N_v \sin\frac{\arg N}{2}\right]$$

Only showing the possible values with the function arg turnning out the reference value,

$$n^{\frac{1}{2}} = |N|^{\frac{1}{2}}\left[\cos\frac{\arg N}{2} + \operatorname{sgn} N_v \sin\frac{\arg N}{2}\right] \quad or \quad |N|^{\frac{1}{2}}\left[\cos\left(\frac{\arg N}{2} + \pi\right) + \operatorname{sgn} N_v \sin\left(\frac{\arg N}{2} + \pi\right)\right]$$

The last result turns out,

$$n^{\frac{1}{2}} = |N|^{\frac{1}{2}}\left[-\cos\frac{\arg N}{2} - \operatorname{sgn} N_v \sin\frac{\arg N}{2}\right]$$

Simplifying the whole expression, we obtain

$$n^{\frac{1}{2}} = \pm|N|^{\frac{1}{2}}\left[\cos\frac{\arg N}{2} + \operatorname{sgn} N_v \sin\frac{\arg N}{2}\right] (\#4)$$

So the plus-minus signs are never mind in general as we extended our numeric system. When you solve the quadratics, we just put in a plus sign is good.

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} (\#5)$$

The possibilities are incorporated to the root form as if the context of complexes or even higher dimensional numbers whether the structure of that is real actually without adding the plus-minus signs is a good approach. You will complain why not just the roots or even the anti-trigonometrics are not multi-valued in the real world, and the last of a minute we can just end up with definition is just definition. You cannot against with the idiomatics.

Lastly, we define the logarithms,

$$\log_a b = \frac{\ln b}{\ln a}, \qquad a, b \neq 0 \quad (1.9)$$

$$\log_u v = e^{\ln(\ln v) - \ln(\ln u)}, \qquad u, v \neq 0 \quad (1.10)$$

Trigonometrics

Now we define

$$\cos x = \cosh ix \, (2.1) \quad and \quad \sin x = -i \sinh ix \, (2.2)$$

Acting with (0.10), it truns out,

$$e^x = \cosh x + \sinh x \, (2.3)$$

By (2.1), we have

$$\cos ix = \cosh x \, (2.4)$$

By (2.2), we have

$$\sin ix = i \sinh x \, (2.5)$$

And we also have

$$\tan ix = i \tanh x \, (2.6) \quad and \quad \tan x = -i \tanh ix \, (2.7)$$

By (0.10), It is obvious that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \, (2.8)$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \, (2.9)$$

$$\tan x = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})} \, (2.10)$$

Putting (2.2) into (2.8), we get

$$\sinh x = \frac{e^x - e^{-x}}{2} \, (2.11)$$

Putting (2.1) into (2.9), we get

$$\cosh x = \frac{e^x + e^{-x}}{2} \, (2.12)$$

Putting (2.7) into (2.10), we get

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \, (2.13)$$

So in the complex world, we know how to estimate

$$\sin(a+bi) = \sin a \cosh b + i \cos a \sinh b \ (3.1)$$

$$\cos(a+bi) = \cos a \cosh b - i \sin a \sinh b \ (3.2)$$

In the circumstances with quaternions, we have

$$\sin(s+v) = \begin{cases} \sin s \cosh|v| + \operatorname{sgn} v \cos s \sinh|v|, & v \neq 0 \\ \sin s, & v = 0 \end{cases} \quad (3.3)$$

$$\cos(s+v) = \begin{cases} \cos s \cosh|v| - \operatorname{sgn} v \sin s \sinh|v|, & v \neq 0 \\ \cos s, & v = 0 \end{cases} \quad (3.4)$$

[Note: *The* $\operatorname{sgn} v$ *acting with the rest factors from* (3.3) *and* (3.4) *is isomorphic extended by* $\pm i$ *satisfying the rule* (&)]

Then we figure out the hyperbolic version of sine and cosine.

By (2.5),

$$\sinh(a+bi) = -i \sin i(a+bi)$$

Therefore,

$$\sinh(a+bi) = i \sin(b - ai)$$

By (3.1),

$$\sinh(a+bi) = i(\sin b \cosh(-a) + i \cos b \sinh(-a))$$

Therefore,

$$\sinh(a+bi) = \sinh a \cos b + i \cosh a \sin b \ (3.5)$$

Extending to quaternions, which is

$$\sinh(s+v) = \begin{cases} \sinh s \cos|v| + \operatorname{sgn} v \cosh s \sin|v|, & v \neq 0 \\ \sinh s, & v = 0 \end{cases} \quad (3.6)$$

[Note: *The* $\operatorname{sgn} v$ *acting with the rest factors from* (3.6) *is isomorphic extended by* $\pm i$ *satisfying the rule* (&)]

By (2.4),

$$\cosh(a+bi) = \cos i(a+bi)$$

And that truns out,

$$\cosh(a+bi) = \cos(b - ai)$$

By (3.2),

$$\cosh(a+bi) = \cos b \cosh(-a) - i \sin b \sinh(-a)$$

Therefore,

$$\cosh(a+bi) = \cosh a \cos b + i \sinh a \sin b \ (3.7)$$

Extending to quaternions, which is

$$\cosh(s+v) = \begin{cases} \cosh s \cos|v| + \operatorname{sgn} v \sinh s \sin|v|, & v \neq 0 \\ \cosh s, & v = 0 \end{cases} \quad (3.8)$$

[Note: *The* $\operatorname{sgn} v$ *acting with the rest factors from* (3.8) *is isomorphic extended by* $\pm i$ *satisfying the rule* (&)]

And then what's more if tangent?

$$\tan(a+bi)=\frac{\tan a+\tan bi}{1-\tan a\tan bi}$$

By (2.6),

$$\tan(a+bi)=\frac{\tan a+i\tanh b}{1-i\tan a\tanh b}\times\frac{1+i\tan a\tanh b}{1+i\tan a\tanh b}$$

It truns out,

$$\tan(a+bi)=\frac{(\tan a)(1-\tanh^2 b)+i(\tanh b)(1+\tan^2 a)}{1+\tan^2 a\tanh^2 b}\quad(3.9)$$

Which is

$$\tan(a+bi)=\frac{\tan a\,\mathrm{sech}^2\, b+i\tanh b\sec^2 a}{1+\tan^2 a\tanh^2 b}$$

Extending (3.9) to quaternions, we get

$$\tan(s+v)=\begin{cases}\dfrac{(\tan s)(1-\tanh^2|v|)+(\mathrm{sgn}\, v)(\tanh|v|)(1+\tan^2 s)}{1+\tan^2 s\tanh^2|v|}, & v\neq 0\\ \tan s, & v=0\end{cases}\quad(3.10)$$

[Note: *The* $\mathrm{sgn}\, v$ *acting with the rest factors from* (3.10) *is isomorphic extended by* $\pm i$ *satisfying the rule* (&)]

What if the hyperbolic version?

$$\tanh(a+bi)=\frac{\tanh a+\tanh bi}{1+\tanh a\tanh bi}$$

By (2.7),

$$\tanh(a+bi)=\frac{\tanh a+i\tan b}{1-i\tanh a\tan b}\times\frac{1+i\tanh a\tan b}{1+i\tanh a\tan b}$$

It truns out,

$$\tanh(a+bi)=\frac{(\tanh a)(1-\tan^2 b)+i(\tan b)(1+\tanh^2 a)}{1+\tanh^2 a\tan^2 b}\quad(3.11)$$

Extending to quaternions, we get

$$\tanh(s+v)=\begin{cases}\dfrac{(\tanh s)(1-\tan^2|v|)+(\mathrm{sgn}\, v)(\tan|v|)(1+\tanh^2 s)}{1+\tanh^2 s\tan^2|v|}, & v\neq 0\\ \tanh s, & v=0\end{cases}\quad(3.12)$$

[Note: *The* $\mathrm{sgn}\, v$ *acting with the rest factors from* (3.12) *is isomorphic extended by* $\pm i$ *satisfying the rule* (&)]

And the rest we just define…

$$\sec n=(\cos n)^{-1}\quad(3.13)$$

$$\csc n=(\sin n)^{-1}\quad(3.14)$$

$$\cot n=(\tan n)^{-1}\quad(3.15)$$

$$\mathrm{sech}\, n=(\cosh n)^{-1}\quad(3.16)$$

$$\mathrm{csch}\, n=(\sinh n)^{-1}\quad(3.17)$$

$$\coth n=(\tanh n)^{-1}\quad(3.18)$$

Anti-Trigonometrics

Now we have to deal with the inverses!

By (2.8), let $x = \sin y$ such that

$$2ix = e^{iy} - e^{-iy}$$

Multiply both side with a factor e^{iy}, then it truns out

$$e^{2iy} - 2ixe^{iy} - 1 = 0$$

Solve for e^{iy} with (#5), that is

$$e^{iy} = \frac{2ix + \sqrt{-4x^2 + 4}}{2}$$

Which is

$$e^{iy} = ix + \sqrt{1 - x^2}$$

So that we get what we would like,

$$\arcsin x = -i \ln\left(ix + \sqrt{1 - x^2}\right) \, (4.1)$$

Extended in quaternions, we get

$$\arcsin(s + v) = \begin{cases} -\operatorname{sgn} v \ln\left((\operatorname{sgn} v)(s + v) + \sqrt{1 - (s + v)^2}\right), & v \neq 0 \\ -i \ln\left(is + \sqrt{1 - s^2}\right), & v = 0 \end{cases} \quad (4.2)$$

[Note: *Some sub-expression of* (4.2) *is single-valued where the common factor with component is* $\operatorname{sgn} v$ *only* $(*)$]

[Note: *The case of 2nd line in* (4.2) *is to specify the default imaginary basis* i *for extending isomorphically* $(+)$]

By (2.9), let $x = \cos y$ such that

$$2x = e^{iy} + e^{-iy}$$

Multiply both side with a factor e^{iy}, then it truns out

$$e^{2iy} - 2xe^{iy} + 1 = 0$$

Solve for e^{iy} with (#5), that is

$$e^{iy} = \frac{2x + \sqrt{4x^2 - 4}}{2}$$

Which is

$$e^{iy} = x + \sqrt{x^2 - 1}$$

So that we get what we want,

$$\arccos x = -i \ln\left(x + \sqrt{x^2 - 1}\right) \, (4.3)$$

Extended in quaternions, we get

$$\arccos(s + v) = \begin{cases} -\operatorname{sgn} v \ln\left(s + v + \sqrt{(s + v)^2 - 1}\right), & v \neq 0 \\ -i \ln\left(s + \sqrt{s^2 - 1}\right), & v = 0 \end{cases} \quad (4.4)$$

[Note: *Some sub-expression of* (4.4) *is single-valued where the common factor with component is* $\operatorname{sgn} v$ *only* $(*)$]

[Note: *The case of 2nd line in* (4.4) *is to specify the default imaginary basis* i *for extending isomorphically* $(+)$]

By (2.10), we assume that $x = \tan y$ so that we have

$$ix\left(e^{iy} + e^{-iy}\right) = e^{iy} - e^{-iy}$$

That is

$$(1 - ix)e^{iy} - (1 + ix)e^{-iy} = 0$$

Multiply both side with a factor e^{iy}, then it truns out

$$(1 - ix)e^{2iy} - (1 + ix) = 0$$

Which is

$$e^{2iy} = \frac{1 + ix}{1 - ix}$$

Solve for e^{2iy}, that is

$$y = \frac{1}{2i}\ln\frac{1 + ix}{1 - ix}$$

So we know

$$\arctan x = -\frac{i}{2}\ln\frac{1 + ix}{1 - ix} \quad (4.5)$$

Extended in quaternions, we obtain

$$\arctan(s + v) = \begin{cases} -\dfrac{\operatorname{sgn} v}{2}\ln\dfrac{1 + (\operatorname{sgn} v)(s + v)}{1 - (\operatorname{sgn} v)(s + v)}, & v \neq 0 \\ -\dfrac{i}{2}\ln\dfrac{1 + is}{1 - is}, & v = 0 \end{cases} \quad (4.6)$$

[Note: *Some sub-expression of* (4.6) *is single-valued where the common factor with component is* $\operatorname{sgn} v$ *only* $(*)$]

[Note: *The case of 2nd line in* (4.6) *is to specify the default imaginary basis* i *for extending isomorphically* $(+)$]

Now we consider with the hyperbolic functions!

Assume that $x = \sinh y$, whereas according to (2.11) we have

$$2x = e^{y} - e^{-y}$$

Multiply both side with a factor e^{y}, which means

$$e^{2y} - 2xe^{y} - 1 = 0$$

Solve for e^{y} with (#5), that is

$$e^{y} = \frac{2x + \sqrt{4x^2 + 4}}{2}$$

Which is

$$e^{y} = x + \sqrt{x^2 + 1}$$

So we know

$$\operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right) \quad (4.7)$$

In quaternion circumstances, we define

$$\operatorname{arcsinh}(s + v) = \ln\left(s + v + \sqrt{(s + v)^2 + 1}\right) \quad (4.8)$$

Then we consider with the consine, when $x = \cosh y$, acting with (2.12) we get

$$2x = e^y + e^{-y}$$

Multiply both side with a factor e^y, which means

$$e^{2y} - 2xe^y + 1 = 0$$

Solve for e^y (#5), that is

$$e^y = \frac{2x + \sqrt{4x^2 - 4}}{2}$$

Which is

$$e^y = x + \sqrt{x^2 - 1}$$

So we know

$$\operatorname{arccosh} x = \ln\left(x + \sqrt{x^2 - 1}\right) \quad (4.9)$$

In quaternion circumstances, we define

$$\operatorname{arccosh}(s + v) = \ln\left(s + v + \sqrt{(s + v)^2 - 1}\right) \quad (4.10)$$

Assume that $x = \tanh y$

From (2.13),

$$x(e^y + e^{-y}) = e^y - e^{-y}$$

That is

$$(1 - x)e^y - (1 + x)e^{-y} = 0$$

Multiply both side with a factor e^y, which means

$$e^{2y} = \frac{1 + x}{1 - x}$$

So we know

$$\operatorname{arctanh} x = \frac{1}{2}\ln\frac{1 + x}{1 - x} \quad (4.11)$$

In quaternions, we define

$$\operatorname{arctanh}(s + v) = \frac{1}{2}\ln\frac{1 + (s + v)}{1 - (s + v)} \quad (4.12)$$

And the rest we just define…

$$\operatorname{arcsec} n = \arccos n^{-1} \quad (4.13)$$

$$\operatorname{arccsc} n = \arcsin n^{-1} \quad (4.14)$$

$$\operatorname{arccot} n = \arctan n^{-1} \quad (4.15)$$

$$\operatorname{arcsech} n = \operatorname{arccosh} n^{-1} \quad (4.16)$$

$$\operatorname{arccsch} n = \operatorname{arcsinh} n^{-1} \quad (4.17)$$

$$\operatorname{arccoth} n = \operatorname{arctanh} n^{-1} \quad (4.18)$$

Periodicity with Reference Angles

We have 2 groups of solutions when solving trigonometrics.

$$\arccos x = -i\ln\left(x+\sqrt{x^2-1}\right)\,(4.3)$$

Consider $2\pi - \arccos x$,

$$2\pi - \arccos x = -i\left[2\pi i - \ln\left(x+\sqrt{x^2-1}\right)\right]$$

Express in self logarithm form,

$$2\pi - \arccos x = -i\left[\ln e^{2\pi i} - \ln\left(x+\sqrt{x^2-1}\right)\right]$$

Which is,

$$2\pi - \arccos x = -i\left[\ln(1) - \ln\left(x+\sqrt{x^2-1}\right)\right]$$

Apply the logarithm property,

$$2\pi - \arccos x = -i\ln\frac{1}{x+\sqrt{x^2-1}}$$

Adjust the fraction with $x-\sqrt{x^2-1}$ so that the denominator without roots,

$$2\pi - \arccos x = -i\ln\left(x-\sqrt{x^2-1}\right)$$

You will notice the reference angles are just swapping those 2 solutions with the related root on the expression.

$$+x \to -x \to +x$$

As you negate a value twice then it comes back to its value, both just form a conjugate in pair (#4).

Then we prove self logarithm form is integrity.

$$\exp n = \begin{cases} (\exp n_s)(\cos|n_v| + \operatorname{sgn} n_v \sin|n_v|), & n_v \neq 0 \\ \exp n_s\,, & n_v = 0 \end{cases} \quad (0.11)$$

$$\ln n = \begin{cases} \ln|n| + \operatorname{sgn} n_v \arg n\,, & n_v \neq 0 \\ \ln(-n_s) + (2k+1)\pi i, & n_s < 0 \text{ and } n_v = 0 \text{ where } k \in Z \\ \ln n_s + 2k\pi i, & n_s > 0 \text{ and } n_v = 0 \text{ where } k \in Z \end{cases} \quad (0.13)$$

Apply (0.11) and (0.13), so that,

$$\ln(e^n) = \ln|e^n| + \operatorname{sgn} e_v^n \arg e^n \ln(e^n)$$

$$= \ln|e^{n_s}|\left|e^{|n_v|\operatorname{sgn} n_v}\right| + \operatorname{sgn}(e^{n_s}\operatorname{sgn} n_v \sin|n_v|)_v \arccos\frac{e^{n_s}\cos|n_v|}{|e^{n_s}|\left|e^{|n_v|\operatorname{sgn} n_v}\right|} = \ln|e^{n_s}| + \operatorname{sgn} n_v \arccos(\cos|n_v|)$$

We need to introduce a variable,

$$\ln(e^n) = n_s + |n_v|\operatorname{sgn} n_v + 2\pi k \operatorname{sgn} n_v = n_s + n_v + 2\pi k \operatorname{sgn} n_v = n + 2\pi k \operatorname{sgn} n_v \quad where \quad k \in \mathbb{Z}$$

So the self logarithm form is,

$$\ln(e^n) = n + 2\pi k \operatorname{sgn} n_v \quad where \quad k \in \mathbb{Z}$$

When you answer another group of solutions with hyperbolics you should look out imaginary.

$$\operatorname{arcsinh}(s+v) = \ln\left(s+v+\sqrt{(s+v)^2+1}\right)\,(4.8)$$

Another group of solutions should be…

$$\begin{cases} \pi \operatorname{sgn} v - \operatorname{arcsinh}(s+v)\,, & v \neq 0 \\ \pi i - \operatorname{arcsinh} s\,, & v = 0 \end{cases} \quad (5.1)$$

[Note: *The case of 2nd line in* (5.1) *is to specify the default imaginary basis* i *for extending isomorphically* $(+)$]

Likewise how we define the trigonometrics, but now is flipped to imaginary, that is a terrific symmetrics.

The periodicity is just served by the natural logarithms.

$$\operatorname{arctanh}(s+v)=\frac{1}{2}\ln\frac{1+(s+v)}{1-(s+v)}\,(4.12)$$

Please be careful with the above logarithms the multi-valued term will be halved and the periodicity of tangent is generally π no matter which is hyperbolics or not also halved from 2π.

$$\frac{1+(s+v)}{1-(s+v)}=(1+s+v)\frac{\operatorname{conjg}(1-s-v)}{(1-s-v)\cdot(1-s-v)}=(1+s+v)\frac{1-s+v}{(1-s)^2+|v|^2}$$

Obviously it matches,

$$n\to\sum_{i\in\mathbb{Z}}x_i\,\mathrm{sgn}^i\,n_v$$

The expression produces single-valued outcome satisfying a power associative rule so that you can use division.