Calculating Orbital Parameters

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I. SUMMARY OF CALCULATIONS

We assume the stellar components have identical masses. Our code will still perform about as efficiently if we do not make this assumption. We will have to modify the expressions for $\Delta \vec{v_p}$ if this is true. The relative orientation between the trajectory of a perturber and the separation vector of a binary system is completely specified by (p, θ, ϕ, v_p), where p is the impact parameter corresponding to the center of mass of the binary and v is the speed of the perturber relative to the binary. In this version of the code, we account for the eccentricity of the binary and the linear contribution to the energy injection. To do this, we must track the stars. To specify the plane of orbit and the direction of rotation, we specify the direction of the angular momentum $\vec{l}$ with an angle γ. Denote the separation unit vector, true anomaly velocity unit vector, and the angular momentum unit vector by $(\hat{r}, \hat{\eta}, \hat{l})$. In the coordinate system where $(\hat{r}, \hat{\eta}, \hat{l}) = (\hat{x}, \hat{y}, \hat{z})$ that is preferred for the binary, γ corresponds to the angle by which $\hat{l}$ deviates from these coordinates. Thus, encounters are now parameterized by $(p, \theta, \phi, \gamma, v_p)$.

Instead of our binaries being distinguished only by semimajor axis. They are now distinguished by semimajor axis, eccentricity, and orbital phase. As before, we let generate a uniformly from a specified bin. The eccentricity is drawn from a thermal distribution as consistent with the recent work of eccentricities using the ERB data. The orbital phase is obtained from the time of orbit, which obeys a uniform distribution. The time is related to the eccentric anomaly by Kepler's Equation

$$M \equiv \omega t = \psi - e \sin \psi$$

Instead of specifying a time of orbit, we specify an effective time (mean anomaly divided by 2π. It is defined by

$$\tau = \frac{t}{2\pi P(a)} = \frac{1}{2\pi} (\psi - e \sin \psi).$$

This is desirable since $\tau \in U[0, 1)$ and it is independent of the binary semimajor axis.

At the inital timestep, the binaries are generated and evolved to the timestep of the first encounter, δt. Consider one of the binaries with orbital parameters (a_i, e_i, τ_i). The encounter injects energy into the binary and perturbs its angular momentum according to the following equations

$$\Delta \vec{v_i} = \left(\frac{2GM_p}{v_p} \right) \frac{U(p_i)}{p_i^2} \vec{p_i}$$

$$\Delta E = \Delta v^2/2 + \vec{v} \cdot \Delta \vec{v}$$

$$\Delta \vec{l} = \vec{r} \times \Delta \vec{v},$$

where the unindexed variables denote the quantities corresponding to relative quantities of the reduced mass of the binary system. Suppose the above quantities are known. Then, the evolve dynamical variables are given by

$$E_f = E_i + \Delta E$$

$$\vec{l_f} = \vec{l_i} + \Delta \vec{l}$$

The orbital parameters are related to the dynamical variables by

$$E = \frac{GM}{2a}$$

$$l = \sqrt{ka(1 - e^2)}$$

Therefore, we evolve binaries by using these relations. The binary orbital parameters (a_i, e_i) specify the dynamical quantities (E_i, l_i). The encounter changes the dynamical quantities to $(E_f, l_f) \rightarrow (a_f, e_f)$.

To calculate the dynamical quantities from the orbital parameters, we need the velocity $\vec{v}$. This quantity is very simply given in the binary-centered coordinate system with $\hat{l} = \hat{z}$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\eta}\hat{\eta},$$

where

$$\dot{\eta} = \frac{l}{r^2}$$

and

$$\dot{r} = \pm\sqrt{2\left(E + \frac{k}{r} - \frac{1}{2}r^2\dot{\eta}^2\right)},$$

where $\dot{r}$ is positive when $0 \geq \psi < \pi$ and negative when $\pi \geq \psi < 2\pi$. Not taking care of this sign can cause issues. The final quantity needed to specify the dynamical variables is the radius r. It is related to the eccentric anomaly by

$$r = a(1 - e\cos\psi).$$

Thus, our code returns evolved orbital parameters from initial orbital parameters as follows. First, we evolve the binary effective orbital times by using

$$\tau_i = \tau_0 + \frac{\delta t}{P(a)}$$

we use the orbital parameters (a_i, e_i, τ_i) to determine the separation r and the eccentric anomaly ψ from the effective time matrix $t(\psi, e)$. The orbital parameters, separation, and the eccentric anomaly can be used to calculate a_f and then e_f. The eccentric anomaly ψ corresponding to the initial keplerian orbit can be converted into an eccentric anomaly corresponding to the final keplerian orbit ψ_f by using the fact that the binary separations do not change significantly during the encounter.

$$r(a_i, e_i, \psi_i) = r(a_f, e_f, \psi_f).$$

This specifies two possible values for ψ_f. By noting that both values are different in the sign of $\dot{r}$, we can use the velocity information to single out the actual ψ_f. I.e.,

$$\psi_f = \begin{cases} \arccos(...) & \dot{r} \geq 0 \\ -\arccos(...) + 2\pi & \dot{r} < 0 \end{cases}$$

This can now be converted to an orbital time using Kepler's Equation.

We finally convert our physical separations r to projected separations s by simply picking out an inclination angle $p(i) \propto \sin(i)$.

II. COMPUTATIONS IN PERTURBER-CENTRIC COORDINATES

We work with two coordinate systems during our computations. In the perturber-centric coordinate system, the perturber trajectory coinicides with the $z-axis$. The binary center of mass is displaced from the origin by the impact parameter $\vec{p} = p\hat{x}$. The $y-axis$ is situated so that the coordinate system is right-handed. Describing the orientation of the binary through the polar and azimuthal angles θ, ϕ, the position of the stellar components relative to the center of the binary is given by $r_i = \pm(r/2)\hat{r}$, where positive applies to star 1 and negative applies to star 2 and

$$\hat{r} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}.$$

Thus, the impact parameters correponding to each star are given by

$$\vec{p_i} = \begin{pmatrix} p \pm (r/2)\sin\theta\cos\phi \\ \pm(r/2)\sin\theta\sin\phi \\ 0 \end{pmatrix}.$$

From this, we can compute the familiar expression

$$\Delta v^2 = \left(\frac{2GM_p}{v_p}\right)^2 \left[\left(\frac{U(p_1)}{p_1}\right)^2 + \left(\frac{U(p_2)}{p_2}\right)^2 - \frac{2U(p_1)U(p_2)}{p_1^2 p_2^2}\left(p^2 - (r/2)^2 \sin^2(\theta)\right)\right].$$

Similarly, the change in the angular momentum simplifies to

$$\Delta l_x = \left(\frac{2GM_p}{v_p}\right) \times \frac{r^2}{2}\sin(\theta)\sin(\phi)\cos(\theta) \times \left(\frac{U(p_1)}{p_1^2} + \frac{U(p_2)}{p_2^2}\right)$$

$$\Delta l_y = \left(\frac{2GM_p}{v_p}\right) \times r\cos(\theta) \times \left[p\left(-\frac{U(p_1)}{p_1^2} + \frac{U(p_2)}{p_2^2})\right) - \frac{r}{2}\sin(\theta)\cos(\phi)\left(\frac{U(p_1)}{p_1^2} + \frac{U(p_2)}{p_2^2}\right)\right]$$

$$\Delta l_z = \left(\frac{2GM_p}{v_p}\right) \times pr^2 \sin(\theta)\sin(\phi) \times \left(\frac{U(p_1)}{p_1^2} - \frac{U(p_2)}{p_2^2}\right)$$

To compute the linear velocity term in the energy injection and the angular momentum, we start with the binary-centric coordinate system, where the origin corresponds to the center of mass. The $x-axis$ coincides with the radius vector of the orbit and the $z-axis$ aligns with the angular momentum of the orbit. The $y-axis$ is chosen so that the coordinates are right-handed. Thus, the Keplerian orbit is confined to the xy-plane and the velocity of the reduced mass is given in cylindrival coordinates by

$$\vec{v} = \dot{r}\hat{x} + r\dot{\eta}\hat{y}.$$

To convert this quantity into perturber-centric coordinates, we must align the separation vector in the binary-centric coordinate system with the separation vector in the perturber-centric coordinate system. That is, we must rotate $\hat{x}$ to the position vector $\hat{r}$ for specified polar and azimuthal angles θ, ϕ. This can be done by first rotating about the $y-axis$ by $\theta - \pi/2$ and then rotating along the untransformed $z-axis$ by ϕ. I.e.,

$$\begin{aligned}\hat{r} &= \mathrm{R}(\phi;\hat{x})\ \mathrm{R}(\theta-\pi/2;\hat{y})\ \hat{x} \\ &= \begin{pmatrix}\sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta\end{pmatrix}\end{aligned}$$

Applying these rotation matrices to $y-hat$ and $z-hat$ give perturber-centric expressions for $e\hat{t}a$ and $\hat{l}$ for the case that the plane of orbit coincides with the xy-plane. I.e.,

$$\hat{\eta}_0 == \begin{pmatrix}-\sin\phi \\ \cos\phi \\ 0\end{pmatrix}$$

$$\hat{l}_0 = \begin{pmatrix}-\cos\theta\cos\phi \\ -\cos\theta\sin\phi \\ \sin\theta\end{pmatrix}.$$

To a produce a general kepler orbit, we simply need to rotate the orbital plane. I.e., we rotate by an angle γ along the separation unit vector $\hat{r}$ to give

$$\begin{aligned}\hat{\eta} &= \mathrm{R}(\gamma;\hat{r})\hat{\eta}_0 \\ &= \begin{pmatrix}-\cos(\theta)\cos(\phi)\sin(\gamma) - \sin(\phi)\cos(\gamma) \\ -\cos(\theta)\sin(\phi)\sin(\gamma) + \cos(\phi)\cos(\gamma) \\ \sin(\theta)\sin(\gamma)\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\hat{l} &= \mathrm{R}(\gamma;\hat{r})\hat{l}_0 \\ &= \begin{pmatrix}-\cos(\theta)\cos(\phi)\cos(\gamma) + \sin(\phi)\sin(\gamma) \\ -\cos(\theta)\sin(\phi)\cos(\gamma) - \cos(\phi)\sin(\gamma) \\ \sin(\theta)\cos(\gamma)\end{pmatrix}\end{aligned}$$

Since we know $\vec{l_i} = l_i\hat{l}$, we can compute l_f from

$$l_f = |\vec{l_i} + \Delta\vec{l}|$$

by performing the vector sum in the perturber-centric coordinates. Moreover, we can similarly express velocity in terms of perturber-centric coordinates and compute the linear term in the velocity by using the expressions for $\vec{v}$ and $\Delta\vec{v}$ in perturber-centric coordinates.