Structure Function Derivations

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I. STRUCTURE FUNCTION FOR POWER LAW DENSITY PROFILES

Suppose we have a perturber with density given by

$$\rho(r) \propto r^{\alpha}\ \Theta(R-r):\ \ \alpha > -3$$

This gives a mass profile given by

$$\mu(r) = \left(\frac{r}{R}\right)^{\alpha+3} \Theta(R-r) + \Theta(r-R).$$

Then, the structure function $U(p)$ is given by

$$U(p) \overset{def}{=} \int_1^{\infty} d\xi\ \frac{\mu(p\xi)}{\xi^2\sqrt{\xi^2-1}} \overset{split}{=} \underbrace{\int_1^{R/p} d\xi\ \frac{\mu(p\xi)}{\xi^2\sqrt{\xi^2-1}}}_{(ii)} + \underbrace{\int_{R/p}^{\infty} \frac{d\xi}{\xi^2\sqrt{\xi^2-1}}}_{(i)}$$

Unlike for the NFW case, this is very simple to calculate for the cases $\alpha = -2, -1, 0, 1$.

$$\begin{aligned} (i)\ \int_{R/p}^{\infty} \frac{d\xi}{\xi^2\sqrt{\xi^2-1}} = \left.\frac{\sqrt{\xi^2-1}}{\xi}\right|_{R/p}^{\infty} &= 1 - \frac{\sqrt{\left(\frac{R}{p}\right)^2 - 1}}{R/p} \\ &= 1 - \sqrt{1-\left(\frac{p}{R}\right)^2} \end{aligned}$$

$$\begin{aligned} (ii)\ \int_1^{R/p} d\xi\ \frac{\mu(p\xi)}{\xi^2\sqrt{\xi^2-1}} &\overset{def}{=} \left(\frac{p}{R}\right)^{\alpha+3} \int_1^{R/p} d\xi\ \frac{\xi^{\alpha+3-2}}{\sqrt{\xi^2-1}} \\ &= \left(\frac{p}{R}\right)^{\alpha+3} \int_0^{\sec^{-1}(R/p)} d\theta\ \sec^{\alpha+2}\theta \\ &= \left.\frac{\sqrt{\xi^2-1}}{\xi^{\alpha+4}} \times {}_2F_1\left(\frac{1}{2}, \frac{3+\alpha}{2}, \frac{3}{2}, \frac{\sqrt{\xi^2-1}}{\xi^2}\right)\right|_{\xi=1}^{\xi=R/p} \\ &= \left.\frac{\sqrt{\xi^2-1}}{\xi^{\alpha+4}} \times {}_2F_1\left(\frac{1}{2}, \frac{3+\alpha}{2}, \frac{3}{2}, \frac{\sqrt{\xi^2-1}}{\xi^2}\right)\right|_{\xi=R/p} \end{aligned}$$

For integer powers of α, the secant integral can be computed using the appropriate reduction formulas. Thus, an expression for the structure function valid for $\alpha > -3$ is given by

$$U^{\alpha}(p,R) = \left.\frac{\sqrt{\xi^2-1}}{\xi^{\alpha+4}} \times {}_2F_1\left(\frac{1}{2}, \frac{3+\alpha}{2}, \frac{3}{2}, \frac{\sqrt{\xi^2-1}}{\xi^2}\right)\right|_{\xi=R/p} + 1 - \sqrt{1-\left(\frac{p}{R}\right)^2}$$

II. NFW STRUCTURE FUNCTION DERIVATION

In this section, we rederive the NFW structure function $U(p)$. Unlike the approach used to derive the structure function used in our code, where (1) compute the structure function in Mathematica using the *Integrate* function and no assumptions, (2) stabilize expression by cancelling out divergent terms, and (3) manually account for the emergence of complex quantities in Python. As we will see, both results match.

Recall, the density profile for the NFW profile is given by

$$\mu(r; r_s, c) = \begin{cases} \frac{\log(r/r_s+1)-\frac{r/r_s}{1+r/r_s}}{\log(1+c)-\frac{c}{1+c}} & (\frac{r}{r_s} < c) \\ 1 & (\frac{r}{r_s} \geq c). \end{cases}$$

Let the denominator be denoted by

$$\mathcal{C}^{-1} = \log(1+c) - \frac{c}{1+c}.$$

Suppose $p/r_s < 1$, $p/r_v < 1$, and $p > 0$. Then, we have

$$U(p) \stackrel{def}{=} \int_1^\infty d\xi \, \frac{\mu(p\xi)}{\xi^2\sqrt{\xi^2-1}} \stackrel{split}{=} \underbrace{\int_1^{r_v/p} d\xi \, \frac{\mu(p\xi)}{\xi^2\sqrt{\xi^2-1}}}_{(ii)} + \underbrace{\int_{r_v/p}^\infty \frac{d\xi}{\xi^2\sqrt{\xi^2-1}}}_{(i)}$$

$$(i) \int_{r_v/p}^\infty \frac{d\xi}{\xi^2\sqrt{\xi^2-1}} = \left.\frac{\sqrt{\xi^2-1}}{\xi}\right|_{r_v/p}^\infty = 1 - \frac{\sqrt{\left(\frac{r_v}{p}\right)^2-1}}{r_v/p}$$

$$= 1 - \sqrt{1-\left(\frac{p}{r_v}\right)^2}$$

$$(ii) \int_1^{r_v/p} d\xi \, \frac{\mu(p\xi)}{\xi^2\sqrt{\xi^2-1}} \stackrel{NFW}{=} \frac{1}{\mathcal{C}}\left[\underbrace{\int_1^{r_v/p} \frac{\log(1+\frac{p\xi}{r_s})}{\xi^2\sqrt{\xi^2-1}}}_{(a)} - \underbrace{\int_1^{r_v/p} \frac{p\xi/r_s}{1+p\xi/r_s}}_{(b)}\right]$$

$$(a) \int_1^{r_v/p} \frac{\log(1+\frac{p\xi}{r_s})}{\xi^2\sqrt{\xi^2-1}} \stackrel{IBP}{=}$$

$$= \left.\frac{\sqrt{\xi^2-1}}{\xi}\log\left(1+\frac{p\xi}{r_s}\right)\right|_1^{r_v/p} - \int_1^{r_v/p} d\xi \, \frac{\sqrt{\xi^2-1}}{\xi} \, \frac{p/r_s}{1+p\xi/r_s}$$

$$= \sqrt{1-\left(\frac{p}{r_v}\right)^2}\log\left(1+\frac{r_v}{r_s}\right) - \underbrace{\int_1^{r_v/p} d\xi \, \frac{\xi^2-1}{\xi^2\sqrt{\xi^2-1}} \, \frac{p\xi/r_s}{1+p\xi/r_s}}_{cancels\ (b)}$$

$$\therefore \ (ii) \int_1^{r_v/p} d\xi \, \frac{\mu(p\xi)}{\xi^2\sqrt{\xi^2-1}} \stackrel{(a)\to(ii)}{=} \frac{1}{\mathcal{C}}\left[\sqrt{1-\left(\frac{p}{r_v}\right)^2}\log\left(1+\frac{r_v}{r_s}\right) - \underbrace{\int_1^{r_v/p} \frac{d\xi}{\sqrt{\xi^2-1}} \, \frac{p\xi/r_s}{1+p\xi/r_s}}_{(c)}\right]$$

$$(c) \int_1^{r_v/p} \frac{d\xi}{\sqrt{\xi^2-1}} \, \frac{p\xi/r_s}{1+p\xi/r_s} = \int_1^{r_v/p} \frac{d\xi}{\sqrt{\xi^2-1}} \, \frac{\xi}{\frac{r_s}{p}+\xi}$$

$$\stackrel{\xi=\sec\theta,\ a=p/r_s}{=} \int d\theta \frac{\sec^2\theta}{\sec\theta+a} = \int \frac{d\theta}{\cos\theta + a\cos^2\theta}.$$

This latter integral was computed using Mathematica's *Integrate* function. Note that our integration domain lies within $0 < \theta < \pi/2$, so we added that fact as an assumption to the integrator. We now work towards simplifying this

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In[43]:= Assuming[{a > 0, 0 < θ < π / 2}, ∫ 1/(Cos[θ] + a * (Cos[θ])^2) dθ]
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$$\text{Out[43]=}\ -\frac{2\,a\,\text{ArcTanh}\left[\frac{(-1+a)\,\text{Tan}\left[\frac{\theta}{2}\right]}{\sqrt{-1+a^2}}\right]}{\sqrt{-1+a^2}} - \text{Log}\left[\text{Cos}\left[\frac{\theta}{2}\right] - \text{Sin}\left[\frac{\theta}{2}\right]\right] + \text{Log}\left[\text{Cos}\left[\frac{\theta}{2}\right] + \text{Sin}\left[\frac{\theta}{2}\right]\right]$$

result as much as possible. Let's begin with the the logarithms to the right.

$$\begin{aligned}
&-\log\left[\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right] + \log\left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\right] \\
&\overset{0<\theta<\pi/2}{=} -\log\left[\sqrt{\frac{1+\cos\theta}{2}} - \sqrt{\frac{1-\cos\theta}{2}}\right] + \log\left[\sqrt{\frac{1+\cos\theta}{2}} + \sqrt{\frac{1-\cos\theta}{2}}\right] \\
&= \log\left[\frac{\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta} - \sqrt{1-\cos\theta}}\right] \\
&\overset{conj}{=} \log\left[\frac{(1+\cos\theta) + (1-\cos\theta) + 2\sqrt{(1-\cos^2\theta)}}{(1+\cos\theta) - (1-\cos\theta)}\right] \\
&= \log\left[\frac{1+\sin\theta}{\cos\theta}\right] \overset{\theta\to\xi}{=} \log\left[\frac{1+\frac{\xi^2-1}{\xi}}{1/\xi}\right] \\
&= \log\left[\xi + \sqrt{\xi^2-1}\right]
\end{aligned}$$

Taking the difference of this quantity from $\xi = r_v/p$ and $\xi = 1$, we have

$$\log\left[\xi + \sqrt{\xi^2-1}\right]\Big|_1^{r_v/p} = \log\left[\frac{r_v}{p} + \sqrt{\left(\frac{r_v}{p}\right)^2 - 1}\right] \tag{1}$$

The final quantity to compute is the inverse hyperbolic tangent, which has the following logarithmic representation:

$$\tanh^{-1}(\chi) = \frac{1}{2}\ \log\left(\frac{\chi+1}{1-\chi}\right) \qquad (0 < \chi < 1)$$

That is, the part of the integral written in terms of the inverse hyperbolic tangent is given as

$$-\frac{a}{\sqrt{-1+a^2}}\log\left(\frac{\chi+1}{1-\chi}\right),$$

where

$$\chi = \frac{(-1+a)\tan\left(\frac{\theta}{2}\right)}{\sqrt{-1+a^2}}.$$

Note the following about $\tan(\theta/2)$:

$$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta} = \frac{\xi-1}{\sqrt{\xi^2-1}} = \sqrt{\frac{\xi-1}{\xi+1}}$$

Evaluating at the relevant limits,

$$\begin{cases} \tan\left(\frac{\theta}{2}\right)|_{\xi=1} = 0 \\ \tan\left(\frac{\theta}{2}\right)|_{\xi=r_v/p} = \sqrt{\frac{r_v/p-1}{r_v/p+1}} \end{cases}$$

$$\implies -\frac{a}{\sqrt{-1+a^2}}\log\left(\frac{\chi+1}{1-\chi}\right)\Bigg|_{\xi=1}^{\xi=r_v/p} \overset{a=r_s/p}{=} -\frac{r_s/p}{\sqrt{-1+(\frac{r_s}{p})^2}}\log\left[\frac{\omega+1}{1-\omega}\right]$$
$$:\omega=\sqrt{\frac{-1+\frac{r_v}{p}}{1+\frac{r_v}{p}}}\sqrt{\frac{-1+\frac{r_s}{p}}{1+\frac{r_s}{p}}} \tag{2}$$

Thus, the integral (c) is given by the sum of the results given by Equations 1 and 2. I.e., working backwards,

$$(c)\int_1^{r_v/p}\frac{d\xi}{\sqrt{\xi^2-1}}\frac{p\xi/r_s}{1+p\xi/r_s}=$$
$$=-\frac{r_s/p}{\sqrt{-1+(\frac{r_s}{p})^2}}\log\left[\frac{\omega+1}{1-\omega}\right]+\log\left[\frac{r_v}{p}+\sqrt{\left(\frac{r_v}{p}\right)^2-1}\right],$$

which allows us to write integral (ii) as

$$(ii)\int_1^{r_v/p}d\xi\,\frac{\mu(p\xi)}{\xi^2\sqrt{\xi^2-1}}$$
$$\overset{(c)\to(ii)}{=}\frac{1}{\mathcal{C}}\left\{\sqrt{1-\left(\frac{p}{r_v}\right)^2}\log\left(1+\frac{r_v}{r_s}\right)+\frac{r_s/p}{\sqrt{-1+(\frac{r_s}{p})^2}}\log\left[\frac{\omega+1}{1-\omega}\right]+\log\left[\frac{r_v}{p}+\sqrt{\left(\frac{r_v}{p}\right)^2-1}\right]\right\}$$

Putting our results for integrals (i) and (ii) together, we obtain our final result for the structure function.

$$U(p)\overset{(i),(ii)}{=}1-\sqrt{1-\left(\frac{p}{r_v}\right)^2}+$$
$$+\frac{1}{\mathcal{C}}\left\{\sqrt{1-\left(\frac{p}{r_v}\right)^2}\log\left(1+\frac{r_v}{r_s}\right)+\frac{r_s/p}{\sqrt{-1+(\frac{r_s}{p})^2}}\log\left[\frac{\omega+1}{1-\omega}\right]-\log\left[\frac{r_v}{p}+\sqrt{\left(\frac{r_v}{p}\right)^2-1}\right]\right\}$$

For the sake of taking the low-p limit, we further simplify the expression to produce

$$U(p)\overset{(i),(ii)}{=}1-\sqrt{1-\left(\frac{p}{cr_s}\right)^2}+$$
$$+\frac{1}{\mathcal{C}}\left\{\sqrt{1-\left(\frac{p}{cr_s}\right)^2}\log(1+c)+\frac{r_s}{\sqrt{-p^2+r_s^2}}\log\left[\frac{\omega+1}{1-\omega}\right]-\log\left[\frac{cr_s}{p}+\sqrt{\left(\frac{cr_s}{p}\right)^2-1}\right]\right\}$$

$$:\omega=\sqrt{\frac{-p+cr_s}{p+cr_s}}\sqrt{\frac{-p+r_s}{p+r_s}}.$$

A. Low-P Behavior

We have checked that the structure function above tends to unity as $p\to r_v$. Here, we give its leading-order behavior as $p\to 0^+$. Code and plots are given in Figure II A.

B. Behavior when $p > r_s$

Note the only issue with the derived $U(p)$ function are the imaginary quantities appearing when $p>r_s$. For easy implementation to Python, we prefer to express all our quantities in terms of floats, so it will be useful to remove these imaginary quantities.

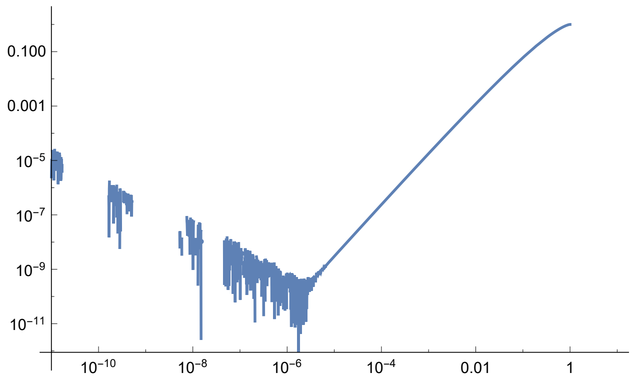
In[12]:= $\omega[p\_, rs\_, c\_] := \sqrt{\frac{-p + c * rs}{p + c * rs}} * \sqrt{\frac{-p + rs}{p + rs}}$

$arg[p\_, rs\_, c\_] := \frac{\omega[p, rs, c] + 1}{1 - \omega[p, rs, c]}$

In[44]:= $norm[c\_] := \frac{1}{Log[1 + c] - \frac{c}{1+c}}$

$$U[p\_, rs\_, c\_] := 1 - \sqrt{1 - \left(\frac{p}{c * rs}\right)^2} + norm[c] * \left(\sqrt{1 - \left(\frac{p}{c * rs}\right)^2} * Log[1 + c] + \frac{rs}{\sqrt{-p^2 + rs^2}} * Log[arg[p, rs, c]] - Log\left[\frac{c * rs}{p} + \sqrt{\left(\frac{c * rs}{p}\right)^2 - 1}\right]\right)$$

In[69]:= LogLogPlot[U[p, 1, 1], {p, 0.00000000001, 10}]

Out[69]=


In[61]:= Assuming[{rs > p > 0, c > 0}, Series[U[p, rs, c], {p, 0, 2}]]

Out[61]= $\left(\left(1 - c + 2\,Log[2] + 2\,c\,Log[2] + 2\,Log[c] + 2\,c\,Log[c] - 2\,Log[1 + c] - 2\,c\,Log[1 + c] - 2\,Log[p] - 2\,c\,Log[p] + 2\,Log[rs] + 2\,c\,Log[rs]\right) p^2\right) \Big/ \left(4\,rs^2 \left(-c + Log[1 + c] + c\,Log[1 + c]\right)\right) + O[p]^3$

In[62]:= $Ulow[p\_, rs\_, c\_] := \left(\left(1 - c + 2\,Log[2] + 2\,c\,Log[2] + 2\,Log[c] + 2\,c\,Log[c] - 2\,Log[1 + c] - 2\,c\,Log[1 + c] - 2\,Log[p] - 2\,c\,Log[p] + 2\,Log[rs] + 2\,c\,Log[rs]\right) p^2\right) \Big/ \left(4\,rs^2 \left(-c + Log[1 + c] + c\,Log[1 + c]\right)\right)$

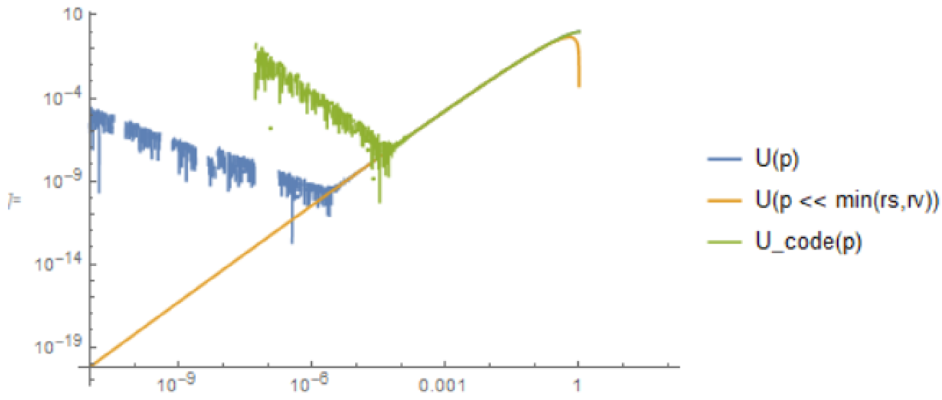


FIG. 1: Plot shows the structure function for $r_s = c = 1$. The low-p behavior obtained using the Series function agrees with the structure function computed at low values of p. The structure function used in our code is given by U_code. This indicates that the structure function we have used in our code is correct, but it suffers from greater numerical instabilities.

Suppose $p > r_s$ and $p < cr_s$. The only term becoming complex is

$$\frac{r_s}{\sqrt{-p^2+r_s^2}} \log\left[\frac{\omega+1}{1-\omega}\right] \overset{p \geq r_s}{=} \frac{r_s}{i\sqrt{p^2-r_s^2}} \log\left[\frac{i\tilde{\omega}+1}{1-i\tilde{\omega}}\right]$$
$$: \tilde{\omega} = \sqrt{\frac{-p+cr_s}{p+cr_s}}\sqrt{\frac{p-r_s}{p+r_s}}.$$

$$\log\left[\frac{i\tilde{\omega}+1}{1-i\tilde{\omega}}\right] = \log\left[\frac{1-\tilde{\omega}^2+2i\tilde{\omega}^2}{1+\tilde{\omega}^2}\right]$$
$$= i\arctan\left(\frac{2\tilde{\omega}}{1+\tilde{\omega}^2}\right)$$

Thus, the alternative expression for $U(p)$ expressed in terms of only real quantities for $p > r_s$ is given by

$$U(p) \overset{(i),(ii)}{=} 1 - \sqrt{1-\left(\frac{p}{cr_s}\right)^2} +$$
$$+\frac{1}{\mathcal{C}}\left\{\sqrt{1-\left(\frac{p}{cr_s}\right)^2}\log(1+c) + \frac{r_s}{\sqrt{p^2-r_s^2}} \arctan\left(\frac{2\tilde{\omega}}{1+\tilde{\omega}^2}\right) - \log\left[\frac{cr_s}{p} - \sqrt{\left(\frac{cr_s}{p}\right)^2-1}\right]\right\}$$

$$: \omega = \sqrt{\frac{-p+cr_s}{p+cr_s}}\sqrt{\frac{-p+r_s}{p+r_s}}.$$

Figure II B shows that this expression agrees with the original expression for the structure function. Interestingly, this expression inherits numerical instabilities very similar to the expression for the structure function that we have previously used in our code.

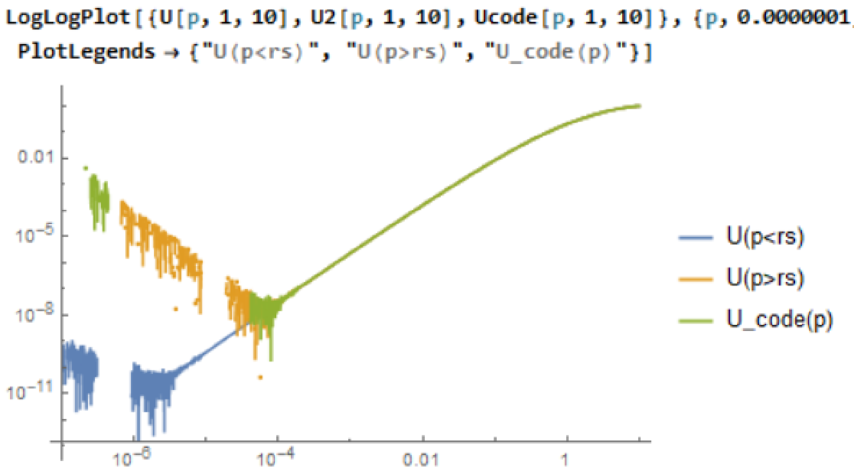


FIG. 2: Another comparison of the structure functions. $U(p < r_s)$ denotes the original structure function derived in the Section 1; $U(p > r_s)$ denotes the expression of the structure function whose terms are purely real when $p > r_s$; $U\_code(p)$ denotes the form of the structure function used in our code.