Maximizing algebraic connectivity of a graph

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November 2020

1 Introduction

Convex optimization methods provide tractable formulations for solving various difficult and interesting problems. One example of such problem is algebraic connectivity of a graph, where the goal is to find optimal edge weights such that the algebraic connectivity is maximized.

1.1 Graphs

A graph is a way of encoding pairwise relationships among a set of objects. A graph G = (V, E) consists of $|V|$ vertices and $|E|$ edges, where each edge is a connection between two nodes. Graphs can be used to model many interesting problems, e.g.: transportation networks, communication networks, social networks etc. Graphs can be undirected or directed and unweighted or weighted. The edges in an undirected graph can be traversed in either direction and represent a two-way relationship, whereas directed graphs have edges with specific directions. In case of an unweighted graph, edges do not carry weights, while weighted graphs have edges with associated weights that can be positive, negative, or zero.

1.2 Graph Representations

In computer science, a graph G = (V, E) with n nodes and m edges is commonly represented with an adjacency matrix. The adjacency matrix encodes the graph as an n x n matrix A, where A[u, v] is equal to 1 if the graph contains the edge (u, v) and 0 otherwise. In this exercise, the graph is undirected, i.e.: (u, v) = (v, u) and therefore A[u, v] = A[v, u], which means that the adjacency matrix A is symmetric. However, this is not the only possible graph representation. In this exercise, we study algebraic graph connectivity that measures how well a graph is connected for which the Laplacian is a convenient graph representation.

2 Algebraic Connectivity

The algebraic connectivity (also known as Fiedler value or Fiedler eigenvalue) of a graph G is the second smallest eigenvalue of the Laplacian matrix of G.

2.1 The Laplacian and Algebraic Connectivity

A graph is poorly connected if one can cut off many vertices by removing only a few edges. Therefore, algebraic connectivity measures the robustness of a graph. Fiedler (1973) showed that algebraic connectivity is only equal to zero if G is disconnected, and that the multiplicity of zero as an eigenvalue of the Laplacian is equal to the number of disconnected components of G. The Laplacian matrix L represents an undirected weighted graph G(V, E, w), which is a product of the incidence matrix, the diagonal matrix formed from w, and the transpose of the incidence matrix.

$$L = A * \operatorname{diag}(w) * A^\top \tag{1}$$

The incidence matrix A is an n x m matrix, where the each row corresponds to a vertex and each column corresponds to an edge. Therefore, A[i, k] = 1 if vertex i is incident upon edge k and i is the initial vertex of edge k, and A [i, k] = -1 if vertex i is incident upon edge k and i is the terminal vertex of edge k, otherwise A[i, k] = 0. Multiplying an n x m matrix, a vector of length m, and an m x n matrix yields an n x n matrix. Therefore, the Laplacian is an n x n symmetric matrix. Moreover, nonnegative weights imply that the Laplacian is a positive semidefinite matrix. The minimum eigenvalue λ_1 of the Laplacian is 0, and the second smallest eigenvalue λ_2 is the algebraic connectivity of graph G (Fiedler, 1973).

$$0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \tag{2}$$

Therefore, the optimization problem of maximizing algebraic connectivity can be formulated as follows:

$$\begin{array}{ll} \text{maximize} & (\lambda_2) \\ \text{subject to} & w \succeq 0 \\ & Fw \preceq g \end{array} \tag{3}$$

2.2 Convex Optimization Formulation of Maximum Algebraic Connectivity

We know that the smallest eigenvalue of L is $\lambda_1 = 0$. Therefore, for all a_k in incidence matrix $A = [a_1...a_m] : \mathbf{1}^\top a_k = 0$, which implies that the eigenvector 1 corresponds to eignevalue λ_1. Hence, we can set up the optimization problem as minimum eigenvalue of L in the subspace of orthogonal complement of **1**, which is a vector corresponding to all ones. Therefore, the objective function max $\lambda_2(L)$ is transformed into $\lambda_{min}(S)$, where $S = Q^\top LQ$. Matrix Q is a n by n-1 matrix and its columns are unit norm vectors, orthogonal to **1** and to each other. The eigenvalues of matrix S are $\lambda_2...\lambda_n$. And $\lambda_{min}(S)$ is a concave function of w, because $\lambda_{min}(S)$ is the infimum over affine functions. We are maximizing over a concave function; therefore, the problem is convex and can be formulated as follows:

$$\begin{array}{ll} \text{maximize} & (\lambda_{min}(S)) \\ \text{subject to} & w \succeq 0 \\ & Fw \preceq g \end{array} \tag{4}$$

2.3 Numerical Example of Maximum Algebraic Connectivity

Using the data provided in max_alg_conn_data.m, the algebraic connectivity with uniform constant weights is 0.002204, while with optimal weights it is 0.005018. The larger the number the better connectivity and robustness; therefore, we can see that optimal edge weights yield a greater algebraic connectivity than uniform constant edge weights. As Figure 1 and Figure 2 demonstrate, a graph that contains fewer edges is not necessarily better connected than a graph with more edges. The graph topology with optimal weights shown in Figure 2 has fewer edges than the graph with uniform weights shown in Figure 1. Despite of that, the graph with optimal weights is better connected (has greater algebraic connectivity), because the edges that are more important for connectivity have greater weight, which significantly improves the overall connectivity of the graph. This is illustrated by Figure 3, which shows a weighted graph with optimal weights such that the thickness of the edges represents their weights. Therefore, the following statement is incorrect: "The more edges a graph has, the more connected it is, so the optimal weight assignment should make use of all available edges."

3 Conclusion

In summary, convex optimization techniques can be applied to many problems. In this exercise we investigated algebraic connectivity of graphs. First, we formulated the problem of finding edge weights that maximize algebraic connectivity as a convex optimization problem. And second, we used a numerical example to calculate and compare the algebraic connectivity of a graph with optimal edge weights versus uniform constant edge weights. We concluded that optimal edge weights lead to better algebraic connectivity, because important edges carry greater weight and therefore, significantly improve the overall graph connectivity. Using this numerical example we demonstrated that greater number of edges does not necessarily lead to greater algebraic connectivity - instead, optimal edge weights are more important.

References

[1] Miroslav Fiedler, Czechoslovak Mathematical Journal 23(98):298-305, 1973.

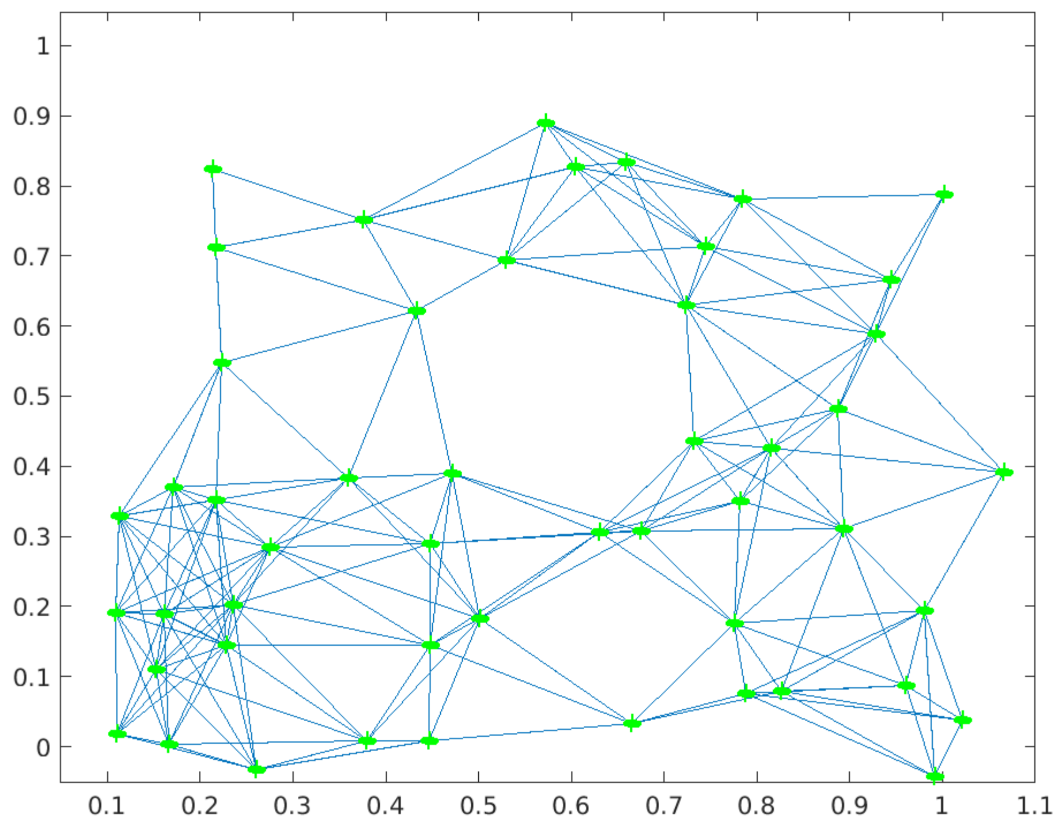

Figure 1: Graph topology with uniform constant weights

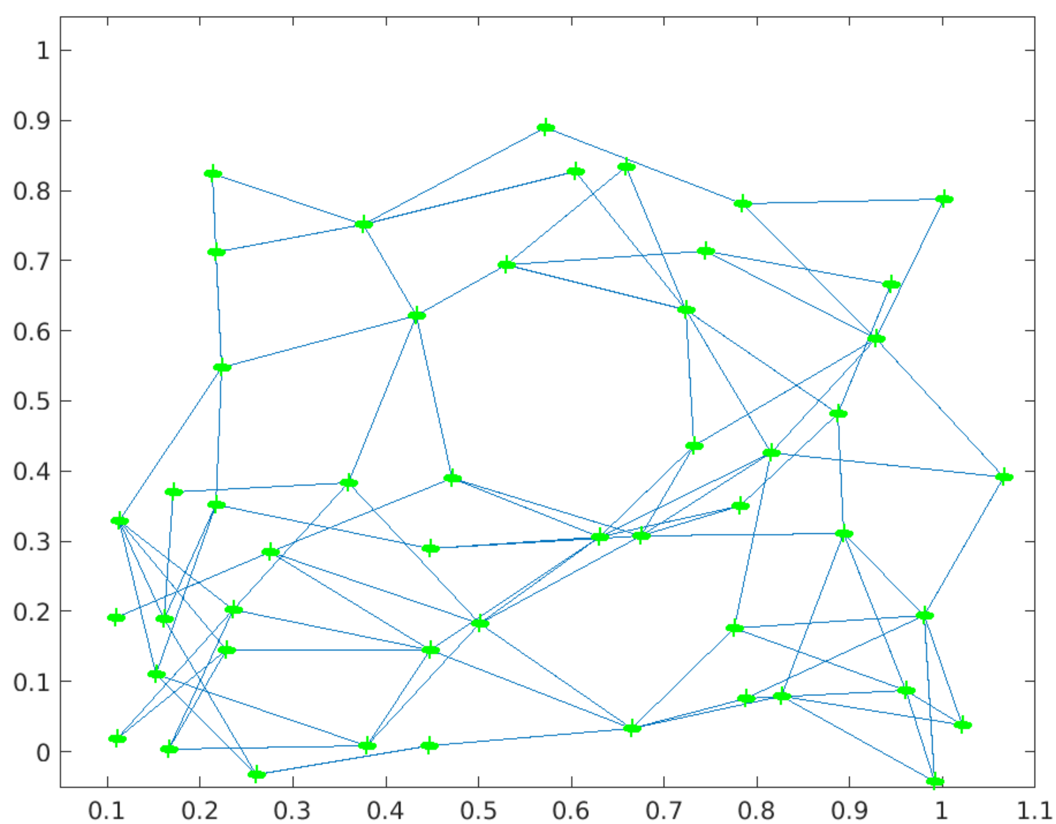

Figure 2: Graph topology with optimal weights

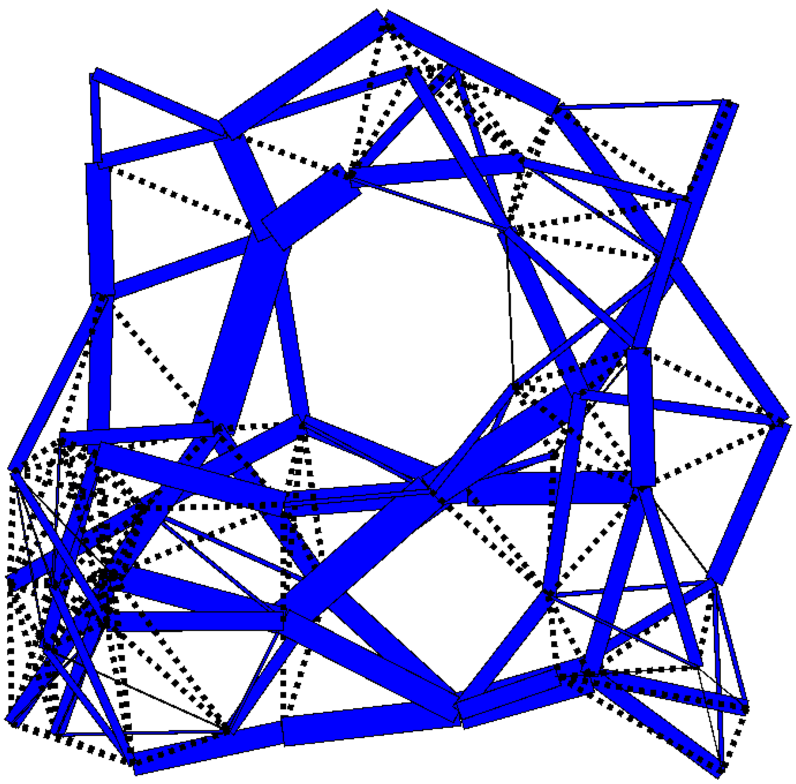

Figure 3: Weighted Graph with optimal weights