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# About Chaotic Dynamics in the Twisted Horseshoe Map

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The twisted horseshoe map was developed in order to study a class of density dependent Leslie population models with two age classes. From the beginning, scientists have tried to prove that this map presents chaotic dynamics. Some demonstrations that have appeared in mathematical literature present some difficulties or delicate issues. In this paper, we give a simple and rigorous proof based on a different approach. We also highlight the possibility of getting chaotic dynamics for a broader class of maps.

*Keywords*: Chaotic dynamics; discrete dynamical systems; stretching along the paths method; twisted horseshoe map.

## 1. Introduction

The mathematical phenomenon of chaos has attracted and continues to attract several researchers from different scientific fields such as astronomy, meteorology, ecology, biology and economics.

Everyone has a natural idea about the meaning of the term "chaos" in the common language. Conversely, it is a delicate question to define it formally in a mathematical context. Indeed, there is no consensus about a unique definition of chaos, although many different points of view have some typical features in common (see the next section for a brief survey on the topic).

On the other hand, it is widely accepted that the birth of *deterministic chaos theory* is attributed to Henry Poincaré as a consequence of his work on the three body problem. In that context, he introduced the notion of "homoclinic point" and he observed the presence of complex dynamical behaviors explained in [Poincaré, 1899, 1908].

However, until 1975, with the celebrated paper "Period three implies chaos" [Li & Yorke, 1975], no one had apparently used the term "chaotic" in a formal proper and explicit manner in mathematical literature. According to the survey on that article available in [Abraham & Ueda, 2000], Li and Yorke were motivated by the aim to explain Lorenz's result. Then they pointed out a sufficient condition for chaos in the case of one-dimensional difference equations. For completeness, we recall their main result which is itself important and it is also relevant to present the central object of this work: the twisted horseshoe map.

To this purpose, we set the general notation $f^{n+1} = f(f^n)$ to represent the composition of a function $f : \mathbb{R}^N \to \mathbb{R}^N$ with itself $(n+1)$-times. Furthermore, given a point $x \in \mathbb{R}^N$, we call it a *fixed point* if $x = f(x)$ or a *periodic point with period* $k$ if $x = f^k(x)$ and $x \neq f^n(x)$ for $1 \leq n < k$. In particular a periodic point of period 1 is a fixed point.

**Theorem 1** [Li & Yorke, 1975]. *Let* $J \subseteq \mathbb{R}$ *be an interval and let* $F : J \to J$ *be continuous. Assume there is a point* $a \in J$ *for which the points* $b = F(a)$, $c = F^2(a)$ *and* $d = F^3(a)$ *satisfy* $d \leq a < b < c$ *(or* $d \geq a > b > c$*). Then, for every* $k = 1, 2, \ldots$ *there* *is a periodic point in $J$ having period $k$. Furthermore, there is an uncountable set $\mathcal{S} \subset J$ (containing no periodic points), which satisfies the following conditions*:

- *For every $p, q \in \mathcal{S}$ with $p \neq q$,*

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0$$

$$\liminf_{n\to\infty} |F^n(p) - F^n(q)| = 0.$$

- *For every $p \in \mathcal{S}$ and any periodic point $q \in J$,*

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0.$$

We observe that such a set $\mathcal{S}$ as in Theorem 1 is nowadays commonly called *scrambled set* (see [Smítal, 1983; Blanchard *et al.*, 2002]). With this notion, the concept of chaos introduced by Li and Yorke reads as follows: let $x_{n+1} = F(x_n)$ be a sequence of points generated by a continuous self-mapping $F$ defined on $J$, then $F$ is named chaotic if there exist both periodic points with period any natural number and an uncountable scrambled set.

In the same period of [Li & Yorke, 1975], Robert May observed the presence of chaotic behaviors in some discrete dynamical systems which describe simple demographic models (like the logistic map). For such a reason, in [May, 1974, 1976], he underlined that the origin of the term "chaotic" is connected to the work of Li and Yorke. In this way the innovative concept of chaos began to become very popular.

In 1977, just after May's works, Guckenheimer, Oster and Ipaktchi faced demographic studies considering density dependent Leslie population models from the viewpoint of chaotic dynamics (see [Guckenheimer *et al.*, 1977]). They investigated a discrete dynamical system modeling a single population with two age classes. Indeed the difference equation involved was the following one:

$$(x_{n+1}, y_{n+1}) = (m_1 x_n + m_2 y_n, S x_n),$$

where $m_1(\cdot)$ and $m_2(\cdot)$ are the per-capita birthrates for each age group and $S$ is the survival fraction of individuals from the first age class to the second one. In that paper the authors had not considered only $m_i(x, y) := b_i \mathrm{e}^{-a(x+y)}$ for $i \in \{1, 2\}$ where $b_1$, $b_2$, $a$ are positive coefficients, but also $S = 1$, $a = 0.1$ and $b_1 = r = b_2$. Consequently they obtained the following equation:

$$(x_{n+1}, y_{n+1}) = (r(x_n + y_n)\mathrm{e}^{-0.1(x_n+y_n)}, x_n). \quad (1)$$

In order to study this model they introduced an analogous system, the so-called *twisted horseshoe map* defined as follows:

$$(x_{n+1}, y_{n+1}) = \left(\frac{x_n}{10} + \frac{y_n}{2} + \frac{1}{4}, \min(2y_n, 2(1 - y_n))\right). \quad (2)$$

The importance of this map is due to a remark inside [Guckenheimer *et al.*, 1977] which asserts that, by choosing certain values of the parameter $r$, the sixth iterate of the map in Eq. (1) has behaviors like the map in Eq. (2), or the following one:

$$(x_{n+1}, y_{n+1}) = \left(\frac{x_n}{10} + \frac{y_n}{2} + \frac{1}{4}, 4y_n(1 - y_n)\right).$$

It is interesting to dwell on the name of the map in (2) to understand the peculiarity that makes it different from the "standard" horseshoe introduced by Stephen Smale (see for instance [Smale, 1965, 1967] and also [Smale, 1998] with a mainly informative and descriptive feature). In fact, the twisted horseshoe map deforms the unit rectangle $\mathcal{R} := [0, 1] \times [0, 1]$ in the following manner:

(a) it stretches $\mathcal{R}$ vertically twice its width and compresses it horizontally,
(b) it makes an *half turn twist* of the image obtained in step (a),
(c) it bends the image in step (b) on itself, so that this deformation is contained in the starting rectangle $\mathcal{R}$.

The approach followed by the authors to explain the existence of chaotic behaviors in the twisted horseshoe involves Markov partitions. However, they did not give a complete analytical proof of this claim, although they underlined the fact that their conclusions would *remain generally valid when the "state of art" progresses to the point of being able to handle maps of this sort* (see [Guckenheimer *et al.*, 1977, p. 120]).

After that, in 1981, Kloeden studied the following map:

$$(x_{n+1}, y_{n+1}) = \left(\min(2x_n, 2(1 - x_n)), \frac{x_n}{2} + \frac{y_n}{10} + \frac{1}{4}\right), \quad (3)$$

which is the same as the one considered in [Guckenheimer *et al.*, 1977] via a change of variables.

In [Kloeden, 1981; Kloeden & Li, 2006a, 2006b], the authors used a theorem developed by Kloeden on chaotic dynamics, which generalizes the result obtained by Li and Yorke to the case of first order $N$-dimensional difference equations:

$$x_{n+1} = f(x_n), \quad n = 0, 1, \ldots$$

where $x_n \in \mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}^N$ is a nonlinear and continuous function. Such a theorem (see [Kloeden & Li, 2006b, p. 255, Theorem 4]) guarantees the existence of periodic points with period any natural number greater than a fixed natural and an uncountable scrambled set. However, the application of this theorem presents a difficulty in a step concerning the verification of the covering condition of a particular set of the plane. More in detail, let the segment $B$ and the line $L_1$ be as in these papers (see [Kloeden & Li, 2006b, p. 259]), then the image of $B$ under the third iteration of the map defined in the difference equation (3) is not contained in $L_1$. Hence the hypotheses of the theorem are not all fulfilled.

Another proof of chaos for (3) is presented in [Diamond *et al.*, 2008; Diamond *et al.*, 2012] with an approach that is applicable only to the class of Lipschitz mappings and so the model in consideration fits well with this method. However, it would be nice to have a method that makes weaker requests.

The purpose of this work is to prove that the twisted horseshoe map (3) presents chaotic dynamics with a topological tool that is also applicable to maps which are not necessarily Lipschitz or not diffeomorphisms. Moreover, we extend this result to more general difference equations of the following type:

$$(x_{n+1}, y_{n+1}) = (\phi_1(x_n), \phi_2(x_n, y_n)) \tag{4}$$

where $\phi_1 : [0, 1] \to [0, 1]$ is a continuous map which takes zero value on the boundary and it assumes value 1 in at least an internal point and $\phi_2 : [0, 1]^2 \to [0, 1]$ is any continuous map.

Concerning the twisted horseshoe, as just observed, the name of this map reminds the reader of the well known Smale's geometric construction. On the other hand, the classical horseshoe theory of Smale (see [Moser, 1973, Chapter III]) is not directly applicable here since the map is not a diffeomorphism. Clearly, the turn twist at step (b) makes the map not injective.

In recent years, the Smale's idea was generalized by the development of the theory of the so-called "topological horseshoes" (see [Burns & Weiss, 1995; Kennedy & Yorke, 2001; Kennedy *et al.*, 2001; Zgliczyński, 1996]). In this frame, only the continuity of the map is required. In [Liz & Ruiz-Herrera, 2012], the authors follow an approach based on topological horseshoes in order to prove the existence of chaotic dynamics for maps as in (1). More in detail, the approach followed is the one developed in [Zgliczyński & Gidea, 2004] concerning the notion of "covering relations". We have reason to believe that the map (3) can be studied with this method, but we will use a slightly different technique because it allows us to generalize the result to the case of maps as in (4) in a natural way and with minimal changes.

The plan of the paper is as follows. In Sec. 2 we first of all recall some definitions of chaos available in mathematical literature in order to exploit relations between them. Furthermore, we give the basic theorems and definitions which are extensively used in the work. In particular, within the frame of the topological horseshoe theory, we will recall a technique developed by Medio, Pireddu and Zanolin in [Medio *et al.*, 2009] and based on the concept of *stretching along the paths* (SAP). In Sec. 3, we prove analytically that the twisted horseshoe map is chaotic by applying the SAP method. In this way we also show how it is possible to obtain other chaotic maps.

## 2. Preliminaries, Definitions and Main Results

### 2.1. *Review on chaos definitions*

It is important to observe that the notion of chaos introduced by Li and Yorke is not the only one in mathematical literature. Quoting the words of Kolyada: "*So many authors, so many definitions*" (see [Kolyada, 2004, p. 1]). Since there are really many definitions, we focus the following presentation only on those concepts which are relevant to our application.

For the remainder of this section, if not differently stated, we consider $(X, d)$, with $X$ a compact metric space and $f : X \to X$ a continuous map. Moreover, the term chaotic will be used without distinction with reference to dynamical systems, i.e. a couple $(X, f)$, as well as only to maps $f$.

Recalling Theorem 1 of Li and Yorke, we observe that different authors consider a map chaotic if it admits only an uncountable scrambled set. This is due to the possibility to include a wider range of cases. Hence, in agreement with [Aulbach & Kieninger, 2001; Kirchgraber & Stoffer, 1989], we give the following definition of chaos in the sense of Li and Yorke.

**Definition 2.1** [LY-chaos]. A map $f$ is called *chaotic in the sense of Li and Yorke* (*or LY-chaotic*) if there exists an uncountable subset $\mathcal{S}$ of $X$ such that:

- $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$ for all $x, y \in \mathcal{S}$, $x \neq y$,
- $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ for all $x, y \in \mathcal{S}$, $x \neq y$,
- $\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0$ for all $x \in \mathcal{S}$, $p \in X \backslash \mathcal{S}$, $p$-periodic.

Another contribution to the development of the chaos theory concerns the definition given in 1992 by Block and Coppel (see [Block & Coppel, 1992, p. 127]).

**Definition 2.2** [BC-chaos]. A map $f$ is called *chaotic in the sense of Block and Coppel* (*or BC-chaotic*) if there exist nonempty disjoint compact subsets $X_0, X_1$ of $X$ and a positive integer $m$ such that, if $Y = X_0 \cup X_1$ and $g = f^m$, then:

- $g(Y) \subseteq Y$,
- for every sequence $\alpha = (\alpha_0, \alpha_1, \dots) \in \{0,1\}^{\mathbb{N}}$ there exists a point $x \in Y$ such that $g^k(x) \in X_{\alpha_k}$ for all $k \geq 0$.

This is an equivalent manner to express the fact that $g$ is semi-conjugate to the Bernoulli shift on two symbols. Usually, if in the previous definition $m = 1$, then we can refer to *chaos in the sense of coin tossing* (see [Kirchgraber & Stoffer, 1989]). Indeed, a diagonal argument (see [Kennedy *et al.*, 2001, Lemma 3]) allows us to extend the one-side sequences in Definition 2.2 to two-side ones, involved in [Kirchgraber & Stoffer, 1989, Definition 1]. In 2009 Medio, Pireddu and Zanolin modified the notion of chaos in agreement with Definition 2.2 by introducing the case of periodic points (see [Medio *et al.*, 2009, p. 3288]). This leads to the following concept:

**Definition 2.3.** A map $f$ induces *chaotic dynamics on two symbols on the set* $\mathcal{D} \subseteq X$ if there exist two disjoint compact sets $\mathcal{K}_0, \mathcal{K}_1 \subseteq \mathcal{D}$ such that for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ there exists a corresponding sequence $(w_i)_{i\in\mathbb{Z}} \subseteq \mathcal{D}^{\mathbb{Z}}$ such that

$$w_i \in \mathcal{K}_{s_i} \quad \text{and} \quad w_{i+1} = f(w_i) \quad \text{for all } i \in \mathbb{Z} \tag{5}$$

and, whenever $(s_i)_{i\in\mathbb{Z}}$ is a $k$-periodic sequence, i.e. $s_{i+k} = s_i$ for all $i \in \mathbb{Z}$, for some $k \geq 1$, there exists a $k$-periodic sequence $(w_i)_{i\in\mathbb{Z}} \subseteq \mathcal{D}^{\mathbb{Z}}$ satisfying condition in (5).

Maps with similar features (i.e. coin tossing property and also periodic points) have been previously considered also in [Srzednicki, 2000; Srzednicki & Wójcik, 1997; Zgliczyński, 1996, 2001; Zgliczyński & Gidea, 2004].

A natural question at this point is if these different definitions have some relationship among them. The answer is affirmative. More in detail, if a map $f$ is chaotic according to Definition 2.3 then it is immediately BC-chaotic too. The converse does not hold (see [Kennedy & Yorke, 2001; Cian, 2014]). Instead, if $f$ is BC-chaotic then it is straightforward to show that it has positive topological entropy. Furthermore, thanks to [Blanchard *et al.*, 2002], if $f$ is onto and it has positive topological entropy, then it is LY-chaotic (instead the converse is not true [Smítal, 1986]). In this way, with regard to these three definitions, we can affirm that the chaos in the sense of Li and Yorke is the weakest.

In the sequel we will prove that the map in (3) is chaotic according to Definition 2.3, hence it is BC-chaotic and also LY-chaotic.

### 2.2. *Theoretical background on chaotic dynamics*

We observe that theorems in [Kloeden, 1981; Kloeden & Li, 2006a, 2006b] require some conditions which include "covering relations". On the contrary, there are different approaches that involve topological tools which are about "crossing relations". These kinds of conditions are sometimes easier to verify than previous ones. Roughly speaking, in that case, one studies the intersections between the image of a set under the iterates of a map and the original set. Usually fixed points and interesting dynamics appear when such *crossings* occur in the right manner. Certainly, the most famous one is the horseshoe introduced by Smale in the case of a planar diffeomorphism. The existence of an horseshoe structure is a very useful technique to prove rigorously the presence of chaos in the sense of Block and Coppel (Definition 2.2) or in that of dynamics on two symbols (Definition 2.3).

Since the twisted horseshoe map is not a diffeomorphism, we are interested in the search of a topological horseshoe. In view of this, we present the *stretching along the paths* (SAP) method with reference to paper [Medio *et al.*, 2009]. Furthermore, we note that this is not the only one method based on the topological horseshoe, other approaches are stated for example in [Zgliczyński, 1996, 2001; Zgliczyński & Gidea, 2004].

Borrowing notations from [Medio *et al.*, 2009], we recall the theory of the SAP method adapted to our context. By path $\gamma$ we mean a continuous mapping $\gamma : [t_0, t_1] \to \mathbb{R}^2$ and without loss of generality we will usually take $[t_0, t_1] = [0, 1]$. By a subpath of $\gamma$ we mean just the restriction of $\gamma$ to a compact subinterval of its domain. We call $\mathcal{R} := [0, 1] \times [0, 1]$ the unit rectangle in the plane and we define the pair $\tilde{\mathcal{R}} = (\mathcal{R}, \mathcal{R}^-)$, where $\mathcal{R}^- := \mathcal{R}_l^- \cup \mathcal{R}_r^-$ is the union of the left and right vertical sides of $\mathcal{R}$, to be the unit oriented rectangle.

The SAP property for maps between oriented rectangles is thus stated as follows.

**Definition 2.4.** Let $\mathcal{R}$ and $\tilde{\mathcal{R}}$ exist as above. Suppose that $\varphi : \mathbb{R}^2 \supseteq D_\varphi \to \mathbb{R}^2$ is a map defined on a set $D_\varphi$. Let $K \subseteq \mathcal{R} \cap D_\varphi$ be a compact set. We say that $(K, \varphi)$ *stretches* $\tilde{\mathcal{R}}$ *(to* $\tilde{\mathcal{R}}$*) along the paths* and we write $(K, \varphi) : \tilde{\mathcal{R}} \Leftrightarrow\!\!\!\to \tilde{\mathcal{R}}$ if the following conditions hold:

(i) $\varphi$ is continuous on $K$,
(ii) for every path $\gamma : [0, 1] \to \mathcal{R} \subseteq \mathbb{R}^2$ such that $\gamma(0) \in \mathcal{R}_l^-$ and $\gamma(1) \in \mathcal{R}_r^-$ (or vice-versa), there exists a subinterval $[t', t''] \subseteq [0, 1]$ such that
  (a) $\gamma(t) \in K$ for all $t \in [t', t'']$,
  (b) $\varphi(\gamma(t)) \in \mathcal{R}$ for all $t \in [t', t'']$,
  (c) $\varphi(\gamma(t'))$ and $\varphi(\gamma(t''))$ belong to different components of $\mathcal{R}^-$.

We are now ready to make the link between SAP property and chaotic dynamics on two symbols (Definition 2.3) by means of the *SAP method*.

**Theorem 2** [Medio *et al.*, 2009]. *Let $\mathcal{R}$ and $\tilde{\mathcal{R}}$ exist as above. Assume $\mathcal{D} \subseteq \mathcal{R} \cap D_\varphi$, with $D_\varphi$ the domain of $\varphi : \mathbb{R}^2 \supseteq D_\varphi \to \mathbb{R}^2$. If $K_0$ and $K_1$ are two disjoint compact sets with $K_0 \cup K_1 \subseteq \mathcal{D}$ and*

$$(K_i, \varphi) : \tilde{\mathcal{R}} \Leftrightarrow\!\!\!\to \tilde{\mathcal{R}} \quad \text{for } i \in \{0, 1\},$$

*then $\varphi$ induces chaotic dynamics on two symbols on $\mathcal{D}$ relatively to $K_0$ and $K_1$.*

In this way it is clear that our task is to verify the SAP property for two disjoint compact sets, in order to prove that the map in (3) exhibits chaos in the sense of Definition 2.3.

## 3. Twisted Horseshoe Map

Let $\mathcal{R} \subseteq \mathbb{R}^2$ be the unit rectangle. We denote with $T$ the *tent map* defined as follows:

$$T : [0, 1] \to [0, 1]$$
$$x \mapsto T(x) := \min(2x, 2(1 - x)).$$

In this way, we consider the following continuous function:

$$\begin{aligned} f &: \mathcal{R} \to \mathcal{R} \\ (x, y) &\mapsto f(x, y) := (T(x), f_2(x, y)), \end{aligned} \tag{6}$$

where the second component of the application is defined by

$$f_2(x, y) := \frac{x}{2} + \frac{y}{10} + \frac{1}{4}.$$

The discrete dynamical system identified by the function $f$ as in (6) is just the twisted horseshoe map introduced in (3), i.e. the following planar difference equation:

$$\begin{aligned} (x_{n+1}, y_{n+1}) &= (T(x_n), f_2(x_n, y_n)) \\ &= \left(T(x_n), \frac{x_n}{2} + \frac{y_n}{10} + \frac{1}{4}\right). \end{aligned} \tag{7}$$

We are now ready to prove that the twisted horseshoe map, namely Eq. (7), induces chaotic dynamics on two symbols and consequently it is also LY-chaotic.

Firstly, we start by observing in Fig. 1 how the region $\mathcal{R}$ is deformed by the action of the map $f$. Moreover, we show that horizontal paths connecting the left vertical side of $\mathcal{R}$, i.e. $\mathcal{R}_l^-$, with the

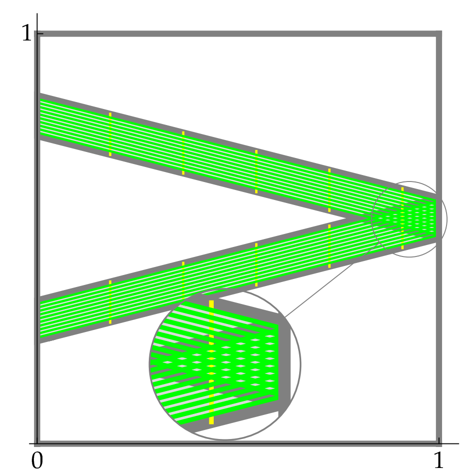


Fig. 1. Image of the unit rectangle $\mathcal{R}$ through $f$. Transformations of straight horizontal lines (green) and vertical ones (yellow) under the action of $f$.

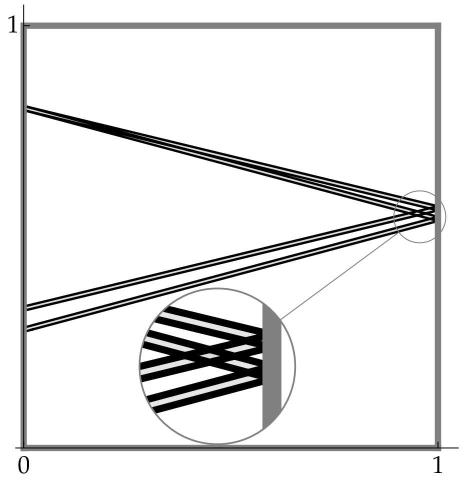


Fig. 2. Image of the unit rectangle $\mathcal{R}$ through the second iteration of the map $f$. We also highlighted the crossings of the image set of $\mathcal{R}$ under $f^2$.

right one, i.e. $\mathcal{R}_r^-$, are mapped in horizontal paths connecting the vertical sides of $\mathcal{R}$ twice. Despite the horizontal paths being correctly stretched, we see graphically that it is not possible to find two disjoint compact subsets of $\mathcal{R}$ as required by Theorem 2. The fact that the subsets have to be disjoint is a key assumption in this theorem, in order to obtain a semi-conjugacy to the Bernoulli shift map of two symbols.

The second iteration of $f$ is defined by the following function:

$$\begin{aligned} f^2 &: \mathcal{R} \to \mathcal{R} \\ (x, y) &\mapsto f^2(x, y) := (T^2(x), \psi_2(x, y)), \end{aligned} \tag{8}$$

where $\psi_2$ is the second component of $f^2$ and it is such that

$$\begin{aligned} \psi_2(x, y) &:= f_2(T(x), f_2(x, y)) \\ &= \frac{T(x)}{2} + \frac{f_2(x, y)}{10} + \frac{1}{4}. \end{aligned}$$

Furthermore, just as done for $f$, we represent in Fig. 2 the image of $\mathcal{R}$ under the action of the map $f^2$. Now we focus our attention on the crossings and we see that the set $f^2(\mathcal{R})$ crosses four times the set $\mathcal{R}$. By choosing in the correct way two crossings, we are allowed to find two disjoint compact subsets with the required properties and so we assert that it makes sense to try to implement SAP method in order to prove the presence of chaotic dynamics for the second iterate of the twisted horseshoe map.

Indeed we prove that the following result holds.

**Theorem 3.** *Let $\mathcal{R}$ be the unit rectangle and $\tilde{\mathcal{R}}$ the unit oriented rectangle. Let $g := f^2$ be a continuous function as defined in (8). Then, the hypotheses of Theorem 2 are all satisfied for $g$, namely there exist two disjoint compact sets $K_0, K_1 \subseteq \mathcal{R}$ such that $K_0 \cup K_1 \subseteq \mathcal{R}$ and*

$$(K_i, g) : \tilde{\mathcal{R}} \Leftarrow\!\Rightarrow \tilde{\mathcal{R}} \quad \textit{for } i \in \{0, 1\}. \tag{9}$$

*Proof.* Let $I := [0, 1]$ be the unit interval and let $\gamma = (\gamma_1, \gamma_2) : I \to \mathcal{R}$ be a generic path such that $\gamma(0) \in \mathcal{R}_l^-$ and $\gamma(1) \in \mathcal{R}_r^-$. Let

$$I_{00} := \left[0, \frac{1}{4}\right], \quad I_{01} := \left[\frac{1}{4}, \frac{1}{2}\right],$$

$$I_{10} := \left[\frac{1}{2}, \frac{3}{4}\right], \quad I_{11} := \left[\frac{3}{4}, 1\right]$$

be subintervals of $I$. We define also the following subrectangles of $\mathcal{R}$:

$$K_{ij} := I_{ij} \times I \quad \text{for } i, j \in \{0, 1\}. \tag{10}$$

We note that each $K_{ij}$ is a compact subset of $\mathcal{R}$ and, moreover,

$$K_{ij} \cup K_{lm} \subseteq \mathcal{R} \quad \text{for } i, j, l, m \in \{0, 1\}.$$

The idea of the proof is to verify that each $K_{ij}$, with $i, j \in \{0, 1\}$, is such that $(K_{ij}, g)$ stretches $\tilde{\mathcal{R}}$ along the paths. If we use graphical tools, this result becomes clear (see for instance Fig. 3). After that, the trick is to select two sets $K_{ij}$, for $i, j \in \{0, 1\}$,

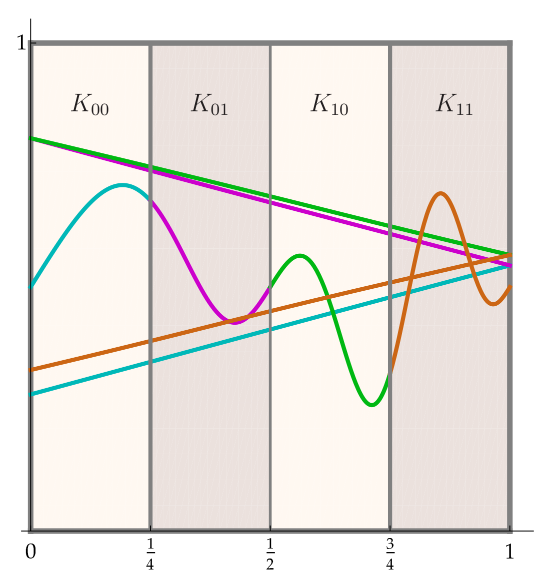


Fig. 3. Image of a generic path connecting $\mathcal{R}_l^-$ to $\mathcal{R}_r^-$ and its transformation under the action of $f^2$. With different colors are highlighted the subpaths (relatively to sets $K_{ij}$ for $i, j \in \{0, 1\}$) and their corresponding images through $f^2$.

in such a way they are disjoint and call them $K_0$ and $K_1$, respectively. At this point, $K_0$ and $K_1$ are two disjoint compact sets which immediately satisfy condition (9).

Thus, let $K_{ij}$, with $i, j \in \{0,1\}$, one of the sets defined as in (10). In order to prove that $(K_{ij}, g) : \tilde{\mathcal{R}} \Longleftrightarrow\!\!\!\rightarrow \tilde{\mathcal{R}}$, we have to verify the conditions in Definition 2.4 which are the following ones:

(i) $g$ is continuous on $K_{ij}$,
(ii) for all paths $\gamma : I \to \mathcal{R}$ joining $\mathcal{R}_l^-$ and $\mathcal{R}_r^-$ there exists a subinterval $[t', t''] \subseteq I$ such that
  (a) $\gamma(t) \in K_{ij}$ for all $t \in [t', t'']$,
  (b) $g(\gamma(t)) \in \mathcal{R}$ for all $t \in [t', t'']$,
  (c) $g(\gamma(t'))$ and $g(\gamma(t''))$ belong to different components of $\mathcal{R}^-$.

Condition (i) is implied by continuity of the function $f$ on $\mathcal{R}$.

Regarding the condition (ii), we show that there exists $J_{ij} := [t', t''] \subseteq I$ such that

(A) $\gamma_1(t') = \min I_{ij}$, $\gamma_1(t'') = \max I_{ij}$ (or vice-versa) and $\gamma(J_{ij}) \subseteq K_{ij}$,
(B) $\gamma_1(t) \notin \partial I_{ij}$ for all $t \in (t', t'')$.

Suppose by contradiction, there does not exist $J_{ij}$ satisfying the first condition (A). Accordingly, the path $\gamma$ will not connect $\mathcal{R}_l^-$ to $\mathcal{R}_r^-$ continuously, which contradicts the hypothesis on $\gamma$. Hence, let $\overline{J}_{ij} := [\overline{t'}, \overline{t''}]$ be an interval which satisfies such a property. Moreover, without loss of generality, we assume $\gamma_1(\overline{t'}) = \min I_{ij}$ and $\gamma_1(\overline{t''}) = \max I_{ij}$. We also define the following sets:

$$L := \{t \in [\overline{t'}, \overline{t''}] \,|\, \gamma_1(t) = \min I_{ij}\},$$

$$R := \{t \in [\overline{t'}, \overline{t''}] \,|\, \gamma_1(t) = \max I_{ij}\}.$$

Denoting with $t''$ the minimum of $R$, we define the set

$$\overline{L} := \{t \in L \,|\, t \leq t''\}.$$

Let $t'$ be the maximum of $\overline{L}$. By setting $J_{ij} := [t', t'']$ we prove that there exists a subinterval of $I$ verifying both conditions (A) and (B) above.

In this way, we have just shown that there exists $J_{ij} \subseteq I$ satisfying (a).

Furthermore, by definition of $f$, it is straightforward to prove (b) because we have $g(\gamma(t)) \in \mathcal{R}$ for all $t \in J_{ij}$.

To verify (c) it is enough to show that for all $y \in I$, $g(\gamma(t')) = (0, y)$ and $g(\gamma(t'')) = (1, y)$, or vice versa. In order to do this, we evaluate $f$ along the subpath $\gamma_{|J_{ij}}$. Indeed,

$$(f \circ \gamma)(J_{ij}) = (T(\gamma_1(J_{ij})), f_2(\gamma(J_{ij}))),$$

where $T(\gamma_1(J_{ij})) = T(I_{ij})$ and $f_2(\gamma(J_{ij})) \subseteq I$. Moreover, we observe that

$$T(I_{00}) = T(I_{11}) = I_{00} \cup I_{01} = \left[0, \frac{1}{2}\right],$$

$$T(I_{10}) = T(I_{01}) = I_{10} \cup I_{11} = \left[\frac{1}{2}, 1\right].$$

So that, for $i, j \in \{0,1\}$, the interval $T(I_{ij})$ is exactly $[0, 1/2]$ or $[1/2, 1]$. At the second iteration,

$$\begin{aligned}(g \circ \gamma)(J_{ij}) &= (f^2 \circ \gamma)(J_{ij}) \\ &= (T^2(\gamma_1(J_{ij})), f_2(T(\gamma_1(J_{ij})), \\ &\qquad f_2(\gamma(J_{ij})))),\end{aligned}$$

where

$$T^2(\gamma_1(J_{ij})) = T(T(I_{ij})) = I$$

because $T([0, 1/2]) = T([1/2, 1]) = I$.

Whereas $\gamma_1(t')$ and $\gamma_1(t'')$ are consecutive elements in $\{0, 1/4, 1/2, 3/4, 1\}$ and $T^2(1/2) = T^2(0) = T^2(1) = 0$, $T^2(1/4) = T^2(3/4) = 1$, we have $T^2(\gamma_1(t')) = 0$ and $T^2(\gamma_1(t'')) = 1$ or vice versa. Moreover, let $g_2$ be the second component of the map $g$, then

$$g_2(\gamma(J_{ij})) = f_2(T(\gamma_1(J_{ij})), f_2(\gamma(J_{ij}))) \subseteq I.$$

Hence the condition (c) is verified. In this way we have found a subinterval $J_{ij}$ which satisfies (ii).

Therefore, for all possible choices of $i, j \in \{0,1\}$, $(K_{ij}, g) : \tilde{\mathcal{R}} \Longleftrightarrow\!\!\!\rightarrow \tilde{\mathcal{R}}$ holds.

Thus, given one of the couples

$$K_{00} - K_{10}, \quad K_{00} - K_{11}, \quad K_{01} - K_{11},$$

if we set the chosen one as $K_0 - K_1$, then the following result holds:

$$(K_0, g) : \tilde{\mathcal{R}} \Longleftrightarrow\!\!\!\rightarrow \tilde{\mathcal{R}}, \quad (K_1, g) : \tilde{\mathcal{R}} \Longleftrightarrow\!\!\!\rightarrow \tilde{\mathcal{R}},$$

which is exactly the condition (9). ■

By Theorem 2, we can conclude that $g\,(= f^2)$ induces chaotic dynamics on two symbols on $\mathcal{R}$. Therefore, explanation in Sec. 2.1 leads to the following chain of implications: $g$ chaotic in the sense of Definition 2.3 implies that $f$ is BC-chaotic and also that $g$ is LY-chaotic.

More generally, this methodology can be applied in order to prove the following extension of Theorem 3 concerning maps described as in (4).

**Corollary 3.1.** *Let $\mathcal{R}$ and $\tilde{\mathcal{R}}$ be such as in the previous theorem. Let*

$$h : \mathcal{R} \to \mathcal{R}$$

$$(x, y) \mapsto h(x, y) := (h_1(x), h_2(x, y))$$

*be a function where $h_2 : \mathcal{R} \to I$ is a continuous map and $h_1 : I \to I$ is a continuous map for which there exists $c \in (0, 1)$ such that $h_1(c) = 1$ and also $h_1(0) = 0 = h_1(1)$. Then, for $h^2$ the hypotheses of Theorem 2 are all satisfied, namely there exist two disjoint compact sets $C_0, C_1 \subseteq \mathcal{R}$ such that $C_0 \cup C_1 \subseteq \mathcal{R}$ and*

$$(C_i, h^2) : \tilde{\mathcal{R}} \Longleftrightarrow \tilde{\mathcal{R}} \quad \text{for } i \in \{0, 1\}. \tag{11}$$

*Proof.* By the Intermediate Value Theorem, the continuity of $h_1$ together with the hypotheses that the function $h_1$ is zero at the endpoints of $I$ and it takes the value 1 in at least a point interior of $I$ lead to the existence of both a point $\alpha \in (0, c)$ and $\beta \in (c, 1)$ such that

$$h_1(\alpha) = c = h_1(\beta)$$

and also

$$[0, c] \subseteq h_1([0, \alpha]), \quad [c, 1] \subseteq h_1([\alpha, c]),$$

$$[c, 1] \subseteq h_1([c, \beta]), \quad [0, c] \subseteq h_1([\beta, 1]).$$

At this point, let

$$I_{00} := [0, \alpha], \quad I_{01} := [\alpha, c],$$

$$I_{10} := [c, \beta], \quad I_{11} := [\beta, 1],$$

be subintervals of $I$ and let $K_{ij}$ be subrectangles of $\mathcal{R}$ defined analogously to (10). By taking $C_0$ and $C_1$ from $K_{ij}$ in such a way that they are disjoint, the result in (11) holds with the same proof of the previous theorem. ■

By applying Corollary 3.1 to such a map $h$ it follows that the difference equation, represented by $(x_{n+1}, y_{n+1}) = h^2(x_n, y_n)$, induces chaotic dynamics on two symbols (Definition 2.3) and thus the map $h^2$ is LY-chaotic.

*Remark 3.1.* Let $x_{n+1} = L(x_n)$ be the logistic map, where the function $L$ is defined on the unit interval as follows

$$L : I \to I$$

$$x \mapsto L(x) := 4x(1 - x).$$

The behavior of the logistic map is closely linked with the one of tent map, in fact these maps are topologically conjugate. If in the definition of the first component of the twisted horseshoe map we consider the logistic map instead of the tent map, we can apply Corollary 3.1 by taking $c = 1/2$.

**Example 3.1.** Let $h$ be a continuous non-Lipschitz function defined as follows:

$$h : \mathcal{R} \to \mathcal{R}$$

$$(x, y) \mapsto h(x, y) := (h_1(x), h_2(x, y))$$

such that

$$h_1(x) = \min\left(\sqrt{3x}, \frac{3}{4} + \frac{3x}{2} - \frac{9x^2}{4}\right),$$

$$h_2(x, y) = \frac{x}{4} + \frac{y}{4} + \frac{1}{2}.$$

Since $h_1$ is a continuous function such that $h_1(0) = 0 = h_1(1)$ and $h_1(1/3) = 1$, then there exist $\alpha \in (0, 1/3)$ and $\beta \in (1/3, 1)$ for which $h_1(\alpha) = 1/3 = h_1(\beta)$. Such points are

$$\alpha = \frac{1}{27}, \quad \beta = \frac{1}{3} + \frac{2\sqrt{6}}{9}.$$

By choosing the sets in the previous theorem as follows $C_0 = [0, \alpha] \times I$ and $C_1 = [1/3, \beta] \times I$, we can prove the condition (11) with an argument similar to the one followed in the proof of Theorem 3. Hence $h^2$ is chaotic in the sense of Definition 2.3.

**Example 3.2.** Beside the case of unimodal maps, like the tent map or the logistic map, even the truncated tent maps are studied in literature (see [Burns & Hasselblatt, 2011]). Let us take the generic truncated tent map, which is a piecewise-linear map with a flat top defined as follows:

$$T_\delta = \min\left(\frac{1}{\delta}x, 1, \frac{1}{\delta}(1 - x)\right), \quad \text{with } 0 < \delta < \frac{1}{2}.$$

We define the following family of maps

$$\begin{aligned} &h_\delta : \mathcal{R} \to \mathcal{R} \\ &(x, y) \mapsto h_\delta(x, y) := (T_\delta(x), h_2(x, y)), \end{aligned} \tag{12}$$

where the map $h_2$ is any continuous map such that $0 \leq h_2(x, y) \leq 1$ for all $(x, y) \in \mathcal{R}$. Then, we can apply Corollary 3.1 by taking $c \in (\delta, 1 - \delta)$ and so we prove that the map $h_\delta^2$ induces chaotic dynamics on two symbols.

Nevertheless, for the dynamics of this class of maps we can provide a sharper result regarding $h_\delta$, without considering its iterate $h_\delta^2$. In fact, it is straightforward to prove that the map $h_\delta$ also induces chaotic dynamics on two symbols, because

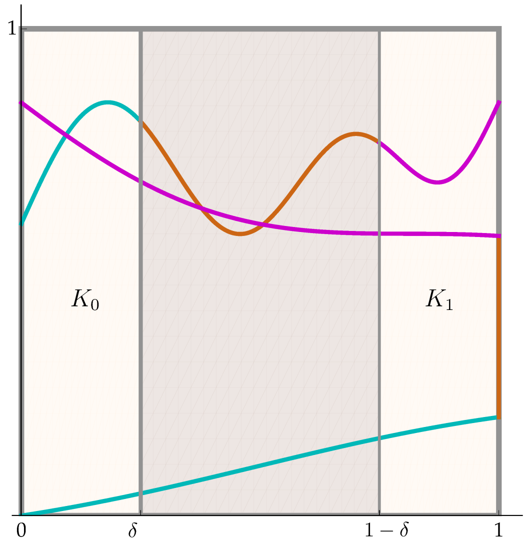


Fig. 4. Image of a generic path connecting $\mathcal{R}_l^-$ to $\mathcal{R}_r^-$. Behavior of its transformation under the action of a generic map $h_\delta$ as defined in (12). With different colors are highlighted the subpaths (relatively to sets $K_0$ and $K_1$) and their corresponding images through $h_\delta$.

Theorem 2 holds with the choice of the following two disjoint compact sets:

$$K_0 = [0, \delta] \times [0, 1], \quad K_1 = [1 - \delta, 1] \times [0, 1].$$

For completeness, a graphical proof of the previous statement is shown in Fig. 4.

Actually, this family of maps represents the reason to use in this work an argument of "stretching" instead of the one about "$h_\delta$-covering". The latter argument is a very useful and powerful technique adopted in similar situations by many authors, as in [Liz & Ruiz-Herrera, 2012, Theorem 5.1]. In this special context, a natural application of this method (see [Liz & Ruiz-Herrera, 2012, Theorem 3.4]) is difficult because it is not possible to prove the existence of two disjoint compact sets of the plane (homeomorphic to rectangles) such that they $h_\delta$-cover each other (in accord with [Liz & Ruiz-Herrera, 2012, Definition 3.3]).

*Remark 3.2.* With respect to the treatment of higher-dimensional maps we can take into account [Zgliczyński, 1999; Zgliczyński & Gidea, 2004; Pireddu, 2015; Ruiz-Herrera & Zanolin, 2015]. Since the tool behind the proof of Theorem 3 works also in an $N$-dimensional setting with one expansive direction (see [Medio *et al.*, 2009]), these papers motivate us to deal with three-dimensional difference equations, instead of two-dimensional difference equations as the one in (4).

Let $\phi = (\phi_1, \phi_2, \phi_3) : [0,1]^3 \to [0,1]^3$ be a continuous function such that the first component $\phi_1 : [0,1] \to [0,1]$ satisfies $\phi_1(0) = 0 = \phi_1(1)$ and there exists at least a point $c \in (0,1)$ such that $\phi_1(c) = 1$. Take $C_0$ and $C_1$ as in Corollary 3.1, then $\phi^2$ induces chaotic dynamics on two symbols relatively to $C_0 \times [0,1]$ and $C_1 \times [0,1]$.

We conclude the analysis of the twisted horseshoe map by studying the existence of periodic points. The reason is that the Kloeden's theory involves not only the existence of a scrambled set but also this concept. Unluckily, Theorem 3 implies the existence of periodic points with period any natural number only for the second iteration $f^2$ of the twisted horseshoe map. Therefore, an immediate question is what happens for $f$. In that case, the analysis is more involved because the SAP method cannot be applied directly.

Let

$$R_0 := \left[0, \frac{1}{2}\right] \times I, \quad R_1 := \left[\frac{1}{2}, 1\right] \times I$$

be subrectangles of $\mathcal{R}$. With a slightly different proof from that of Theorem 3 it is simple to verify that both $R_0$ and $R_1$ satisfy the condition (9), namely they stretch $\tilde{\mathcal{R}}$ along the paths. However $R_0$ and $R_1$ are not disjoint because $R_0 \cap R_1 = \{1/2\} \times I$ and so the hypotheses of Theorem 2 are not all satisfied.

Although the map $h$ is not semi-conjugate to the Bernoulli shift map on two symbols, anyhow it is possible to recover some features of the SAP's approach. More in detail, we observe that no sequence of points $p_n = (x_n, y_n) \in \mathbb{R}^2$ with $n = 0, 1, \ldots,$ starting from $x_0 = 1/2$ and $y_0 \in \mathbb{R}$, is periodic. Indeed, if $p_0 = (1/2, y_0)$ then $p_1 = (1, 1/2 + y_0/10)$, $p_2 = (0, 16/20 + y_0/100)$, $p_3 = (0, 66/200 + y_0/1000)$, and so on. In this way, we observe that $x_n \neq 1/2$ for all $n > 0$. Thus the point $(1/2, y)$, with $y \in \mathbb{R}$, is never a periodic point for $f$. Since on the segment $\{1/2\} \times I$ there is no periodic point and both $(R_0, f) : \tilde{\mathcal{R}} \Longleftrightarrow\!\!\!\!\!\!\rightarrow \tilde{\mathcal{R}}$ and $(R_1, f) : \tilde{\mathcal{R}} \Longleftrightarrow\!\!\!\!\!\!\rightarrow \tilde{\mathcal{R}}$ hold, then also the twisted horseshoe map $f$ admits periodic points with period any natural number.


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