Second-order differential equations

Tuesday, November 9, 2021 7:18 AM

A second-order differential equation has the form

$$F\left(x, y, y', y''\right) = 0$$

We may have an ordinary differential equation without initial conditions or we may have an initial value problem (IVP).

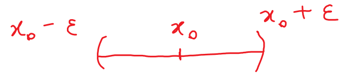
<u>Cauchy's problem:</u>

$$\begin{cases} y'' = f(x, y, y') \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$$

<u>Theorem:</u> We assume that $f(x, y, y')$, $\frac{\partial f}{\partial y}(x, y, y')$, $\frac{\partial f}{\partial y'}(x, y, y')$ are continuous on a domain D in $\mathbb{R}^3$; suppose $(x_0, y_0, y_1) \in D$.

Then there is an open neighborhood around x_0 such that there exists a unique solution to Cauchy's problem in this neighborhood.

"local solution"


x₀ - ε
x₀
x₀ + ε

Linear second-order differential equations

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$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

This is a linear second-order differential equation.

$p(x)$, $q(x)$: coefficients

If $f(x) = 0$: homogeneous equation

If $f(x) \neq 0$: inhomogeneous equation

For homogeneous equation

$$y'' + p(x)\, y' + q(x)\, y = 0$$

If y_1, y_2 are solutions. $\begin{cases} y_1'' + p(x)\, y_1' + q(x)\, y_1 = 0 \\ y_2'' + p(x)\, y_2' + q(x)\, y_2 = 0 \end{cases}$

$$\Rightarrow (y_1 + y_2)'' + p(x)(y_1 + y_2)' + q(x)(y_1 + y_2) = 0$$

$\Rightarrow$ $y_1 + y_2$ is a solution

More generally, $k_1 y_1 + k_2 y_2$ is a solution (k_1, k_2 constants)

<u>Example</u> $y'' + y = 0$

- $y_1 = \cos x$, $y_2 = \sin x$
- $y = k_1 \cos x + k_2 \sin x$ is a general solution.

How about inhomogeneous equation.

$$y'' + p(x)\,y' + q(x)\,y = f(x) \qquad (I)$$

We want to solve (I)

$$y'' + p(x)\,y' + q(x)\,y = 0 \qquad (H)$$

the complementary equation of (I)

(phương trình bổ trợ)

Suppose y_p is a solution of (I) and y_c of (H)

then $y_c + y_p$ is a solution of (I).

Linear 2nd order ODE, homogeneous, constant coefficients

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$$y'' + p(x)\,y' + q(x)\,y = 0$$

$p(x)$, $q(x)$. coefficients

We assume that $p(x)$, $q(x)$ are constants

We have

$$\boxed{\begin{array}{c} ay'' + by' + cy = 0 \\ \text{linear, 2nd order, constant coefficients} \end{array}}$$

Auxiliary quadratic equation :

$$ar^2 + br + c = 0$$

Auxiliary quadratic equation	General solution to DE
2 distinct real zeros r_1, r_2	$y = k_1 e^{r_1 x} + k_2 e^{r_2 x}$
a double real zero $r = r_1 = r_2$	$y = k_2 e^{rx} + k_2 x e^{rx}$
2 complex zeros $\alpha \pm i\beta$	$y = k_1 e^{\alpha x} \cos \beta x + k_2 e^{\alpha x} \sin \beta x$

We can check that these are solutions very easily by just plugging in the ODE.

Example. $y'' + y = 0$

Auxiliary equation $r^2 + 1 = 0$

$$r = \pm i$$

General solution : $y = k_1 \cos x + k_2 \sin x$

Example. solve $4y'' + 12y' + 9y = 0$

Example. solve $4y'' + 12y' + 9y = 0$

Auxiliary equation $4r^2 + 12r + 9 = 0$

$$(2r+3)^2 = 0$$

$$r = -3/2$$

$$\langle e^{rx} , x \cdot e^{rx} \rangle$$

General solution $y = k_1 e^{rx} + k_2 x e^{rx}$.

Linear 2nd order ODE, inhomogeneous, constant coefficients

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2 methods: $\begin{cases} \text{undetermined coefficients} \\ \text{variation of parameters} \end{cases}$

$$y'' + py' + qy = f(x) \qquad (\mathrm{I})$$

The above 2 methods are used to find a particular solution y_p to (I).

Method of Undetermined Coefficients

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$$y'' + py' + qy = f(x)$$

(1) If $f(x) = e^{kx} \cdot P(x)$, where $P(x)$ is a polynomial of degree n,

then we try to find a solution

$$y_p = e^{kx} \cdot Q(x)$$ where $Q(x)$ is a polynomial of degree n.

In some cases, we may have to find

$$y_p = e^{kx} \, x\,Q(x) \quad \text{or} \quad e^{kx} \cdot x^2 Q(x)$$

(2) If $f(x) = e^{kx} \, \underline{P}(x) \cdot \cos(mx)$

or $f(x) = e^{kx} \cdot \underline{P}(x) \cdot \sin(mx)$

with $P(x)$ a polynomial of degree n,

then we try $y_p = e^{kx} \cdot Q(x) \cdot \cos(mx) + e^{kx} R(x) \cdot \sin(mx)$

with $Q(x)$, $R(x)$ polynomials of degree n

Sometimes we may have to multiply $Q(x)$ and $R(x)$

by either x or x^2

Examples

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Example: solve $y'' + y' - 2y = x^2$ (I)

Here $f(x) = x^2 = e^{0x} \cdot x^2$

$y'' + y' - 2y = 0$ (H)

Auxiliary equation $r^2 + r - 2 = 0$

$(r-1)(r+2) = 0$

$r = 1, -2$

General solution to (H) $y_c = k_1 e^{x} + k_2 e^{-2x}$

(I) $y'' + y' - 2y = x^2$

we find $y_p = e^{0x} \underbrace{Q(x)}$

a polynomial of degree 2

$y_p = Ax^2 + Bx + C$

A, B, C : undetermined coefficients

$y_p' = 2Ax + B$

$y_p'' = 2A$

$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$

$\Rightarrow A = -\frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = -\frac{3}{4}$

$y_p = -\frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}$

So the general solution to (I) is

$y = y_c + y_p = k_1 e^{x} + k_2 e^{-2x} + \left(-\frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}\right)$

Example: solve $y'' - 4y' + 13y = \underline{e^{2x}\cos 3x}$ (I)

Complementary equation:

$$y'' - 4y' + 13y = 0 \quad (H)$$

Auxiliary equation $r^2 - 4r + 13 = 0$

$$r = 2 \pm 3i$$

General solution to (H) $\quad y_c = k_1 e^{2x} \cos 3x + k_2 e^{2x} \sin 3x$

Note that $f(x) = \underline{e^{2x} \cos 3x}$ is a solution to (H)

For (I) $\quad y'' - 4y' + 13y = e^{2x} \cos 3x$,

we find $\quad y_p = x \left(e^{2x} \cos 3x \cdot \underbrace{Q(x)}_{\text{polynomial of degree } 0} + e^{2x} \sin 3x \cdot \underbrace{R(x)}_{\text{polynomial of degree } 0} \right)$

(If we had a double zero, we multiply by x^2.)

So we have $\quad y_p = x \left(Q \cdot e^{2x} \cos 3x + R \cdot e^{2x} \sin 3x \right)$

$(Q, R$ constants$)$

is a solution of $y'' - 4y' + 13y = e^{2x} \cos 3x$

Then we find y_p', y_p'', plug in to compute Q, R

Method of Variation of Parameters

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$$y'' + p y' + q y = f(x) \qquad (I)$$

Complementary equation

$$y'' + p y' + q y = 0 \qquad (H)$$

with general solution

$$y_c = k_1 \cdot y_1(x) + k_2 \, y_2(x)$$

$$(k_1, k_2 \text{ constants})$$

We will find a solution y_p to (I) of the form

$$y_p = u_1(x) \, y_1(x) + u_2(x) \, y_2(x)$$

$$(u_1(x), u_2(x) \text{ functions})$$

Example of Variation of Parameters

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Example: solve $y'' + y = \tan x \quad \left(0 < x < \frac{\pi}{2}\right) \quad (I)$

Complementary equation $y'' + y = 0 \quad (H)$

$$y_c = k_1 \cos x + k_2 \sin x$$

We find a solution y_p of (I)

$$y_p = u_1(x) \cos x + u_2(x) \sin x$$

$$y_p'' + y_p = \tan x$$

$$y_p' = \underbrace{\left(u_1' \cos x + u_2' \sin x\right)}_{\text{we set this to be zero}} + \left(u_1(-\sin x) + u_2 \cos x\right)$$

$$u_1' \cos x + u_2' \sin x = 0$$

Then $y_p' = -u_1 \sin x + u_2 \cos x$

$$y_p'' = \left(-u_1' \sin x + u_2' \cos x\right) + \left(-u_1 \cos x - u_2 \sin x\right)$$

$$y_p = u_1 \cos x + u_2 \sin x$$

$$y_p'' + y_p = \tan x,$$

so $\begin{cases} -u_1' \sin x + u_2' \cos x = \tan x \\ u_1' \cos x + u_2' \sin x = 0 \end{cases}$

Solve for u_1', u_2'

Integrate to find u_1, u_2

$$u_2' = \sin x, \quad u_1' = -\frac{\sin^2 x}{\cos x}$$

$$u_2 = -\cos x, \quad u_1 = \sin x - \ln\left(\frac{1}{\cos x} + \tan x\right)$$

$$y_p = u_1 \cos x + u_2 \sin x \quad \text{solution of (I)}$$

General solution to (I) is $y = y_c + y_p$

2nd order ODE: without y and y'

Tuesday, November 9, 2021 8:58 AM

$$F(x, y, y', y'') = 0$$

If there is no y and y'.

$$F(x, y'') = 0$$

<u>Case 1:</u> $y'' = g(x)$

Integrate to find y'

Integrate to find y

<u>Case 2</u> $x = h(y'')$

Idea. convert 2nd order ODE to 1st order ODE

Put $p = y'$, so $p' = y''$

We have $x = h(p')$

this is first-order!

2nd order ODE: without y

Tuesday, November 9, 2021 9:04 AM

$$F(x, y, y', y'') = 0$$

If there is no y

$$F(x, y', y'') = 0$$

We convert it to first-order!

Put $p = y'$, so $p' = y''$

$$F(x, p, p') = 0$$

This is first-order ODE of $p = p(x)$

2nd order ODE: without x

Tuesday, November 9, 2021 9:06 AM

$$F(x, y, y', y'') = 0$$

If there is no x

$$F(y, y', y'') = 0$$

We want to make it first-order!

Put $p = y'$, so $p'(x) = y''$.

$$y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \underbrace{\frac{dy}{dx}}_{= y' = p} = \frac{dp}{dy} \cdot y' = \frac{dp}{dy} \cdot p$$

$$F\left(y, p, \frac{dp}{dy} \cdot p\right) = 0$$

This is first-order! Think of p as a function of y, and you have a relation of

$$y, p, \frac{dp}{dy}.$$

Review - Linear 2nd order differential equations

Tuesday, November 16, 2021 7:22 AM

Linear second-order differential equation:

$$y'' + p(x)\,y' + q(x) = f(x)$$

linear: y'' $\cancel{y''^2}$ . . .

y' $\cancel{y'^2}$

y $\cancel{y^2}$

second-order: y, y', y''

$$y'' + p(x)\,y' + q(x) = 0 \qquad (H)$$

$$y'' + p(x)\,y' + q(x) = f(x) \qquad (I)$$

We call (H) the complementary equation of (I)

The general solution of (H) is denoted by y_c

To solve (I):

1. find y_c of (H)
2. find a particular solution y_p of (I)
3. conclude the general solution of (I) is $y_c + y_p$

Linear second-order differential equation with constant coefficients:

- $p(x)$ and $q(x)$ are not general functions, but they are constant functions

$$y'' + py' + qy = f(x) \quad (I)$$

$$y'' + py' + q = 0 \quad (H)$$

$$z^2 + pz + q = 0 \quad \text{characteristic equation}$$

Solving characteristic equation, we get y_c.

How to find a particular solution.

- method of undetermined coefficients (UC)
- method of variation of parameters (VP)

Today we will cover special cases of UC.

Note: $y'' + p(x)y' + q(x)y = f(x) + g(x) \quad (I)$

If $y'' + p(x)y' + q(x)y = f(x)$ has a particular solution y_{p1}

and $y'' + p(x)y' + q(x)y = g(x)$ has a particular solution y_{p2},

then (I) has a particular solution $y_{p1} + y_{p2}$

Method of undetermined coefficients :

① $f(x) = e^{kx} \cdot Poly_n(x)$

② $f(x) = e^{kx} \; Poly_n(x) \cdot \cos(hx)$

or $e^{kx} \; Poly_n(x) \cdot \sin(hx)$

$Poly_n(x)$ is a polynomial of degree n.

① Let $y_p = e^{kx} \cdot Poly_n^*(x)$

sometimes $y_p = e^{kx} \; Poly_n^*(x) \cdot x$ or $y_p = e^{kx} \cdot Poly_n^*(x) \cdot x^2$

② Let $y_p = e^{kx} \; Poly_n^*(x) \cdot \cos(hx) + e^{kx} \; Poly_n^{**}(x) \; \sin(hx)$

sometimes we multiply by an extra x or x^2

The coefficients of $Poly^*$, $Poly^{**}$ are undetermined

We solve for these coefficients to find y_p

More examples - Method of undetermined coefficients

Tuesday, November 16, 2021 8:02 AM

Example: Solve $y'' + y' = x$ (I)

- $y'' + y' = 0$ (H)

$z^2 + z = 0$ has solutions $z = 0, -1$

$y_c = k_1 e^{0x} + k_2 e^{-x}$

- Find y_p for (I): $f(x) = x = \underbrace{x}_{\text{polynomial of degree } 1} \cdot e^{0x}$

Put $y_p = (Ax + B) \cdot e^{0x} \cdot x$

$= Ax^2 + Bx$

$y'' + y' = x$

Solve for A, B

$y' = 2Ax + B$

$y'' = 2A$

Plug in to get $2A + (2Ax + B) = x$

$\Rightarrow A = \frac{1}{2}, \quad B = -1$

$\Rightarrow y_p = \frac{x^2}{2} - x$

So (I) has general solution

$y_c + y_p = k_1 + k_2 e^{-x} + \frac{x^2}{2} - x.$

Example. solve $y'' - 6y' + 9y = xe^{3x}$. (I)

$y'' - 6y' + 9y = 0$ (H)

$z^2 - 6z + 9 = 0$ has a double root $z = 3$

$(z-3)^2 = 0$

$y_c = k_1 e^{3x} + k_2 x \cdot e^{3x}$

. To find y_p for (I), note that $f(x) = \underbrace{x}_{\text{poly of degree 1}} \cdot e^{3x}$

Put $y_p = (Ax + B) \cdot e^{3x} \cdot x^2$

$y_p = (Ax^3 + Bx^2) e^{3x}$, $y'' - 6y' + 9y = xe^{3x}$.

Compute y_p', y_p'' as before and plug in.

$\Rightarrow A = \frac{1}{6}$, $B = 0$

$y_p = \frac{1}{6} x^3 e^{3x}$.

The general solution of (I) is

$$y_c + y_p = k_1 e^{3x} + k_2 x e^{3x} + \frac{1}{6} x^3 e^{3x}.$$

2nd order differential equations without some quantities

Tuesday, November 16, 2021 8:23 AM

- First order ODE without some quantity :

$$F(x, y, y') = 0$$

① There is no x $\quad F(y, y') = 0$

Put $t = y'$. Transform $F(y, y') = 0$ into a separable equation in terms of x, t.

② There is no y $\quad F(x, y') = 0$

2.1 solve $y' = g(x)$, then integrate to find y

2.2 solve $x = h(y')$

Put $t = y'$ Transform $x = h(y')$ into a separable equation in terms of y, t.

$\rightarrow$ A parametric solution $\begin{cases} x = h(t) \\ y = k(t) \end{cases}$

2nd order ODE without some quantities.

put new variable to transform the equation from second-order to first-order.

Example: solve $x = y''^2 + y'' + 1$.

- 2nd order, without y and y'

Put $p = y'$. So $p' = y''$ and $\underbrace{x = p'^2 + p' + 1}_{\text{1st order, without } p}$.

Put $t = p'$. Change to a separable equation in p, t.

$$x = t^2 + t + 1 \Rightarrow dx = (2t+1)\,dt$$

$$t = \frac{dp}{dx} \Rightarrow dp = t\,dx$$

$$dp = t(2t+1)\,dt$$

$$p = \int t(2t+1)\,dt$$

$$\begin{cases} p = \frac{2}{3}t^3 + \frac{t^2}{2} + k & (k \text{ constant}) \\ x = t^2 + t + 1 \end{cases}$$

This is the solution for 1st order ODE

Back to 2nd order ODE:

$$p = y' = \frac{dy}{dx}$$

$$\Rightarrow dy = p\,dx = \left(\frac{2t^3}{3} + \frac{t^2}{2} + k\right)(2t+1)\,dt$$

Solve for y. $y = \int \left(\frac{2t^3}{3} + \frac{t^2}{2} + k\right)(2t+1)\,dt$

$$\begin{cases} y = \frac{4}{15}t^5 + \frac{5}{12}t^4 + \frac{1}{6}t^3 + kt^2 + kt + k' \end{cases}$$

$$\begin{cases} \quad \\ x = t^2 + t + 1 \end{cases}$$

(k, k' constants)

This is the solution for 2nd order ODE

Example: 2nd order ODE without y

$$F(x, y', y'') = 0$$

Solve $y'' + y' + 1 = x$

reduce to 1st order! Put $p = y'$

$x = p' + p + 1$.

$p' + p = x - 1 \quad (I)$ 1st order, linear

variation of parameters

$p' + p = 0 \quad (H)$ has general solution $p_c = k e^{-x}$

So we put $p = K(x) e^{-x}$

$$p' + p = x - 1$$

To find $K(x)$: $p' = K' \cdot e^{-x} - K \cdot e^{-x}$

$$p = K \cdot e^{-x}$$

$$\Rightarrow p' + p = K' \cdot e^{-x} = x - 1$$

$$\Rightarrow K' = e^{x}(x-1)$$

$$\Rightarrow K = \int e^{x}(x-1)\,dx$$

$$= (x-2)e^{x} + c \quad (c \text{ constant})$$

$$p = x - 2 + c \cdot e^{-x}$$

But $p = y' = \frac{dy}{dx}$

$$\Rightarrow dy = \left(x - 2 + c\,e^{-x}\right) dx$$

$$\Rightarrow y = \int \left(x - 2 + c\,e^{-x}\right) dx$$

$$y = \frac{x^2}{2} - 2x - c\,e^{-x} + c'$$

$(c, c'$ constants$)$

Example: 2nd order ODE without x

$$F(y, y', y'') = 0$$

$$F\left(y, y', y''\right) = 0$$

Solve $2yy'' = y'^2 + 1$. NOT linear!

Reduce the order!

Put $p = y'$, so $y'' = p' = \dfrac{dp}{dx} = \dfrac{dp}{dy} \cdot \dfrac{dy}{dx}$

$$= \frac{dp}{dy} \cdot p$$

$$2y \cdot \frac{dp}{dy} \cdot p = p^2 + 1.$$

- This is 1st order in p, y
- Separable!

$$\int \frac{dy}{y} = \int \frac{2p}{p^2+1}\, dp$$

$$\ln|y| = \ln(p^2+1) + k \quad (k \text{ constant})$$

$$y = k_1 \cdot (p^2+1) \quad (k_1 \text{ constant})$$

But $p = \dfrac{dy}{dx}$

$$y = k_1 \left(\left(\frac{dy}{dx}\right)^2 + 1\right) = k_1 (p^2+1)$$

We have $dx = \dfrac{dy}{p} = \dfrac{k_1 \cdot 2p\, dp}{p} = 2k_1\, dp$

$$p = \int \frac{1}{2k_1}\, dx = k_2 x + k_3$$

$(k_2, k_3 \text{ constants}, k_2 \neq 0)$

$$k_2 = \frac{1}{2k_1}$$

$$\frac{dy}{dx} = k_2 x + k_3$$

$$dy = (k_2 x + k_3)\, dx$$

$$y = \frac{k_2 x^2}{2} + k_3 x + k_4 .$$