A brief introduction to

Partial Differential Equation

Version: 5.2

Edward J. Xu
Energy System Dynamics and Analysis
Department of Management Engineering
Technical University of Denmark, Denmark

December, 2018

1 Introduction to PDE

Mathematical modelling plays a big role in the description of a large part of phe- nomena in the applied sciences and in several aspects of technical and industrial activity.

By a "mathematical model" we mean a set of equations and/or other mathe- matical relations capable of capturing the essential features of a complex natural or artificial system, in order to describe, forecast and control its evolution. The applied sciences are not confined to the classical ones; in addition to physics and chemistry, the practice of mathematical modelling heavily affects disciplines like finance, biology, ecology, medicine, sociology.

In the industrial activity (e.g. for aerospace or naval projects, nuclear reactors, combustion problems, production and distribution of electricity, traffic control, etc.) the mathematical modelling, involving first the analysis and the numerical simulation and followed by experimental tests, has become a common procedure, necessary for innovation, and also motivated by economic factors. It is clear that all of this is made possible by the enormous computational power now available.

In general, the construction of a mathematical model is based on two main ingredients: general laws and constitutive relations.

In this book we shall deal with general laws coming from continuum mechanics and appearing as conservation or balance laws (e.g. of mass, energy, linear momentum, etc.).

The constitutive relations are of an experimental nature and strongly depend on the features of the phenomena under examination. Examples are the Fourier law of heat conduction, the Fick's law for the diffusion of a substance or the way the speed of a driver depends on the density of cars ahead.

The outcome of the combination of the two ingredients is usually *a partial differential equation or a system of them.*

1.1 Partial Differential Equations

1.2 Well Posed Problems

Usually, in the construction of a mathematical model, only some of the general laws of continuum mechanics are relevant, while the others are eliminated through the constitutive laws or suitably simplified according to the current situation. In general, additional information are necessary to select or to predict the existence of a unique solution. These information are commonly supplied in the form of initial and/or boundary data, although other forms are possible. For instance, typical boundary conditions prescribe the value of the solution or of its normal derivative, or a combination of the two, at the boundary of the relevant domain. A main goal of a theory is to establish suitable conditions on the data in order to have a problem with the following features:

1.3 Basic Facts and Notations

1.3.1 Well Posed Problems

Usually, in the construction of a mathematical model, only some of the general laws of continuum mechanics are relevant, while the others are eliminated through the constitutive laws or suitably simplified according to the current situation. In general, additional information are necessary to select or to predict the existence of a unique solution. These information are commonly supplied in the form of *initial and/or boundary data*, although other forms are possible. For instance, typical boundary conditions prescribe the value of the solution or of its normal derivative, or a combination of the two, at the boundary of the relevant domain. A main goal of a theory is to establish suitable conditions on the data in order to have a problem with the following features:

1. There exists at least one solution.
2. There exists at most one solution.
3. The solution depends continuously on the data.

This property is extremely important and may be expressed as a **local stability of the solution with respect to the data**. Think for instance of using a computer to find an approximate solution: the insertion of the data and the computation algorithms entail approximation errors of various type. A significant sensitivity of the solution on small variations of the data would produce an unacceptable result.

The notion of continuity and the error measurements, both in the data and in the solution, are made precise by introducing a suitable notion of distance. In dealing with a numerical or a finite dimensional set of data, an appropriate distance may be the usual euclidean distance

1.4 Important Methods and Issues

1.4.1 The Method of Characteristics

Subsubsection 2.2 / P189, 4.3.2 The method of characteristics, 4.3 Traffic Dynamics, Salsa (2016)

To solve the problem, which means to compute the density ρ at a point (x, t) , we follow the idea we already exploited in the linear transport case without external sources:to connect the point (x, t) with a point $(x_0, 0)$ on the x-axis, through a curve along which ρ is constant (Fig. 2.2).

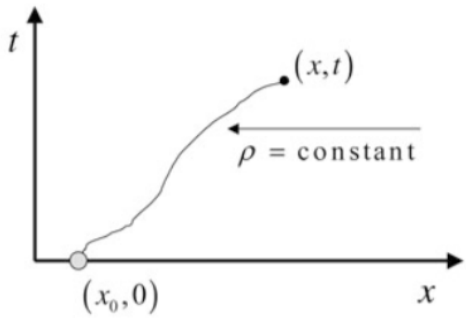


Figure 2.2, Characteristic Curve

Clearly, if we manage to find such a curve, that we call characteristic based at $(x_0, 0)$, the value of ρ at (x, t) is given by $\rho(x_0, 0) = g(x_0)$. Moreover, if this procedure can be repeated for every point (x, t), $x \in \mathbb{R}$, $t > 0$, then we can compute ρ everywhere in the upper half-plane and the problem is completely solved. This is the *method of characteristics*.

1.4.2 Separation of Variables

> Mirza Karamehmedović: Many linear PDE problems can be solved using the so-called separation of variables that reduces the PDE problem to a set of ODE problems in the independent variables. This classical method dates back to Fourier (1812), who in fact developed what is now known as Fourier series to solve PDE problems.

Subsection 6.1.5 / P23, 2.1.4 A solution by separation of variables, Salsa (2016).

$$\begin{cases} U_t - \mathrm{D}U_{xx} = 0 \\ U(x,0) = u^{st}(x) - g(x) \\ U(0,t) = 0 \\ U(L,t) = 0 \end{cases}$$

P269, 5.3.2 Separation of Variables, 5.3 The one-dimensional Wave Equation, Salsa (2016).

$$\begin{cases} u_{tt} - c\, u_{xx} = 0 \end{cases}$$

P303, Square membrane, 5.7.1 Small vibrations of an elastic membrane, 5.7 Two Classical Models

Problem 3, Assignment 4: Solve the inital-boundary problem with wave equation:

$$\begin{cases} u_{tt} - 9 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right) = 0 \ , \ 0 < x < 3 \ , \ 0 < y < 2 \ , \ t > 0 \\ u(x,y,0) = -\sin\left(2\,\pi\, x\right)\, \sin\left(\dfrac{\pi}{4} y\right) \ , \ 0 \leqslant x \leqslant 3 \ , \ 0 \leqslant y \leqslant 2 \\ u_t(x,y,0) = \sin\left(\dfrac{\pi}{3} x\right)\, \sin\left(\dfrac{7\,\pi}{4} y\right) \ , \ 0 \leqslant x \leqslant 3 \ , \ 0 \leqslant y \leqslant 2 \\ u(0,y,t) = u(3,y,t) = 0 \ , \ 0 \leqslant y \leqslant 2 \ , \ t \geqslant 0 \\ u(x,0,t) = 0 \ , \ 0 \leqslant x \leqslant 3 \ , \ t \geqslant 0 \\ u_y(x,2,t) = 0 \ , \ 0 \leqslant x \leqslant 3 \ , \ t \geqslant 0 \end{cases}$$

1.4.3 The Global Cauchy Problem

In important applications, for instance in financial mathematics, x varies over unbounded intervals, typically $(0, \infty)$ or $\mathbb{R}$. In these cases one has to require that the solution does not grow too much at infinity.

Section 6.4 / P76, 2.8.1 The homogeneous case, 2.8 The Global Cauchy Problem, 2 Diffusion, Salsa (2016).

Problem B, Assignment 2: Consider the global Cauchy problem:

$$\begin{cases} u_t - u_{xx} = 0 & x \in \mathbb{R} \ , \ t > 0 \\ u(x,0) = g_u(x) = 3\, \cos x & x \in \mathbb{R} \end{cases} \tag{1}$$

P275, 5.4.1 The homogeneous equation, 5.4 The d'Alembert Formula, 5 Waves and Vibrations, Salsa (2016), the equation in the book:

$$\begin{cases} u_{tt} - c^2\, u_{xx} = 0 & , \ x \in \mathbb{R} \ , \ t > 0 \\ u(x,0) = g(x) & , \ x \in \mathbb{R} \\ u_t(x,0) = h(x) & , \ x \in \mathbb{R} \end{cases}$$

Problem 2, Assignment 4: Assume a function $u(x,t)$ is in $C^2(\mathbf{R} \times [0, \infty[)$, and satisfies:

$$\begin{cases} u_{tt} - 4\, u_{xx} = 0 \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$$

which is a Cauchay-Dirichlet problem with a wave equation, and

$$
\begin{aligned}
g(6) &= 6 \\
g(2) &= 8 \\
\int_0^2 h(y)\,\mathrm{d}y &= -12 \\
\int_0^6 h(y)\,\mathrm{d}y &= 8
\end{aligned}
$$

P310, 5.8 The Global Cauchy Problem, 5 Waves and Vibrations, Salsa (2016), the equation in the book:

$$
\begin{cases}
u_{tt} - c^2\,\Delta u = 0 & , \vec{x} \in \mathbb{R}^3 \ , t > 0 \\
u(\vec{x}, 0) = g(\vec{x}) & , \vec{x} \in \mathbb{R} \\
u_t(\vec{x}, 0) = h(\vec{x}) & , \vec{x} \in \mathbb{R}
\end{cases}
$$

1.4.4 Well-posed Problem and Uniqueness

As mentioned in P5, 1.3 Well posed problem, Salsa (2016), the governing equations in a mathematical model have to be supplemented by additional information in order to obtain a well posed problem, i.e. a problem that has exactly one solution, depending continuously on the data.

Subsubsection 6.1.4 / P20, 2.1.3 Well posed problems $(n = 1)$**, 2.1 The Diffusion Equation**.

Subsection 8.2 / P116, 3.2 Well Posed Problems. Uniqueness, 3 The Laplace Equation, Salsa (2016)

P298, 5.6.2 Well posed problems. Uniqueness, 5.6 The Multi-dimensional Wave Equation $(n > 1)$**, Salsa (2016)**.

2 Scalar Conservation Laws and Firs Order Equations

2.1 Macroscopic model

Macroscopic model describing the intense traffic in one way road as a fluid flow contains three macroscopic variables: the density of cars ρ, the average speed v, and the flux function q, which three can be expressed by the simple convection relation, with the assumption that average speed depends on the density, under condition $t \geqslant 0$ and $x \in \mathbb{R}$:

$$q(\rho) = v(\rho)\,\rho \tag{2}$$

N(t) is the number of cars in a particular section (x_1, x_2) of the road, and can be expressed be the following equation, with its derivative on time t:

$$N(t) = \int_{x_1}^{x_2} \rho(x,t)\,\mathrm{d}x \tag{3}$$

$$N‘(t) = \frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,\mathrm{d}x = \int_{x_1}^{x_2} \rho_t(x,t)\,\mathrm{d}x \tag{4}$$

Because it's assumed that there is no overtaking, "sources", and "sinks", we can derive the conservation law, which summarizes the flux of two borders to get $N‘(t)$:

$$N‘(t) = q(\rho(x_1,t)) - q(\rho(x_2,t)) = -\int_{x_1}^{x_2} q(\rho)_x(x,t)\,\mathrm{d}x \tag{5}$$

So we can get **conservation law for three variables** by eq.4 minus eq.5:

$$\int_{x_1}^{x_2} [\rho_t(x,t) + q(\rho)_x(x,t)]\,\mathrm{d}x = 0 \tag{6}$$

$$\begin{aligned} \rho_t(x,t) + q(\rho(x,t))_x &= 0 \\ \rho_t(x,t) + q_x(\rho(x,t))\,\rho_x &= 0 \\ \rho_t(x,t) + [v(\rho)\,\rho]_x &= 0 \end{aligned} \tag{7}$$

One of the most simplest relationship between ρ and v can be assumed:

$$v(\rho) = v_m\left(1 - \frac{\rho}{\rho_m}\right) \tag{8}$$

So that q can be expressed by put v into equation: 2,

$$q(\rho) = v_m\,\rho\left(1 - \frac{\rho}{\rho_m}\right) \tag{9}$$

And we can get $q(\rho)_x$ for eq.7 to get eq.12:

$$\begin{aligned} q(\rho)_x &= q_\rho(\rho)\,\rho_x \\ \frac{\partial\, q(\rho)}{\partial\, x} &= \frac{\partial\, q(\rho)}{\partial\,\rho}\,\frac{\partial\,\rho}{\partial\, x} \\ &= \left[v_m\,\rho\left(1 - \frac{\rho}{\rho_m}\right)_\rho + (v_m\,\rho)_\rho\left(1 - \frac{\rho}{\rho_m}\right)\right]\rho_x \\ &= \left[v_m\,\rho\left(-\frac{1}{\rho_m}\right) + v_m\left(1 - \frac{\rho}{\rho_m}\right)\right]\rho_x \\ &= v_m\left(1 - \frac{2\rho}{\rho_m}\right)\rho_x \end{aligned} \tag{10}$$

$q(\rho)_x$ can also be expressed:

$$q(\rho)_x = [v(\rho)\,\rho]_x = v\,\rho_x + v_\rho\,\rho_x\rho \tag{11}$$

Above all, we get the partial differential equation problem, of which equation is the only quasi-linear equation for this course, with initial condition assumed:

$$\begin{cases} \rho_t + v_m\left(1 - \frac{2\rho}{\rho_m}\right)\rho_x = 0 \\ \rho(x,0) = g(x) \\ \mathrm{t} \geqslant 0\ ,\ x \in \mathbb{R} \end{cases} \tag{12}$$

2.2 The method of characteristics

To solve the problem, which means to compute the density ρ at a point (x,t) , we follow the idea we already exploited in the linear transport case without external sources:to connect the point (x,t) with a point $(x_0,0)$ on the x-axis, through a curve along which ρ is constant (Fig. 2.2).

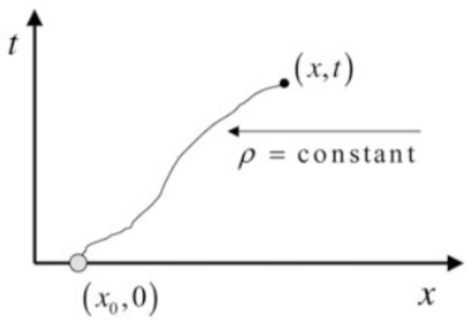


Figure 2.2, Characteristic Curve

Clearly, if we manage to find such a curve, that we call characteristic based at $(x_0,0)$, the value of ρ at (x,t) is given by $\rho(x_0,0) = g(x_0)$. Moreover, if this procedure can be repeated for every point (x,t), $x \in \mathbb{R}$, $t > 0$, then we can compute ρ everywhere in the upper half-plane and the problem is completely solved. This is the *method of characteristics*.

Adopting a slightly different point of view, we can implement the above idea as follows: assume that $x = x(t)$ is the equation of the characteristic based at the point $(x_0,0)$ and that along $x = x(t)$ we always observe the same initial density $g(x_0)$. In other words, we have:

$$\rho(x(t),t) = g(x_0) \tag{13}$$

for every $t > 0$. If we differentiate the identity from equation 13, we get:

$$\frac{d}{dt}\rho(x(t),t) = \rho_x(x(t),t)\,x_t(t) + \rho_t(x(t),t) = 0 \tag{14}$$

On the other hand, equation 7 yields the following equation, and the one after substitution of equation 13:

$$\begin{aligned} \rho_t(x(t),t) + q_x(\rho(x(t),t))\,\rho_x(x(t),t) = 0 \\ \rho_t(x(t),t) + q_x(g(x_0))\,\rho_x(x(t),t) = 0 \end{aligned} \tag{15}$$

After subtracting equation 14 and equation 15, we obtain:

$$\rho_x(x(t),t)\,[x_t(t) - q_x(g(x_0))] = 0 \tag{16}$$

Assuming $\rho_x(x(t),t) \neq 0$, otherwise there is no car on the road, we deduce:

$$x_t(t) = q_x(g(x_0)) \tag{17}$$

After indefinite integration and $x(0) = x_0$, we get $x(t)$:

$$x(t) = \int x_t(t)\,\mathrm{d}t = q_x(g(x_0))\,t + x_0 \tag{18}$$

Thus, the characteristics are straight lines with slope $q_x(g(x_0))$. Different values of x_0 give, in general, different values of the slope (Fig. 2).

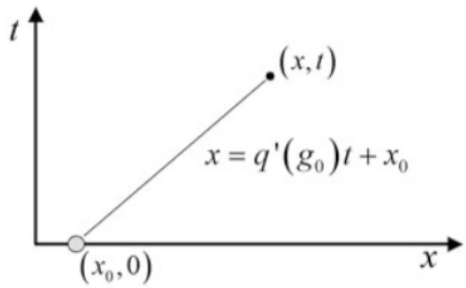


Figure 2, Outcome Characteristic Curve $g_0 = g(x_0)$

We can now derive a formula for $\rho(x,t)$, so we move back in time along the characteristic through (x,t) until its base point $(x_0, 0)$. Then we can get the following equation with initial condition from equation 12:

$$\rho(x,t) = \rho(x_0, 0) = g(x_0) \tag{19}$$

From equation 18 we have:

$$x_0 = x(t) - q_x(g(x_0))\,t \tag{20}$$

And we finally get formula for $\rho(x,t)$ after substitute equation 20 in to equation 19:

$$\rho(x,t) = g(x(t) - q_x(g(x_0))) \tag{21}$$

This equation represents a *travelling wave (or a signal, a disturbance) propagating with speed* $q_x(g(x_0))$ along the positive x-direction.

We emphasize that $q_x(g(x_0))$ is the local wave speed:

$$q_x(g(x_0)) = v_m\left(1 - \frac{2\,g(x_0)}{\rho_m}\right) \tag{22}$$

And it must not be confused with the traffic velocity. In fact, in general,

$$\frac{d\,q}{d\,\rho} = \frac{d\,(\rho\,v)}{d\,\rho} = v + \rho\,\frac{d\,v}{d\,\rho} \leqslant v \tag{23}$$

The different nature of the two speeds becomes more evident if we observe that the wave speed may be negative as well. This means that, while the traffic advances along the positive x-direction, the disturbance given by the travelling wave may propagate in the opposite direction. Indeed, in our model equation 9, $dq < 0$ when $\rho > \frac{\rho_m}{2}$.

Formula 21 seems to be rather satisfactory, since, apparently, it gives the solution of the initial value problem 12 at every point. Actually, a more accurate analysis shows that, even if the initial datum g is smooth, the solution may develop a singularity in finite time (e.g. a jump discontinuity). When this occurs, the method of characteristics does not work anymore and formula 21 is not effective. A typical case is described in Fig. 3: two characteristics based at different points $(x_1, 0)$ and $(x_2, 0)$ intersect at the point (x, t) and the value u (x, t) is not uniquely determined as soon as $g(x_1) = g(x_2)$.

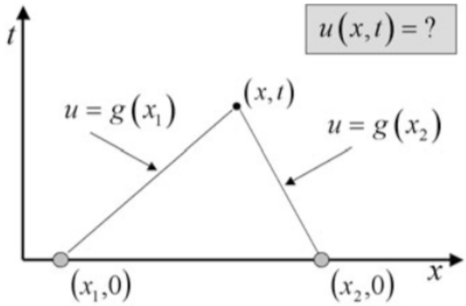


Figure 3, Intersection of Characteristics

In this case we have to weaken the concept of solution and the computation technique. We will come back on these questions later. For the moment, we analyze the method of characteristics in some remarkable cases.

2.3 The Green Light Problem

Suppose that bumper-to-bumper traffic is standing at a red light, placed at $x = 0$, while the road ahead is empty. Accordingly, the initial density profile is:

$$g(x) = \begin{cases} \rho_m & \text{for } x \leqslant 0 \\ 0 & \text{for } x > 0 \end{cases} \tag{24}$$

At time $t = 0$, the traffic light turns green and we want to describe the car flow evolution for $t > 0$. At the beginning, only the cars closer to the light start moving while most of them remain standing.

The local wave speed is given by equation 22 after substituting equation 24:

$$q_x(g(x_0)) = \begin{cases} -\,v_m & \text{for } x_0 \leqslant 0 \\ v_m & \text{for } x_0 > 0 \end{cases} \tag{25}$$

And the characteristics are the straight lines:

$$x = \begin{cases} -\,v_m\, t + x_0 & \text{for } x_0 \leqslant 0 \\ v_m\, t + x_0 & \text{for } x_0 > 0 \end{cases} \tag{26}$$

The lines $x = v_m\, t$ and $x = v_m\, t$ partition the upper half-plane in the three regions R, S and T, shown in Fig. 4.

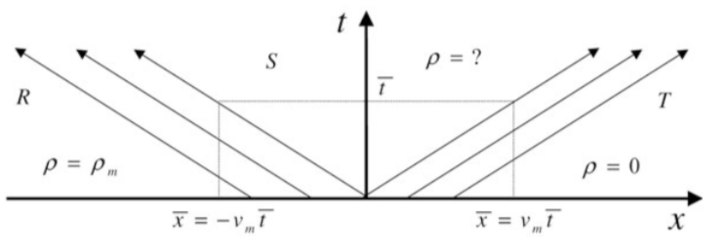


Figure 4, Characteristic for the Green Light Problem

Inside R we have $\rho(x,t) = \rho_m$, while inside T we have $\rho(x,t) = 0$. Consider the points on the horizontal line $t = t_1$. At the points $(x, t_1) \in T$ the density is zero: the traffic has not yet arrived in that area. The front car is located at the point:

$$x_{front} = v_m\, t_1 \tag{27}$$

which moves at the maximum speed, since ahead the road is empty.

The cars placed at the points $(x, t_1) \in R$ are still standing. The first car that starts moving at time $t = t_1$ is at the point:

$$x_{back} = -v_m\, t_1 \tag{28}$$

In particular, it follows that the green light signal propagates back through the traffic at speed v_m.

What is the value of the density inside the sector S? No characteristic extends into S, due to the discontinuity of the initial data at the origin, and the method as it stands does not give any information on the value of ρ inside S.

2.4 Smoothing in the Green Light Problem

A strategy that may give a reasonable answer is the following:

1. Approximate the initial data by a continuous function g_ε, which converges to g as $\varepsilon \to 0$ at every point x, except 0.
2. Construct the solution ρ_ε of the ε-problem by the method of characteristics.
3. Let $\varepsilon \to 0$ and check that the limit of ρ_ε is a solution of the original problem.

Clearly we run the risk of constructing many solutions, each one depending on the way we regularize the initial data, but for the moment we are satisfied if we construct at least one solution.

Let us choose as g_ε the function (Fig. 5):

$$g_\varepsilon = \begin{cases} \rho_m & x \leqslant 0 \\ \rho_m \left(1 - \frac{x}{\varepsilon}\right) & 0 < x < \varepsilon \\ 0 & x \geqslant \varepsilon \end{cases} \tag{29}$$

When $\varepsilon \to 0$, $g_\varepsilon(x) \to g(x)$ for every $x \neq 0$.

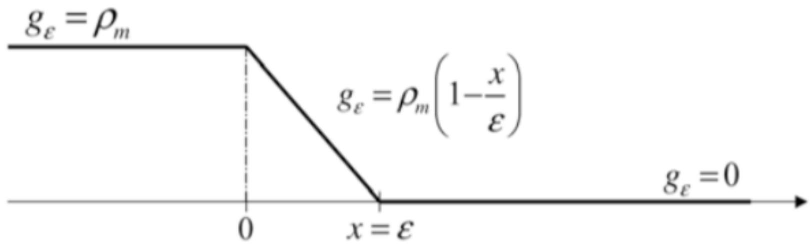


Figure 5, Smoothing of the Initial Data

The characteristics for the ε-problem are thus different from equation 26:

$$\begin{aligned} x &= -\,v_m\, t + x_0 && \text{if } x_0 < 0 \\ x &= -\,v_m \left(1 - 2\,\frac{x_0}{\varepsilon}\, t + x_0\right) && \text{if } 0 \leqslant x_0 < \varepsilon \\ x &= -\,v_m\, t + x_0 && \text{if } x_0 \geqslant \varepsilon \end{aligned} \tag{30}$$

Hence, for $0 \leqslant x_0 < \varepsilon$, the local wave speed (equation 22) is given by:

$$q_x(g(x_0)) = -\, v_m \left(1 - 2\,\frac{x_0}{\varepsilon}\right) \tag{31}$$

We say that the characteristics in the region $v_m\, t < x < v_m\, t + \varepsilon$ form a rarefaction fan (Fig. 6).

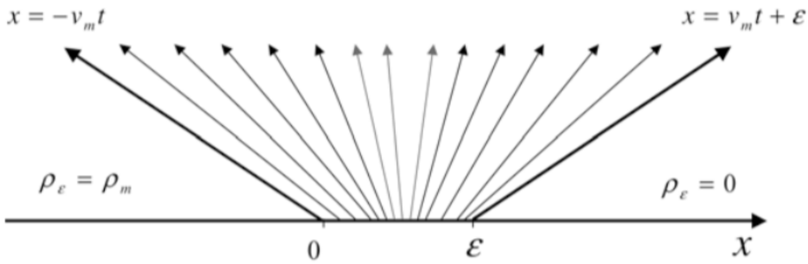


Figure 6, Fanlike Characteristics

Solving for x0 in the equation of the characteristic (eq. 30), we get:

$$x_0 = \varepsilon\,\frac{x + v_m\, t}{x\ t + \varepsilon} \tag{32}$$

Then we substitute eq. 29 and eq. 32 into eq. 19 to get:

$$\begin{aligned}\rho_\varepsilon(x,t) &= g_\varepsilon(x_0)\\ &= \rho_m \left(1 - \frac{x_0}{\varepsilon}\right)\\ &= \rho_m \left(1 - \frac{x + v_m\, t}{2\, v_m\, t + \varepsilon}\right)\end{aligned} \tag{33}$$

Let $\varepsilon \to 0$ in eq. 33, we obtain:

$$\rho(x,t) = \begin{cases} \rho_m & \text{for } x \leqslant -v_m\, t\\ \frac{\rho_m}{2}\left(1 - \frac{x}{v_m\ t}\right) & \text{for } -v_m\, t < x < v_m\, t\\ 0 & \text{for } x \geqslant v_m\, t\end{cases} \tag{34}$$

It is easy to check that ρ is a solution of the equation (4.23) in the regions R, S, T. For fixed t, the function ρ decreases linearly from ρ_m to 0, as x varies from $v_m\, t$ to $v_m\, t$. Moreover, ρ is constant on the fan of straight lines:

$$x = h\, t \qquad -v_m < h < v_m \tag{35}$$

These type of solutions are called rarefaction or simple waves (centered at the origin). The characteristics and a typical profile are shown in Figs. 7 and Fig. 8.

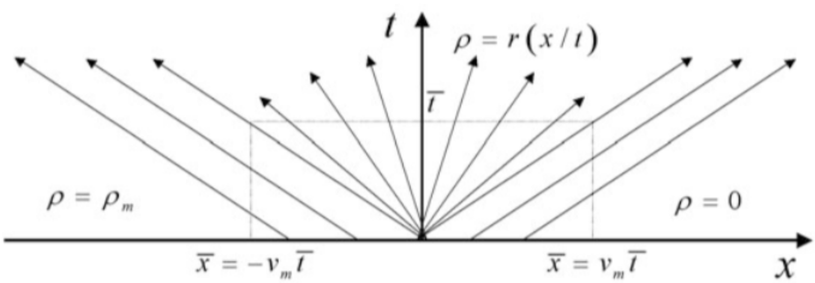


Figure 7, Characteristics in a Rarefaction Wave

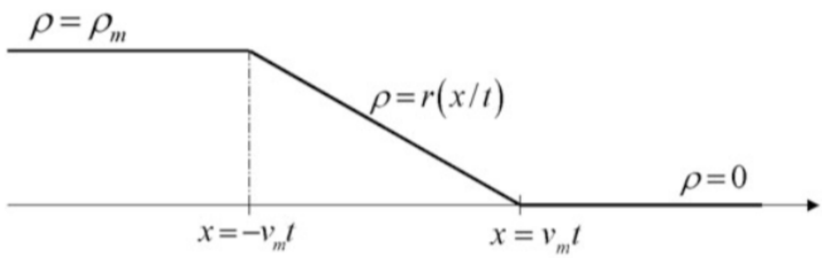


Figure 8, Profile of a Rarefaction Wave at t

3 Assignment 1: Traffic Dynamics

DTU01418, Introduction to Partial Differential Equations, Fall 2018

3.1 Problem 1: Polluted Water in Pipe

3.1.1 1.1

It's assumed that the flux function in macroscopic model of pollution in pipe line is expressed as:

$$q(c) = v(c)\, c \tag{36}$$

By focusing on the accumulated concentration from (x_1, x_2) part of the pipe, we can derive the following equations:

$$N(t) = \int_{x_1}^{x_2} c(x,t)\,\mathrm{d}x \tag{37}$$

$$N`(t) = \frac{d}{dt}\int_{x_1}^{x_2} c(x,t)\,\mathrm{d}x = \int_{x_1}^{x_2} c_t(x,t)\,\mathrm{d}x \tag{38}$$

$$N`(t) = q(c(x_1,t)) - q(c(x_2,t)) = -\int_{x_1}^{x_2} q(c)_x(x,t)\,\mathrm{d}x \tag{39}$$

we can get **conservation law for three variables** by eq.38 minus eq.39:

$$\int_{x_1}^{x_2} \left[c_t(x,t) + q(c)_x(x,t)\right]\mathrm{d}x = 0 \tag{40}$$

$$c_t(x,t) + [v(c)\, c]_x = 0 \tag{41}$$

Since $v(c) = 0.5$, which is independent of c. We can write the initial value problem:

$$\begin{cases} c_t + 0.5\, c_x = 0 \\ c(x,0) = g(x) = 1 + \cos x\, \sin x \\ \mathrm{t} \geqslant 0\ ,\ x \in \mathbb{R} \end{cases} \tag{42}$$

But we assume that the profile of polluted water from factory is the same as that is already let out.

3.1.2 1.2

Assume there is line expressing the relation between t and x on which the concentration are equal to the intersection $(x_0, 0)$, and that means:

$$c(x(t),t) = c(x_0,t) = g(x_0) \tag{43}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t} c(x(t),t) = c_x\, x_t + c_t = 0 \tag{44}$$

We can get the following equation by eq.44 minus eq.42:

$$c_x\,(x_t - 0.5) = 0 \tag{45}$$

$$\begin{cases} c_x & = 0 \\ x_t & = 0.5\ ,\ x = 0.5\, t + x_0 \end{cases} \tag{46}$$

From eq.46, we get:

$$x_0 = x - 0.5\, t \tag{47}$$

Substitute eq.47 into eq.43 to get:

$$\begin{aligned} c(x(t),t) &= c(x_0,0) = g(x_0) \\ &= g(x-0.5\,t) = 1+\sin\left(x-0.5\,t\right)\,\cos\left(x-0.5\,t\right) \end{aligned} \tag{48}$$

Actually, we can only get, but we admit the eq.48 because the last assumption in the above section:

$$c(x(t),t) = 1+\sin\left(x-0.5\,t\right)\,\cos\left(x-0.5\,t\right)\ ,\ x \geqslant 0.5\,t \tag{49}$$

3.1.3 1.3

When $t = 10$, the concentration yields:

$$c(x,10) = 1+0.5\,\sin\left(2\left(x-5\right)\right) \tag{50}$$

This means the maximum value is 1.5 where $x = 5+\frac{\pi}{4}$.

3.1.4 1.4

let's say when $t = t_1$, the polluted water starts flowing back toward the factory. We change the initial value problem of PDE to:

$$\begin{cases} c_t - 0.5\,c_x = 0 \\ c(x,0) = 1+0.5\,\sin\left(2\left(x-0.5\,t_1\right)\right) \\ \mathrm{t} \geqslant 0\ ,\ x\in\mathbb{R} \end{cases} \tag{51}$$

3.2 Problem 2: Solve the Initial Value Problem

$$\begin{cases} u_t - 3/4\,u_x = x^2 t\ ,\ x\in\mathbb{R}\ ,\ t\geqslant 0 \\ u(x,0) = g(x) = x\ ,\ x\in\mathbb{R}\ , \end{cases} \tag{52}$$

This is a linear transport equation with distributed source and initial value problem.
The $N`(t)$ can be expressed in two ways, one of which is the integration of u_t in eq.52:

$$N`(t) = \frac{\mathrm{d}}{\mathrm{d}\,t}\int_{x_1}^{x_2} u\,\mathrm{d}x = \int_{x_1}^{x_2} u_t\,\mathrm{d}x = \int_{x_1}^{x_2} \left(3/4\,u_x + x^2\right)\,\mathrm{d}x \tag{53}$$

$$\begin{aligned} N`(t) &= q[u(x_1,t)] - q[u(x_2,t)] + \int_{x_1}^{x_2} f(x,t)\,\mathrm{d}x \\ &= -\int_{x_1}^{x_2} q(u)_x\,\mathrm{d}x + \int_{x_1}^{x_2} f(x,t)\,\mathrm{d}x \end{aligned} \tag{54}$$

......

According to proposition 4.1, the unique solution of the initial value problem is:

$$u(x,t) = g(x-v\,t) + \int_0^t f(x-v\,(t-s),s)\,\mathrm{d}s \tag{55}$$

From eq.52, we can get:

$$v = -\frac{3}{4} \tag{56}$$

$$f(x) = x^2 \tag{57}$$

$$g(x) = x \tag{58}$$

Then, eq.**??** can be derived:

$$u(x,t) = (x + 3/4\,t) + \int_0^t [x + 3/4\,(t-s)]^2\,\mathrm{d}s \tag{59}$$

$$= \ldots\ldots \tag{60}$$

Remember to check the correctness of the solution, by substituting into the problem.

3.3 Burger's Equation

3.3.1 3.1

$$\begin{cases} u_t + u\,u_x = 0 \\ u(x,0) = g(x) = \begin{cases} -1/3 & , x < 1 \\ 2 & , x > 1 \end{cases} \end{cases} \tag{61}$$

In a similar way as subsection 1.2, we get:

$$u_x(x(t),t)\,[u(x(t),t) - x_t(t)] = 0 \tag{62}$$

So,

$$x_t(t) = u(x(t),t) = g(x_0) \tag{63}$$

$$x(t) = g(x_0)\,t + x_0 \tag{64}$$

$$= \begin{cases} x_0 - 1/3\,t & , x_0 < 1 \\ x_0 + 2\,t & , x_0 > 1 \end{cases} \tag{65}$$

$$\begin{aligned} x_0 &= x - g(x_0)\,t \\ &= \begin{cases} x + 1/3\,t & , x < 1 + 1/3\,t \\ x - 2\,t & , x > 1 - 2\,t \end{cases} \end{aligned} \tag{66}$$

$$\begin{aligned} u(x(t),t) &= u(x_0,0) = g(x_0) \\ &= \begin{cases} -1/3 & \text{when } x < 1 + 1/3\,t \\ 2 & \text{when } x > 1 - 2\,t \end{cases} \end{aligned} \tag{67}$$

3.3.2 1.3

$$\begin{cases} u^\epsilon_t + u^\epsilon\,u^\epsilon_x = 0 \\ u^\epsilon(x,0) = g^\epsilon(x) = \begin{cases} -1/3 & , x < 1 \\ \frac{7\,x - \epsilon - 7}{3\,\epsilon} & , 1 \leqslant x \leqslant 1 + \epsilon \\ 2 & , x > 1 + \epsilon \end{cases} \end{cases} \tag{68}$$

Similar as eq.64:

$$\begin{aligned} x(t) &= g(x_0)\,t + x_0 \\ &= \begin{cases} x_0 - 1/3\,t & \text{when } x_0 < 1 \\ (\frac{7\,t}{3\,\epsilon} + 1)\,x_0 - \frac{(\epsilon+7)\,t}{3\,\epsilon} & \text{when } 1 \leqslant x_0 \leqslant 1 + \epsilon \\ x_0 + 2\,t & \text{when } x_0 > 1 \end{cases} \end{aligned} \tag{69}$$

$$
\begin{aligned}
x_0 &= x - g(x_0)\, t \\
&= \begin{cases} x + 1/3\, t & \text{when } x < 1 + 1/3\, t \\ \frac{3\,\epsilon}{7\,t+3\,\epsilon}\, x + \frac{(7+\epsilon)\,t}{7\,t+3\,\epsilon} & \text{when } 1 - \frac{t}{3} \leqslant x \leqslant 2\,t + \epsilon + 1 \\ x - 2\,t & \text{when } x > 1 - 2\,t \end{cases}
\end{aligned}
\tag{70}
$$

4 Finite Difference

4.1 Opportunities and Dangers of Finite Difference

Assume a function $u(x)$. Choose a mesh size Δx, and approximate the value $u(j\,\Delta x)$ for $x = j\,\Delta x$ (divide x to j parts of Δx) by a number u_j indexed by an integer j:

$$u_j \sim u(j\,\Delta x) \sim u(x)$$
$$\text{Similarly, } u_{j+1} \sim u((j+1)\,\Delta x) \sim u(x+\Delta x)$$

Assume $u(j\,\Delta x) \in C^2(I)$, then the Taylor expansion of $u(x+\Delta x)$ and $u(x-\Delta x)$ are:

$$u(x+\Delta x) = u(x) + u'(x)\,\Delta x + O\,(\Delta x)^2$$
$$u(x-\Delta x) = u(x) - u'(x)\,\Delta x + O\,(\Delta x)^2$$

Then the three standard approximation for the first derivative $\frac{\partial u}{\partial x}\,(j\,\Delta x)$ are shown. Pay attention to the change of the order of omission term $O\,(\Delta x)$, which causes truncation error mentioned below and has an impact on the final outcome.

$$\text{The backward differences: } u'(j\,\Delta x) \approx \frac{u_j - u_{j-1}}{\Delta x} + \cancel{O(\Delta x)} \tag{71}$$

$$\text{The forward differences: } u'(j\,\Delta x) \approx \frac{u_{j+1} - u_j}{\Delta x} + \cancel{O(\Delta x)} \tag{72}$$

$$\text{The centered differences: } u'(j\,\Delta x) \approx \frac{u_{j+1} - u_{j-1}}{2\,\Delta x} + \cancel{O(\Delta x)} \tag{73}$$

Assume $u(j\,\Delta x) \in C^3(I)$, then the Taylor expansion can be more detailed:

$$u(x+\Delta x) = u(x) + u'(x)\,\Delta x + \frac{1}{2}\,u''(x)\,(\Delta x)^2 + O\,(\Delta x)^3$$

For the second derivative, the simplest approximation is the centered second difference:

$$u''(j\,\Delta x) \approx \frac{u_{j+1} - 2\,u_j + u_{j-1}}{(\Delta x)^2} + \cancel{O(\Delta x)^2} \tag{74}$$

For functions of two variables $u(x,t)$, we choose a mesh size for two variables. We write:

$$u_j^n \sim u(j\,\Delta x, n\,\Delta t) \sim u(x,t)$$

And we can write similar expressions like equation (71) - (74).
Two kinds of errors can be introduced in a computation using such approximation:

- **Truncation Error**: The error in the solutions by the approximation itself.
- **Round-off Error**: The limitation that only a certain number of digits, typically 8 or 16, are obtained by the computer at each step of the computation.

4.2 Approximations of Diffusion

The choice of the mesh Δt relative to the mesh Δx is called Courant Number:

$$s = \frac{\Delta t}{(\Delta x)^2} \tag{75}$$

Let's see a simple problem:

$$u_t - u_{xx} = 0 \;,\; u(x,0) = \phi(x) \tag{76}$$

The finite difference equation of eq.76 can be write with local truncation error:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \cancel{O(\Delta t)} = \frac{u_{j+1}^n - 2\, u_j^n + u_{j-1}^n}{(\Delta x)^2} + \cancel{O(\Delta x)^2} \tag{77}$$

Substitute eq.75, we can get:

$$u_j^{(n+1)} = s\,(u_{j+1}^n + u_{j-1}^n) + (1 - 2\,s)\,u_j^n \tag{78}$$

And the following diagram can be a visual explanation of eq.78.

t					
(n+1)			*		
n		s	(1-2s)	s	
...					
1					
0	...	(j-1)	j	(j+1)	x

So, from initial value $u(x, 0) = \phi(x)$ in eq.75, we can get any $u(x, j)$. There may be need of other necessary additional boundary condition when come down to the value along the boundary.

But there is a large difference in the outcome of the above method because of chosen value of s, This is called the stability criterion which is discussed below.

4.2.1 Stability Criterion

To actually demonstrate that this is the stability condition, we separate the variables in the difference equation. Thus we look for solutions of equation 78 of the form:

$$\begin{aligned} u_j^n &= X_j\,T_n \\ &= \frac{u_j^{n+1} - s\,(u_{j+1}^n + u_{j-1}^n)}{(1 - 2\,s)} \end{aligned} \tag{79}$$

Thus,

$$\begin{aligned} \frac{T_{n+1}}{T_n} &= \frac{u_j^{n+1}}{u_j^n} \\ &= s\,\frac{X_{j+1} + X_{j-1}}{X_j} + 1 - 2\,s \end{aligned} \tag{80}$$

Both sides of equation 79 must be a constant ξ independent of j and n. Therefore,

$$\begin{aligned} T_n &= \xi^n\,T_0 \\ \frac{T_{n+1}}{T_n} &= \xi \end{aligned} \tag{81}$$

So,

$$s\,\frac{X_{j+1}+X_{j-1}}{X_j}+1-2\,s=\xi \tag{82}$$

To solve the spatial eq.82, we argue that it is a discrete version of a second-order ODE which has sine and cosine solutions. Therefore, we guess solutions of eq.82 of the form:

$$X_j = A\,\cos\left(j\,\theta\right)+B\,\sin\left(j\,\theta\right) \tag{83}$$

for some θ, where A and B are arbitrary. The boundary condition $X_0 = 0$ at $j\,\theta = 0$ implies that $A = 0$. So we can freely set $B = 1$. Then $X_j = \sin\left(j\,\theta\right)$.

Furthermore, the boundary condition $X_J = 0$ at $j = J$ implies that $\sin\left(j\,\theta\right) = 0$. Thus $J\,\theta = k\,\pi$ for some integer k. But the discretization into J equal intervals of length Δx meas that $J = \pi/\Delta x$. Therefore, $\theta = k\,\Delta x$ and

$$X_j = \sin\left(j\,k\,\Delta x\right) \tag{84}$$

Now eq.82 takes the form:

$$s\,\frac{\sin\left(\left(j+1\right)k\,\Delta x\right)+\sin\left(\left(j-1\right)k\,\Delta x\right)}{\sin\left(j\,k\,\Delta x\right)}+1-2\,s=\xi$$

or

$$\xi = \xi(k) = 1-2\,s\left[1-\cos\left(k\,\Delta x\right)\right] \tag{85}$$

According to eq.81, the growth in time $t = n\,\Delta t$ at the wave number k is governed by the powers $\xi(k)^n$. So, **unless** $|\xi(k)| \leqslant 1$ **for all k the scheme is unstable** and could not possibly approximate the true (exact) solution. (Recall that the true solution tends to zero as $t \to$ inf.) Now we analyze eq.85 to determine whether $|\xi(k)| \leqslant 1$ or not. Since factor $1-\cos\left(k\,\Delta x\right)$ ranges between 0 and 2, we have $1-4\,s \leqslant \xi(k) \leqslant 1$. So stability requires that $1-4\,s \geqslant -1$, which means that

$$\frac{\Delta t}{(\Delta x)^2} = s \leqslant \frac{1}{2} \tag{86}$$

5 Finite Element Method

5.1 Introduction to Finite Element Method

Finite difference method has two main limitations cause by the "rectangle" it utilizes. The size of rectangles are similar so there is no way to handle curved/irregularly shaped domain ,or emphasize some area for more detail.

So there comes **Finite Element Method (FEM)**, it subdivides a large problem domain into smaller and simpler pieces (polygons) that are called finite elements. The simple equations that model these finite elements are then assembled into a larger system of equations that models the entire problem. FEM then uses variational methods from the calculus of variations to approximate a solution by minimizing an associated error function.

FEM is best understood from its practical application, known as **Finite Element Analysis (FEA)**. FEA as applied in engineering is a computational tool for performing engineering analysis. It includes the use of mesh generation techniques for dividing a complex problem into small elements, as well as the use of software program coded with FEM algorithm. In applying FEA, the complex problem is usually a physical system with the underlying physics such as the Euler-Bernoulli beam equation, the heat equation, or the Navier-Stokes equations expressed in either PDE or integral equations, while the divided small elements of the complex problem represent different areas in the physical system.

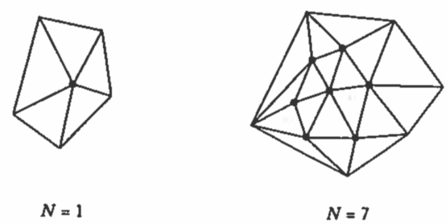
FEA is a good choice for analyzing problems over complicated domains (like cars and oil pipelines), when the domain changes (as during a solid state reaction with a moving boundary), when the desired precision varies over the entire domain, or when the solution lacks smoothness. FEA simulations provide a valuable resource as they remove multiple instances of creation and testing of hard prototypes for various high fidelity situations. For instance, in a frontal crash simulation it is possible to increase prediction accuracy in "important" areas like the front of the car and reduce it in its rear (thus reducing cost of the simulation).

Let's consider the Dirichlet problem for Poisson's equation in the plane:

$$\begin{aligned} -\Delta u &= f \text{ in } D \\ u &= 0 \text{ on bdy } D \end{aligned} \tag{87}$$

D is triangulated by be approximated by a region D_N, which is the union of a finite number of triangles (see fig.1). Let the interior vertices be donated by $V_1, ..., V_N$.

Figure 1: Illustration of triangles when $N = 1$ and $N = 7$)

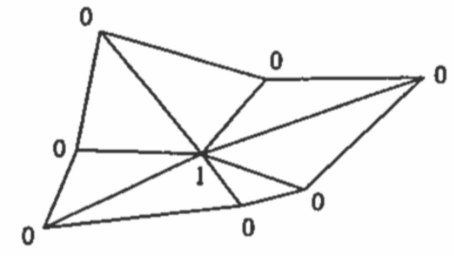


Next, we pick N trial functions, $v_1(x, y), ..., v_N(x, y)$, one for each interior vertex. Each trial function $v_i(x, y)$ is chosen to equal 1 at its vertex V_i and to equal 0 at all the other vertices (see fig.2). Inside each triangle, each trial function is a linear function: $v_i(x, y) = a + b\,x + c\,y$. (The coefficients a, b, c are different for each trial function and for each triangle.) This prescription determines $v_i(x, y)$ uniquely. In fact, its graph is simply a pyramid of unit height with its summit at V_i and it is identically zero on all the triangles that do not touch V_i.

We shall approximate the solution $u(x, y)$ by a linear combination of the $v_i(x, y)$:

$$u_N(x, y) = U_1\, v_1(x, y) + ... + U_N\, v_N(x, y) \tag{88}$$

Figure 2: Illustration of triangles when $N = 1$ and $N = 7$)



To motivate our choice we need a degression. Let's rewrite problem in eq.87 using Green's first identity [formula (GI) from Section 7.1]. We multiply Poisson's equation by any function $v(x, y)$ that vanishes on the boundary. Then:

$$\iint_D \nabla u \, \nabla v \, \mathrm{d}x \, \mathrm{d}y = \iint_D f \, v \, \mathrm{d}x \, \mathrm{d}y \tag{89}$$

Rather than requiring eq.89 to be valid for $u_N(x, y)$ for all functions $v(x, y)$, we ask only that it be valid for the first N special trial functions $v = v_j$ ($j = 1, ..., N$). Thus, with $u(x, y) = u_N(x, y)$ and $v(x, y) = v_j(x, y)$, eq.89 becomes:

$$\sum_{i=1}^{N} U_i \left(\iint_D \nabla v_i \, \nabla v_j \, \mathrm{d}x \, \mathrm{d}y \right) = \iint_D f \, v_j \, \mathrm{d}x \, \mathrm{d}y \tag{90}$$

This is a system of N linear equation ($j = 1, ..., N$) in the N unknowns $U_1, ..., U_N$. If we denote:

$$m_{ij} = \iint_D \nabla v_i \, \nabla v_j \, \mathrm{d}x \, \mathrm{d}y \quad \text{and} \quad f_j = \iint_D f \, v_j \, \mathrm{d}x \, \mathrm{d}y \tag{91}$$

then the system takes the form:

$$\sum_{i=1}^{N} m_{ij} \, U_i = f_i \quad (j = 1, ..., N) \tag{92}$$

The finite element method consists of calculating m_{ij} and f_j from eq.91 and solving eq.92. The approximate solution u_N automatically vanishes on the boundary of D_N. Notice also that, at a vertex $V_k = (x_k, y_k)$,

$$u_N(x_k, y_k) = U_1 \, v_1(x_k, y_k) + ... + U_N \, v_N(x_k, y_k) = U_k \tag{93}$$

since $v_i(x_k, y_k)$ equals 0 for $i \neq k$. Thus the coefficients are precisely the values of the qpproximate solution at the vertices.

6 Diffusion

6.1 Diffusion Equation

6.1.1 Introduction to Diffusion Equation

$$u_t - \mathrm{D}\,\Delta u = f \tag{94}$$

where Δ denotes the **Laplace Operator**:

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} \tag{95}$$

When $f \equiv 0$, the equation is said to be **homogeneous** and in this case the **Superposition Principle** holds: if u and v are solutions of eq.94 and a, b are real (or complex) numbers, $a\,u + b\,v$ also is a solution of eq.94. More generally, if $u_k(\mathbf{x}, t)$ is a family of solutions depending on the parameter k (integer or real) and $g = g(k)$ is a function rapidly vanishing at infinity, then

$$\sum_{k=1}^{\infty} u_k(\mathbf{x}, t)\, g(k) \quad \text{and} \quad \int_{-\infty}^{+\infty} u_k(\mathbf{x}, t)\, g(k)\, \mathrm{d}k$$

are still solutions.

6.1.2 General Diffusion Equation

On the other hand eq.94 constitutes a much more general diffusion model, where by diffusion we mean, for instance, the transport of a substance due to the molecular motion of the surrounding medium. In this case, u could represent the concentration of a polluting material or of a solute in a liquid or a gas (dye in a liquid, smoke in the atmosphere) or even a probability density. We may say that the diffusion equation unifies at a macroscopic scale a variety of phenomena, that look quite different when observed at a microscopic scale.

Through eq.94 and some of its variants we will explore the deep connection between probabilistic and deterministic models, according (roughly) to the scheme:

diffusion processes $\leftrightarrow$ probability density $\leftrightarrow$ differential equations

The star in this field is Brownian motion, derived from the name of the botanist Brown, who observed in the middle of the 19th century, the apparently chaotic behavior of certain particles on a water surface, due to the molecular motion. This irregular motion is now modeled as a *Stochastic Process* under the terminology of *Wiener Process / Brownian Motion.* The operator

$$\frac{1}{2}\Delta$$

is strictly related to Brownian motion [1] and indeed it captures and synthesizes the microscopic features of that process.

Under equilibrium conditions, that is when there is no time evolution, the solution u depends only on the space variable and satisfies the stationary version of the diffusion equation (letting $D = 1$):

$$-\Delta u = f \tag{96}$$

($u_{xx} = f$, in dimension $n = 1$). Eq.96 is known as the *Poisson Equation.* When $f = 0$, it is called *Laplace's Equation* and its solutions are so important in so many fields that they have deserved the special name of **harmonic functions**. This equations will be considered in the next chapter.

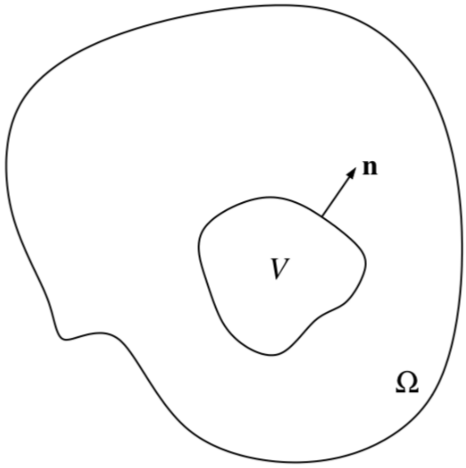


Figure 3: Illustration for Basic Law of Heat Conduction

6.1.3 Heat conduction in a solid body

A common example of diffusion is given by heat conduction in a solid body. Conduction comes from molecular collision, transferring heat by kinetic energy, without macroscopic material movement. If the medium is homogeneous and isotropic with respect to the heat propagation, the evolution of the temperature is described by eq.94; f represents the intensity of an external distributed source. For this reason eq.94 is also known as the **heat equation**.

Since heat is a form of energy, it is natural to use the law of conservation of energy, that we can formulate in the following way: Let V be an arbitrary control volume inside the body. *The time rate of change of thermal energy in V equals the net flux of heat through the boundary ∂V of V, due to the conduction, plus the time rate at which heat is supplied by the external sources.*

If we denote by $e = e(\vec{x}, t)$ the thermal energy per unit mass, the total quantity of thermal energy $M(t)$ and its time rate of change $M‘(t)$ inside V is given by:

$$M(t) = \int_V \rho\, e(\vec{x}, t)\, \mathrm{d}\vec{x} \tag{97}$$

$$M‘(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho\, e(\vec{x}, t)\, \mathrm{d}\vec{x} = \int \rho\, e_t(\vec{x}, t)\, \mathrm{d}\vec{x} \tag{98}$$

$$M‘(t) = -\int_{\partial V} \vec{q}(\vec{x}, t)\, \hat{\nu}\, \mathrm{d}\sigma + \int_V \rho\, r(\vec{x}, t)\, \mathrm{d}\vec{x} \tag{99}$$

Divergence theorem:

$$M‘(t) = -\int_V \nabla\vec{q}(\vec{x}, t)\, \mathrm{d}\vec{x} + \int_V \rho\, r(\vec{x}, t)\, \mathrm{d}\vec{x} \tag{100}$$

Equation:

$$\int \rho\, e_t(\vec{x}, t)\, \mathrm{d}\vec{x} = -\int_V \nabla\vec{q}(\vec{x}, t)\, \mathrm{d}\vec{x} + \int_V \rho\, r(\vec{x}, t)\, \mathrm{d}\vec{x} \tag{101}$$

The arbitrariness of V allows us to convert the integral eq. into the pointwise relation:

$$\rho\, e_t(\vec{x}, t) = -\nabla\vec{q}(\vec{x}, t) + \rho\, r(\vec{x}, t) \tag{102}$$

Final

$$u_t - \frac{\kappa}{c_v\, \rho}\, \Delta u = \frac{1}{c_v}\, r \tag{103}$$

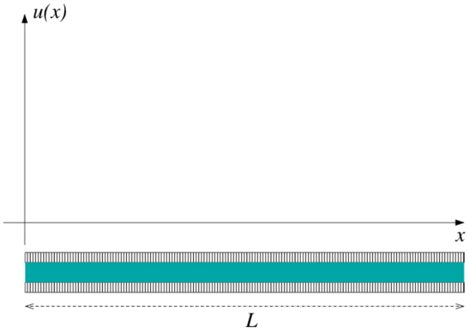


Figure 4: Illustration for Heat Conduction in a Long Bar

6.1.4 Well Posed Problems ($n = 1$)

The problems associated with the above boundary conditions have a corresponding nomenclature. Summarizing, we can state the most common problems for the one dimensional heat equation as follows: given $f = f(x,t)$ (external source) and $g = g(x)$ (initial or Cauchy data), determine $u = u(x,t)$ such that:

$$\begin{cases} u_t - \mathrm{D}\,\Delta u = f & 0 < x < L,\ 0 < t < T \\ u(x,0) = g(x) & 0 \leqslant x \leqslant L \\ \text{boundary condition} & 0 < t \leqslant T \end{cases} \tag{104}$$

where the boundary conditions may be:
1. Dirichlet: $u(0,t) = h_1(t)$, $u(L,t) = h_2(t)$
2. Neumann: $-u_x(0,t) = h_1(t)$, $u_x(L,t) = h_2(t)$
3. Robin or Radiation: $-u_x(0,t) + \alpha\, u(0,t) = h_1(t)$, $u_x(L,t) + \alpha\, u(L,t) = h_2(t)$
or mixed conditions. Accordingly, we have the initial-Dirichlet problem, the initial-Neumann problem and so on. When $h_1 = h_2 = 0$, we say that the boundary conditions are homogeneous.

6.1.5 A solution by separation of variables

> Mirza Karamehmedović: Many linear PDE problems can be solved using the so-called separation of variables that reduces the PDE problem to a set of ODE problems in the independent variables. This classical method dates back to Fourier (1812), who in fact developed what is now known as Fourier series to solve PDE problems.

We will prove that, under reasonable hypotheses, the initial Dirichlet, Neumann or Robin and mixed problems are well posed. Sometimes this can be shown using elementary techniques like *the separation of variables method* that we describe below through a simple example of heat conduction. We emphasize that the reduction to homogeneous boundary conditions is crucial to carry on the computations.

$$\begin{cases} U_t - \mathrm{D}U_{xx} = 0 \\ U(x,0) = u^{st}(x) - g(x) \\ U(0,t) = 0 \\ U(L,t) = 0 \end{cases} \tag{105}$$

[1]In the theory of stochastic processes, $1/2\,\Delta$ represents the *infinitesimal generator of the Brownian motion.*

Assume $U(x,t) = X(x)\,T(t)$, then

$$X\,T' - \mathrm{D} X''\,T = 0 \tag{106}$$

$$\frac{T'}{\mathrm{D}\,T} - \frac{X''}{X} = 0 \tag{107}$$

$$\frac{1}{\mathrm{D}}\,\frac{T'(t)}{T(t)} = \frac{X''}{X} = \lambda \quad \text{(constant) ???} \tag{108}$$

So, we get Eigen-Value Problem:

$$X''(x) = \lambda\,X(x) \tag{109}$$

$$X(0) = X(L) = 0 \tag{110}$$

and ODC ???

$$T'(t) = \mathrm{D}\,\lambda\,T(t) \tag{111}$$

To solve Eigen-Value Problem:
When we assume $\lambda > 0$:

$$X(x) = \mathcal{C}_1\,e^{\sqrt{\lambda}\,x} + \mathcal{C}_2\,e^{-\sqrt{\lambda}\,x} \tag{112}$$

$$\begin{cases} X(0) = \mathcal{C}_1 + \mathcal{C}_2 = 0 \quad \text{and } \mathcal{C}_1 \neq 0,\, \mathcal{C}_2 \neq 0 \\ X(L) = \mathcal{C}_1\,e^{\sqrt{\lambda}\,L} + \mathcal{C}_2\,e^{-\sqrt{\lambda}\,L} \end{cases} \tag{113}$$

According to eq. :

$$\mathcal{C}_2(-e^{\sqrt{\lambda}\,L} + e^{-\sqrt{\lambda}\,L}) = 0 \tag{114}$$

$$-\,e^{\sqrt{\lambda}\,L} + e^{-\sqrt{\lambda}\,L} = 0 \tag{115}$$

$$\sqrt{\lambda}\,L = -\sqrt{\lambda}\,L \tag{116}$$

$$L = -L \tag{117}$$

This is not true, so the assumption of $\lambda > 0$ is wrong.
When we assume $\lambda = 0$:

$$X''(x) = 0 \tag{118}$$

$$X(x) = \mathcal{C}_1\,x + \mathcal{C}_2 \tag{119}$$

$$\begin{cases} X(0) = \mathcal{C}_2 = 0 \quad \mathcal{C}_1 \neq 0 \\ X(L) = \mathcal{C}_1\,L + \mathcal{C}_2 = \mathcal{C}_1\,L = 0 \end{cases} \tag{120}$$

This is not true, so the assumption of $\lambda = 0$ is wrong.
When we assume $\lambda < 0$, we set $\lambda = -\mu^2$, $\mu > 0$:

$$X(x) = \mathcal{C}_1\,\cos(\mu\,x) + \mathcal{C}_2\,\sin(\mu\,x) \tag{121}$$

$$\begin{cases} X(0) = \mathcal{C}_1 = 0 \quad \text{and } \mathcal{C}_2 \neq 0 \\ X(L) = \mathcal{C}_1\,\cos(\mu\,L) + \mathcal{C}_2\,\sin(\mu\,L) = \mathcal{C}_2\,\sin(\mu\,L) = 0 \end{cases} \tag{122}$$

So, we get:

$$\sin(\mu\,L) = 0 \tag{123}$$

$$\mu_m\,L = m\,\pi\ ,\ m = 1,\,2,\,\ldots \tag{124}$$

$$\mu_m = \frac{m\,\pi}{L}\ ,\ m = 1,\,2,\,\ldots \tag{125}$$

$$\lambda_m = -\left(\frac{m\,\pi}{L}\right)^2\ ,\ m = 1,\,2,\,\ldots \tag{126}$$

So,

$$X_m(x) = \mathrm{C}_2 \,\sin\left(\frac{m\,\pi}{L}\,x\right)\ ,\ m = 1,\ 2,\ ... \tag{127}$$

And

$$T`(t) = D\,\lambda\,T(t) = -\mu^2\,D\,T(t) \tag{128}$$

$$T_m(t) = \mathcal{B}\,e^{-\mu_m^2\,t} = \mathcal{B}\,e^{-\left(\frac{m\,\pi}{L}\right)^2\,t}\ ,\ m = 1,\ 2,\ ... \tag{129}$$

And

$$U_m(x,t) = \mathcal{C}_2 \sin\left(\frac{m\,\pi}{L}\,x\right)\mathcal{B}\,e^{-\left(\frac{m\,\pi}{L}\right)^2\,t}\ ,\ m = 1,\ 2,\ ... \tag{130}$$

$$U(x,t) = \sum_{m=1}^{\infty}\ \mathcal{A}\,\sin\left(\frac{m\,\pi}{L}\,x\right)e^{-\left(\frac{m\,\pi}{L}\right)^2\,t} \tag{131}$$

Then

$$U(x,0) = u^{st}(x) - g(x) = \sum_{m=1}^{\infty}\ \mathcal{A}\,\sin\left(\frac{m\,\pi}{L}\,x\right)e^{-\left(\frac{m\,\pi}{L}\right)^2\,t} \tag{132}$$

$$u(x,t) = u^{st}(x) - U(x,t) = u_0 + \frac{u_1 - u_0}{L}\,x - \sum_{m=1}^{\infty}\ \mathcal{A}\,\sin\left(\frac{m\,\pi}{L}\,x\right)e^{-\left(\frac{m\,\pi}{L}\right)^2\,t} \tag{133}$$

Identify constants via initial condition, and validate the solution.

6.1.6 Problems in Dimension $n > 1$

Summarizing, we have the following typical problems: given $f = f(\vec{x},t)$ and $g = g(\vec{x})$, determine $u = u(\vec{x},t)$ such that:

$$\begin{cases} u_t - \mathrm{D}\,\Delta\,u = f & \text{in } Q_T \\ u(\vec{x},0) = g(\vec{x}) & \text{in } \overline{\Omega} \\ \text{boundary conditions on } \partial\Omega\times(0,T] \end{cases} \tag{134}$$

where the boundary conditons could be:

1. Dirichlet: $u = h$
2. Neumann: $\partial_{\hat{\mu}}\,u = h$
3. Robin or Radiation: $\partial_{\hat{\mu}}\,u + \alpha\,u = h \quad (\alpha > 0)$
4. Mixed, for instance: $u = h_1$ on Γ_{D}, $\partial_{\hat{\mu}}\,u = h_2$ on Γ_{N}

Also in dimension $n > 1$, the *global Cauchy problem* is important:

$$\begin{cases} u_t - \mathrm{D}\,\Delta\,u = f & \vec{x}\in\mathbb{R}^n\ ,\ 0 < t < T \\ u(\vec{x},0) = g(\vec{x}) & \vec{x}\in\mathbb{R}^n \\ \text{conditions as } |\vec{x}|\to\infty \end{cases} \tag{135}$$

We again emphasize that no final condition (for $t = T$, $x \in \Omega$) is required. The data is assigned on the *parabolic boundary* $\partial_p\,Q_T$ of Q_T , given by the union of the bottom points $\overline{\Omega}\times t = 0$ and the side points $S_T = \partial\Omega\times(0,T]$:

$$\partial_p\,Q_T = (\overline{Q}\times t = 0)\cup S_T \tag{136}$$

6.2 Uniqueness and Maximum Principle

Mirza Karamehmedović: Results regarding the existence and uniqueness of solution are essential in the analysis of PDEs. These results can tell us, e.g., whether a given model of a physical process is coherent and gives rise to a well-posed problem, and whether we should trust numerical solutions of the problem. The maximum principles that we shall consider are surprisingly strong and useful characterizations of solutions to the diffusion equation.

6.2.1 Integral method

Generalizing the energy method used in Subsect. 2.1.4, it is easy to show that all the problems we have formulated in the previous section have at most one solution under reasonable conditions on the data. Suppose u and v are solutions of one of those problems, sharing the same boundary conditions, and let $w = uv$; we want to show that $w \equiv 0$.

The above calculations are completely justified if Ω is a sufficiently smooth domain and, for instance, we require that u and v are continuous in $\overline{Q}_T = \overline{\Omega} \times [0, T]$, together with their first and second spatial derivatives and their first order time derivatives. We denote the set of these functions by the symbol $C^{2,1}(\overline{Q}_T)$ and summarize everything in the following statement.

Theorem 2.2 The initial Dirichlet, Neumann, Robin and mixed problems have at most one solution belonging to $C^{2,1}(\overline{Q}_T)$.

6.2.2 Maximum principles

The fact that heat flows from higher to lower temperature regions implies that a solution of the homogeneous heat equation attains its maximum and minimum values on $\partial_p Q_T$. This result is known as the *maximum principle* and reflects an aspect of the time irreversibility of the phenomena described by the heat equation, in the sense that the future cannot have an influence on the past (*causality principle*). In other words, the value of a solution u at time t is independent of any change of the data after t.

Definition 2.3 A function $w \in C2,1(Q_T)$ such that $w_t \mathrm{D}\,\Delta w \leqslant 0 (\geqslant 0)$ in Q_T is called a sub-solution (super-solution) of the diffusion equation.

Theorem 2.4 Let $w \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ such that ...

As an immediate consequence of Theorem 2.4, we have that, if

$$w_t - \mathrm{D}\,\Delta w = 0 \quad \text{in } Q_T \tag{137}$$

then w attains its maximum and its minimum on $\partial_p Q_T$. In particular,

$$\min_{\partial_p Q_T} w \leqslant w(\vec{x}, t) \leqslant \max_{\partial_p Q_T} w \quad \text{for every } (\vec{x}, t) \in Q_T \tag{138}$$

Moreover (for the proof see Problem 2.6):

6.3 The Fundamental Solution

Mirza Karamehmedović: The fundamental solution is interesting since it is a special solution that can be used to construct other solutions.

There are privileged solutions of the diffusion equation that can be used to construct many other ones. In this section we are going to discover one of these special building blocks, the most important one. First we point out some features of the heat equation.

6.3.1 Invariant transformations

6.3.2 The fundamental solution ($n = 1$)

Fundamental solution of $u_t - D\,\Delta u = 0$:

$$\Gamma_D(x,t) = \frac{1}{\sqrt{4\,\pi\,D\,t}}\,e^{-\frac{x^2}{4\,D\,t}} \tag{139}$$

6.3.3 The Dirac distribution

6.3.4 The fundamental solution ($n > 1$)

In space dimension greater than 1, we can more or less repeat the same arguments.

6.4 The Global Cauchy Problem ($n = 1$)

> Rasmus Dalgas Kongskov: Global Cauchy problem means that the equation is posed on the whole real line, i.e., not on a bounded domain. Since there are no boundaries in the traditional sense, the only data supplied is the initial condition.

6.4.1 The homogeneous case

The global Cauchy problem:

$$\begin{cases} u_t - \mathrm{D}u_{xx} = 0 & \text{in } \mathbb{R} * (0,\infty) \\ u(x,0) = g(x) & \text{in } \mathbb{R} \end{cases} \tag{140}$$

where g, the *initial datum*, is given.

Thanks to the linearity of the diffusion equation, we can use the superposition principle and compute the solution as the sum of all contributions. In this way, we get the formula:

$$u(x,t) = \int_{\mathbb{R}} \Gamma_{\mathrm{D}}(x-y,t)\,g(y)\mathrm{d}y = \frac{1}{\sqrt{4\,\pi\,\mathrm{D}\,t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\,\mathrm{D}\,t}}\,g(y)\,\mathrm{d}y \tag{141}$$

Clearly, one has to check rigorously that, under reasonable hypotheses on the initial datum g, eq. really gives the unique solution of the Cauchy problem.

6.4.2 Existence of a solution

Theorem 2.12 Assume that there exist positive numbers a and c such that:

$$|g(x)| \leqslant c\,e^{a\,x^2} \qquad \text{for all } x \in \mathbb{R} \tag{142}$$

Let u be given by eq.2.135 and $T < \frac{1}{4\,a\,D}$. Then the following properties hold.

1) There are positive numbers C and A such that

$$|u(x,t)| \leqslant C\,e^{A\,x^2} \qquad \text{for all } (x,t) \in \mathbb{R}\times(0,T] \tag{143}$$

2) $u \in C^\infty(\mathbb{R}\times(0,T])$ and in the strip $\mathbb{R}\times(0,T]$

$$u_t - D\,u_{xx} = 0 \tag{144}$$

3) Let $(x,t)(x_0,0^+)$. If g is continuous at x_0 then $u(x,t) \to g(x_0)$

Remark 2.13 The theorem says that, if we allow an initial data with a controlled exponential growth at infinity expressed by eq.2.136, then eq.2.135 is a solution in the strip $\mathbb{R}\times(0,T)$. We will see that, under the stated conditions, eq.2.135 is actually the unique solution.

6.4.3 Global maximum principles and uniqueness.

The uniqueness of the solution to the global Cauchy problem is still to be discussed. This is not a trivial question since the following counterexample of Tychonov shows that there could be several solutions of the homogeneous problem.

Among the class of functions with growth at infinity controlled by an exponential of the type $C\,e^{A\,x^2}$ for any $t \geqslant 0$ (the so called Tychonov class), the solution to the homogeneous Cauchy problem is unique.

This is a consequence of the following maximum principle.

Theorem 2.16 Global maximum principle Let z be continuous in $\mathbb{R} \times [0, T]$, with derivatives z_x, z_{xx}, z_t continuous in $\mathbb{R} \times (0, T)$, such that, in $\mathbb{R} \times (0, T)$:

$$\begin{aligned} &z_t - D\, z_{xx} \leqslant 0 \quad (\text{resp.} \geqslant 0) \\ &z(x,t) \leqslant C\, e^{A\,x^2} \quad \left(\text{resp.} \geqslant -e^{A\,x^2}\right) \end{aligned} \tag{145}$$

where $C > 0$.

Then

$$\sup_{\mathbb{R}\times[0,T]} z(x,t) \leqslant \sup_{\mathbb{R}} z(x,0) \qquad \left(\text{resp.} \inf_{\mathbb{R}\times[0,T]} z(x,t) \geqslant \inf_{\mathbb{R}} z(x,0)\right) \tag{146}$$

The proof is rather difficult [2], but if we assume that z is bounded from above or below ($A = 0$ in (2.145)), then the proof relies on a simple application of the weak maximum principle, Theorem 2.4. In Problem 2.15 we ask the reader to fill in the details of the proof.

We now are in position to prove the following uniqueness result.

Corollary 2.17 Uniqueness 1

Corollary 2.18 Uniqueness 2 Let g be continuous in $\mathbb{R}$, satisfying eq.2.146, and let f be as in Theorem 2.15. Then the Cauchy problem eq.2.144 has a unique solution u in $\mathbb{R} \times [0, T]$, for $T < \frac{1}{4\,D\,a}$, belonging to the Tychonov class. This solution is given by eq.2.135 and moreover

$$\inf_{\mathbb{R}} + t \inf_{\mathbb{R}\times[0,T)} f \leqslant u(x,t) \leqslant \sup_{\mathbb{R}} g + t \sup_{\mathbb{R}\times[0,T)} f \tag{147}$$

6.4.4 Example of a homogeneous global Cauchy problem

$$\begin{cases} u_t - 2\,u_{xx} = 0 & x \in \mathbb{R}\ ,\ t > 0 \\ u(x,0) = g(x) = 4\,e^{3\,x^2} & x \in \mathbb{R} \end{cases} \tag{148}$$

Q1 For what values of x and t are we guaranteed the existence and the uniqueness of a Tychonov-class solution of this problem?

$$D = 2 \tag{149}$$

Assume that there exist positive numbers a and c such that:

$$|g(x)| = 4\,e^{3\,x^2} \leqslant c\,e^{a\,x^2} \quad \text{for all } x \in \mathbb{R} \tag{150}$$

Then $c = 4$, $a = 3$.

So we know from Theorem 2.12 that there is a unique solution:

$$u(x,t) = \int_{\mathbb{R}} \Gamma_{\mathrm{D}}(x-y,t)\, g(y) \mathrm{d}y \tag{151}$$

[2]See [7], F. John, 1982.

satisfies the estimate:

$$|u(x,t)| \leqslant C\, e^{A\,x^2} \qquad \text{for all } (x,t) \in \mathbb{R} \times (0,T] \tag{152}$$

where C and A are positive numbers.

And therefore it belongs to the Tychonov class in $\mathbb{R} \times [0,T]$, for $T < \frac{1}{4\,D\,a}$, according to Corollary 2.18.

Q2 Due to a measurement error, we are given the imperfect initial datum $\tilde{g}(x) = 4\,e^{3\,x^2} + e^{|x|}$, $x \in \mathbb{R}$, instead of the correct initial datum g. For what values of x and t are we now guaranteed the existence and the uniqueness of a Tychonov-class solution of this problem?

Assume that there exist positive numbers a_2 and c_2 such that:

$$\tilde{g}(x) = 4\,e^{3\,x^2} + e^{|x|} \leqslant 4\,e^{3\,x^2} + 1 \leqslant 5\,e^{3\,x^2} \tag{153}$$

Then $c_2 = 5$, $a_2 = 3$.

We can say the error in the initial data does not alter the domain for which the solution is of Tychonov class.

Q3 Let u and $\tilde{u}$ be the unique Tychonov-class solutions of the above global Cauchy problem, with initial data g and $\tilde{g}$, respectively. We keep the independent variables x and t in the domain where the solutions u and $\tilde{u}$ exist and are unique. Estimate the maximal solution error $\sup_{x,t} |u(x,t) - \tilde{u}(x,t)|$ that can result due to the imperfect initial datum g?

Due to the linearity of the diffusion equation, the difference $z(x,t) = u(x,t) - \tilde{u}(x,t)$ leads to $z_t - D\,z_{xx} \leqslant 0$, but with $g_z(x) = g(x) - \tilde{g}(x) = -e^{-|x|}$.

Since z then is of Tychonov class, Theorem 2.12 shows that $z \in C^\infty(\mathbb{R} \times (0,T])$.

Then, according to Theorem 2.16:

$$\sup_{x,t} |z(x,t)| \leqslant \sup_{x,t} |z(x,0)| = 1 \tag{154}$$

6.5 An Example of Reaction-Diffusion in Dimension $n = 3$

In this section we examine a model of reaction-diffusion in a fissionable material. Although we deal with a greatly simplified model, some interesting implications can be drawn.

By shooting neutrons into an uranium nucleus it may happen that the nucleus breaks into two parts, releasing other neutrons already present in the nucleus and causing a chain reaction. Some macroscopic aspects of this phenomenon can be described by means of an elementary model.

Suppose that a cylinder with height h and radius R is made of a fissionable material of constant density ρ. At a macroscopic level, the free neutrons diffuse like a chemical in a porous medium, with a flux proportional and opposite to its density gradient. In other terms, if $N = N(x,y,z,t)$ is the *neutron density* and no fission occurs, the *flux of neutrons is equal to* $\kappa\,\Delta N$, where κ is a positive constant depending on the material. The mass conservation law then gives $N_t - \kappa\,\Delta N = 0$. When fission occurs at a constant rate $\gamma > 0$, we get the *reaction-diffusion equation* from eq.94:

$$N_t - \kappa\,\Delta N = \gamma\,N \tag{155}$$

We look for bounded solutions satisfying a homogeneous Dirichlet condition on the boundary of the cylinder, with the idea that the density is higher at the center of the cylinder and very low near the boundary. Then, it is reasonable to assume that N has a radial distribution with respect to the axis of the cylinder. More precisely, using the cylindrical coordinates (r,θ,z), of which the definition is in appendix, we write eq.155 to:

$$N_t - \kappa\,[N_{rr} + \frac{1}{r}\,N_r + N_{zz}] = \gamma\,N \tag{156}$$

And, we can write $N = N(r,z,t)$ and the homogeneous Dirichlet condition on the boundary of the cylinder translates into:

$$\begin{aligned} N(R,z,t) = 0\ , \quad 0 < z < h\ , \quad t > 0 \\ N(r,0,t) = N(r,h,t) = 0\ , \quad 0 < r < R\ , \quad t > 0 \end{aligned} \tag{157}$$

Accordingly we prescribe an initial condition

$$N(r,z,0) = N_0(r,z) \tag{158}$$

such that

$$\begin{aligned} N_0(R,z) &= 0, \quad 0 < z < h \\ N_0(r,0) &= N_0(r,h) = 0 \end{aligned} \tag{159}$$

To solve problem of eq.156, eq.157, eq.158, first we get rid of the reaction term by setting:

$$N(r,z,t) = \mathcal{N}(r,z,t)\, e^{\gamma\, t} \tag{160}$$

By substituting eq.160 into eq.156, we get:

$$(e^{\gamma\, t}\,\mathcal{N}_t + \gamma\, e^{\gamma\, t}\,\mathcal{N}) - \kappa\, [e^{\gamma\, t}\,\mathcal{N}_{rr} + \frac{1}{r}\, e^{\gamma\, t}\,\mathcal{N}_r + e^{\gamma\, t}\,\mathcal{N}_{zz}] = \gamma\, e^{\gamma\, t}\,\mathcal{N}$$

$$\text{simplified to} \quad \mathcal{N}_t - \kappa\, [N_{rr} + \frac{1}{r}\,\mathcal{N}_r + \mathcal{N}_{zz}] = 0 \tag{161}$$

with the same initial and boundary conditions of N. So the problem becomes:

$$\begin{cases} \mathcal{N}_t - \kappa\, [\mathcal{N}_{rr} + \frac{1}{r}\,\mathcal{N}_r + \mathcal{N}_{zz}] = 0 & \\ \mathcal{N}_0(R,z) = 0 & , 0 < z < h \\ \mathcal{N}_0(r,0) = \mathcal{N}_0(r,h) = 0 & , 0 < r < R \\ \mathcal{N}(R,z,t) = 0 & , 0 < z < h \ , \ t > 0 \\ \mathcal{N}(r,0,t) = \mathcal{N}(r,h,t) = 0 & , 0 < r < R \ , \ t > 0 \end{cases} \tag{162}$$

By maximum principle, we know that there exists only one solution, continuous up to the boundary of the cylinder. To find an explicit formula for the solution, we use the method of sepa- ration of variables, first searching for bounded solutions of the form:

$$\mathcal{N}(r,z,t) = P(r)\, Z(z)\, T(t) \tag{163}$$

satisfying the homogeneous Dirichlet conditions $u(R) = 0$ and $v(0) = v(h) = 0$.

6.6 Appendix: Cylindrical Coordinates

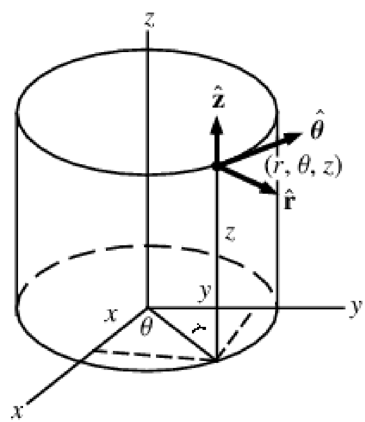


Figure 5: Illustration of the Cylinder Coordinates

$$x = r\,\cos\theta,\, y = r\,\sin\theta,\, z = z \qquad (r > 0,\, 0 \leqslant \theta \leqslant 2\,\pi)$$
$$\mathbf{e}_r = \cos(\theta\,\mathbf{i}) + \sin(\theta\,\mathbf{j}),\, \mathbf{e}_\theta = -\sin(\theta\,\mathbf{i}) + \cos(\theta\,\mathbf{j}),\, \mathbf{e}_z = \mathbf{k}$$

Laplacian:

$$\begin{aligned}\Delta f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}\end{aligned}$$

7 Assignment 2: Diffusion

DTU01418, Introduction to Partial Differential Equations, Fall 2018

7.1 Problem A

$$\begin{cases} u_t - D\,u_{xx} = 0 \\ u(x,0) = 235 - 7\,\sin\frac{\pi\,x}{2\,L} + 4\,\sin\frac{7\,\pi\,x}{2\,L}\ ,\ x \in [0,L] \\ u(0,t) = 235 \\ u_x(L,t) = 0 \end{cases} \tag{164}$$

7.1.1 1/3, Reduction to Homogeneous Boundary Condition

To reduce the problem eq.164 to well posed problem with homogeneous boundary condition, we set:

$$u(x,t) = u^{st}(x) - U(x,t) \tag{165}$$

where $u^{st}(x)$ represents the steady-state solution and $U(x,t)$ represents the transit-state solution.

For steady-state solution $u^{st}(x)$, according to eq.164 and our assumption:

$$\begin{cases} u^{st}_t(x) - D\,u^{st}_{xx}(x) = 0 \\ u^{st}(x) = \mathcal{E}\,x + \mathcal{F} \\ u^{st}(0) = \mathcal{F} = 235 \\ u^{st}_x(L) = \mathcal{E} = 0 \end{cases} \tag{166}$$

So, we get:

$$u^{st}(x) = 235 \tag{167}$$

Then we can get the well posed problem of eq.164 based on $U(x,t)$:

$$U(x,t) = u^{st}(x) - u(x,t) \tag{168}$$

$$U_t = \cancel{u^{st}_t} - u_t \tag{169}$$

$$U_x = u^{st}_x(x) - u_x = 0 - u_x \tag{170}$$

$$U_{xx} = -u_{xx} \tag{171}$$

$$U_t - D\,U_{xx} = -(u_t - D\,u_{xx}) = 0 \tag{172}$$

and

$$U(x,0) = u^{st}(x) - u(x,0) = 7\,\sin\frac{\pi\,x}{2\,L} + 4\,\sin\frac{7\,\pi\,x}{2\,L}\ ,\ x \in [0,L] \tag{173}$$

$$U(0,t) = u^{st}(0) - u(0,t) = 235 - 235 = 0 \tag{174}$$

$$U_x(L,t) = u^{st}_x(L) - u_x(L,t) = 0 - 0 = 0 \tag{175}$$

Summary, we get the well posed diffusion problem with homogeneous boundary condition:

$$\begin{cases} U_t - D\,U_{xx} = 0 \\ U(x,0) = 7\,\sin\frac{\pi\,x}{2\,L} + 4\,\sin\frac{7\,\pi\,x}{2\,L}\ ,\ x \in [0,L] \\ U(0,t) = 0 \\ U_x(L,t) = 0 \end{cases} \tag{176}$$

7.1.2 2/3, Solution by Separation of Variables

We first make the following assumption:

$$U(x,t) = X(x)\,T(t) \tag{177}$$

Then, we get:

$$\begin{aligned} X\,T` - \mathrm{D}X``T &= 0 \\ \frac{T`}{\mathrm{D}\,T} - \frac{X``}{X} &= 0 \\ \frac{1}{\mathrm{D}}\,\frac{T`(t)}{T(t)} = \frac{X``}{X} &= \lambda \quad \text{(constant)} \end{aligned} \tag{178}$$

So, we get Eigen-Value Problem:

$$\begin{cases} X``(x) = \lambda\,X(x) \\ X(0) = 0 \\ X`(L) = 0 \end{cases} \tag{179}$$

and ODE:

$$\begin{cases} T`(t) & = \mathrm{D}\,\lambda\,T(t) \\ T(0) & = 7\,\sin\frac{\pi\,x}{2\,L} + 4\,\sin\frac{7\,\pi\,x}{2\,L} \;\;,\; x \in [0,L] \end{cases} \tag{180}$$

To solve Eigen-Value Problem:
When we assume $\lambda > 0$:

$$\begin{cases} X(x) = \mathcal{C}_1\,e^{\sqrt{\lambda}\,x} + \mathcal{C}_2\,e^{-\sqrt{\lambda}\,x} \\ X`(x) = \mathcal{C}_1\,\sqrt{\lambda}\,e^{\sqrt{\lambda}\,x} - \mathcal{C}_2\,\sqrt{\lambda}\,e^{-\sqrt{\lambda}\,x} \\ X(0) = \mathcal{C}_1 + \mathcal{C}_2 = 0 \quad \text{and } \mathcal{C}_1 \neq 0,\, \mathcal{C}_2 \neq 0 \\ X`(L) = \mathcal{C}_1\,\sqrt{\lambda}\,e^{\sqrt{\lambda}\,L} - \mathcal{C}_2\,\sqrt{\lambda}\,e^{-\sqrt{\lambda}\,L} = 0 \end{cases} \tag{181}$$

So, we can say:

$$\begin{aligned} \mathcal{C}_2(e^{\sqrt{\lambda}\,L} + e^{-\sqrt{\lambda}\,L}) = 0 \\ e^{\sqrt{\lambda}\,L} + e^{-\sqrt{\lambda}\,L} = 0 \end{aligned} \tag{182}$$

Because $e^x > 0$, this is not true, so the assumption of $\lambda > 0$ is wrong.
When we assume $\lambda = 0$:

$$\begin{cases} X``(x) = 0 \\ X`(x) = \mathcal{C}_1 \\ X(x) = \mathcal{C}_1\,x + \mathcal{C}_2 \\ X(0) = \mathcal{C}_2 = 0 \quad \text{and } \mathcal{C}_1 \neq 0 \\ X`(L) = \mathcal{C}_1 = 0 \end{cases} \tag{183}$$

This is not true, so the assumption of $\lambda = 0$ is wrong.
When we assume $\lambda < 0$, we set $\lambda = -\mu^2$, $\mu > 0$:

$$\begin{cases} X(x) = \mathcal{C}_1\,\cos(\mu\,x) + \mathcal{C}_2\,\sin(\mu\,x) \\ X`(x) = -\mu\,\mathcal{C}_1\,\sin(\mu\,x) + \mu\,\mathcal{C}_2\,\cos(\mu\,x) \\ X(0) = \mathcal{C}_1 = 0 \quad \text{and } \mathcal{C}_2 \neq 0 \\ X`(L) = -\mu\,\mathcal{C}_1\,\sin(\mu\,L) + \mu\,\mathcal{C}_2\,\cos(\mu\,L) = 0 \end{cases} \tag{184}$$

So, we can say:

$$\cos(\mu L) = 0 \ , \ \mu L > 0 \tag{185}$$

$$\mu_m L = \frac{\pi}{2} + m\pi \ , \ m = 0, 1, 2, ...$$

$$\mu_m = \frac{(2m+1)\pi}{2L} \ , \ m = 0, 1, 2, ...$$

$$\lambda_m = -\left(\frac{(2m+1)\pi}{2L}\right)^2 \ , \ m = 0, 1, 2, ... \tag{186}$$

So we can get the solution of $X_m(x)$,

$$X_m(x) = \mathcal{C}_2 \sin\left(\frac{(2m+1)\pi}{2L}x\right) \ , \ m = 0, 1, 2, ... \tag{187}$$

And **the general solution of** $T(t)$ is:

$$T_m(t) = \mathcal{B} e^{-\mu_m^2 t} \tag{188}$$

$$= \mathcal{B} \exp\left[-\left(\frac{(2m+1)\pi}{2L}\right)^2 t\right] \ , \ m = 0, 1, 2, ... \tag{189}$$

Then, according to eq.177, the solution of $U(x,t)$:

$$U_m(x,t) = \mathcal{C}_2 \sin\left(\frac{(2m+1)\pi}{2L}x\right) \mathcal{B} \exp\left[-\left(\frac{(2m+1)\pi}{2L}\right)^2 t\right] \ , \ m = 0, 1, 2, ... \tag{190}$$

$$U(x,t) = \sum_{m=0}^{\infty} \mathcal{A}_m \sin\left(\frac{(2m+1)\pi}{2L}x\right) \exp\left[-\left(\frac{(2m+1)\pi}{2L}\right)^2 t\right] \tag{191}$$

Then, according to eq.165 and eq.167:

$$u(x,t) = u^{st}(x) - U(x,t) \tag{192}$$

$$= 235 - \sum_{m=1}^{\infty} \mathcal{A}_m \sin\left(\frac{(2m+1)\pi}{2L}x\right) \exp\left[-\left(\frac{(2m+1)\pi}{2L}\right)^2 t\right] , \tag{193}$$

$$x \in [0, L] \ , \ t > 0 \tag{194}$$

7.1.3 3/3, Constants Identification and Solution Validation

Identify constants via initial condition, and validate the solution.

$$u(x,0) = 235 - \sum_{m=1}^{\infty} \mathcal{A}_m \sin\left(\frac{(2m+1)\pi}{2L}x\right) \tag{195}$$

$$= 235 - 7\sin\frac{\pi x}{2L} + 4\sin\frac{7\pi x}{2L} \ , \ x \in [0, L] \tag{196}$$

So we get:

$$u(x,t) = 235 - \sum_{m=1}^{\infty} \mathcal{A}_m \sin\left(\frac{(2m+1)\pi}{2L}x\right) \exp\left[-\left(\frac{(2m+1)\pi}{2L}\right)^2 t\right] , \tag{197}$$

$$\mathcal{A}_0 = 7, \mathcal{A}_3 = -4, \mathcal{A}_m = 0 \text{ for } m \neq 0 \text{ or } 3 \ ,$$

$$x \in [0, L] \ , \ t > 0$$

7.2 Problem B

Consider the global Cauchy problem:

$$\begin{cases} u_t - u_{xx} = 0 & x \in \mathbb{R}\ ,\ t > 0 \\ u(x,0) = g_u(x) = 3\,\cos x & x \in \mathbb{R} \end{cases} \tag{198}$$

Q1 For what values of x and t (that is, in which domain in the $(x,t) \in \mathbb{R} \times [0,\infty]$ half-plane) are we guaranteed the existence and uniqueness of a Tychonov-class solution of the problem? Explain.

$$D = 2 \tag{199}$$

Assume that there exist positive numbers a and c such that:

$$|g(x)| = 3\,\cos x \leqslant c\,e^{a\,x^2} \qquad \text{for all } x \in \mathbb{R} \tag{200}$$

To look for the minimum value of $p = c\,e^{a\,x^2}$, we differentiate p:

$$p` = 2\,a\,c\,x\,e^{a\,x^2} \tag{201}$$

and we find when $x < 0$, $p` < 0$, when $x = 0$, $p` = 0$, and when $x > 0$, $p` > 0$. So $p(0)$ is the minimum values:

$$\min_{x\in\mathbb{R}} c\,e^{a\,x^2} = c \tag{202}$$

And we know that $g(x) = 3\,\cos x$ reaches its maximum value at $x = 2\,n\,\pi$, and all the maximum values are 3.

Then as long as $c \geqslant 3$, the eq.200 can be satisfied. And $a > 0$ is the only restriction for a. So, the restriction of T is $\frac{1}{4\,D\,a} = \infty$, which means $t \in (0,\infty)$.

So according to Theorem 2.12, for all $(x,t) \in \mathbb{R} \times [0,\infty]$ half-plane, we can guaranteed the existence and uniqueness of a Tychonov-class solution of the problem.

Q2 Let u be the unique Tychonov-class solution of the problem. Write down an expression for u.

$$u(x,t) = \int_{\mathbb{R}} \Gamma_{\mathrm{D}}(x-y,t)\,g(y)\mathrm{d}y \tag{203}$$

$$= \frac{3}{\sqrt{4\,\pi\,\mathrm{D}\,t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\,\mathrm{D}\,t}} \cos y\,\mathrm{d}y \tag{204}$$

Q3 Assume v is the Tychonov-class solution of the global Cauchy problem:

$$\begin{cases} v_t - v_{xx} = 0 & x \in \mathbb{R}\ ,\ t > 0 \\ v(x,0) = g_v(x) = 2\,\cos x & x \in \mathbb{R} \end{cases} \tag{205}$$

Estimate the smallest and the largest possible value attained by the difference $u(x,t) - v(x,t)$ for $x \in \mathbb{R}$ and $t > 0$. Explain how you arrived at your estimates.

Due to the linearity of the diffusion equation, the difference $z(x,t) = u(x,t) - v(x,t)$ leads to $z_t - D\,z_{xx} \leqslant 0$ and $z_t - D\,z_{xx} \geqslant 0$, but with $g_z(x) = g_u(x) - g_v(x) = 3\,\cos x - 2\,\sin x$.

Since z is of Tychonov class, Theorem 2.12 shows that $z \in C^\infty(\mathbb{R} \times (0,T])$.

Then, according to Theorem 2.16:

$$\sup_{x,t}\, z(x,t) \leqslant \sup_{x,t}\, z(x,0) = \sup_{x,t}\,(3\,\cos x - 2\,\sin x) = \sqrt{13} \tag{206}$$

and

$$\inf_{x,t}\, z(x,t) \geqslant \inf_{\mathbb{R}}\, z(x,0) = \inf_{x,t}\,(3\,\cos x - 2\,\sin x) = -\sqrt{13} \tag{207}$$

So $(u(x,t) - v(x,t)) \in [-\sqrt{13}, \sqrt{13}]$ for $x \in \mathbb{R}$ and $t > 0$.

7.3 Problem C

Consider the global Cauchy problem:

$$\begin{cases} u_t - \frac{1}{2}\, u_{xx} = 0 & x \in \mathbb{R}\ ,\ t \in (0,3) \\ u(x,0) = 4\ \exp\left(1/7\, x^2\right) & x \in \mathbb{R} \end{cases} \tag{208}$$

(You do not need to evaluate any integrals occurring in the solution.) Are there other solutions to the above problem, and why? Describe in short how you would evaluate/simplify any integrals occurring in your solution.

Assume that there exist positive numbers a and c such that:

$$|g(x)| \leqslant c\, e^{a\, x^2} \qquad \text{for all } x \in \mathbb{R} \tag{209}$$

We can solve:

$$c = 4 \tag{210}$$

$$a = 1/7 \tag{211}$$

Then, we can get:

$$3 < T < \frac{1}{4\, a\, D} \tag{212}$$

According to theorem 2.12, the following question is the unique solution:

$$u(x,t) = \int_{\mathbb{R}} \Gamma_{\mathrm{D}}(x-y,t)\, g(y)\mathrm{d}y = \frac{1}{\sqrt{4\,\pi\,\mathrm{D}\,t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\,\mathrm{D}\,t}}\, g(y)\, \mathrm{d}y \tag{213}$$

8 The Laplace Equation

8.1 3.1, Introduction

The Laplace equation $\Delta u = 0$ occurs frequently in the applied sciences, in particular in the study of the *steady state phenomena*. Its solutions are called *harmonic functions*.

Assume u is holomorphic, $x + i\,y$:

$$\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)\,u = 0 \tag{214}$$

which means:

$$\bar{\partial} u = 0 \tag{215}$$
$$\partial(\bar{\partial} u) = 0 \tag{216}$$

So, eq.216 becomes:

$$\left(\frac{\partial}{\partial x} + -i\,\frac{\partial}{\partial y}\right)\,\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)\,u = 0 \tag{217}$$
$$\left(i\,\cancel{\frac{\partial}{\partial x}\frac{\partial}{\partial y}} - i\,\cancel{\frac{\partial}{\partial y}\frac{\partial}{\partial x}} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\,u = 0 \tag{218}$$
$$\Delta u = 0 \tag{219}$$

Then, eq.219 becomes:

$$\Delta(\Re\,u + i\,\Im\,u) = 0 \tag{220}$$
$$\Delta(\Re\,u) + i\,\Delta(\Im\,u) = 0$$
$$\Delta(\Re\,u) = 0\ ,\ \Delta(\Im\,u) = 0 \tag{221}$$

Slightly more generally, Poisson's equation $\Delta u = f$ plays an important role in the theory of conservative fields (electrical, magnetic, gravitational, ...), where the vector field is derived from the gradient of a potential. So, if $\vec{u} = -\nabla\varphi$, then:

$$\mathrm{div}\vec{u} = -\mathrm{div}\nabla\varphi = -\Delta\varphi \tag{222}$$
$$\Delta\varphi = -\mathrm{div}\vec{u} = f \tag{223}$$

where $\vec{u}$ is a conservative field and φ is the associated potential.

8.2 3.2, Well Posed Problems. Uniqueness

Consider the Poisson equation:

$$\Delta u = f \quad \text{ in } \Omega \tag{224}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, which means Ω is an open simply connected subset of $\mathbb{R}^n$.

The well posed problems associated with eq.224 are the stationary counterparts of the corresponding problems for the diffusion equation. Clearly here there is no initial condition. On the boundary $\partial\Omega$ we may assign:

Dirichlet data:	$u\vert\partial\,\Omega = g$	(225)
or Neumann data:	$\partial_{\vec{v}}\,u\vert\partial\,\Omega = h$	(226)
or Robin (radiation) condition:	$(\partial_{\vec{v}}\,u + \alpha\,u)\vert\partial\,\Omega = h \quad (\alpha > 0)$	(227)
or mixed condition:	$u\vert\partial\,\Omega_D = g\ ,\ \partial_{\vec{v}}\,u\vert\partial\,\Omega_N = h$	(228)

where $\vec{v}$ is the outward normal unit vector to $\partial\,\Omega$, besides, Ω_D and Ω_N are relatively open regular subsets of $\partial\,\Omega$, $\overline{\partial\,\Omega_D} \cup \overline{\partial\,\Omega_N} = \partial\,\Omega$ (the overline means including the boundary), $\Omega_D \cap \Omega_N = \varnothing$.

When $g = h = 0$, we say that the above boundary conditions are *homogeneous.*

To prove the uniqueness of the result of Possion equation, we assume u and v both solves the above problem. Then, we get:

$$\begin{cases} \Delta u = f \quad \text{ in } \Omega \\ u|\partial\,\Omega = g \quad \text{, which is Dirichlet data, or} \\ \partial_{\vec{v}}\, u|\partial\,\Omega = h \quad \text{, which is Neumann data, or} \\ (\partial_{\vec{v}}\, u + \alpha\, u)|\partial\,\Omega = 0 \quad (\alpha > 0) \quad \text{, which is Robin (radiation) condition, or} \\ u|\partial\,\Omega_D = g \text{ , } \partial_{\vec{v}}\, u|\partial\,\Omega_N = h \quad \text{, which is mixed condition} \end{cases} \tag{229}$$

$$\begin{cases} \Delta v = f \quad \text{ in } \Omega \\ v|\partial\,\Omega = g \quad \text{, which is Dirichlet data, or} \\ \partial_{\vec{v}}\, v|\partial\,\Omega = h \quad \text{, which is Neumann data, or} \\ (\partial_{\vec{v}}\, v + \alpha\, v)|\partial\,\Omega = 0 \quad (\alpha > 0) \quad \text{, which is Robin (radiation) condition, or} \\ v|\partial\,\Omega_D = g \text{ , } \partial_{\vec{v}}\, v|\partial\,\Omega_N = h \quad \text{, which is mixed condition} \end{cases} \tag{230}$$

Define $w = u - v$, we get:

$$\begin{cases} \Delta w = 0 \quad \text{ in } \Omega \\ w|\partial\,\Omega = 0 \quad \text{, which is Dirichlet data, or} \\ \partial_{\vec{v}}\, w|\partial\,\Omega = 0 \quad \text{, which is Neumann data, or} \\ (\partial_{\vec{v}}\, w + \alpha\, w)|\partial\,\Omega = 0 \quad (\alpha > 0) \quad \text{, which is Robin (radiation) condition, or} \\ w|\partial\,\Omega_D = 0 \text{ , } \partial_{\vec{v}}\, w|\partial\,\Omega_N = 0 \quad \text{, which is mixed condition} \end{cases} \tag{231}$$

From eq.231, we can get:

$$\int_\Omega \Delta w = 0 \tag{232}$$

According to Green's First Identity (1.13), we can write eq.232 to:

$$\int_\Omega w\, \Delta w \,\mathrm{d}\vec{x} = 0 \tag{233}$$

$$\int_{\partial\,\Omega} w\, \partial_{\vec{v}}\, w \,\mathrm{d}\vec{\sigma} - \int_\Omega \nabla w\, \nabla w \,\mathrm{d}\vec{x} = 0$$

$$\int_{\partial\,\Omega} w\, \partial_{\vec{v}}\, w \,\mathrm{d}\vec{\sigma} - \int_\Omega |\nabla w|^2 \,\mathrm{d}\vec{x} = 0 \tag{234}$$

For Dirichlet, Neumann and mixed boundary condition, eq.234 becomes:

$$\int_\Omega |\nabla w|^2 \,\mathrm{d}\vec{x} = 0 \tag{235}$$

$$\nabla w = 0 \quad \text{ in } \Omega \tag{236}$$

$$w = u - v = \textit{constant} \quad \text{ in } \Omega \tag{237}$$

For Neumann condition, two solutions u and v differ by a constant, which is **Theorem 3.1**. □
For Dirichlet and mixed boundary condition, according to eq.237, we can say:

$$w = 0 \quad \text{ in } \Omega \tag{238}$$

For Robin boundary condition in eq.231, we can get:

$$\partial_{\vec{v}}\, w|\partial\,\Omega = -\alpha\, w|\partial\,\Omega \quad (\alpha > 0) \tag{239}$$

Then, we substitute eq.239 in to eq.234:

$$-\alpha \int_{\partial\,\Omega} w^2 - \int_{\Omega} |\nabla w|^2 = 0 \tag{240}$$

And we can say:

$$0 \leqslant \int_{\Omega} |\nabla w|^2 = -\alpha \int_{\partial\,\Omega} w^2 \leqslant 0 \tag{241}$$

so, we can get the following two conclusions:

$$\int_{\Omega} |\nabla w|^2 = 0 \tag{242}$$

$$-\alpha \int_{\partial\,\Omega} w^2 = 0 \tag{243}$$

Eq.242 indicates:

$$\nabla w = 0 \quad \text{ in } \Omega \tag{244}$$

$$w = \text{constant} \quad \text{ in } \Omega \tag{245}$$

Additionally, Robin boundary condition $(\partial_{\vec{w}}\, w + \alpha\, w)|\partial\,\Omega = 0 \quad (\alpha > 0)$ means:

$$\lim_{x \to \partial\,\Omega} (\partial_{\vec{v}}\, w(x) + \alpha\, w(x)) = 0 \tag{246}$$

$$\lim_{x \to \partial\,\Omega} \alpha\, w = \alpha\, w = 0 \tag{247}$$

$$w = 0 \tag{248}$$

□

Remark 3.2 Consider the Neumann problem. Integrate the equation on Ω, and using Gauss' formula, we find:

$$\int_{\Omega} f \mathrm{d}\vec{x} = \int_{\partial\Omega} h\, \mathrm{d}\vec{\sigma} \tag{249}$$

The relation eq.249 appears as a *compatibility* condition on the data f and h, that has *necessarily* to be satisfied in order for the Neumann problem to admit a solution. Thus, when having to solve a Neumann problem, the first thing to do is to check the validity of eq.249. If it does not hold, the problem does not have any solution. We will examine later the physical meaning of eq.249.

Interpretation: for a steady state to be possible, the heat production in Ω (due to f) must be perfectly balanced with the heat outflow from Ω (due to h). □

8.3 3.3, Harmonic Functions

8.3.1 3.3.2, Proof of Mean Value Properties

Theorem 3.4 Let u be harmonic in $\Omega \subseteq \mathcal{R}^n$. Then, for any ball $B_R(\vec{x}) \subset\subset \Omega$, the following mean value formulas hold:

$$u(\vec{x}) = \frac{n}{\omega_n\, R^n} \int_{B_R(\vec{x})} u(\vec{y})\, \mathrm{d}\vec{y} \tag{250}$$

$$u(\vec{x}) = \frac{1}{\omega_n\, R^{n-1}} \int_{\partial B_R(\vec{x})} u(\vec{\sigma})\, \mathrm{d}\sigma \tag{251}$$

where σ_n is the surface measure of ∂B_1. □

Proof Let us start from the second formula. For $r < R$ define:

$$g(r) = \frac{1}{\omega_n\, r^{n-1}} \int_{\partial B_r(\vec{x})} u(\vec{\sigma})\, \mathrm{d}\sigma \tag{252}$$

Perform the change of variables $\vec{\sigma} = \vec{x} + r\,\vec{\sigma}'$. Then:

$$\vec{\sigma}' \in \partial B_1(\vec{0})\ ,\ \mathrm{d}\sigma = r^{n-1}\, \mathrm{d}\sigma' \tag{253}$$

$$g(r) = \frac{1}{\omega_n} \int_{\partial B_1(\vec{0})} u(\vec{x} + r\,\vec{\sigma}')\, \mathrm{d}\sigma' \tag{254}$$

Let $v(\vec{y}) = u(\vec{x} + r\,\vec{y})$ and observe that:

$$\nabla v(\vec{y}) = r\,\nabla u(\vec{x} + r\,\vec{y}) \tag{255}$$

$$\Delta v(\vec{y}) = r^2\,\Delta u(\vec{x} + r\,\vec{y}) \tag{256}$$

Then we have:

$$\begin{aligned}
g'(r) &= \frac{1}{\omega_n} \int_{\partial B_1(\vec{0})} \frac{\mathrm{d}}{\mathrm{d}r}\, u(\vec{x} + r\,\vec{\sigma}')\, \mathrm{d}\sigma' && (257)\\
&= \frac{1}{\omega_n} \int_{\partial B_1(\vec{0})} \nabla\, u(\vec{x} + r\,\vec{\sigma}')\, \vec{\sigma}'\, \mathrm{d}\sigma' && (258)\\
&= \frac{1}{\omega_n\, r} \int_{\partial B_1(\vec{0})} \nabla\, v(\vec{\sigma}')\, \vec{\sigma}'\, \mathrm{d}\sigma' && (259)\\
&= \frac{1}{\omega_n\, r} \int_{\partial B_1(\vec{0})} \Delta\, v(\vec{y})\, \mathrm{d}\vec{y} \quad \text{(divergence theorem)} && (260)\\
&= \frac{r}{\omega_n} \int_{\partial B_1(\vec{0})} \Delta\, u(\vec{x} + r\,\vec{y})\, \mathrm{d}\vec{y} && (261)
\end{aligned}$$

Thus, g is constant and since $g(r) \to u(\vec{x})$ for $r \to 0$, we get eq.251.

To obtain eq.250, let $R = r$ in eq.251, multiply by r and integrate both sides between 0 and R. We find:

$$\begin{aligned}
\frac{R^n}{n}\, u(\vec{x}) &= \frac{1}{\omega_n} \int_0^R \mathrm{d}r \int_{\partial B_r(\vec{x})} u(\vec{\sigma})\, \mathrm{d}\sigma && (262)\\
&= \frac{1}{\omega_n} \int_{B_R(\vec{x})} u(\vec{y})\, \mathrm{d}\vec{y} && (263)
\end{aligned}$$

from which eq.250 follows. □

8.3.2 3.3.3, Maximum Principles

P124, Salsa (2016).

8.4 3.4, A Probabilistic Solution of the Dirichlet Problem

Theorem 3.19 Let Ω be a bounded Lipschitz domain and $g \in C(\partial\Omega)$. The unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of Possion equation with Dirichlet boundary condition is given by:

$$u(\vec{x}) = E^{\vec{x}}[g(\vec{X}(\tau))] = \int_{\partial\Omega} g(\vec{\sigma})\, P(\vec{x}, \tau(\vec{x}), \mathrm{d}\sigma) \tag{264}$$

8.5 3.6, Fundamental Solution and Newtonian Potential

8.5.1 3.6.1, The Fundamental Solution

Eq.264 is not the only representation formula for the solution of the Dirichlet problem. We shall derive deterministic formulas involving various types of potentials, constructed using a special function, called the fundamental solution of the Laplace operator.

As we did for the diffusion equation, let us look at the invariance properties characterizing the operator Δ: the invariances by translations and by rotations.

Let $u = u(x)$ be harmonic in $\mathbb{R}^n$. **Invariance by translations** means that the function $v(\vec{x}) = u(\vec{x} - \vec{y})$, for each fixed $\vec{y}$, is also harmonic, as it is immediate to check.

Invariance by rotations means that, given a rotation in $\mathbb{R}^n$, represented by an orthogonal matrix $\mathbf{M}$ (i.e. $\mathbf{M}^\top = \mathbf{M}^1$), also $v(\vec{x}) = u(\mathbf{M}\,\vec{x})$ is harmonic in $\mathbb{R}^n$. To check it, observe that, if we denote by D^2u the Hessian of u, we have:

$$\Delta u = \mathrm{Tr} D^2 u = \text{trace of the Hessian of } u \tag{265}$$

Since

$$D^2 v(\vec{x}) = \mathbf{M}^\top\, D^2 u(\mathbf{M}\vec{x})\, \mathbf{M} \tag{266}$$

and $\mathbf{M}$ is orthogonal, we have:

$$\begin{aligned} \Delta v(\vec{x}) &= \mathrm{Tr}[\mathbf{M}^\top\, D^2 u(\mathbf{M}\vec{x})\, \mathbf{M}] \\ &= \mathrm{Tr} D^2 u(\mathbf{M}\vec{x}) \\ &= \Delta u(\mathbf{M}\vec{x}) = 0 \end{aligned} \tag{267}$$

and therefore v is harmonic.

Now, a typical rotationally invariant quantity is *the distance function from a point*, for instance from the origin, that is $r = |\vec{x}|$. Thus, let us look for *radially symmetric* harmonic functions $u = u(r)$.

Consider first $n = 2$; using polar coordinates and recalling (3.23), we find:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r} = 0 \tag{268}$$

so that:

$$u(r) = C\,\log r + C_1 \tag{269}$$

In dimension $n = 3$ using spherical coordinates (r, ψ, θ), $r > 0$, $0 < \psi < \pi$, $0 < \theta < 2\pi$, the operator Δ has the following expression[3]:

$$\Delta = \underbrace{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\,\frac{\partial}{\partial r}}_{\text{radial part}} + \frac{1}{r}\,\underbrace{\left\{\frac{1}{(\sin\psi)^2}\,\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \psi^2} + \cot\psi\,\frac{\partial}{\partial \psi}\right\}}_{spherical part (Laplace-Beltrami operator)} \tag{270}$$

The Laplace equation for $u = u(r)$ becomes:

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\,\frac{\partial u}{\partial r} = 0 \tag{271}$$

whose general integral is:

$$u(r) = \frac{C}{r} + C_1 \quad , C,\, C_1 \text{ arbitrary constants} \tag{272}$$

[3] Appendix D

Choose $C_1 = 0$ and $C = \frac{1}{4\pi}$ if $n = 3$, $C = \frac{1}{2\pi}$ if $n = 2$. The function:

$$\Phi(\vec{x}) = \begin{cases} -\frac{1}{2\pi} \log |\vec{x}| & n = 2 \\ \frac{1}{4\pi\,|\vec{x}|} & n = 3 \end{cases} \tag{273}$$

is called the **fundamental solution** for the Laplace operator Δ. As we shall prove in Chap. 7, the above choice of the constant C is made in order to have:

$$\Delta\Phi(\vec{x}) = -\delta_n(\vec{x}) \tag{274}$$

where $\delta_n(\vec{x})$ denotes the $n$dimensional Dirac measure at $\vec{x} = \vec{0}$.

8.6 3.7, The Green Function

P155, Salsa (2016).

9 Assignment 3: The Laplace Equation

DTU01418, Introduction to Partial Differential Equations, Fall 2018

9.1 Problem A

To prove the above statement, we firstly set $w(\vec{x}) = u_1(\vec{x}) - u_2(\vec{x})$, so we have to prove there is a point $\vec{O} \in \Omega$, so that $u_1(\vec{O}) - u_2(\vec{O}) = 0$.

Let's make the first assumption that $\vec{p}$ is a minimum point for $w(\vec{x})$, which means:

$$w(\vec{p}) \leqslant u(\vec{y}) \qquad , \forall \vec{y} \in \Omega \tag{275}$$

Let $\vec{q}$ be another arbitrary point in Ω. Since Ω is connected, it is possible to find a finite sequence of balls $B(\vec{x}_j) \subset\subset \Omega$, $j = 0, ..., N$, such that:

$$\vec{x}_j \in B(\vec{x}_{j-1}) \ , \ j = 0, ..., N \tag{276}$$

$$\vec{x}_0 = \vec{p} \ , \ \vec{x}_N = \vec{q} \tag{277}$$

The mean value property gives:

$$u_1(\vec{p}) = \frac{1}{|B(\vec{p})|} \int_{B(\vec{p})} u_1(\vec{y}) \, \mathrm{d}\vec{y} \tag{278}$$

$$u_2(\vec{p}) = \frac{1}{|B(\vec{p})|} \int_{B(\vec{p})} u_2(\vec{y}) \, \mathrm{d}\vec{y} \tag{279}$$

So,

$$w(\vec{p}) = \frac{1}{|B(\vec{p})|} \int_{B(\vec{p})} [u_1(\vec{y}) - u_2(\vec{y})] \, \mathrm{d}\vec{y} \tag{280}$$

Let's make a second assumption that there exists $\vec{z} \in B(\vec{p})$ such that:

$$w(\vec{z}) > w(\vec{p}) \tag{281}$$

which, means:

$$u_1(\vec{z}) - u_2(\vec{z}) > u_1(\vec{p}) - u_2(\vec{p}) \tag{282}$$

Then, given a circle $B_r(\vec{z}) \subset B(\vec{p})$, we can write:

$$u_1(\vec{p}) = \frac{1}{|B(\vec{p})|} \left\{ \int_{B(\vec{p}) \setminus B_r(\vec{z})} u_1(\vec{y}) \, \mathrm{d}\vec{y} + \int_{B_r(\vec{z})} u_1(\vec{y}) \, \mathrm{d}\vec{y} \right\} \tag{283}$$

$$u_2(\vec{p}) = \frac{1}{|B(\vec{p})|} \left\{ \int_{B(\vec{p}) \setminus B_r(\vec{z})} u_2(\vec{y}) \, \mathrm{d}\vec{y} + \int_{B_r(\vec{z})} u_2(\vec{y}) \, \mathrm{d}\vec{y} \right\} \tag{284}$$

and

$$w(\vec{p}) = \frac{1}{|B(\vec{p})|} \left\{ \left(\int_{B(\vec{p}) \setminus B_r(\vec{z})} u_1(\vec{y}) \, \mathrm{d}\vec{y} + \int_{B_r(\vec{z})} u_1(\vec{y}) \, \mathrm{d}\vec{y} \right) - \left(\int_{B(\vec{p}) \setminus B_r(\vec{z})} u_2(\vec{y}) \, \mathrm{d}\vec{y} + \int_{B_r(\vec{z})} u_2(\vec{y}) \, \mathrm{d}\vec{y} \right) \right\} \tag{285}$$

$$w(\vec{p}) = \frac{1}{|B(\vec{p})|} \left\{ \int_{B(\vec{p}) \setminus B_r(\vec{z})} [u_1(\vec{y}) - u_2(\vec{y})] \, \mathrm{d}\vec{y} + \int_{B_r(\vec{z})} [u_1(\vec{y}) - u_2(\vec{y})] \, \mathrm{d}\vec{y} \right\} \tag{286}$$

Since $w(\vec{y}) \geqslant w(\vec{p})$ for every y

$$w(\vec{z}) \, |B_r(\vec{z})| > w(\vec{p}) \, |B_r(\vec{z})| \tag{287}$$

and, by the mean value again, we can get:

$$u_1(\vec{z}) = \frac{1}{|B_r(\vec{z})|} \int_{B_r(\vec{z})} u_1(\vec{y})\,\mathrm{d}\vec{y} \tag{288}$$

$$u_2(\vec{z}) = \frac{1}{|B_r(\vec{z})|} \int_{B_r(\vec{z})} u_2(\vec{y})\,\mathrm{d}\vec{y} \tag{289}$$

so that:

$$\int_{B_r(\vec{z})} [u_1(\vec{y}) - u_2(\vec{y})]\,\mathrm{d}\vec{y} = w(\vec{z})\,|B_r(\vec{z})| \tag{290}$$

which means:

$$\int_{B_r(\vec{z})} [u_1(\vec{y}) - u_2(\vec{y})]\,\mathrm{d}\vec{y} > w(\vec{p})\,|B_r(\vec{z})| \tag{291}$$

From eq.12, we can get:

$$\int_{B(\vec{p})\backslash B_r(\vec{z})} [u_1(\vec{y}) - u_2(\vec{y})]\,\mathrm{d}\vec{y} + \int_{B_r(\vec{z})} [u_1(\vec{y}) - u_2(\vec{y})]\,\mathrm{d}\vec{y} > \int_{B(\vec{p})\backslash B_r(\vec{z})} [u_1(\vec{y}) - u_2(\vec{y})]\,\mathrm{d}\vec{y} + w(\vec{p})\,|B_r(\vec{z})| \tag{292}$$

$$w(\vec{p})\,|B(\vec{p})| > \int_{B(\vec{p})\backslash B_r(\vec{z})} [u_1(\vec{y}) - u_2(\vec{y})]\,\mathrm{d}\vec{y} + w(\vec{p})\,|B_r(\vec{z})| \tag{293}$$

$$w(\vec{p})\,|B(\vec{p}) - B_r(\vec{z})| > \int_{B(\vec{p})\backslash B_r(\vec{z})} [u_1(\vec{y}) - u_2(\vec{y})]\,\mathrm{d}\vec{y} \tag{294}$$

$$\frac{|B(\vec{p}) - B_r(\vec{z})|}{|B(\vec{p})|} \int_{B(\vec{p})} [u_1(\vec{y}) - u_2(\vec{y})]\,\mathrm{d}\vec{y} > \int_{B(\vec{p})\backslash B_r(\vec{z})} [u_1(\vec{y}) - u_2(\vec{y})]\,\mathrm{d}\vec{y} \tag{295}$$

which is not true, so we must say the second assumption is not true.

And this means $w \equiv w(\vec{p})$ in $B(\vec{p})$ and in particular $w(\vec{x}_1) = w(\vec{p})$. We repeat now the same argument with x_1 in place of $\vec{p}$ to show that $w \equiv w(\vec{p})$ in $B(\vec{x}_1)$ and in particular $w(\vec{x}_2) = w(\vec{p})$. Iterating the procedure we eventually deduce that $w(\vec{x}_N) = w(\vec{p})$. Since $\vec{q}$ is an arbitrary point of Ω, we conclude that $w \equiv w(\vec{p})$ in Ω.

So we must say that the first assumption is not true. For the same reason, we can prove that the assumption of a existing maximum point for w in Ω. Thus the maximum point and minimum point of w must exist at $\partial\Omega$. In other words:

$$w(\vec{x}) < \max_{\partial\Omega} w \tag{296}$$

$$w(\vec{x}) > \min_{\partial\Omega} w \tag{297}$$

And we know:

$$w = u_1 - u_2 = -1 \qquad \text{on } \partial\Omega \tag{298}$$

so that:

$$-1 \leqslant w(\vec{x}) \leqslant -1 \qquad \text{in } \overline{\Omega} \tag{299}$$

$$w(\vec{x}) = -1 \qquad \text{in } \overline{\Omega} \tag{300}$$

So we must there is no point $\vec{x} \in \Omega$ such that $u_1(\vec{x}) = u_2(\vec{x})$.

9.2 Problem B

Let $\partial B_R(\vec{0})$ be the open disk in the plane $\mathbf{R}^2$, with radius $R > 0$ and centered at the origin, and write for the unit outward normal to the boundary $\partial B_R(\vec{0})$ of the disk. Also, let (r, θ) be the polar coordinates in $\mathbf{R}^2$. Is there a (smooth) solution u of the Poisson problem:

$$\begin{cases} \Delta u & = r \quad \text{in } B_R(\vec{0}) \\ \partial_{\vec{v}} u & = 8/3 \quad \text{on } \partial B_R(\vec{0}) \end{cases} \tag{301}$$

if $R = 3$? How about if $R = \sqrt{8}$? Explain.

According to Remark 3.2, if there is a solution for this problem, the compatibility condition between r and $8/3$ must be satisfies, which is:

$$\int_\Omega f \,\mathrm{d}\vec{x} = \int_{\partial\Omega} h \mathrm{d}\sigma \tag{302}$$

When in polar coordinates, we have $x = r\,\cos\theta$, $y = r\,\sin\theta$, and we can get:

$$\mathrm{d}\vec{x} = r\,\mathrm{d}r\,\mathrm{d}\theta \tag{303}$$

$$\mathrm{d}\sigma = R\,\mathrm{d}\phi \tag{304}$$

So

$$\int_\Omega r\,r\,\mathrm{d}r\,\mathrm{d}\theta = \int_{\partial\Omega} 8/3\,R\,\mathrm{d}\phi \tag{305}$$

$$\int_0^{2\,\pi} (1/3\,r^3|_0^R)\,\mathrm{d}\theta = \int_0^{2\,\pi} 8/3\,R\,\mathrm{d}\phi \tag{306}$$

$$16/3\,\pi\,R = 2/3\,\pi\,R^3 \tag{307}$$

So we can get $R = \sqrt{8}$ can satisfy the compatibility condition, that is when $R = \sqrt{8}$, we can solve the Poisson problem with Neumann boundary condition.

9.3 Problem C

If u_1 and u_2 are (smooth) functions satisfying:

$$\begin{cases} \Delta u_1 = r^2/3 + 2 & \text{in } B_R(\vec{0}) \\ u_1 = \sin\theta & \text{on } \partial B_R(\vec{0}) \end{cases} \quad \begin{cases} \Delta u_2 = r^2/3 + 2 & \text{in } B_R(\vec{0}) \\ u_2 = \cos\theta & \text{on } \partial B_R(\vec{0}) \end{cases} \tag{308}$$

Let $w = u_1 - u_2$, then w is harmonic, which satisfies:

$$\begin{cases} \Delta w = 0 & \text{in } B_R(\vec{0}) \\ w = \sin\theta - \cos\theta & \text{on } \partial B_R(\vec{0}) \end{cases} \tag{309}$$

According to stability characteristics of Poisson problem with Dirichlet boundary condition in corollary 3.8, we know:

$$|w| \leqslant \max_{\partial\Omega} |\sin\theta - \cos\theta| \tag{310}$$

And we know (Always remember to write the whole process):

$$\begin{aligned} \max_{\partial\Omega} |\sin\theta - \cos\theta| &= \max_{\partial\Omega} |\sqrt{1 - \sin 2\,\theta}| \\ &= \sqrt{2} \end{aligned} \tag{311}$$

so that:

$$|u_1 - u_2| \leqslant \sqrt{2} \tag{312}$$

and the maximum value of $|u_1 - u_2|$ is $\sqrt{2}$.

Be careful. The maximum principle says that the max of a nonconstant harmonic function is attained on the boundary, which, in this case, doesn't belong to $B_R(\vec{0})$. i.e. $\sqrt{(2)}$ is the least upper bound, for $|u_1 - u_2|$.

9.4 Problem D

Write an integral representation formula for a solution of the boundary problem 301. If your formula involves a Green's function or a Neumann function, you do not need to find these functions explicitly. Also, you do not need to evaluate any integrals in your formula explicitly. However, explain in detail all components of your integral representation formula.

The equation is:

$$\begin{cases} \Delta u & = r \quad \text{ in } B_R(\vec{0}) \\ \partial_{\vec{v}}\, u & = 8/3 \quad \text{ on } \partial B_R(\vec{0}) \end{cases}$$

We can transform the problem to that in polar coordinate first:

$$\begin{cases} U_{rr} + \frac{1}{r}\, U_r + \frac{1}{r^2}\, U_{\theta\theta} = r & , 0 < r < \sqrt{8}\ , 0 \leqslant \theta \leqslant 2\,\pi \\ U_r(R, \theta) = 8/3 & , 0 \leqslant \theta \leqslant 2\,\pi \end{cases} \tag{313}$$

According to theorem 3.39, the solution of the Neumann problem can written as:

$$u = \underbrace{\frac{1}{|\partial\Omega|} \int_{\partial\Omega} u(\vec{\sigma})\, \mathrm{d}\sigma}_{\text{mean value of } u} + \underbrace{\int_{\partial\Omega} h(\vec{\sigma})\, N(\vec{x}, \vec{\sigma})\, \mathrm{d}\sigma}_{\text{single layer potential of } \partial_{\vec{v}}\, u} - \underbrace{\int_{\Omega} f(\vec{y})\, N(\vec{x}, \vec{y})\, \mathrm{d}\vec{y}}_{\text{double layer potential of } u} \tag{314}$$

$$= \frac{1}{2\,\pi\, R} \int_{\partial\Omega} R\, u\, \mathrm{d}\phi + \int_{\partial\Omega} R\, 8/3\, N(\vec{x}, \vec{\sigma})\, \mathrm{d}\phi - \int_{\Omega} r^2\, N(\vec{x}, \vec{y})\, \mathrm{d}r\, \mathrm{d}\theta \tag{315}$$

$$= 8/3 + 16\sqrt{2}/3 \int_{\partial\Omega} N(\vec{x}, \vec{\sigma})\, \mathrm{d}\phi - \int_{\Omega} r^2\, N(\vec{x}, \vec{y})\, \mathrm{d}r\, \mathrm{d}\theta \tag{316}$$

in which the Neumann function:

$$N(\vec{x}, \vec{\sigma}) = \Phi(\vec{x} - \vec{\sigma}) - \psi(\vec{x}, \vec{\sigma}) \tag{317}$$

$$= -\frac{1}{2\,\pi}\, \log \sqrt{8 - r^2} - \psi(\vec{x}, \vec{\sigma}) \tag{318}$$

$$N(\vec{x}, \vec{y}) = \Phi(\vec{x} - \vec{y}) - \psi(\vec{x}, \vec{y}) \tag{319}$$

$$= -\frac{1}{2\,\pi}\, \log \sqrt{|r_1^2 - r_2^2|} - \psi(\vec{x}, \vec{y}) \tag{320}$$

And the function $\Phi(\vec{x})$:

$$\Phi(\vec{x}) = \begin{cases} -\frac{1}{2\pi}\, \log |\vec{x}| & n = 2 \\ \frac{1}{4\pi\, |\vec{x}|} & n = 3 \end{cases} \tag{321}$$

is called the **fundamental solution** for the Laplace operator Δ.

10 Waves and Vibrations

10.1 Non-homogeneous Equation, Duhamel's Method

To solve the non-homogeneous problem:

$$\begin{cases} u_{tt} - c^2\, u_{xx} = f(x,t) & x \in \mathcal{R}\ ,\ t > 0 \\ u(x,0) = 0\ ,\ u_t(x,0) = 0 & x \in \mathcal{R} \end{cases} \tag{322}$$

we use the Duhamel's method (see Subsect. 2.2.8). For $s \geqslant 0$ fixed, let $w = w(x,t;s)$ be the solution of problem:

$$\begin{cases} w^s_{tt} - c^2\, w^s_{xx} = 0 \quad x \in \mathcal{R}\ ,\ t > 0 \\ w^s(x,s) = 0 \quad x \in \mathcal{R} \\ w^s_t(x,s) = f(x,s) \end{cases} \tag{323}$$

Since the wave equation is invariant under (time) translations, from the d'Alembert formula, we get:

$$w^s(x,t) = \frac{1}{2\,c} \int_{x-c\,(t-s)}^{x+c\,(t-s)} f(y,s)\,\mathrm{d}y \tag{324}$$

Then, the solution of eq.322 is given by:

$$u(x,t) = \int_{s=0}^{t} w^s(x,t)\,\mathrm{d}s \tag{325}$$

$$= \frac{1}{2\,c} \int_{s=0}^{t} \mathrm{d}s \int_{x-c\,(t-s)}^{x+c\,(t-s)} f(y,s)\,\mathrm{d}y \tag{326}$$

$$= \frac{1}{2\,c} \int_{S_{x,t}} f(y,s)\,\mathrm{d}y\,\mathrm{d}s \tag{327}$$

where $S_{x,t}$ is the triangular sector in fig. 5.5. In fact,

$$u(x,0) = \int_{s=0}^{0} w^s(x,0)\,\mathrm{d}s = 0 \tag{328}$$

and

$$u_t(x,t) = \lim_{\Delta t\to 0} \frac{1}{\Delta t} \left[\int_{s=0}^{t+\Delta t} w^s(x,t+\Delta t)\,\mathrm{d}s - \int_{s=0}^{t} w^s(x,t)\,\mathrm{d}s\right] \tag{329}$$

$$= \lim_{\Delta t\to 0} \frac{1}{\Delta t} \left[\int_{s=0}^{t} w^s(x,t+\Delta t)\,\mathrm{d}s + \int_{s=t}^{t+\Delta t} w^s(x,t+\Delta t)\,\mathrm{d}s - \int_{s=0}^{t} w^s(x,t)\,\mathrm{d}s\right] \tag{330}$$

$$= \lim_{\Delta t\to 0} \frac{1}{\Delta t} \left[\int_{s=0}^{t} w^s(x,t+\Delta t)\,\mathrm{d}s - \int_{s=0}^{t} w^s(x,t)\,\mathrm{d}s\right] + \lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{s=t}^{t+\Delta t} w^s(x,t+\Delta t)\,\mathrm{d}s \tag{331}$$

$$= \lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{s=0}^{t} \left[w^s(x,t+\Delta t)\,\mathrm{d}s - w^s(x,t)\right]\,\mathrm{d}s + \lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{s=t}^{t+\Delta t} w^s(x,t+\Delta t)\,\mathrm{d}s \tag{332}$$

$$= \int_{s=0}^{t} \lim_{\Delta t\to 0} \frac{1}{\Delta t} \left[w^s(x,t+\Delta t)\mathrm{d}s - w^s(x,t)\right]\,\mathrm{d}s + \lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{s=t}^{t+\Delta t} w^s(x,t)\,\mathrm{d}s \tag{333}$$

$$= \int_{s=0}^{t} w_t^s(x,t)\,\mathrm{d}s + \lim_{\Delta t\to 0} \frac{1}{\Delta t}\, w^t(x,t) \int_{s=t}^{t+\Delta t} \mathrm{d}s \tag{334}$$

$$= \int_{s=0}^{t} w_t^s(x,t)\,\mathrm{d}s + w^t(x,t) \tag{335}$$

$$= \int_{s=0}^{t} w_t^s(x,t)\,\mathrm{d}s \tag{336}$$

since $w^t(x,t) = 0$.

Thus

$$u_t(x,0) = \int_{s=0}^{0} w_t^s(x,0)\mathrm{d}s = 0 \tag{337}$$

Moreover,

$$u_{tt}(x,t) = \int_{s=0}^{t} w_{tt}^s(x,t)\,\mathrm{d}s + w_t^t(x,t) \tag{338}$$

$$= \int_{s=0}^{t} w_{tt}^s(x,t)\,\mathrm{d}s + f(x,t) \tag{339}$$

$$u_{xx}(x,t) = \frac{\partial^2}{\partial x^2} \int_{s=0}^{t} w^s(x,t)\,\mathrm{d}s \tag{340}$$

$$= \int_{s=0}^{t} w_{xx}^s(x,t)\,\mathrm{d}s \tag{341}$$

Therefore,

$$u_{tt} - c^2\,u_{xx} = f(x,t) + \int_{s=0}^{t} w_{tt}^s(x,t)\,\mathrm{d}s - c^2 \int_{s=0}^{t} w_{xx}^s(x,t)\,\mathrm{d}s \tag{342}$$

$$= f(x,t) + \int_{s=0}^{t} \left[w_{tt}^s(x,t) - c^2\,w_{xx}^s(x,t)\right]\mathrm{d}s \tag{343}$$

$$= f(x,t) \tag{344}$$

Everything works and gives the unique solution in $C^2(R\times[0,+\infty))$, under rather natural hypotheses on f: we require f and f_x to be continuous in $R\times[0,+\infty)$.

Finally note from (5.57) that the value of u at the point (x,t) depends on the values of the forcing term f in all the triangular sector $S_{x,t}$.

11 Assignment 4: Wave and Vibration

DTU01418, Introduction to Partial Differential Equations, Fall 2018

11.1 Problem 1.1

At time $t = 0$ we take a fish out from a refrigerator at absolute temperature $T_{fish,0} = 278.15$ Kelvin (5 degrees Celsius) and throw it into a frying pan filled with cooking oil at absolute temperature $T_{oil} = 473.15$ Kelvin (200 degrees Celsius). The fish is approximately cylindrical in shape, of length L and of an approximately circular cross section with radius R. The thermal diffusivity (heat diffusion coefficient) of the fish meat is D, and the fish is fully immersed in the cooking oil.

Using a cylindrical coordinate system (r, θ, z) such that the volume occupied by the fish is given by $0 \leqslant r \leqslant R$, $0 \leqslant \theta < 2\pi$ and $0 \leqslant z \leqslant L$, set up an initial-boundary problem that models the temporal evolution of the temperature profile u within the fish for $t > 0$. Write the PDE in your model explicitly with respect to the above cylindrical coordinate system. Explain your modelling choices and any simplifications you introduce into the model.

$$\begin{cases} u_t - D\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{\partial^2 u}{\partial z^2}\right) = 0 \\ u_0(r,z) = 278.15 \ , \ 0 \leqslant r \leqslant R \ , \ 0 \leqslant z \leqslant L \\ u(R,z,t) = 473.15 \ , \ 0 \leqslant z \leqslant L \ , \ t > 0 \\ u(r,0,t) = u(r,L,t) = 473.15 \ , \ 0 \leqslant r < R \ , \ t > 0 \end{cases} \tag{345}$$

11.2 Problem 1.2

Show that the temperature profile can in cylindrical coordinates be written:

$$u(r,z,t) = 473.15 + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \exp\left(-\mu_{mn}\, D\, t\right) J_0\left(\lambda_m\, r/R\right) \sin(n\,\pi\, z/L) \tag{346}$$

where $(\lambda_m)_{m\in\mathbf{N}}$ is the increasing sequence of positive zeros of the Bessel function J_0. Give a formula for the μ_{mn}. You do not need to find the coefficients C_{mn}.

11.2.1 Reduction to Homogeneous Boundary Condition

To reduce the problem eq.345 to well posed problem with homogeneous boundary condition, we set:

$$u(r,\theta,z,t) = u^{st}(r,\theta,z) - U(r,\theta,z,t) \tag{347}$$

where $u^{st}(r,\theta,z)$ represents the steady-state solution and $U(r,\theta,z,t)$ represents the transit-state solution.

For steady-state solution $u^{st}(r,\theta,z)$, according to eq.345 and our assumption:

$$\begin{cases} -D\left(\dfrac{\partial^2 u^{st}}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u^{st}}{\partial r} + \dfrac{\partial^2 u^{st}}{\partial z^2}\right) = 0 \\ u^{st}(R,z) = 473.15 \ , \ 0 \leqslant z \leqslant L \\ u^{st}(r,0) = u^{st}(r,L) = 473.15 \ , \ 0 \leqslant r < R \end{cases} \tag{348}$$

which is a Laplace equation with Dirichlet boundary condition.

According to theorem 3.4 mean value property and theorem 3.7 maximum principle of harmonic function, we get:

$$u(R,\theta,z) \leqslant \max_{\partial\Omega} u = 473.15$$
$$u(R,\theta,z) \geqslant \min_{\partial\Omega} u = 473.15$$

so, we can say:

$$u^{st}(R,\theta,z) = 473.15 \tag{349}$$

For transit-state solution $U(r,\theta,z,t)$, we have the differential equation with homogeneous Dirichlet boundary condition:

$$\begin{cases} U_t - D\left(\dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r} + \dfrac{\partial^2 U}{\partial z^2}\right) = 0 \\ U_0(r,z) = 278.15 \ , \ 0 \leqslant r \leqslant R \ , \ 0 \leqslant z \leqslant L \\ U(R,z,t) = 0 \ , \ 0 \leqslant z \leqslant L \ , \ t > 0 \\ U(r,0,t) = U(r,L,t) = 0 \ , \ 0 \leqslant r < R \ , \ t > 0 \end{cases} \tag{350}$$

11.2.2 Method of separation of variables

Now, it's in a position to find an explicit formula for U using the method of separation of variables. The main idea is to exploit the linear nature of the problem constructing the solution by super-position of simpler solutions of the form $P(r)\,Z(z)\,T(t)$ in which the variables r, z and t appear in separated form:

$$U(r,z,t) = P(r)\,Z(z)\,T(t) \tag{351}$$

For the same reason in Page 72-73, Salsa (2016), we get the following solution of $U(r,z,t)$:

$$\begin{aligned} U_{mn}(r,z,t) &= P_m(r)\,Z_n(z)\,T_{mn}(t) \\ &= J_0(\frac{\lambda_m\,r}{R})\,\sin(\nu_n\,z)\,\exp\left[-D\,(\nu_n^2 + \frac{\lambda_m^2}{R^2})\,t\right] \end{aligned} \tag{352}$$

$$U(r,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_0(\frac{\lambda_m\,r}{R})\,\sin(\nu_n\,z)\,\exp\left[-D\,(\nu_n^2 + \frac{\lambda_m^2}{R^2})\,t\right] \tag{353}$$

with

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2\,k} \tag{354}$$

$$\nu_n = \frac{n\,\pi}{L} \tag{355}$$

So substitute eq.349 and eq.353 to eq.347, we get:

$$u(r,\theta,z,t) = 473.15 - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\,J_0(\frac{\lambda_m\,r}{R})\,\sin(\nu_n\,z)\,\exp\left[-D\,(\nu_n^2 + \frac{\lambda_m^2}{R^2})\,t\right] \tag{356}$$

Compared with eq.346, we find:

$$\mu_{mn} = \frac{n^2\,\pi^2}{L^2} + \frac{\lambda_m^2}{R^2} \tag{357}$$

11.3 Problem 2

Assume a function $u(x,t)$ is in $C^2(\mathbf{R}\times[0,\infty[)$, and satisfies:

$$\begin{cases} u_{tt} - 4\,u_{xx} = 0 \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases} \tag{358}$$

which is a Cauchay-Dirichlet problem with a wave equation, and

$$g(6) = 6 \tag{359}$$

$$g(2) = 8 \tag{360}$$

$$\int_0^2 h(y)\,\mathrm{d}y = -12 \tag{361}$$

$$\int_0^6 h(y)\,\mathrm{d}y = 8 \tag{362}$$

Find the value of u for $t=1$, $x=4$.

From d'Alembert formula and $c=2$, we can get the solution for the problem 358:

$$u(x,t) = \frac{1}{2}[g(x+2\,t) + g(x-2\,t)] + \frac{1}{4}\int_{x-2\,t}^{x+2\,t} h(y)\,\mathrm{d}y \tag{363}$$

which can also be written in:

$$\begin{cases} u(x,t) = F(x+2\,t) + G(x-2\,t) \\ F(x+2\,t) = \frac{1}{2}g(x+2\,t) + \frac{1}{4}\int_0^{x+2\,t} h(y)\,\mathrm{d}y \\ G(x+2\,t) = \frac{1}{2}g(x-2\,t) - \frac{1}{4}\int_0^{x-2\,t} h(y)\,\mathrm{d}y \end{cases} \tag{364}$$

According to the characteristic parallelogram, we can get:

$$\begin{cases} u(4,1) = F(6) + G(2) \\ F(6) = \frac{1}{2}g(6) + \frac{1}{4}\int_0^6 h(y)\,\mathrm{d}y = 5 \\ G(2) = \frac{1}{2}g(2) - \frac{1}{4}\int_0^2 h(y)\,\mathrm{d}y = 7 \end{cases} \tag{365}$$

and we get:

$$u(4,1) = 12 \tag{366}$$

11.4 Problem 3

Solve the inital-boundary problem with wave equation:

$$\begin{cases} u_{tt} - 9\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) = 0\ ,\ 0 < x < 3\ ,\ 0 < y < 2\ ,\ t > 0 \\ u(x,y,0) = -\sin(2\,\pi\,x)\,\sin\left(\dfrac{\pi}{4}y\right)\ ,\ 0 \leqslant x \leqslant 3\ ,\ 0 \leqslant y \leqslant 2 \\ u_t(x,y,0) = \sin\left(\dfrac{\pi}{3}x\right)\,\sin\left(\dfrac{7\,\pi}{4}y\right)\ ,\ 0 \leqslant x \leqslant 3\ ,\ 0 \leqslant y \leqslant 2 \\ u(0,y,t) = u(3,y,t) = 0\ ,\ 0 \leqslant y \leqslant 2\ ,\ t \geqslant 0 \\ u(x,0,t) = 0\ ,\ 0 \leqslant x \leqslant 3\ ,\ t \geqslant 0 \\ u_y(x,2,t) = 0\ ,\ 0 \leqslant x \leqslant 3\ ,\ t \geqslant 0 \end{cases} \tag{367}$$

The square shape of the membrane and the homogeneous boundary conditions suggest the use of the method of separation of variables. Let us seek for a solution under the form:

$$u(x,y,t) = v(x,y)\,q(t) \tag{368}$$

By substituting, problem eq.367 becomes:

$$\begin{cases} q''(t)\,v(x,y) - 3^2\,q(t)\Delta v(x,y) = 0 \ , \ 0 < x < 3 \ , \ 0 < y < 2 \ , \ t > 0 \\ q(0) = -\sin\left(2\,\pi\,x\right)\,\sin\left(\dfrac{\pi}{4}y\right) \ , \ 0 \leqslant x \leqslant 3 \ , \ 0 \leqslant y \leqslant 2 \\ q'(0) = \sin\left(\dfrac{\pi}{3}x\right)\,\sin\left(\dfrac{7\,\pi}{4}y\right) \ , \ 0 \leqslant x \leqslant 3 \ , \ 0 \leqslant y \leqslant 2 \\ v(0,y) = v(3,y) = 0 \ , \ 0 \leqslant y \leqslant 2 \\ v(x,0) = 0 \ , \ 0 \leqslant x \leqslant 3 \\ v_y(x,2) = 0 \ , \ 0 \leqslant x \leqslant 3 \end{cases} \tag{369}$$

The PDE in eq.369 can written in the following expression, and separation of variables can be applied:

$$\frac{q''(t)}{3^2 q(t)} = \frac{\Delta v(x,y)}{v(x,y)} = A \tag{370}$$

which means the usual argument – the left-hand side of the equation depends only on t, while the right-hand side depends only on (x, y) – leads us to conclude that both sides must equal some constant A.

Then we get the ODE problem:

$$\begin{cases} q''(t) - 3^2\,A\,q(t) = 0 \ , \ t > 0 \\ q(0) = -\sin\left(2\,\pi\,x\right)\,\sin\left(\dfrac{\pi}{4}y\right) \ , \ 0 \leqslant x \leqslant 3 \ , \ 0 \leqslant y \leqslant 2 \\ q'(0) = \sin\left(\dfrac{\pi}{3}x\right)\,\sin\left(\dfrac{7\,\pi}{4}y\right) \ , \ 0 \leqslant x \leqslant 3 \ , \ 0 \leqslant y \leqslant 2 \end{cases} \tag{371}$$

and eigen-value problem:

$$\begin{cases} \Delta v - A\,v = 0 \ , \ 0 < x < 3 \ , \ 0 < y < 2 \\ v(0,y) = v(3,y) = 0 \ , \ 0 \leqslant y \leqslant 2 \\ v(x,0) = 0 \ , \ 0 \leqslant x \leqslant 3 \\ v_y(x,2) = 0 \ , \ 0 \leqslant x \leqslant 3 \end{cases} \tag{372}$$

And assuming $v(x,y) = X(x)\,Y(y)$, we rewrite the PDE for v in eigen-value problem:

$$X''(x)\,Y(y) + X(x)\,Y''(y) = A\,X(x)\,Y(y) \tag{373}$$

that is the following equation and separation of variables is applied again,

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} - A = C \tag{374}$$

since the left-hand side of this equation only depends on x, and the right-hand side only depends on y, both sides must equal a constant C.

Then we get two eigen-value problems for $X(x)$ and $Y(y)$:

$$\begin{cases} X''(x) + C\,X(x) = 0\ ,\ 0 < x < 3 \\ X(0) = X(3) = 0 \end{cases} \tag{375}$$

$$\begin{cases} Y''(y) - (A + C)\,Y(y) = 0\ ,\ 0 < y < 2 \\ Y(0) = 0 \\ Y'(2) = 0 \end{cases} \tag{376}$$

We have seen the problem eq.375 before, and we know that only when $C > 0$, it has nontrivial solutions, which is:

$$X_n(x) = \mathcal{H}\,\sin\left(\frac{n\,\pi}{3}x\right)\ ,\ n = 1, 2, 3, ... \tag{377}$$

As for problem eq.376, let's first **assume** $(A + C) > 0$, we get:

$$Y(y) = \mathcal{E}\,\exp\left(\sqrt{(A + C)}\,y\right) + \mathcal{F}\,\exp\left(-\sqrt{(A + C)}\,y\right) \tag{378}$$

$$\begin{cases} Y(0) = \mathcal{E} + \mathcal{F} = 0\ ,\ \text{and}\ \mathcal{E} \neq 0\ ,\ \mathcal{F} \neq 0 \\ Y'(2) = \sqrt{(A + C)}\,\left[\mathcal{E}\,\exp\left(2\sqrt{(A + C)}\right) - \mathcal{F}\,\exp\left(-2\sqrt{(A + C)}\right)\right] = 0 \end{cases} \tag{379}$$

So we get:

$$\exp\left(2\sqrt{(A + C)}\right) + \exp\left(-2\sqrt{(A + C)}\right) = 0 \tag{380}$$

which is not true, because $e^x > 0$ for all x.

When we assume $(A + C) = 0$, we get:

$$\begin{aligned} &Y''(y) = 0 \\ &Y'(y) = \mathcal{E} \\ &Y(y) = \mathcal{E}x + \mathcal{F} \end{aligned} \tag{381}$$

$$\begin{cases} Y(0) = \mathcal{F} = 0\ ,\ \text{and}\ \mathcal{E} \neq 0 \\ Y'(2) = \mathcal{E} = 0 \end{cases} \tag{382}$$

which is not true.

When we assume $(A + C) < 0$, so $A < 0$. Then, we set $B = A + C = -\mu^2$, and get:

$$Y(y) = \mathcal{E}\,\cos(\mu\,y) + \mathcal{F}\,\sin(\mu\,y) \tag{383}$$

$$Y'(y) = -\mu\,\mathcal{E}\,\sin(\mu\,y) + \mu\,\mathcal{F}\,\cos(\mu\,y) \tag{384}$$

$$\begin{cases} Y(0) = \mathcal{E} = 0\ \text{and}\ \mathcal{F} \neq 0 \\ Y'(2) = \mu\,\mathcal{F}\,\cos(2\,\mu) = 0 \end{cases} \tag{385}$$

so we get:

$$2\,\mu_m = \frac{\pi}{2} + m\,\pi\ ,\ m = 0, 1, 2, ... \tag{386}$$

$$\mu_m = \frac{(2\,m + 1)\,\pi}{4}\ ,\ m = 0, 1, 2, ... \tag{387}$$

$$\lambda_m = -\left(\frac{(2\,m + 1)\,\pi}{4}\right)^2\ ,\ m = 0, 1, 2, ... \tag{388}$$

So we can get the solution of $Y_m(y)$:

$$Y_m(y) = \mathcal{F}\, \sin\left(\frac{(2\,m+1)\,\pi}{4}\,x\right)\ ,\ m = 0, 1, 2, \ldots \tag{389}$$

and:

$$C_n = \left(\frac{n\,\pi}{3}\right)^2 > 0 \tag{390}$$

$$(A+C)_m = -\left(\frac{(2\,m+1)\,\pi}{4}\right)^2 < 0 \tag{391}$$

$$A_{mn} = -\left(\frac{(2\,m+1)\,\pi}{4}\right)^2 - \left(\frac{n\,\pi}{3}\right)^2 < 0 \tag{392}$$

So:

$$\begin{aligned} v_{mn} &= X_n(x)\,Y_m(y) \\ &= \mathcal{J}\, \sin\left(\frac{n\,\pi}{3}\,x\right)\, \sin\left(\frac{(2\,m+1)\,\pi}{4}\,x\right)\ ,\ n = 1, 2, 3, \ldots\ ,\ m = 0, 1, 2, \ldots \end{aligned} \tag{393}$$

The general integral of ODE problem eq.371 is:

$$q_{mn}(t) = \mathcal{K}\, \cos\left[3\,t\,\sqrt{\left(\frac{(2\,m+1)\,\pi}{4}\right)^2 + \left(\frac{n\,\pi}{3}\right)^2}\right] + \mathcal{L}\, \sin\left[3\,t\,\sqrt{\left(\frac{(2\,m+1)\,\pi}{4}\right)^2 + \left(\frac{n\,\pi}{3}\right)^2}\right] \tag{394}$$
$$,\ n = 1, 2, 3, \ldots\ ,\ m = 0, 1, 2, \ldots$$

So, the solution of problem eq.367:

$$\begin{aligned} u_{mn}(x, y, t) &= q_{mn}(t)\, v_{mn}(x, y) \\ = \Bigg\{ \mathcal{R}\, \cos\left[3\,t\,\sqrt{\left(\frac{(2\,m+1)\,\pi}{4}\right)^2 + \left(\frac{n\,\pi}{3}\right)^2}\right] &+ \mathcal{S}\, \sin\left[3\,t\,\sqrt{\left(\frac{(2\,m+1)\,\pi}{4}\right)^2 + \left(\frac{n\,\pi}{3}\right)^2}\right] \Bigg\} \\ &\times \sin\left(\frac{n\,\pi}{3}\,x\right)\, \sin\left(\frac{(2\,m+1)\,\pi}{4}\,x\right)\ ,\ n = 1, 2, 3, \ldots\ ,\ m = 0, 1, 2, \ldots \end{aligned} \tag{395}$$

$$\begin{aligned} u(x, y, t) &= q(t)\, v(x, y) \\ = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \Bigg\{ \mathcal{R}\, \cos\left[3\,t\,\sqrt{\left(\frac{(2\,m+1)\,\pi}{4}\right)^2 + \left(\frac{n\,\pi}{3}\right)^2}\right] &+ \mathcal{S}\, \sin\left[3\,t\,\sqrt{\left(\frac{(2\,m+1)\,\pi}{4}\right)^2 + \left(\frac{n\,\pi}{3}\right)^2}\right] \Bigg\} \\ &\times \sin\left(\frac{n\,\pi}{3}\,x\right)\, \sin\left(\frac{(2\,m+1)\,\pi}{4}\,x\right)\ ,\ n = 1, 2, 3, \ldots\ ,\ m = 0, 1, 2, \ldots \end{aligned} \tag{396}$$

11.4.1 Constants Identification and Solution Validation

Determine $\mathcal{R}$ and $\mathcal{S}$ via initial condition

11.5 Problem 5: Cylindrical Wave

Let D be the unit disk in $\mathbb{R}^2$ defined in polar coordinates (r, θ) by $0 \leqslant r < 1$, $\theta \in \mathbb{R}$. Consider the PDE problem:

$$\begin{cases} -\Delta v = \lambda\, v \ , \ x \in D \\ v = 0 \ , \ x \in \partial D \end{cases} \tag{397}$$

Let J_1 denote the Bessel function of order 1 and let β_{11} denote the first zero of J_1. Show that:

$$v(r, \theta) = J_1(\beta_{11}\, r)\, cos(\theta) \tag{398}$$

satisfies eq.397 for some λ. What is λ?

We can write the Poisson equation in cylindrical coordinates:

$$-\Delta v = \lambda\, v \tag{399}$$

$$-\frac{\partial^2 v}{\partial r^2} - \frac{1}{r}\frac{\partial v}{\partial r} = \lambda v \tag{400}$$

which is Bessel equation of order zero with parameter λ.

References

Salsa, S. (2016). *Partial differential equations in action: from modelling to theory*, volume 99. Springer.