# Chapter 1

# VECTORS AND THE GEOMETRY OF SPACE

**Reference:** James Stewart. *Calculus*, sixth edition. Thomson, USA 2008.

## 1.1 Three-dimensional coordinate systems

**1.** Find the lengths of the sides of the triangle $PQR$. Is it a right triangle? Is it an isosceles triangle?

a) $P(3;-2;-3)$, $Q(7;0;1)$, $R(1;2;1)$.

b) $P(2;-1;0)$, $Q(4;1;1)$, $R(4;-5;4)$.

**2.** Find an equation of the sphere with center $(1;-4;3)$ and radius 5. Describe its intersection with each of the coordinate planes.

**3.** Find an equation of the sphere that passes through the origin and whose center is $(1;2;3)$.

**4.** Find an equation of a sphere if one of its diameters has end points $(2;1;4)$ and $(4;3;10)$.

**5.** Find an equation of the largest sphere with center $(5,4,9)$ that is contained in the first octant.

**6.** Write inequalities to describe the following regions

a) The region consisting of all points between (but not on) the spheres of radius $r$ and $R$ centered at the origin, where $r < R$.

b) The solid upper hemisphere of the sphere of radius 2 centered at the origin.

**7.** Consider the points $P$ such that the distance from $P$ to $A(-1;5;3)$ is twice the distance from $P$ to $B(6;2;-2)$. Show that the set of all such points is a sphere, and find its center and radius.

**8.** Find an equation of the set of all points equidistant from the points $A(-1;5;3)$ and $B(6;2;-2)$. Describe the set.

## 1.2 Vectors

**9.** Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2;4)$.

**10.** Find the unit vectors that are parallel to the tangent line to the curve $y = 2\sin x$ at the point $(\pi/6;1)$.

**11.** Find the unit vectors that are perpendicular to the tangent line to the curve $y = 2\sin x$ at the point $(\pi/6;1)$.

**12.** Let $C$ be the point on the line segment $AB$ that is twice as far from $B$ as it is from $A$. If $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$, show that $\mathbf{c} = \mathbf{\frac{2}{3}a} + \mathbf{\frac{1}{3}b}$.

## 1.3 The dot product

**13.** Determine whether the given vectors are orthogonal, parallel, or neither

a) $a = (-5;3;7), \quad b = (6;-8;2)$

b) $a = (4;6), \quad b = (-3;2)$

c) $a = -i + 2j + 5k, \quad b = 3i + 4j - k$

d) $u = (a,b,c), \quad v = (-b;a;0)$

**14.** For what values of $b$ are the vectors $(-6;b;2)$ and $(b;b^2;b)$ orthogonal?

**15.** Find two unit vectors that make an angle of $60^o$ with $v = (3;4)$.

**16.** If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$, find the third direction angle $\gamma$.

**17.** Find the angle between a diagonal of a cube and one of its edges.

**18.** Find the angle between a diagonal of a cube and a diagonal of one of its faces.

## 1.4 The cross product

**19.** Find the area of the parallelogram with vertices $A(-2;1)$, $B(0;4)$, $C(4;2)$, and $D(2;-1)$.

**20.** Find the area of the parallelogram with vertices $K(1;2;3)$, $L(1;3;6)$, $M(3;8;6)$ and $N(3;7;3)$.

**21.** Find the volume of the parallelepiped determined by the vectors $a$, $b$, and $c$.

a) $a = (6;3;-1), \quad b = (0;1;2), \quad c = (4;-2;5).$

b) $a = i + j - k, \quad b = i - j + k, \quad c = -i + j + k.$

**22.** Let $v = 5j$ and let $u$ be a vector with length 3 that starts at the origin and rotates in the $xy$-plane. Find the maximum and minimum values of the length of the vector $u \times v$. In what direction does $u \times v$ point?

## 1.5 Equations of lines and planes

**23.** Determine whether each statement is true or false.

a) Two lines parallel to a third line are parallel.

b) Two lines perpendicular to a third line are parallel.

c) Two planes parallel to a third plane are parallel.

d) Two planes perpendicular to a third plane are parallel.

e) Two lines parallel to a plane are parallel.

f) Two lines perpendicular to a plane are parallel.

g) Two planes parallel to a line are parallel.

h) Two planes perpendicular to a line are parallel.

i) Two planes either intersect or are parallel.

j) Two lines either intersect or are parallel.

k) A plane and a line either intersect or are parallel.

**24.** Find a vector equation and parametric equations for the line.

a) The line through the point $(6;-5;2)$ and parallel to the vector $(1;3;-2/3)$.

b) The line through the point $(0;14;-10)$ and parallel to the line $x = -1+2t; y = 6-3t; z = 3+9t$.

c) The line through the point $(1,0,6)$ and perpendicular to the plane $x + 3y + z = 5$.

**25.** Find parametric equations and symmetric equations for the line of intersection of the plane $x+y+z=1$ and $x+z=0$.

**26.** Find a vector equation for the line segment from $(2;-1;4)$ to $(4;6;1)$.

**27.** Determine whether the lines $L_1$ and $L_2$ are parallel, skew, or intersecting. If they intersect, find the point of intersection.

a) $L_1 : x = -6t, y = 1+9t, z = -3t; \quad L_2 : x = 1+2s, y = 4-3s, z = s.$

b) $L_1 : \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}; \quad L_2 : \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}.$

**28.** Find an equation of the plane.

a) The plane through the point $(6;3;2)$ and perpendicular to the vector $(-2;1;5)$

b) The plane through the point $(-2;8;10)$ and perpendicular to the line $x = 1+t, y = 2t, z = 4-3t$.

c) The plane that contains the line $x = 3+2t, y = t, z = 8-t$ and is parallel to the plane $2x+4y+8z = 17$.

**29.** Find the cosine of the angle between the planes $x+y+z = 0$ and $x+2y+3z = 1$.

**30.** Find parametric equations for the line through the point $(0;1;2)$ that is perpendicular to the line $x = 1+t, y = 1-t, z = 2t$, and intersects this line.

**31.** Find the distance between the skew lines with parametric equations $x = 1+t, y = 1+6t, z = 2t$ and $x = 1+2s, y = 5+15s, z = -2+6s$.

## 1.6 Quadric surfaces

**32.** Find an equation for the surface obtained by rotating the parabola $y = x^2$ about the $y$-axis.

**33.** Find an equation for the surface consisting of all points that are equidistant from the point $(-1;0;0)$ and the plane $x = 1$. Identify the surface.

# Chapter 2

# VECTOR FUNCTIONS

**Reference:** James Stewart. *Calculus*, sixth edition. Thomson, USA 2008.

## 2.1 Vector functions

**34.** Find the domain of the vector function.

a) $r(t) = (\sqrt{4-t^2}, e^{-3t}, \ln(t+1))$

b) $r(t) = \frac{t-2}{t+2}i + \sin t j + \ln(9-t^2)k$

**35.** Find the limit

a) $\lim\limits_{t\to 0}(\frac{e^t-1}{t}, \frac{\sqrt{1+t}-1}{t}, \frac{3}{t+1})$

b) $\lim\limits_{t\to\infty}(\arctan t, e^{-2t}, \frac{\ln t}{t+1})$

**36.** Find a vector function that represents the curve of intersection of the two surfaces.

a) The cylinder $x^2+y^2=4$ and the surface $z=xy$.

b) The paraboloid $z=4x^2+y^2$ and the parabolic cylinder $y=x^2$.

**37.** Suppose $u$ and $v$ are vector functions that possess limits as $t\to a$ and let $c$ be a constant. Prove the following properties of limits.

a) $\lim\limits_{t\to a}[u(t)+v(t)] = \lim\limits_{t\to a} u(t) + \lim\limits_{t\to a} v(t)$

b) $\lim\limits_{t\to a} cu(t) = c\lim\limits_{t\to a} u(t)$

c) $\lim\limits_{t\to a}[u(t).v(t)] = \lim\limits_{t\to a} u(t). \lim\limits_{t\to a} v(t)$

d) $\lim\limits_{t\to a}[u(t)\times v(t)] = \lim\limits_{t\to a} u(t) \times \lim\limits_{t\to a} v(t)$

**38.** Find the derivative of the vector function.

a) $r(t) = (t\sin t, t^3, t\cos 2t)$.

b) $r(t) = \arcsin t i + \sqrt{1-t^2} j + k$

c) $r(t) = e^{t^2} i - \sin^2 t j + \ln(1+3t)$

**39.** Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point. Illustrate by graphing both the curve and the tangent line on a common screen.

a) $x = t, y = e^{-t}, z = 2t - t^2; (0; 1; 0)$

b) $x = 2\cos t, y = 2\sin t, z = 4\cos 2t; (\sqrt{3}, 1, 2)$

c) $x = t\cos t, y = t, z = t\sin t; (-\pi, \pi, 0)$

**40.** Find the point of intersection of the tangent lines to the curve $r(t) = (\sin \pi t, 2\sin \pi t, \cos \pi t)$ at the points where $t = 0$ and $t = 0.5$

**41.** Evaluate the integral

a) $\int_0^{\pi/2} (3\sin^2 t\cos t\, i + 3\sin t\cos^2 t\, j + 2\sin t\cos t\, k)dt$

b) $\int_1^2 (t^2\, i + t\sqrt{t-1}\, j + t\sin \pi t\, k)dt$

c) $\int (e^t\, i + 2t\, j + \ln t\, k)dt$

d) $\int (\cos \pi t\, i + \sin \pi t\, j + t^2\, k)dt$

**42.** If a curve has the property that the position vector $r(t)$ is always perpendicular to the tangent vector $r'(t)$, show that the curve lies on a sphere with center the origin.

## 2.2 Arc length and curvature

**43.** Find the length of the curve.

a) $r(t) = (2\sin t, 5t, 2\cos t), \quad -10 \le t \le 10$

b) $r(t) = (2t, t^2, \frac{1}{3}t^3), \quad 0 \le t \le 1$

c) $r(t) = \cos t\, i + \sin t\, j + \ln\cos t\, k, \quad 0 \le t \le \pi/4$

**44.** Let $C$ be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of $C$ from the origin to the point $(6; 18; 36)$.

**45.** Suppose you start at the point $(0; 0; 3)$ and move 5 units along the curve $x = 3\sin t, y = 4t, z = 3\cos t$ in the positive direction. Where are you now?

**46.** Reparametrize the curve

$$r(t) = \left(\frac{2}{t^2+1} - 1\right) i + \frac{2t}{t^2+1} j$$

with respect to arc length measured from the point $(1; 0)$ in the direction of increasing . Express the reparametrization in its simplest form. What can you conclude about the curve?

**47.** Find the curvature

a) $r(t) = t^2\, i + t\, k$

b) $r(t) = t\, i + t\, j + (1 + t^2)\, k$

c) $r(t) = 3t\, i + 4\sin t\, j + 4\cos t\, k$

d) $x = e^t \cos t, y = e^t \sin t$

e) $x = t^3 + 1, y = t^2 + 1$

**48.** Find the curvature of $r(r) = (e^t \cos t, e^t \sin t, t)$ at the point (1, 0, 0).

**49.** Find the curvature of $r(r) = (t, t^2, t^3)$ at the point (1, 1, 1).

**50.** Find the curvature

a) $y = 2x - x^2$, b) $y = \cos x$, c) $y = 4x^{5/2}$.

**51.** At what point does the curve have maximum curvature? What happens to the curvature as $x \to \infty$?

a) $y = \ln x$, b) $y = e^x$.

**52.** Find an equation of a parabola that has curvature 4 at the origin.

# Chapter 3

# DOUBLE INTEGRALS

**Reference:** James Stewart. *Calculus*, sixth edition. Thomson, USA 2008.

## 3.1 Double integrals

**53.** Calculate the iterated integral

a) $\int_1^3 \int_0^1 (1+4xy)dxdy$ b) $\int_0^2 \int_0^1 (2x+y)^8 dxdy$ c) $\int_0^1 \int_1^2 \frac{xe^x}{y} dydx$

d) $\int_1^4 \int_1^2 \left(\frac{x}{y}+\frac{y}{x}\right) dxdy$ e) $\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}dxdy$ f) $\int_0^2 \int_0^\pi r\sin^2\varphi d\varphi dr.$

**54.** Calculate the double integral

a) $\iint_D \frac{1+x^2}{1+y^2} dxdy, \quad D=\{(x,y)|0\le x\le 1, 0\le y\le 1\}$

b) $\iint_D \frac{x}{1+xy} dxdy, \quad D=\{(x,y)|0\le x\le 1, 0\le y\le 1\}$

c) $\iint_D \frac{x}{x^2+y^2} dxdy, \quad D=[1,2]\times[0,1]$

d) $\iint_D xye^{x^2y} dxdy, \quad D=[0,1]\times[0,2]$

**55.** Find the volume of the solid that lies under the hyperbolic paraboloid $z=4+x^2-y^2$ and above the square $D=[-1;1]\times[0;2]$

**56.** Find the volume of the solid enclosed by the surface $z=1+e^x\sin y$ and the planes $x=\pm 1, y=0, y=\pi$ and $z=0$.

**57.** Find the volume of the solid in the first octant bounded by the cylinder $z=16-x^2$ and the plane $y=5$.

**58.** Evaluate the iterated integral

a) $\int_0^4 \int_0^{\sqrt{y}} xy^2 dxdy,$ b) $\int_0^2 \int_y^{2y} xydxdy,$ c) $\int_0^1 \int_0^v \sqrt{1-v^2}dudv.$

**59.** Evaluate the double integral

a) $\iint_D \frac{y}{1+x^5} dxdy, \quad D = \{(x, y)|0 \le x \le 1, 0 \le y \le x^2\}$

b) $\iint_D y^2 e^{xy} dxdy, \quad D = \{(x, y)|0 \le y \le 4, 0 \le x \le y\}$

c) $\iint_D x\sqrt{y^2 - x^2} dxdy, \quad D = \{(x, y)|0 \le y \le 1, 0 \le x \le y\}$

d) $\iint_D (x + y)dxdy$, $D$ is bounded by $y = \sqrt{x}$ and $y = x^2$

e) $\iint_D y^3 dxdy$, $D$ is the triangle region with vertices $(0; 2)$, $(1; 1)$ and $(3; 2)$

f) $\iint_D xy^2 dxdy$, $D$ is enclosed by $x = 0$ and $x = \sqrt{1 - y^2}$

**60.** Find the volume of the given solid

a) Under the surface $z = 2x + y^2$ and above the region bounded by $x = y^2$ and $x = y^3$.

b) Enclosed by the paraboloid $z = x^2 + 3y^2$ and the planes $x = 0$, $y = 1$, $y = x$, $z = 0$

c) Enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$.

d) Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant

e) Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$.

f) The solid enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$.

**61.** Sketch the region of integration and change the order of integration.

a) $\displaystyle\int_0^4 \int_0^{\sqrt{x}} f(x, y)dydx$, b) $\displaystyle\int_0^1 \int_{4x}^4 f(x, y)dydx$, c) $\displaystyle\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x, y)dxdy$.

d) $\displaystyle\int_0^3 \int_0^{\sqrt{9-y}} f(x, y)dxdy$, e) $\displaystyle\int_1^2 \int_0^{\ln x} f(x, y)dydx$, f) $\displaystyle\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y)dydx$.

**62.** Evaluate the integral by reversing the order of integration

a) $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2} dxdy$ b) $\displaystyle\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)dxdy$ c) $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3 + 1} dydx$

d) $\displaystyle\int_0^1 \int_x^1 e^{x/y} dydx$ e) $\displaystyle\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x} dxdy$ f) $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4} dxdy$.

## 3.2 Double integrals in polar coordinates

**63.** Evaluate the given integral by changing to polar coordinates.

a) $\iint_D (x+y)dxdy$ where $D$ is the region that lies to the left of the $y$-axis, between the circles $x^2+y^2=1$, and $x^2+y^2=4$.

b) $\iint_D \cos(x^2+y^2)dxdy$ where $D$ is the region that lies above the $x$-axis within the circle $x^2+y^2=9$.

c) $\iint_D \sqrt{4-x^2-y^2}dxdy$ where $D=\{(x,y)|x^2+y^2\leq 4, x\geq 0\}$.

d) $\iint_D ye^x dxdy$ where $D$ is the region in the first quadrant enclosed by the circle $x^2+y^2=25$.

e) $\iint_D \arctan(y/x)dxdy$ where $D=\{(x,y)|1\leq x^2+y^2\leq 4, 0\leq y\leq x\}$.

f) $\iint_D xdxdy$ where $D$ is the region in the first quadrant that lies between the circles $x^2+y^2=4$ and $x^2+y^2=2x$.

**64.** Use a double integral to find the area of the region.

a) The region enclosed by the curve $r=4+3\cos\varphi$

b) The region inside the cardioid $r=1+\cos\varphi$ and outside the circle $r=3\cos\varphi$.

**65.** Use polar coordinates to find the volume of the given solid.

a) Below the paraboloid $z=18-2x^2-2y^2$ and above the $xy$-plane

b) Bounded by the paraboloid $z=1+2x^2+2y^2$ and the plane $z=7$ in the first octant.

c) Above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$

d) Bounded by the paraboloids $z=3x^2+3y^2$ and $z=4-x^2-y^2$.

**66.** Evaluate the iterated integral by converting to polar coordinates

a) $\displaystyle\int_0^a\int_{-\sqrt{a^2-y^2}}^{0} x^2y\,dxdy$, b) $\displaystyle\int_0^1\int_y^{\sqrt{2y-y^2}}(x+y)dxdy$, c) $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}dydx$.

## 3.3 Applications of double integrals

**67.** Find the mass and center of mass of the lamina that occupies the region $D$ and has the given density function $\rho$.

a) $D$ is the triangular region enclosed by the lines $x = 0$, $y = x$ and $2x + y = 6$, $\rho(x, y) = x^2$.

b) $D$ is bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$, $\rho(x, y) = y$.

c) $D$ is bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$, $\rho(x, y) = x$.

d) $D$ is bounded by the parabolas $y = x^2$, and $x = y^2$, $\rho(x, y) = x$.

**68.** A lamina occupies the region inside the circle $x^2 + y^2 = 2y$ but outside the circle $x^2 + y^2 = 1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

# Chapter 4

# TRIPLE INTEGRALS

**Reference:** James Stewart. *Calculus*, sixth edition. Thomson, USA 2008.

**69.** Evaluate the iterated integral.

a) $\int_0^1 \int_x^{2x} \int_0^y 2xyz dz dy dx$, b) $\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y dx dz dy$, c) $\int_0^1 \int_0^z \int_0^y ze^{-y^2} dx dy dz$.

d) $\int_0^{\pi/2} \int_0^y \int_0^x \cos(x+y+z) dz dx dy$, e) $\int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \sin y dy dz dx$.

**70.** Evaluate the triple integral

a) $\iiint\limits_E y dV$, where $E$ is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 4$

b) $\iiint\limits_E x^2 e^y dV$, where $E$ is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes, $z = 0$, $x = 1$, and $x = -1$.

c) $\iiint\limits_E xy dV$, where $E$ is bounded by the parabolic cylinder $y = x^2$ and $x = y^2$ and the planes, $z = 0$ and $z = x + y$.

d) $\iiint\limits_E xyz dV$, where $E$ is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

e) $\iiint\limits_E x dV$, where $E$ is the bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.

f) $\iiint\limits_E z dV$, where $E$ is the bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $y = 3x$, and $z = 0$ in the first octant.

**71.** Find the volume of the given solid

a) The solid bounded by the cylinder $y = x^2$ and the planes $z = 0$, $z = 4$, and $y = 9$.

b) The solid enclosed by the cylinder $x^2 + y^2 = 9$and the planes $y + z = 5$ and $z = 1$.

c) The solid enclosed by the paraboloid $x = y^2 + z^2$ and the plane $x = 16$.

**72.** Evaluate $\iiint\limits_E (x^3 + xy^2)dV$, where $E$ is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$.

**73.** Evaluate $\iiint\limits_E e^z dV$, where $E$ is enclosed by the paraboloid $z = 1 + x^2 + y^2$, the cylinder $x^2 + y^2 = 5$, and the xy-plane.

**74.** Evaluate $\iiint\limits_E x dV$, where $E$ is enclosed by the planes $z = 0$ and $z = x + y + 5$ and by the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

**75.** Find the volume of the solid that lies within both the cylinder $x^2+y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

**76.** Find the volume of the region $E$ bounded by the paraboloids $z = x^2+y^2$ and $z = 36 - 3x^2 - 3y^2$.

**77.** Evaluate the integral by changing to cylindrical coordinates

a) $\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz dz dx dy$.

b) $\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} \sqrt{x^2 + y^2} dz dy dx$.

**78.** A solid lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. Write a description of the solid in terms of inequalities involving spherical coordinates.

**79.** Use spherical coordinates

a) Evaluate $\iiint\limits_H (9 - x^2 - y^2)dV$, where $H$ is the solid hemisphere $x^2 + y^2 + z^2 \leq 9$, $z \geq 0$.

b) Evaluate $\iiint\limits_E z dV$, where $E$ lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ in the first octant.

c) Evaluate $\iiint\limits_E e^{\sqrt{x^2+y^2+z^2}} dV$, where $E$ is enclosed by the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

d) Evaluate $\iiint\limits_E x^2 dV$, where $E$ is bounded by the $xz$-plane and the hemispheres $y = \sqrt{9 - x^2 - z^2}$ and $y = \sqrt{16 - x^2 - z^2}$.

**80.** Evaluate the integral by changing to spherical coordinates.

a) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} xydzdydx$.

b) $\int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2z + y^2z + z^3)dzdxdy$..

**81.** Calculate $\iiint\limits_E y^2z^2 dV$, where $E$ is bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$.

**82.** Evaluate the triple integral $\iiint\limits_V ydxdydz$, where $V$ is bounded by the cone $y = \sqrt{x^2 + z^2}$ and the plane $y = h$, $(h > 0)$.

**83.** Evaluate the triple integral

$$\iiint\limits_V \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dxdydz, \quad \text{where } V : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, (a, b, c > 0).$$

**84.** Evaluate $\iiint\limits_V \sqrt{x^2 + y^2 + z^2} dxdydz$, where $V$ is defined by $x^2 + y^2 + z^2 \leq z$.

**85.** Evaluate $\iiint\limits_V \sqrt{(6x - x^2 - y^2 - z^2)^3} dxdydz$, where $V$ is the sphere defined by $x^2 + y^2 + z^2 \leq 6x$.

**86.** Evaluate $\iiint\limits_V \frac{z}{1 + x^2 + y^2} dxdydz$, where $V$ is bounded by $z = 6 - \sqrt{x^2 + y^2}$, $z = 5$.

# Chapter 5

# Line integrals

**87.** Evaluate the line integral, where $C$ is the given curve

a) $\int\limits_C x \sin y ds$, $C$ is the line segment from $(0, 3)$ to $(4, 6)$.

b) $\int\limits_C (x^2y^3 - \sqrt{x})dy$, $C$ is the arc of the curve $y = \sqrt{x}$ from $(1, 1)$ to $(4, 2)$.

c) $\int\limits_C xe^y dx$, $C$ is the arc of the curve $x = e^y$ from $(1, 0)$ to $(e, 1)$.

d) $\int\limits_C \sin x dx + \cos y dy$, $C$ consists of the top half of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$ and the line segment from $(-1, 0)$ to $(-2, 3)$.

e) $\int\limits_C xyz ds$, $C : x = 2 \sin t, y = t, z = -2 \cos t,\ 0 \le t \le \pi$.

f) $\int\limits_C xyz^2 ds$, $C$ is the line segment from $(-1, 5, 0)$ to $(1, 6, 4)$.

g) $\int\limits_C x^2y\sqrt{z} dz$, $C : x = t^3, y = t, z = t^2,\ 0 \le t \le 1$.

h) $\int\limits_C z dx + x dy + y dz$, $C : x = t^2, y = t^3, z = t^2,\ 0 \le t \le 1$.

k) $\int\limits_C (x + yz)dx + 2x dy + xyz dz$, $C$ consists of line segments from $(1, 0, 1)$ to $(2, 3, 1)$ and from $(2, 3, 1)$ to $(2, 5, 2)$.

l) $\int\limits_C x^2 dx + y^2 dy + z^2 dz$, $C$ consists of line segments from $(0, 0, 0)$ to $(1, 2, -1)$ and from $(1, 2, -1)$ to $(3, 2, 0)$.

**88.** Evaluate the following line integrals

a) $\int\limits_C (x - y)ds$, where $C$ is the circle $x^2 + y^2 = 2x$.

b) $\int\limits_C (x^2 + y^2 + z^2) ds$, where $C$ is the helix $x = a\cos t$, $y = a\sin t$, $z = bt$, $(0 \le t \le 2\pi)$.

**89.** Evaluate the line integral $\int_C F \cdot dr$, where $F(x, y, z) = xi - zj + yk$ and $C$ is given by $r(t) = 2ti + 3tj - t^2k$, $-1 \le t \le 1$.

**90.** Find the work done by the force field $F(x, y, z) = (y + z, x + z, x + y)$ on a particle that moves along the line segment from $(1; 0; 0)$ to $(3; 4; 2)$.

**91.** Evaluate the line integral by two methods: (a) directly and using Green's Theorem

a) $\oint_C (x - y)dx + (x + y)dy$, $C$ is the circle with center the origin and radius 2.

b) $\oint_C xydx + x^2dy$, $C$ is the rectangle with vertices $(0; 0)$, $(3; 0)$, $(3; 1)$, and $(0; 1)$.

c) $\oint_C ydx + xdy$, $C$ consists of the line segments from $(0; 1)$ to $(0; 0)$ and from $(0; 0)$ to $(1; 0)$ and the parabola $y = 1 - x^2$ from $(1; 0)$ to $(0; 1)$.

**92.** Use Green's Theorem to evaluate the line integral along given positively oriented curve

a) $\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y)dy$, $C$ is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

b) $\int_C xe^{-2x}dx + (x^4 + 2x^2y^2)dy$, $C$ is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

c) $\int_C (e^x + x^2y)dx + (e^y - xy^2)dy$, $C$ is the circle $x^2 + y^2 = 25$.

d) $\int_C (2x - x^3y^5)dx + x^3y^8dy$, $C$ is the ellipse $4x^2 + y^2 = 4$.

**93.** Show that the line integral is independent of path and evaluate the integral

a) $\int_C (1 - ye^{-x})dx + e^{-x}dy$, $C$ is any path from $(0, 1)$ to $(1, 2)$.

b) $\int_C 2y^{3/2}dx + 3x\sqrt{y}dy$, $C$ is any path from $(1, 1)$ to $(2, 4)$.

**Curl and Divergence**

**94.** Determine whether or not $F$ is a conservative vector field. If it is, find a function $f$ such that $F = \nabla f$.

a) $F(x,y) = (2x - 3y)i + (-3x + 4y - 8)j$

b) $F(x,y) = e^x \cos y i + e^x \sin y j$

c) $F(x,y) = (xy \cos xy + \sin xy)i + (x^2 \cos xy)j$

d) $F(x,y) = (\ln y + 2xy^3)i + (3x^2y^2 + x/y)j$

e) $F(x,y) = (ye^x + \sin y)i + (e^x + x \cos y)j$

**95.** Find a function $f$ such that $F = \nabla f$ and then evaluate $\int_C F \cdot dr$ along the given curve $C$.

a) $F(x,y) = xy^2 i + x^2 y j$, $C:\ r(t) = (t + \sin \frac{1}{2}\pi t, t + \cos \frac{1}{2}\pi t)$, $0 \le t \le 1$.

b) $F(x,y) = \dfrac{y^2}{1+x^2} i + 2y \arctan x j$, $C:\ r(t) = t^2 i + 2t j$, $0 \le t \le 1$.

c) $F(x,y) = (2xz + y^2)i + 2xyj + (x^2 + 3z^2)k$, $C:\ x = t^2, y = t+1, z = 2t-1$, $0 \le t \le 1$.

d) $F(x,y) = e^y i + xe^y j + (z+1)e^z k$, $C:\ x = t, y = t^2, z = t^3$, $0 \le t \le 1$.

# Chapter 6

# Surface Integrals

**96.** Evaluate the surface integral

a) $\iint\limits_S xy dS$, $S$ is the triangular region with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,2)$.

b) $\iint\limits_S yz dS$, $S$ is the part of the plane $x+y+z=1$ that lies in the first octant.

c) $\iint\limits_S yz dS$, $S$ is the surface with parametric equations $x=u^2$, $y=u\sin v$, $z=u\cos v$, $0\le u\le 1, 0\le v\le \pi/2$.

d) $\iint\limits_S z dS$, $S$ is the surface $x=y+2z^2$, $0\le y\le 1$, $0\le z\le 1$.

e) $\iint\limits_S y^2 dS$, $S$ is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the $xy-$plane.

**97.** Evaluate the surface integral $\iint\limits_S F\cdot dS$ for the given vector field $F$ and the oriented surface $S$. In other words, find the flux of $F$ across $S$. For closed surfaces, use the positive (outward) orientation.

a) $F(x,y,z)=xze^y i-xze^y j+zk$, $S$ is the part of the plane $x+y+z=1$ in the first octant and has downward orientation.

b) $F(x,y,z)=xi+yj+z^4k$, $S$ is the part of the cone $z=\sqrt{x^2+y^2}$ beneath the plane $z=1$ with downward orientation.

c) $F(x,y,z)=xzi+xj+yk$, $S$ is the hemisphere $x^2+y^2+z^2=25, y\ge 0$, oriented in the direction of the positive $y-$axis.

d) $F(x,y,z) = xyi + 4x^2j + yzk$, $S$ is the surface $z = xe^y$, $0 \le x \le 1, 0 \le y \le 1$, with upward orientation.

e) $F(x,y,z) = x^2i + y^2j + z^2k$, $S$ is the boundary of the solid half-cylinder $0 \le z \le \sqrt{1-y^2}$, $0 \le x \le 2$.

**98.** a) Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$, if it has constant density.
b) Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2+y^2}$, $1 \le z \le 4$, if its density function is $\rho(x,y,z) = 10 - z$.

**Stokes Theorem**

**99.** Use Stokes Theorem to evaluate $\iint\limits_S \mathrm{curl} F \cdot dS$

a) $F(x,y,z) = 2y\cos zi + e^x \sin zj + xe^y k$, $S$ is the hemisphere $x^2+y^2+z^2 = 9$, $z \ge 0$, oriented upward.

b) $F(x,y,z) = x^2z^2i + y^2z^2j + xyzk$, $S$ is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$, oriented upward.

**The Divergence Theorem**

**100.** Use the Divergence Theorem to calculate the surface integral $\iint\limits_S F \cdot dS$; that is, calculate the flux of $F$ across $S$

a) $F(x,y,z) = x^3yi - x^2y^2j - x^2yzk$, $S$ is the surface of the solid bounded by the hyperboloid $x^2 + y^2 - z^2 = 1$ and the planes $z = -2$ and $z = 2$.

b) $F(x,y,z) = (\cos z + xy^2)i + xe^{-z}j + (\sin y + x^2z)k$, $S$ is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

c) $F(x,y,z) = 4x^3zi + 4y^3zj + 3z^4k$, $S$ is the sphere with radius $R$ and center the origin.