Optimizing Linear Algebra Performance for Programmers

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1 Multiplying Many Matrices and One Vector

Consider the following matrix operation:

$$Result = \underbrace{\mathbf{A} * \mathbf{B} * \mathbf{C} * \mathbf{D} * \mathbf{E}}_{\text{100 by 100}} * \underbrace{\vec{x}}_{\text{100 by 1}} \tag{1}$$

If this operation was done programmatically, with a compiler that evaluated each operation left-to-right, there would be over 4 million multiplication operations:

$$\underbrace{100 * 100 * 100 * \mathbf{4}}_{\substack{\text{4 instances of multiply-}\\\text{ing a 100x100 matrix}\\\text{by a 100x100 matrix}}} + \underbrace{100 * 100}_{\substack{\text{1 instance of}\\\text{multiplying a}\\\text{100x100 ma-}\\\text{trix by a 100x1}\\\text{vector}}} = 4,010,000 \tag{2}$$

However, if we instead first evaluate $\mathbf{E} * \vec{x}$, then evaluate $\mathbf{D} * (\mathbf{E} * \vec{x})$, etc. there will be just over 1 million multiplication operations:

$$\underbrace{100 * 100 * \mathbf{4}}_{\substack{\text{4 instances of multiply-}\\\text{ing a 100x100 matrix}\\\text{by a 100x1 vector}}} + \underbrace{100 * 100 * 100}_{\substack{\text{1 instance of}\\\text{multiplying a}\\\text{100x100 matrix}\\\text{by a 100x100}\\\text{matrix}}} = 1,040,000 \tag{3}$$

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I used Eigen and C++ to practically test this, timing how long it took to do the approach from Equation 2 vs. the approach from Equation 3 with various matrix sizes.

Matrix Dimension	Slow Method Time (μs)	Fast Method Time (μs)
10	162	52
50	13,291	738
100	32,515	2561
300	537,397	17,399
1000	19,730,814	173,000

Table 1: Timing the dumb way vs. the smart way

For each dimension, the 'smart' method is orders of magnitude faster, becoming better as the dimension increases.

2 Calculating the Inverse of a Matrix

Consider the following matrix operation:

$$Result = \vec{x}^T * \mathbf{A}^T * \mathbf{B}^{-1} * \mathbf{A} * \vec{x} \tag{4}$$

One approach is to simply calculate the inverse of $\mathbf{B}$ then multiply it with the rest of the terms. However, calculating the inverse of a matrix is a very expensive operation. Instead, consider rearranging the equation like so:

$$\mathbf{B}b = \mathbf{A} * \vec{x}$$
$$Result = \vec{x}^T * \mathbf{A}^T * b \tag{5}$$

It is a much cheaper operation to solve a $\mathbf{A}\vec{x} = \mathbf{B}$ matrix problem than it is to find an inverse of a matrix. I used Eigen and C++ to practically test this, timing how long it takes to do the approach from Equation 4 vs. the approach from Equation 5.

Table 3 is an example of Pop-Art.

The parameter $\boxed{filename}$ should contain the name of the file.

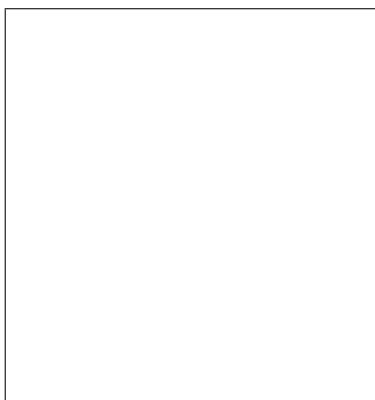| Matrix Dimension	Slow Method Time (μs)	Fast Method Time (μs)
10	162	52
50	3824	691
100	32,515	2561
300	537,397	17,399
1000	20,126,148	169,915

Table 2: Using *inverse*() vs. using *solve*()

Table 3: Five by Five in Centimetres.