Hanoi University of Science and Technology **Dr. Bui Xuan Dieu**

School of Applied Mathematics and Informatics **Class ICT K62**

Math4 Exercises

1 Multiple Integrals

1.1 Double Integrals

1.1.1 Double Integrals in Cartesian coordinate

Exercise 1.1. *Evaluate*

a) $\iint\limits_{D} x\sin(x+y)dxdy$, *where* $D = \left\{(x,y) \in \mathbb{R}^2 : 0 \le y \le \frac{\pi}{2}, 0 \le x \le \frac{\pi}{2}\right\}$

b) $\iint\limits_{D} x^2\left(y-x\right)dxdy$ *where* D *is the region bounded by* $y = x^2$ *and* $x = y^2$.

c) $\iint\limits_{D} |x+y|dxdy, D := \left\{(x,y) \in \mathbb{R}^2 \left| |x \le 1|, |y| \le 1\right.\right\}$

d) $\iint\limits_{D} \sqrt{|y-x^2|}dxdy, D := \left\{(x,y) \in \mathbb{R}^2 \left| |x| \le 1, 0 \le y \le 1\right.\right\}$

e) $\iint\limits_{[0,1]\times[0,1]} \frac{ydxdy}{(1+x^2+y^2)^{\frac{3}{2}}}$

f) $\iint\limits_{D} \frac{x^2}{y^2}dxdy$, *where* D *is bounded by the lines* $x = 2, y = x$ *and the hyperbola* $xy = 1$.

1.1.2 Change the order of integration

Exercise 1.2. *Change the order of integration*

a) $\int\limits_{-1}^{1} dx \int\limits_{-\sqrt{1-x^2}}^{1-x^2} f\left(x,y\right)dy$.

b) $\int\limits_{0}^{1} dy \int\limits_{2-y}^{1+\sqrt{1-y^2}} f\left(x,y\right)dx$.

c) $\int\limits_{0}^{2} dx \int\limits_{\sqrt{2x-x^2}}^{\sqrt{2x}} f\left(x,y\right)dx$.

d) $\int\limits_{0}^{\sqrt{2}} dy \int\limits_{0}^{y} f\left(x,y\right)dx + \int\limits_{\sqrt{2}}^{2} dy \int\limits_{0}^{\sqrt{4-y^2}} f\left(x,y\right)dx$.

1.1.3 Change of variables

Exercise 1.3. *Evaluate* $I = \iint\limits_{D} \left(4x^2-2y^2\right)dxdy$, *where* $D : \begin{cases} 1 \le xy \le 4 \\ x \le y \le 4x. \end{cases}$

Exercise 1.4. *Evaluate*

$$I = \iint\limits_{D} \frac{x^2\sin xy}{y}dxdy,$$

where D *is bounded by parabolas*

$$x^2 = ay, x^2 = by, y^2 = px, y^2 = qx,\ (0 < a < b, 0 < p < q).$$

Exercise 1.5. *Evaluate* $I = \iint\limits_D xydxdy$*, where* D *is bounded by the curves*

$$y = ax^3, y = bx^3, y^2 = px, y^2 = qx, \ (0 < b < a, 0 < p < q).$$

Hint: Change of variables $u = \frac{x^3}{y}, v = \frac{y^2}{x}$.

Exercise 1.6. *Prove that*

$$\int\limits_0^1 dx \int\limits_0^{1-x} e^{\frac{y}{x+y}} dy = \frac{e-1}{2}.$$

Hint: Change of variables $u = x + y, v = y$.

Exercise 1.7. *Find the area of the domain bounded by* $xy = 4, xy = 8, xy^3 = 5, xy^3 = 15$.

Hint: Change of variables $u = xy, v = xy^3$, $(S = 2\ln 3)$.

Exercise 1.8. *Find the area of the domain bounded by* $y^2 = x, y^2 = 8x, x^2 = y, x^2 = 8y$.

Hint: Change of variables $u = \frac{y^2}{x}, v = \frac{x^2}{y}$, $(S = \frac{279\pi}{2})$.

Exercise 1.9. *Hint: Change of variables* $y = x^3, y = 4x^3, x = y^3, x = 4y^3$.

Exercise 1.10. *Prove that*

$$\iint\limits_{x+y\le 1, x\ge 0, y\ge 0} \cos\left(\frac{x-y}{x+y}\right) dxdy = \frac{\sin 1}{2}.$$

Hint: Change of variables $u = x - y, v = x + y$.

Exercise 1.11. *Evaluate*

$$I = \iint\limits_D \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}}\right) dxdy,$$

where D *is bounded by the axes and the parabola* $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$.

1.1.4 Double Integrals in polar coordinate

Exercise 1.12. *Express the double integral* $I = \iint\limits_D f(x, y)\, dxdy$ *in terms of polar coordinates, where* D *is given by* $x^2 + y^2 \ge 4x, x^2 + y^2 \le 8x, y \ge x, y \le \sqrt{3}x$.

Exercise 1.13. *Evaluate* $\iint\limits_D xy^2 dxdym$ *where* D *is bounded by* $\begin{cases} x^2 + (y-1)^2 = 1 \\ x^2 + y^2 - 4y = 0. \end{cases}$

Exercise 1.14. *Evaluate*

a) $\iint\limits_D |x + y| dxdy$, *b)* $\iint\limits_D |x - y| dxdy$,

where $D : x^2 + y^2 \le 1$.

Exercise 1.15. *Evaluate* $\iint\limits_D \frac{dxdy}{(x^2+y^2)^2}$*, where* $D : \begin{cases} 4y \le x^2 + y^2 \le 8y \\ x \le y \le x\sqrt{3}. \end{cases}$

Exercise 1.16. *Evaluate* $\iint\limits_D \frac{xy}{x^2+y^2} dxdy$*, where* $D : \begin{cases} x^2 + y^2 \le 12, x^2 + y^2 \ge 2x \\ x^2 + y^2 \ge 2\sqrt{3}y, x \ge 0, y \ge 0. \end{cases}$

1.2 Applications of Double Integrals

Exercise 1.17. *Compute the area of the domain D bounded by*

a) $\begin{cases} y = 2^x, y = 2^{-x}, \\ y = 4. \end{cases}$

b) $\begin{cases} y^2 = x, y^2 = 2x \\ x^2 = y, x^2 = 2y. \end{cases}$

c) $\begin{cases} y = 0, y^2 = 4ax \\ x + y = 3a, \quad (a > 0). \end{cases}$

d) $\begin{cases} x^2 + y^2 = 2x, x^2 + y^2 = 4x \\ x = y, y = 0. \end{cases}$

e) $r = 1, r = \frac{2}{\sqrt{3}} \cos\varphi.$

f) $\left(x^2 + y^2\right)^2 = 2a^2 xy \;\; (a > 0).$

g) $x^3 + y^3 = axy \;\; (a > 0)$ *(Descartes leaf)*

h) $r = a\left(1 + \cos\varphi\right) \;\; (a > 0)$ *(Cardioids)*

Exercise 1.18. *Compute the volume of the object given by*

a) $\begin{cases} 3x + y \geq 1, y \geq 0 \\ 3x + 2y \leq 2, \\ 0 \leq z \leq 1 - x - y \end{cases}$

b) $V : \begin{cases} 0 \leq z \leq 1 - x^2 - y^2 \\ y \geq x, y \leq \sqrt{3}x \end{cases}$

c) $V : \begin{cases} x^2 + y^2 + z^2 \leq 4a^2 \\ x^2 + y^2 - 2ay \leq 0. \end{cases}$

Exercise 1.19. *Compute the volume of the object bounded by the surfaces*

a) $\begin{cases} z = 4 - x^2 - y^2 \\ 2z = 2 + x^2 + y^2 \end{cases}$

b) $\begin{cases} z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}, z = 0 \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{2x}{a} \end{cases}$

c) $\begin{cases} az = x^2 + y^2 \\ z = \sqrt{x^2 + y^2}. \end{cases}$

Exercise 1.20. *Find the area of the part of the paraboloid $x = y^2 + z^2$ that satisfies $x \leq 1$.*

1.3 Triple Integrals

1.3.1 Triple Integrals in Cartesian coordinate

Exercise 1.21. *Evaluate $\iiint\limits_V \left(x^2 + y^2\right) dxdydz$, where V is bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the cone $x^2 + y^2 - z^2 = 0$.*

1.3.2 Change of variables

Exercise 1.22. *Evaluate*

a) $\iiint\limits_V (x + y + z) dxdydz$, *where V is bounded by* $\begin{cases} x + y + z = \pm 3 \\ x + 2y - z = \pm 1 \\ x + 4y + z = \pm 2 \end{cases}.$

b) $\iiint\limits_V (3x^2 + 2y + z) dxdydz$, *where $V : |x - y| \leq 1, |y - z| \leq 1, |z + x| \leq 1$.*

c) $\iiint\limits_V dxdydz$, *where $V : |x - y| + |x + 3y| + |x + y + z| \leq 1$.*

1.3.3 Triple Integrals in Cylindrical Coordinates

Exercise 1.23. *Evaluate* $\iiint\limits_V \left(x^2+y^2\right) dxdydz$, *where* $V : \begin{cases} x^2+y^2 \le 1 \\ 1 \le z \le 2 \end{cases}$

Exercise 1.24. *Evaluate* $\iiint\limits_V z\sqrt{x^2+y^2}dxdydz$, *where:*

a) V *is bounded by:* $x^2+y^2=2x$ *and* $z=0, z=a \ \ (a>0)$.

b) V *is a half of the sphere* $x^2+y^2+z^2 \le a^2, z \ge 0 \ \ (a>0)$

Exercise 1.25. *Evaluate* $I=\iiint\limits_V \sqrt{x^2+y^2}dxdydz$ *where* V *is bounded by:* $\begin{cases} x^2+y^2=z^2 \\ z=1. \end{cases}$

Exercise 1.26. *Evaluate* $\iiint\limits_V \frac{dxdydz}{\sqrt{x^2+y^2+(z-2)^2}}$, *where* $V : \begin{cases} x^2+y^2 \le 1 \\ |z| \le 1. \end{cases}$

1.3.4 Triple Integrals in Spherical Coordinates

Exercise 1.27. *Evaluate* $\iiint\limits_V \left(x^2+y^2+z^2\right) dxdydz$, *where* $V : \begin{cases} 1 \le x^2+y^2+z^2 \le 4 \\ x^2+y^2 \le z^2. \end{cases}$

Exercise 1.28. *Evaluate* $\iiint\limits_V \sqrt{x^2+y^2+z^2}dxdydz$, *where* $V: \ x^2+y^2+z^2 \le z$.

Exercise 1.29. *Evaluate* $\iiint\limits_V z\sqrt{x^2+y^2}dxdydz$, *where* V *is a half of the ellipsoid* $\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2} \le 1, z \ge 0, (a,b>0)$.

Exercise 1.30. *Evaluate* $\iiint\limits_V \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dxdydz$, *where* $V : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1, (a,b,c>0)$.

Exercise 1.31. *Evaluate* $\iiint\limits_V \sqrt{z-x^2-y^2-z^2}dxdydz$, *where* $V: \ x^2+y^2+z^2 \le z$.

Exercise 1.32. *Evaluate* $\iiint\limits_V (4z-x^2-y^2-z^2)dxdydz$, *where* V *is the sphere* $x^2+y^2+z^2 \le 4z$.

Exercise 1.33. *Evaluate* $\iiint\limits_V xzdxdydz$, *where* V *is the domain* $x^2+y^2+z^2-2x-2y-2z \le -2$.

Exercise 1.34. *Evaluate*

$$I=\iiint\limits_V \frac{dxdydz}{(1+x+y+z)^3},$$

where V *is bounded by* $x=0, y=0, z=0$ *và* $x+y+z=1$.

Exercise 1.35. *Evaluate*

$$\iiint\limits_V zdxdydz,$$

where V *is a half of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2} \le 1, (z \ge 0).$$

Exercise 1.36. *Evaluate*

a) $I_1=\iiint\limits_B \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$, *where* B *is the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$.

b) $I_2 = \iiint\limits_C z dx dy dz$, *where* C *is the domain bounded by the cone* $z^2 = \frac{h^2}{R^2}(x^2+y^2)$ *and the plane* $z = h$.

c) $I_3 = \iiint\limits_D z^2 dx dy dz$, *where* D *is bounded by the sphere* $x^2+y^2+z^2 \leq R^2$ *and the sphere* $x^2+y^2+z^2 \leq 2Rz$.

d) $I_4 = \iiint\limits_V (x+y+z)^2 dx dy dz$, *where* V *is bounded by the paraboloid* $x^2+y^2 \leq 2az$ *and the sphere* $x^2+y^2+z^2 \leq 3a^2$.

Exercise 1.37. *Find the volume of the object bounded by the planes* $0xy, x = 0, x = a, y = 0, y = b$, *and the paraboloid elliptic*

$$z = \frac{x^2}{2p} + \frac{y^2}{2y}, \ (p > 0, q > 0).$$

Exercise 1.38. *Evaluate*

$$I = \iiint\limits_V \sqrt{x^2+y^2+z^2} dx dy dz,$$

where V *is the domain bounged by* $x^2+y^2+z^2 = z$.

Exercise 1.39. *Evaluate*

$$I = \iiint\limits_V z dx dy dz,$$

where V *is the domain bounded by the surfaces* $z = x^2+y^2$ *and* $x^2+y^2+z^2 = 6$.

Exercise 1.40. *Evaluate*

$$I = \iiint\limits_V \frac{xyz}{x^2+y^2} dx dy dz,$$

where V *is the domain bounded by the surface* $(x^2+y^2+z^2)^2 = a^2xy$ *and the plane* $z = 0$.

2 Integrals depending on a parameter

2.1 Definite Integrals depending on a parameter

Exercise 2.1. *Compute*

a) $\lim\limits_{y\to 0} \int\limits_y^{1+y} \frac{dx}{1+x^2+y^2}$.

b) $\lim\limits_{y\to 0} \int\limits_0^2 x^2 \cos xy dx$.

Exercise 2.2. *Evaluate*

a) $I(y) = \int\limits_0^1 \arctan \frac{x}{y} dx$.

b) $J(y) = \int\limits_0^1 \ln\left(x^2+y^2\right) dx$.

c) $K = \int\limits_0^1 \frac{x^b - x^a}{\ln x}$, $(0 < a < b)$.

2.2 Improper Integrals depending on a parameter

Exercise 2.3. *Show that the integral*

a) $I(y) = \int\limits_1^\infty \sin(yx) dx$ *is convergent if* $y = 0$ *and is divergent if* $y \neq 0$.

b) $I(y) = \int\limits_0^\infty \frac{\cos \alpha x}{x^2+1}$ *is uniformly convergent on* $\mathbb{R}$.

c) $I(y) = \int\limits_0^1 x^{-y} dx = \int\limits_1^\infty t^{y-2} dt$ *is convergent if* $y < 1$ *and divergent if* $y \geq 1$.

d) $I(y) = \int\limits_0^{+\infty} e^{-yx} \frac{\sin x}{x}$ *is uniformly convergent on* $[0, +\infty)$.

e) $I(y) = \int\limits_0^\infty \frac{\cos \alpha x}{x^2+1}$ *is uniformly convergent on* $\mathbb{R}$.

Exercise 2.4. *a) Evaluate* $I(y) = \int\limits_0^{+\infty} y e^{-yx} dx \quad (y > 0)$.

b) Prove that $I(y)$ *converges to* 1 *uniformly on* $[y_0, +\infty)$ *for all* $y_0 > 0$.

c) Explain why $I(y)$ *is not uniformly convergent on* $(0, +\infty)$.

Exercise 2.5. *Prove that*

a) $\int\limits_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

b) $\int\limits_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

c) $\int\limits_0^\infty \sin(x^2) dx = \int\limits_0^\infty \cos(x^2) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

d) $\int\limits_0^{+\infty} e^{-yx} \frac{\sin x}{x} = \frac{\pi}{2} - \arctan y$.

e) $\int\limits_0^\infty \frac{\sin yx}{x(1+x^2)} dx = \frac{\pi}{2}(1 - e^{-y}), \quad y \geq 0$.

f) $\int\limits_0^\infty \frac{1-\cos yx}{x^2} = \frac{\pi}{2}|y|$.

g) $\int\limits_0^\infty \frac{x \sin yx}{a^2+x^2} dx = \frac{\pi}{2} e^{-ay}, \quad a, y \geq 0$.

h) $\int\limits_0^\infty e^{-yx^2} dx = \frac{\sqrt{\pi}}{2\sqrt{y}}, \quad y > 0$.

i) $\int\limits_0^{+\infty} \left(e^{-\frac{a}{x^2}} - e^{-\frac{b}{x^2}} \right) dx = \sqrt{\pi b} - \sqrt{\pi a}, (a, b > 0)$.

j) $\int\limits_0^{+\infty} \frac{\arctan \frac{x}{a} - \arctan \frac{x}{b}}{x} dx = \frac{\pi}{2} \ln \frac{b}{a}, \quad (a, b > 0)$.

k) $\lim\limits_{y \to 0^+} \left(\int\limits_0^{+\infty} y e^{-yx} dx \right) \neq \int\limits_0^{+\infty} \left(\lim\limits_{y \to 0^+} y e^{-yx} \right) dx$ *and explain why?*

Exercise 2.6. *Evaluate* $(a, b, \alpha, \beta > 0)$*:*

a) $\int\limits_0^{+\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx$.

b) $\int\limits_0^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2} dx$.

c) $\int\limits_0^{+\infty} \frac{dx}{(x^2+y)^{n+1}}$.

d) $\int\limits_0^{+\infty} e^{-ax} \frac{\sin bx - \sin cx}{x}$.

e) $\int\limits_0^{+\infty} e^{-ax} \frac{\cos bx - \cos cx}{x}, \ (a > 0)$.

f) $\int\limits_0^{+\infty} e^{-ax} \cos yx$.

g) $\int\limits_0^{+\infty} e^{-x^2} \cos(yx) dx$.

h) $\int\limits_{-\infty}^{+\infty} \frac{\arctan(x+y)}{1+x^2} dx$.

i) $\int\limits_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} dx$, *where* $a, b > 0$.
$\int\limits_0^{+\infty} \frac{e^{-ax^3} - e^{-bx^3}}{x} dx$, *where* $a, b > 0$.

j) $\int\limits_0^\infty \frac{e^{-ax^2} - \cos bx}{x^2} dx, (a > 0)$

k) $\int\limits_0^\pi \ln(1 + y \cos x) dx$,

l) $\int\limits_0^\infty e^{-x^2} \sin ax dx$,

m) $\int\limits_0^\infty \frac{\sin xy}{x} dx, y \geq 0$,

n) $\int\limits_0^{\infty} e^{-ax^2}\cos bx dx \ (a>0)$,

o) $\int\limits_0^{\infty} x^{2n}e^{-x^2}\cos bx dx, n\in\mathbb{N}$.

p) $\int\limits_0^{\infty} \frac{\sin ax\cos bx}{x}dx$,

q) $\int\limits_0^{\infty} \frac{\sin ax\sin bx}{x}dx$.

2.3 Euler Integral

Exercise 2.7. *Evaluate*

a) $\int\limits_0^{\frac{\pi}{2}} \sin^6 x\cos^4 x dx$.

b) $\int\limits_0^{a} x^{2n}\sqrt{a^2-x^2}dx \ (a>0)$.

c) $\int\limits_0^{+\infty} x^{10}e^{-x^2}dx$.

d) $\int\limits_0^{+\infty} \frac{\sqrt{x}}{(1+x^2)^2}dx$.

e) $\int\limits_0^{+\infty} \frac{1}{1+x^3}dx$.

f) $\int\limits_0^{+\infty} \frac{x^{n+1}}{(1+x^n)}dx, \ (2<n\in\mathbb{N})$.

g) $\int\limits_0^{1} \frac{1}{\sqrt[n]{1-x^n}}dx, \ n\in\mathbb{N}^*$.

h) $\int\limits_0^{+\infty} \dfrac{x^4}{(1+x^3)^2}dx$,

3 Line Integrals

3.1 Line Integrals of scalar Fields

Exercise 3.1. *Evaluate*

a) $\int\limits_C (x-y)\,ds$, *where C is the circle* $x^2+y^2=2x$.

b) $\int\limits_C y^2 ds$, *where C is the curve* $\begin{cases} x=a\,(t-\sin t) \\ y=a\,(1-\cos t)\end{cases}, 0\le t\le 2\pi, a>0$.

c) $\int\limits_C \sqrt{x^2+y^2}ds$, *where C is the curve* $\begin{cases} x=(\cos t+t\sin t) \\ y=(\sin t-t\cos t)\end{cases}, 0\le t\le 2\pi$.

d) $\int\limits_C (x+y)ds$, *where C is the circle* $x^2+y^2=2y$.

e) $\int\limits_L xyds$, *where L is the part of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, x\ge 0, y\ge 0$.

f) $I=\int\limits_L |y|ds$, *where L is the Cardioid curve* $r=a(1+\cos\varphi)\ (a>0)$.

g) $I=\int\limits_L |y|ds$, *where L is the Lemniscate curve* $(x^2+y^2)^2=a^2(x^2-y^2)$.

3.2 Line Integrals of vector Fields

Exercise 3.2. *Evaluate* $\int\limits_{ABCA} 2\left(x^2+y^2\right)dx+x\,(4y+3)\,dy$, *where $ABCA$ is the quadrangular curve,* $A(0,0), B(1,1), C(0,2)$.

Exercise 3.3. *Evaluate* $\int\limits_{ABCDA} \frac{dx+dy}{|x|+|y|}$, *where $ABCDA$ is the triangular curve,* $A(1,0), B(0,1), C(-1,0), D(0,-1)$.

3.2.1 Green's Theorem

Exercise 3.4. *Evaluate the integral* $\int\limits_C (xy+x+y)\,dx + (xy+x-y)\,dy$*, where* C *is the positively oriented circle* $x^2+y^2=R^2$ *by*

i) computing it directly and

ii) Green's Theorem, then compare the results,

Exercise 3.5. *Evaluate the following integrals, where* C *is a half the circle* $x^2+y^2=2x$*, traced from* $O(0,0)$ *to* $A(2,0)$*.*

a) $\int\limits_C (xy+x+y)\,dx + (xy+x-y)\,dy$

b) $\int\limits_C x^2\left(y+\frac{x}{4}\right)dy - y^2\left(x+\frac{y}{4}\right)dx.$

c) $\int\limits_C (xy+e^x\sin x+x+y)\,dx - (xy-e^{-y}+x-\sin y)\,dy.$

Exercise 3.6. *Evaluate* $\oint\limits_{OABO} e^x\left[(1-\cos y)\,dx-(y-\sin y)\,dy\right]$*, where* $OABO$ *is the triangle,* $O(0,0), A(1,1), B(0,2)$*.*

3.2.2 Applications of Line Integrals

Exercise 3.7. *Find the area of the domain bounded by an arch of the cycloid* $\begin{cases} x=a(\theta-\sin\theta) \\ y=a(1-\cos\theta) \end{cases}$ *and* Ox $(a>0)$*.*

3.2.3 Independence of Path

Exercise 3.8. *Evaluate* $\int\limits_{(-2,1)}^{(3,0)} \left(x^4+4xy^3\right)dx + \left(6x^2y^2-5y^4\right)dy.$

Exercise 3.9. *Evaluate* $\int\limits_{(1,\pi)}^{(2,\pi)} \left(1-\frac{y^2}{x^2}\cos\frac{y}{x}\right)dx + \left(\sin\frac{y}{x}+\frac{y}{x}\cos\frac{y}{x}\right)dy.$

4 Surface Integrals

4.1 Surface Integrals of scalar Fields

Exercise 4.1. *Evaluate* $\iint\limits_S \left(z+2x+\frac{4y}{3}\right)dS$*, where* $S=\left\{(x,y,z)\,|\,\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1, x,y,z\geq 0\right\}.$

Exercise 4.2. *Evaluate* $\iint\limits_S \left(x^2+y^2\right)dS$*, where* $S=\left\{(x,y,z)\,|\,z=x^2+y^2, 0\leq z\leq 1\right\}.$

Exercise 4.3. *Evaluate* $\iint\limits_S x^2y^2z\,dS$*, where* S *is the part of the cone* $z=\sqrt{x^2+y^2}$ *lies below the plane* $z=1$*.*

Exercise 4.4. *Evaluate* $\iint\limits_S \frac{dS}{(2+x+y+z)^2}$*, where* S *is the boundary of the triangular pyramid* $x+y+z\leq 1, x\geq 0, y\geq 0, z\geq 0.$

4.2 Surface Integrals of vector Fields

Exercise 4.5. *Evaluate* $\iint\limits_S z\left(x^2+y^2\right)dxdy$, *where* S *is a half of the sphere* $x^2+y^2+z^2=1, z\geq 0$, *with the outward normal vector.*

Exercise 4.6. *Evaluate* $\iint\limits_S ydxdz+z^2dxdy$, *where* S *is the surface* $x^2+\frac{y^2}{4}+z^2=1, x\geq 0, y\geq 0, z\geq 0$, *and is oriented downward.*

Exercise 4.7. *Evaluate* $\iint\limits_S x^2y^2zdxdy$, *where* S *is the surface* $x^2+y^2+z^2=R^2, z\leq 0$ *and is oriented upward.*

4.2.1 The Divergence Theorem

Exercise 4.8. *Evaluate the following integrals, where* S *is the surface* $x^2+y^2+z^2=a^2$ *with outward orientation.*

a. $\iint\limits_S xdydz+ydzdx+zdxdy$

b. $\iint\limits_S x^3dydz+y^3dzdx+z^3dxdy.$

Exercise 4.9. *Evaluate* $\iint\limits_S y^2zdxdy+xzdydz+x^2ydxdz$, *where* S *is the boundary of the domain* $x\geq 0, y\geq 0, x^2+y^2\leq 1, 0\leq z\leq x^2+y^2$ *which is outward oriented.*

Exercise 4.10. *Evaluate* $\iint\limits_S xdydz+ydzdx+zdxdy$, *where* S *the boundary of the domain* $(z-1)^2\leq x^2+y^2, a\leq z\leq 1, a>0$ *which is outward oriented.*

4.2.2 Stokes' Theorem

Exercise 4.11. *Use Stokes' Theorem to evaluate* $\int\limits_C F\cdot dr=\int\limits_C Pdx+Qdy+Rdz$. *In each case* C *is oriented counterclockwise as viewed from above.*

1. $F(x,y,z)=(x+y^2)\mathbf{i}+(y+z^2)\mathbf{j}+(z+x^2)\mathbf{k}$, *C is the triangle with vertices* $(1,0,0),(0,1,0)$ *and* $(0,0,1)$.
2. $F(x,y,z)=\mathbf{i}+(x+yz)\mathbf{k}+(xy-\sqrt{z})\mathbf{k}$, *C is the boundary of the part of the plane* $3x+2y+z=1$ *in the first octant.*
3. $F(x,y,z)=yz\mathbf{i}+2xz\mathbf{j}+e^{xy}\mathbf{k}$, *C is the circle* $x^2+y^2=16, z=5$.
4. $F(x,y,z)=xy\mathbf{i}+2z\mathbf{j}+3y\mathbf{k}$, *C is the curve of intersection of the plane* $x+z=5$ *and the cylinder* $y^2+y^2=9$.

5 Vector Calculus

5.1 Scalar Fields

Exercise 5.1. *Find the directional derivative of the function* $f(x,y,z)=x^2y^3z^4$ *at the point* $M(1,1,1)$ *in the direction of the vector* $\vec{l}=(1,1,1)$.

Exercise 5.2. *Find ∇u, where $u = r^2 + \frac{1}{r} + \ln r$ and $r = \sqrt{x^2 + y^2 + z^2}$.*

Exercise 5.3. *In what direction from $O(0,0,0)$ does $f = x \sin z - y \cos z$ have the maximum rate of change.*

5.2 Vector Fields

Exercise 5.4. *Let $F = xz^2 \overrightarrow{i} + yx^2 \overrightarrow{j} + zy^2 \overrightarrow{k}$. Find the flux of F across the surface $S : x^2 + y^2 + z^2 = 1$ with the outward direction.*

Exercise 5.5. *Let $F = x(y+z)\overrightarrow{i} + y(z+x)\overrightarrow{j} + z(x+y)\overrightarrow{k}$ and L is the intersection between the quatity $x^2 + y^2 + y = 0$ and a half of the sphere $x^2 + y^2 + z^2 = 2, z \geq 0$. Prove that the circulation of F across L is equal to 0.*

Exercise 5.6. *Prove that F is a conservative vector field on Ω if and only if $\operatorname{curl} F(M) = 0 \; \forall M \in \Omega$.*

Exercise 5.7. *Which of the following fields are conservative and find their potential functions.*

a. $F = 5(x^2 - 4xy)\overrightarrow{i} + (3x^2 - 2y)\overrightarrow{j} + \overrightarrow{k}$.

b. $G = yz\overrightarrow{i} + xz\overrightarrow{j} + xy\overrightarrow{k}$.

c. $H = (x+y)\overrightarrow{i} + (x+z)\overrightarrow{j} + (z+y)\overrightarrow{k}$.

6 Series

6.1 Infinite series

Exercise 6.1. *Show that the harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent.*

Exercise 6.2. *Find the sum of the series $\sum\limits_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.*

Exercise 6.3. *Test for convergence or divergence of the series*

a) $\sum\limits_{n=1}^{\infty} \sin \frac{n+\sin n}{3n+1}$ *b)* $\sum\limits_{n=1}^{\infty} \cos \frac{1}{n}$ *c)* $\sum\limits_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$.

6.1.1 The Integral Test

Exercise 6.4. *Show that the series $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is convergent iff $p > 1$.*

Exercise 6.5. *Test for convergence or divergence of the series*

a) $\sum\limits_{n=1}^{\infty} \frac{\ln \frac{1}{n}}{(n+2)^2}$ *d)* $\sum\limits_{n=1}^{\infty} \frac{\ln(1+n)}{(n+3)^2}$ *g)* $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n^p}$ *j)* $\sum\limits_{n=2}^{\infty} \frac{1}{\ln(2n-1)}$

b) $\sum\limits_{n=1}^{\infty} n^2 e^{-n^3}$ *e)* $\sum\limits_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ *h)* $\sum\limits_{n=1}^{\infty} \frac{\ln n}{3n^2}$ *k)* $\sum\limits_{n=1}^{\infty} \frac{\cos n}{\sqrt{n^3+1}}$

c) $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n^3}$ *f)* $\sum\limits_{n=1}^{\infty} \frac{n^2}{e^n}$ *i)* $\sum\limits_{n=1}^{\infty} \frac{1}{\ln(2n+1)}$ *l)* $\sum\limits_{n=1}^{\infty} \frac{\sin n}{\sqrt{n^3+1}}$.

6.1.2 The Comparison Test

Exercise 6.6. *Test for convergence or divergence of the series*

1) $\sum_{n=1}^{\infty} \frac{n^3}{(n+2)^4}$

2) $\sum_{n=1}^{\infty} \frac{2016^n}{2015^n+2017^n}$

3) $\sum_{n=1}^{\infty} \frac{n\sin^2 n}{1+n^3}$

4) $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{\sqrt{n+3}}$

5) $\sum_{n=1}^{\infty} \sin(\sqrt{n+1}-\sqrt{n})$

6) $\sum_{n=1}^{\infty} \frac{n+\sin n}{\sqrt[3]{n^7+1}}$

7) $\sum_{n=1}^{\infty} \sin\frac{n+1}{n^3+n+1}$

8) $\sum_{n=1}^{\infty} \ln\left[1+\frac{1}{3n^2}\right]$

9) $\sum_{n=1}^{\infty} \frac{1}{\ln(2n+1)}$

10) $\sum_{n=1}^{\infty} \frac{\cos n}{\sqrt{n^3+1}}$

11) $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\sin\frac{1}{n}\right)$

12) $\sum_{n=1}^{\infty} \left(1-\cos\frac{1}{n}\right)$

13) $\sum_{n=1}^{\infty} \left(\sqrt[n]{e}-1-\frac{1}{n}\right)$

14) $\sum_{n=1}^{\infty} \arcsin\frac{n-1}{n^2-n+1}$

15) $\sum_{n=2}^{\infty} \frac{1}{[\ln(\ln(n+1))]^{\ln n}}$

16) $\sum_{n=1}^{\infty} n\left(e^{\frac{1}{n}}-1\right)^2,$

17) $\sum_{n=2}^{\infty} \frac{(-1)^n+1}{n-\ln n},$

18) $\sum_{n=1}^{\infty} \arcsin(e^{-n}),$

19) $\sum_{n=1}^{\infty} \sin(\pi\sqrt{n^2+a^2}),$

20) $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{3^n n!},$

21) $\sum_{n=1}^{\infty} \left(\cos\frac{a}{n}\right)^{n^3},$

22) $\sum_{n=1}^{\infty} \frac{n^{n^2}2^n}{(n+1)^{n^2}},$

23) $\sum_{n=3}^{\infty} \frac{1}{n^{\alpha}(\ln n)^{\beta}}, (\alpha,\beta>0),$

24) $\sum_{n=3}^{\infty} \frac{(-1)^n+2\cos n\alpha}{n(\ln n)^{\frac{3}{2}}},$

25) $\sum_{n=1}^{\infty} \frac{na}{(1-a^2)^n}, 0<|a|\neq 1$

26) $\sum_{n=1}^{\infty} \frac{(n!)^2}{4^{n^2}},$

27) $\sum_{n=1}^{\infty} \left(\cos\frac{1}{n+1}-\cos\frac{1}{n}\right).$

6.1.3 Alternating Series

Exercise 6.7. *Test for convergence or divergence of the following series*

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+1}$

b) $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n^2}{n^3+1}.$

c) $\sum_{n=1}^{\infty} (-1)^n\frac{2n+1}{3n+2n},$

d) $\sum_{n=1}^{\infty} (-1)^n\frac{n^2}{n^3+4},$

e) $\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+n+1)}{2^n(n+1)},$

f) $\sum_{n=1}^{\infty} (-1)^n\sin\left(\frac{\pi}{n}\right),$

g) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{\pi^n},$

h) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!},$

i) $\sum_{n=1}^{\infty} (-1)^n\left(\frac{n+1}{n+2}\right)^n,$

j) $\sum_{n=1}^{\infty} (-1)^n\sin\frac{1}{n\sqrt{n}},$

k) $\sum_{n=1}^{\infty} (-1)^n\frac{\ln n}{n}.$

6.1.4 The ratio (d'Alambert) Test

Exercise 6.8. *Test for convergence or divergence of the series.*

a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

b) $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

c) $\sum_{n=1}^{\infty} \frac{5^n(n!)^2}{n^{2n}}$

d) $\sum_{n=1}^{\infty} \frac{(2n+1)!!}{n^n}$

e) $\sum_{n=1}^{\infty} \frac{(n^2+n+1)}{2^n(n+1)}$

f) $\sum_{n=1}^{\infty} \frac{(2n)!!}{n^n}$

g) $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{5^n\ln(n+1)}$

h) $\sum_{n=1}^{\infty} \ln\left[1+\frac{n+1}{2^n+1}\right]$

6.1.5 The root (Cauchy) Test

Exercise 6.9. *Test for convergence or divergence of the series* *a)* $\sum\limits_{n=1}^{\infty}\left(\frac{n^2+n+1}{3n^2+n+1}\right)^n$ *c)* $\sum\limits_{n=1}^{\infty}\frac{n^{n^2}5^n}{2^n(n+1)^{n^2}}$ *e)* $\sum\limits_{n=1}^{\infty}\left(\frac{n+3}{n+2}\right)^{n(n+4)}$ *g)* $\sum\limits_{n=1}^{\infty}\left(\frac{n^2+\sqrt{n}+\sin n}{2n^2+1}\right)^{3n}$

b) $\sum\limits_{n=1}^{\infty}\left(\frac{n}{n+2}\right)^{n^2}$ *d)* $\sum\limits_{n=1}^{\infty}\left(\frac{n+2}{n+3}\right)^{n(n+4)}$ *f)* $\sum\limits_{n=1}^{\infty}\left(\frac{2n+1}{3n+1}\right)^n$ *h)* $\sum\limits_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$.

Exercise 6.10. *Test for convergence or divergence of the series*

(a) $\sum\limits_{n=1}^{\infty}\frac{n}{10n^2+1}$, *(e)* $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1+n}{n}\right)^n$, *(i)* $\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}-\ln\frac{1+n}{n}\right)$,

(b) $\sum\limits_{n=2}^{\infty}\frac{n}{\sqrt{(n-1)(n+2)}}$, *(f)* $\sum\limits_{n=2}^{\infty}\frac{1}{\ln n}$, *(j)* $\sum\limits_{n=2}^{\infty}\ln\frac{n^2+\sqrt{n}}{n^2-n}\tan\frac{1}{n^2}$,

(c) $\sum\limits_{n=2}^{\infty}\left(\frac{1+n}{n^2-1}\right)^2$, *(g)* $\sum\limits_{n=2}^{\infty}\frac{\ln n}{\sqrt{n}}$, *(k)* $\sum\limits_{n=1}^{\infty}\frac{(3n+1)!}{n^2 8^n}$,

(d) $\sum\limits_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n^{\frac{3}{4}}}$, *(h)* $\sum\limits_{n=2}^{\infty}\frac{1}{\sqrt{n}}\ln\frac{1+n}{n-1}$, *(l)* $\sum\limits_{n=2}^{\infty}\frac{1.3.5\ldots(2n-1)}{2^{2n}(n-1)!}$.

6.1.6 Absolute and Conditional Convergence

Exercise 6.11. *Test for absolute or conditional convergence of the series*

a) $\sum\limits_{n=1}^{\infty}\frac{\sin n}{\sqrt{n^3}}$. *d)* $\sum\limits_{n=1}^{\infty}\frac{(-1)^n(n^2+n+1)}{2^n(n+1)}$ *g)* $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{3^n n!}$ *j)* $\sum\limits_{n=1}^{\infty}(-1)^n\frac{\ln n}{n}$.

b) $\sum\limits_{n=1}^{\infty}(-1)^n\frac{2n+1}{3n+2n^2}$ *e)* $\sum\limits_{n=1}^{\infty}(-1)^n\sin\left(\frac{\pi}{n}\right)$ *h)* $\sum\limits_{n=1}^{\infty}(-1)^n\left(\frac{n+1}{n+2}\right)^n$

c) $\sum\limits_{n=1}^{\infty}(-1)^n\frac{n^2}{n^3+4}$ *f)* $\sum\limits_{n=1}^{\infty}\frac{(-1)^n n^2}{\pi^n}$ *i)* $\sum\limits_{n=1}^{\infty}(-1)^n\sin\frac{1}{n\sqrt{n}}$

Exercise 6.12. *Prove that the series* $\sum\limits_{n=1}^{\infty}\frac{\sin n}{n}$ *is a conditionally convergent.*

Exercise 6.13. *Prove that the series* $\sum\limits_{n=1}^{\infty}\frac{\sin n}{n^p}$ *is*

a) absolutely convergent if $p>1$, *b) conditionally convergent if* $0<p\leq 1$.

6.2 Series of Functions

6.2.1 Domain of convergence

Exercise 6.14. *Find the domain of convergence of the series*

a) $\sum\limits_{n=1}^{\infty}x^n$ *c)* $\sum\limits_{n=1}^{\infty}\frac{\sin x+\cos x}{n^2+x^2}$ *e)* $\sum\limits_{n=1}^{\infty}\frac{\sin nx}{2^n(n+1)}$ *g)* $\sum\limits_{n=1}^{\infty}\frac{2^{2n+1}x^n}{5^n}$

b) $\sum\limits_{n=1}^{\infty}\frac{1}{n^x}$ *d)* $\sum\limits_{n=1}^{\infty}\frac{x^n}{n!}$ *f)* $\sum\limits_{n=1}^{\infty}\frac{(2n)!!}{n^n}x^n$ *h)* $\sum\limits_{n=1}^{\infty}\sin\frac{n+\sin x}{3n+1}$.

6.2.2 Uniform convergence

Exercise 6.15. *Test for uniform convergence of the following series* *a)* $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^2+n^2}, x \in \mathbb{R}.$

b) $\sum\limits_{n=1}^{\infty} \frac{\sin nx}{n^2+x^2}, x \in \mathbb{R}.$

c) $\sum\limits_{n=1}^{\infty} \frac{x^n}{2^n n \sqrt[3]{n}}, x \in [-2, 2].$

d) $\sum\limits_{n=1}^{\infty} \frac{1}{2^{n-1}} \left(\frac{2x+1}{x+2}\right)^n, x \in [-1, 1].$

Exercise 6.16. *Test for continuity of the series of functions* $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} \arctan \frac{x}{\sqrt{n+1}}.$

Exercise 6.17. *Find the domain of convergence and its sum*

a) $\sum\limits_{n=1}^{\infty} (-1)^{n-1}(n+1)(x-1)^n$

b) $\sum\limits_{n=1}^{\infty} (-1)^n (2n+1) x^{2n}$

c) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x+1)^n$

d) $\sum\limits_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1}.$

Exercise 6.18. *Prove that*

a) $\arctan x = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots, x \in [-1, 1].$

b) $\frac{\pi}{4} = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$

6.3 Power Series

Exercise 6.19. *Find interval of convergence of the series*

a) $\sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

b) $\sum\limits_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

c) $\sum\limits_{n=0}^{\infty} \frac{n(x+1)^n}{4^n}$

d) $\sum\limits_{n=0}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n+1}}$

e) $\sum\limits_{n=1}^{\infty} n!(2x-1)^n$

f) $\sum\limits_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

Exercise 6.20. *Find a power series representation for*

a) $f(x) = \ln(1+x)$

b) $f(x) = \ln(2+x)$

c) $f(x) = \frac{1}{1+x^2}$

d) $f(x) = \arctan x$

e) $f(x) = \frac{2}{3-x}$

f) $f(x) = \frac{5}{1-4x^2}$

g) $f(x) = \frac{1-x}{1+x}$

h) $f(x) = \frac{2}{x^2-x-2}$

i) $f(x) = \frac{x+2}{2x^2-x-1}$

j) $f(x) = \frac{x^2+x}{(1-x)^3}$

k) $f(x) = \sin^2 x$

l) $f(x) = e^x \sin x$

m) $f(x) = \int\limits_0^x e^{-t^2} dt$

n) $f(x) = \ln \frac{1+x}{1-x}$

o) $f(x) = \int\limits_0^x \frac{\sin t}{t} dt.$

6.4 Fourier Series

Exercise 6.21. *Find the Fourier series of the* 2π*-periodic function defined as*

a) $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ -1, & -\pi \le x < 0. \end{cases}$

b) $f(x) = \begin{cases} x, & 0 \le x \le \pi \\ -1, & -\pi \le x < 0. \end{cases}$

c) $f(x) = x^2, -\pi < x < \pi.$

d) $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & -\pi \le x < 0. \end{cases}$

Exercise 6.22. *Find the Fourier cosine series and Fourier sine series of the following functions*

a) $f(x) = \begin{cases} 1, & 0 \le x \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x \le \pi \end{cases}.$

b) $f(x) = 1 - x, 0 \le x \le \pi.$

c) $f(x) = \pi + x, 0 \le x \le \pi.$

d) $f(x) = x(\pi - x), 0 < x < \pi.$

Exercise 6.23. *Find the Fourier series of the function* $f(x) = x^2, -2 \le x \le 2$ *which is periodic with period* $2L = 4$*.*

Exercise 6.24. *Find the Fourier expansion of*

a) $f(x) = |x|, |x| < 1$

b) $f(x) = 2x, 0 < x < 1$

c) $f(x) = 10 - x, 5 < x < 15.$