Relative Velocity Distribution for 2 Maxwell Boltzmann Dist

$$f(\vec v_i) \propto \exp\left(-\frac{\vec v_i^{\,2}}{2\sigma_i^2}\right) \;:\; |\vec v_i| \in [0, \underbrace{v_{esc}}_{R}).$$

$$\Rightarrow f(\vec v') \propto \int_{S_R^3 \times S_R^3} d^3v_1\, d^3v_2 \exp\left(-\frac{\vec v_1^{\,2} + \vec v_2^{\,2}}{2\sigma^2}\right) \underbrace{\delta(\vec v' - (\vec v_1 - \vec v_2))}_{\vec v_1 = \vec v_2 + \vec v'}$$

$$= \int_{S_R^3} d^3v_2 \exp\left[-\frac{(\vec v_2 + \vec v')^2 + \vec v_2^{\,2}}{2\sigma^2}\right] \Theta(-|\vec v_2 + \vec v'| + R)$$

$\vec u = \vec v_2 + \vec v'$

$$= \int_{S_R^3 + \vec v'} d^3u \exp\left[-\frac{\vec u^{\,2} + (\vec u - \vec v')^2}{2\sigma^2}\right] \Theta(R - u)$$

$$= \int_{(S_R^3 + \vec v') \cap (S_R^3) \equiv \Omega'} d^3u \exp\left[-\frac{\vec u^{\,2} - \vec u \cdot \vec v'}{\sigma^2}\right] \exp\left[-\frac{\vec v'^{\,2}}{2\sigma^2}\right]$$

$$= \int_{\Omega'} d^3u \exp\left[-\frac{(\vec u - \frac{\vec v'}{2})^2 + \frac{\vec v'^{\,2}}{4}}{\sigma^2}\right] \exp\left[-\frac{\vec v'^{\,2}}{2\sigma^2}\right]$$

$\vec v = \vec u - \frac{\vec v'}{2}$

$$= \exp\left[-\frac{\vec v'^{\,2}}{4\sigma^2}\right] \times \int_{(S_R^3 + \frac{\vec v'}{2}) \cap (S_R^3 - \frac{\vec v'}{2}) \equiv \Omega(\vec v')} d^3v \exp\left[-\frac{\vec v^{\,2}}{\sigma^2}\right]$$

$$\Rightarrow f(v') \propto \int d^3v'\, f(\vec v')\, \delta(v' - |\vec v'|)$$

$$= \int d\Omega\, v'^2 f(\vec v')$$

Note: $f(\vec v')$ is symmetric w.r.t. rotation.

$$\Rightarrow f(v') \propto v'^2 \exp\left[-\frac{\vec v'^{\,2}}{4\sigma^2}\right] \int_{\Omega(v'\hat z)} d^3v \exp\left[-\frac{\vec v^{\,2}}{\sigma^2}\right]$$

Note:

$$\int_{\Omega(v'\hat z)} d^3v \exp\left[-\frac{\vec v^{\,2}}{\sigma^2}\right] \propto \int_{-1}^{1} d(\cos\theta) \int_0^{v_\theta} v^2 \exp\left[-\frac{v^2}{\sigma^2}\right]$$

$$\overset{(\psi)}{=} \int_{-1}^{1} d(\cos\theta) \int_0^{v_\theta(\cos\theta)} v^2 \exp\left[-\frac{v^2}{\sigma^2}\right]$$

$$= \int_{-1}^{1} du \int_0^{v_\theta(u)} v^2 \exp\left[-\frac{v^2}{\sigma^2}\right]$$

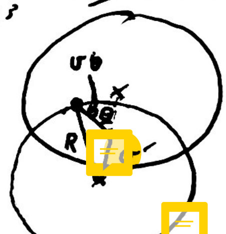
From here, need Mathematica. $\begin{cases} l = v' \\ R = v_{esc} \end{cases}$

$\mathbb{R}^3$

$\Omega(\vec v')$

Parameterize $\Omega(v'\hat z)$:

$(\psi)\ \mathbb{R}^3$



Law of Cosines:

$$R^2 = v_\theta^2 + v'^2 + 2 v_\theta v' \cos\theta$$

$$\Rightarrow v_\theta^2 + (2v'\cos\theta) v_\theta + (v'^2 - R^2) = 0$$

$$\Rightarrow v_\theta = -v'\cos\theta \pm \sqrt{v'^2\cos^2\theta + (R^2 - v'^2)}$$

$$\Rightarrow v_\theta = -v'\cos\theta + \sqrt{v'^2\cos^2\theta + (R^2 - v'^2)}$$