Power Series

Tuesday, October 12, 2021 7:07 AM

$$\sum_{n=1}^{\infty} u_n(x)$$

$$s_n(x) = \sum_{j=1}^{n} u_j(x)$$

$$\lim_{n \to \infty} s_n(x) = \sum_{n=1}^{\infty} u_n(x)$$

Question: can we represent an arbitrary function as a series of functions?

Answer No!

But we can represent many useful functions as a series of functions

- power series (Taylor series / Maclaurin series)
- Fourier series

Definition (power series)

A power series is a series of the form

$$\sum_{n=1}^{\infty} c_n (x - x_0)^n$$

c_n = coefficient

x_0 = center

Examples ① $\sum_{n=0}^{\infty} x^n$ geometric series

$= \sum_{n=0}^{\infty} c_n x^n$ center $x_0 = 0$

coefficient $c_n = 1$

Domain of Convergence: $|x| < 1$

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② $\sum_{n=0}^{\infty} n!\, x^n$. Put $u_n(x) = n! \cdot x^n$

Ratio Test

$$\frac{u_{n+1}(x)}{u_n(x)} = \frac{(n+1)!\, x^{n+1}}{n!\, x^n} = (n+1)\,x$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right| = \lim_{n\to\infty} \left|(n+1)\,x\right| = \begin{cases} 0\ (<1) & (x = 0) \\ +\infty & (x \neq 0) \end{cases}$$

So $\sum_{n=0}^{\infty} n!\, x^n$ converges when $x = 0$
diverges when $x \neq 0$

Domain of Convergence $\{0\} = \{\text{center}\}$

③ $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$. Put $u_n(x) = \frac{x^n}{n!}$

Ratio Test

$$\frac{u_{n+1}(x)}{u_n(x)} = \frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x}{n+1}$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 < 1$$

So $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for $x \in \mathbb{R}$

" $= e^x$ "

④ $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ center $x_0 = 3$

Put $u_n(x) = \frac{(x-3)^n}{n}$

Ratio Test

$$\frac{u_{n+1}(x)}{u_n(x)} = \frac{(x-3)^{n+1}/(n+1)}{(x-3)^n/n} = (x-3) \cdot \frac{n}{n+1}$$

$$\lim_{n\to\infty} \left| (x-3)\,\frac{n}{n+1} \right| = |x-3|$$

$$\lim_{n \to \infty} \left| (x-3) \frac{\quad}{n+1} \right| = |x-3|$$

$$|x - 3| < 1 \iff 2 < x < 4$$

The series converges for $2 \leq x < 4$,

diverges when $x < 2$ or $x \geq 4$

Power series	Center	Domain of Convergence
$\sum_{n=0}^{\infty} x^n$	0	$(-1, 1)$
$\sum_{n=0}^{\infty} n!\, x^n$	0	$\{0\}$
$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	0	$\mathbb{R}$
$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	3	$[2, 4)$

Domain of Convergence

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Theorem (Abel)

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n .$$

There are 3 possibilities:

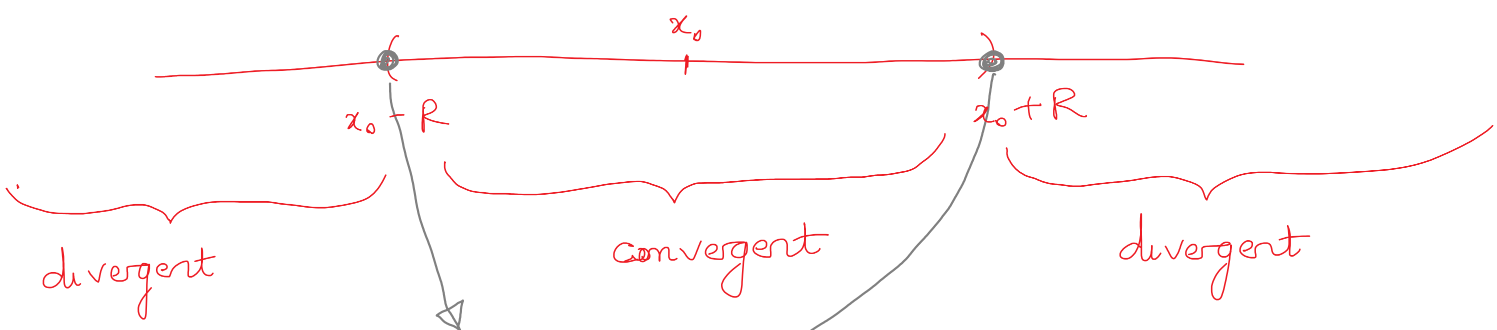
(1) the series converges only when $x = x_0$

(2) the series converges when $|x - x_0| < R$ and diverges when $|x - x_0| > R$

(3) the series converges for all $x \in \mathbb{R}$

In case (2), we say R is the radius of convergence

Domain of Convergence (R can be 0 or $+\infty$)



check the convergence

Theorem: Suppose we have a power series

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

with radius of convergence $R \in [0, +\infty]$

Then $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ converges uniformly on $(x_0 - R, x_0 + R)$

interval of convergence

Derivative and Integral of Power Series

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Theorem: Suppose $p(x) = \sum_{n=1}^{\infty} c_n (x - x_0)^n$ has radius of convergence $R > 0$.

Then: ① $p(x)$ is continuous (hence integrable), and differentiable in $(x_0 - R, x_0 + R)$

② . $p'(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{n-1}$

. $p'(x)$ has radius of convergence R

(so one can differentiate a power series as many times as one likes)

③ $\int p(x)\, dx = c_0 + \frac{1}{2} c_1 (x - x_0)^2 + \frac{1}{3} c_2 (x - x_0)^3 + \cdots$

$$= \sum_{n=1}^{\infty} c_n \cdot \frac{(x - x_0)^{n+1}}{n+1} + c_0$$

Example Find a power series representation for $\ln(1+x)$

$f(x) = \ln(1+x)$

$f'(x) = \dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad (|x| < 1)$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

$$f(x) = \int \frac{1}{1+x}\, dx$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + c_0$$

To compute c_0, we take $x = 0$ and deduce that

$0 = \sum^{\infty} (-1)^n \frac{0^{n+1}}{\ldots} + c_0$ so $c_0 = 0$

To compute c_0, we take $x = 0$ and [illegible]

$$0 = \sum_{n=0}^{\infty} (-1)^n \frac{0^{n+1}}{n+1} + c_0 \text{, so } c_0 = 0$$

Thus
$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{when } |x| < 1$$

This is the power series representation centered at $x=0$ for $\ln(1+x)$

Taylor Series and Maclaurin Series

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formal computation

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

so $f'(0) = c_1$

$$f''(x) = 2c_2 + 6c_3 x + \cdots$$

so $\frac{1}{2} f''(0) = c_2$

$$f'''(x) = 6c_3 + \cdots$$

so $\frac{1}{6} f'''(0) = c_3$

General formula: $c_n = \frac{1}{n!} f^{(n)}(0)$

Theorem: Let $f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ be a power series with radius of convergence $R > 0$.

Then $c_n = \frac{f^{(n)}(x_0)}{n!}$

So $f(x) = c_0 + c_1 (x - x_0) + c_2 (x - x_0)^2 + \cdots$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n}$$ Taylor's formula

Definition. Let $f(x)$ be an infinitely differentiable function

Definition. Let $f(x)$ be an $\underbrace{\text{infinitely differentiable}}_{\text{smooth}}$ function

We call $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

the **Taylor series expansion** of $f(x)$ at $x = x_0$

If $x_0 = 0$, we call this

the **Maclaurin series** of $f(x)$

Examples $f(x) = e^x$

- **Maclaurin series** $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n$

$$f^{(n)}(x) = e^x$$

$$f^{(n)}(0) = 1$$

- **Taylor series at $x_0 = 2021$**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2021)}{n!}(x-2021)^n = \sum_{n=0}^{\infty} \frac{e^{2021}}{n!}(x-2021)^n$$

$$f^{(n)}(x) = e^x$$

$$f^{(n)}(2021) = e^{2021}$$

Questions. ① Does the Taylor series of e^x converge ?

② Does it converge to e^x ?

Theorem: Suppose $\left| f^{(n)}(x) \right| < L \quad (L > 0)$

for all $n \in \mathbb{N}$ and for $|x - x_0| < d$ $(d > 0)$.

Then $$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = f(x)$$

when $|x - x_0| < d$

Example $$e^x \rightsquigarrow \sum_{n=0}^{\infty} \frac{e^{2021}}{n!}(x - 2021)^n$$

$$f^{(n)}(x) = e^x$$

when $|x - 2021| < 1$, $|f^{(n)}(x)| = e^x < e^{2022}$

so $$\sum_{n=0}^{\infty} \frac{e^{2021}}{n!}(x - 2021)^n = e^x$$

when $|x - 2021| < 1$

Repeating this argument, we deduce that

$$\sum_{n=0}^{\infty} \frac{e^{x_0}}{n!}(x - x_0)^n = e^x$$

for all x_0 and for all x

In particular, $$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$$

Examples:

- $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (x \in \mathbb{R})$$
- $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (x \in \mathbb{R})$$

· Maclaurin series of

$$\frac{1}{1-x}$$

$$\ln(1+x)$$

$$(1+x)^k$$

$$\arctan x$$

(wikipedia)