Tangent-Plane Projection of the Sphere

Thursday, March 30, 2023 4:27 PM

goal: project a small region of the sphere into a plane.

$\Rightarrow \begin{cases} \text{flat} \to \text{the mesh cwt well-defined} \\ \text{small} \to \text{the flat metric on the plane is similar to the angular metric on } S^2. \end{cases}$

Coordinate Space "Projection"

review: mapping S^2 to coordinate space.

i.e., $(x, y, z) \to (\theta, \phi)$

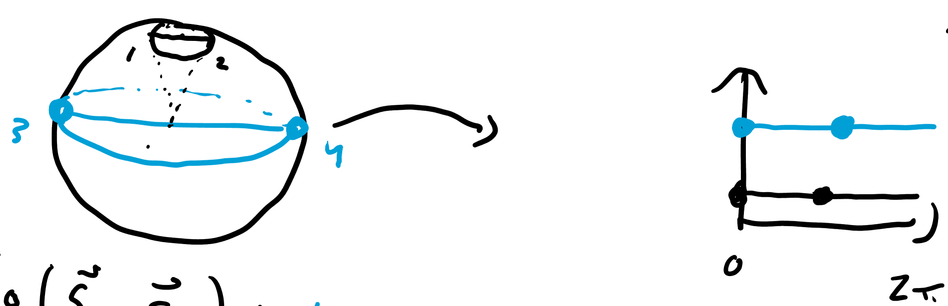
- we have used this to test the efficacy of the CWT in detecting point sources.

Issue: (S^2, d_0) does not agree with $(\mathbb{R}^2, d_2)$ near the poles

d_0: great-circle distance

d_2: Euclidean metric

ex:



$d_0(\vec{s}_1, \vec{s}_2) < d_0(\vec{s}_3, \vec{s}_4) = \pi \quad d_0(\vec{s}_1, \vec{s}_2) = d_0(s_3, s_4) = \pi$

remark: We considered small regions near $(\theta, \phi) \sim (\frac{\pi}{2}, 0)$.
It is clear that both metrics agree here.

Pf(Issue): $\vec{s}_1, \vec{s}_2 \in S^2$: $\Delta\theta, \Delta\phi \ll 1$.

$d_0(\vec{s}_1, \vec{s}_2) = acos(\cos\theta_1 \cos\theta \ldots$

$\Delta\theta, \Delta\phi \ll 1$.

$$d_0(\vec{s}_1, \vec{s}_2) = \operatorname{acos}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\Delta\phi)\right)$$

$$\sim \operatorname{acos}\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 - \sin\theta_1\sin\theta_2\frac{(\Delta\phi)^2}{2}\right)$$

$$= \operatorname{acos}\left(\cos(\Delta\theta) - \sin\theta_1\sin\theta_2\frac{(\Delta\phi)^2}{2}\right)$$

$$\sim \operatorname{acos}\left[1 - \frac{\Delta\theta^2}{2} - \sin\theta_1\sin\theta_2\frac{(\Delta\phi)^2}{2}\right)$$

$$\sim \sqrt{\Delta\theta^2 + \sin\theta_1\sin\theta_2\,\Delta\phi^2}$$

$$\neq \sqrt{\Delta\theta^2 + \Delta\phi^2} \quad (\sin\theta < 1)$$

d_0 defined in wiki: great-circle distance.

Note: $(\phi, \lambda) \to (\frac{\pi}{2} - \theta, \phi)$.

Identity: $\operatorname{acos}(1-x) = \sqrt{2x} + \frac{(2x)^{3/2}}{24} + O(x^2)$

- Note, this also shows the validity of the remark.

Tangent-Plane Projection

Following Vielva 07, we use a tangent plane projection.

Projection

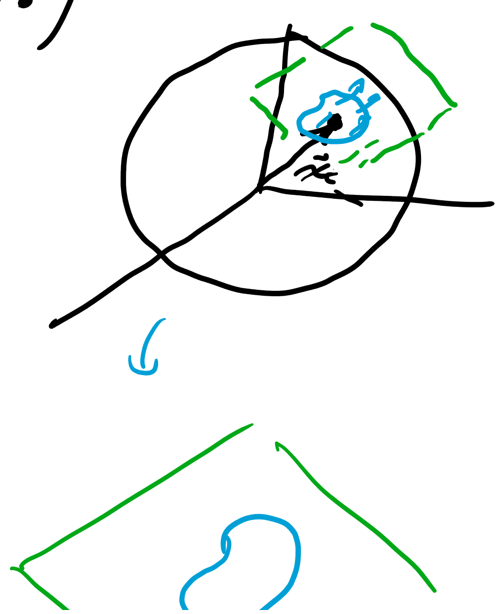
Def: Suppose we want to project the patch $\Omega \to \mathbb{R}^2$.

Let $\vec{s} \in \Omega$ and $\vec{x}_c$ be at the "center" of Ω.

The plane tangent to $\vec{x}_c = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$ is given by ($\hat{n}_{plane} = \vec{x}_c$)

$$x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta = 1$$

$$= \vec{r}\cdot\vec{x}_c = 1$$



We project $\vec{s}$ to the plane by translating it along $\hat{n}_{plane}$ until it intersects the plane.

I.e., $\vec{s} \to (\vec{s} + b\vec{x}_c)$ s.t.

$(\vec{s} + b\vec{x}_c)\cdot\vec{x}_c = 1$

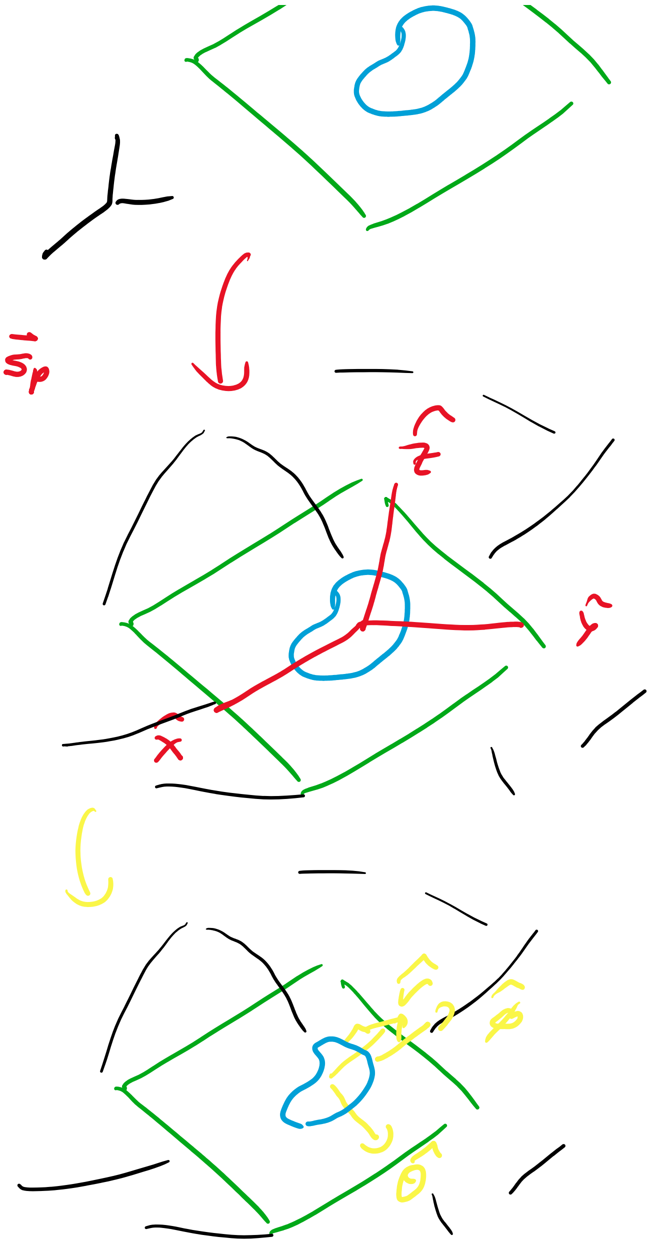
$\Rightarrow \vec{s}\cdot\vec{x}_c + b \ldots$

$$\ldots + b\vec{x}_c)\cdot\vec{x}_c \ldots \text{ s.t.}$$

$$\Rightarrow \vec{s}\cdot\vec{x}_c + b(\underbrace{\vec{x}_c\cdot\vec{x}_c}_{1}) = 1$$

$$\Rightarrow b = 1 - \vec{s}\cdot\vec{x}_c$$

$$\therefore \vec{s} \to \vec{s} + (1 - \vec{s}\cdot\vec{x}_c)\vec{x}_c = \vec{s}_p$$



Coordinate Transformation

We have our projected set. However, we still are using 3 coordinates. We want to reduce this to 2 coords so $\Omega \to \bar{\Omega} \subset \mathbb{R}^2$.

1. Set $\vec{x}_c$ as origin

$$\Rightarrow \vec{s}_p \to \vec{s}_p - \vec{x}_c$$
$$= \vec{s} - (\vec{s}\cdot\vec{x}_c)\vec{x}_c$$

2. Change of basis

$(\hat{x}, \hat{y}, \hat{z}) \to (\hat{r}, \hat{\theta}, \hat{\phi})$.

At $\vec{x}_c$, $\{\hat{\theta}, \hat{\phi}\}$ are an orthogonal basis of the tangent plane. So, we change to this basis by:

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} = \underbrace{\begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}}_{\overleftrightarrow{\Theta}(\theta, \phi)} \begin{bmatrix} \hat{\boldsymbol{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{bmatrix}$$

$$\vec{s} = s_i\hat{e}_i = s_i'\hat{e}_i'$$

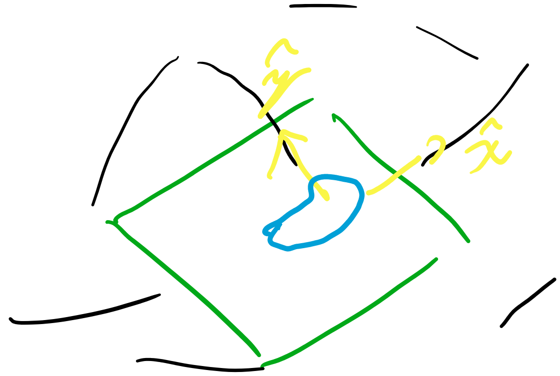
$$= (\hat{x}, \hat{y}, \hat{z})\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = (\hat{r}, \hat{\theta}, \hat{\phi})\begin{pmatrix} s_1' \\ s_2' \\ s_3' \end{pmatrix}$$

$$= (\hat{r}, \hat{\theta}, \hat{\phi})\,\overleftrightarrow{\Theta}^T(\theta, \phi)\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix} s_1' \\ s_2' \\ s_3' \end{pmatrix} = \overleftrightarrow{\Theta}^T(\theta, \phi)\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}}$$

(s_3') $\quad (\theta, \phi) \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$

Checked in Mathematica!
$s_1' = 0$.



3. Convenience (Setting x-y coords).
Note: $\hat{\theta}$ points towards the South Pole. Since we want to flip this, we set

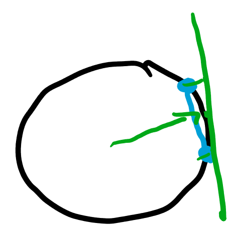
$$(\hat{\phi}, -\hat{\theta}) \to (\hat{x}, \hat{y})$$

OR $(\hat{l}, \hat{b}) \to (\hat{x}, \hat{y})$.

Q: Is (S^2, d_0) compatible w/ $(\mathbb{R}^2, d_2)$ under this projection?

ANS: Yes. But only for nearby points.

Pf (Informal): Clearly, d_2 in the projected plane agree w/ d_2 for nearby points. So, let's compare d_2 in $\mathbb{R}^3$ to d_0 for nearby points. $(\Delta\theta, \Delta\phi \ll 1)$.



$$(d_2(\vec{s}_1, \vec{s}_2))^2 = [(\sin\theta_1 \cos\phi_1 - \sin\theta_2 \cos\phi_2)^2 + (\sin\theta_1 \sin\phi_1 - \sin\theta_2 \sin\phi_2)^2 + (\cos\theta_1 - \cos\theta_2)^2$$

$$= \sin^2\theta_1 \cos^2\phi_1 - 2\sin\theta_1 \sin\theta_2 \cos\phi_1 \cos\phi_2 + \sin^2\theta_2 \cos^2\phi_2$$
$$+ \sin^2\theta_1 \sin^2\phi_1 - 2\sin\theta_1 \sin\theta_2 \sin\phi_1 \sin\phi_2 + \sin^2\theta_2 \sin^2\phi_2$$
$$+ \cos^2\theta_1 - 2\cos\theta_1 \cos\theta_2 + \cos^2\theta_2$$

$$= \sin^2\theta_1 + \sin^2\theta_2 - 2\sin\theta_1 \sin\theta_2 \underbrace{\cos(\Delta\phi)}_{1 - \frac{\Delta\phi^2}{2}} - 2\cos\theta_1 \cos\theta_2 + \cos^2\theta_1 + \cos^2\theta_2$$

$$= 2 - \underbrace{\underbrace{2\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2}_{-2\cos(\theta_1 - \theta_2)}}_{\sim -2 + \frac{\Delta\theta^2}{2}} + \sin\theta_1 \sin\theta_2 \Delta\phi^2$$

$\approx \Delta$ ...

$$\overline{\cos(\theta_1 - \theta_2)} \qquad \sin\theta_1 \sin\theta_2 \Delta\phi^2$$

$$\sim \overline{-2 + \frac{\Delta\theta^2}{2}}$$

$$\sim \Delta\theta^2 + \sin\theta_1 \sin\theta_2 \Delta\phi^2$$

$$\sim d_0(\vec{s}_1, \vec{s}_2)$$

$\therefore$ The distances agree!

$\Rightarrow$ It is safe to assume MHW wavelet scales in the tangent plane!