# Notes on Risk

**James Holland Jones** *Stanford University*

### Video Notes

I have created three videos to introduce some key concepts in the theory of risk to go along with these notes:

1. Variance Discounting
2. Risk Aversion and Risk Premium
3. Arrow-Pratt Index of Risk Aversion

### Fitness/Utility Functions

A fitness function that maps the fitness values to the outcomes of decisions should have two properties: (1) it should be increasing in the payoffs, and (2) it should (probably) show diminishing marginal returns to these payoffs. Consider the case discussed in Jones and Bliege Bird (2014) of the fitness consequences of fertility. Each child that a woman has increases her fitness (i.e., either her proportionate representation in the population or the growth rate of her share of the population), but her later births increase her fitness less and less. Why is that? We can get some intuition for this by thinking about what happens to a population over time. Suppose a woman lives in a population of size $N$ growing at average annual per capita rate $r$. At the outset, her contribution to the population is $1/N$. In ten years, her contribution will be $1/(Ne^{r\cdot 10})$. This shows that over time, an individual woman's contribution to the size of the population decreases exponentially. Births that happen later in a woman's life will thus increase her fitness less and less as she gets older. This is actually the fundamental idea behind Fisher's (1958) concept of reproductive value.

The way to get a fitness curve that fulfills these two properties is to have a positive first derivative and a negative second derivative. That is: $f'(x) > 0$ and $f''(x) < 0$. While the terminology can be confusing, such a function is often called *concave*. There are many functions that fulfill these criteria. One that is commonly employed in economics is a negative exponential:

$$f(x) = 1 - e^{-rx}$$

We can plot this fitness function (note that this function can always be re-scaled – what we care about right now is its shape), and inspect the plot for what the consequences of its concavity are for risk. Suppose you need to make a decision about something that has a variable outcome. Without loss of generality, we can assume that you start from some average value of the outcome of your decisions, $x$ (this could be wealth or energy or number of offspring, etc.). We represent risk as a *lottery*, which is just a discrete probability distribution on a set of outcomes. For simplicity, we will assume that you have even odds (i.e., a 50/50 chance) of either going up one unit to $x_1$ or down one unit to $x_0$. In figure 1, I've drawn these increments with red arrows.

```
x <- seq(0,5,length=1000)
r <- 0.75
```

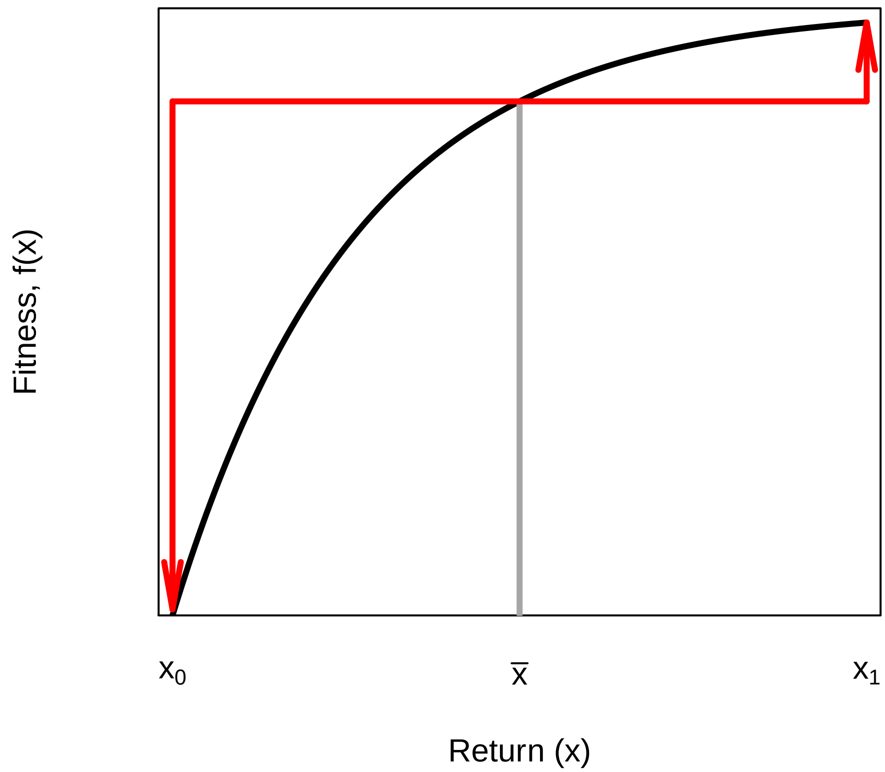


Figure 1: Concave fitness function showing the asymmetry between the downside and upside risk of a lottery played at the mean outcome

```
fx <- 1-exp(-r*x)
## for part deux
aaa <- (fx-0.4882412)^2
which(aaa==min(aaa))
```

```
## [1] 179
```

```
##
plot(x,fx, type="l", lwd=3, axes=FALSE, frame=TRUE,
     xlab="Return (x)", ylab="Fitness, f(x)",
     xaxs="i", yaxs="i", xlim=c(-0.1,5.1), ylim=c(0,1))
axis(1, at=c(0,2.5,5), labels=c(expression(x[0]), expression(bar(x)),
                           expression(x[1])), tick=FALSE)
segments(2.5,0,2.5,0.846645, lwd=3, lty=1, col=grey(0.65))
segments(2.5,0.846645,0,0.846645, lwd=3, lty=1, col="red")
arrows(0,0.846645,0,0.01, lwd=3, lty=1, col="red", length=.25,angle=10)
segments(2.5,0.846645,5,0.846645, lwd=3, lty=1, col="red")
arrows(5,0.846645,5,fx[1000], lwd=3, lty=1, col="red", length=.25,angle=10)
```

There is a clear asymmetry between what we call the *downside risk* and the *upside risk*. Because of the concavity of $f(x)$, the downside risk is substantially greater than the upside risk. In such a case, we say that our decision maker is *risk averse*. In effect, this means that she would prefer to pay for a certain outcome than to play the lottery. The amount she would be willing to pay for certainty is known as the *certainty premium*, denoted $\pi$, which we discuss below.

### *Expected Utility*

The theory of decision-making under risk takes as its objective function *expected utility*. Remember that mathematically, the word "expected" is synonymous with "average". Suppose that there is some variable resource, the acquisition of which is associated with some utility or fitness. Following standard statistical convention, we denote our random variable (i.e., the resource in question) using uppercase Roman letter, $X$, and denote specific realizations of this random variable in lowercase, $x$. Expected fitness is simply the average of the fitnesses $f(x)$ associated with each value of $x$:

$$E(f) = \int_{\Omega} u(x)\, g(x) dx,$$

where $E()$ indicates mathematical expectation (i.e., the arithmetic mean), $g(x)$ is the probability density of outcome $x$, $\Omega$ is the set of all possible outcomes being averaged over. The limits of $\Omega$ are based on the specifics of the problem but are defined such that $g(x)$ is a true probability distribution – i.e., $\int_{\Omega} g(x) dx = 1$.

### *An Approximation*

We often don't know the full probability distribution $g(x)$. Perhaps more importantly, this formula for the expected utility doesn't provide us with any analytical insights that help us develop intuition about how features of the environment are likely to affect preferences. To help us develop such intuition, we do the thing that one (almost) always does in such a situation: we perform a Taylor series expansion!

We will use $\bar{x}$, the mean outcome as the point around which we expand expected fitness The Taylor series expansion around the mean reward looks something like this (at least for the first three terms):

$$f(x) = f(\bar{x}) + f'(x - \bar{x}) + \frac{1}{2} f''(x - \bar{x})^2 + \frac{1}{6} f'''(x - \bar{x})^3 + \ldots$$

We will focus on the first two terms, and take expectations of both sides of the equation, yielding:

$$\overline{f(x)} \approx f(\bar{x}) + \frac{1}{2} f'' \,\mathrm{Var}(x)$$

This result follows from the following three definitions:

1. The expected value of a single number is that number, so $E[f(\bar{x})] = f(\bar{x})$
2. The expected value of $(x - \bar{x})$ is zero since $E[(x - \bar{x})] = E[x] - \bar{x} = \bar{x} - \bar{x} = 0$
3. $E[(x - \bar{x})^2]$ is, by definition, equal to the variance of $x$, since variance is defined as the expected value of the squared deviations from the mean

Now let's think about this result a bit. The second derivative is simply a measure of the curvature of the utility function. In this case, it is the curvature in a very particular point, namely, in the vicinity of the average environment $\bar{x}$. In the figure 1, this is the point where the vertical grey line meets the function. If we look to the left side of this point, we see that the curvature is far steeper than on the right side of the point. This means that as we get larger and larger values, the increment in fitness, while still positive (more is generally better), gets smaller. This is what we mean by *diminishing marginal utility*, specified above. In this context, the term "marginal" simply means the next increment. The term marginal is generally used to mean the derivative, or the local rate of change in the slope of a function. When marginal fitness is diminishing, it means that the fitness curve bends downward or has a negative second derivative, $f'' < 0$. When $f'' < 0$, variance in $x$ will reduce the expected fitness from the fitness of the average $x$.

The fundamental insight provided by our Taylor series analysis of expected fitness is that when there is diminishing marginal fitness, variance is bad for the decision-maker. This is because in such a case $f'' < 0$, so the second term in the equation is subtracted from the fitness of the mean. Of course, if marginal fitness increases, then $f'' > 0$ and the second term is positive. When there is increasing marginal utility, people are variance loving or risk prone.

*Certainty Premium*

To find the certain payoff that the risk-averse individual would be willing to accept in lieu of the average (risky) payoff $\bar{x}$, we draw a chord connecting the points of the utility curve at $x_0$ and $x_1$. Because our lottery is a 50/50 chance of winning or losing, the expected value (i.e., average) is the midpoint of this chord. From this point we move horizontally to the left until we hit the utility curve (note that moving horizontally means that we are holding utility constant). Because this utility is equivalent to the expected utility of the lottery, this intersection indicates the certain payoff that you would be willing to receive in lieu of the risky mean. Drawing a line segment down to the x-axis gives us the value of $x$, which we call $x_C$. The difference between $\bar{x}$ and $x_C$ is known as the certainty premium: $\pi = \bar{x} - x_C$.

```
## part deux
plot(x,fx, type="l", lwd=3, axes=FALSE, frame=TRUE,
     xlab="Return (x)", ylab="",
     xaxs="i", yaxs="i", xlim=c(0,5.1), ylim=c(0,1))
segments(0,0,5,fx[1000], lwd=3, col=grey(0.75))
axis(1, at=c(0.05,x[179],2.5,5), labels=c(expression(x[0]), expression(x[C]),
                                expression(bar(x)), expression(x[1])),
     tick=FALSE)
mtext("Fitness, f(x)", side=2, line=2, adj=0.75)
par(las=2)
mtext(expression(f (x[C])), side=2, line=1, adj=0.488)
#axis(2, at=0.4882412, labels=expression(f (x[C])), tick=FALSE)
segments(2.5,0,2.5,0.4882412,lwd=3, lty=1, col="red")
segments(2.5,0.4882412,x[179],0.4882412,lwd=3, lty=1, col="red")
segments(x[179],0.4882412,x[179],0, lwd=3, lty=1, col="green")
segments(x[179],0.4882412,0,0.4882412,lwd=3, lty=2, col="red")
```

**Arrow-Pratt Index of Risk-Aversion**

Since the amount by which variance detracts from the fitness of the mean outcome is scaled by $f''(x)$, this suggests that greater curvature (i.e., more negative second derivative) should induce greater risk aversion. We move back further from the utility of the mean to the utility curve when

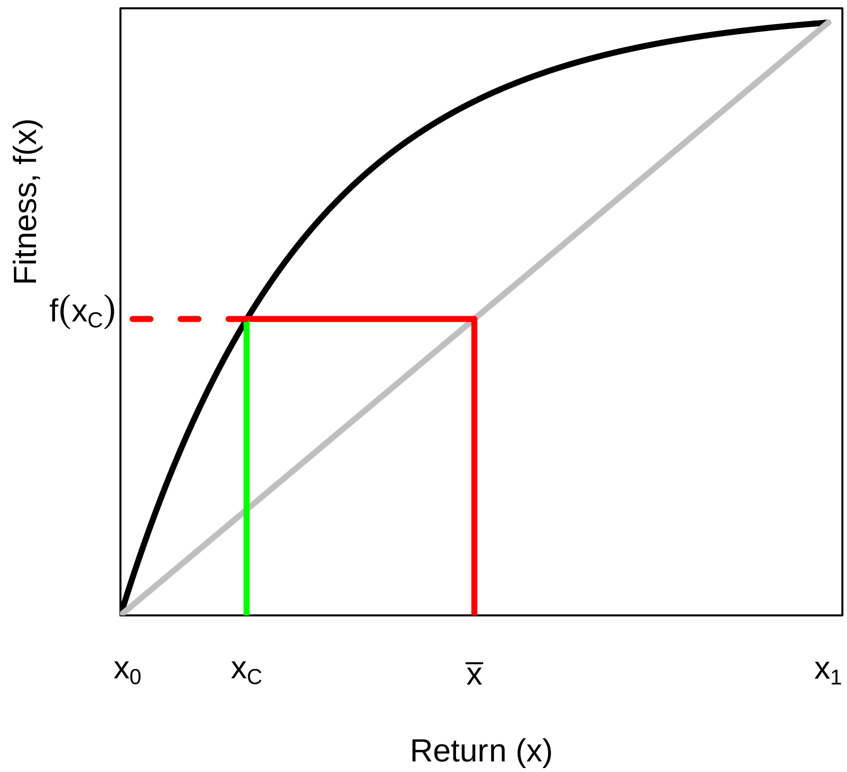


Figure 2: Illustration of the certainty premium for a lottery given the fitness fuction.

the function is more curved. For a linear utility function – i.e., one with no curve – the points are identical and the person is risk-neutral. The upside and downside risk are equal. We can measure the extent of a person's risk aversion by measuring the curvature of her utility curve. Of course, this is a big "if." In general, utility curves are theoretical constructs that give us insight into behavior, though this need not be the case (see (Jones and Bliege Bird, 2014) for an empirical measurement of a human fitness curve as a function of parity). For our fitness curve with diminishing marginal fitness, we can see that toward the middle of $x$ (i.e., near $\bar{x}$, the function $f(x)$ has a high degree of curvature whereas at the upper-end of the range of $x$, it is quite flat. Arrow (1965) used this observation to note that wealthy people tend to be less risk-averse on the margin. A wealthy person is willing to take a financial gamble like a chimpanzee sated on fruit is willing to take a risk hunting red colobus monkeys.

Since the degree of risk-aversion, as measured by the certainty premium, depends on the degree of curvature of the fitness function. This suggests that a measure of risk using second derivatives of $f(x)$ might be in order. This is exactly what the Arrow-Pratt measure of risk-aversion does. The Arrow-Pratt metric measures the curvature of $f(x)$ through the second derivative, $f''(x)$ and scales this by the first derivative, $f'(x)$. Scaling by the first derivative removes the effect of any multiplicative constant on the fitness function (since both the first and second derivatives will contain this constant). This is important because utility in economics is typically defined only up to an affine transform.

The so-called *Arrow-Pratt Index of Absolute Risk Aversion* is:

$$A_f(x) = -\frac{f''(x)}{f'(x)}.$$

A related measure is known as the *Index of Relative Risk Aversion*:

$$R_f(x) = -\frac{xf''(x)}{f'(x)}.$$

Another way to think about the relative risk aversion is that it is an elasticity of the marginal utility. Elasticities measure the proportionate change in a quantity for some small change in an input. For example, by what percentage does mean fitness change if we improve juvenile survivorship by 10%? How much does garden production increase if we increase irrigation by 2%? Define an elasticity $e(f[x])$ as

$$e(f[x]) = \frac{d \log f}{d \log x} = \frac{df}{dx}\frac{x}{f}.$$

It is not difficult to see that, in fact,

$$R_f(x) = -e(f'(x)).$$

That is, how does marginal utility of a resource – such as food energy – change with an increase in energy availability? If this number is high, people are highly risk averse and should be willing to pay a premium for certainty.

The elasticity of marginal utility/index of relative risk aversion also plays an important role in deriving the social discount rate for determining time preferences.

*Approximating the Risk Premium*

We can derive an analytical approximation for the certainty premium. Since the risk premium is difference between expected fitness and risk-free fitness, we can approximate $\pi$ by equating respective Taylor series approximations for two functions. It is conventional to employ the second-order Taylor series approximation for expected utility, but to use a first-order Taylor series approximation of the risk-free utility. This bit of mathematical trickery allows us to write a simple formulation for $\pi$, but is limited in its applicability to small gambles. Note that expected utility is, naturally, an expectation. It therefore makes sense that we'd want to use a second-order Taylor series to approximate it since we want to be able to account (at least) for variance (which is the second moment of a random variable). If we keep the variation small around the mean, then approximating risk-free fitness using the local slope of the fitness function at its mean (i.e., the geometric interpretation of a first-order Taylor series) should be okay.

Assume a small gamble where your final outcome is $x + \epsilon$. We assume that $\epsilon$ is a random variable with zero mean and variance $\sigma^2$. Since you would rather take the certainty equivalent to the gamble because you are risk averse, forcing you to take the gamble is effectively the equivalent of of subtracting $\pi$ from your wealth. This sets up the following equivalence:

$$E\left[f(x+\epsilon)\right] = f(x - \pi).$$

We then perform a second-order Taylor expansion of the left side and a first-order Taylor expansion of the right-hand side:

$$E\left[f(x) + \epsilon f'(x) + (\epsilon^2/2)f''(x)\right] = f(x) - \pi f'(x).$$

Again, the $E(\epsilon) = 0$ and $E(\epsilon^2) = \mathrm{Var}(x)$. Note also that $-f''(x)/f'(x) = A_f(x)$, the Arrow-Pratt index of absolute risk aversion. Rearranging (and reminding ourselves this is only true for small gambles), we get

$$\pi \approx \frac{\sigma^2}{2} A_f(x).$$

This says that, for small gambles, the amount that a risk-averse agent is willing to pay for certainty is proportional to the curvature of her utility curve and the degree of variability, measured by the variance of the resource.

*Non-Monotonic Fitness Functions*

So far, all the examples of fitness curves have had the same concavity throughout the range of resource values. We have seen that concave fitness curves imply risk-aversion. However, there are certainly conditions in which we should expect people to be risk-prone. For example, at very low levels of resource holding (extreme destitution or near-starvation), people might be willing to take a risk to get a reward since, in effect, they have nothing to lose, but then at more average levels of resource holding, people become risk-averse. This is an insight applied in the classic paper by Friedman & Savage (1948) to suggest the origins of middle-class sensibilities. This insight has been applied more recently in an anthropological context by Kuznar (2002) and Kuznar & Frederick (2003). This is more likely to apply to utility curves than to fitness curves (strictly), so we'll switch to utility for the following plot.

The results of Milner-Gulland and colleagues (1996) provide an excellent demonstration of the relationship to risk changing with wealth in an apparently sigmoid manner. Farmers nearest destitution welcomed the high-variance crop (maize) since scoring a bumper crop of maize was their only chance of raising themselves out of destitution. As farmers become richer, they switch over to millet, despite its lower potential yields. Wealthy farmers do not like variance since they are risk-averse. In contrast, the destitute farmers welcome variance – they have convex utility. Their only hope of emerging from destitution is to get lucky with the rains and score a bumper crop of maize.

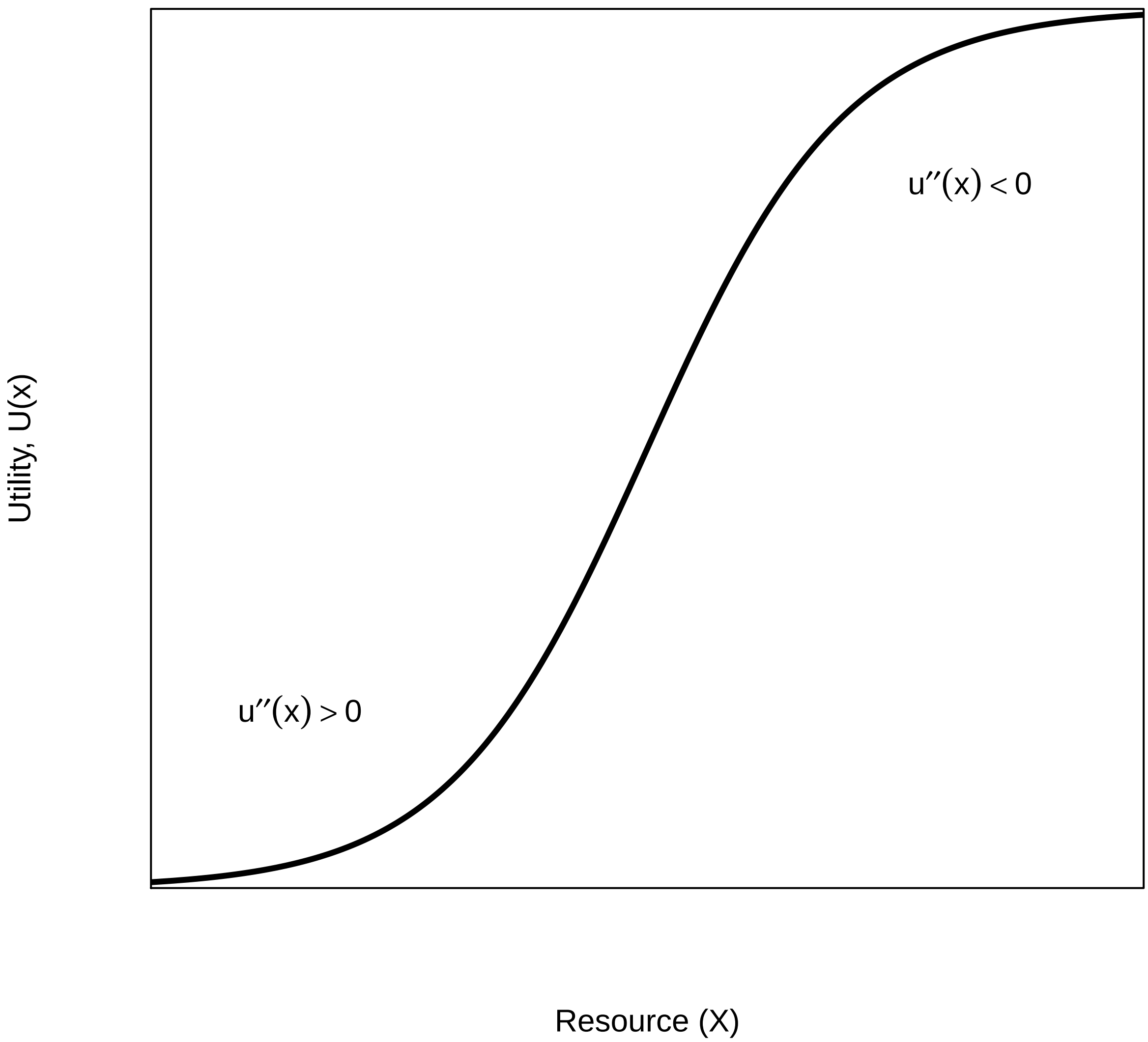


Figure 3: Sigmoid utility curve. Regions where the decision-maker is risk-prone vs. risk-averse indicated by the corresponding second derivatives.

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