

Chapter 1

VECTORS AND THE GEOMETRY OF SPACE

Reference: James Stewart. *Calculus*, sixth edition. Thomson, USA 2008.

1.1 Three-dimensional coordinate systems

1. Find the lengths of the sides of the triangle PQR . Is it a right triangle? Is it an isosceles triangle?

a) $P(3; -2; -3), \quad Q(7; 0; 1), \quad R(1; 2; 1).$

b) $P(2; -1; 0), \quad Q(4; 1; 1), \quad R(4; -5; 4).$

2. Find an equation of the sphere with center $(1; -4; 3)$ and radius 5. Describe its intersection with each of the coordinate planes.

3. Find an equation of the sphere that passes through the origin and whose center is $(1; 2; 3)$.

4. Find an equation of a sphere if one of its diameters has end points $(2; 1; 4)$ and $(4; 3; 10)$.

5. Find an equation of the largest sphere with center $(5, 4, 9)$ that is contained in the first octant.

6. Write inequalities to describe the following regions

a) The region consisting of all points between (but not on) the spheres of radius r and R centered at the origin, where $r < R$.

b) The solid upper hemisphere of the sphere of radius 2 centered at the origin.

7. Consider the points P such that the distance from P to $A(-1; 5; 3)$ is twice the distance from P to $B(6; 2; -2)$. Show that the set of all such points is a sphere, and find its center and radius.

8. Find an equation of the set of all points equidistant from the points $A(-1; 5; 3)$ and $B(6; 2; -2)$. Describe the set.

1.2 Vectors

9. Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2; 4)$.

10. Find the unit vectors that are parallel to the tangent line to the curve $y = 2 \sin x$ at the point $(\pi/6; 1)$.

11. Find the unit vectors that are perpendicular to the tangent line to the curve $y = 2 \sin x$ at the point $(\pi/6; 1)$.

12. Let C be the point on the line segment AB that is twice as far from B as it is from A . If $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$, show that $\mathbf{c} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.

1.3 The dot product

13. Determine whether the given vectors are orthogonal, parallel, or neither

a) $a = (-5; 3; 7)$, $b = (6; -8; 2)$

b) $a = (4; 6)$, $b = (-3; 2)$

c) $a = -i + 2j + 5k$, $b = 3i + 4j - k$

d) $u = (a, b, c)$, $v = (-b; a; 0)$

14. For what values of b are the vectors $(-6; b; 2)$ and $(b; b^2; b)$ orthogonal?

15. Find two unit vectors that make an angle of 60° with $v = (3; 4)$.

16. If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$, find the third direction angle γ .

17. Find the angle between a diagonal of a cube and one of its edges.

18. Find the angle between a diagonal of a cube and a diagonal of one of its faces.

1.4 The cross product

19. Find the area of the parallelogram with vertices $A(-2; 1)$, $B(0; 4)$, $C(4; 2)$, and $D(2; -1)$.

20. Find the area of the parallelogram with vertices $K(1; 2; 3)$, $L(1; 3; 6)$, $M(3; 8; 6)$ and $N(3; 7; 3)$.

21. Find the volume of the parallelepiped determined by the vectors a , b , and c .

a) $a = (6; 3; -1)$, $b = (0; 1; 2)$, $c = (4; -2; 5)$.

b) $a = i + j - k$, $b = i - j + k$, $c = -i + j + k$.

22. Let $v = 5j$ and let u be a vector with length 3 that starts at the origin and rotates in the xy -plane. Find the maximum and minimum values of the length of the vector $u \times v$. In what direction does $u \times v$ point?

1.5 Equations of lines and planes

23. Determine whether each statement is true or false.

- a) Two lines parallel to a third line are parallel.
- b) Two lines perpendicular to a third line are parallel.
- c) Two planes parallel to a third plane are parallel.
- d) Two planes perpendicular to a third plane are parallel.
- e) Two lines parallel to a plane are parallel.
- f) Two lines perpendicular to a plane are parallel.
- g) Two planes parallel to a line are parallel.
- h) Two planes perpendicular to a line are parallel.
- i) Two planes either intersect or are parallel.
- j) Two lines either intersect or are parallel.
- k) A plane and a line either intersect or are parallel.

24. Find a vector equation and parametric equations for the line.

- a) The line through the point $(6; -5; 2)$ and parallel to the vector $(1; 3; -2/3)$.
- b) The line through the point $(0; 14; -10)$ and parallel to the line $x = -1 + 2t; y = 6 - 3t; z = 3 + 9t$.
- c) The line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$.

25. Find parametric equations and symmetric equations for the line of intersection of the plane $x + y + z = 1$ and $x + z = 0$.

26. Find a vector equation for the line segment from $(2; -1; 4)$ to $(4; 6; 1)$.

27. Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

- a) $L_1 : x = -6t, y = 1 + 9t, z = -3t; \quad L_2 : x = 1 + 2s, y = 4 - 3s, z = s$.
- b) $L_1 : \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}; \quad L_2 : \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$.

28. Find an equation of the plane.

- a) The plane through the point $(6; 3; 2)$ and perpendicular to the vector $(-2; 1; 5)$
- b) The plane through the point $(-2; 8; 10)$ and perpendicular to the line $x = 1 + t, y = 2t, z = 4 - 3t$.
- c) The plane that contains the line $x = 3 + 2t, y = t, z = 8 - t$ and is parallel to the plane $2x + 4y + 8z = 17$.

29. Find the cosine of the angle between the planes $x + y + z = 0$ and $x + 2y + 3z = 1$.

30. Find parametric equations for the line through the point $(0; 1; 2)$ that is perpendicular to the line $x = 1 + t, y = 1 - t, z = 2t$, and intersects this line.

31. Find the distance between the skew lines with parametric equations $x = 1 + t, y = 1 + 6t, z = 2t$ and $x = 1 + 2s, y = 5 + 15s, z = -2 + 6s$.

1.6 Quadric surfaces

32. Find an equation for the surface obtained by rotating the parabola $y = x^2$ about the y -axis.

33. Find an equation for the surface consisting of all points that are equidistant from the point $(-1; 0; 0)$ and the plane $x = 1$. Identify the surface.

Chapter 2

VECTOR FUNCTIONS

Reference: James Stewart. *Calculus*, sixth edition. Thomson, USA 2008.

2.1 Vector functions

34. Find the domain of the vector function.

a) $r(t) = (\sqrt{4-t^2}, e^{-3t}, \ln(t+1))$

b) $r(t) = \frac{t-2}{t+2}i + \sin tj + \ln(9-t^2)k$

35. Find the limit

a) $\lim_{t \rightarrow 0} (\frac{e^t-1}{t}, \frac{\sqrt{1+t}-1}{t}, \frac{3}{t+1})$

b) $\lim_{t \rightarrow \infty} (\arctan t, e^{-2t}, \frac{\ln t}{t+1})$

36. Find a vector function that represents the curve of intersection of the two surfaces.

a) The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

b) The paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$.

37. Suppose u and v are vector functions that possess limits as $t \rightarrow a$ and let c be a constant. Prove the following properties of limits.

a) $\lim_{t \rightarrow a} [u(t) + v(t)] = \lim_{t \rightarrow a} u(t) + \lim_{t \rightarrow a} v(t)$

b) $\lim_{t \rightarrow a} cu(t) = c \lim_{t \rightarrow a} u(t)$

c) $\lim_{t \rightarrow a} [u(t) \cdot v(t)] = \lim_{t \rightarrow a} u(t) \cdot \lim_{t \rightarrow a} v(t)$

d) $\lim_{t \rightarrow a} [u(t) \times v(t)] = \lim_{t \rightarrow a} u(t) \times \lim_{t \rightarrow a} v(t)$

38. Find the derivative of the vector function.

a) $r(t) = (t \sin t, t^3, t \cos 2t).$

b) $r(t) = \arcsin ti + \sqrt{1 - t^2}j + k$

c) $r(t) = e^{t^2}i - \sin^2 tj + \ln(1 + 3t)$

39. Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point. Illustrate by graphing both the curve and the tangent line on a common screen.

a) $x = t, y = e^{-t}, z = 2t - t^2; (0; 1; 0)$

b) $x = 2 \cos t, y = 2 \sin t, z = 4 \cos 2t; (\sqrt{3}, 1, 2)$

c) $x = t \cos t, y = t, z = t \sin t; (-\pi, \pi, 0)$

40. Find the point of intersection of the tangent lines to the curve $r(t) = (\sin \pi t, 2 \sin \pi t, \cos \pi t)$ at the points where $t = 0$ and $t = 0.5$

41. Evaluate the integral

a) $\int_0^{\pi/2} (3 \sin^2 t \cos t i + 3 \sin t \cos^2 t j + 2 \sin t \cos t k) dt$

b) $\int_1^2 (t^2 i + t\sqrt{t-1} j + t \sin \pi t k) dt$

c) $\int (e^t i + 2t j + \ln t k) dt$

d) $\int (\cos \pi t i + \sin \pi t j + t^2 k) dt$

42. If a curve has the property that the position vector $r(t)$ is always perpendicular to the tangent vector $r'(t)$, show that the curve lies on a sphere with center the origin.

2.2 Arc length and curvature

43. Find the length of the curve.

a) $r(t) = (2 \sin t, 5t, 2 \cos t), \quad -10 \leq t \leq 10$

b) $r(t) = (2t, t^2, \frac{1}{3}t^3), \quad 0 \leq t \leq 1$

c) $r(t) = \cos t \, i + \sin t \, j + \ln \cos t \, k, \quad 0 \leq t \leq \pi/4$

44. Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point $(6; 18; 36)$.

45. Suppose you start at the point $(0; 0; 3)$ and move 5 units along the curve $x = 3 \sin t, y = 4t, z = 3 \cos t$ in the positive direction. Where are you now?

46. Reparametrize the curve

$$r(t) = \left(\frac{2}{t^2 + 1} - 1 \right) i + \frac{2t}{t^2 + 1} j$$

with respect to arc length measured from the point $(1; 0)$ in the direction of increasing t . Express the reparametrization in its simplest form. What can you conclude about the curve?

47. Find the curvature

a) $r(t) = t^2 i + t k$

b) $r(t) = t i + t j + (1 + t^2) k$

c) $r(t) = 3t i + 4 \sin t j + 4 \cos t k$

d) $x = e^t \cos t, y = e^t \sin t$

e) $x = t^3 + 1, y = t^2 + 1$

48. Find the curvature of $r(t) = (e^t \cos t, e^t \sin t, t)$ at the point $(1, 0, 0)$.

49. Find the curvature of $r(t) = (t, t^2, t^3)$ at the point $(1, 1, 1)$.

50. Find the curvature

a) $y = 2x - x^2,$

b) $y = \cos x,$

c) $y = 4x^{5/2}.$

51. At what point does the curve have maximum curvature? What happens to the curvature as $x \rightarrow \infty$?

a) $y = \ln x,$

b) $y = e^x.$

52. Find an equation of a parabola that has curvature 4 at the origin.

Chapter 3

Multiple Integrals

3.1 Double Integrals

3.1.1 Double Integrals in Cartesian coordinate

53. Evaluate

$$a) \iint_{[0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]} x \sin(x + y) dx dy,$$

$$g) \iint_{[0, 1] \times [0, 1]} \frac{1+x^2}{1+y^2} dx dy,$$

$$b) \iint_{[0, 2] \times [1, 2]} (x - 3y^2) dx dy,$$

$$h) \iint_{[0, \frac{\pi}{6}] \times [0, \frac{\pi}{3}]} x \sin(x + y) dx dy,$$

$$c) \iint_{[1, 2] \times [0, \pi]} y \sin(xy) dx dy,$$

$$i) \iint_{[0, 1] \times [0, 1]} \frac{x}{1+xy} dx dy,$$

$$d) \iint_{[0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]} \sin(x - y) dx dy,$$

$$j) \iint_{[0, 2] \times [0, 3]} ye^{-xy} dx dy,$$

$$e) \iint_{[0, 2] \times [1, 2]} (y + xy^{-2}) dx dy,$$

$$k) \iint_{[1, 3] \times [1, 2]} \frac{1}{1+x+y} dx dy.$$

$$f) \iint_{[0, 1] \times [-3, 2]} \frac{xy^2}{x^2+1} dx dy,$$

54. Evaluate

$$a) \iint_D x^2 (y - x) dx dy \text{ where } D \text{ is the region bounded by } y = x^2 \text{ and } x = y^2.$$

$$b) \iint_D |x + y| dx dy, D := \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1, |y| \leq 1\}$$

$$c) \iint_D \sqrt{|y - x^2|} dx dy, D := \{(x, y) \in \mathbb{R}^2 \mid |x| \leq 1, 0 \leq y \leq 1\}$$

$$d) \iint_{[0, 1] \times [0, 1]} \frac{y dx dy}{(1+x^2+y^2)^{\frac{3}{2}}}$$

- e) $\iint_D \frac{x^2}{y^2} dx dy$, where D is bounded by the lines $x = 2, y = x$ and the hyperbola $xy = 1$.
- f) $\iint_D \frac{y}{1+x^5} dx dy$, where $D = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x^2\}$,
- g) $\iint_D y^2 e^{xy} dx dy$, where $D = \{(x, y) | 0 \leq y \leq 4, 0 \leq x \leq y\}$,
- h) $\iint_D x \sqrt{y^2 - x^2} dx dy$, where $D = \{(x, y) | 0 \leq y \leq 1, 0 \leq x \leq y\}$,
- i) $\iint_D (x + y) dx dy$, where D is bounded by $y = \sqrt{x}$ and $y = x^2$,
- j) $\iint_D y^3 dx dy$, where D is the triangle region with vertices $(0, 2), (1, 1)$ and $(3, 2)$,
- k) $\iint_D xy^2 dx dy$, where D is enclosed by $x = 0$ and $x = \sqrt{1 - y^2}$.

Change the order of integration

55. Change the order of integration

- | | |
|---|--|
| a) $\int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{1-x^2} f(x, y) dy.$ | f) $\int_0^3 dy \int_{-\sqrt{9-y^2}}^{9-y^2} f(x, y) dx,$ |
| b) $\int_0^1 dy \int_{2-y}^{1+\sqrt{1-y^2}} f(x, y) dx.$ | g) $\int_0^3 dy \int_0^{\sqrt{9-y}} f(x, y) dx,$ |
| c) $\int_0^2 dx \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} f(x, y) dx.$ | h) $\int_0^2 dx \int_0^{\ln x} f(x, y) dy,$ |
| d) $\int_0^4 dx \int_0^{\sqrt{x}} f(x, y) dy,$ | i) $\int_0^1 dx \int_{\arctan x}^{\frac{\pi}{4}} f(x, y) dy,$ |
| e) $\int_0^1 dx \int_{4x}^4 f(x, y) dy,$ | j) $\int_0^{\sqrt{2}} dy \int_0^y f(x, y) dx + \int_{\sqrt{2}}^2 dy \int_0^{\sqrt{4-y^2}} f(x, y) dx.$ |

56. Evaluate the integral by reversing the order of integration

- | | |
|---|--|
| a) $\int_0^1 dy \int_{3y}^3 e^{x^2} dx,$ | d) $\int_0^1 dx \int_x^1 e^{\frac{x}{y}} dy,$ |
| b) $\int_0^{\sqrt{\pi}} dy \int_y^{\sqrt{\pi}} \cos(x^2) dx,$ | e) $\int_0^1 dy \int_{\arcsin y}^{\frac{\pi}{2}} \cos x \sqrt{1 + \cos^2 x} dx,$ |
| c) $\int_0^4 dx \int_{\sqrt{x}}^2 \frac{1}{y^3+1} dy,$ | f) $\int_0^8 dy \int_{\sqrt[3]{y}}^2 e^{x^4} dx.$ |

Change of variables

57. Evaluate $I = \iint_D (4x^2 - 2y^2) dx dy$, where $D : \begin{cases} 1 \leq xy \leq 4 \\ x \leq y \leq 4x. \end{cases}$

58. Evaluate

$$I = \iint_D \frac{x^2 \sin xy}{y} dx dy,$$

where D is bounded by parabolas

$$x^2 = ay, x^2 = by, y^2 = px, y^2 = qx, \quad (0 < a < b, 0 < p < q).$$

59. Evaluate $I = \iint_D xy dx dy$, where D is bounded by the curves

$$y = ax^3, y = bx^3, y^2 = px, y^2 = qx, \quad (0 < b < a, 0 < p < q).$$

Hint: Change of variables $u = \frac{x^3}{y}, v = \frac{y^2}{x}$.

60. Prove that

$$\int_0^1 dx \int_0^{1-x} e^{\frac{y}{x+y}} dy = \frac{e-1}{2}.$$

Hint: Change of variables $u = x + y, v = y$.

61. Find the area of the domain bounded by $xy = 4, xy = 8, xy^3 = 5, xy^3 = 15$.

Hint: Change of variables $u = xy, v = xy^3, (S = 2 \ln 3)$.

62. Find the area of the domain bounded by $y^2 = x, y^2 = 8x, x^2 = y, x^2 = 8y$.

Hint: Change of variables $u = \frac{y^2}{x}, v = \frac{x^2}{y}, (S = \frac{279\pi}{2})$.

63. Hint: Change of variables $y = x^3, y = 4x^3, x = y^3, x = 4y^3$.

64. Prove that

$$\iint_{x+y \leq 1, x \geq 0, y \geq 0} \cos\left(\frac{x-y}{x+y}\right) dx dy = \frac{\sin 1}{2}.$$

Hint: Change of variables $u = x - y, v = x + y$.

65. Evaluate

$$I = \iint_D \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} \right) dx dy,$$

where D is bounded by the axes and the parabola $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$.

Double Integrals in polar coordinate

66. Express the double integral $I = \iint_D f(x, y) dx dy$ in terms of polar coordinates, where D is given by $x^2 + y^2 \geq 4x, x^2 + y^2 \leq 8x, y \geq x, y \leq \sqrt{3}x$.

67. Evaluate $\iint_D xy^2 dx dy$ where D is bounded by $\begin{cases} x^2 + (y - 1)^2 = 1 \\ x^2 + y^2 - 4y = 0. \end{cases}$

68. Evaluate

a) $\iint_D |x + y| dx dy,$

b) $\iint_D |x - y| dx dy,$

where $D : x^2 + y^2 \leq 1$.

69. Evaluate $\iint_D \frac{dx dy}{(x^2 + y^2)^2}$, where $D : \begin{cases} 4y \leq x^2 + y^2 \leq 8y \\ x \leq y \leq x\sqrt{3}. \end{cases}$

70. Evaluate $\iint_D \frac{xy}{x^2 + y^2} dx dy$, where $D : \begin{cases} x^2 + y^2 \leq 12, x^2 + y^2 \geq 2x \\ x^2 + y^2 \geq 2\sqrt{3}y, x \geq 0, y \geq 0. \end{cases}$

71. Evaluate $\iint_D (x + y) dx dy$, where D is the region that lies to the left of the y -axis, between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

72. Evaluate $\iint_D \cos(x^2 + y^2) dx dy$, where D is the region that lies above the x -axis within the circle $x^2 + y^2 = 9$.

Evaluate $\iint_D \sqrt{4 - x^2 - y^2} dx dy$, where $D = \{(x, y) | x^2 + y^2 \leq 4, x \geq 0\}$.

73. Evaluate $\iint_D ye^x dx dy$, where D is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 25$.

74. Evaluate $\iint_D \arctan \frac{y}{x} dx dy$, where $D = \{(x, y) | 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}$.

75. Evaluate $\iint_D x dx dy$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.

3.1.2 Applications of Double Integrals

76. Compute the area of the domain D bounded by

$$a) \begin{cases} y = 2^x, y = 2^{-x}, \\ y = 4. \end{cases}$$

$$d) \begin{cases} x^2 + y^2 = 2x, x^2 + y^2 = 4x \\ x = y, y = 0. \end{cases}$$

$$b) \begin{cases} y^2 = x, y^2 = 2x \\ x^2 = y, x^2 = 2y. \end{cases}$$

$$e) r = 1, r = \frac{2}{\sqrt{3}} \cos \varphi.$$

$$f) (x^2 + y^2)^2 = 2a^2xy \quad (a > 0).$$

$$c) \begin{cases} y = 0, y^2 = 4ax \\ x + y = 3a, \quad (a > 0). \end{cases}$$

$$g) x^3 + y^3 = axy \quad (a > 0) \text{ (Descartes leaf)}$$

$$h) r = a(1 + \cos \varphi) \quad (a > 0) \text{ (Cardioids)}$$

77. Compute the volume of the object given by

$$a) \begin{cases} 3x + y \geq 1, y \geq 0 \\ 3x + 2y \leq 2, \\ 0 \leq z \leq 1 - x - y. \end{cases}$$

$$b) \begin{cases} 0 \leq z \leq 1 - x^2 - y^2, \\ x \leq y \leq x\sqrt{3}. \end{cases}$$

78. Compute the volume of the object bounded by the surfaces

$$a) \begin{cases} z = 4 - x^2 - y^2 \\ 2z = 2 + x^2 + y^2 \end{cases} \quad b) \begin{cases} z = \frac{x^2}{a^2} + \frac{y^2}{b^2}, z = 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2x}{a} \end{cases} \quad c) \begin{cases} az = x^2 + y^2 \\ z = \sqrt{x^2 + y^2}. \end{cases}$$

79. Find the area of the part of the paraboloid $x = y^2 + z^2$ that satisfies $x \leq 1$.

3.1.3 Triple Integrals

Triple Integrals in Cartesian coordinate

80. Evaluate

$$a) \iiint_V (x^2 + y^2) dx dy dz, \text{ where } V \text{ is bounded by the sphere } x^2 + y^2 + z^2 = 1 \text{ and the cone } x^2 + y^2 - z^2 = 0.$$

$$b) \iiint_E y dx dy dz, \text{ where } E \text{ is bounded by the planes } x = 0, y = 0, z = 0 \text{ and } 2x + 2y + z = 4.$$

- c) $\iiint_E x^2 e^y dx dy dz$, where E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0, x = 1$ and $x = -1$.
- d) $\iiint_E xy dx dy dz$, where E is bounded by the parabolic cylinder $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = x + y$.
- e) $\iiint_E xyz dx dy dz$, where E is the solid tetrahedron with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$.
- f) $\iiint_E x dx dy dz$, where E is the bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.
- g) $\iiint_E z dx dy dz$, where E is the bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0, y = 3x$ and $z = 0$ in the first octant.

Change of variables

81. Evaluate

- a) $\iiint_V (x + y + z) dx dy dz$, where V is bounded by
$$\begin{cases} x + y + z = \pm 3 \\ x + 2y - z = \pm 1 \\ x + 4y + z = \pm 2 \end{cases}$$
- b) $\iiint_V (3x^2 + 2y + z) dx dy dz$, where $V : |x - y| \leq 1, |y - z| \leq 1, |z + x| \leq 1$.
- c) $\iiint_V dx dy dz$, where $V : |x - y| + |x + 3y| + |x + y + z| \leq 1$.

Triple Integrals in Cylindrical Coordinates

82. Evaluate $\iiint_V (x^2 + y^2) dx dy dz$, where $V : \begin{cases} x^2 + y^2 \leq 1 \\ 1 \leq z \leq 2 \end{cases}$
83. Evaluate $\iiint_V z \sqrt{x^2 + y^2} dx dy dz$, where:
- a) V is bounded by: $x^2 + y^2 = 2x$ and $z = 0, z = a$ ($a > 0$).
- b) V is a half of the sphere $x^2 + y^2 + z^2 \leq a^2, z \geq 0$ ($a > 0$)
84. Evaluate $I = \iiint_V \sqrt{x^2 + y^2} dx dy dz$ where V is bounded by:
$$\begin{cases} x^2 + y^2 = z^2 \\ z = 1. \end{cases}$$
85. Evaluate $\iiint_V \frac{dx dy dz}{\sqrt{x^2 + y^2 + (z-2)^2}}$, where $V : \begin{cases} x^2 + y^2 \leq 1 \\ |z| \leq 1. \end{cases}$

Triple Integrals in Spherical Coordinates

86. Evaluate $\iiint_V (x^2 + y^2 + z^2) dx dy dz$, where $V : \begin{cases} 1 \leq x^2 + y^2 + z^2 \leq 4 \\ x^2 + y^2 \leq z^2. \end{cases}$

87. Evaluate $\iiint_V \sqrt{x^2 + y^2 + z^2} dx dy dz$, where $V : x^2 + y^2 + z^2 \leq z$.

88. Evaluate $\iiint_V z \sqrt{x^2 + y^2} dx dy dz$, where V is a half of the ellipsoid $\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leq 1, z \geq 0, (a, b > 0)$.

89. Evaluate $\iiint_V \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dx dy dz$, where $V : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, (a, b, c > 0)$.

90. Evaluate $\iiint_V \sqrt{z - x^2 - y^2 - z^2} dx dy dz$, where $V : x^2 + y^2 + z^2 \leq z$.

91. Evaluate $\iiint_V (4z - x^2 - y^2 - z^2) dx dy dz$, where V is the sphere $x^2 + y^2 + z^2 \leq 4z$.

92. Evaluate $\iiint_V xz dx dy dz$, where V is the domain $x^2 + y^2 + z^2 - 2x - 2y - 2z \leq -2$.

93. Evaluate

$$I = \iiint_V \frac{dx dy dz}{(1 + x + y + z)^3},$$

where V is bounded by $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

94. Evaluate

$$\iiint_V z dx dy dz,$$

where V is a half of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} \leq 1, (z \geq 0).$$

95. Evaluate

a) $I_1 = \iiint_B \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)$, where B is the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$.

b) $I_2 = \iiint_C z dx dy dz$, where C is the domain bounded by the cone $z^2 = \frac{h^2}{R^2}(x^2 + y^2)$ and the plane $z = h$.

c) $I_3 = \iiint_D z^2 dx dy dz$, where D is bounded by the sphere $x^2 + y^2 + z^2 \leq R^2$ and the sphere $x^2 + y^2 + z^2 \leq 2Rz$.

d) $I_4 = \iiint_V (x + y + z)^2 dx dy dz$, where V is bounded by the paraboloid $x^2 + y^2 \leq 2az$ and the sphere $x^2 + y^2 + z^2 \leq 3a^2$.

96. Find the volume of the object bounded by the planes Oxy , $x = 0$, $x = a$, $y = 0$, $y = b$, and the paraboloid elliptic

$$z = \frac{x^2}{2p} + \frac{y^2}{2q}, \quad (p > 0, q > 0).$$

97. Evaluate

$$I = \iiint_V \sqrt{x^2 + y^2 + z^2} dx dy dz,$$

where V is the domain bounded by $x^2 + y^2 + z^2 = z$.

98. Evaluate

$$I = \iiint_V z dx dy dz,$$

where V is the domain bounded by the surfaces $z = x^2 + y^2$ and $x^2 + y^2 + z^2 = 6$.

99. Evaluate

$$I = \iiint_V \frac{xyz}{x^2 + y^2} dx dy dz,$$

where V is the domain bounded by the surface $(x^2 + y^2 + z^2)^2 = a^2 xy$ and the plane $z = 0$.

Chapter 4

Integrals depending on a parameter

4.1 Definite Integrals depending on a parameter

100. *Compute*

$$a) \lim_{y \rightarrow 0} \int_y^{1+y} \frac{dx}{1+x^2+y^2}.$$

$$b) \lim_{y \rightarrow 0} \int_0^2 x^2 \cos xy dx.$$

101. *Evaluate*

$$a) I(y) = \int_0^1 \arctan \frac{x}{y} dx. \quad b) J(y) = \int_0^1 \ln(x^2 + y^2) dx. \quad c) K = \int_0^1 \frac{x^b - x^a}{\ln x}, \quad (0 < a < b).$$

4.2 Improper Integrals depending on a parameter

102. *Show that the integral*

$$a) I(y) = \int_1^\infty \sin(yx) dx \text{ is convergent if } y = 0 \text{ and is divergent if } y \neq 0.$$

$$b) I(y) = \int_0^\infty \frac{\cos \alpha x}{x^2+1} \text{ is uniformly convergent on } \mathbb{R}.$$

$$c) I(y) = \int_0^1 x^{-y} dx = \int_1^\infty t^{y-2} dt \text{ is convergent if } y < 1 \text{ and divergent if } y \geq 1.$$

$$d) I(y) = \int_0^{+\infty} e^{-yx} \frac{\sin x}{x} \text{ is uniformly convergent on } [0, +\infty).$$

$$e) I(y) = \int_0^\infty \frac{\cos \alpha x}{x^2+1} \text{ is uniformly convergent on } \mathbb{R}.$$

103. a) Evaluate $I(y) = \int_0^{+\infty} ye^{-yx} dx$ ($y > 0$).

b) Prove that $I(y)$ converges to 1 uniformly on $[y_0, +\infty)$ for all $y_0 > 0$.

c) Explain why $I(y)$ is not uniformly convergent on $(0, +\infty)$.

104. Prove that

$$a) \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

$$g) \int_0^{\infty} \frac{x \sin yx}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ay}, \quad a, y \geq 0.$$

$$b) \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

$$h) \int_0^{\infty} e^{-yx^2} dx = \frac{\sqrt{\pi}}{2\sqrt{y}}, \quad y > 0.$$

$$c) \int_0^{\infty} \sin(x^2) dx = \int_0^{\infty} \cos(x^2) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

$$d) \int_0^{+\infty} e^{-yx} \frac{\sin x}{x} dx = \frac{\pi}{2} - \arctan y.$$

$$i) \int_0^{+\infty} \left(e^{-\frac{a}{x^2}} - e^{-\frac{b}{x^2}} \right) dx = \sqrt{\pi b} - \sqrt{\pi a}, \quad (a, b > 0).$$

$$e) \int_0^{\infty} \frac{\sin yx}{x(1+x^2)} dx = \frac{\pi}{2}(1 - e^{-y}), \quad y \geq 0.$$

$$j) \int_0^{+\infty} \frac{\arctan \frac{x}{a} - \arctan \frac{x}{b}}{x} dx = \frac{\pi}{2} \ln \frac{b}{a}, \quad (a, b > 0).$$

$$f) \int_0^{\infty} \frac{1 - \cos yx}{x^2} dx = \frac{\pi}{2} |y|.$$

$$k) \lim_{y \rightarrow 0^+} \left(\int_0^{+\infty} ye^{-yx} dx \right) \neq \int_0^{+\infty} \left(\lim_{y \rightarrow 0^+} ye^{-yx} \right) dx \text{ and explain why?}$$

105. Evaluate $(a, b, \alpha, \beta > 0)$:

$$a) \int_0^{+\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx.$$

$$h) \int_{-\infty}^{+\infty} \frac{\arctan(x+y)}{1+x^2} dx.$$

$$b) \int_0^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2} dx.$$

$$i) \int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} dx, \text{ where } a, b > 0.$$

$$c) \int_0^{+\infty} \frac{dx}{(x^2+y)^{n+1}}.$$

$$\int_0^{+\infty} \frac{e^{-ax^3} - e^{-bx^3}}{x} dx, \text{ where } a, b > 0.$$

$$d) \int_0^{+\infty} e^{-ax} \frac{\sin bx - \sin cx}{x} dx.$$

$$j) \int_0^{\infty} \frac{e^{-ax^2} - \cos bx}{x^2} dx, \quad (a > 0)$$

$$e) \int_0^{+\infty} e^{-ax} \frac{\cos bx - \cos cx}{x} dx, \quad (a > 0).$$

$$k) \int_0^{\pi} \ln(1 + y \cos x) dx,$$

$$f) \int_0^{+\infty} e^{-ax} \cos yx dx.$$

$$l) \int_0^{\infty} e^{-x^2} \sin ax dx,$$

$$g) \int_0^{+\infty} e^{-x^2} \cos(yx) dx.$$

$$m) \int_0^{\infty} \frac{\sin xy}{x} dx, \quad y \geq 0,$$

$$n) \int_0^{\infty} e^{-ax^2} \cos bxdx \quad (a > 0),$$

$$p) \int_0^{\infty} \frac{\sin ax \cos bx}{x} dx,$$

$$o) \int_0^{\infty} x^{2n} e^{-x^2} \cos bxdx, n \in \mathbb{N}.$$

$$q) \int_0^{\infty} \frac{\sin ax \sin bx}{x} dx.$$

4.3 Euler Integral

106. *Evaluate*

$$a) \int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x dx.$$

$$e) \int_0^{+\infty} \frac{1}{1+x^3} dx.$$

$$b) \int_0^a x^{2n} \sqrt{a^2 - x^2} dx \quad (a > 0).$$

$$f) \int_0^{+\infty} \frac{x^{n+1}}{(1+x^n)} dx, \quad (2 < n \in \mathbb{N}).$$

$$c) \int_0^{+\infty} x^{10} e^{-x^2} dx.$$

$$g) \int_0^1 \frac{1}{\sqrt[n]{1-x^n}} dx, \quad n \in \mathbb{N}^*.$$

$$d) \int_0^{+\infty} \frac{\sqrt{x}}{(1+x^2)^2} dx.$$

$$h) \int_0^{+\infty} \frac{x^4}{(1+x^3)^2} dx,$$

Chapter 5

Line Integrals

5.1 Line Integrals of scalar Fields

107. Evaluate

a) $\int_C (x - y) ds$, where C is the circle $x^2 + y^2 = 2x$.

b) $\int_C y^2 ds$, where C is the curve $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, 0 \leq t \leq 2\pi, a > 0$.

c) $\int_C \sqrt{x^2 + y^2} ds$, where C is the curve $\begin{cases} x = (\cos t + t \sin t) \\ y = (\sin t - t \cos t) \end{cases}, 0 \leq t \leq 2\pi$.

d) $\int_C (x + y) ds$, where C is the circle $x^2 + y^2 = 2y$.

e) $\int_L xy ds$, where L is the part of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x \geq 0, y \geq 0$.

f) $I = \int_L |y| ds$, where L is the Cardioid curve $r = a(1 + \cos \varphi)$ ($a > 0$).

g) $I = \int_L |y| ds$, where L is the Lemniscate curve $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

5.2 Line Integrals of vector Fields

108. Evaluate $\int_{ABCA} 2(x^2 + y^2) dx + x(4y + 3) dy$, where $ABCA$ is the quadrangular curve, $A(0, 0), B(1, 1), C(0, 2)$.

109. Evaluate $\int_{ABCD} \frac{dx + dy}{|x| + |y|}$, where $ABCD$ is the triangular curve, $A(1, 0), B(0, 1), C(-1, 0), D(0, -1)$.

Green's Theorem

110. Evaluate the integral $\int_C (xy + x + y) dx + (xy + x - y) dy$, where C is the positively oriented circle $x^2 + y^2 = R^2$ by

i) computing it directly and

ii) Green's Theorem, then compare the results,

111. Evaluate the following integrals, where C is a half the circle $x^2 + y^2 = 2x$, traced from $O(0, 0)$ to $A(2, 0)$.

a) $\int_C (xy + x + y) dx + (xy + x - y) dy$

b) $\int_C x^2 \left(y + \frac{x}{4}\right) dy - y^2 \left(x + \frac{y}{4}\right) dx.$

c) $\int_C (xy + e^x \sin x + x + y) dx - (xy - e^{-y} + x - \sin y) dy.$

112. Evaluate $\oint_{OABO} e^x [(1 - \cos y) dx - (y - \sin y) dy]$, where $OABO$ is the triangle, $O(0, 0)$, $A(1, 1)$, $B(1, 0)$.

Applications of Line Integrals

113. Find the area of the domain bounded by an arch of the cycloid $\begin{cases} x = a(\theta - \sin \theta) \\ y = a(1 - \cos \theta) \end{cases}$ and Ox ($a > 0$).

Independence of Path

114. Evaluate $\int_{(-2,1)}^{(3,0)} (x^4 + 4xy^3) dx + (6x^2y^2 - 5y^4) dy.$

115. Evaluate $\int_{(1,\pi)}^{(2,2\pi)} \left(1 - \frac{y^2}{x^2} \cos \frac{y}{x}\right) dx + \left(\sin \frac{y}{x} + \frac{y}{x} \cos \frac{y}{x}\right) dy.$

Chapter 6

Surface Integrals

6.1 Surface Integrals of scalar Fields

116. Evaluate $\iint_S (z + 2x + \frac{4y}{3}) dS$, where $S = \{(x, y, z) \mid \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1, x, y, z \geq 0\}$.

117. Evaluate $\iint_S (x^2 + y^2) dS$, where $S = \{(x, y, z) \mid z = x^2 + y^2, 0 \leq z \leq 1\}$.

118. Evaluate $\iint_S x^2 y^2 z dS$, where S is the part of the cone $z = \sqrt{x^2 + y^2}$ lies below the plane $z = 1$.

119. Evaluate $\iint_S \frac{dS}{(2 + x + y + z)^2}$, where S is the boundary of the triangular pyramid $x + y + z \leq 1, x \geq 0, y \geq 0, z \geq 0$.

6.2 Surface Integrals of vector Fields

120. Evaluate $\iint_S z(x^2 + y^2) dx dy$, where S is a half of the sphere $x^2 + y^2 + z^2 = 1, z \geq 0$, with the outward normal vector.

121. Evaluate $\iint_S y dx dz + z^2 dx dy$, where S is the surface $x^2 + \frac{y^2}{4} + z^2 = 1, x \geq 0, y \geq 0, z \geq 0$, and is oriented downward.

122. Evaluate $\iint_S x^2 y^2 z dx dy$, where S is the surface $x^2 + y^2 + z^2 = R^2, z \leq 0$ and is oriented upward.

The Divergence Theorem

123. Evaluate the following integrals, where S is the surface $x^2 + y^2 + z^2 = a^2$ with outward orientation.

a. $\iint_S xdydz + ydzdx + zdxdy$

b. $\iint_S x^3dydz + y^3dzdx + z^3dxdy$.

124. Evaluate $\iint_S y^2zdx dy + xzdydz + x^2ydx dz$, where S is the boundary of the domain $x \geq 0, y \geq 0, x^2 + y^2 \leq 1, 0 \leq z \leq x^2 + y^2$ which is outward oriented.

125. Evaluate $\iint_S xdydz + ydzdx + zdxdy$, where S the boundary of the domain $(z - 1)^2 \leq x^2 + y^2, a \leq z \leq 1, a > 0$ which is outward oriented.

Stokes' Theorem

126. Use Stokes' Theorem to evaluate $\int_C F \cdot dr = \int_C Pdx + Qdy + Rdz$. In each case C is oriented counterclockwise as viewed from above.

1. $F(x, y, z) = (x + y^2)\mathbf{i} + (y + z^2)\mathbf{j} + (z + x^2)\mathbf{k}$, C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

2. $F(x, y, z) = \mathbf{i} + (x + yz)\mathbf{j} + (xy - \sqrt{z})\mathbf{k}$, C is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant.

3. $F(x, y, z) = yz\mathbf{i} + 2xz\mathbf{j} + e^{xy}\mathbf{k}$, C is the circle $x^2 + y^2 = 16, z = 5$.

4. $F(x, y, z) = xy\mathbf{i} + 2z\mathbf{j} + 3y\mathbf{k}$, C is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$.