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Indefinite weight nonlinear problems with Neumann boundary conditions ☆

Elisa Sovrano *, Fabio Zanolin [1]

Department of Mathematics, Computer Science and Physics, University of Udine, via delle Scienze 206, 33100 Udine, Italy

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A B S T R A C T

We present a multiplicity result of positive solutions for the Neumann problem associated with a second order nonlinear differential equation of the following form $u'' + a(t)g(u) = 0$, where the weight function $a(t)$ has indefinite sign. The only assumption we make for the nonlinear term $g(u)$ is that its primitive $G(u)$ presents some oscillations at infinity, expressed by the condition involving $\underline{\lim}\, G(u)/u^2 = 0 < \overline{\lim}\, G(u)/u^2$. As an application, we obtain multiple radially symmetric solutions for Neumann problems associated with $\Delta u + a(x)g(u) = 0$.

1. Introduction

We deal with the study of positive solutions for indefinite ordinary differential equations with Neumann boundary conditions on a compact interval $[0, T]$. More in detail the problem analyzed is the following:

$$\begin{cases} u'' + a(t)g(u) = 0, \\ u(t) > 0, \quad \forall\, t \in [0, T], \\ u'(0) = u'(T) = 0. \end{cases} \tag{$\mathscr{P}$}$$

Setting $\mathbb{R}^+ := [0, +\infty)$, we assume that the nonlinear term $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying the conditions:

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* Corresponding author.
E-mail addresses: sovrano.elisa@spes.uniud.it (E. Sovrano), fabio.zanolin@uniud.it (F. Zanolin).

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(g_*) $g(0) = 0$ *and* $g(s) > 0$ *for all* $s > 0$,

(G_*) $G_\infty := \liminf_{s\to+\infty} \dfrac{2G(s)}{s^2} = 0 < G^\infty := \limsup_{s\to+\infty} \dfrac{2G(s)}{s^2}$,

where

$$G(s) := \int_0^s g(\xi)d\xi.$$

The requirements on the nonlinearity as in (G_*) can be traced back to 1930 with a classical paper of Hammerstein [18]. In that work, the Author proved the existence of solutions to a nonlinear integral equation (nowadays called "Hammerstein equation") of the form

$$\psi(x) = \int_B K(x,y) f(y, \psi(y))\, dy,$$

under a linear growth assumption on the function f defined on $B \times \mathbb{R}$ and a non-resonance condition, which can be equivalently written as

$$\limsup_{u\to\pm\infty} \frac{2F(x,u)}{u^2} < \lambda_1, \text{ uniformly for } x \in B,$$

where

$$F(x,u) := \int_0^u f(x,s)\, ds.$$

Without entering in all the technical details, we recall that in [18], B is a one-dimensional or multi-dimensional bounded domain, $K(x,y)$ is a bounded symmetric and positive definite kernel and λ_1 is the first eigenvalue of the associated linear problem. The pioneering work of Hammerstein stimulated further researches about the solvability of nonlinear boundary value problems "below the first eigenvalue", by imposing conditions on the primitive of the nonlinearity (see [10,16,17,21] and the references therein). Applications to the Dirichlet problem, involving these kind of conditions, guarantee the existence of at least one solution for

$$\begin{cases} \Delta u + g(u) = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{D}$}$$

if $h \in L^\infty(\Omega)$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function with a suitable polynomial growth (depending on the Sobolev embeddings) such that

$$\limsup_{s\to\pm\infty} \frac{2G(s)}{s^2} < \lambda_1^{\mathscr{D}}(\Omega).$$

As usual, $\Omega \subseteq \mathbb{R}^N$ is assumed be a bounded domain with a sufficiently smooth boundary and we denote by $\lambda_1^{\mathscr{D}}(\Omega)$ the first eigenvalue of $-\Delta$ with the Dirichlet boundary conditions.

In the one-dimensional case $\Omega =]0, T[$, an improvement of this result was obtained in [13, Theorem 1], by replacing the Hammerstein type condition with

$$\liminf_{s\to\pm\infty} \frac{2G(s)}{s^2} < \left(\frac{\pi}{T}\right)^2 = \lambda_1^{\mathscr{D}}(\Omega).$$

Moreover, in that paper, the study of the one-dimensional Dirichlet boundary value problem, under the assumptions $g(s) \to +\infty$ for $s \to +\infty$ and

$$G_\infty < \left(\frac{\pi}{T}\right)^2 < G^\infty,$$

leads to the existence of infinitely many solutions $u(t) > 0$ for all $t \in]0, T[$ (see [13, Theorem 3]). Concerning the multiplicity of positive solutions for Dirichlet problems, further investigations have been performed from different points of view, considering also in [20,22,24] more general (nonlinear) differential operators.

In [24], for the weighted nonlinear problem on a general bounded domain $\Omega \subseteq \mathbb{R}^N$,

$$\begin{cases} \Delta u + a(x)g(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{D}_w$}$$

is guaranteed the existence of a sequence of solutions $u_n \geq 0$ in Ω such that $\max_{\overline{\Omega}} u_n \to +\infty$, if the nonlinearity $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies (g_*) and (G_*) with $G^\infty = +\infty$, provided that the weight function belongs to $L^\infty(\Omega)$ and $\operatorname{essinf}_\Omega a(x) > 0$. A further extension has been achieved by Obersnel and Omari in [23, Theorem 2.2], by proving the existence of two sequences of solutions $(u_n)_n$ and $(v_n)_n$ which are strictly positive on Ω and such that $\lim_{n\to+\infty} u_n(x)/\operatorname{dist}(x, \partial\Omega) = \lim_{n\to+\infty} v_n(x)/\operatorname{dist}(x, \partial\Omega) = +\infty$. The theorem of Obersnel and Omari applies to a sign-changing weight as well.

The treatment of these kind of problems, with respect to the Neumann boundary conditions, presents some peculiar features, due to the fact that the first eigenvalue of the associated linear problem is $\lambda_1^{\mathscr{N}}(\Omega) = 0$. Thus, dealing with the Neumann problem

$$\begin{cases} \Delta u + g(u) = h(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{N}$}$$

the Hammerstein non-resonance condition with respect to the first eigenvalue, expressed by $\limsup_{s\to\pm\infty} 2G(s)/s^2 < 0$, implies the existence of two sequences of real numbers $(w_n)_n$ and $(v_n)_n$ such that $w_n \to -\infty$ and $g(w_n) \to +\infty$, as well as, $v_n \to +\infty$ and $g(v_n) \to -\infty$. Hence, given any $h \in L^\infty(\Omega)$ we can find a pair (α, β) of constant lower- and upper-solutions with $\alpha < 0 < \beta$. This way, the problem becomes easily affordable via the theory of lower- and upper-solutions [9]. The interesting and more difficult question arises, whether the solvability of the Neumann problem occurs under a Hammerstein type non-resonance condition with respect to the second eigenvalue $\lambda_2^{\mathscr{N}}(\Omega)$, which is the first one positive. Existence results in this direction were carried out by Mawhin, Ward and Willem [21, Theorem 2] for a nonlinearity of the form $f(x, u)$ which satisfies a Hammerstein condition, without the need of uniformity and by Gossez and Omari [16,17] for the problem $(\mathscr{N})$ under non-resonance conditions with respect to the eigenvalue $\lambda_2^{\mathscr{N}}(\Omega)$ involving a combination of hypotheses on $g(s)/s$ and $2G(s)/s^2$.

As far as we know, in literature there aren't works about multiple positive solutions for the analogous of problem $(\mathscr{D}_w)$ with Neumann boundary conditions. More precisely, the study of a nonlinearity g satisfying (g_*) and (G_*) is still open for problem

$$\begin{cases} \Delta u + a(x)g(u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{N}_w$}$$

with $u > 0$ in $\overline{\Omega}$ even in the one-dimensional case.

On the other hand, several results of multiplicity can be found for Neumann problems associated with

$$\Delta u - k(x)u + a(x)g(u) = 0,$$

where $k(x) > 0$, or even for more general p-Laplacian type equations (see [2] and the references therein). The structure of this latter equation is however completely different to the one treated here.

If we look for *positive* solutions for the problem $(\mathscr{N}_w)$ under the assumptions (g_*) and $a(\cdot) \not\equiv 0$, then a necessary condition is that $a(\cdot)$ must change its sign (see [1]). This way, we are interested in problems with a *sign-indefinite weight* and a positive nonlinearity with an *oscillatory potential*, that, for $N = 1$ and $\Omega =]0, T[$, lead to problem $(\mathscr{P})$. For ease of discussion, we will focus our study to the simplified situation where the weight has a "positive hump" followed by a "negative hump". Actually we can consider more general cases, by allowing the existence of subintervals where the weight function identically vanishes. Namely, to fix our framework, we assume that there exists $\sigma \in]0, T[$ such that

(a_*) *$a(t) \geq 0$, $a \not\equiv 0$ for a.e. $t \in [0, \sigma]$, $a(t) \leq 0$, $a \not\equiv 0$ for a.e. $t \in [\sigma, T]$.*

Generally speaking, not any sign-indefinite weight is suitable to guarantee the existence of solutions to $(\mathscr{P})$. For instance, if g is continuously differentiable in $\mathbb{R}^+_0 :=]0, +\infty[$, with $g'(s) > 0$ for all $s > 0$, it is a well-known fact that a positive solution of the Neumann problem on $[0, T]$ may exist only if $\int_0^T a(t)\, dt < 0$. Moreover, other features connected to the graph of $g(\cdot)$, can require further conditions on the positive or negative part of $a(\cdot)$. Hence, it is convenient to consider a weight of the form

$$a_{\lambda,\mu}(t) := \lambda a^+(t) - \mu a^-(t), \tag{1.1}$$

for λ and μ given real positive parameters. In this manner, problem $(\mathscr{P})$ reads as

$$\begin{cases} u'' + a_{\lambda,\mu}(t)g(u) = 0, \\ u(t) > 0, \quad \forall\, t \in [0, T], \\ u'(0) = u'(T) = 0. \end{cases} \tag{$\mathscr{P}_{\lambda,\mu}$}$$

Notice that solutions of the differential equation in $(\mathscr{P}_{\lambda,\mu})$ will be considered in the Carathéodory sense and, clearly, are classical C^2-solutions when the weight function is continuous. As a corollary of our main result (see Theorem 3.1 in Section 3), the following theorem holds.

Theorem 1.1. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and (G_*) with $s \mapsto g(s)/s$ upper bounded in a right neighborhood of 0. Let $a : [0, T] \to \mathbb{R}$ be a bounded piecewise continuous function satisfying (a_*). Then, there exists $\lambda^* \geq 0$ such that, for each $\lambda > \lambda^*$, $r > 0$ and for every integer $k \geq 1$, there is a constant $\mu^* = \mu^*(\lambda, r, k) > 0$ such that for each $\mu > \mu^*$ the problem $(\mathscr{P}_{\lambda,\mu})$ has at least k solutions which are nonincreasing on $[0, T]$ and satisfy $0 < u(t) \leq r$, for each $t \in [\sigma, T]$. Moreover, if $G^\infty = +\infty$, the result holds with $\lambda^* = 0$.*

The method of the proof is based on a careful analysis of the trajectories of the associated phase-plane system

$$\begin{cases} x' = y, \\ y' = -(\lambda a^+(t) - \mu a^-(t))g_M(x), \end{cases} \tag{$S_{\lambda,\mu}$}$$

where, given a fixed constant $M > 0$, we have denoted by $g_M(x)$ the truncated function

$$g_M(x) = \begin{cases} 0, & \text{if } x < 0, \\ g(x), & \text{if } 0 \leq x \leq M, \\ g(M), & \text{if } x > M. \end{cases} \tag{1.2}$$

Positive solutions of the Neumann problem will be obtained by means of the shooting-type method applied to system $(S_{\lambda,\mu})$, starting from the positive half-axis $X^+ := \{(x, 0) : x > 0\}$ and hitting again X^+ at the time $t = T$. Notice that, by construction, the solutions $(x(t), y(t))$ we find are such that $x'(t) = y(t) \leq 0$ on $[0, T]$. Hence, $u(t) = x(t)$ is nonincreasing on $[0, T]$ and therefore is a solution of $(\mathscr{P}_{\lambda,\mu})$ provided that $u(0) \leq M$.

The plan of the paper is the following. In Section 2 we introduce and prove some technical lemmas for understanding the behavior of the solutions of the equations $u'' + \lambda a^+(t)g(u) = 0$ and $u'' - \mu a^-(t)g(u) = 0$, separately. By these results, in Section 3, we prove a theorem about multiplicity of solutions for $(\mathscr{P}_{\lambda,\mu})$. Some variants and consequence will be discussed as well. In Section 4 we present how our approach extends to the study of positive solutions of problem $(\mathscr{N}_w)$ for radially symmetric domains with a weight function $w(x) = a(|x|)$. In Section 5 we conclude with few comments about the possibility of extending our main results to more general equations.

2. Basic tools and technical estimates

In this section, to describe our approach in a more transparent way, preliminary results will be done for a *locally Lipschitz* continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying (g_*). We tacitly extend the function $g(\cdot)$ to the whole real line, by setting $g(s) = 0$ for all $s < 0$ and the extension is still denoted by $g(\cdot)$. Let $a \in L^1([0, T])$ be a weight function such that conditions in (a_*) are satisfied.

Let $(x(\cdot\,; t_0, x_0, y_0), y(\cdot\,; t_0, x_0, y_0))$ be the solution of the system

$$\begin{cases} x' = y, \\ y' = -\lambda a^+(t)g(x), \end{cases} \tag{S_λ^+}$$

satisfying the initial condition

$$x(t_0) = x_0, \qquad y(t_0) = y_0,$$

for $t_0 \in [0, \sigma]$. By the concavity of $x(t)$ and the assumption $g(s) = 0$ for $s < 0$, it is straightforward to check that the solution $(x(t), y(t))$ is globally defined on $[0, \sigma]$.

Lemma 2.1. *Let $r > 0$ be fixed. If $(x(t), y(t))$ is any solution of (S_λ^+) with $x(0) > r$ and $y(0) = 0$, then $y(t) \leq 0$ for all $t \in [0, \sigma]$. Furthermore, there exists $\bar{t} \in [0, \sigma[$ such that $y(t) = 0$ for all $0 \leq t \leq \bar{t}$ and $y(t) < 0$ for all $t \in]\bar{t}, \sigma]$. If, moreover, $x(0) > (1 + \sigma)r$, then*

$$x(t)^2 + y(t)^2 > r^2, \quad \forall\, t \in [0, \sigma].$$

Proof. To prove the first part of the claim, it is sufficient to observe that

$$x'(t) = y(t) = -\lambda \int_0^t a^+(\xi)g(x(\xi))\, d\xi \leq 0, \quad \forall\, t \in [0, \sigma].$$

Furthermore, if $x(t^o) = 0$ for some $t^o \in]0,\sigma]$, there exists $\xi^o \in]0,t^o[$ such that $x'(\xi^o) < 0$ and therefore, $y(t) \leq y(\xi^o) < 0$ for all $t \in [\xi^o,\sigma]$. On the other hand, if $x(t) > 0$ for all $t \in [0,\sigma]$, then the same conclusion holds since $\int_0^\sigma a^+(\xi)\,d\xi > 0$. Thus, our assertion follows by taking $\bar{t} := \inf\{t \in]0,\sigma] : y(t) < 0\}$.

To prove the last part of the claim, suppose, by contradiction, that there exists $t^\# \in]0,\sigma]$ such that $x(t^\#)^2 + y(t^\#)^2 \leq r^2$. Given $B(0,r) := \{(x,y) : x^2 + y^2 < r^2\}$, since $(x(0),y(0)) \notin \mathrm{cl}B(0,r)$, let $\tilde{t} \in]0,\sigma]$ be the minimum of the t such that $(x(t),y(t)) \in \partial B(0,r)$. This way, $(x(\tilde{t}),y(\tilde{t})) \in \partial B(0,r)$ and $(x(t),y(t)) \notin \mathrm{cl}B(0,r)$ for all $t \in [0,\tilde{t}[$. Recalling that $g(s) = 0$ for $s < 0$, we easily deduce that $x(t) \geq 0$ for all $t \in [0,\tilde{t}]$. The monotonicity of $y(t)$ implies that $|y(t)| \leq |y(\tilde{t})| \leq r$ for all $t \in [0,\tilde{t}]$. From $x' = y$, we have

$$x(t) = x(0) + \int_0^t y(\xi)\,d\xi > (1+\sigma)r - \int_0^\sigma |y(\xi)|\,d\xi$$
$$\geq (1+\sigma)r - \sigma r = r, \quad \forall\, t \in [0,\tilde{t}].$$

Hence, for $t = \tilde{t}$, we obtain the contradiction $r \geq x(\tilde{t}) > r$. The result is thus proved. □

The lemma just proved does not require any special condition on $a^+(\cdot)$ and $g(\cdot)$. On the contrary, in the next results qualitative information about the solutions will be provided under some additional hypotheses on the weight function and the nonlinearity.

Lemma 2.2. *Suppose that there exists an interval $[t_1,t_2] \subseteq [0,\sigma]$ and a constant $\delta > 0$ such that $a^+(t) \geq \delta$ for a.e. $t \in [t_1,t_2]$. If*

$$\lambda\delta G^\infty > \left(\frac{\pi}{2(t_2-t_1)}\right)^2, \tag{2.1}$$

then, for any fixed constant ρ with

$$\lambda\delta G^\infty > \lambda\delta\rho > \left(\frac{\pi}{2(t_2-t_1)}\right)^2, \tag{2.2}$$

there exists an increasing sequence of positive real numbers $(d_j)_j$ with $d_j \nearrow +\infty$ for which the following property holds: If $(x(t),y(t))$ is any solution of (S_λ^+) with $x(0) \geq d_j$, $y(0) = 0$ and $x(t_1) = d_j$, then there is $\tilde{t} \in]t_1,t_2[$ such that

- $x(\tilde{t}) = 0$,
- $\dfrac{y(t)^2}{\lambda\delta\rho} + x(t)^2 \geq d_j^2, \quad \forall\, t \in [t_1,\tilde{t}]$.

Proof. By fixing in (2.2) a positive constant ρ with $\rho < G^\infty$, from [13] by $\limsup_{s\to+\infty}(2G(s) - \rho s^2) = +\infty$, there exists an increasing sequence of positive real numbers $(d_j)_j$ with $d_j \nearrow +\infty$ such that the following inequality holds

$$2\,(G(d_j) - G(s)) > \rho(d_j^2 - s^2), \quad \forall\, s \in [0,d_j[. \tag{2.3}$$

Assume that $(x(t),y(t))$ is a solution of (S_λ^+) with $x(0) \geq d_j$, $y(0) = 0$ and $x(t_1) = d_j$. Note also that $y(t_1) \leq 0$ (cf. Lemma 2.1). Let $[t_1,\tilde{t}] \subseteq [t_1,t_2]$ be the maximal closed subinterval of $[t_1,t_2]$ where $x(t) \geq 0$ (and, necessarily, also $y(t) \leq 0$). From system (S_λ^+), using the fact that $a^+(\xi) \geq \delta$ for a.e. $\xi \in [t_1,t]$, we have

$$yy' + \lambda\delta g(x)x' \geq 0, \quad \text{a.e in } [t_1,\tilde{t}],$$

which yields a map $\xi \mapsto \frac{1}{2}y(\xi)^2 + \lambda\delta G(x(\xi))$ nondecreasing in $[t_1, \tilde{t}]$. This in turn implies that, for all $\xi \in [t_1, \tilde{t}]$,

$$y(\xi)^2 + 2\lambda\delta G(x(\xi)) \geq y(t_1)^2 + 2\lambda\delta G(x(t_1)) \geq 2\lambda\delta G(x(t_1)) = 2\lambda\delta G(d_j).$$

Using (2.3), in the above inequality, we obtain

$$x'(\xi)^2 = y(\xi)^2 \geq \lambda\delta\rho(d_j^2 - x(\xi)^2), \quad \forall\, \xi \in [t_1, \tilde{t}] \tag{2.4}$$

and, as a further consequence, we also deduce

$$\int_{x(\tilde{t})}^{d_j} \frac{1}{\sqrt{d_j^2 - x^2}}\, dx = \int_{t_1}^{\tilde{t}} \frac{-x'(\xi)}{\sqrt{d_j^2 - x(\xi)^2}}\, d\xi \geq (\tilde{t} - t_1)\sqrt{\lambda\delta\rho}.$$

Notice that $x(\xi) < d_j$ for all $t_1 < \xi \leq \tilde{t}$ as $x' = y$ is strictly decreasing on $[t_1, \tilde{t}]$ and hence also $x(t)$ is strictly decreasing as $y(t_1) \leq 0$.

We claim that $\tilde{t} < t_2$. Indeed, otherwise,

$$\int_{0}^{d_j} \frac{1}{\sqrt{d_j^2 - x^2}}\, dx \geq \int_{x(t_2)}^{d_j} \frac{1}{\sqrt{d_j^2 - x^2}}\, dx \geq (t_2 - t_1)\sqrt{\lambda\delta\rho}.$$

This provides a contradiction because $\int_0^{d_j} 1/\sqrt{d_j^2 - x^2}\, dx = \pi/2$, while, according to the choice of ρ in (2.2), we have $(t_2 - t_1)\sqrt{\lambda\delta\rho} > \pi/2$.

We have thus proved that $x(t)$ vanishes at some time $\tilde{t} \in]t_1, t_2[$. The inequality $y(t)^2/\lambda\delta\rho + x(t)^2 \geq d_j^2$, for all $t \in [t_1, \tilde{t}\,]$, follows from (2.4). □

Lemma 2.3. *Suppose that $a^+ \in L^\infty([0, \sigma])$ and let $G_\infty = 0$. For any fixed $0 < \theta < 1$ and $0 < \nu < \pi/2$, there exists an increasing sequence of positive numbers $(\beta_j)_j$ with $\lim \beta_j = +\infty$ for which the following property holds: If $(x(t), y(t))$ is any solution of (S_λ^+) with $x(0) = \beta_j$, $y(0) = 0$, then*

- $\theta\beta_j \leq x(t) \leq \beta_j, \ \ \forall\, t \in [0, \sigma]$,
- $\tan(|y(t)|/x(t)) < \tan(\nu), \ \ \forall\, t \in [0, \sigma]$.

Proof. Let $\theta \in]0, 1[$ and $\nu \in]0, \pi/2[$ be two fixed constants. The assumption $G_\infty = 0$ implies that $\limsup_{s\to+\infty}(\varepsilon s^2 - 2G(s)) = +\infty$, for every $\varepsilon > 0$. Hence, following [13], there exists an increasing sequence of positive real numbers $(\beta_j^\varepsilon)_j$ with $\beta_j^\varepsilon \nearrow +\infty$ such that the following inequality holds

$$2\left(G(\beta_j^\varepsilon) - G(s)\right) < \varepsilon((\beta_j^\varepsilon)^2 - s^2), \quad \forall\, s \in [0, \beta_j^\varepsilon[\,. \tag{2.5}$$

Assume that $(x(t), y(t))$ is a solution of (S_λ^+) with $x(0) = \beta_j^\varepsilon$ and $y(0) = 0$. Recall from Lemma 2.1 also that $y(t) \leq 0$ for all $t \in [0, \sigma]$, so that $x(t) \leq \beta_j^\varepsilon$ for all $t \in [0, \sigma]$.

We claim that $x(t) \geq \theta\beta_j^\varepsilon$ for all $t \in [0, \sigma]$. To prove this claim, suppose, by contradiction that there exists a maximal interval $[0, \hat{t}\,] \subset [0, \sigma[$ such that

$$\theta\beta_j^\varepsilon \leq x(\xi) \leq \beta_j^\varepsilon, \quad \forall\, \xi \in [0, \hat{t}\,], \text{ with } x(\hat{t}) = \theta\beta_j^\varepsilon\,. \tag{2.6}$$

From system (S^+_λ), using the fact that $a^+(\xi) \leq \|a^+\|_\infty$ for a.e. $\xi \in [0,\sigma]$, we have

$$yy' + \lambda \|a^+\|_\infty g(x)x' \leq 0, \quad \text{a.e in } [0,\hat{t}\,],$$

which yields a map $\xi \mapsto \frac{1}{2}y(\xi)^2 + \lambda\|a^+\|_\infty G(x(\xi))$ nonincreasing in $[0,\hat{t}\,]$. This in turn implies that, for all $\xi \in [0,\hat{t}\,]$,

$$y(\xi)^2 + 2\lambda\|a^+\|_\infty G(x(\xi)) \leq y(0)^2 + 2\lambda\|a^+\|_\infty G(x(0)) = 2\lambda\|a^+\|_\infty G(\beta^\varepsilon_j).$$

Using (2.5), in the above inequality, we obtain that

$$x'(\xi)^2 = y(\xi)^2 \leq \lambda\|a^+\|_\infty \varepsilon((\beta^\varepsilon_j)^2 - x(\xi)^2) \tag{2.7}$$

holds for all $\xi \in [0,\hat{t}\,]$. As a further consequence, we have

$$\begin{aligned}\int_{\theta\beta^\varepsilon_j}^{\beta^\varepsilon_j} \frac{1}{\sqrt{(\beta^\varepsilon_j)^2 - x^2}}\,dx &= \int_0^{\hat{t}} \frac{-x'(\xi)}{\sqrt{(\beta^\varepsilon_j)^2 - x(\xi)^2}}\,d\xi \\ &\leq \hat{t}\sqrt{\lambda\|a^+\|_\infty\varepsilon} \leq \sigma\sqrt{\lambda\|a^+\|_\infty\varepsilon}.\end{aligned}$$

Since the left integral in the above inequality can be explicitly computed, as $(\pi/2) - \arcsin\theta$ (independently on β^ε_j), we obtain

$$\frac{\pi}{2} < \arcsin\theta + \sigma\sqrt{\lambda\|a^+\|_\infty\varepsilon},$$

which is clearly false if ε is chosen sufficiently small, namely

$$0 < \varepsilon < \frac{(\frac{\pi}{2} - \arcsin\theta)^2}{\sigma^2\lambda\|a^+\|_\infty}. \tag{2.8}$$

For such a choice of $\varepsilon > 0$ we can find a sequence $(\beta^\varepsilon_j)_j$ such that $\theta\beta^\varepsilon_j \leq x(t) \leq \beta^\varepsilon_j$ for all $t \in [0,\sigma]$. As a consequence, we also know that condition (2.7) holds for all $\xi \in [0,\sigma]$ and therefore, recalling that $y(t) \leq 0$, we deduce

$$|y(t)| \leq \beta^\varepsilon_j\sqrt{\lambda\|a^+\|_\infty\varepsilon}, \quad \forall\, t \in [0,\sigma].$$

This in turn implies that $\tan(|y(t)|/x(t)) < \tan(\nu)$, for all $t \in [0,\sigma]$, provided that

$$0 < \varepsilon < \frac{(\theta\tan(\nu))^2}{\lambda\|a^+\|_\infty}. \tag{2.9}$$

This way the theorem is proved by choosing a sequence $(\beta^\varepsilon_j)_j$ for a constant ε satisfying (2.8) and (2.9). $\square$

Lemma 2.4. *Given $a^+ \in L^\infty([0,\sigma])$, suppose that there exist an interval $[t_1,t_2] \subseteq [0,\sigma]$ and a constant $\delta > 0$ such that $a^+(t) \geq \delta$ for a.e. $t \in [t_1,t_2]$. Assume also (G_*) and let $\lambda > 0$ be such that (2.1) holds. Let also $0 < \theta < 1$, $0 < \nu < \pi/2$ be fixed. Then, there exist two increasing sequences of positive numbers $(\alpha_j)_j$ and $(\beta_j)_j$ with $\lim \alpha_j = \lim \beta_j = +\infty$ and*

$$r < \alpha_1 < \theta\beta_1 < \beta_1 < \alpha_2 < \ldots \alpha_j < \theta\beta_j < \beta_j < \alpha_{j+1} < \ldots \tag{2.10}$$

for which the following properties hold:

- $x(t;0,\alpha_j,0)$ *vanishes at some* $t < t_2$,
- $\theta\beta_j \leq x(t;0,\beta_j,0) \leq \beta_j$, $\tan(|y(t;0,\beta_j,0)|/x(t;0,\beta_j,0)) < \tan(\nu)$ $\forall t \in [0,\sigma]$.

Proof. We choose a constant $\rho > 0$ in accord to (2.2) and consider a corresponding sequence $(d_j)_j$ as in Lemma 2.2. Next, we apply Lemma 2.3 and find a sequence $(\beta_j)_j$. We can also suppose that

$$r < d_1 < \theta\beta_1 < \beta_1 < d_2 < \ldots d_j < \theta\beta_j < \beta_j < d_{j+1} < \ldots$$

up to a subsequence, if necessary. By the intermediate value theorem and the continuous dependence of the solutions on the initial data, for each j, there exists α_j with $d_j \leq \alpha_j < \beta_j$ such that $x(t_1;0,\alpha_j,0) = d_j$. At this point, a direct application of Lemma 2.2 and Lemma 2.3 allows to conclude the proof of the theorem. □

Until now we have analyzed the behavior of the solutions in the interval $[0,\sigma]$ where $a_{\lambda,\mu}(t) \geq 0$ for a.e. t. As a next step, we are going to consider the solutions on the interval $[\sigma,T]$. Due to the sign of $a_{\lambda,\mu}(t)g(x(t))$ which implies the convexity of $x(t)$ in the interval $[\sigma,T]$, in general, we cannot guarantee that the solutions are defined on the whole interval. For this reason, we introduce a truncation on the nonlinear term of the form

$$g_M(x) = \begin{cases} g(x), & \text{if } x \leq M, \\ g(M), & \text{if } x > M, \end{cases}$$

where $M > 0$ is a given constant. Accordingly, we study the system

$$\begin{cases} x' = y, \\ y' = \mu a^-(t) g_M(x), \end{cases} \tag{S_μ^-}$$

on the interval $[\sigma,T]$. In the foregoing results we shall require a further technical condition on the weight function, namely that $a(t)$ is not identically zero a.e. in each right neighborhood of σ. This can be equivalently expressed by the following condition:

$$A^-(t) > 0, \quad \forall\, t \in\,]\sigma,T],$$

where we have set

$$A^-(t) := \int_\sigma^t a^-(\xi)\, d\xi. \tag{2.11}$$

This hypothesis is not restrictive in view of (a_*) (see [4, Remark 2.2] where an analogous situation is treated). In this framework, we obtain the following result.

Lemma 2.5. *For any fixed* $r > 0$, $q \in]0,1[$ *and* $C > 0$, *there is a constant* $\hat{\mu} > 0$ *such that for each* $\mu > \hat{\mu}$ *the following holds: If* $(x(t),y(t))$ *is any solution of* (S_μ^-) *with* $x(\sigma) = r$ *and* $0 > y(\sigma) \geq -C$, *then*

- $x(t) > qr$ *for all* $t \in [\sigma,T]$,
- $y(t)$ *vanishes at some* $t \in]\sigma,T[$.

Proof. First of all, notice that there exists $0 < \varepsilon \leq r(1-q)/C$ such that $x(t) > qr$ for all $t \in [\sigma, \sigma+\varepsilon[$. Indeed,

$$x(t) = x(\sigma) + \int_{\sigma}^{t} y(\xi)\, d\xi \geq r - \int_{\sigma}^{t} C\, d\xi = r - C(t-\sigma)$$
$$> r - C\varepsilon \geq qr, \quad \forall t \in [\sigma, \sigma+\varepsilon[.$$

Therefore, let us fix ε as above and assume by contradiction that there is $\tilde{t} \in [\sigma+\varepsilon, T]$ such that $x(\tilde{t}) = qr$ and $x(t) > qr$ for all $t \in [\sigma, \tilde{t}[$. By denoting with $\kappa_{g,r} := \min\{g_M(s) : qr \leq s \leq r\}$, we have

$$x''(t) = y'(t) = \mu a^-(t) g(x(t)) \geq \mu a^-(t)\kappa_{g,r}, \quad \text{for a.e. } t \in [\sigma, \tilde{t}].$$

After a first integration on $[\sigma, t]$, we get

$$x'(t) = y(t) \geq y(\sigma) + \mu\kappa_{g,r} A^-(t) \geq -C + \mu\kappa_{g,r} A^-(t), \quad \forall\, t \in [\sigma, \tilde{t}].$$

Integrating again in the same interval we have

$$x(t) \geq x(\sigma) - C(t-\sigma) + \mu\kappa_{g,r} \int_{\sigma}^{t} A^-(\xi)\, d\xi$$
$$\geq r - C(T-\sigma) + \mu\kappa_{g,r} \int_{\sigma}^{\sigma+\varepsilon} A^-(\xi)\, d\xi.$$

The evaluation of the above inequality for $t = \tilde{t}$ yields to a contradiction if μ is sufficiently large, namely

$$\mu \geq \mu_1 := \frac{C(T-\sigma)}{\kappa_{g,r} \int_{\sigma}^{t} A^-(\xi)\, d\xi}\,.$$

At this step, we have proved that $x(t) > qr$ for all $t \in [\sigma, T]$.

Suppose now, by contradiction that $y(t)$ never vanishes on $]\sigma, T]$. Then, since $y(\sigma) < 0$, we have $x'(t) = y(t) < 0$ for all $t \in [\sigma, T]$. Hence the function $x(t)$ is decreasing on $[\sigma, T]$ and, therefore, $qr < x(t) < r$ for all $t \in]\sigma, T]$. Accordingly, the inequality $y'(t) \geq \mu a^-(t)\kappa_{g,r}$ holds for a.e. $t \in [\sigma, T]$. With an integration on $[\sigma, t]$ we obtain

$$y(t) \geq -C + \mu\kappa_{g,r} A^-(t), \quad \forall\, t \in [\sigma, T].$$

So that

$$0 > y(T) \geq -C + \mu\kappa_{g,r} A^-(T).$$

A contradiction occurs whenever μ is sufficiently large, namely

$$\mu \geq \mu_2 := \frac{C}{\kappa_{g,r} \int_{\sigma}^{\sigma+T} a^-(\xi)\, d\xi}\,.$$

At this point, the conclusion follows by taking $\hat{\mu} \geq \max\{\mu_1, \mu_2\}$. $\square$

3. Main results

As said in the Introduction, our goal is to prove the existence of positive solutions for the Neumann problem

$$\begin{cases} u'' + a_{\lambda,\mu}(t)g(u) = 0, \\ u(t) > 0, \quad \forall\, t \in [0,T], \\ u'(0) = u'(T) = 0, \end{cases} \tag{$\mathscr{P}_{\lambda,\mu}$}$$

where the continuous map $g : \mathbb{R}^+ \to \mathbb{R}^+$ verifies (g_*) and the weight term $a_{\lambda,\mu}$ is defined as in (1.1) for a function $a \in L^1([0,T])$ satisfying (a_*). Our method of proof is based on the shooting method and therefore we need to analyze the Poincaré map associated with the planar system

$$\begin{cases} x' = y, \\ y' = -(\lambda a^+(t) - \mu a^-(t))g_M(x), \end{cases} \tag{$S_{\lambda,\mu}$}$$

where g_M is defined as in (1.2) for a suitable constant $M > 0$. In order to have the Poincaré map well defined, we shall implicitly assume the uniqueness of the solutions for the associated initial value problems. Obviously, this is guaranteed if g is locally Lipschitz continuous as we have assumed for convenience in the exposition in Section 2. However, this condition can be removed and this will be discussed at the end of the proof of Theorem 3.1. Recall that, given an interval $[\tau_0, \tau_1] \subseteq [0,T]$, the Poincaré map for $(S_{\lambda,\mu})$ on the interval $[\tau_0,\tau_1]$ is the planar map which, to any point $z_0 = (x_0, y_0) \in \mathbb{R}^2$, associates the point $(x(\tau_1), y(\tau_1))$ where $(x(t), y(t))$ is the solution of $(S_{\lambda,\mu})$ with $(x(\tau_0), y(\tau_0)) = z_0$. Such map will be denoted by $\Phi_{\tau_0}^{\tau_1}$.

A solution of $(\mathscr{P}_{\lambda,\mu})$ can be found by looking for a point $P_0 = (x_0, 0) \in X^+ := \{(x, 0) : x > 0\}$ such that $x_0 \le M$ and $\Phi_0^T(P_0) \in X^+$. In this case, the first component $u(t)$ of the map $t \mapsto \Phi_0^t(P_0)$ is a solution of $(\mathscr{P}_{\lambda,\mu})$ with $u(0) = x_0$. More formally, we can state the following lemma.

Lemma 3.1. *Suppose that there is $P_0 = (x_0, 0) \in X^+$ with $x_0 \le M$ such that $\Phi_0^T(P_0) \in X^+$. Let also $(x(t), y(t))$ be the solution of $(S_{\lambda,\mu})$ with $(x(0), y(0)) = P_0$. Then, $u(t) := x(t)$ is a solution of $(\mathscr{P}_{\lambda,\mu})$ with $u(t) \le M$ and $u'(t) = y(t) \le 0$ for all $t \in [0,T]$.*

Proof. Consider at first the solution in the interval $[0,\sigma]$. As $x(t)$ is concave in such interval, we have that $x(t) \le x(0) \le M$ and we also claim that $x(t) > 0$ for all $t \in [0,\sigma]$. Indeed, if by contradiction $x(t)$ vanishes somewhere, we take $\hat{t}$, with $0 < \hat{t} \le \sigma$, as its first zero. As a consequence of the concavity, $y(\hat{t}) = x'(\hat{t}) < 0$ and then, $x'(t) = x'(\hat{t}) < 0$ for all $t \in [\hat{t}, T]$, because $g_M(s) = 0$ for $s \le 0$. Thus, we have the contradiction $\Phi_0^T(P_0) \notin X^+$. From $y'(t) = -\lambda a^+(t)g(x(t))$, with $g(x(t)) > 0$ for all $t \in [0,\sigma]$ and $a^+ \not\equiv 0$, we deduce that $x'(\sigma) = y(\sigma) < 0$. On the other hand, the function $x(t)$ is convex on $[\sigma, T]$ with $x(T) > 0$ and $x'(T) = 0$. Hence, $0 < x(T) \le x(t) < x(\sigma)$ for all $t \in [\sigma, T]$ and this concludes the proof. □

In view of the hypothesis on the weight function, which state that it assumes different sign on the intervals $[0,\sigma]$ and $[\sigma, T]$, it will be convenient to split the Poincaré map as

$$\Phi_0^T := \Phi_\sigma^T \circ \Phi_0^\sigma,$$

where Φ_0^σ and Φ_σ^T are the Poincaré maps associated with systems (S_λ^+) and (S_μ^-), respectively. Consistently with the notation introduced at the beginning of Section 2, we notice that for any point $P_0 = (x_0, 0) \in X^+$ with $x_0 \le M$, we have

$$\Phi_0^t(P_0) = (x(t; 0, x_0, 0), y(t; 0, x_0, 0)), \quad \forall\, t \in [0,\sigma].$$

To formulate the next result, we introduce the following notation. For any real number η, we denote by $L_\eta := \{(\eta, y) \in \mathbb{R}^2 : y < 0\}$ the negative half-line $x = \eta$. Given two points $(A, 0), (B, 0) \in X^+$, the segment contained in X^+ and joining the two points is denoted by $\overline{AB}$.

Proposition 3.1. *Given $a^+ \in L^\infty([0, \sigma])$, suppose that there exist an interval $[t_1, t_2] \subseteq [0, \sigma]$ and a constant $\delta > 0$ such that $a^+(t) \geq \delta$ for a.e. $t \in [t_1, t_2]$. Assume also (G_*) and let $\lambda > 0$ be such that (2.1) holds. Furthermore, let $r > 0$ be fixed. Then, for any given integer $k \geq 1$ there are constants $M > r$, $C_M > r$ and points*

$$r < A'_1 < B'_1 < B''_1 < A''_1 < A'_2 < \cdots < A'_k < B'_k < B''_k < A''_k < M,$$

such that, setting

$$\Gamma'_j := \Phi_0^\sigma\left(\overline{A'_j B'_j}\right), \qquad \Gamma''_j := \Phi_0^\sigma\left(\overline{B''_j A''_j}\right),$$

we have

$$\Gamma'_j, \Gamma''_j \subseteq \Big([0, r] \times [-C_M, 0[\Big), \tag{3.1}$$

with

$$\Gamma'_j \cap L_0^- \neq \emptyset \neq L_r^- \cap \Gamma'_j, \qquad \Gamma''_j \cap L_0^- \neq \emptyset \neq L_r^- \cap \Gamma''_j, \tag{3.2}$$

for all $j = 1, \ldots, k$.

Proof. Given $\lambda > 0$ and $r > 0$, we choose $0 < \theta < 1$ and $0 < \nu < \pi/2$. So, an application of Lemma 2.4 provides two sequences $(\alpha_j)_j$ and $(\beta_j)_j$ which satisfy (2.10). Moreover, for any integer $k \geq 1$, we take a constant M such that

$$M > \alpha_k\,. \tag{3.3}$$

Since M is now fixed, follows that also the vector field in the system $(S_{\lambda,\mu})$ is so. The constant $C_M > 0$ will be chosen so that any possible solution $(x(t), y(t))$ of $(S_{\lambda,\mu})$ with $0 < x(0) \leq M$ and $y(0) = 0$, satisfies

$$-C_M \leq y(t) \leq 0, \quad \forall\, t \in [0, \sigma].$$

Notice that the constant C_M depends on the function a^+ and the constants λ and M, but does not depend on the parameter μ. In fact, we can estimate C_M as follows:

$$C_M := \lambda \|a^+\|_{L^1} \max_{s \in [0,M]} g(s).$$

For the rest of the proof we consider the solutions of the system $(S_{\lambda,\mu})$ on the interval $[0, \sigma]$, with an initial point $(c, 0)$ such that $0 < c \leq M$. These are exactly the solutions $(x(\cdot\,; 0, c, 0), y(\cdot\,; 0, c, 0))$ of the system (S_λ^+).

As a first step, for $j = 1, \ldots, k$, we suppose that $\alpha_j \leq c \leq \beta_j$. By Lemma 2.4, it follows that

$$x(\sigma; 0, \alpha_j, 0) < 0, \qquad x(\sigma; 0, \beta_j, 0) \geq \theta \beta_j > r.$$

By continuity, we can determine a sub-interval $[A'_j, B'_j] \subseteq\,]\alpha_j, \beta_j[$ such that $x(\sigma; 0, A'_j, 0) = 0$, $x(\sigma; 0, B'_j, 0) = r$ and $0 < x(\cdot\,; 0, c, 0) < r$ for all $c \in\,]A'_j, B'_j[$.

As a second step, for $j = 1, \dots, k$, we suppose that $\beta_j \leq c \leq \alpha_{j+1}$. By Lemma 2.4, it follows that

$$x(\sigma; 0, \alpha_{j+1}, 0) < 0, \qquad x(\sigma; 0, \beta_j, 0) \geq \theta\beta_j > r.$$

Again, by continuity, we can determine a sub-interval $[B''_j, A''_j] \subseteq]\beta_j, \alpha_{j+1}[$ such that $x(\sigma; 0, B''_j, 0) = r$, $x(\sigma; 0, A''_j, 0) = 0$ and $0 < x(\sigma\,; 0, c, 0) < r$ for all $c \in]B''_j, A''_j[$. Moreover, $-C_M \leq y(\sigma\,; 0, c, 0) < 0$ (recalling also Lemma 2.1).

To conclude, we define

$$\Gamma'_j := \Phi_0^\sigma\left(\overline{A'_j B'_j}\right), \quad \Gamma''_j := \Phi_0^\sigma\left(\overline{B''_j A''_j}\right), \quad \forall\, j = 1, \dots, k.$$

This way each arc, Γ'_j and Γ''_j with $j \in \{1, \dots, k\}$, satisfies all the desired properties. $\square$

Remark 3.1. We observe that the constants β_j are precisely determined in Lemma 2.3 by means of (2.5), instead of the constants α_j, for which we know only that they belong to $[d_j, \beta_j[$. With this respect, it might be more convenient to fix the constant M in terms of the values β_j. For this reason, one could prefer to replace the condition in (3.3) with $M > \beta_{k+1}$.

Under this latter choice, notice that a further arc, $\Gamma'_{k+1} := \Phi_0^\sigma\left(\overline{A'_{k+1} B'_{k+1}}\right)$ with $[A'_{k+1} B'_{k+1}] \subseteq]\alpha_{k+1}, \beta_{k+1}[$ defined as in the proof, can be determined. Finally, if we assume $M > \beta_{k+1}$, we have $2k + 1$ arcs defined as images through the Poincaré map of pairwise disjoint compact sub-intervals of X^+.

The next result deals with the solutions of the system $(S_{\lambda,\mu})$ in the time interval $[\sigma, T]$, or equivalently, the ones of (S_μ^-). As previously observed, we will suppose that σ is chosen so that $A^-(t) > 0$ for all $\sigma < t \leq T$, where $A^-(t)$ is defined according to (2.11).

Proposition 3.2. *Given $r > 0$ and $C > r$, there exists a constant $\bar{\mu} > 0$ such that for each $\mu > \bar{\mu}$ the following holds: For any connected set Γ with*

$$\Gamma \subseteq [0, r] \times [-C, 0[, \qquad \Gamma \cap L_0^- \neq \emptyset \neq L_r^- \cap \Gamma,$$

there exists at least a solution $(x(t), y(t))$ of the system (S_μ^-) with $(x(\sigma), y(\sigma)) \in \Gamma$, $(x(T), y(T)) \in X^+$ such that $r \geq x(t) > 0$ and $y(t) \leq 0$ for all $t \in [\sigma, T]$.

Proof. For r and C given as above, let us fix a parameter q with $0 < q < 1$. From Lemma 2.5, we have that for each μ sufficiently large (i.e. $\mu > \hat{\mu}$), any solution $(x(t), y(t))$ of (S_μ^-) with $x(\sigma) = r$ and $-C \leq y(\sigma) < 0$ is such that $x(t) \geq qr$ for all $t \in [\sigma, T]$ and $y(t) = 0$ for some $t \in]\sigma, T]$. Let us fix now $\mu > \hat{\mu}$.

We choose a point $Q \in L_r^- \cap \Gamma$ and denote by $(x_Q(t), y_Q(t))$ the solution of (S_μ^-) having Q as initial point at the time $t = \sigma$. By Lemma 2.5 there exists a first time $t_Q \in]\sigma, T]$ such that $y(t_Q) = 0$. If $t_Q = T$, we are done. Otherwise, $y_Q(t_Q) = 0$ for $\sigma < t_Q < T$ and, by the convexity of $x_Q(t)$ in the interval $[\sigma, T]$, we have $y_Q(T) \geq y_Q(t_Q) = 0$.

Similarly, we select a point $P \in \Gamma \cap L_0^-$ and denote by $(x_P(t), y_P(t))$ the solution of (S_μ^-) which has P as initial point at the time $t = \sigma$. We have $x_P(\sigma) = 0$ and $x'_P(\sigma) = y_P(\sigma) < 0$. Moreover, $g(s) = 0$ for all $s \leq 0$. Hence, $y_P(t) = y_P(\sigma)$ for all $t \in [\sigma, T]$ and, therefore, $y_P(T) < 0$.

The continuous dependence of the solutions on the initial data and the connectedness of Γ imply that there exists a point in $\Gamma \setminus L_0^-$ from which starts (at the time $t = \sigma$) a solution $(x(t), y(t))$ of (S_μ^-) such that $y(T) = 0$. This way, it follows also that $x(t) > 0$ for all $t \in [\sigma, T]$ (in fact, if not, we obtain a contradiction from $g(s) = 0$ for all $s \leq 0$). Finally, we also observe that $y(t) \leq 0$ for all $t \in]\sigma, T]$ (otherwise, if we suppose that $y(t) > 0$ for some $t \in]\sigma, T[$, then a contradiction is reached by a convexity argument). Thus the thesis is achieved by choosing any $\bar{\mu} \geq \hat{\mu}$. $\square$

We are now in position to prove our main result that establishes the existence and the multiplicity of positive solutions for problem $(\mathscr{P}_{\lambda,\mu})$.

Theorem 3.1. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*) and (G_*) with $s \mapsto g(s)/s$ upper bounded in a right neighborhood of 0. Let $a \in L^1([0,T])$ satisfying (a_*) with $a^+ \in L^\infty([0,\sigma])$. Suppose also that there are an interval $[t_1,t_2] \subseteq [0,\sigma]$ and a constant $\delta > 0$ such that $a^+(t) \geq \delta$ for a.e. $t \in [t_1,t_2]$. Then, there exists $\lambda^* \geq 0$ such that, for each $\lambda > \lambda^*$, $r > 0$ and for every integer $k \geq 1$, there is a constant $\mu^* = \mu^*(\lambda,r,k) > 0$ such that for each $\mu > \mu^*$ the problem $(\mathscr{P}_{\lambda,\mu})$ has at least $2k$ solutions which are nonincreasing on $[0,T]$ and satisfy $0 < u(t) \leq r$ for each $t \in [\sigma,T]$.*

Proof. Our demonstration will be divided into two parts. In the first one we let the shooting method work within its classical framework, by assuming $g(\cdot)$ locally Lipschitz continuous. In the second part, we present two possible ways in order to extend the result obtained in the previous step to the case in which $g(\cdot)$ is only continuous.

Step 1. We suppose that $g(\cdot)$ is locally Lipschitz continuous. Notice that, under this condition it follows immediately $s \mapsto g(s)/s$ upper bounded in a right neighborhood of 0.

First of all, we define a constant $\lambda^* \geq 0$ as $\lambda^* = 0$ if $G^\infty = +\infty$ or $\lambda^* = \pi^2/4(t_2-t_1)^2\delta G^\infty$ if $G^\infty < +\infty$. In this manner, the inequality in (2.1) is satisfied for each $\lambda > \lambda^*$.

We fix now $\lambda > \lambda^*$, $r > 0$ and an integer $k \geq 1$. In accord to Proposition 3.1, there are constants $M > r$, $C_M > r$ and points

$$r < A_1' < B_1' < B_1'' < A_1'' < A_2' < \cdots < A_k' < B_k' < B_k'' < A_k'' < M,$$

such that conditions in (3.1) and (3.2) are satisfied for the arcs

$$\Gamma_j' := \Phi_0^\sigma\left(\overline{A_j'B_j'}\right), \qquad \Gamma_j'' := \Phi_0^\sigma\left(\overline{B_j''A_j''}\right).$$

At this step we apply Proposition 3.2 for $C := C_M$ and determine a constants $\bar{\mu}$ such that, for each $\mu > \bar{\mu}$, the following holds: for each Γ_j', Γ_j'' with $j \in \{1,\dots,k\}$ there exist points $\zeta_j' \in \Gamma_j'$ and $\zeta_j'' \in \Gamma_j''$ such that

$$\Phi_\sigma^T(\zeta_j'),\ \Phi_\sigma^T(\zeta_j'') \in X^+.$$

Notice that the constant $\bar{\mu}$ does not depend on the particular choice of the arcs Γ_j' or Γ_j''. It depends only on r and C_M. The last constant, in turn, depends on M and therefore it is derived from λ and k.

On the other hand, ζ_j' and ζ_j'' are images through the Poincaré map Φ_0^σ of the initial points $Z_j' \in \overline{A_j'B_j'}$ and $Z_j'' \in \overline{B_j''A_j''}$, respectively. Then, we have found $2k$ points $Z_j', Z_j'' \in X^+$ such that $\Phi_0^T(Z_j'), \Phi_0^T(Z_j'') \in X^+$. From Lemma 3.1 follows that all the solutions $(x(t),y(t))$ starting from these initial points are such that $0 < x(t) < M$ and $y(t) \leq 0$, for all $t \in [0,T]$. Hence, they are solutions of the system

$$\begin{cases} x' = y, \\ y' = -a_{\lambda,\mu}(t)g(x(t)). \end{cases}$$

In particular, they correspond to solutions of the problem $(\mathscr{P}_{\lambda,\mu})$ with initial conditions $(u(0),u'(0)) = Z_j'$ or $(u(0),u'(0)) = Z_j''$, respectively. All these solutions are decreasing in $[0,T]$ by construction and, from Proposition 3.2, they satisfy the condition $0 < u(t) \leq r$, for all $t \in [\sigma,T]$. Thus, the result is proved by choosing any $\mu^* \geq \bar{\mu}$ and $g(\cdot)$ locally Lipschitz continuous.

Step 2. At this point the Lipschitz condition is no more assumed. Usually, one can follow two possible ways in order to achieve the result for a nonlinearity which is only continuous. A first approach consists in approximating the nonlinear term $g(\cdot)$ with a sequence of locally Lipschitz functions $g_n : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying (g_*) and such that $g_n \to g$ uniformly on compact sets, for example, using mollifiers as in [26, p. 294]. Then, one can prove that each approximating equation has a solution u_n with $(u_n(t), u_n'(t)) \in \mathcal{K}, \forall\, t \in [0,T]$, where $\mathcal{K}$ is a compact set which can be chosen independently on n. At last, from the Ascoli–Arzelà Theorem we obtain a solution $(u(t), u'(t)) \in \mathcal{K}, \forall\, t \in [0,T]$ of the original equation, passing to the limit along a subsequence. This is a standard procedure well described in the book of Krasnosel'skiĭ [19]. Moreover, this approach is also exploited in [26,27] where some specific results of existence and multiplicity of solutions are obtained via the shooting method without uniqueness of the Cauchy problems. In our case, this method can be safely applied by choosing the compact intervals $\overline{A_j' B_j'}$ and $\overline{B_j'' A_j''}$ for $j = 1, \ldots, k$ pairwise disjoint and observing that the initial points of the solutions of the approximating problems belong to these intervals (at least for n sufficiently large).

A second possible point of view involves a procedure of "shooting without uniqueness", that gives up from the beginning to the hypothesis of uniqueness for the Cauchy problems. In these frameworks we can apply a generalized version of the Hukuhara–Kneser result, as presented in [7] or in [8, Section 2]. It is based on the following observation. Let $[\tau_0, \tau_1] \subseteq [0,T]$. Given a set $E_0 \subseteq \mathbb{R}^2$, let us consider the set E_1 made by all the points of $\mathbb{R}^2$ of the form $(x(\tau_1), y(\tau_1))$, where $(x(t), y(t))$ is *any* solution of the system such that $(x(\tau_0), y(\tau_0)) \in E_0$. Then, E_1 is a compact/connected (or both) provided that E_0 is a compact/connected (or both), respectively (cf. [7, p. 22]). In this context, for all $j = 1, \ldots, k$ the sets Γ_j' and Γ_j'' given as in Proposition 3.1 are well defined continua (instead of arcs). Moreover, to prove Proposition 3.2, instead of using Bolzano Theorem, on the function $y(t)$ we just observe that a connected set Γ at the time $t = \sigma$ is transported into a connected set at the time $t = T$, whose projection on the y-axis contains $y = 0$.

In conclusion, we have found $2k$ *non-negative* solutions of

$$u'' + a_{\lambda,\mu}(t) g(u) = 0, \quad u'(0) = u'(T) = 0,$$

which are nonincreasing on $[0,T]$ and satisfy $0 \leq u(t) \leq r$ for each $t \in [\sigma, T]$. Since $s \mapsto g(s)/s$ is upper bounded in a right neighborhood of 0, a maximum principle argument applies and the *positivity* of the solutions on $[0,T]$ is guaranteed. □

Without the condition

$$\limsup_{s \to 0^+} \frac{g(s)}{s} < +\infty \tag{g_0}$$

we can prove that any solution found satisfies

$$u(t) \geq r,\ \forall\, t \in [0,\sigma] \quad \text{and} \quad 0 \leq u(t) \leq r,\ \forall\, t \in [\sigma, T].$$

Nevertheless, without assuming (g_0), we cannot guarantee, in general, that $u(t)$ does not vanish at some point of the interval when the weight is negative. Examples in this direction are given in [1,5] and they show that (g_*) along with (g_0) represent the minimal requests needed to get the positivity of the solutions. For this reason, the main hypothesis which characterizes our result is the "oscillatory condition" (G_*).

Remark 3.2. As a main assumption on the weight term, we have supposed that the function $a(\cdot)$ goes from positive to negative values. One could also consider a dual condition instead of (a_*), namely

(a_{**}) $a(t) \leq 0,\ a \not\equiv 0$ *for a.e.* $t \in [0,\sigma]$, $\ a(t) \geq 0,\ a \not\equiv 0$ *for a.e.* $t \in [\sigma, T]$.

In this case, we derive a different version of the Theorem 3.1 in which the hypotheses have to be modified by assuming $a^+ \in L^\infty([\sigma, T])$ and $a^+(t) \geq \delta$ for a.e. t in a suitable subinterval of $[\sigma, T]$. As a conclusion, the existence of $2k$ solutions to problem $(\mathscr{P}_{\lambda,\mu})$ is still guaranteed. Such solutions, in this case, are nondecreasing on $[0, T]$ and satisfy $0 < u(t) \leq r$ for each $t \in [0, \sigma]$. To prove this assertion, we can either apply Theorem 3.1 with the change of variable $t \mapsto T - t$, or apply the shooting method backward in time from $t = T$ to $t = 0$.

4. Radially symmetric solutions for PDEs

In this section we extend the preceding results to the case of some Neumann problems in $\mathbb{R}^N$, for $N \geq 2$. So, we consider

$$\begin{cases} \Delta u + w_{\lambda,\mu}(x) g(u) = 0 & \text{in } \Omega, \\ u(x) > 0 & \text{in } \overline{\Omega}, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{N}_{\lambda,\mu}$}$$

where the weight function depends on the real positive parameters λ, μ and is defined as

$$w_{\lambda,\mu}(x) := \lambda w^+(x) - \mu w^-(x),$$

for $w \in L^1(\Omega)$. We shall focus our study to the case when the domain Ω is an open ball $B(0, R)$ or an open annulus $B(0, R_2) \setminus B[0, R_1]$, where with $B[0, r]$ we denote the closed ball of center the origin and radius $r > 0$. As usual, in these situations the problem can be reduced to a Neumann boundary value problem with an ordinary differential equation if $w(x)$ has a radial symmetry. Accordingly, from now on we suppose that

$$w(x) = \mathcal{Q}(|x|), \tag{4.1}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$.

We look for radially symmetric positive solutions of $(\mathscr{N}_{\lambda,\mu})$, namely solutions of the form

$$u(x) = U(\varrho), \quad \text{with } \; \varrho := |x|, \tag{4.2}$$

and we discuss separately the two cases of our interest.

4.1. Neumann problem for an annular domain

Let $R_2 > R_1 > 0$ be two fixed radii and let us consider the Neumann problem $(\mathscr{N}_{\lambda,\mu})$ for the domain

$$\Omega := B(0, R_2) \setminus B[0, R_1].$$

We suppose that $w(\cdot)$ is as in (4.1), with $\mathcal{Q} \in L^1([R_1, R_2])$. By means of (4.2) our problem is reduced to the study of

$$\begin{cases} U''(\varrho) + \dfrac{N-1}{\varrho} U'(\varrho) + \mathcal{Q}_{\lambda,\mu}(\varrho) g(U(\varrho)) = 0, \\ U(x) > 0, \quad \forall\, \varrho \in [R_1, R_2], \\ U'(R_1) = U'(R_2) = 0. \end{cases} \tag{4.3}$$

By the classical change of variable $t = h(\varrho) := \int_{R_1}^{\varrho} \xi^{1-N}\, d\xi$, $\varrho = \varrho(t) := h^{-1}(t)$, we set

$$v(t) := U(\varrho(t)), \quad a(t) := \varrho(t)^{2(N-1)}\mathcal{Q}(\varrho(t)) \quad \text{and} \quad T := \int_{R_1}^{R_2} \xi^{1-N}\, d\xi,$$

this way it follows that problem (4.3) is equivalent to

$$\begin{cases} v''(t) + a_{\lambda,\mu}(t)g(v(t)) = 0, \\ v(t) > 0, \quad \forall\, t \in [0,T], \\ v'(0) = v'(T) = 0, \end{cases} \tag{4.4}$$

see for instance [3,12]. Hence, we enter in the framework of problem $(\mathscr{P})$ and we can apply directly Theorem 3.1 to the system (4.4). Therefore we can state the following result.

Theorem 4.1. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*), (g_0) and (G_*). Let $\mathcal{Q} \in L^1([R_1,R_2])$ with $\mathcal{Q}^+ \in L^\infty$ and suppose there exists $\sigma \in]R_1,R_2[$ such that*

$$\mathcal{Q}(\varrho) \geq 0,\ \mathcal{Q} \not\equiv 0 \text{ for a.e. } \varrho \in [R_1,\sigma],\ \mathcal{Q}(\varrho) \leq 0,\ \mathcal{Q} \not\equiv 0 \text{ for a.e. } \varrho \in [\sigma,R_2].$$

Suppose also that there are an interval $[t_1,t_2] \subseteq [R_1,\sigma]$ and a constant $\delta > 0$ such that $\mathcal{Q}^+(\varrho) \geq \delta$ for a.e. $\varrho \in [t_1,t_2]$. Then, there exists $\lambda^* \geq 0$ such that, for each $\lambda > \lambda^*$, $r > 0$ and for every integer $k \geq 1$, there is a constant $\mu^* = \mu^*(\lambda,r,k) > 0$ such that for each $\mu > \mu^*$ the problem $(\mathscr{N}_{\lambda,\mu})$ has at least $2k$ radially symmetric solutions which are nonincreasing in ϱ on $[R_1,R_2]$ and satisfy $0 < u(x) \leq r$ for each x with $|x| \in [\sigma,R_2]$.

4.2. Neumann problem for a ball

Let $R > 0$ be a fixed radius and let us consider the Neumann problem $(\mathscr{N}_{\lambda,\mu})$ for the domain

$$\Omega := B(0,R).$$

We suppose that $w(\cdot)$ is as in (4.1), with $\mathcal{Q} \in L^1([0,R])$. By means of (4.2), our problem is reduced to

$$\begin{cases} U''(\varrho) + \dfrac{N-1}{\varrho}U'(\varrho) + \mathcal{Q}_{\lambda,\mu}(\varrho)g(U(\varrho)) = 0, \quad 0 < \varrho \leq R, \\ U(x) > 0, \quad \forall\, \varrho \in [0,R], \\ U'(0) = U'(R) = 0, \end{cases} \tag{4.5}$$

which has a singularity at $\varrho = 0$. The previous problem is in its turn equivalent to

$$\begin{cases} \big(\varrho^{N-1}U'(\varrho)\big)' + \varrho^{N-1}\mathcal{Q}_{\lambda,\mu}(\varrho)g(U(\varrho)) = 0, \quad 0 < \varrho \leq R, \\ U(x) > 0, \quad \forall\, \varrho \in [0,R], \\ U'(0) = U'(R) = 0. \end{cases} \tag{4.6}$$

In this case, the following result holds.

Theorem 4.2. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (g_*), (g_0) and (G_*). Let $\mathcal{Q} \in L^1([0,R])$ with $\mathcal{Q}^+ \in L^\infty$ and suppose there exists $\sigma\,]0,R[$ such that*

$$\mathcal{Q}(\varrho) \geq 0,\ \mathcal{Q} \not\equiv 0 \text{ for a.e. } \varrho \in [0,\sigma],\ \mathcal{Q}(\varrho) \leq 0,\ \mathcal{Q} \not\equiv 0 \text{ for a.e. } \varrho \in [\sigma,R].$$

Suppose also that there are an interval $[t_1,t_2] \subseteq [0,\sigma]$ and a constant $\delta > 0$ such that $\mathcal{Q}^+(\varrho) \geq \delta$ for a.e. $\varrho \in [t_1,t_2]$. Then, there exists $\lambda^* \geq 0$ such that, for each $\lambda > \lambda^*$, $r > 0$ and for every integer $k \geq 1$, there is a constant $\mu^* = \mu^*(\lambda,r,k) > 0$ such that for each $\mu > \mu^*$ the problem $(\mathscr{N}_{\lambda,\mu})$ has at least $2k$ radially symmetric solutions which are nonincreasing in ϱ on $[0,R]$ and satisfy $0 < u(x) \leq r$ for each x with $|x| \in [\sigma,R]$.

Proof. Our proof follows verbatim that of Theorem 3.1. For this reason, we focus our attention only to those points which require some technical adjustments due to the presence of the singularity at $\rho = 0$. In particular, we will split our proof in two steps.

Step 1. Let $g(\cdot)$ be a locally Lipschitz continuous function. We also truncate $g(\cdot)$ as in (1.2) at the level $M > 0$, so that the differential equation in (4.5) can be read in the phase-plane equivalently as

$$\begin{cases} x' = y, \\ y' = -\frac{N-1}{t}y - \mathcal{Q}_{\lambda,\mu}(t)g_M(x), \end{cases} \tag{4.7}$$

with $t = \varrho > 0$.

Notice that the associated initial value problem has a local solution (see the Appendix). The Lipschitz condition for $g(\cdot)$ and the boundedness of $g_M(\cdot)$ also imply that the local solution is unique and it can globally extended to the all interval $[0,R]$. For this reason, as is known in the literature (cf. [6]), the shooting method can be applied also in this context.

In order to recover the results in Section 2 and Section 3, we discuss now the qualitative behavior of the solutions in both the intervals $[0,\sigma]$ and $[\sigma,R]$.

Analysis of the solutions for $t \in [0,\sigma]$. We will suppose, without loss of generality, that $[t_1,t_2] \subseteq]0,\sigma]$. From

$$x'(t) = y(t) = -\lambda \frac{\int_0^t \xi^{N-1} a^+(\xi) g_M(x(\xi))\,d\xi}{t^{N-1}},$$

we obtain $y(t) \leq 0$ for all $t \in [0,\sigma]$. Furthermore, analogously as in Lemma 2.1, there exists $\bar{t} \in [0,\sigma[$ such that $y(t) = 0$ for all $0 \leq t \leq \bar{t}$ and $y(t) < 0$ for all $t \in]\bar{t},\sigma]$. We also find immediately a constant $C_M > 0$ such that any possible solution $(x(t),y(t))$ of (4.7) with $0 < x(0) \leq M$ and $y(0) = 0$, satisfies

$$-C_M \leq y(t) \leq 0, \quad \forall\, t \in [0,\sigma].$$

Now we give an analogous result of Lemma 2.2. Indeed, within the same framework of that lemma and, in particular for d_j and ρ satisfying (2.3), we proceed as follows. Suppose that $(x(t),y(t))$ is a solution of (4.7) with $M \geq x(0) \geq d_j$, $y(0) = 0$ and $x(t_1) = d_j$. As in Lemma 2.2, we denote by $[t_1,\tilde{t}] \subseteq [t_1,t_2]$ the maximal closed subinterval of $[t_1,t_2]$ where $x(\cdot) \geq 0$ (and, necessarily, also $y(\cdot) \leq 0$). From the equation

$$x'' + \frac{N-1}{t}x' + \lambda a^+(t)g(x) = 0, \tag{4.8}$$

with the position $z(t) := y(t)t^{N-1}$, we have

$$z'z + \lambda a^+(t)t^{2(N-1)}g(x)x' = 0.$$

Hence, it follows

$$z'(t)z(t) + \lambda\delta {t_1}^{2(N-1)} g(x)x' \geq 0, \quad \text{for a. e. } t \in [t_1, t_2],$$

which implies that the function $\xi \mapsto z(\xi)^2 + 2\lambda\delta {t_1}^{2(N-1)} G(x(\xi))$ is nondecreasing in $[t_1, \tilde{t}]$. From this, we obtain

$$-x'(\xi) = |y(\xi)| \geq \left(\frac{t_1}{t_2}\right)^{N-1} \sqrt{\lambda\delta\rho}\sqrt{d_j^2 - x(\xi)^2}, \quad \forall\, \xi \in [t_1, \tilde{t}].$$

Apart from a multiplicative constant, notice that the above inequality is like the one in (2.4), so that the same conclusion is achieved, if λ is taken sufficiently large, namely

$$\lambda\delta G^\infty > \left(\frac{t_2}{t_1}\right)^{2(N-1)} \left(\frac{\pi}{2(t_2 - t_1)}\right)^2.$$

Finally, we give an analogous result of Lemma 2.3. Indeed, within the sane framework of that lemma and, in particular for a given $\vartheta \in]0, 1[$, and for ε and β_j^ε satisfying (2.5), we proceed as follows. Assume that $(x(t), y(t))$ is a solution of (4.7) with $0 < x(0) = \beta_j^\varepsilon \leq M$ and $y(0) = 0$. As in Lemma 2.2 we suppose by contradiction that it is not true that $x(t) \geq \theta\beta_j^\varepsilon$ and then consider a maximal interval $[0, \hat{t}\,] \subset [0, \sigma]$ such that (2.6) holds.

From the equation (4.8), we obtain

$$x''x' + \lambda\|a^+\|_\infty g(x)x' \leq x''x' + \lambda a^+(t)g(x)x' = -\frac{N-1}{t}(x')^2 \leq 0$$

which implies that the function $\xi \mapsto x'(\xi)^2 + 2\lambda\|a^+\|_\infty G(x(\xi))$ is nonincreasing in $[0, \hat{t}\,]$. From now on we have only to repeat the same proof of Lemma 2.3.

With these results at hand and since the shooting method is working, we can recover Lemma 2.4 and Proposition 3.1 without difficulty.

Analysis of the solutions for $t \in [\sigma, R]$. In this case, we are far from the singularity (which is at $t = 0$) and so, via minor changes in the constants, we can repeat the same analysis previously performed in Lemma 2.5. This way, the Proposition 3.2 can be re-established again.

Having achieved all the preliminary results in Proposition 3.1 and in Proposition 3.2, we get the same conclusion of the proof of Step 1 of Theorem 3.1.

Step 2. Assume now that $g(\cdot)$ is only continuous (and not necessarily locally Lipschitz). Then, in this case we can apply the standard techniques already recalled in Step 2 of the proof of Theorem 3.1. □

5. Final remarks

In the presents paper we have studied only the case of linear second order differential operators u'' and Δu. The technical tools we have developed for proving our main results (Lemma 2.2 and Lemma 2.3) rely essentially on time-mapping estimates associated to the autonomous equation

$$u'' + g(u) = 0. \tag{5.1}$$

This fact suggest different directions along which we could provide extensions of our results.

On the one hand, we can replace the condition (G_*) with an hypothesis of the form

$$0 \leq \mathcal{T}_\infty := \liminf_{c \to +\infty} \mathcal{T}(c) < \mathcal{T}^\infty := \limsup_{c \to +\infty} \mathcal{T}(c) = +\infty, \tag{$\mathcal{T}_*$}$$

where, for $c > 0$, $\mathcal{T}(c)$ is the time-mapping associated to (5.1) defined as

$$\mathcal{T}(c) := 2 \int_0^c \frac{ds}{\sqrt{2(G(c) - G(s))}}.$$

Within $(\mathcal{T}_*)$, we can deal with more general linear differential operators such as $u'' + m(t)u'$. We refer to [25] for analogous considerations.

It is worth noticing that the notion of time-mapping, associated to second order autonomous differential equations, plays an important role in the study of different nonlinear problems. Indeed, this technique has been successfully employed in different contexts. In the present paper, the time-mapping is defined in a compact interval $[0, c]$. There are, however, significant applications also when the time-mapping is considered in a neighborhood of infinity. With this respect, we refer to the interesting work [11] where the use of time-mappings has been effectively applied in the search of blow up solutions for nonlinear PDEs with non-monotone nonlinearities.

On the other hand, there is also a great deal of interest in studying differential equations involving nonlinear differential operators, such as p-Laplacians. A condition analogous to (G_*) is considered in [2] for Neumann problems in the p-Laplacian setting. With this respect, we remark that the work done in Section 2 addresses to use our technique to study the problem

$$\begin{cases} (\phi(u'))' + a(t)g(u) = 0, \\ u(t) > 0, \quad \forall\, t \in [0, T], \\ u'(0) = u'(T) = 0, \end{cases}$$

using information about the time-mapping associated with

$$(\phi(u'))' + g(u) = 0.$$

In this case, estimates for the time-mappings are already done in [15,20,22,24] and could be fruitfully exploited to extend Theorem 3.1 as well as Theorem 4.1 and Theorem 4.2 to the case of more general differential operators, such as p-Laplacians or ϕ-Laplacians.

6. Appendix

In this section we provide a local existence result for the initial value problem associated with equation

$$\big(p(t)\phi(u')\big)' + q(t)g(u) = 0. \tag{6.1}$$

We assume that $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous function. For the weight functions, we suppose that $p : [0, R] \to \mathbb{R}$ is continuous and positive on $]0, R]$ and $q \in L^1([0, R])$.

We are interested to find solutions of (6.1) satisfying the initial condition

$$u(0) = d, \quad u'(0) = 0. \tag{6.2}$$

By a local solution of (6.1)–(6.2) we mean a function $u(t)$ defined on a interval $[0, \delta] \subseteq [0, R]$ for some $\delta > 0$ with $u \in C^1([0, \delta])$, $p(t)\phi(u'(t))$ absolutely continuous on $[0, \delta]$ and satisfying equation (6.1) for a.e. t, as well as the initial condition (6.2). If $\delta = R$ we say that the solution is globally defined.

Proposition 6.1. *Suppose that*

$$\lim_{t \to 0^+} \frac{\int_0^t |q(s)|\, ds}{p(t)} = 0. \tag{6.3}$$

Then, for every $b \in \mathbb{R}$ there is at least a local solution of (6.1)–(6.2).

Proof. Let us fix a constant $\delta \in]0, R]$ and consider the operator $\mathscr{H}$ defined by

$$\mathscr{H}(u)(t) := b + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)} \int_0^s q(\xi) g(u(\xi))\, d\xi \right) ds$$

and acting on the Banach space $C([0, \delta])$ endowed with the sup-norm. For every $u \in C([0, \delta])$ it holds that the function $z(t) := \frac{1}{p(s)} \int_0^s q(\xi) g(u(\xi))\, d\xi$ is continuous on $]0, \delta]$ and, thanks to condition (6.3), can be continuously extended to $t = 0$, by setting $z(0) = 0$. Hence $\mathscr{H} : C([0, \delta]) \to C^1([0, \delta])$. Moreover, $\mathscr{H} : C([0, \delta]) \to C^0([0, \delta])$ is completely continuous. Via standard estimates, one can easily prove that for $\delta > 0$ sufficiently small $\mathscr{H}$ maps the closed unit ball around b, namely $\mathscr{B}[b, 1] := \{u \in C^0([0, \delta]) : \|u(\cdot) - b\|_\infty \leq 1\}$ to itself. The Schauder theorem guarantees the existence of a fixed point $\tilde{u}$ for $\mathscr{H}$ which is a local solution of (6.1)–(6.2). $\square$

In the special case $\phi(s) = s$ for all $s \in \mathbb{R}$ and $g(\cdot)$ locally Lipschitz continuous, one can apply the contraction mapping principle to prove the (local) uniqueness of the solution.

Proposition 6.1 applies to the initial value problem associated to the equation (4.6) or to the equation in (4.5) and, therefore, to system (4.7), where $\phi(s) = s$. A similar result is already given in [22]. See also [14] and the references therein for general results involving the initial value problems for ϕ-Laplacian type differential operators.

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