DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **11**, Number **2**, April **2018**

doi:10.3934/dcdss.2018019

pp. **345–355**

AMBROSETTI-PRODI TYPE RESULT TO A NEUMANN PROBLEM VIA A TOPOLOGICAL APPROACH

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Abstract. We prove an Ambrosetti-Prodi type result for a Neumann problem associated to the equation $u'' + f(x,u(x)) = \mu$ when the nonlinearity has the following form: $f(x,u) := a(x)g(u) - p(x)$. The assumptions considered generalize the classical one, $f(x,u) \to +\infty$ as $|u| \to +\infty$, without requiring any uniformity condition in x. The multiplicity result which characterizes these kind of problems will be proved by means of the shooting method.

1. **Introduction.** We are interested in studying a problem of Ambrosetti-Prodi type under Neumann boundary conditions. In more detail, the problem that we take into account is associated with a second order ordinary differential equation characterized by the presence, in the nonlinear term, of a weight function not necessarily strictly positive. As far as we know, the usual assumptions in such kind of problems do not include the previous case (see [5, 15, 19, 20]).

More precisely, we consider here the following Neumann problem, depending on a real parameter μ,

$$(\mathcal{P}_\mu) \qquad \begin{cases} u'' + a(x)g(u) = \mu + p(x), \\ u'(0) = u'(\mathrm{T}) = 0, \end{cases}$$

with $p \in L^\infty(0,\mathrm{T})$ and $g\colon \mathbb{R} \to \mathbb{R}$ a function of class C^1 which satisfies

$$-\infty < g'(-\infty) < 0 < g'(+\infty) < +\infty, \qquad (g_\infty)$$

where, by definition,

$$g'(-\infty) := \lim_{s\to-\infty} g'(s) \quad \text{and} \quad g'(+\infty) := \lim_{s\to+\infty} g'(s).$$

The novelty of our work is to consider a weight term $a \in L^\infty(0,\mathrm{T})$ such that

$$a(x) \geq 0 \text{ for a.e. } x \in [0,\mathrm{T}] \text{ with } \int_0^{\mathrm{T}} a(x)\,dx > 0. \qquad (a_*)$$

2010 *Mathematics Subject Classification.* 34B15.

Key words and phrases. Ambrosetti-Prodi problems, Neumman boundary conditions, multiplicity results, shooting method.

Work partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). Progetto di Ricerca 2016: "Problemi differenziali non lineari: esistenza, molteplicità e proprietà qualitative delle soluzioni".

Our main intent is to prove that, problem $(\mathcal{P}_\mu)$ preserves the multiplicity result which typifies the weaker form of the classical scheme, *zero, one or two solutions*, introduced for the first time in [3] by A. Ambrosetti and G. Prodi.

1.1. **Motivation in historical perspective.** First of all, we propose a brief review on Ambrosetti and Prodi problems. The intent is to outline chronologically how the classical assumptions in these problems have changed, in order to still guarantee results of multiplicity of solutions.

In [3, 4], Ambrosetti and Prodi considered the following Dirichlet problem:

$$\begin{cases} \Delta u + h(u) = v(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded open set, $v \in C^{0,\alpha}(\overline{\Omega})$ with $\alpha \in]0,1[$ and $h \in C^2(\mathbb{R})$ is a strictly convex function such that $h(0) = 0$ and

$$0 < h'(-\infty) < \lambda_1^\Omega(-\Delta) < h'(+\infty) < \lambda_2^\Omega(-\Delta) \tag{$\mathrm{Hp_{AP}}$}$$

with $\lambda_1^\Omega(-\Delta)$, $\lambda_2^\Omega(-\Delta)$ the first two eigenvalues of $-\Delta$ with Dirichlet boundary conditions on $\partial\Omega$. Under these assumptions, they proved that there exists a C^1 manifold $\mathcal{M}$ of codimension one which separates $C^{0,\alpha}(\overline{\Omega})$ into two disjoint open regions A_1 and A_2 such that $C^{0,\alpha}(\overline{\Omega}) = A_1 \cup \mathcal{M} \cup A_2$ and problem (1.1) has *zero solutions* if $v \in A_1$, *exactly one solution* if $v \in \mathcal{M}$ and *exactly two solutions* if $v \in A_2$. This pioneering result was followed by that of Manes and Micheletti [14], where the same statement was achieved by requiring, instead of $(\mathrm{Hp_{AP}})$, that

$$-\infty \le h'(-\infty) < \lambda_1^\Omega(-\Delta) < h'(+\infty) < \lambda_2^\Omega(-\Delta). \tag{$\mathrm{Hp_{MM}}$}$$

From the conditions in $(\mathrm{Hp_{MM}})$, we can thus observe that the positivity of $h'(-\infty)$ is not necessary. On the contrary, the main assumption is that the derivative of the nonlinearity has to cross the first eigenvalue when u goes from $-\infty$ to $+\infty$ (from which the name of "asymmetric crossing nonlinearity", see [12]).

Another seminal work in that context was done by Berger and Podolak [6], where the previous abstract description of the solution set was proposed by the Authors in a more explicit manner. This was made possible by writing the function v as $v(x) = \mu\phi_1(x) + w(x)$, with $\int_\Omega \phi_1(x)w(x)\,dx = 0$ where ϕ_1 is the first positive eigenfunction associated with $\lambda_1^\Omega(-\Delta)$. This way, they proved that there exists $\mu^* \in \mathbb{R}$ such that problem (1.1) has zero solutions if $\mu < \mu^*$, exactly one solution if $\mu = \mu^*$ and exactly two solutions if $\mu > \mu^*$. The next important contribution comes form the work [13] of Kazdan and Warner, who exploited the technique of upper and lower solutions, to obtain an existence result generalizing the assumptions on the nonlinear term. Indeed, they considered a problem of the following form

$$\begin{cases} \Delta u + f(x,u) = \mu\phi_1(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where f is sufficiently smooth and satisfies

$$-\infty \le \limsup_{s \to -\infty} \frac{f(x,s)}{s} < \lambda_1^\Omega(-\Delta) < \liminf_{s \to +\infty} \frac{f(x,s)}{s} \le +\infty, \tag{$\mathrm{Hp_{KW}}$}$$

uniformly in $x \in \overline{\Omega}$. Notice that the assumptions in $(\mathrm{Hp_{KW}})$ are weaker than the ones in $(\mathrm{Hp_{MM}})$. In [13], one of the main results states that there exists $\mu^* \in \mathbb{R}$ such that problem (1.2) has zero solutions if $\mu < \mu^*$ and at least one solution if $\mu > \mu^*$. Multiplicity results of Ambrosetti-Prodi type were then obtained by many other authors combining this latter tool with the degree theory or the fixed point index theory (see [1, 7, 5] and the survey [9] for a complete list of references).

In parallel, issues concerning periodic boundary conditions or Neumann boundary conditions were addressed as well, see [10, 11, 15, 16, 17, 18]. In a $2\mathrm{T}-$periodic environment, the Dirichlet problem (1.2) can be transposed as follows

$$\begin{cases} u'' + f(x,u) = \mu, \\ u(0) - u(2\mathrm{T}) = u'(0) - u'(2\mathrm{T}) = 0. \end{cases} \tag{1.3}$$

In that context, the results of Fabry, Mawhin and Nkashama [10], which are in particular applied to Liénard differential equations, show that the set of solutions of problem (1.3) follows a scheme of Ambrosetti-Prodi type. In such a case, a weaker hyphotesis than that assumed in ($\mathrm{Hp_{KW}}$) leads to the announced result, namely if the continuous function $f\colon [0,2\mathrm{T}]\times\mathbb{R}\to\mathbb{R}$ is such that

$$\lim_{|s|\to+\infty} f(x,s) = +\infty \quad \text{uniformly on } [0,2\mathrm{T}], \tag{$\mathrm{Hp_{FMN}}$}$$

then there exists $\mu^* \in \mathbb{R}$ such that problem (1.3) has zero solutions if $\mu < \mu^*$, at least one solution if $\mu = \mu^*$ and at least two solutions if $\mu > \mu^*$ (see [15, p. 296]). In comparison to this work, in [18] Ortega considered the problem

$$\begin{cases} u'' + cu' + h(u) = v(x), \\ u(0) - u(2\mathrm{T}) = u'(0) - u'(2\mathrm{T}) = 0, \end{cases} \tag{1.4}$$

where $h \in C^2(\mathbb{R})$ is a strictly convex function which satisfies a condition analogous to ($\mathrm{Hp_{MM}}$):

$$-\infty \le h'(-\infty) < 0 < h'(+\infty) \le \Gamma_1, \tag{$\mathrm{Hp_O}$}$$

with $\Gamma_1 = (\pi/\mathrm{T})^2 + c^2/4$. This way, the existence of a closed connected C^1 manifold of codimension one in the space of the $2\mathrm{T}-$periodic and continuous solutions was ensured with the proprieties related to the ones that come from [3].

Notwithstanding these original contributions, the periodic case is still a problem that deserves to be studied, as observed by Ambrosetti in his recent note devoted to "some global inversion theorems with applications to semilinear elliptic equations", in memory of Prodi (see [2, p. 13]).

The present work, continues the study of the Ambrosetti-Prodi periodic problem that we started in [21]. In fact, in [21] we proved, with several approaches, that the periodic problem (1.4) may present a set of periodic solutions with a very complicated behavior. In view of the stability results achieved by Ortega in [17, 18], we have shown that different levels of "chaos" appear when $h'(+\infty)$ in ($\mathrm{Hp_O}$) skips away from Γ_1, which plays the role of the second eigenvalue of $-d^2/dx^2$ with periodic boundary conditions. Now, instead, we focus on condition ($\mathrm{Hp_{FMN}}$), addressing the question whether the uniformity condition on the interval $[0,2\mathrm{T}]$ could be removed.

Let us observe that the Neumann problem on the interval $[0,\mathrm{T}]$ can be viewed as a subproblem of the periodic problem on the interval $[0,2\mathrm{T}]$, since one can find a $2\mathrm{T}-$periodic solution starting from a solution to the Neumann problem on the interval $[0,\mathrm{T}]$ via a even reflection and a periodic extension.

The contribution of Fabry, Mawhin and Nkashama is adapted to Neumann boundary conditions in [15]. Thus, an analogous result to the periodic one holds for the following problem:

$$\begin{cases} u'' + f(x,u) = \mu, \\ u'(0) = u'(\mathrm{T}) = 0. \end{cases} \tag{1.5}$$

Theorem 1.1 (Mawhin, [15]). *Assume that $f\colon [0,\mathrm{T}]\times\mathbb{R}\to\mathbb{R}$ is a continuous function which satisfies*

$$\lim_{|s|\to+\infty} f(x,s) = +\infty \quad \textit{uniformly on } [0,\mathrm{T}] \tag{$\mathrm{Hp_M}$}$$

then, there exists $\mu^*\in\mathbb{R}$ such that problem (1.5) has zero solutions if $\mu<\mu^*$, at least one solution if $\mu=\mu^*$ and at least two solutions if $\mu>\mu^*$.

Remark. Problem $(\mathcal{P}_\mu)$, belongs to the setting of the work of Fabry, Mawhin and Nkashama, by assuming that $p\in L^\infty(0,\mathrm{T})$, the weight term $a\in L^\infty(0,\mathrm{T})$ is such that

$$\operatorname{ess\,inf}_{x\in[0,\mathrm{T}]} a(x) \geq a_0 > 0$$

and the function $g\in C(\mathbb{R})$ satisfies

$$\lim_{|s|\to+\infty} g(s) = +\infty. \tag{g_*}$$

The remark above is the reason why we have proposed a possible generalization of the result carried out in [15].

1.2. **Main result and structure of the paper.** In the present paper we generalize the condition in $(\mathrm{Hp_M})$ to one without uniformity requirement, when the function $f\colon[0,\mathrm{T}]\times\mathbb{R}\to\mathbb{R}$ in problem (1.5) is defined as

$$f(x,u) := a(x)g(u) - p(x).$$

In more detail, we study the problem $(\mathcal{P}_\mu)$ allowing the $\operatorname{ess\,inf} a$ to be zero. This is done by assuming that the weight term $a(x)$ satisfies (a_*). Actually, for $f(x,u)$ having the special form as above, the uniform requirement on $(\mathrm{Hp_M})$ is considerably relaxed. Indeed, $f(x,u)$ can even vanish identically on sets of positive measure. Meanwhile, we suppose that the function $g(u)$ is a crossing nonlinearity. Assuming (g_∞), here is the statement of our main result.

Theorem 1.2. *Let $p\in L^\infty(0,\mathrm{T})$. Assume that $g\in C^1(\mathbb{R})$ is a function which satisfies (g_∞). Moreover, suppose that $a\in L^\infty(0,\mathrm{T})$ is such that conditions in (a_*) hold. Then, there exists $\mu^*\in\mathbb{R}$ such that problem $(\mathcal{P}_\mu)$ has at least two solutions for all $\mu>\mu^*$.*

In the sequel, without loss of generality we suppose also that

$$g(0)=0, \qquad g'(s)<0 \quad \forall s<0. \tag{g_0}$$

In fact, from (g_∞) there exists $r_0<0$ such that $g'(s)<0$ for each $s<r_0$. Therefore, taking $z:=u-r_0$, we can consider the equivalent Neumann problem associated with $z''+a(x)\tilde{g}(z)=\mu+\tilde{p}(x)$ where $\tilde{p}(x)=p(x)-a(x)g(r_0)$ and $\tilde{g}(z):=g(z+r_0)-g(r_0)$ satisfies (g_0).

The plan of the paper is as follows. In Section 2, we give an existence result for a suitable truncated problem and then we show that the solution of the auxiliary problem is actually a solution of the original one. In Section 3, we carry out the proof of Theorem 1.2 and we show the existence of a second solution. The main technique adopted in both sections to find the solutions of $(\mathcal{P}_\mu)$ is the so called shooting method. Finally, in Section 4, we conclude with a discussion of further lines of work.

2. **Existence result.** In this section, we consider the Neumann problem associated with the equation

$$u'' + \varphi_\mu(x,u) = 0, \tag{2.1}$$

where

$$\varphi_\mu(x,s) := \begin{cases} a(x)g(s) - \mu - p(x) & s \leq 0, \\ -\mu - p(x) & s > 0, \end{cases}$$

which coincides with $(\mathcal{P}_\mu)$ when $u(x) \leq 0$. We are going to prove that the modified Neumann problem has at least one negative solution when μ exceeds some value μ^* which will therefore solve problem $(\mathcal{P}_\mu)$.

In our framework both uniqueness and global existence for the solutions of the associated Cauchy problems is guaranteed. Thus, let $u(\cdot\,; u_0, u_1)$ be the unique and globally defined on $[0,\mathrm{T}]$ solution of the equation (2.1) satisfying the initial values

$$u(0) = u_0 \in \mathbb{R}, \quad u'(0) = u_1 \in \mathbb{R}. \tag{2.2}$$

We recall that, for every $\mu \in \mathbb{R}$ fixed, the Poincaré map for (2.1) on the interval $[0,\mathrm{T}]$ is the well defined map

$$\Phi_0^{\mathrm{T}} \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (u_0,u_1) \mapsto (u(\mathrm{T}), u'(\mathrm{T}))$$

where $u(x)$ is the solution of (2.1) with the initial values (2.2). Moreover, the standard theory of ordinary differential equations guarantees that the Poincaré map is actually a global diffeomorphism of the plane onto itself.

The recipe of the shooting method states that a solution of the Neumann problem, associated with the equation (2.1), can be found by looking for a point $(u_0, 0) \in \mathbb{R}^2$ such that $\Phi_0^{\mathrm{T}}(u_0,0) \in \mathbb{R} \times \{0\}$. The strategy is thus to prove that, for any $A > 0$, the Poincaré map associated with (2.1) is such that $\Phi_0^{\mathrm{T}}(A,0) \in \mathbb{R}^+ \times \mathbb{R}^+$, while, for any $-B \ll 0$, we have $\Phi_0^{\mathrm{T}}(-B,0) \in \mathbb{R}_0^- \times \mathbb{R}^-$. A continuity argument then leads to the existence of $C \in]-B, A[$ such that $\Phi_0^{\mathrm{T}}(C,0) \in \mathbb{R} \times \{0\}$.

Accordingly, we state the following preliminary lemmas.

Lemma 2.1. *Let $\mu > \operatorname{ess\,sup}_{x\in[0,\mathrm{T}]} -p(x)$. Then, for any fixed $u_0 > 0$, the solution of (2.1) with initial values $u(0) = u_0$ and $u'(0) = 0$ is such that $u(x) > 0$ for all $x \in [0,\mathrm{T}]$ and $u'(x) > 0$ for all $x \in]0,\mathrm{T}]$.*

The proof of Lemma 2.1 requires a straightforward argument by contradiction.

Lemma 2.2. *Let $\mu \in \mathbb{R}$, $p \in L^\infty(0,\mathrm{T})$ and $a \in L^\infty(0,\mathrm{T})$ satisfies (a_*). Assume that $g \in C^1(\mathbb{R})$ satisfies (g_∞) and (g_0). Then, there exists $r_\mu < 0$ such that for any fixed $u_0 < r_\mu$, if u is a solution of (2.1) with initial values $u(0) = u_0$ and $u'(0) = 0$, then $u(x) < 0$ for each $x \in [0,\mathrm{T}]$ and, moreover, $u'(\mathrm{T}) < 0$.*

Proof. As long as $u(x)$ is negative, integrating equation (2.1) two times with respect to x we obtain

$$u(x) = u_0 - \int_0^x \left(\int_0^t a(\xi) g(u(\xi))\, d\xi \right) dt + \frac{\mu x^2}{2} + \int_0^x P(\xi)\, d\xi, \tag{2.3}$$

where $P(x) := \int_0^x p(\xi)\, d\xi$. Since the function g satisfies conditions in (g_0), from equation (2.3), we get

$$u(x) \leq u_0 + \frac{\mu \mathrm{T}^2}{2} + \|P\|_{L^1}.$$

For any $\mu \in \mathbb{R}$, we define

$$M_\mu = M(\mu) := \frac{\mu \mathrm{T}^2}{2} + \|P\|_{L^1}. \tag{2.4}$$

Then, from the choice of $u_0 < -M_\mu$, we have that for each $x \in [0, \mathrm{T}]$

$$u(x) \leq u_0 + M_\mu < 0. \tag{2.5}$$

Now we prove $u'(\mathrm{T}) < 0$. An integration on $[0, \mathrm{T}]$ of equation (2.1) leads to the following inequality:

$$u'(\mathrm{T}) \leq -\int_0^{\mathrm{T}} a(\xi) g(u(\xi))\, d\xi + \mu \mathrm{T} + \|p\|_{L^1}.$$

Recalling (2.5) and (g_0), which implies that the function g is strictly decreasing on $[0, -\infty)$, by using the previous inequality we obtain

$$u'(\mathrm{T}) \leq -g(u_0 + M_\mu)\|a\|_{L^1} + \mu \mathrm{T} + \|p\|_{L^1}.$$

From assumption (g_∞) follows $g(s) \to +\infty$ as $s \to -\infty$. Therefore, there exists $m_\mu = m(\mu) > 0$ such that

$$g(u) > \alpha := \frac{\mu \mathrm{T} + \|p\|_{L^1}}{\|a\|_{L^1}}, \quad \forall u < -m_\mu.$$

Now, choose $r_\mu = r(\mu) := -(m_\mu + M_\mu)$. Then, $g(u_0 + M_\mu) > \alpha$ for each $u_0 < r_\mu$. Consequently, taking $u_0 < r_\mu$, we achieve the thesis, since $u'(T) < 0$ and $u(x) < 0$ for all $x \in [0, \mathrm{T}]$. □

Let us state the main result of this section.

Theorem 2.3. *Let $p \in L^\infty(0, \mathrm{T})$ and $a \in L^\infty(0, \mathrm{T})$ satisfies (a_*). Assume that $g \in C^1(\mathbb{R})$ satisfies (g_∞) and (g_0). Then, there exists $\mu^* \in \mathbb{R}$ such that for each $\mu > \mu^*$ problem $(\mathcal{P}_\mu)$ has at least one solution.*

Proof. Let us set

$$\mu^* := \operatorname{ess\,sup}_{x \in [0,\mathrm{T}]} - p(x). \tag{2.6}$$

We perform the proof in two steps. In the first step, we will show that for every fixed $\mu > \mu^*$ the Neumann problem associated with the truncated equation (2.1) has at least a solution u. This will be proved using the shooting method. In more detail, we will provide initial conditions $u(0) = u_0$ and $u'(0) = 0$, to the Cauchy problem associated with (2.1), such that the corresponding solution u verifies $u(\mathrm{T}) = 0$. In the second step, we will prove that the solution of the truncated problem (2.1) under Neumann boundary conditions is a solution of $(\mathcal{P}_\mu)$. Hence, we will show that this solution is negative, namely $u(x) < 0$ for all $x \in [0, \mathrm{T}]$.

Step 1. We claim that for every $\mu > \mu^*$ there exists $C_1 \in \mathbb{R} \times \{0\}$ such that

$$\Phi_0^{\mathrm{T}}(C_1, 0) \in \mathbb{R} \times \{0\}.$$

Let us fix $\mu > \mu^*$. We choose a point $(A, 0) \in \mathbb{R}^+ \times \{0\}$ and we denote by u_A the solution of (2.1) with initial conditions $u(0) = A$ and $u'(0) = 0$. An application of Lemma 2.1 leads to

$$\Phi_0^{\mathrm{T}}(A, 0) = \big(u_A(\mathrm{T}), u_A'(\mathrm{T})\big) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

On the other hand, thanks to Lemma 2.2 there exists a value $r_\mu < 0$ such that, if we select a point $(-B, 0) \in \mathbb{R}^- \times \{0\}$ with $-B < r_\mu$ and we denote by u_B the solution of (2.1) with initial conditions $u(0) = -B$ and $u'(0) = 0$ then

$$\Phi_0^{\mathrm{T}}(-B, 0) = \big(u_B(\mathrm{T}), u_B'(\mathrm{T})\big) \in \mathbb{R}^- \times \mathbb{R}^-.$$

At this point, the continuous dependence of the solutions upon the initial data implies that there exists $C_1 \in]-B, A[$ such that the solution u_{C_1} of (2.1) with initial conditions $u(0) = C_1$ and $u'(0) = 0$ verifies $\Phi_0^{\mathrm{T}}(C_1, 0) = \big(u_{C_1}(\mathrm{T}), u'_{C_1}(\mathrm{T})\big) \in \mathbb{R}\times\{0\}$, and thus the claim is proved. The solution u_{C_1} is in turn a solution of the truncated equation (2.1) with Neumann boundary conditions since $u'_{C_1}(\mathrm{T}) = 0$.

Step 2. We claim that $u_{C_1}(x) < 0$ for every $x \in [0, \mathrm{T}]$.

First of all, if $u_{C_1}(x) \geq 0$ for all $x \in [0, \mathrm{T}]$, then we achieve a contradiction since $\varphi_\mu(x, u_{C_1}(x)) < 0$ for a.e. $x \in [0, \mathrm{T}]$. Now, using a standard maximum principle argument, it is easy to prove that there exists $\delta > 0$ (which depends on the fixed parameter μ) such that $u_{C_1}(x) \leq -\delta$ for all $x \in [0, \mathrm{T}]$. The same conclusion can be derived also from standard facts from the theory of upper and lower solutions (see [8]).

The proof is thus concluded because we have found a solution of the Neumann problem $(\mathcal{P}_\mu)$. $\square$

3. **Multiplicity result.**

The results achieved in Section 2 suggest to define $\mu^* \in \mathbb{R}$ as in (2.6), and so, from now on, we take a fixed value μ with $\mu > \mu^*$. Therefore, Theorem 2.3 ensures the existence of at least a solution of $(\mathcal{P}_\mu)$. Let us call it $\tilde{u}$.

In this section, we provide the proof of Theorem 1.2.

Lemma 3.1. *Let $\mu > \mu^*$, $p \in L^\infty(0, \mathrm{T})$ and $a \in L^\infty(0, \mathrm{T})$ satisfies (a_*). Assume that $g \in C^1(\mathbb{R})$ satisfies (g_∞) and (g_0). Then, there exists $\varepsilon > 0$ such that if u_ε is a solution of $u'' + a(x)g(u) = \mu + p(x)$ with initial values $u(0) = \tilde{u}(0) + \varepsilon$ and $u'(0) = 0$, then $u'_\varepsilon(\mathrm{T}) > 0$.*

Proof. Let us take $v_\varepsilon(x) := u_\varepsilon(x) - \tilde{u}(x)$. The Cauchy problem considered here can be equivalently described by the differential equation

$$v''_\varepsilon + a(x)\big(g(v_\varepsilon + \tilde{u}(x)) - g(\tilde{u}(x))\big) = 0 \tag{3.1}$$

with initial conditions $v_\varepsilon(0) = \varepsilon$ and $v'_\varepsilon(0) = 0$.

We claim that there exists $\varepsilon > 0$ such that $v'_\varepsilon(\mathrm{T}) > 0$. To check this assertion, since $g \in C^1(\mathbb{R})$, from equation (3.1) we have

$$v''_\varepsilon + a(x)\mathcal{B}_\varepsilon(x)v_\varepsilon = 0 \tag{3.2}$$

where

$$\mathcal{B}_\varepsilon(x) := \int_0^1 g'\big(\tilde{u}(x) + \theta v_\varepsilon(x)\big)\, d\theta.$$

Next, by the continuous dependence of the solutions upon the initial data, it follows that $v_\varepsilon(x) \to 0$ uniformly in x as $\varepsilon \to 0^+$. As a consequence, there exists $\varepsilon^* \ll 1$ such that, for each $0 < \varepsilon < \varepsilon^*$ we have $\tilde{u}(x) + \theta v_\varepsilon(x) < 0$ for all $x \in [0, \mathrm{T}]$ and for all $\theta \in [0, 1]$. This way, recalling (g_0), we obtain that $\mathcal{B}_\varepsilon(x) < 0$ for each $0 < \varepsilon < \varepsilon^*$.

Thus, if we prove that $v_\varepsilon(x) > 0$ for every $x \in [0, \mathrm{T}]$, then the claim is verified. Arguing by contradiction, let us suppose that there exists a first point $x^* \in]0, \mathrm{T}]$ such that $v_\varepsilon(x^*) = 0$. Then, from (3.2) we deduce that $v_\varepsilon(x) \geq v_\varepsilon(0) = \varepsilon > 0$ for all $x \in [0, x^*]$, a contradiction with respect to $v_\varepsilon(x^*) = 0$. The proof is concluded since $u'_\varepsilon(\mathrm{T}) = v'_\varepsilon(\mathrm{T}) > 0$. $\square$

Lemma 3.2. *Let $\mu > \mu^*$, $p \in L^\infty(0, \mathrm{T})$ and $a \in L^\infty(0, \mathrm{T})$ satisfies (a_*). Assume that $g \in C^1(\mathbb{R})$ satisfies (g_∞) and (g_0). Then, there exists $R_\mu > 0$ such that for any fixed $u_0 > R_\mu$, if u is a solution of $u'' + a(x)g(u) = \mu + p(x)$ with initial values $u(0) = u_0$ and $u'(0) = 0$, then $u'(\mathrm{T}) < 0$.*

Proof. From assumptions (g_0) and (g_∞), it follows that there exists a global minimum g_{min} of g on $\mathbb{R}$ such that

$$g_{min} = \min_{s\in[0,+\infty[} g(s) \leq g(0) = 0.$$

Accordingly, from

$$u'' = -a(x)g(u) + \mu + p(x), \tag{3.3}$$

we get the differential inequality

$$u'' \leq -a(x)g_{min} + \mu + p(x).$$

Now, integrating on $[x_1, x_2] \subseteq [0, \mathrm{T}]$, we have

$$u'(x_2) \leq u'(x_1) - g_{min}\|a\|_{L^1} + \mu\mathrm{T} + \|p\|_{L^1}.$$

Then, we fix a constant $K_\mu = K(\mu) > 0$ such that

$$K_\mu > -g_{min}\|a\|_{L^1} + \mu\mathrm{T} + \|p\|_{L^1}.$$

We claim that there exists $x_1 \in [0, \mathrm{T}]$ such that $u'(x_1) < -K_\mu$. From this fact, it immediately follows that $u'(\mathrm{T}) < 0$. To check the claim, suppose by contradiction that $u'(x) \geq -K_\mu$ for every $x \in [0, \mathrm{T}]$. It clearly follows that $u(x) \geq u_0 - K_\mu\mathrm{T}$ for every $x \in [0, \mathrm{T}]$.

From assumption (g_∞) we deduce that $g(s) \to +\infty$ as $s \to +\infty$, which implies that there exists $k_\mu = k(\mu) > 0$ such that

$$g(s) > \beta := \frac{K_\mu + \mu\mathrm{T} + \|p\|_{L^1}}{\|a\|_{L^1}}, \quad \forall s > k_\mu.$$

Now, choose $R_\mu = R(\mu) := k_\mu + K_\mu\mathrm{T} > 0$ and take $u_0 > R_\mu$. In this manner we obtain $u(x) > k_\mu$ for every $x \in [0, \mathrm{T}]$. An integration on $[0, \mathrm{T}]$ of (3.3) yields to a contradiction, since

$$u'(\mathrm{T}) \leq -\inf_{u>k_\mu} g(u)\|a\|_{L^1} + \mu\mathrm{T} + \|p\|_{L^1} < -\beta\|a\|_{L^1} + \mu\mathrm{T} + \|p\|_{L^1} = -K_\mu.$$

Our claim is thus verified and this completes the proof. □

We remark that the assumption that the function g is of class C^1 is crucial only in the proof of Lemma 3.1. For all the other auxiliary lemmas, the condition (g_*) is enough to achieve the conclusions. On the other hand, due to the nature of our approach based on the shooting method, we need to require uniqueness of the solutions for the initial value problems and their global continuability. Accordingly, in place of (g_∞), one can recover the main results of Theorem 1.2 and Theorem 2.3, by assuming that the nonlinearity $g \in C^1(\mathbb{R})$ satisfies (g_0) coupled with (g_*) and the global continuability for the Cauchy problems associated with the differential equation in $(\mathcal{P}_\mu)$.

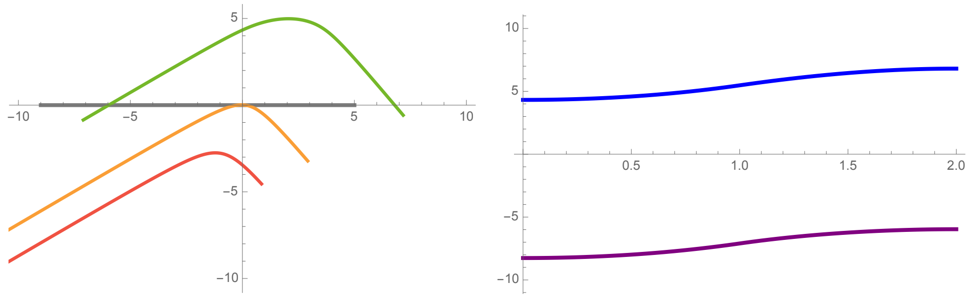
Now, we are ready to complete the proof of our multiplicity result, looking for a second solution of $(\mathcal{P}_\mu)$. For ease of notation, we still denote by Φ_0^{T} the Poincaré map associated with the differential equation in $(\mathcal{P}_\mu)$.

Proof of Theorem 1.2. Let $\mu > \mu^*$ be fixed. Theorem 2.3 guarantees the existence of a solution $\tilde{u}$ to problem $(\mathcal{P}_\mu)$. In view of Lemma 3.1, there exists a sufficiently small constant ε such that, if we choose a point $(D, 0) \in \mathbb{R} \times \{0\}$ with $D := \tilde{u}(0) + \varepsilon$ and we denote by u_D the solution of $u'' + a(x)g(u) = \mu + p(x)$ with initial conditions $u(0) = D$ and $u'(0) = 0$, then we have $\Phi_0^{\mathrm{T}}(D, 0) = \big(u_D(\mathrm{T}), u_D'(\mathrm{T})\big) \in \mathbb{R} \times \mathbb{R}^+$. Clearly, we can take $D < 0$.

On the other hand, in view of Lemma 3.2, we can find a sufficiently large constant E depending on μ such that, if we choose the point $(E,0)\in\mathbb{R}^+\times\{0\}$ and we denote by u_E the solution of $u''+a(x)g(u)=\mu+p(x)$ with initial conditions $u(0)=E$ and $u'(0)=0$, then it follows $\Phi_0^{\mathrm{T}}(E,0)=\big(u_E(\mathrm{T}),u_E'(\mathrm{T})\big)\in\mathbb{R}\times\mathbb{R}^-$.

Finally, the existence of a second solution to problem $(\mathcal{P}_\mu)$ follows again by the continuous dependence of the solutions upon the initial data. Indeed, there exists $C_2\in]D,E[$ such that the solution u_{C_2} of $u''+a(x)g(u)=\mu+p(x)$ with initial conditions $u(0)=C_2$ and $u'(0)=0$ verifies $\Phi_0^{\mathrm{T}}(C_2,0)=\big(u_{C_2}(\mathrm{T}),u_{C_2}'(\mathrm{T})\big)\in\mathbb{R}\times\{0\}$. The proof is thus completed. $\square$

We conclude this section with an example that illustrates the results reached.



(A) In the phase plane (u,u'), images of the segment $[-9,5]\subseteq\mathbb{R}\times\{0\}$ (gray) through the action of the Poincaré map varying the parameter μ. Considered values are $\mu=-2$ (red), $\mu=-0.7$ (yellow) and $\mu=2$ (green). Consistently with Theorem 1.2, having chosen μ sufficiently large, the green line intersects the u-axis two times.

(B) Approximation of two solution of the Neumann problem $(\mathcal{P}_\mu)$ with $\mu=2$. The profiles are sketched via an estimate of the intersection points between $\mathbb{R}\times\{0\}$ and the image of the segment $[-9,5]\subseteq\mathbb{R}\times\{0\}$ through the action of the Poincaré map. As expected from the proof of Theorem 2.3, one solution is negative.

FIGURE 1. Numerical simulations for the Neumann problem $(\mathcal{P}_\mu)$ defined as in Example.

Example. Consider the equation $u''+a(x)g(u)=\mu+p(x)$ with Neumann boundary conditions on $[0,2]$, where $g(s)=\sqrt{1+s^2}-1$, $p(x)=\sin(x)$ and

$$a(x)=\begin{cases}0 & \text{for } x\in[0,1[,\\ 1 & \text{for } x\in[1,2].\end{cases}$$

The assumptions of Theorem 1.2 are all fulfilled and so there exists $\mu^*\in\mathbb{R}$ such that for every $\mu>\mu^*$ the Neumann problem has at least two solution. This is a typical example that can not be treated within the framework built up in [15], since $f(x,u)=a(x)g(u)$ does not tend uniformly to infinity.

In Figure 1, we show how these two solutions could be found by applying the shooting method. We stress that the present work was limited to prove a result of existence of at least two solutions for problem $(\mathcal{P}_\mu)$. Nonetheless, by means of some numerical simulations, for this example we point out that the solution set could be here described by the classical scheme of zero, one or two solutions, which is a typical feature in Ambrosetti-Prodi results.

4. **Conclusion.** We end the present paper with some ideas for future work. In particular, we list two problems that we propose to address.

Taking into account the results achieved by Fonda and Sfecci in [11], we argue whether our result still holds even if the nonlinear term g in $(\mathcal{P}_\mu)$ has one or two singularities. It could be interesting to further understand the following problem:

> Extend the result of Theorem 1.2 to a nonlinearity $g\colon (a,b) \to \mathbb{R}$ where $-\infty \leq a < b \leq +\infty$ and such that $\lim_{s\to a^+} g(s) = \lim_{s\to b^-} g(s) = +\infty$.

On the other hand, as observed in the Introduction, the structure of differential problems under Neumann boundary conditions is very similar to that under periodic boundary conditions. Accordingly, it seems interesting to ask if the classical result of Fabry, Mawhin and Nkashama [10] can be generalized in view of Theorem 1.2. In their work, the Authors considered the Liénard differential equation

$$(\mathcal{E}_\mu) \qquad u'' + q(u)u' + f(x,u) = \mu$$

where $\mu \in \mathbb{R}$, $q \in C(\mathbb{R})$ and f is a continuous function $2\pi-$periodic in x. Their main result reads as follow. If the nonlinearity f satisfies

$$\lim_{|s|\to+\infty} f(x,s) = +\infty \quad \text{uniformly on } [0,2\pi], \qquad (\mathrm{Hp_{FMN}})$$

then, there exists a number μ^* such that equation $(\mathcal{E}_\mu)$ has zero periodic solutions if $\mu < \mu^*$, at least one solution if $\mu = \mu^*$ and at least two solutions if $\mu > \mu^*$ (see [10, Theorem 3]). Thus, the following question arises.

> *Open problem.* Can we provide the same result of Fabry, Mawhin and Nkashama without assuming uniformity in $(\mathrm{Hp_{FMN}})$?

Acknowledgments. The author is grateful to F. Zanolin for all the helpful discussions and his invaluable support and to the anonymous reviewers of this paper which helped to improve it. The author would also like to thank P. Candito, R. Livrea, S. Marano and G. D'Aguì for the invitation at the special session SS92 of the 2016 AIMS Conference in Orlando, which was one of the main motivations to continue studying and stay focused on the topic.

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Received February 2017; revised May 2017.

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