Exercise Sheet 1

You can download my lecture and exercise sheets at the address

http://sami.hust.edu.vn/giang-vien/?name=huynt

1) Let A, B be sets. What does the statement "A is not a subset of B " mean?

2) Let A, B, C, X be sets with A, B, and C are subsets of X. Prove the following set equalities

a) $(A \cap B)' = A' \cup B'$

b) $(A \cup B)' = A' \cap B'$

where the complements are taken in X.

c) $A \setminus B = A \cap B'$

d) $A\cap(B \setminus C) = (A\cap B)\setminus(A\cap C)$

3) Let $A = \{x \in \mathbf{R} \mid x^2-5x+4 \leq 0\}$; $B = \{x \in \mathbf{R} \mid |x^2- 1/2| \leq 1\}$ and $C = \{x \in \mathbf{R} \mid x^2-7x+12 < 0\}$. Determine $(A \cup B) \cap C$.

4) Let $f: X \to Y$ be a mapping; Let $A, B \subset X$; $C, D \subset Y$. Prove that :

a) $f(A \cap B) \subset f(A) \cap f(B)$; Find examples of A, B, and f such that $f(A \cap B) \neq f(A) \cap f(B)$

b) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

5) Let $f : \mathbf{R} \setminus \{0\} \to \mathbf{R}$, $x \mapsto 1/x$ and $g : \mathbf{R} \to \mathbf{R}$, $x \mapsto 2x/(1+x^2)$

be mappings.

a) Determine $f \circ g$ and $g \circ f$.

b) Find the image $g(\mathbf{R})$. Is g injective? surjective? (Answer the same question for f.)

6) Let $f : A \to C$ and $g : B \to D$ be two mappings. Consider the mapping $h : A\text{x}B \to C\text{x}D$ defined by $h(a, b) = (f(a), g(b))$ for all $(a, b) \in A\text{x}B$.

a) Prove that, f and g are both injective if and only if h is injective.

b) Prove that, f and g are both surjective if and only if h is surjective.

Exercise Sheet 2

1) Suppose G is the set of all bijective functions from **Z** to **Z** with multiplication defined by composition, i.e., $f \cdot g = f \circ g$. Prove that, $(G, \circ)$ is a group but not an abelian group.

2) Suppose G is the set of all real functions defined on the interval [0,1] (i.e., all functions of the form $f : [0, 1] \to \mathbf{R}$). Define an addition on G by $(f+g)(t) = f(t) + g(t)$ for all $t \in [0, 1]$ and all f and $g \in G$. Show that (G, +) is an abelian group.

3) Which set of the following sets is a ring? a field?

a) $2\mathbf{Z} = \{2m \mid m \in \mathbf{Z}\}$; b) $2\mathbf{Z}+1=\{2m+1 \mid m \in \mathbf{Z}\}$

c) $X = \{a+ b\sqrt{2} \mid a, b \in \mathbf{Z}\}$; d) $Y = \{a+ b\sqrt{2} \mid a, b \in \mathbf{Q}\}$

where, the addition and multiplication are the common addition and multiplication.

4) Solve the following exercises on complex numbers:

1. (Multiplication by i) Show that multiplication of a complex number by i corresponds to a **counterclockwise rotation** of the corresponding vector through the angle $\pi/2$.

Find

2. $|-0.2i|$ **3.** $|1.5 + 2i|$ **4.** $|z|^4, |z^4|$ **5.** $|\cos\theta + i\sin\theta|$

6. $\left|\frac{\bar{z}}{z}\right|$ **7.** $\left|\frac{5+7i}{7-5i}\right|$ **8.** $\left|\frac{z+1}{z-1}\right|$ **9.** $\left|\frac{(1+i)^6}{i^3(1+4i)^2}\right|$

Represent in polar form:

10. $2i, -2i$ **11.** $1 + i$ **12.** -3 **13.** $6 + 8i$

14. $\frac{1+i}{1-i}$ **15.** $\frac{i\sqrt{2}}{4+4i}$ **16.** $\frac{3\sqrt{2}+2i}{-\sqrt{2}-2i/3}$ **17.** $\frac{2+3i}{5+4i}$

Find all values of the following roots and plot them in the complex plane.

26. $\sqrt{i}$ **27.** $\sqrt{-8i}$ **28.** $\sqrt{-7-24i}$ **29.** $\sqrt[8]{1}$

30. $\sqrt[4]{-7+24i}$ **31.** $\sqrt[4]{-1}$ **32.** $\sqrt[5]{-1}$ **33.** $\sqrt[3]{1+i}$

Solve the equations:

34. $z^2 + z + 1 - i = 0$ **35.** $z^2 - (5+i)z + 8 + i = 0$

36. $z^4 - 3(1+2i)z^2 - 8 + 6i = 0$

37. Prove the following useful inequalities, which we shall need occasionally:

$$|\text{Re } z| \leqq |z|, \qquad |\text{Im } z| \leqq |z|. \tag{19}$$

38. Verify the triangle inequality for $z_1 = 4 + 5i$, $z_2 = -2 + 1.5i$.

39. Prove the triangle inequality.

40. (Parallelogram equality) Show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$. This is called the *parallelogram equality*. Can you see why?

Exercise Sheet 3

Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & -2 \\ 4 & 5 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 3 & 0 & 2 \\ 4 & 0 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 6 & 1 & -5 \\ 5 & -2 & 13 \end{bmatrix}$.

Find the following expressions or give reasons why they are undefined.

1. $\mathbf{A} + \mathbf{B}$, $\mathbf{B} + \mathbf{A}$
2. $4\mathbf{A}$, $-3\mathbf{C}$, $3\mathbf{A} - 3\mathbf{B}$, $3(\mathbf{A} - \mathbf{B})$
3. $2\mathbf{C} + 2\mathbf{D}$, $2(\mathbf{C} + \mathbf{D})$
4. $\mathbf{A} + \mathbf{B} + \mathbf{C}$, $\mathbf{C} - \mathbf{D}$
5. $\mathbf{A} - \mathbf{C}$, $\mathbf{A} + 0\mathbf{C}$, $\mathbf{C} + 0\mathbf{A}$
6. $\mathbf{A} + \mathbf{A}^\mathsf{T}$, $(\mathbf{A} + \mathbf{B})^\mathsf{T}$, $\mathbf{A}^\mathsf{T} + \mathbf{B}^\mathsf{T}$, $(\mathbf{A}^\mathsf{T})^\mathsf{T}$
7. $4\mathbf{B} + 8\mathbf{B}^\mathsf{T}$, $4(\mathbf{B} + 2\mathbf{B}^\mathsf{T})$
8. $(2\mathbf{C})^\mathsf{T}$, $2\mathbf{C}^\mathsf{T}$, $\mathbf{C} + \mathbf{C}^\mathsf{T}$, $\mathbf{C}^\mathsf{T} - 2\mathbf{D}^\mathsf{T}$

Let $\mathbf{K} = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 5 \end{bmatrix}$, $\mathbf{L} = \begin{bmatrix} 0 & 2 & -8 \\ -2 & 0 & 6 \\ 8 & -6 & 0 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 9 \\ 0 \\ 5 \end{bmatrix}$.

Find the following expressions or give reasons why they are undefined.

9. $\mathbf{K} + \mathbf{L}$, $\mathbf{K} - \mathbf{L}$
10. $3(\mathbf{a} - 4\mathbf{b})$, $3\mathbf{a} - 12\mathbf{b}$, $\mathbf{K} + \mathbf{a}$, $\mathbf{a} + \mathbf{a}^\mathsf{T}$
11. $\mathbf{K} - \mathbf{K}^\mathsf{T}$, $\mathbf{L} + \mathbf{L}^\mathsf{T}$, $\mathbf{a}^\mathsf{T} + \mathbf{b}^\mathsf{T}$
12. $3\mathbf{K} + 4\mathbf{L}$, $6\mathbf{K} + 8\mathbf{L}$
13. $\mathbf{K} + \mathbf{K}^\mathsf{T} + \mathbf{L} - \mathbf{L}^\mathsf{T}$
14. $6\mathbf{a}^\mathsf{T} - 9\mathbf{b}^\mathsf{T}$, $3(2\mathbf{a} - 3\mathbf{b})^\mathsf{T}$, $3(3\mathbf{b}^\mathsf{T} - 2\mathbf{a}^\mathsf{T})$

Symmetric and skew-symmetric matrices

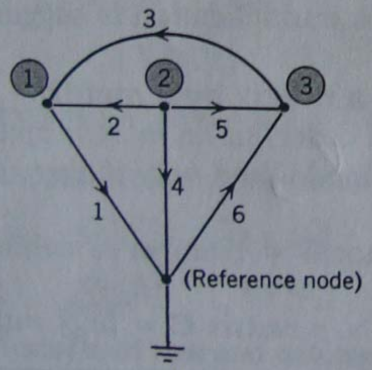
15. Show that $\mathbf{K}$ is symmetric and $\mathbf{L}$ is skew-symmetric.
16. Show that for a symmetric matrix $\mathbf{A} = [a_{jk}]$ we have $a_{jk} = a_{kj}$.
17. Show that if $\mathbf{A} = [a_{jk}]$ is skew-symmetric, then $a_{jk} = -a_{kj}$, in particular, $a_{jj} = 0$.
18. Write $\mathbf{A}$ (in Probs. 1–8) as the sum of a symmetric and a skew-symmetric matrix.
19. Write $\mathbf{B}$ as the sum of a symmetric and a skew-symmetric matrix.
20. Show that if $\mathbf{A}$ is any square matrix, then $\mathbf{S} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^\mathsf{T})$ is symmetric, $\mathbf{T} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^\mathsf{T})$ is skew-symmetric, and $\mathbf{A} = \mathbf{S} + \mathbf{T}$.
21. Prove (3) and (4) in Sec. 7.2 as well as $(\mathbf{A}^\mathsf{T})^\mathsf{T} = \mathbf{A}$.

Use of matrices in modeling networks. Matrices have various engineering applications, as we shall see. For instance, they may be used to characterize connections (in electrical networks, in nets of roads connecting cities, in production processes, etc.), as follows.

22. **(Nodal incidence matrix)** Figure 131 shows an electrical network having 6 *branches* (connections) and 4 *nodes* (points where two or more branches come together). One node is the *reference node* (grounded node, whose voltage is zero). We number the other nodes and number and direct the branches. This we do arbitrarily. The network can now be described by a matrix $\mathbf{A} = [a_{jk}]$, where

$$a_{jk} = \begin{cases} +1 \text{ if branch } k \text{ leaves node } (j) \\ -1 \text{ if branch } k \text{ enters node } (j) \\ 0 \text{ if branch } k \text{ does not touch node } (j) \,. \end{cases}$$

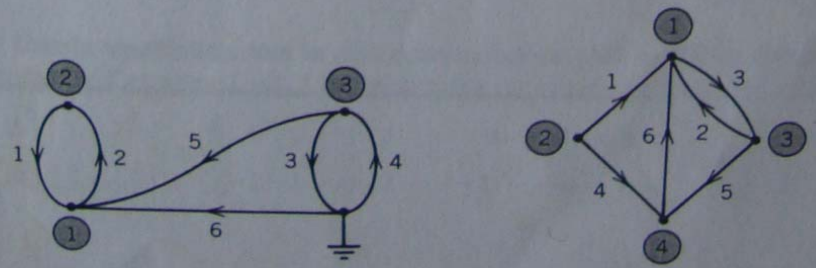
A is called the *nodal incidence matrix* of the network. Show that for the network in Fig. 131, **A** has the given form.



$$\begin{array}{c} \text{Branch} \\ \text{Node } (1) \\ \text{Node } (2) \\ \text{Node } (3) \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Fig. 131. Network and nodal incidence matrix in Prob. 22

23. Find the nodal incidence matrix of the electrical network in Fig. 132A.



(A) Problem 23 (B) Problem 24

Fig. 132. Electrical network and net of one-way streets

Exercise Sheet 4

Let $\mathbf{a} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \\ 2 & 3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 3 & 4 \end{bmatrix}$, $\mathbf{d} = [1 \quad 0 \quad 2]$.

Find those of the following expressions that are defined.

1. $\mathbf{CB}$, $\mathbf{B}^\mathsf{T}\mathbf{C}^\mathsf{T}$, $\mathbf{BC}^\mathsf{T}$
2. $\mathbf{C}^2$, $\mathbf{C}^3$, $\mathbf{CC}^\mathsf{T}$, $\mathbf{C}^\mathsf{T}\mathbf{C}$
3. $\mathbf{Ca}$, $\mathbf{Cd}^\mathsf{T}$, $\mathbf{C}^\mathsf{T}\mathbf{d}^\mathsf{T}$
4. $\mathbf{B}^\mathsf{T}\mathbf{a}$, $\mathbf{Bd}$, $\mathbf{dB}$, $\mathbf{ad}$
5. $\mathbf{B}^\mathsf{T}\mathbf{C}$, $\mathbf{B}^\mathsf{T}\mathbf{B}$
6. $\mathbf{BB}^\mathsf{T}$, $\mathbf{BB}^\mathsf{T}\mathbf{C}$, $\mathbf{BB}^\mathsf{T}\mathbf{a}$
7. $\mathbf{a}^\mathsf{T}\mathbf{a}$, $\mathbf{a}^\mathsf{T}\mathbf{Ca}$, $\mathbf{dCd}^\mathsf{T}$
8. $\mathbf{dd}^\mathsf{T}$, $\mathbf{d}^\mathsf{T}\mathbf{d}$, $\mathbf{adB}$, $\mathbf{adBB}^\mathsf{T}$

9. Prove (5).
10. Find real 2×2 matrices (as many as you can) whose square is $\mathbf{I}$, the unit matrix.
11. Find a 2×2 matrix $\mathbf{A} \neq \mathbf{0}$ such that $\mathbf{A}^2 = \mathbf{0}$.
12. Find two 2×2 matrices $\mathbf{A}$, $\mathbf{B}$ such that $(\mathbf{A} + \mathbf{B})^2 \neq \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.
13. **(Idempotent matrix)** A matrix $\mathbf{A}$ is said to be *idempotent* if $\mathbf{A}^2 = \mathbf{A}$. Give examples of idempotent matrices, different from the zero or unit matrix.
14. Show that $\mathbf{AA}^\mathsf{T}$ is symmetric.
15. Find all real square matrices that are both symmetric and skew-symmetric.
16. Show that the product of symmetric matrices $\mathbf{A}$, $\mathbf{B}$ is symmetric if and only if $\mathbf{A}$ and $\mathbf{B}$ commute, $\mathbf{AB} = \mathbf{BA}$.

Special linear transformations were used in the text to motivate matrix multiplication, and we add some problems of practical interest. (Linear transformations in general follow in Sec. 7.15.)

17. **(Rotation)** Show that the linear transformation $\mathbf{y} = \mathbf{Ax}$ with matrix

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

is a counterclockwise rotation of the Cartesian x_1x_2-coordinate system in the plane about the origin, where θ is the angle of rotation.

18. Show that in Prob. 17,

$$\mathbf{A}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

What does this result mean geometrically?

19. **(Computer graphics)** To visualize a three-dimensional object with plane faces (e.g., a cube), we may store the position vectors of the vertices with respect to a suitable $x_1x_2x_3$-coordinate system (and a list of the connecting edges) and then obtain a two-dimensional image on a video screen by projecting the object onto a coordinate plane, for instance, onto the x_1x_2-plane by setting $x_3 = 0$. To change the appearance of the image, we can impose a linear transformation on the position vectors stored. Show that a diagonal matrix $\mathbf{D}$ with main diagonal entries 3, 1, $\frac{1}{2}$ gives from an $\mathbf{x} = [x_j]$ the new position vector $\mathbf{y} = \mathbf{Dx}$, where $y_1 = 3x_1$ (stretch in the x_1-direction by a factor 3), $y_2 = x_2$ (unchanged), $y_3 = \frac{1}{2}x_3$ (contraction in the x_3-direction). What effect would a scalar matrix have?

20. (Rotations in space in computer graphics) What effect would the following matrices have in the situation described in Prob. 19?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix}, \quad \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise Sheet 5

Solve the following systems of linear equations:

4.
$$\begin{aligned} x + 2y - 8z &= 0 \\ 2x - 3y + 5z &= 0 \\ 3x + 2y - 12z &= 0 \end{aligned}$$

5.
$$\begin{aligned} 3x - y + z &= -2 \\ x + 5y + 2z &= 6 \\ 2x + 3y + z &= 0 \end{aligned}$$

6.
$$\begin{aligned} 7x - y - 2z &= 0 \\ 9x - y - 3z &= 0 \\ 2x + 4y - 7z &= 0 \end{aligned}$$

7.
$$\begin{aligned} x + y + z &= -1 \\ 4y + 6z &= 6 \\ y + z &= 1 \end{aligned}$$

8.
$$\begin{aligned} 5x + 3y &= 22 \\ -4x + 7y &= 20 \\ 9x - 2y &= 15 \end{aligned}$$

9.
$$\begin{aligned} 4y + 3z &= 13 \\ x - 2y + z &= 3 \\ 3x + 5y &= 11 \end{aligned}$$

10.
$$\begin{aligned} 7x - 4y - 2z &= -6 \\ 16x + 2y + z &= 3 \end{aligned}$$

11.
$$\begin{aligned} x - 3y + 2z &= 2 \\ 5x - 15y + 7z &= 10 \end{aligned}$$

12.
$$\begin{aligned} 3x - 3y - 7z &= -4 \\ x - y + 2z &= 3 \end{aligned}$$

13.
$$\begin{aligned} 3w - 6x - y - z &= 0 \\ w - 2x + 5y - 3z &= 0 \\ 2w - 4x + 3y - z &= 3 \end{aligned}$$

14.
$$\begin{aligned} 4w + 3x - 9y + z &= 1 \\ -w + 2x - 13y + 3z &= 3 \\ 3w - x + 8y - 2z &= -2 \end{aligned}$$

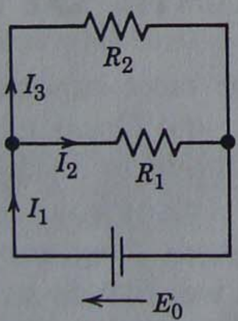
15.
$$\begin{aligned} w + x + y &= 6 \\ -3w - 17x + y + 2z &= 2 \\ 4w - 17x + 8y - 5z &= 2 \\ -5x - 2y + z &= 2 \end{aligned}$$

16.
$$\begin{aligned} w - x + 3y - 3z &= 3 \\ -5w + 2x - 5y + 4z &= -5 \\ -3w - 4x + 7y - 2z &= 7 \\ 2w + 3x + y - 11z &= 1 \end{aligned}$$

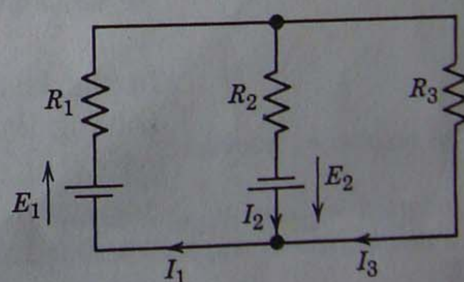
Models of electrical networks

Using Kirchhoff's laws (see Example 2), find the currents in the following networks.

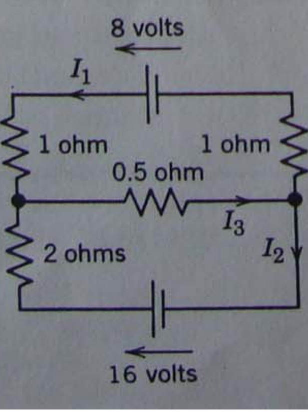
17.



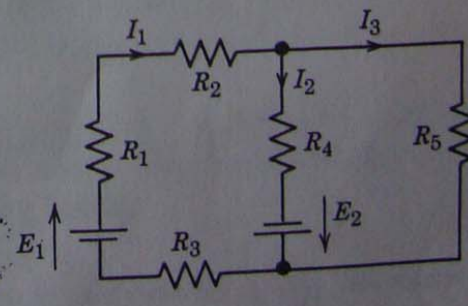
18.



19.



20.



24. (Equivalence relation) By definition, an *equivalence relation* on a set is a relation satisfying three conditions:

(1) Each element A of the set is equivalent to itself.

(2) If A is equivalent to B, then B is equivalent to A.

(3) If A is equivalent to B and B is equivalent to C, then A is equivalent to C.

For instance, equality is an equivalence relation on the set of real numbers. Show that row equivalence satisfies these three conditions.

Exercise Sheet 6

1) Show that for any scalar k and any vectors u and v of a vector space V we have
k(u - v) = ku-kv.

2)
Let V be the set of all functions from a nonempty set X into a field K. For any functions $f, g \in V$ and any scalar $k \in K$, let $f + g$ and kf be the functions in V defined as follows:

$$(f+g)(x) = f(x) + g(x) \qquad \text{and} \qquad (kf)(x) = kf(x) \qquad \forall x \in X$$

Prove that V is a vector space over K.

3) Show that W is a subspace of R^3 where W = {(a, b, c) | a + b + c = 0}.

4) Express v =(1 , —2, 5) in R^3 as a linear combination of the vectors u_1, u_2, u_3 where
u_1 = (1, -3, 2), u_2 = (2, -4, - 1), u_3 = (1, - 5, 7).

5)
Express the polynomial $v = t^2 + 4t - 3$ over **R** as a linear combination of the polynomials $p_1 = t^2 - 2t + 5$, $p_2 = 2t^2 - 3t$, $p_3 = t + 3$.

6)
Find a condition on a, b, c so that $w = (a, b, c)$ is a linear combination of $u = (1, -3, 2)$ and $v = (2, -1, 1)$, that is, so that w belongs to span (u, v).

7)
Determine whether or not the following vectors in $\mathbf{R}^3$ are linearly dependent:

$$u = (1, -2, 1), \; v = (2, 1, -1), \; w = (7, -4, 1).$$

8)
Consider the vector space $\mathbf{P}(t)$ of polynomials over **R**. Determine whether the polynomials u, v, and w are linearly dependent where $u = t^3 + 4t^2 - 2t + 3$, $v = t^3 + 6t^2 - t + 4$, $w = 3t^3 + 8t^2 - 8t + 7$.

9)
Let V be the vector space of functions from **R** into **R**. Show that $f, g, h \in V$ are linearly independent, where $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = t$.

10)
Let V be the vector space of 2×2 matrices over **R**. Determine whether the matrices $A, B, C \in V$ are linearly dependent, where:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

11)
Suppose u, v, and w are linearly independent vectors. Show that $u + v$, $u - v$, and $u - 2v + w$ are also linearly independent.

12)
Show that the vectors $v = (1 + i, 2i)$ and $w = (1, 1 + i)$ in $\mathbf{C}^2$ are linearly dependent over the complex field **C** but are linearly independent over the real field **R**.

13) Determine whether (1, 1, 1), (1, 2, 3), and (2, — 1, 1) form a basis for the vector space R^3.

14)
Consider the vector space $\mathbf{P}_n(t)$ of polynomials in t of degree $\leq n$. Determine whether or not $1 + t, t + t^2, t^2 + t^3, \ldots, t^{n-1} + t^n$ form a basis of $\mathbf{P}_n(t)$.

15)

Let V be the vector space of real 2×2 matrices. Determine whether

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis for V.

16)

Let V be the vector space of 2×2 symmetric matrices over K. Show that $\dim V = 3$. [Recall that $A = (a_{ij})$ is symmetric iff $A = A^T$ or, equivalently, $a_{ij} = a_{ji}$.]

17) Find a basis and the dimension of the subspace W of R^3 where:
(a) W={(a,b,c) | a + b + c = 0}, (b) W = {{a, b, c) | a = b = c},
(c) W = (a, b, c) | c = 3a}

18)

Find a basis and the dimension of the subspace W of $\mathbf{R}^4$ spanned by

$$u_1 = (1, -4, -2, 1), \quad u_2 = (1, -3, -1, 2). \quad u_3 = (3, -8, -2, 7).$$

19)

Let W be the subspace of $\mathbf{R}^4$ spanned by the vectors

$$u_1 = (1, -2, 5, -3), \quad u_2 = (2, 3, 1, -4) \quad u_3 = (3, 8, -3, -5).$$

(*a*) Find a basis and the dimension of W. (*b*) Extend the basis of W to a basis of the whole space $\mathbf{R}^4$.

20)

Let V be the vector space of real 2×2 matrices. Find the dimension and a basis of the subspace W of V spanned by

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 5 & 12 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 4 \\ -2 & 5 \end{pmatrix}$$

21)

Determine whether the following matrices have the same row space:

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{pmatrix}$$

22)

Show that $A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 4 & 3 \\ 1 & 1 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \\ 7 & 12 & 17 \end{pmatrix}$ have the same column space.

23)

Consider the subspace $U = \text{span}(u_1, u_2, u_3)$ and $W = \text{span}(w_1, w_2, w_3)$ of $\mathbf{R}^3$ where:

$$u_1 = (1, 1, -1), \qquad u_2 = (2, 3, -1), \qquad u_3 = (3, 1, -5)$$
$$w_1 = (1, -1, -3), \qquad w_2 = (3, -2, -8), \qquad w_3 = (2, 1, -3)$$

Show that $U = W$.

24)

Find the rank of the matrix A where:

$$(a) \quad A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{pmatrix}, \qquad (b) \quad A = \begin{pmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{pmatrix}$$

25)

Suppose R is a row vector and A and B are matrices such that RB and AB are defined. Prove:

(*a*) RB is a linear combination of the rows of B.

(*b*) Row space of AB is contained in the row space of B.

(*c*) Column space of AB is contained in the column space of A.

(*d*) rank $AB \leq$ rank B and rank $AB \leq$ rank A.

26)

Find the dimension and a basis of the solution space W of the system

$$\begin{aligned} x + 2y + z - 3t &= 0 \\ 2x + 4y + 4z - t &= 0. \\ 3x + 6y + 7z + t &= 0 \end{aligned}$$

27) Let U and W be subspaces of a vector space V. We define the sum

$$U + W = \{u + w \mid u \in U;\ w \in W\}.$$

Show that:

(a) U and W are each contained in U + W;

(b) U + W is a subspace of V, and it is the smallest subspace of V containing U and W, that is, if there is any subspace L of V such that L contains U and W then L contains U + W.

(c) W + W = W.

28) Let U and W be subspaces of a vector space V. We say that V is the direct sum of U and W if V = U + W and $U \cap W = \{0\}$. In this case we write $V = U \oplus W$.

a) Let U and W be the subspaces of R^3 defined by $U = \{(a, b, c) \mid a = b = c\}$ and $W = \{(0, b, c)\}$, Prove that $R^3 = U \oplus W$.

b) Let V be the vector space of n-square matrices over **R**.

Show that $V = U \oplus W$ where U and W are the subspaces of symmetric and antisymmetric matrices, respectively. (Recall M is symmetric iff $M = M^T$, and M is antisymmetric iff $M^T = -M$.)

Exercise Sheet 7

Find the inverse and check the result, or state that the inverse does not exist, giving a reason.

4. $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ **5.** $\begin{bmatrix} 6 & -2 & \frac{1}{2} \\ 1 & 5 & 2 \\ -8 & 24 & 7 \end{bmatrix}$ **6.** $\begin{bmatrix} 0 & 0 & 5 \\ 0 & -1 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 0.5 & 0 & -0.5 \\ -0.1 & 0.2 & 0.3 \\ 0.5 & 0 & -1.5 \end{bmatrix}$ **8.** $\begin{bmatrix} 7 & 9 & 11 \\ 8 & -8 & 5 \\ 4 & 60 & 29 \end{bmatrix}$ **9.** $\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & 5 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 2 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ **11.** $\begin{bmatrix} 1 & 4 & 8 \\ 0 & 5 & 2 \\ 0 & 0 & 10 \end{bmatrix}$ **12.** $\begin{bmatrix} 10 & 0 & 0 \\ 0 & 9 & 17 \\ 0 & 4 & 8 \end{bmatrix}$

13. $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ **14.** $\begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$ **15.** $\begin{bmatrix} 371 & -76 & -40 \\ 36 & -7 & -4 \\ -176 & 36 & 19 \end{bmatrix}$

16) Show that $(A^{-1})^{-1} = A$; and $(AB)^{-1} = B^{-1}A^{-1}$

17) Show that $(A^{-1})^{T} = (A^{T})^{-1}$

19. Show that the inverse of a nonsingular symmetric matrix is symmetric.

20. Show that $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$. Find $(\mathbf{A}^2)^{-1}$ for the matrix in Prob. 14.

Find ranks of the following matrices:

1. $\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}$ **2.** $\begin{bmatrix} 30 & -70 & 50 \\ -36 & 84 & -60 \end{bmatrix}$ **3.** $\begin{bmatrix} 3 & 6 & 12 & 10 \\ 2 & 4 & 8 & 7 \end{bmatrix}$

4. $\begin{bmatrix} 0.4 & 2.0 \\ 3.2 & 1.6 \\ 0 & 1.1 \end{bmatrix}$ **5.** $\begin{bmatrix} 0 & 5 & 2 \\ 3 & 0 & -1 \\ 7 & 9 & 0 \end{bmatrix}$ **6.** $\begin{bmatrix} 21 & -3 & 17 & 13 \\ 46 & 11 & 52 & 14 \\ 33 & 48 & 71 & -23 \end{bmatrix}$

Using cofactor matrices, find the inverse of the matrices:

7. $\begin{bmatrix} 9 & 5 \\ 25 & 14 \end{bmatrix}$ **8.** $\begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}$ **9.** $\begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix}$

10. $\begin{bmatrix} -3 & 1 & -1 \\ 15 & -6 & 5 \\ -5 & 2 & -2 \end{bmatrix}$ **11.** $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}$ **12.** $\begin{bmatrix} 2 & 5 & 4 \\ 0 & 1 & 8 \\ 0 & 0 & 10 \end{bmatrix}$

13. $\begin{bmatrix} 0 & -0.4 & 0.2 \\ 0.1 & 0.1 & -0.1 \\ -0.2 & 0.4 & 0 \end{bmatrix}$ **14.** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **15.** $\begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

Solve by Cramer's rule and by the Gauss elimination:

16. $\begin{aligned} 4x - y &= 3 \\ -2x + 5y &= 21 \end{aligned}$

17. $\begin{aligned} -x + 3y - 2z &= 7 \\ 3x + 3z &= -3 \\ 2x + y + 2z &= -1 \end{aligned}$

18. $\begin{aligned} 2x + 5y + 3z &= 1 \\ -x + 2y + z &= 2 \\ x + y + z &= 0 \end{aligned}$

Geometrical applications: Using Cramer's Theorem show that:

22. The **plane through three points** (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) in space is given by the formula below.

23. The **circle through three points** (x_1, y_1), (x_2, y_2), (x_3, y_3) in the plane is given by the formula below.

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Problem 22

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Problem 23

24. Find the plane through (1, 1, 1), (5, 0, 5), (3, 2, 6).

25. Find the circle through (2, 6), (6, 4), (7, 1).

26)

Let S be the basis of $\mathbf{R}^2$ consisting of $u_1 = (2, 1)$ and $u_2 = (1, -1)$. Find the coordinate vector $[v]$ of v relative to S where $v = (a, b)$.

27)

Consider the vector space $\mathbf{P}_3(t)$ of real polynomials in t of degree ≤ 3.

(*a*) Show that $S = \{1, 1 - t, (1 - t)^2, (1 - t)^3\}$ is a basis of $\mathbf{P}_3(t)$.

(*b*) Find the coordinate vector $[u]$ of $u = 2 - 3t + t^2 + 2t^3$ relative to S.

28)

Consider the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix}$ in the vector space V of 2×2 real matrices. Find the coordinate vector $[A]$ of the matrix A relative to $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$, the usual basis of V.

29)

Let S be the following basis of the vector space W of 2×2 real symmetric matrices:

$$\left\{\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}\right\}$$

Find the coordinate vector of the matrix $A \in W$ relative to the above basis S where (a) $A = \begin{pmatrix} 1 & -5 \\ -5 & 5 \end{pmatrix}$ and (b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

30)

Consider the following bases of $\mathbf{R}^2$:

$$S_1 = \{u_1 = (1, -2), u_2 = (3, -4)\} \qquad \text{and} \qquad S_2 = \{v_1 = (1, 3), v_2 = (3, 8)\}$$

(a) Find the coordinates of an arbitrary vector $v = (a, b)$ in $\mathbf{R}^2$ relative to the basis $S_1 = \{u_1, u_2\}$.

(b) Find the change-of-basis matrix P from S_1 to S_2.

(c) Find the coordinates of an arbitrary vector $v = (a, b)$ in $\mathbf{R}^2$ relative to the basis $S_2 = \{v_1, v_2\}$.

(d) Find the change-of-basis matrix Q from S_2 back to S_1.

31)

Consider the basis $S = \{u_1 = (1, 2, 0), u_2 = (1, 3, 2), u_3 = (0, 1, 3)\}$ of $\mathbf{R}^3$. Find:

(a) The change-of-basis matrix P from the usual basis $E = \{e_1, e_2, e_3\}$ of $\mathbf{R}^3$ to the basis S,

(b) The change-of-basis matrix Q from the above basis S back to the usual basis E of $\mathbf{R}^3$.

32)

Suppose P is the change-of-basis matrix from a basis $\{u_i\}$ to a basis $\{w_i\}$, and suppose Q is the change-of-basis matrix from the basis $\{w_i\}$ back to the basis $\{u_i\}$. Prove that P is invertible and $Q = P^{-1}$.

Exercise Sheet 8

Evaluate

1. $\begin{vmatrix} 17 & 9 \\ -4 & 13 \end{vmatrix}$ **2.** $\begin{vmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{vmatrix}$ **3.** $\begin{vmatrix} 4.3 & 0.7 \\ 0.8 & -9.2 \end{vmatrix}$

4. $\begin{vmatrix} 1.0 & 0.2 & 1.6 \\ 3.0 & 0.6 & 1.2 \\ 2.0 & 0.8 & 0.4 \end{vmatrix}$ **5.** $\begin{vmatrix} 5 & 1 & 8 \\ 15 & 3 & 6 \\ 10 & 4 & 2 \end{vmatrix}$ **6.** $\begin{vmatrix} 4 & 6 & 5 \\ 0 & 1 & -7 \\ 0 & 0 & 6 \end{vmatrix}$

7. $\begin{vmatrix} 16 & 22 & 4 \\ 4 & -3 & 2 \\ 12 & 25 & 2 \end{vmatrix}$ **8.** $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ **9.** $\begin{vmatrix} 1.1 & 8.7 & 3.6 \\ 0 & 9.1 & -1.7 \\ 0 & 0 & 4.5 \end{vmatrix}$

10. $\begin{vmatrix} 2 & 8 & 0 & 0 \\ 9 & -4 & 0 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 6 & -2 \end{vmatrix}$ **11.** $\begin{vmatrix} 3 & 2 & 0 & 0 \\ 6 & 8 & 0 & 0 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 2 & 5 \end{vmatrix}$ **12.** $\begin{vmatrix} -6 & 4 & 5 & 6 \\ 2 & 7 & 2 & 1 \\ -1 & 7 & 2 & 4 \\ -7 & 4 & 5 & 7 \end{vmatrix}$

Evaluate

13. $\begin{vmatrix} 4 & 3 & 9 & 9 \\ -8 & 3 & 5 & -4 \\ -8 & 0 & -2 & -8 \\ -16 & 6 & 14 & -5 \end{vmatrix}$ **14.** $\begin{vmatrix} 4 & 3 & 0 & 0 \\ -8 & 1 & 2 & 0 \\ 0 & -7 & 3 & -6 \\ 0 & 0 & 5 & -5 \end{vmatrix}$ **15.** $\begin{vmatrix} 12 & 6 & 1 & 11 \\ 4 & 4 & 1 & 4 \\ 7 & 4 & 3 & 7 \\ 8 & 2 & 3 & 9 \end{vmatrix}$

16. Show that det $(k\mathbf{A}) = k^n$ det $\mathbf{A}$ (*not* k det $\mathbf{A}$), where $\mathbf{A}$ is any $n \times n$ matrix.

17. Write the product of the determinants in Probs. 5 and 6 as a determinant.

18. Do the same task as in Prob. 17, taking the determinants in reverse order.

19. Verify that the answer to Prob. 11 equals the product of the determinants of the 2×2 submatrices containing no zero entries. Explain why.

20. Show that the straight line through two points P_1: (x_1, y_1) and P_2: (x_2, y_2) in the xy-plane is given by formula (a) (below), and derive from (a) the familiar formula (b).

(a) $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ (b) $\dfrac{x - x_1}{x_1 - x_2} = \dfrac{y - y_1}{y_1 - y_2}$

Exercise Sheet 9

1)

Show that the following mapping is linear: $F: \mathbf{R}^3 \to \mathbf{R}$ defined by $F(x, y, z) = 2x - 3y + 4z$.

2)

A mapping $F: \mathbf{R}^2 \to \mathbf{R}^3$ is defined by $F(x, y) = (x + 1, 2y, x + y)$. Is F linear?

3) Let K be a field (often, K = **R** or **C**) .

Let V be the vector space of n-square matrices over K. Let M be an arbitrary but fixed matrix in V. Let $T: V \to V$ be defined by $T(A) = AM + MA$, where $A \in V$. Show that T is linear.

4) Let $\mathbf{F} : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear mapping for which $\mathbf{F}(1, 2) = (2, 3)$ and $\mathbf{F}(0, 1) = (1, 4)$. Find a formula for $\mathbf{F}$, that is, find $\mathbf{F}(\mathbf{a}, \mathbf{b})$ for arbitrary $\mathbf{a}$ and $\mathbf{b}$.

5)

Let $T: V \to U$ be linear, and suppose $v_1, \ldots, v_n \in V$ have the property that their images $T(v_1), \ldots, T(v_n)$ are linearly independent. Show that the vectors $v_1, \ldots, v_n$ are also linearly independent.

6)

Let $F: \mathbf{R}^5 \to \mathbf{R}^3$ be the linear mapping defined by

$$F(x, y, z, s, t) = (x + 2y + z - 3s + 4t,\ 2x + 5y + 4z - 5s + 5t,\ x + 4y + 5z - s - 2t)$$

Find a basis and the dimension of the image of F.

7)

Let $G: \mathbf{R}^3 \to \mathbf{R}^3$ be the linear mapping defined by $G(x, y, z) = (x + 2y - z,\ y + z,\ x + y - 2z)$. Find a basis and the dimension of the kernel of G.

8) Consider the linear mapping $\mathbf{A}: \mathbf{R}^3 \to \mathbf{R}^4$ which has the matrix representation corr. to the usual bases of $\mathbf{R}^3$ and $\mathbf{R}^4$ as

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{pmatrix}$$

Find a basis and the dimension of (a) the image of A; (b) the kernel of A.

9)

Find a linear map $F: \mathbf{R}^3 \to \mathbf{R}^4$ whose image is spanned by $(1, 2, 0, -4)$ and $(2, 0, -1, -3)$.

10)

Let V be the vector space of 2 by 2 matrices over $\mathbf{R}$ and let $M = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$. Let $F: V \to V$ be the linear map defined by $F(A) = AM - MA$. Find a basis and the dimension of the kernel W of F.

11)

Suppose $F: V \to U$ and $G: U \to W$ are linear. Prove:

(a) $\operatorname{rank}(G \circ F) \le \operatorname{rank} G$ (b) $\operatorname{rank}(G \circ F) \le \operatorname{rank} F$

12)

Consider the linear operator T on $\mathbf{R}^3$ defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. (a) Show that T is invertible. Find formulas for: (b) T^{-1}, (c) T^2, and (d) T^{-2}.

13)

Let $F: \mathbf{R}^3 \to \mathbf{R}^2$ be the linear mapping defined by $F(x, y, z) = (3x + 2y - 4z, x - 5y + 3z)$.

(*a*) Find the matrix of F in the following bases of $\mathbf{R}^3$ and $\mathbf{R}^2$:

$$S = \{w_1 = (1, 1, 1), w_2 = (1, 1, 0), w_3 = (1, 0, 0)\} \qquad S' = \{u_1 = (1, 3), u_2 = (2, 5)\}$$

(*b*) Verifv that the action of F is preserved by its matrix representation; that is, for any $v \in \mathbf{R}^3$, $[F]_{S'}^{S}[v]_S = [F(v)]_{S'}$

14)

Let G be the linear operator on $\mathbf{R}^3$ defined by $G(x, y, z) = (2y + z, x - 4y, 3x)$.

(*a*) Find the matrix representation of G relative to the basis

$$S = \{w_1 = (1, 1, 1), w_2 = (1, 1, 0), w_3 = (1, 0, 0)\}$$

15)

Each of the sets (*a*) $\{1, t, e^t, te^t\}$ and (*b*) $\{e^{3t}, te^{3t}, t^2e^{3t}\}$ is a basis of a vector space V of functions $f: \mathbf{R} \to \mathbf{R}$. Let $\mathbf{D}$ be the differential operator on V, that is, $\mathbf{D}(f) = df/dt$. Find the matrix of $\mathbf{D}$ in each given basis.

16)

Find the matrix representation of each of the following linear mappings relative to the usual bases of $\mathbf{R}^n$:

$F: \mathbf{R}^2 \to \mathbf{R}^3$ defined by $F(x, y) = (3x - y, 2x + 4y, 5x - 6y)$

$F: \mathbf{R}^4 \to \mathbf{R}^2$ defined by $F(x, y, s, t) = (3x - 4y + 2s - 5t, 5x + 7y - s - 2t)$

$F: \mathbf{R}^3 \to \mathbf{R}^4$ defined by $F(x, y, z) = (2x + 3y - 8z, x + y + z, 4x - 5z, 6y)$

Exercise Sheet 10

Find the eigenvalues and eigenvectors of the following matrices.

1. $\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ **3.** $\begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}$ **4.** $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **7.** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **8.** $\begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$

9. $\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ **10.** $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ **11.** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **12.** $\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$

13. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ **14.** $\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix}$ **15.** $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$ **17.** $\begin{bmatrix} 6 & 10 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 2 \end{bmatrix}$ **18.** $\begin{bmatrix} 32 & -24 & -8 \\ 16 & -11 & -4 \\ 72 & -57 & -18 \end{bmatrix}$

19. $\begin{bmatrix} 8 & 0 & 3 \\ 2 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ **20.** $\begin{bmatrix} 5 & 0 & -15 \\ -3 & -4 & 9 \\ 5 & 0 & -15 \end{bmatrix}$ **21.** $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$

Some general properties of the spectrum: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be all eigenvalues of an n-square matrix $\mathbf{A}=[a_{jk}]$. In each case, prove the proposition and illustrate with an example:

22. **(Trace)** The so-called *trace* of $\mathbf{A}$, given by trace $\mathbf{A} = a_{11} + a_{22} + \cdots + a_{nn}$, is equal to $\lambda_1 + \cdots + \lambda_n$. The constant term of $D(\lambda)$ equals det $\mathbf{A}$.
23. If $\mathbf{A}$ is real, the eigenvalues are real or complex conjugates in pairs.
24. **(Inverse)** The inverse $\mathbf{A}^{-1}$ exists if and only if $\lambda_j \neq 0$ $(j = 1, \cdots, n)$.
25. The inverse $\mathbf{A}^{-1}$ has the eigenvalues $1/\lambda_1, \cdots, 1/\lambda_n$.
26. **(Triangular matrix)** If $\mathbf{A}$ is triangular, the entries on the main diagonal are the eigenvalues of $\mathbf{A}$.
27. **("Spectral shift")** The matrix $\mathbf{A} - k\mathbf{I}$ has the eigenvalues $\lambda_1 - k, \cdots, \lambda_n - k$.
28. The matrix $k\mathbf{A}$ has the eigenvalues $k\lambda_1, \cdots, k\lambda_n$.
29. The matrix $\mathbf{A}^m$ (m a nonnegative integer) has the eigenvalues $\lambda_1^m, \cdots, \lambda_n^m$.

30.

(Spectral mapping theorem) The matrix

$$k_m\mathbf{A}^m + k_{m-1}\mathbf{A}^{m-1} + \cdots + k_1\mathbf{A} + k_0\mathbf{I},$$

which is called a **polynomial matrix,** has the eigenvalues

$$k_m\lambda_j^m + k_{m-1}\lambda_j^{m-1} + \cdots + k_1\lambda_j + k_0 \quad (j = 1, \cdots, n).$$

(This proposition is called the *spectral mapping theorem for polynomial matrices*.) The eigenvectors of that matrix are the same as those of $\mathbf{A}$.

31) Find all eigenvalues and a maximal set S of linearly independent eigenvectors for the following matrices:

$$(a)\quad A = \begin{pmatrix} 5 & 6 \\ 3 & -2 \end{pmatrix} \qquad (b)\quad C = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$$

Which of the matrices can be diagonalized? If so, diagonalize them!

32) Suppose

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$$

Find: (a) the characteristic polynomial of A, (b) the eigenvalues of A, and (c) a maximal set of linearly independent eigenvectors of A. (d) Is A diagonalizable? If yes, find P such that $P^{-1}AP$ is diagonal.

33) Answer the same questions as in Prob. **19** for the matrices:

$$B = \begin{pmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{pmatrix} \quad ; C = \begin{pmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix}.$$

34) Suppose A and B are n-square matrices.
(a) Show that 0 is an eigenvalue of A if and only if A is singular.
(b) Show that AB and BA have the same eigenvalues.
(c) Suppose A is nonsingular (invertible) and *c* is an eigenvalue of A. Show that c^{-1} is an eigenvalue of A^{-1}.
(d) Show that A and its transpose A^T have the same characteristic polynomial.

35) Suppose $A=[a_{jk}]$ is an n-square matrix. Define the trace of A by $\mathrm{Tr}(A)=\sum_{k=1}^{n} a_{kk}$ (the sum of all entries on the main diagonal. Prove that Tr(AB) = Tr(BA) for n-square matrices A and B. Prove also that similar matrices have the same traces.

36) Find the principal directions and corresponding factors of extension or contraction of the elastic deformation $\mathbf{y} = A\mathbf{x}$ where

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

What is the new shape the elastic membrane takes?, given the old shape is a circle.

37)

Let V be the vector space of 2×2 matrices with the usual basis

$$\left\{E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

Let $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and T be the linear operator on V defined by $T(A) = MA$. Find the matrix representation of T relative to the above usual basis of V.
Also, find the matrix of the following linear operator H with respect to the above usual basis of V: $H(A)=MA-AM$

38)

Let G be the linear operator on $\mathbf{R}^3$ defined by $G(x, y, z) = (2y + z, x - 4y, 3x)$ and consider the usual basis E of $\mathbf{R}^3$ and the following basis S of $\mathbf{R}^3$:

$$S = \{w_1 = (1, 1, 1), w_2 = (1, 1, 0), w_3 = (1, 0, 0)\}$$

(*a*) **Find the change-of-basis matrix P from E to S, the change-of-basis matrix Q from S back to E, and verify that $Q = P^{-1}$.**

(*b*) **Verify that $[G]_S = P^{-1}[G]_E P$.**

39)

Find all eigenvalues and a basis of each eigenspace of the operator $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by T(x, y, z) = (2x + 4y + 4z, 4x+2y+ 4z, 4x+ 4y + 2z). Is **T** diagonalizable? If so, find a basis of R^3 such that the matrix representation of **T** with respect to which is diagonal.

40) Suppose v is an eigenvector of an operator T corresponding to the eigenvalue k. Show that for $n > 0$, v is also an eigenvector of T^n corresponding to k^n.

41)

Suppose λ is an eigenvalue of an operator T and $f(t)$ is a polynomial. Show that $f(\lambda)$ is an eigenvalue of $f(T)$.

Exercise Sheet 11

1) Verify that the following is an inner product on $\mathbf{R}^2$ where $u = (x_1, x_2)$ and $v = (y_1, y_2)$:

$$\langle u, v\rangle = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$$

2)

Find the values of k so that the following is an inner product on $\mathbf{R}^2$ where $u = (x_1, x_2)$ and $v = (y_1, y_2)$:

$$\langle u, v\rangle = x_1y_1 - 3x_1y_2 - 3x_2y_1 + kx_2y_2$$

3) Let V be the vector space of mxn matrices over **R**. Show that $\langle A, B\rangle = \operatorname{tr}(B^TA)$ defines an inner product in V.

4) Let V be the vector space of polynomials over **R**. Show that

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$$

defines an inner product in V.

5) Suppose $|\langle u, v\rangle| = \| u\| \| v \|$. (That is, the Cauchy-Schwarz inequality reduces to equality) Show that u and v are linearly dependent.

6) Let V be the vector space of polynomials over **R** of degree $\leq$ **2** with inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt.$$

Find a basis of the subspace W orthogonal to $h(t) = 2t + 1$.

7) Find a basis of the subspace W of $\mathbf{R}^4$ orthogonal to $u = (1, -2, 3, 4)$ and $v = (3, -5, 7, 8)$.

8) Let $w = (1, -2, -1, 3)$ be a vector in $\mathbf{R}^4$**.** Find (a) an orthogonal and (b) an orthonormal basis for $w^\perp$.

9) Let W be the subspace of R^4 orthogonal to $u = (1, 1, 2, 2)$ and $v = (0, 1, 2, -1)$. Find (a) an orthogonal and (b) an orthonormal basis for W.

10) Let S consist of the following vectors in $\mathbf{R}^4$:

$u_1=(1,1,1,1)$, $u_2 = (1, 1, -1, -1)$, $u_3 = (1, -1, 1, -1)$, $u_4 = (1, -1, -1, 1)$

(a) Show that S is orthogonal and a basis of $\mathbf{R}^4$.
(b) Write $v = (1, 3, -5, 6)$ as a linear combination of u_1, u_2, u_3, u_4
(c) Find the coordinates of an arbitrary vector $v = (a, b, c, d)$ in $\mathbf{R}^4$ relative to the basis S.
(d) Normalize S to obtain an orthonormal basis of $\mathbf{R}^4$.

11) Find an orthogonal and an orthonormal basis for the subspace U of $\mathbf{R}^4$ spanned by the vectors $u = (1, 1, 1, 1)$, $v = (1, -1, 2, 2)$, $w=(1, 2, -3, -4)$.

12)

Let V be the vector space of polynomials $f(t)$ with inner product $\langle f, g\rangle = \int_0^2 f(t)g(t)\,dt$. Apply the Gram–Schmidt algorithm to the set $\{1, t, t^2\}$ to obtain an orthogonal set $\{f_0, f_1, f_2\}$ with integer coefficients.

13) Suppose $v = (1, 2, 3,4,6)$. Find the orthogonal projection of v onto W (or find $w \in W$ which minimizes $\| v - w \|$) where W is the subspace of $\mathbf{R}^5$ spanned by:

(a) $u_1 = (1, 2, 1, 2, 1)$ **and** $u_2 = (1, -1, 2, -1, 1)$; **(b)** $v_1 = (1, 2, 1, 2, 1)$ **and** $v_2 = (1, 0, 1, 5, -1)$

14) Orthogonally diagonalize the following symmetric matrix:

$$A=\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

15) Consider the quadratic form $q(x, y, z) = 3x^2 + 2xy + 3y^2 + 2xz + 2yz + 3z^2$. Find:
(a) The symmetric matrix A which represents q and its characteristic polynomial,
(b) The eigenvalues of A,
(c) A maximal set S of nonzero orthogonal eigenvectors of A.
(d) An orthogonal change of coordinates which diagonalizes q.

16)

On $\mathbb{R}^3$ consider the following quadratic form $q = 8x^2 + 8xy - 2xz - 7y^2 + 8yz + 8z^2$.
Find an orthogonal change of coordinates which diagonalizes q.

17)

Let $A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & -4 & 1 \\ 6 & 8 & -3 \end{pmatrix}$; $B = \begin{pmatrix} -2 \\ 7 \\ 5 \end{pmatrix}$. Find a column vector $\tilde{X} \in M_{3\times 1}(\mathbb{R})$ which minimizes the function $f(X) = ||AX - B||$ defined for all $X \in M_{3\times 1}(\mathbb{R})$.

18) Let A, B be n-square symmetric matrices on **R**. Prove that:
a) All the eigenvalues of A are positive if and only if $X^T AX > 0$ for all $X \in M_{nx1}(\mathbf{R}) \setminus \{0\}$.
b) If all the eigenvalues of A and B are positive, then so are the eigenvalues of A + B.