HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY
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Lecture on

MATH 4

MULTIPLE INTEGRAL, INTEGRAL THAT DEPENDS ON A PARAMETER, LINE INTEGRAL, SURFACE INTEGRAL, FIELD THEORY AND SERIES

Summary, Examples, Exercises and Solutions

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CONTENTS

CHAPTER 1

MULTIPLE INTEGRAL

§1. DOUBLE INTEGRAL

1.1 Calculation of a double integral in Cartesian coordinate system

Consider the integral

$$I = \iint\limits_{\mathscr{D}} f(x,y)dxdy. \tag{1.1}$$

1. **(The Corollary of Fubini's theorem)**
   Suppose that $\mathscr{D} = [a,b] \times [c,d]$ and $f: \ \mathscr{D} \to \mathbb{R}$ is a continuous function on $\mathscr{D}$. Then

$$I = \int\limits_a^b dx \int\limits_c^d f(x,y)dy = \int\limits_c^d dy \int\limits_a^b f(x,y)dx$$

2. If $\mathscr{D}$ is described as follows: $\mathscr{D} = \begin{cases} a \le x \le b \\ \varphi(x) \le y \le \psi(x) \end{cases}$,
   where $y = \varphi(x), \ \ y = \psi(x)$ are continuous and have continuous derivatives on $[a,b]$
   then $I = \int\limits_a^b \left[\int\limits_{\varphi(x)}^{\psi(x)} f(x,y)dy \right] dx$ or

$$I = \int\limits_a^b dx \int\limits_{\varphi(x)}^{\psi(x)} f(x,y)dy. \tag{1.2}$$

3. If $\mathscr{D}$ is described as follows: $\mathscr{D} = \begin{cases} c \le y \le d \\ \varphi(y) \le x \le \psi(y) \end{cases}$,
   where $x = \varphi(y),\ \ x = \psi(y)$ are continuous and have continuous derivatives on $[c, d]$ then
$$I = \int\limits_c^d dy \int\limits_{\varphi(y)}^{\psi(y)} f(x, y)dx. \tag{1.3}$$

Example 1.1. *Calculate the double integral*
$$I = \iint\limits_{\mathscr{D}} x^2 y dxdy,$$
where $\mathscr{D} = [0, 1] \times [0, 2]$.

Solution. We have
$$I = \iint\limits_{\mathscr{D}} x^2 y dxdy = \int\limits_0^1 dx \int\limits_0^2 x^2 y dy = \int\limits_0^1 \left(x^2 \frac{y^2}{2} \right) \Big|_0^2 dx$$
$$= \int\limits_0^1 x^2 \frac{4}{2} dx = 2.\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}.$$
■

Example 1.2. *Calculate the double integral* $I = \iint\limits_{\mathscr{D}} \left(x^3 + xy\right) dxdy$ *where* $\mathscr{D}$ *is bounbed by the curves* $y = x^2$ *and* $y = \sqrt{x}$.

Solution. We have the region $\mathscr{D} = \left\{0 \le x \le 1,\ \ x^2 \le y \le \sqrt{x}\right\}$ (Figure 1.1).

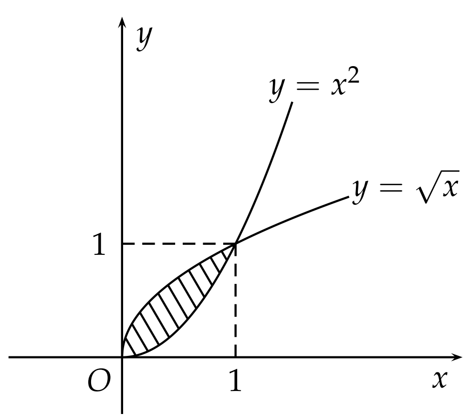

Figure 1.1

Therefore

$$I = \int_0^1 dx \int_{x^2}^{\sqrt{x}} \left(x^3 + xy\right) dy = \int \left(x^3 y + x\frac{y^2}{2}\right) \Big|_{x^2}^{\sqrt{x}} dx$$
$$= \int_0^1 \left(x^3\sqrt{x} - x^5 + \frac{1}{2}x^2 - \frac{1}{2}x^5\right) dx = \frac{5}{36}. \qquad \blacksquare$$

Example 1.3. *Interchange the order of the following integrals:*

i) $I = \int_0^2 dx \int_x^{2x} f(x,y)dy;$ *ii)* $I = \int_1^e dy \int_0^{\ln y} f(x,y)dx.$

Solution. i) We have $\mathscr{D} = \begin{cases} x = 0, x = 2 \\ y = x \\ y = 2x \end{cases}$ (Figure 1.2)

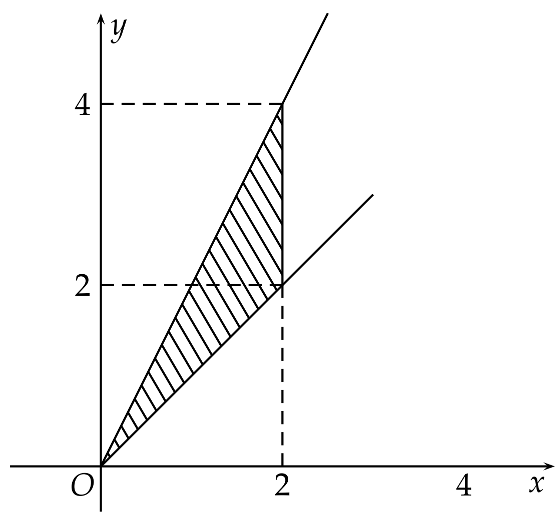

Figure 1.2

From above figure, we have

$$I = \int_0^2 dy \int_{y/2}^{y} f(x,y)dx + \int_2^4 dy \int_{y/2}^{2} f(x,y)dx.$$

ii) We have $\mathscr{D} = \begin{cases} y = 1, y = e \\ x = 0 \\ x = \ln y \end{cases}$ (Figure 1.3).

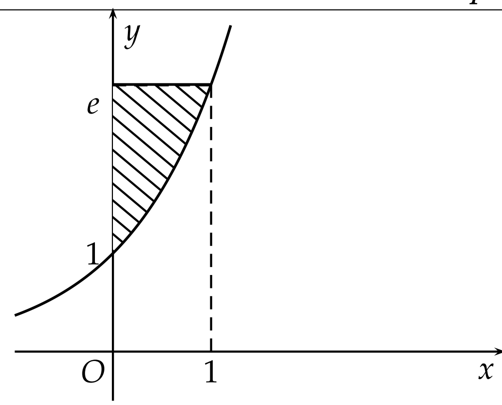

Figure 1.3

Hence

$$I = \int_0^1 dx \int_{e^x}^e f(x,y)dy.$$

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1.2 Change of variables in double integrals, polar coordinate

1. **In general case**
   Put $I = \iint\limits_{\mathscr{D}} f(x,y)dxdy$.
   To calculate I, we can perform the tranformation

$$\begin{cases} x = x(u,v) \\ y = y(u,v). \end{cases} \tag{1.4}$$

The two equations in (1.4) define a mapping which carries a point $(x,y) \in \mathscr{D} \subset O_{xy}$ to $(u,v) \in \overline{\mathscr{D}} \subset O'_{uv}$ (or inversion).
We shall consider mapping for which the functions $x = x(u,v), y = y(u,v)$ are continuous and have continuous partial derivatives on $\overline{\mathscr{D}}$. Then

$$I = \iint\limits_{\mathscr{D}} f(x,y)dxdy = \iint\limits_{\overline{\mathscr{D}}} f\left(x(u,v), y(u,v)\right) |J|\, dudv,$$

where $J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \dfrac{D(x,y)}{D(u,v)} \neq 0.$

2. **Polar coordinate**

   In this case we write r and φ instead of u and v and discrible the mapping by the two equations

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \quad ; \quad |J| = r \geq 0. \tag{1.5}$$

   Then

$$I = \iint\limits_{\mathscr{D}_{Oxy}} f(x,y)dxdy = \iint\limits_{\overline{\mathscr{D}}_{Or\varphi}} f\left(r\cos\varphi, r\sin\varphi\right) rdrd\varphi.$$

Example 1.4. *Calculate* $I = \iint\limits_{\mathscr{D}} \frac{y}{x}dxdy$, *where the region* $\mathscr{D}$ *is bounded by*

$$y = x, \;\; y = 2x, \;\; xy = 1, \;\; xy = 3 \quad (x > 0).$$

Solution. Because $x > 0$, put $\frac{y}{x} = u, \;\; xy = v \;\; (u > 0, v > 0)$. Therefore we perform the tranformation

$$\begin{cases} x &= \dfrac{\sqrt{v}}{\sqrt{u}} \\ y &= \sqrt{v}.\sqrt{u} \end{cases} \Longrightarrow J = \frac{D(x,y)}{D(u,v)} = \begin{vmatrix} -\dfrac{\sqrt{v}}{u}.\dfrac{1}{2\sqrt{u}} & \dfrac{1}{2\sqrt{v}\sqrt{u}} \\ \dfrac{\sqrt{v}}{2\sqrt{u}} & \dfrac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = -\frac{1}{u}.$$

The region $\mathscr{D} \to \overline{\mathscr{D}} = \begin{cases} 1 \leq u \leq 2 \\ 1 \leq v \leq 3 \end{cases}$.

Hence

$$I = \int\limits_1^3 dv \int\limits_1^2 u\left|-\frac{1}{u}\right| du = \left(v\Big|_1^3\right)\left(u\Big|_1^2\right) = 2.$$

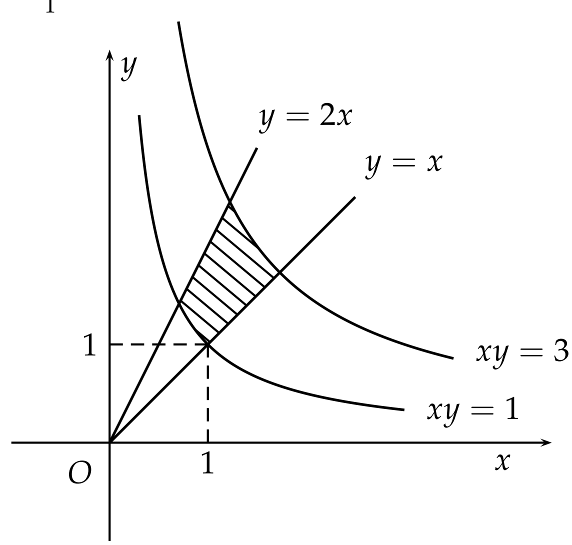

Figure 1.4

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Example 1.5. *Tranform each of the given integrals to one or more interated integrals in polar coordinate*

1. $I = \int\limits_0^1 dx \int\limits_0^1 f(x,y)dy;$ 2. $I = \int\limits_0^1 dx \int\limits_{1-x}^{\sqrt{1-x^2}} f(x,y)dy.$

Solution. 1. We have $\mathscr{D} = [0,1] \times [0,1]$ (Figure 1.5).

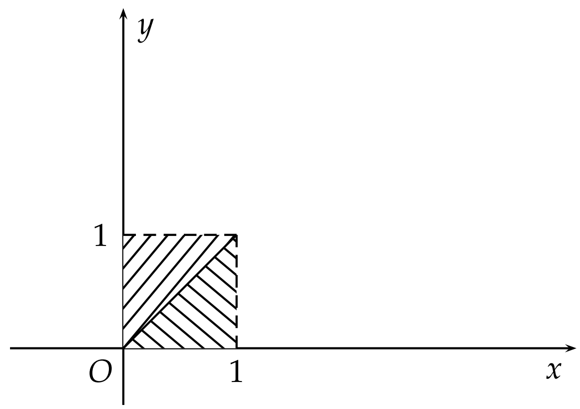

Figure 1.5

To transform to polar coordinate, we put $x = r\cos\varphi;\ \ y = r\sin\varphi$.
We devide the region $\mathscr{D}$ into two subregions by the line $y = x:\ \ \mathscr{D} = \mathscr{D}_1 \cup \mathscr{D}_2$, where

$$\mathscr{D}_1 :\ \ 0 \le \varphi \le \frac{\pi}{4};\ \ 0 \le r \le \frac{1}{\cos\varphi};$$
$$\mathscr{D}_2 :\ \ \frac{\pi}{4} \le \varphi \le \frac{\pi}{2};\ \ 0 \le r \le \frac{1}{\sin\varphi}$$

Therefore

$$I = \int\limits_0^{\pi/4} d\varphi \int\limits_0^{1/\cos\varphi} f\left(r\cos\varphi, r\sin\varphi\right) rdr + \int\limits_{\pi/4}^{\pi/2} d\varphi \int\limits_0^{1/\sin\varphi} f\left(r\cos\varphi, r\sin\varphi\right) rdr$$

2. Rewrite the region $\mathscr{D} = \begin{cases} x = 1,\ \ x = 2 \\ y = 1 - x \\ y = \sqrt{1-x^2} \end{cases}$ (Figure 1.5)

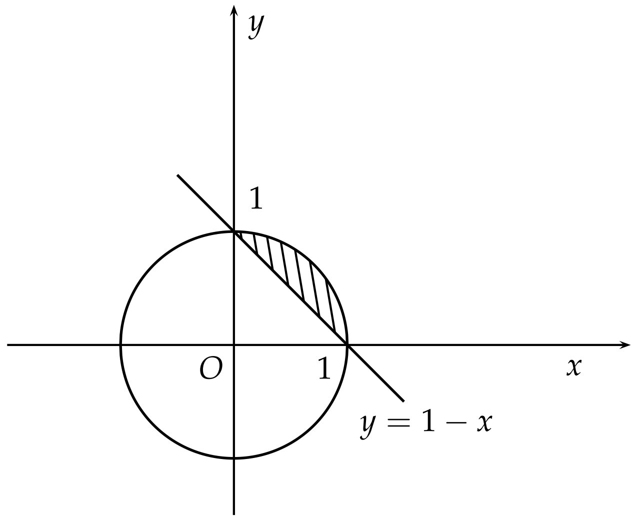

Figure 1.6

Put $\begin{cases} x = r\cos\varphi; \\ y = r\sin\varphi \end{cases}$, we have $\begin{cases} 0 \le \varphi \le \dfrac{\pi}{2} \\ \dfrac{1}{\sin\varphi + \cos\varphi} \le r \le 1 \end{cases}$.

Hence

$$I = \int_0^{\pi/2} d\varphi \int_{1/(\sin\varphi+\cos\varphi)}^{1} f(r\cos\varphi, r\sin\varphi)\, r dr.$$

■

Note: Some regions used frequently in polar coordinate

i) If $\mathscr{D}_{Oxy} = \left\{x^2 + y^2 \le R^2\right\}$ then

$$\overline{\mathscr{D}}_{Or\varphi} = \{0 \le \varphi \le 2\pi;\ \ 0 \le r \le R\}.$$

ii) If $\mathscr{D} = \left\{x^2 + y^2 \le 2ax;\ \ a > 0\right\} = \left\{(x-a)^2 + y^2 \le a^2\right\}$ then

$$\overline{\mathscr{D}} = \begin{cases} -\dfrac{\pi}{2} \le \varphi \le \dfrac{\pi}{2} \\ 0 \le r \le 2a\cos\varphi. \end{cases}$$

iii) If $\mathscr{D} = \left\{x^2 + y^2 \le 2ay;\ \ a > 0\right\} = \left\{x^2 + (y-a)^2 \le a^2\right\}$ then

$$\overline{\mathscr{D}} = \begin{cases} 0 \le \varphi \le \pi \\ 0 \le r \le 2a\sin\varphi. \end{cases}$$

iv) If

$$\begin{aligned} \mathscr{D} &= \left\{x^2 + y^2 \le 2ax + 2ay,\ \ a > 0, b > 0\right\} \\ &= \left\{(x-a)^2 + (y-b)^2 \le a^2 + b^2\right\} \end{aligned}$$

then $\overline{\mathscr{D}} = \begin{cases} -\operatorname{arctg}\dfrac{a}{b} \le \varphi \le -\operatorname{arctg}\dfrac{a}{b} + \pi \\ 0 \le r \le 2\left(a\cos\varphi + b\sin\varphi\right). \end{cases}$

v) If $\mathscr{D} = \left\{x^2 + y^2 = -2ax,\ \ a > 0\right\} = \left\{(x+a)^2 + y^2 = a^2\right\}$ then
$\overline{\mathscr{D}} = \begin{cases} \dfrac{\pi}{2} \le \varphi \le \dfrac{3\pi}{2} \\ 0 \le r \le -2a\cos\varphi \end{cases}$

vi) If $\mathscr{D}$ is ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ then we perform the transformation

$$\begin{cases} x = ar\cos\varphi \\ y = br\sin\varphi \end{cases},\quad |J| = abr.$$

Then $\mathscr{D} \to \overline{\mathscr{D}} = \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le 1 \end{cases}$ and

$$I = \int_0^{2\pi} d\varphi \int_0^1 f\left(ar\cos\varphi, br\sin\varphi\right) abr dr. \tag{1.6}$$

1.3 Applications of double integrals

1. **To compute the area of a plane domain $\mathscr{D}$**

The area of the domain $\mathscr{D}$ in the plane Oxy is computed by the formula:

$$S = \iint_{\mathscr{D}} dxdy \tag{1.7}$$

Example 1.6. *Compute the area of the domain $\mathscr{D}$ bounded by the curves*

$$xy = a^2,\ \ x + y = \frac{5}{2}a.$$

Solution. The curve $xy = a^2$ cuts the line $x + y = \dfrac{5a}{2}$ at two points that have abscissas $x = \dfrac{a}{2}$ and $x = 2a$, respectively (Figure 1.7).
Therefore the area of $\mathscr{D}$ is

$$S = \int_{a/2}^{2a} dx \int_{a^2/x}^{5a/2-x} dy = \left(\frac{15}{8} - 2\ln 2\right)a^2$$

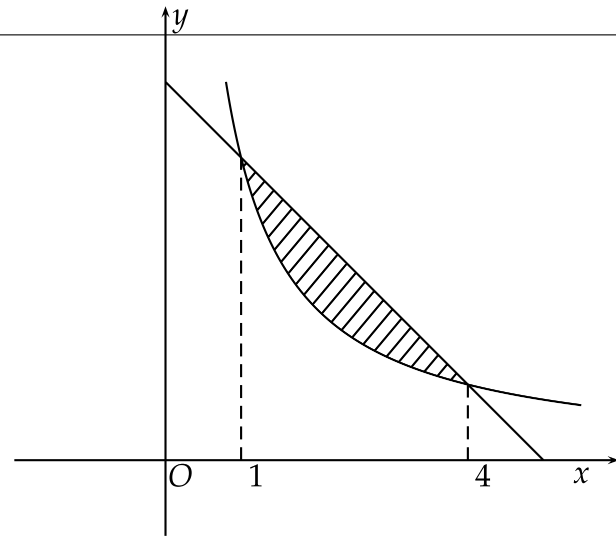

Figure 1.7

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2. **To compute the area of a curved surface**

   Suppose that S is a curved surface whose equation is $z = f(x,y)$ (Figure 1.6) and $\mathscr{D}$ is the projection of S on the plane Oxy. Then the area of S is

$$S = \iint_{\mathscr{D}} \sqrt{1 + z_x'^2 + z_y'^2} dxdy \tag{1.8}$$

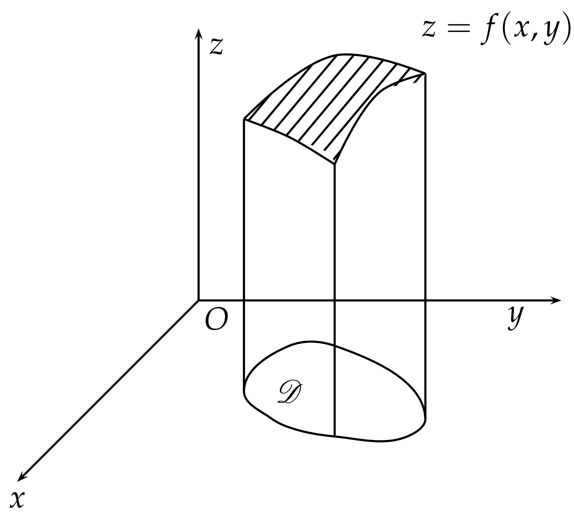

Figure 1.8

3. **To compute the volume of a object**

   Suppose that the object Ω is bounded by the smooth curves

$$z = f(x,y), \ \ z = g(x,y), \ \ f \geq g \ \ \forall (x,y) \in \mathscr{D}$$

and the surrounding cylinder has the directrix that is the border of the region $\mathscr{D}$: $\varphi(x,y)=0$ and has the element that is parallel with Oz.
Then the volume of Ω is

$$V=\iint\limits_{\mathscr{D}}[f(x,y)-g(x,y)]\,dxdy. \tag{1.9}$$

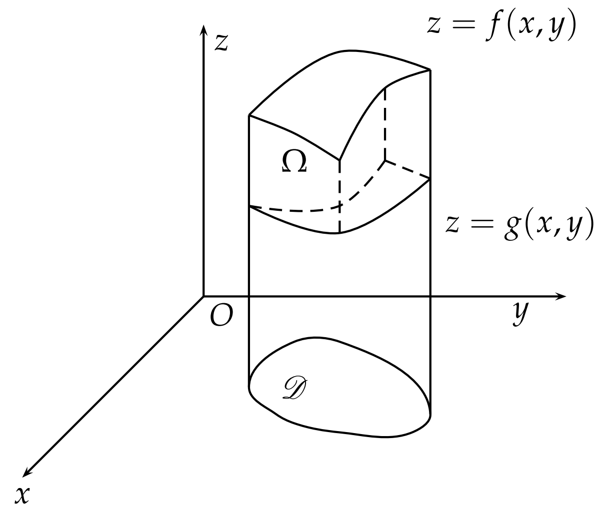

Figure 1.9

Example 1.7. Use the double integral to compute the volume of the object bounded by

$$z=1+x+y,\ \ z=0,\ \ x+y=1,\ \ x=0, y=0.$$

Solution. We have

$$I=\int\limits_0^1 dx\int\limits_0^{1-x}(1+x+y)\,dy=\int\limits_0^1\frac{1}{2}\left[4-(1+x)^2\right]dx=2-\frac{1}{2}.\frac{1}{3}(1+x)^3\Big|_0^1=\frac{5}{6}.$$ ■

1.4 Exercises

Exercise 1.1. Interchange the order of the following integrals

1. $\displaystyle\int\limits_{-1}^{1}dx\int\limits_{-\sqrt{1-x^2}}^{1-x^2}f(x,y)dy;$

2. $\displaystyle\int\limits_{0}^{1}dy\int\limits_{2-y}^{1+\sqrt{1-y^2}}f(x,y)dx;$

3. $\int\limits_0^2 dx \int\limits_{\sqrt{2x-x^2}}^{\sqrt{2x}} f(x,y)dy;$

4. $\int\limits_0^{\sqrt{5}} dy \int_0^y f(x,y)dx + \int\limits_{\sqrt{2}}^{2} dy \int\limits_0^{\sqrt{4-y^2}} f(x,y)dx.$

Exercise 1.2. Calculate the following integrals

1. $\iint\limits_{\mathscr{D}} x\sin(x+y)\,dxdy,\quad \mathscr{D} = \left\{(x,y)\in\mathbb{R}^2 : 0\le x\le \frac{\pi}{2}; 0\le y\le \frac{\pi}{2}\right\};$

2. $\iint\limits_{\mathscr{D}} x^2(y-x)\,dxdy$, where $\mathscr{D}$ is bounded by curves $y=x^2$ and $x=y^2$;

3. $\iint\limits_{\mathscr{D}} |x+y|\,dxdy,\quad \mathscr{D} = \left\{(x,y)\in\mathbb{R}^2 : |x|\le 1, |y|\le 1\right\};$

4. $\iint\limits_{\mathscr{D}} \sqrt{|y-x^2|}dxdx, \mathscr{D} = \{|x|\le 1, 0\le y\le 1\};$

5. $\iint\limits_{|x|+|y|\le 1} (|x|+|y|)\,dxdy.$

Exercise 1.3. Transform the integral to polar coordinate and compute its value

1. $\int\limits_0^R dx \int\limits_0^{\sqrt{R^2-x^2}} \ln\left(1+x^2+y^2\right) dy,\quad (R>0);$

2. $\int\limits_0^R dx \int\limits_{-\sqrt{Rx-x^2}}^{\sqrt{Rx-x^2}} \sqrt{Rx-x^2-y^2}dy,\quad (R>0);$

3. $\iint\limits_{\mathscr{D}} xydxdy$, where $\mathscr{D} = \left\{(x,y)\in\mathbb{R}^2 :\ (x-2)^2+y^2\le 1, y\ge 0\right\}.$

4. $\iint\limits_{\mathscr{D}} xy^2dxdy$, where $\mathscr{D}$ is bounded by the curves $x^2+(y-1)^2=1$ and $x^2+y^2-4y=0$.

Exercise 1.4. Calculate these following integrals:

1. $\iint\limits_{\mathscr{D}} \frac{dxdy}{(x^2+y^2)^2}$, where $\mathscr{D}:\ \begin{cases} 4y\le x^2+y^2\le 8y \\ x\le y\le \sqrt{3}x \end{cases};$

2. $\displaystyle\iint\limits_{\mathscr{D}} \frac{xy}{x^2+y^2}dxdy$, where $\mathscr{D}: \begin{cases} x^2+y^2 \leq 12 \\ x^2+y^2 \geq 2x \\ x^2+y^2 \geq 2\sqrt{3}y \\ x \geq 0, y \geq 0 \end{cases}$;

3. $\displaystyle\iint\limits_{\mathscr{D}} \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}dxdy$, where $\mathscr{D}: x^2+y^2 \leq 1$;

4. $\displaystyle\iint\limits_{\mathscr{D}} \left|9x^2-4y^2\right| dxdy$, where $\mathscr{D}: \dfrac{x^2}{4}+\dfrac{y^2}{9} \leq 1$;

5. $\displaystyle\iint\limits_{\mathscr{D}} \left(4x^2-2y^2\right) dxdy$, where $\mathscr{D}: \begin{cases} 1 \leq xy \leq 4 \\ x \leq y \leq 4x \end{cases}$

Exercise 1.5. Compute the area of the domain $\mathscr{D}$ bounded by $\begin{cases} y = 2^x \\ y = 2^{-x} \\ y = 4. \end{cases}$

Exercise 1.6. Compute the area of the domain $\mathscr{D}$ bounded by $\begin{cases} y = 0; \;\; y^2 = 4ax \\ x + y = 3a; \;\; y \leq 0, \;\; (a > 0). \end{cases}$

Exercise 1.7. Compute the area of the domain $\mathscr{D}$ bounded by $\begin{cases} x^2+y^2 = 2x; \;\; x^2+y^2 = 4x \\ x = y, \;\; y = 0 \end{cases}$

Exercise 1.8. Compute the volume of the object bounded by the surfaces

$$\begin{cases} 3x + y \geq 1 \\ 3x + 2y \leq 2 \\ y \geq 0, \;\; 0 \leq z \leq 1 - x - y. \end{cases}$$

Exercise 1.9. Compute the volume of the object bounded by the surfaces

$$\begin{cases} 0 \leq z \leq 1 - x^2 - y^2 \\ y \geq x, \;\; y \leq x\sqrt{3}. \end{cases}$$

1.5 Solutions

Solution 1.1. 1. $I = \int\limits_{-1}^{0} dy \int\limits_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx + \int\limits_{0}^{1} dy \int\limits_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)dx;$

2. $I = \int\limits_{1}^{2} dx \int\limits_{2-x}^{\sqrt{2x-x^2}} f(x,y)dy;$

3. $\int\limits_{0}^{1} dy \int\limits_{y^2/2}^{1-\sqrt{1-y^2}} f(x,y)dx + \int\limits_{0}^{1} dy \int\limits_{1+\sqrt{1-y^2}}^{2} f(x,y)dx + \int\limits_{1}^{2} dy \int\limits_{y^2/2}^{2} f(x,y)dx;$

4. $I = \int\limits_{0}^{\sqrt{2}} dx \int\limits_{x}^{\sqrt{4-x^2}} f(x,y)dy.$

Solution 1.2. 1. $I = \int\limits_{0}^{\pi/2} dx \int\limits_{0}^{\pi/2} x\sin(x+y)dy = \frac{\pi}{2};$

2. $I = \int\limits_{0}^{1} dx \int\limits_{x^2}^{\sqrt{x}} \left(x^2y - x^3\right) dy = -\frac{1}{504};$

3. Divide $\mathscr{D}$ into two regions $\mathscr{D} = \mathscr{D}_1 \cup \mathscr{D}_2$, where

$$\mathscr{D}_1 = \{-1 \le x \le 1,\ \ -x \le y \le 1\}$$
$$\mathscr{D}_2 = \{-1 \le x \le 1,\ \ -1 \le y \le -x\}.$$

Then

$$I = \int\limits_{-1}^{1} dx \int\limits_{-x}^{1} (x+y)dy - \int\limits_{-1}^{1} dx \int\limits_{-1}^{-x} (x+y)dy = \frac{8}{3}.$$

4. Divide $\mathscr{D}$ into two regions $\mathscr{D} = \mathscr{D}_1 \cup \mathscr{D}_2$, where

$$\mathscr{D}_1 = \left\{-1 \le x \le 1,\ \ x^2 \le y \le 1\right\}$$
$$\mathscr{D}_2 = \left\{-1 \le x \le 1,\ \ 0 \le y \le x^2\right\}.$$

Hence

$$I = \int\limits_{-1}^{1} dx \int\limits_{x^2}^{1} \sqrt{y - x^2}dy + \int\limits_{-1}^{1} dx \int\limits_{0}^{x^2} \sqrt{x^2 - y}dy = \frac{3\pi + 4}{12}.$$

5. Note that $\mathscr{D}$ is axisymetric and the function $f(x,y) = |x| + |y|$ is even with respect to x and y. Therefore $I = 4\iint\limits_{\mathscr{D}_1} (|x| + |y|)\, dxdy$, where

$$\mathscr{D}_1 = \{(x,y) : \ 0 \le x \le 1; \ 0 \le y \le 1 - x\}.$$

Hence

$$I = 4\int\limits_0^1 dx \int\limits_0^{1-x} (x+y)dy = \frac{4}{3}.$$

Solution 1.3. 1. We have $\mathscr{D} = \left\{0 \le x \le R; \ 0 \le y \le \sqrt{R^2 - x^2}\right\}$.

Put $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Longrightarrow 0 \le \varphi \le \frac{\pi}{2}, \ 0 \le r \le R.$

Hence

$$I = \int\limits_0^{\pi/2} d\varphi \int\limits_0^R \ln\left(1 + r^2\right) rdr = \frac{\pi}{4}\left[\left(R^2 + 1\right)\ln(R^2 + 1) - R^2\right].$$

2. We have $\mathscr{D} = \left\{0 \le x \le R, \ -\sqrt{Rx - x^2} \le y \le \sqrt{RX - x^2}\right\}$.

Put $\begin{cases} x = \frac{R}{2} + r\cos\varphi \\ y = r\sin\varphi \end{cases} \Longrightarrow |J| = r, \ \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le \frac{R}{2} \end{cases}.$

Hence

$$I = \int\limits_0^{2\pi} d\varphi \int\limits_0^{R/2} \sqrt{\frac{R^2}{4} - r^2} rdr = \frac{\pi R^3}{12}.$$

3. Put $\begin{cases} x = 2 + r\cos\varphi \\ y = r\sin\varphi \end{cases} \Longrightarrow \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \end{cases}$. Then

$$I = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 (2 + r\cos\varphi)\, r\sin\varphi rdr = 0$$

Note: The domain $\mathscr{D}$ is Ox-axisymetric and the function $f(x,y) = xy$ is odd with respect to y. Therefore we have $I = 0$.

4. Put $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Longrightarrow \begin{cases} 0 \le \varphi \le \pi \\ 2\sin\varphi \le r \le 4\sin\varphi \end{cases}$. Hence

$$I = \int\limits_0^{\pi} d\varphi \int\limits_{2\sin\varphi}^{4\sin\varphi} r\cos\varphi\, (r\sin\varphi)^2\, rdr = 0.$$

Note: The domain $\mathscr{D}$ is symetric to the axis Oy and the function $f(x,y) = xy^2$ is odd with respect to x. Therefore we have $I = 0$.

Solution 1.4. 1. Put $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \implies \begin{cases} \dfrac{\pi}{4} \le \varphi \le \dfrac{\pi}{3} \\ 4\sin\varphi \le r \le 8\sin\varphi \end{cases}$. Hence

$$I = \int\limits_{\pi/4}^{\pi/3} d\varphi \int\limits_{4\sin\varphi}^{8\sin\varphi} \frac{1}{r^4} r dr = \frac{3}{128}\left(1 - \frac{1}{\sqrt{3}}\right).$$

2. Divide $\mathscr{D}$ into two regions $\mathscr{D} = \mathscr{D}_1 \cup \mathscr{D}_2$, where

$$\mathscr{D}_1 = \left\{0 \le \varphi \le \frac{\pi}{6},\ 2\cos\varphi \le r \le 2\sqrt{3}\right\}$$
$$\mathscr{D}_2 = \left\{\frac{\pi}{6} \le \varphi \le \frac{\pi}{2},\ 2\sqrt{3}\sin\varphi \le r \le 2\sqrt{3}\right\}.$$

Then

$$I = \int\limits_0^{\pi/6} d\varphi \int\limits_{2\cos\varphi}^{2\sqrt{3}} \frac{r^2\cos\varphi\sin\varphi}{r^2} r dr + \int\limits_{\pi/6}^{\pi/2} d\varphi \int\limits_{2\sqrt{3}\sin\varphi}^{2\sqrt{3}} \frac{r^2\cos\varphi\sin\varphi}{r^2} r dr = \frac{11}{8}.$$

3. Put $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \implies \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le 1 \end{cases}$. Then

$$I = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 \sqrt{\frac{1-r^2}{1+r^2}} r dr = \frac{\pi^2}{2}.$$

4. Put $\begin{cases} x = 2r\cos\varphi \\ y = 3r\sin\varphi \end{cases} \implies |J| = 6r,\ \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le 1 \end{cases}$. Then

$$I = 6 \times 36 \int\limits_0^{2\pi} |\cos 2\varphi|\, d\varphi \int\limits_0^1 r^3 dr = 216.$$

5. Put $\begin{cases} u = xy \\ v = \dfrac{y}{x} \end{cases} \implies \begin{cases} 1 \le u \le 4 \\ 1 \le v \le 4 \end{cases}$ and $x = \sqrt{\dfrac{u}{v}},\ y = \sqrt{uv}$. Therefore

$$I = \int\limits_1^4 du \int\limits_1^4 \left[4\frac{u}{v} - 2uv\right] . \frac{1}{2v} dv = -\frac{45}{4}.$$

Solution 1.5. Divide the domain $\mathscr{D}$ into two subdomain $\mathscr{D} = \mathscr{D}_1 \cup \mathscr{D}_2$, where

$$\mathscr{D}_1 = \left\{-2 \le x \le 0,\ \ 2^{-x} \le y \le 4\right\};\ \ \mathscr{D}_2 = \left\{0 \le x \le 2,\ \ 2^{x} \le y \le 4\right\}.$$

Then

$$S = \int\limits_{-2}^{0} dx \int\limits_{2^{-x}}^{4} dy + \int\limits_{0}^{2} dx \int\limits_{2^{x}}^{4} dy = 2\left(8 - \frac{3}{\ln 2}\right).$$

Solution 1.6. We have

$$S = \int\limits_{-6a}^{0} dy \int\limits_{y^2/4a}^{3a-y} dx = \int\limits_{-6a}^{0} \left(3a - y - \frac{y^2}{4a}\right) dy = 18a^2.$$

Solution 1.7. Change to polar coordinate $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Longrightarrow \begin{cases} 0 \le \varphi \le \dfrac{\pi}{4}, \\ 2\cos\varphi \le r \le 4\cos\varphi \end{cases}.$

Hence

$$S = \int\limits_{0}^{\pi/4} d\varphi \int\limits_{2\cos\varphi}^{4\cos\varphi} rdr = \frac{3\pi}{4} + \frac{3}{2}.$$

Solution 1.8. We have

$$V = \int\limits_{0}^{1} dy \int\limits_{(1-y)/3}^{(2-2y)/3} (1 - x - y)\, dx = \frac{1}{6}\int\limits_{0}^{1} \left(1 - 2y + y^2\right) dy = \frac{1}{18}.$$

Solution 1.9. We have $V = 2V_1$, where

$$V_1 = \int\limits_{\pi/4}^{\pi/3} d\varphi \int\limits_{0}^{1} \left(1 - r^2\right) rdr = \frac{\pi}{48}.$$

Hence $V = \dfrac{\pi}{24}$.

§2. Triple Integral

2.1 Calculation of a triple integral in Cartesian coordinate system

Consider the integral

$$I = \iiint\limits_{V} f(x, y, z)dxdydz, \tag{1.10}$$

where $f(x,y,z)$ is three-variables function that is continuous on V.
If

$$V = \left\{(x,y,z) \in \mathbb{R}^3 | (x,y) \in \mathscr{D};\ \ z_1(x,y) \le z \le z_2(x,y)\right\},$$

where $\mathscr{D}$ is the projection of V on the plane Oxy and z_1, z_2 are continuous on $\mathscr{D}$ then

$$I = \iint\limits_{\mathscr{D}} dxdy \left(\int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz \right). \tag{1.11}$$

Example 2.1. *Calculate the integral* $I = \iiint\limits_V \dfrac{dxdydz}{(x+y+z)^3}$, *where* V *is bounded by the planes* $x = 0, y = 0, z = 0$ *and* $x + y + z = 1$.

Solution. V is tetrahedron bounded by two planes $z = 0$ and $z = 1 - x - y,\ \ (x,y) \in \mathscr{D}$, where $\mathscr{D}$ is the triangle OAB in the plane Oxy (Figure 2.1). Hence we have

$$\begin{aligned}
I &= \iint\limits_{\mathscr{D}} dxdy \int\limits_0^{1-x-y} \frac{dz}{(x+y+z)^3} = \iint\limits_{\mathscr{D}} \frac{(1+x+y+z)^{-2}}{-2}\Big|_{z=0}^{z=1-x-y} dxdy \\
&= -\frac{1}{2}\int\limits_0^1 dx \int\limits_0^{1-x} \left(\frac{1}{4} - \frac{1}{(1+x+y)^2}\right) dy = -\frac{1}{2}\int\limits_0^1 \left(\frac{1-x}{4} + \frac{1}{2} - \frac{1}{1+x}\right) dx \\
&= -\frac{1}{2}\int\limits_0^1 \left(\frac{3}{4} - \frac{x}{4} - \frac{1}{1+x}\right) dx = \frac{1}{2}\ln 2 - \frac{5}{16}.
\end{aligned}$$

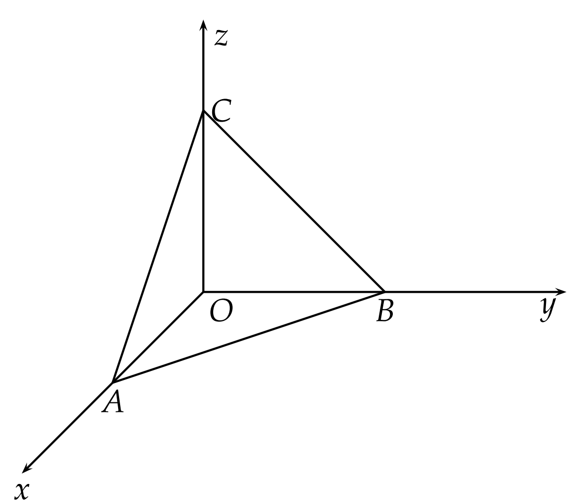

Figure 2.1

■

2.2 Change of variables in triple integrals

Consider the tranformation:

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w). \end{cases}$$

Suppose that the following conditions are satisfied:

i) $(u,v,w) \in V'$ in $O'uvw$-plane and $x(u,v,w), y(u,v,w), z(u,v,w)$ are continuous and have continuous partial derivatives on V'.

ii) The vecto-valued mapping $\Phi: \quad V' \to V$ is one-to-one.

iii) The Jacobian determinant

$$J = \frac{D(x,y,z)}{D(u,v,w)} = \begin{vmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{vmatrix} \neq 0 \text{ in } V'.$$

Then

$$I = \iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V'} f\left(x(u,v,w), y(u,v,w), z(u,v,w)\right)|J|\, dudvdw. \tag{1.12}$$

2.3 Calculate the triple integrals in cylindrical coordinate

Here we write r, φ, z for u, v, w and define the mapping by the equations:

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z. \tag{1.13}$$

In other words, we replace x and y by their polar coordinate in the plane Oxy and retain z.

Again, to get a one-to-one mapping we must keep $r > 0$ and restrict φ to be in an interval of the form: $\varphi_o \leq \varphi < \varphi_o + 2\pi$.

The Jacobian determinant of the mapping in (1.13) is

$$J = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\left(\cos^2\varphi + \sin^2\varphi\right) = r > 0$$

and therefore we have the tranformation formula

$$\iiint_V f(x,y,z)dxdydz = \iiint_{V'} f\left(r\cos\varphi, r\sin\varphi, z\right) rdrd\varphi dz. \tag{1.14}$$

Note: In some cases we use the generalized cyclindrical coordinate

$$x = ar\cos\varphi,\ \ y = br\sin\varphi,\ \ z = z$$

and $J = abr$.

Example 2.2. Transform the integral $I = \iiint_V \left(x^2 + y^2\right) dxdydz$ to cylindrical coordinate and compute its value, where V is the region bounded by the surfaces $x^2 + y^2 = 2z$ and $z = 2$.

Solution. Transform to cylindrical coordinate : $x = r\cos\varphi,\ y = r\sin\varphi, z = z,\ 0 \leq \varphi \leq 2\pi$.

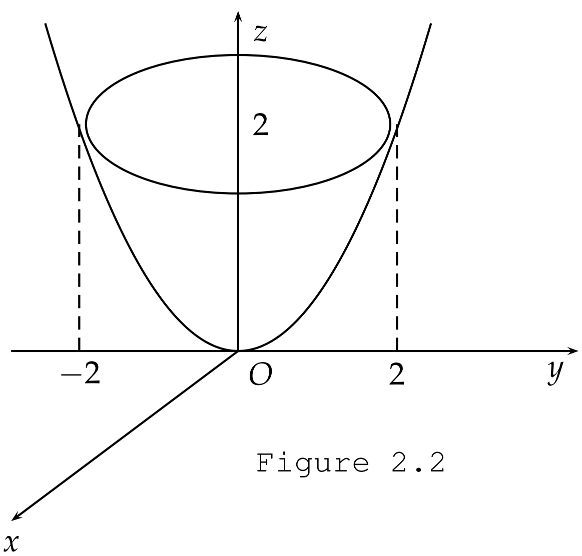

Figure 2.2

We note that the paraboloid $x^2 + y^2 = 2z$ cuts the plane $x^2 + y^2 = 4$ by the circle $x^2 + y^2 = 4$, therefore $0 \leq r \leq 2$.
On the other hand on the paraboloid we have $r^2\cos^2\varphi + r^2\sin^2\varphi = 2z \Longrightarrow z = \dfrac{r^2}{2}$.
So that in B we have $\dfrac{r^2}{2} \leq z \leq 2$.
Therefore we have

$$I = \int_0^{2\pi} d\varphi \int_0^2 dr \int_{r^2/2}^2 r^3 dz = 2\pi \int_0^2 r^3 \left(2 - \frac{r^2}{2}\right) dr = \frac{16\pi}{3}.$$

■

2.4 Calculate the triple integrals in spherical coordinate

In this case the symbols r, θ, φ are used instead of u, v, w and the mapping is defined by the equations

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \tag{1.15}$$

To get a one-to-one mapping we keep $r > 0,\ \ 0 \leq \varphi < 2\pi$ and $0 \leq \theta < \pi$.
The Jacobian determinant of the mapping is

$$J = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = -r^2\sin\theta$$

Therefore we have the tranformation formula

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V'} f\left(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta\right) r^2\sin\theta drd\theta d\varphi \tag{1.16}$$

Example 2.3. Transform the integral $I = \iiint\limits_V \sqrt{x^2+y^2+z^2}dxdydz$ to spherical coordinate and compute its value, where V is the sphere $x^2+y^2+z^2 \leq z$.

Solution. We have

$$x^2+y^2+z^2-z = x^2+y^2+\left(z-\frac{1}{2}\right)^2 - \frac{1}{4}.$$

So that $V = \left\{(x,y,z):\ \ x^2+y^2+\left(z-\frac{1}{2}\right)^2 \leq \frac{1}{4}\right\}$, i.e V is sphere whose center is the point $\left(0,0,\frac{1}{2}\right)$ and radius $R = \frac{1}{2}$.
Transform to spherical coordinate. ■

2.5 Exercises

Exercise 1.10. Calculate the following triple integrals:

1. $\iiint\limits_V zdxdydz$, where the region V is defined by $\begin{cases} 0 \leq x \leq 4 \\ x \leq y \leq 2x \\ 0 \leq z \leq \sqrt{1-x^2-y^2} \end{cases}$.

2. $\iiint\limits_V \left(x^2+y^2\right) dxdydz$, where $V: \begin{cases} x^2+y^2+z^2 \le 1 \\ x^2+y^2-z^2 \le 0 \end{cases}$.

3. $\iiint\limits_V \left(x^2+y^2\right) zdxdydz$, where $V: \begin{cases} x^2+y^2 \le 1 \\ 1 \le z \le 2 \end{cases}$.

4. $\iiint\limits_V \left(x^2+y^2\right) zdxdydz$, where

   (a) V is the region bounded by the cylinder $x^2+y^2=2x$ and the planes $z=0,\ z=a\ (a>0)$.

   (b) V is a half of the sphere $x^2+y^2+z^2 \le a^2,\ z \ge 0\ (a>0)$.

   (c) V is a half of the ellipsoid $\dfrac{x^2+y^2}{a^2}+\dfrac{z^2}{b^2} \le 1,\ z \ge 0,\ (a,b>0)$.

5. $\iiint\limits_V ydxdydz$, where V is bounded by the cone $y=\sqrt{x^2+z^2}$ and the plane $y=h,\ (h>0)$.

6. $\iiint\limits_V \left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right) dxdydz$, where V is bounded by $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1\ (a,b,c>0)$.

2.6 Solutions

Solution 1.10. 1. $I=\int\limits_1^{1/4} dx \int\limits_x^{2x} dy \int\limits_0^{\sqrt{1-x^2-y^2}} zdz = \dfrac{43}{3072}$.

2. Transform to spherical coordinate $\begin{cases} x=r\sin\theta\cos\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\theta \end{cases} \implies |J|=r^2\sin\theta$.

   We have $0 \le \varphi \le 2\pi,\ 0 \le r \le 1$ and $0 \le \theta \le \dfrac{\pi}{4}$. Hence

$$I=\int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi/4} d\theta \int\limits_0^1 r^2\sin^2\theta . r^2\sin\theta dr = \frac{2\pi}{5}.\frac{\left(8-5\sqrt{2}\right)}{12}.$$

3. Transform to cylindrical coordinate $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \Longrightarrow \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le 1 \\ 1 \le z \le 2 \end{cases}$. Hence

$$I = \int_0^{2\pi} d\varphi \int_0^1 dr \int_1^2 r^2 zrdz = \frac{3\pi}{4}.$$

4. (a) Transform to cylindrical coordinate:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \Longrightarrow \begin{cases} -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\varphi \\ 0 \le z \le a \end{cases}$$

Then

$$I = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2\cos\varphi} dr \int_0^a zrrdz = \frac{16a^2}{9}.$$

(b) Transform to cylindrical coordinate:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \Longrightarrow \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le a \\ 0 \le z \le \sqrt{a^2 - r^2} \end{cases}$$

Hence

$$I = \int_0^{2\pi} d\varphi \int_0^a dr \int_0^{\sqrt{a^2-r^2}} zrrdz = \frac{2\pi a^5}{15}.$$

(c) Transform to generalized cylindrical coordinate

$$\begin{cases} x = ar\cos\varphi \\ y = ar\sin\varphi \\ z = bz' \end{cases} \Longrightarrow |J| = abr, \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le 1 \\ 0 \le z' \le \sqrt{1 - r^2} \end{cases}.$$

Hence

$$I = \int_0^{2\pi} d\varphi \int_0^1 dr \int_0^{\sqrt{1-r^2}} bz'rabrdz' = \frac{2\pi ab^2}{15}.$$

5. Transform to cylindrical coordinate

$$\begin{cases} x = r\sin\varphi \\ z = r\sin\varphi \\ y = y \end{cases} \Longrightarrow |J| = r, \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le h \\ r \le y \le h \end{cases}.$$

Hence

$$I = \int\limits_0^{2\pi} d\varphi \int\limits_0^h dr \int\limits_r^h yrdr = \frac{\pi h^4}{4}.$$

6. Transform to generalized spherical coordinate

$$\begin{cases} x = ar\sin\theta\cos\varphi \\ y = br\sin\theta\sin\varphi \\ z = cr\cos\theta \end{cases} \Longrightarrow |J| = abcr^2\sin\theta \text{ and } \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le \theta \le \pi \\ 0 \le r \le 1 \end{cases}.$$

Hence

$$I = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} d\theta \int\limits_0^1 r^2.abc.r^2\sin\theta dr = \frac{4abc\pi}{5}.$$

CHAPTER 2

INTEGRALS THAT DEPEND ON A PARAMETER

§1. THE DEFINITE INTEGRALS THAT DEPEND ON A PARAMETER

1.1 Definition

Suppose that $f(x,y)$ is a function defined with $x \in [a,b]$ and $y \in Y$ such that for each $y \in Y$, fixed the function $f(x,y)$ is integrable in $[a,b]$.
Then

$$I(y) = \int_a^b f(x,y)dx \tag{2.1}$$

is a function that is defined on Y and called *integral that depends on a parameter* of the function $f(x,y)$ on $[a,b]$.

1.2 Properties

1. **The continuity and limitation under the integral sign**
   If function $f(x,y)$ is defined and continuous on the rectangle $\mathscr{D} = [a,b] \times [c,d]$ then the integral $I(y)$ is continuous on $[c,d]$, i.e

$$\lim_{y \to y_o \in [c,d]} I(y) = \lim_{y \to y_o} \int_a^b f(x,y)dx = \int_a^b \lim_{y \to y_o} f(x,y)dx = \int_a^b f(x,y_o)dx = I(y_o)$$

Example 1.1. Compute $\lim\limits_{y \to 0} \int\limits_0^2 x^2 \cos xy dx$.

Solution. Let $[c,d]$ be any interval that contains the point $y=0$. Then the function $f(x,y) = x^2\cos xy$ is continuous on the rectangle $\mathscr{D} = [0,2]\times[c,d]$. Therefore the integral $I(y) = \int\limits_0^2 x^2\cos xy dx$ is continuous on $[c,d]$ and we have

$$\lim_{y\to 0} I(y) = I(0) = \int\limits_0^2 x^2\cos 0 dx = \int\limits_0^2 x^2 dx = \frac{8}{3}.$$

■

2. **Differentiation under the integral sign**
   Suppose that

   i) $f(x,y)$ is defined on the rectangle $\mathscr{D} = [a,b]\times[c,d]$ and continuous with respect to $x\in[a,b]$ for each $y\in[c,d]$, fixed.

   ii) $f(x,y)$ has the partial derivative $\dfrac{\partial f(x,y)}{\partial y}$ that is continuous on $\mathscr{D}$.

   Then the integral $I(y)$ is differential function on $[c,d]$ and

$$I'(y) = \int\limits_a^b \frac{\partial f}{\partial y}(x,y)dx, \quad y\in[c,d] \quad \text{(Leibniz's rule)}.$$

Example 1.2. Compute the derivative with respect to the parameter of the integral

$$I(a) = \int\limits_0^{\pi/2} \ln\left(a^2 - \sin^2 x\right) dx, \quad (a>1).$$

Solution. The function $f(a,x) = \ln\left(a^2-\sin^2 x\right)$ is continuous in the region $a>1$ and $x\in\left[0,\dfrac{\pi}{2}\right]$ and has partial derivative with respect to a:

$$\frac{\partial f}{\partial a} = \frac{2a}{a^2-\sin^2 x}, \quad a>1$$

is continuous in that region.
Hence we can apply the Leibniz's formula to obtain

$$I'(a) = \int\limits_0^{\pi/2} \frac{2a}{a^2-\sin^2 x}dx = 2a\int\limits_0^{\pi/2} \frac{dx}{(a^2-1)+\cos^2 x}.$$

Change the variable $t = \operatorname{tg} x$ we have

$$I'(a) = 2a\int\limits_0^{+\infty} \frac{dt}{a^2+(a^2-1)t^2} = \frac{2}{\sqrt{a^2-1}}.\operatorname{arctg}\frac{\sqrt{a^2-1}}{a}t\Big|_0^{+\infty} = \frac{\pi}{\sqrt{a^2-1}}.$$

■

3. **Integration under the integral sign**

If $f(x,y)$ is defined and continuous on the rectangle $\mathscr{D} = [a,b] \times [c,d]$ then

$$\int_c^d I(y)dy = \int_c^d dy \int_a^b f(x,y)dx = \int_a^b dx \int_c^d f(x,y)dy.$$

Example 1.3. By integrating under the integral sign, compute the integral

$$I(a,b) = \int_0^1 \sin\left(\ln\frac{1}{x}\right)\frac{x^b - x^a}{\ln x}, \quad a > 0, b > 0.$$

Solution. Assume that $a < b$. Firstly, we note that

$$\frac{x^b - x^a}{\ln x} = \int_a^b x^y dy, \quad 0 < a < b.$$

Therefore we can rewrite

$$I(a,b) = \int_0^1 \sin\left(\ln\frac{1}{x}\right)\frac{x^b - x^a}{\ln x}dx = \int_0^1 dx \int_a^b x^y \sin\left(\frac{1}{x}\right) dy.$$

Let $f(x,y) = x^y \sin\left(\ln\frac{1}{x}\right)$.

Because $\lim\limits_{x\to 0} f(x,y) = \lim\limits_{x\to 0} x^y \sin\left(\ln\frac{1}{x}\right) = 0$, so that we can add the value $f(0,y) = 0$ such that the function $f(x,y)$ is continuous on the rectangle $[0,1] \times [a,b]$.

Therefor we can change the order of integration and get

$$I(a,b) = \int_a^b dy \int_0^1 x^y \sin\left(\sin\frac{1}{x}\right) dx.$$

Change the variable $x = e^{-t}$, we have

$$I(a,b) = \int_a^b dy \int_0^{+\infty} e^{-t(1+y)} \sin t dt,$$

where

$$\int_0^{+\infty} e^{-t(1+y)} \sin t dt = \frac{-(1+y)\sin t - \cos t}{1+(1+y)^2} e^{-t(1+y)}\Big|_0^{+\infty} = \frac{1}{1+(1+y)^2}.$$

In conclusion we have

$$I(a,b) = \int_a^b \frac{dy}{1+(1+y)^2} = \operatorname{arctg}(b+1) - \operatorname{arctg}(a+1) = \operatorname{arctg}\frac{b-a}{1+(1+a)(1+b)}. \quad \blacksquare$$

§2. The generalized integarls that depend on a parameter

2.1 The uniformly convergent integrals

Consider the integral

$$I(y) = \int_a^{+\infty} f(x,y)dx, \quad y \in Y. \tag{2.2}$$

We say that $I(y)$ is uniformly convergent if:

i) For each $y \in Y$, fixed the integral $\int\limits_a^{+\infty} f(x,y)dx$ is convergent.

ii) $\forall \varepsilon > 0, \exists A_o = A_o(\varepsilon) > a$ (only depends on ε) such that $\forall A > A_o$ we have

$$\left| \int_A^{+\infty} f(x,y)dx \right| < \varepsilon, \quad \forall y \in Y.$$

Some uniformly convergent criteria

1. *Cauchy's criterion*
   The necessary and sufficient condition for the integral (2.2) is uniformly convergent on Y is:
   $\forall \varepsilon > 0, \exists A_o = A_o(\varepsilon)$ such that for all $A', A'' > A_o$ we have

$$\left| \int_{A'}^{A''} f(x,y)dx \right| < \varepsilon, \quad \forall y \in Y.$$

2. *Weierstrass' criterion*
   Suppose that $\varphi(x)$ is nonnegative function on $[a, +\infty)$ such that

$$|f(x,y)| \leq \varphi(x), \quad \forall x \in [a, +\infty), \quad \forall y \in Y.$$

   Then if the integral $\int\limits_a^{+\infty} \varphi(x)dx$ is convergent then the integral $\int\limits_a^{+\infty} f(x,y)dx$ is uniformly convergent on Y.

2.2 Properties

1. **The continuity and limitation under the integral sign**
   Suppose that the function $f(x,y)$ is defined and continuous on $[a,+\infty)\times[c,d]$. Then if the integral (2.2) is uniformly convergent on $[c,d]$ then $I(y)$ is continuous on $[c,d]$, i.e

$$\lim_{y\to y_o\in[c,d]} I(y) = \lim_{y\to y_o}\int_a^{+\infty} f(x,y)dx = \int_a^{+\infty}\lim_{y\to y_o} f(x,y)dx = \int_a^{+\infty} f(x,y_o)dx = I(y_o)$$

2. **Differentiation under the integral sign**
   Suppose that

   i) $f(x,y)$ is continuous with respect to $x\in[a,+\infty)$.

   ii) $f(x,y)$ has the partial derivative $\dfrac{\partial f(x,y)}{\partial y}$ that is continuous on $[a,+\infty)\times[c,d]$.

   iii) The integral $\displaystyle\int_a^{+\infty}\frac{\partial f}{\partial y}(x,y)dx$ is uniformly convergent on $[c,d]$.

   Then the integral $I(y)=\displaystyle\int_a^{+\infty} f(x,y)dx$ is differential on $[c,d]$ and

$$I^{'}(y) = \int_a^{+\infty}\frac{\partial f}{\partial y}(x,y)dx,\ \ y\in[c,d].$$

3. **Integration under the integral sign**
   If $f(x,y)$ is defined and continuous on the rectangle $\mathscr{D}=[a,b]\times[c,d]$ then

$$\int_c^d I(y)dy = \int_c^d dy\int_a^b f(x,y)dx = \int_a^b dx\int_c^d f(x,y)dy.$$

2.3 Euler's integrals

1. **The Gamma function Γ**

$$\Gamma(a) = \int_0^{+\infty} x^{a-1}e^{-x}dx,\ \ a>0. \tag{2.3}$$

   The differentiation formula:

$$\Gamma^{(k)}(a) = \int_0^{+\infty} x^{a-1}e^{-x}\left(\ln x\right)^k dx,\ \ k\in\mathbb{N}$$

   Some basic properties

i) $\Gamma(a+1) = a\Gamma(a)$

ii) $\Gamma(n+1) = n!, \ \ n \in \mathbb{N}$

iii) $\Gamma\left(n+\dfrac{1}{2}\right) = \dfrac{(2n-1)!!}{2^n}.\sqrt{\pi}, \ \ n \in \mathbb{N}$

iv) $\Gamma(a).\Gamma(1-a) = \dfrac{\pi}{\sin \pi a}, \ \ 0 < a < 1.$

2. **The Beta function** B

$$B(a,b) = \int\limits_0^1 x^{a-1}(1-x)^{b-1}dx, \ \ a > 0, b > 0. \tag{2.4}$$

Some basic properties

i) $B(a,b) = B(b,a) \ \ \forall a > 0, b > 0$

ii) $B(a,b) = \dfrac{b-1}{a+b-1}B(a,b-1), \ \ a > 0, b > 1$

iii) $B(a,1-a) = \dfrac{\pi}{\sin \pi a}, \ \ 0 < a < 1.$

§3. Exercises

Exercise 2.1. Consider the continuity of the integral $I(y) = \int\limits_0^1 \dfrac{yf(x)}{x^2+y^2}dx$, where $f(x) > 0$ and is continuous on $[0,1]$.

Exercise 2.2. Compute the following integrals

1. $\displaystyle\int\limits_0^1 \frac{x^b - x^a}{\ln x}dx, \ \ (0 < a < b);$

2. $\displaystyle\int\limits_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}dx, \ \ (\alpha > 0, \beta > 0);$

3. $\displaystyle\int\limits_0^{+\infty} e^{-ax}\frac{\sin(bx) - \sin(cx)}{x}dx, \ \ (a,b,c > 0);$

4. $\displaystyle\int\limits_0^{+\infty} e^{-x^2}\cos(yx)dx.$

Exercise 2.3. Express the integral $\int\limits_0^{\pi/2} \sin^m x \cos^n x dx$ through the function $B(m,n)$ $(m,n \in \mathbb{Z}, m,n > 1)$.

Exercise 2.4. Compute the following integrals

1. $\int\limits_0^{\pi/2} \sin^6 x \cos^4 x dx$;

2. $\int\limits_0^{a} x^{2n}\sqrt{a^2-x^2}dx \quad (a>0)$ (Suggest: put $x = a\sqrt{t}$);

3. $\int\limits_0^{+\infty} x^{10}e^{-x^2}dx$;

4. $\int\limits_0^{+\infty} \dfrac{dx}{1+x^3}$.

§4. Solutions

Solution 2.1.
- With $y \neq 0$, the function $g(x,y) = \dfrac{yf(x)}{x^2+y^2}$ is continuous on each rectangle $[0,1]\times[c,d]$ or $[0,1]\times[-d,-c],\ 0<c<d$.
  Because c can be arbitrarily small and d can be arbitrarily great so that $I(y)$ is continuous when $y \neq 0$.

- With $y = 0$.
  Because $f(x) > 0, \forall x \in [0,1]$ so that $\forall m > 0$ such that $f(x) \geq m > 0,\ \ \forall x \in [0,1]$.
  Therefore $\forall \varepsilon > 0$ we have

$$I(\varepsilon) = \int\limits_0^1 \frac{\varepsilon f(x)}{x^2+\varepsilon^2}dx \geq \int\limits_0^1 \frac{m\varepsilon}{x^2+\varepsilon^2}dx = m\,\mathrm{arctg}\,\frac{1}{\varepsilon}$$

$$I(-\varepsilon) = \int\limits_0^1 \frac{-\varepsilon f(x)}{x^2+\varepsilon^2}dx \leq m\,\mathrm{arctg}\,\frac{1}{\varepsilon}$$

$$\Longrightarrow |I(\varepsilon) - I(-\varepsilon)| \geq 2m\,\mathrm{arctg}\,\frac{1}{\varepsilon} \to 2m.\frac{\pi}{2} \text{ khi } \varepsilon \to 0.$$

  Hence $I(y)$ is interrupted at $y = 0$.

Solution 2.2. 1. Consider the function $f(x,y) = x^y$, we have $f(x,y)$ is continuous on $[0,1]\times[c,d]$.

The integral $\int\limits_0^1 x^y dx$ is uniformly convergent because $x^y \le x^b$; $\int\limits_0^1 x^b dx = \dfrac{1}{b+1}$.

Hence

$$I = \int\limits_0^1 dx \int\limits_a^b x^y dy = \int\limits_a^b dy \int\limits_0^1 x^y dx = \ln\frac{b+1}{a+1}.$$

2. Put $F(x,y) = \dfrac{e^{-xy}}{x}$, we have

$$\frac{e^{-\alpha x} - e^{-\beta x}}{x} = F(x,\alpha) - F(x,\beta) = -\int\limits_\alpha^\beta F'_y(x,y)dy = \int\limits_\alpha^\beta e^{-yx}dy$$

Put $f(x,y) = e^{-yx}$, check the uniformly convergent conditions:

i) $f(x,y)$ is continuous on $[0,+\infty)\times[\alpha,\beta]$;

ii) $I(y) = \int\limits_0^{+\infty} e^{-yx}dx$ is uniformly convergent on $[\alpha,\beta]$ as

$$e^{-yx} \le e^{-\alpha x}, \forall (x,y) \in [0,+\infty)\times[\alpha,\beta]$$

and $\int\limits_0^\infty e^{-\alpha x}dx = \dfrac{1}{\alpha}$ - convergent.

Hence

$$I = \int\limits_0^\infty dx \int\limits_\alpha^\beta e^{-yx}dy = \int\limits_\alpha^\beta dy \int\limits_0^\infty e^{-yx}dx = \int\limits_\alpha^\beta \frac{dy}{y} = \ln\frac{\beta}{\alpha}.$$

3. Put $F(x,y) = \dfrac{e^{-ax}\sin yx}{x}$, we have

$$e^{-ax}\frac{\sin bx - \sin cx}{x} = F(x,b) - F(x,c) = \int\limits_c^b F'_y dy.$$

Put $f(x,y) = e^{ax}\cos yx$, we have

i) $f(x,y)$ is continuous on $[0,+\infty)\times[c,b]$;

ii) $\int\limits_0^\infty e^{-ax}\cos yx dx$ is uniformly convergent on $[c,b]$ because $|e^{-ax}\cos yx| \le e^{-ax}$ and the integral $\int\limits_0^\infty e^{-ax}dx$ is convergent.

Hence

$$I=\int_0^{+\infty}dx\int_c^b \cos yx dx=\int_c^b dy\int_0^{\infty}e^{-ax}\cos yx dx=\int_c^b \frac{a}{a^2+y^2}dy=\operatorname{arctg}\frac{b}{a}-\operatorname{arctg}\frac{c}{a}.$$

Solution 2.3. Put $\sin x=\sqrt{t},\;\; 0<t\le 1\Longrightarrow \cos x dx=\dfrac{1}{2\sqrt{t}}dt.$
Hence

$$I=\int_0^{\pi/2}\sin^m x\left(1-\sin^2 x\right)^{n-1/2}\cos x dx=\frac{1}{2}\int_0^1 t^{m/2}\left(1-t\right)^{n-1/2}.t^{-1/2}dt$$
$$=\frac{1}{2}B\left(\frac{m+1}{2},\frac{n+1}{2}\right).$$

Solution 2.4. 1. Use the result of exercise 2.3 we have

$$I=\int_0^{\pi/2}\sin^6 x\cos^4 x dx=\frac{1}{2}B\left(\frac{7}{2},\frac{5}{2}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{7}{2}\right).\Gamma\left(\frac{5}{2}\right)}{\Gamma(6)}=\frac{1}{2}B\left(\frac{7}{2},\frac{5}{2}\right)=$$
$$=\frac{1}{2}\frac{\Gamma\left(3+\frac{1}{2}\right).\Gamma\left(2+\frac{1}{2}\right)}{\Gamma(5+1)}=\frac{1}{2}\frac{\frac{5!!}{2^3}.\sqrt{\pi}.\frac{3!!}{2^2}\sqrt{\pi}}{5!}=\frac{3\pi}{512}.$$

2. Put $x=a\sqrt{t}\Longrightarrow dx=\dfrac{adt}{2\sqrt{t}}$. Then

$$I=\int_0^1 a^{2n}t^n a\left(1-t\right)^{1/2}a\frac{1}{2}t^{-1/2}dt=\frac{a^{2n+2}}{2}B\left(n+\frac{1}{2},\frac{3}{2}\right)=\frac{\pi a^{2n+2}}{2}.\frac{(2n-1)!!}{(2n+2)!!}$$

3. Put $x=\sqrt{t},t\ge 0$ we have $dx=\dfrac{dt}{2\sqrt{t}}$ and

$$I=\int_0^{+\infty}t^5e^{-t}\frac{1}{2}t^{-1/2}dt=\frac{1}{2}\Gamma\left(\frac{11}{2}\right)=\frac{9!!\sqrt{\pi}}{2^6}.$$

4. Put $x^3=t\Longrightarrow dx=\dfrac{1}{3}t^{-2/3}dt$. Hence

$$I=\int_0^{+\infty}\frac{1}{3}\frac{t^{-2/3}dt}{1+t}=\frac{1}{3}B\left(\frac{1}{3},\frac{2}{3}\right)=\frac{2\pi}{3\sqrt{3}}.$$

CHAPTER 3

LINE INTEGRAL

§1. LINE INTEGRAL OF THE FIRST KIND

1.1 Definition

Assume that $f(x, y)$ is a function of two variables which is defined in a plane curve AB. Divide AB into n sub-curves $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. Take an arbitrary point M_i in the curve Δs_i. The limit, if exists, of the sum $\sum\limits_{i=1}^{n} f(M_i)\Delta s_i$ when $n \to \infty$ such that $\max\limits_{1\le i\le n} d(\Delta s_i) \to 0$, is called the line integral of the first kind (path integral) of the function $f(x, y)$ over the line AB. It is denoted by

$$\int\limits_{AB} f(M)ds$$

1.2 Calculation formulae

a) If AB is given by the equation $y = y(x), a \le x \le b$, then

$$\int\limits_{AB} f(x,y)ds = \int\limits_{a}^{b} f(x, y(x))\sqrt{1 + y'^2(x)}dx \tag{1}$$

b) If AB is given by the equation $x = x(y), c \le y \le d$, then

$$\int\limits_{AB} f(x,y)ds = \int\limits_{c}^{d} f(x(y), y)\sqrt{1 + x'^2(y)}dy \tag{2}$$

c) If AB is given by the equation $x = x(t), y = y(t), t_1 \leq t \leq t_2$, then

$$\int\limits_{AB} f(x,y)ds = \int\limits_{t_1}^{t_2} f(x(t),y(t))\sqrt{x'^2(t)+y'^2(t)}dt \tag{3}$$

Example 1.1. Calculate the following line integrals of the first kind

i) $\int\limits_C (x+y)ds$, where C is the circumference of the triangle OAB whose vertices are $O(0,0), A(1,0)$ and $B(0,1)$.

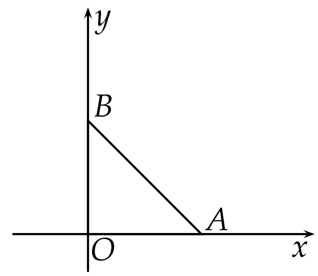

Figure 3.1

$$I = \int\limits_C (x+y)ds = \int\limits_{OA} (x+y)ds + \int\limits_{AB} (x+y)ds + \int\limits_{BO} (x+y)ds$$

- In the line $OA : y = 0, 0 \leq x \leq 1$, hence

$$\int\limits_{OA} (x+y)ds = \int\limits_0^1 (x+0)dx = \frac{1}{2}$$

- In the line $AB : y = 1-x, 0 \leq x \leq 1$, hence

$$\int\limits_{AB} (x+y)ds = \int\limits_0^1 \sqrt{2}dx = \sqrt{2}$$

- In the line $BO : x = 0, 0 \leq y \leq 1$, hence $ds = dy$

$$\int\limits_{BO} (x+y)ds = \int\limits_0^1 ydy = \frac{1}{2}$$

We conclude that $I = 1+\sqrt{2}$.

ii) $\int\limits_C (x-y)ds$, where C is the circle $x^2+y^2 = 2x$.

Solution 1. We can divide C into two curve $C_1 : y = \sqrt{2x-x^2}$ and $C_2 : y = -\sqrt{2x-x^2}$, then apply the formula (1) to integrate $\int\limits_{C_1} (x-y)ds, \int\limits_{C_2} (x-y)ds$.

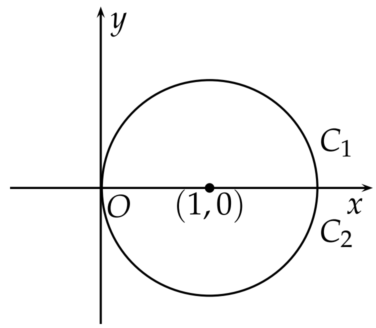

Figure 3.2

Solution 2. We parameterize this circle by $x = 1 + \cos t, y = \sin t, 0 \leq t \leq 2\pi$.

Then $x'^2(t) + y'^2(t) = 1$ and apply formula (3), we obtain

$$\int_C (x - y)ds = \int_0^{2\pi} (1 + \cos t - \sin t)dt = 2\pi$$

§2. Line integral of the second kind

2.1 Definition

Assume that $P(x, y)$ and $Q(x, y)$ are functions of two variables which are defined in a plane curve AB. Divide AB into n sub-curves $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$ whose initial points are $A_0 \equiv A, A_1, \ldots, A_n \equiv B$. A vector $\overrightarrow{A_{i-1}A_i}$ has coordinate $\overrightarrow{A_{i-1}A_i} = \Delta s_i = (\Delta x_i, \Delta y_i)$. Take an arbitrary point M_i in the curve Δs_i . The limit, if exists, of the sum $\sum\limits_{i=1}^{n} [P(M_i)\Delta x_i + Q(M_i)\Delta y_i]$ when $n \to \infty$ such that $\max\limits_{1 \leq i \leq n} \{\Delta x_i, \Delta y_i\} \to 0$, is called the line integral of the second kind of the functions $P(x, y)dx + Q(x, y)dy$ over the line AB. It is denoted by

$$\int_{AB} Pdx + Qdy$$

2.2 Calculation formulae

a) If AB is given by the equation $y = y(x)$; the initial and the end points correspond to $x = a$ and $x = b$ respectively, then

$$\int_{AB} Pdx + Qdy = \int_a^b [P(x, y(x)) + Q(x, y(x))y'(x)]dx \qquad (4)$$

If AB is given by the equation $x = x(y)$; the initial and the end points correspond to $y = c$ and $y = d$ respectively, then

$$\int_{AB} Pdx + Qdy = \int_{c}^{d} [P(x(y),y)x'(y) + Q(x(y),y)]dy \tag{5}$$

If AB is given by the equation $x = x(t), y = y(t)$; the initial and the end points correspond to $t = t_1$ and $t = t_2$ respectively, then

$$\int_{AB} Pdx + Qdy = \int_{t_1}^{t_2} [P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)]dt \tag{6}$$

b) Green's formula:

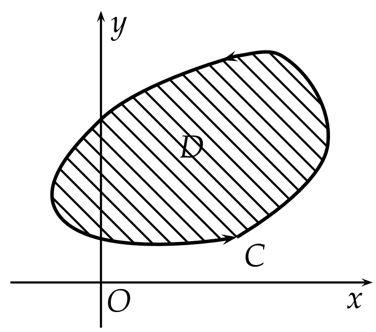

Figure 3.3

Assume that the curve C is closed and restricts a domain D, and when going along C, one will see the domain D on the left. Furthermore, suppose that the functions P, Q together with their partial derivatives are continuous on $\overline{D}$, then

$$\int_{C} Pdx + Qdy = \iint_{D} \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)dxdy$$

The direction of C defined as above is called positive direction. Inverse direction is called negative one.

c) If $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ then the integral $\displaystyle\int_{AB} Pdx + Qdy$ does not depend on the path from A to B. We will choose a special path to calculate $\displaystyle\int_{AB} Pdx + Qdy$.

Example 2.1. Calculate the following line integrals of the second kind

i) $\displaystyle\int_{C} (x^2 + 2y)dx - (x - y)dy$, where C is the parabola $x^2 = 2y + 1$ from $A(1,0)$ to $B(-3,5)$.

$y = \dfrac{x^2 - 1}{2}$, then $dy = xdx$.

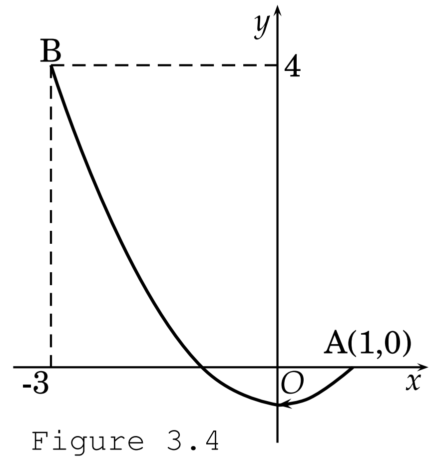

Figure 3.4

We have

$$\int_C (x^2+2y)dx-(x-y)dy = \int_1^{-3}\Big[x^2+x^2-1-\Big(x-\frac{x^2-1}{2}\Big)x\Big]dx$$
$$= \frac{1}{2}\int_1^{-3}\Big(x^3+2x^2-x-2\Big)dx = \frac{1}{2}\Big(\frac{x^4}{4}+\frac{2x^3}{3}-\frac{x^2}{2}-2x\Big)\Big|_1^{-3}$$
$$= \frac{8}{3}$$

ii) $\int_C (x+y)dx+(y-x)dy$, where C is the cycloid $x=t-\sin t, y=1-\cos t, 0\le t\le 2\pi$ whose direction is the increasing direction of the parameter t.

$dx=(1-\cos t)dt, dy=\sin t dt$, we have

$$I=\int_0^{2\pi}\Big[(t-\sin t+1-\cos t)(1-\cos t)+(1-\cos t-t+\sin t)\sin t\Big]dt$$
$$=\int_0^{2\pi}\Big[t-t\cos t-t\sin t+2-2\cos t\Big]dt = \int_0^{2\pi}\Big[t-t\cos t-t\sin t+2\Big]dt$$
$$=\Big(\frac{t^2}{2}-t\sin t-\cos t+t\cos t-\sin t+2t\Big)\Big|_0^{2\pi} = 2\pi^2+6\pi$$

Example 2.2. Calculate the following line integrals of the second kind

i) $\int_C y^2dx-(x^2y-x^3)dy$, where C is the circle $x^2+y^2=4x$ with positive direction.

Because the circle is closed and the functions $P=y^2$, $Q=x^3-x^2y$ are continuous in

$\mathbb{R}^2$, we can apply Green's formula

$$\int\limits_C y^2dx - (x^2y - x^3)dy = \iint\limits_{x^2+y^2\leq 4x} \Big(3x^2 - 2xy - 2y\Big)dxdy$$
$$= 3\iint\limits_{x^2+y^2\leq 4x} x^2dxdy$$

(because $-2xy - 2y$ is odd function with respect to y and the domain of integration is symmetric to the Ox axis, $\iint\limits_{x^2+y^2\leq 4x}(-2xy-2y)dxdy = 0$).

Set $x = 2 + r\cos\varphi, y = r\sin\varphi$ then

$$I = 3\int\limits_{x^2+y^2\leq 4x} x^2dxdy = 3\int\limits_0^2 dr\int\limits_0^{2\pi}(2+r\cos\varphi)^2 rdrd\varphi$$
$$= 3\int\limits_0^2 2\pi\Big(4r + \frac{r^3}{2}\Big)dr = 60\pi$$

ii) $\int\limits_C (y\cos(xy) - 3x^2y)dx + (x\cos(xy) + 2x)dy$, where C is the semi-circle $x = \sqrt{1-y^2}$ from $A(0,1)$ to $B(0,-1)$.

$P = y\cos(xy) - 3x^2y; Q = x\cos(xy) + 2x.$

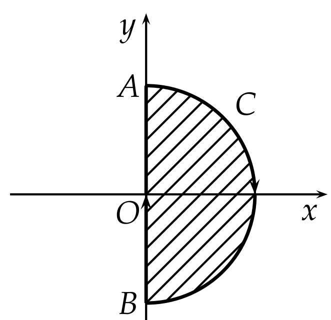

Figure 3.5

The integrating curve is not closed. Add to both sides of the observing integral the integral along the line BOA, we obtain

$$I + \int\limits_{BOA} Pdx + Qdy = \int\limits_L Pdx + Qdy$$

where $L = C \cup BOA$ is a closed curve with negative direction which restricts the domain $D : x^2 + y^2 \leq 1, x \geq 0$.

Apply Green's formula to the right-hand side we have

$$\begin{aligned} I + \int\limits_{BOA} Pdx + Qdy &= -\iint\limits_{D} (2+3x^2)dxdy \\ &= -\frac{1}{2} \iint\limits_{x^2+y^2\le 1} (2+3x^2)dxdy \\ &= -\frac{1}{4} \iint\limits_{x^2+y^2\le 1} (4+3x^2+3y^2)dxdy \\ &= -\frac{1}{4} \int\limits_0^{2\pi} d\varphi \int\limits_0^1 (4+3r^2)rdr = -\frac{11\pi}{8} \end{aligned}$$

In the line BOA: $x = 0$ then $dx = 0, Q = 0$, so $\int\limits_{BOA} Pdx + Qdy = 0$.

In conclusion we have $I = -\dfrac{11\pi}{8}$.

2.3 Theorem of four equivalent propositions

Assume that D is a simply connected domain, P, Q together with their partial derivatives are continuous functions in $\overline{D}$. The four following propositions are equivalent

1. $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ for all $(x, y) \in D$.
2. $\int\limits_L Pdx + Qdy = 0$ for all closed curve L lying in D.
3. $\int\limits_{AB} Pdx + Qdy = 0$ does not depend on the path from A to B, for all the paths L and A, B lying in D.
4. $Pdx + Qdy$ is an exact integrand. That means there exists a function $u(x, y)$ such that $du = Pdx + Qdy$. u can be found by the following formulae

$$u(x,y) = \int\limits_{x_0}^{x} P(x,y_0)dx + \int\limits_{y_0}^{y} Q(x,y)dy = \int\limits_{x_0}^{x} P(x,y)dx + \int\limits_{y_0}^{y} Q(x_0,y)dy$$

Example 2.3. For which α the integral $\int\limits_{AB} \dfrac{xdy - ydx}{(x^2+y^2)^\alpha}$ does not depend on the path from $A(0,-1)$ to $B(0,1)$ which lies in the plane $x < 0$. For this α calculate the line integral $\int\limits_{AB} \dfrac{xdy - ydx}{(x^2+y^2)^\alpha}$.

$P = \dfrac{-y}{(x^2+y^2)^\alpha}, Q = \dfrac{x}{(x^2+y^2)^\alpha}$.

The integral does not depend on the path if and only if

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \text{ for all } (x,y), x<0$$

$$\Leftrightarrow \frac{(x^2+y^2)^\alpha - 2\alpha x^2(x^2+y^2)^{\alpha-1}}{(x^2+y^2)^{2\alpha}} = -\frac{(x^2+y^2)^\alpha - 2\alpha y^2(x^2+y^2)^{\alpha-1}}{(x^2+y^2)^{2\alpha}} \forall (x,y), x<0$$

$$\Leftrightarrow \alpha = 1$$

For $\alpha = 1$, we choose a special path from $A(0,-1)$ to $B(0,1)$, it is the circle $x = \cos t, y = \sin t$, where t is from $\dfrac{3\pi}{2}$ to $\dfrac{\pi}{2}$.

$$\int\limits_{AB} \frac{xdy - ydx}{x^2+y^2} = \int\limits_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{\cos t d(\sin t) - \sin t d(\cos t)}{\sin t^2 + \cos t^2} = \int\limits_{\frac{3\pi}{2}}^{\frac{\pi}{2}} dt = -\pi$$

2.4 Area of a plane domain

Assume that D is the domain restricted by a closed curve C. The area of D is

$$S = \frac{1}{2}\oint\limits_C (xdy - ydx)$$

§3. Exercises

Exercise 3.1. Calculate the following line integrals of the first kind

a) $\int\limits_C xy^2 ds$, where C is the curve $x = \sqrt{1-y^2}$, $-1 \le y \le 1$.

b) $\int\limits_C \sqrt{x^2+y^2} ds$, where C is the circle $x^2+y^2 = ax$.

c) $\int\limits_C (x^{\frac{4}{3}} + y^{\frac{4}{3}}) ds$, where C is the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

d) $\int\limits_C xyds$, where C is the hyperbol $x = a\,\mathrm{ch}t, y = a\mathrm{sh}t$, $(0 \le t \le 1)$.

e) $\int\limits_C (x^2+y^2+z^2) ds$, where C is the helix $x = a\cos t, y = a\sin t, z = bt$, $(0 \le t \le 2\pi)$.

Exercise 3.2. Calculate the following line integrals of the second kind

a) $\int\limits_C ydx + x^2 dy$, where C is the curve $x = \sqrt{1-y^2}$ from $(0,1)$ to $(0,-1)$.

b) $\int\limits_C (x^2+y^2)dx + (x^2-y^2)dy$, where C is the path $y = 1 - |1-x|$, $0 \le x \le 2$, whose direction is increasing direction of the variable x.

c) $\int\limits_C \dfrac{(x+y)dx - (x-y)dy}{x^2+y^2}$, where C is the circle $x^2+y^2 = a^2$, whose direction is counter-clockwise.

d) $\int\limits_L ydx + (2+3x)dy$, L is the curve $\begin{cases} x = 1-\cos t \\ y = t - \sin t \end{cases}$ $,0 \le t \le 2\pi$, whose direction is increasing direction of the parameter t.

Exercise 3.3. Use Green's formula to calculate the following line integrals of the second kind

a) $\oint\limits_C xy^2dy - x^2ydx$, where C is the circle $x^2+y^2=a^2$.

b) $\oint\limits_{x^2+y^2=\pi} \cos(x^2+y^2)\left[x^4dy + (y^3+2y^2)dx\right]$.

c) $\oint\limits_C \dfrac{(\sin x - y)dx + (x+\sin y)dy}{x^2+y^2}$, where C is the circle $x^2+y^2=1$ with positive direction.

d) $\int\limits_{OA} e^y[(\sin x - 1)dx + (1-\cos x)dy]$, where OA is the curve $x = \sin y$ from $O(0,0)$ to $A(0,\pi)$.

e) $\int\limits_C [xy^4 + x^2 - ye^{xy}]dx + [\dfrac{x^3}{3} + xy^2 - x - xe^{xy}]dy$, where C is the semi-circle $x^2+y^2 = 1, y \le 0$ from $A(-1,0)$ to $B(1,0)$.

f) $\oint\limits_C e^{-x}\arcsin(xy)dx + e^y\arccos(xy)dy$, where C is the curve $|x|+|y|=1$.

g) $\oint\limits_{C_a} \dfrac{(x+y)dx-(x-y)dy}{x^2+y^2}$ for an arbitrary number $a>0$, where C_a is the ellip $ax^2 + y^2 = 1$.

Exercise 3.4. Check out that the elements of the integration are exact integrands, then calculate the following line integrals

a) $\int\limits_{(1,\pi)}^{(2,2\pi)} \left(1 - \dfrac{y^2}{x^2}\cos\dfrac{y}{x}\right)dx + \left(\sin\dfrac{y}{x} + \dfrac{y}{x}\cos\dfrac{y}{x}\right)dy$, along the curves lying in the plane $x > 0$.

b) $\int\limits_{(0,1)}^{(6,8)} \dfrac{xdx+ydy}{\sqrt{x^2+y^2}}$, along the curves lying in the plane $y>0$.

c) $\int\limits_{(2,1)}^{(1,2)} \dfrac{ydx - xdy}{x^2}$ along the curves which do not intersect with the axis Oy.

d) $\displaystyle\int\limits_{(2,-1,0)}^{(1,1,1)} 2xdx - 3y^2dy - 4z^3dz.$

Exercise 3.5.

a) Find a, b such that the integral

$$\int\limits_{AB} [axy^3 - 5y^2 + y\cos xy]dx + [3x^2y^2 + bxy + x\cos xy]dy$$

does not depend on the line from A to B. Find the function $u(x, y)$ such that the integrand is du.

b) Find a function $h(x)$ such that the integral

$$\int\limits_{AB} h(x)[(x\sin y + y\cos y)dx + (x\cos y - y\sin y)dy]$$

does not depend on the line from A to B. Use the found function $h(x)$ calculate the above integral when the points $A(0, \pi)$ and $B(1, 2\pi)$.

c) Find a function $h(y)$ such that the integral

$$\int\limits_{AB} h(y)[y(2x + y^3)dx - x(2x - y^3)dy]$$

does not depend on the line from A to B. Use the found function $h(y)$ calculate the above integral when the points $A(0, 1)$ and $B(-3, 2)$.

d) Find a function $h(xy)$ such that the integral

$$\int\limits_{AB} h(xy)[2y(x^3 - y^3)dx + x(y^3 - 4x^3)dy]$$

does not depend on the line from A to B. Use the found function $h(y)$ calculate the above integral when the points $A(1, 1)$ and $B(\frac{1}{2}, 2)$.

Exercise 3.6. Calculate the area of the following plane domain

a) The domain D is restricted by the vertical aixs Oy and the curve $x = x(y)$ whose representation is $x = a(1 - \cos t), y = a(t - \sin t), 0 \le t \le 2\pi$.

b) The domain D is defined by $x^2 + \frac{y^2}{4} \le 1, x \ge 0$.

§4. SOLUTION

Solution 3.1.

a) $x'(y) = -\dfrac{y}{\sqrt{1-y^2}}$, then $ds = \dfrac{dx}{x}$, $\int\limits_C xy^2 ds = \int\limits_{-1}^{1} y^2 dy = \dfrac{2}{3}$

b) $x = \dfrac{a}{2}(1+\cos t), y = \dfrac{a}{2}\sin t, 0 \le t \le 2\pi.$

$$\int\limits_C \sqrt{x^2+y^2}ds = \frac{a^2}{2}\int\limits_0^{2\pi} |\cos\frac{t}{2}|dt = 2a^2$$

c) $x = a\cos^3 t, y = a\sin^3 t, 0 \le t \le 2\pi$

$$\int\limits_C (x^{\frac{4}{3}} + y^{\frac{4}{3}})ds = 3a^{\frac{7}{3}}\int\limits_0^{2\pi}(\cos^4 t + \sin^4 t)|\sin t\cos t|dt = 4a^{\frac{7}{3}}$$

d) $\int\limits_C xyds = \int\limits_0^1 a^2\dfrac{\mathbf{sh}2t}{2}a\sqrt{\mathbf{ch}2t}dt = \dfrac{a^3}{6}\left[(\mathbf{ch}2)^{\frac{3}{2}} - 1\right]$

e) $\int\limits_C (x^2+y^2+z^2)ds = \int\limits_0^{2\pi}(a^2+b^2t^2)\sqrt{a^2+b^2}dt = \sqrt{a^2+b^2}\left(2\pi a^2 + \dfrac{8}{3}\pi^3 b^2\right)$

Solution 3.2.

a) $x = \sqrt{1-y^2}$ from $(0,1)$ to $(0,-1)$.
Set $x = \cos t, y = \sin t$, t is from $\dfrac{\pi}{2}$ to $-\dfrac{\pi}{2}$, and

$$\int\limits_C ydx + x^2dy = \int\limits_{\frac{\pi}{2}}^{-\frac{\pi}{2}}(-\sin^2 t + \cos^3 t)dt = \frac{3\pi - 10}{6}$$

b) $OA : 0 \le x \le 1, y = x$: $\int\limits_0^1 2x^2dx = \dfrac{2}{3}$

$AB : 1 \le x \le 2, y = 2-x$: $\int\limits_1^2 2(2-x)^2dx = \dfrac{2}{3}$

$$\int\limits_C (x^2+y^2)dx + (x^2-y^2)dy = \frac{4}{3}$$

c) $x = a\cos t, y = a\sin t$, t is from 2π to 0 (counter-clockwise direction).

$$\int\limits_C \frac{(x+y)dx - (x-y)dy}{x^2+y^2} = \int\limits_{2\pi}^{0}(-dt) = 2\pi$$

d) $\int\limits_{L} ydx + (2+3x)dy = \int\limits_{0}^{2\pi} [(t-\sin t)\sin t + (2+3-3\cos t)(1-\cos t)]dt = 10\pi$

Solution 3.3.

a) $\oint\limits_{C} xy^2dy - x^2ydx = \iint\limits_{x^2+y^2\le a^2} (x^2+y^2)dxdy = \int\limits_{0}^{a} r^3dr \int\limits_{0}^{2\pi} d\varphi = \frac{\pi}{2}a^4$

b) $I_b = -\oint\limits_{x^2+y^2=\pi} \left[x^4dy + (y^3+2y^2)dx\right] = -\iint\limits_{x^2+y^2\le\pi} (4x^3-3y^2-4y)dxdy$

$I_b = 3\iint\limits_{x^2+y^2\le\pi} y^2dxdy = \frac{3}{2}\iint\limits_{x^2+y^2\le\pi} (x^2+y^2)dxdy = \frac{3}{4}\pi^5$

c) $I_c = \oint\limits_{C} (\sin x - y)dx + (x+\sin y)dy = 2\iint\limits_{x^2+y^2\le 1} dxdy = 2\pi$

d) $I_d + \int\limits_{AO} e^y[(\sin x-1)dx + (1-\cos x)dy] = \oint\limits_{C} [(\sin x-1)e^ydx + e^y(1-\cos x)dy] = J$

$J = \int\limits_{0}^{\pi} dy \int\limits_{0}^{\sin y} e^ydx = \frac{1}{2}(e^\pi+1)$

$AO: x=0$ *then* $dx=0, Q=0, \int\limits_{AO} e^y[(\sin x-1)dx + (1-\cos x)dy] = 0.$

$I_d = \frac{1}{2}(e^\pi+1)$

e) $I_e + \int\limits_{BOA} [xy^4+x^2-ye^{xy}]dx + [\frac{x^3}{3}+xy^2-x-xe^{xy}]dy = \int\limits_{AmBOA} = J$ *($AmBOA$ has negative direction).*

$BOA: y=0, dy=0$: $\int\limits_{BOA} [xy^4+x^2-ye^{xy}]dx + [\frac{x^3}{3}+xy^2-x-xe^{xy}]dy = \int\limits_{1}^{-1} x^2dx = -\frac{2}{3}$

$J = -\iint\limits_{D} (x^2+y^2-1-3xy^2-2x)dxdy = -\iint\limits_{D} (x^2+y^2-1)dxdy$, *($D: x^2+y^2\le 1, y\le 0$)*

$J = \int\limits_{\pi}^{2\pi} d\varphi \int\limits_{0}^{1} r(1-r^2)dr = \frac{\pi}{4}, I_e = \frac{\pi}{4}+\frac{2}{3}$

f) $I_f = \iint\limits_{|x|+|y|\le 1} \frac{-e^yy - e^{-x}x}{\sqrt{1-x^2y^2}}dxdy = \iint\limits_{|x|+|y|\le 1} \frac{-(e^x+e^{-x})x}{\sqrt{1-x^2y^2}}dxdy = 0$

(we use symmetric role of x, y in the domain then odd property with respect to x of the function $\frac{-(e^x+e^{-x})x}{\sqrt{1-x^2y^2}}$)

g) $\exists r > 0 : B = B(O,r) \subset \{ax^2 + y^2 \leq 1\}$

$$\int\limits_{C_a \cup B^-} \frac{(x+y)dx - (x-y)dy}{x^2+y^2} = 0, \text{ so}$$

$$I_g = \int\limits_{B^+} \frac{(x+y)dx - (x-y)dy}{x^2+y^2}, B(O,r) : x = r\cos\varphi, y = r\sin\varphi, 0 \leq \varphi \leq 2\pi, \text{ then}$$

$$I_g = \int\limits_{B^+} \frac{(x+y)dx - (x-y)dy}{x^2+y^2} = -2\pi$$

Solution 3.4.

a) $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y} = -\dfrac{2y}{x^2}\cos\dfrac{y}{x} - \dfrac{y^2}{x^3}\sin\dfrac{y}{x}$

Choose the path $y = \pi x, 1 \leq x \leq 2$, *then* $I = 1$.

b) $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y} = \dfrac{-xy}{(x^2+y^2)^{\frac{3}{2}}}$

Choose the path AB, BC*:* $A(0,1); B(0,8); C(6,8)$, *then* $I = 9$.

c) $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y} = \dfrac{1}{x^2}$

Choose the path AB, BC*:* $A(2,1); B(2,2); C(1,2)$, *then* $I = -\dfrac{3}{2}$.

d) $u = x^2 - y^3 - z^4$ *satisfies* $du = 2xdx - 3y^2dy - 4z^3dz$

$I = u(2,-1,0) - u(1,1,1) = 6$

Solution 3.5.

a) $a = 2, b = -10; u = x^2y^3 - 5xy^2 + \sin xy + C$

b) $h'(x) = h(x); h(x) = Ce^x$. *Choose* $C = 1, h(x) = e^x$.

$$I_b = \int\limits_{(0,\pi)}^{(1,2\pi)} = \int\limits_{(0,\pi)}^{(1,\pi)} + \int\limits_{(1,\pi)}^{(1,2\pi)} = (2\pi+1)e$$

c) $h'(y)y + 3h(y) = 0, h(y) = \dfrac{1}{y^3}, u(x,y) = \dfrac{x^2}{y^2} + xy$

$I_c = u(-3,2) - u(0,1) = -\dfrac{15}{4}$

d) $h'(xy)xy + 3h(xy) = 0, h(xy) = \dfrac{1}{x^3y^3}, u(x,y) = \dfrac{2x}{y^2} + \dfrac{y}{x^2}$

$I_d = u(1,1) - u(\frac{1}{2},2) = -\dfrac{21}{4}$

Solution 3.6.

a) $S_a = \dfrac{1}{2} \displaystyle\int\limits_{C \cup AO} (xdy - ydx)$

$AO : x = 0$, *then:* $\dfrac{1}{2} \displaystyle\int\limits_{AO} (xdy - ydx) = 0.$

The cycloids $(C) : x = a(1 - \cos t), y = a(t - \sin t), 0 \leq t \leq 2\pi$,

$$S_a = \frac{1}{2} \int\limits_{C} (xdy - ydx) = 3\pi a^2$$

b) $S_b = \dfrac{1}{2} \displaystyle\int\limits_{\mathscr{E} \cup BOA} (xdy - ydx)$

$BOA : x = 0$, *then:* $\dfrac{1}{2} \displaystyle\int\limits_{BOA} (xdy - ydx) = 0.$

The ellip $x = \cos t, y = 2 \sin t, -\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$,

$$S_b = \frac{1}{2} \int\limits_{\mathscr{E}} (xdy - ydx) = \pi$$

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CHAPTER 4

SURFACE INTEGRAL

§1. SURFACE INTEGRAL OF THE FIRST KIND

1.1 Definition

Given a function $f(x,y,z)$ which defines in a surface S. Divide S into n sub-surfaces $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$. In each ΔS_i take an arbitrary point M_i. The limit, if exists, of the sum $\sum\limits_{i=1}^{n} f(M_i)\Delta S_i$ when $n \to \infty$ and $\max\limits_{1\le i\le n} d(\Delta S_i) \to 0$ is called the surface integral of the first kind of the function $f(M)$ in the surface S. This integral is denoted by

$$\iint\limits_S f(x,y,z)dS$$

1.2 Calculation formulae

Assume that S is the surface

$$z = z(x,y); ((x,y) \in D \subset \mathbb{R}^2),$$

where $z(x,y)$ is a continuously differentiable function then

$$\iint\limits_S f(x,y,z)dS = \iint\limits_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy$$

Example 1.1. Calculate the surface integral of the first kind $I_1 = \iint\limits_S (x^2+y^2+z^2)dS$, where S is the sphere $x^2+y^2+z^2 = a^2$.

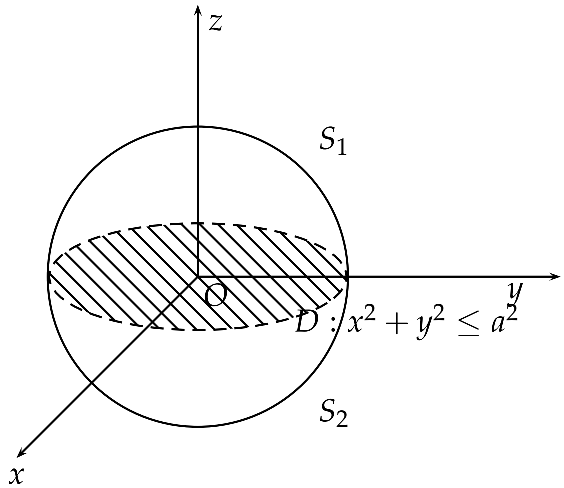

Figure 4.1

We divide S into two pieces $S = S_1 \cup S_2$,

$$(S_1) : \begin{cases} z = \sqrt{a^2 - x^2 - y^2} \\ x^2 + y^2 \leq a^2 \end{cases} \qquad (S_2) : \begin{cases} z = -\sqrt{a^2 - x^2 - y^2} \\ x^2 + y^2 \leq a^2 \end{cases}$$

We can calculate that $\sqrt{1 + (z'_x)^2 + (z'_y)^2} = \dfrac{a}{\sqrt{a^2 - x^2 - y^2}}$, then

$$I_1 = \iint\limits_{S_1} + \iint\limits_{S_2} = 2 \iint\limits_{x^2+y^2 \leq a^2} a^2 \frac{a}{\sqrt{a^2 - x^2 - y^2}} dxdy$$

Change into polar coordinate $x = r\cos\varphi, y = r\sin\varphi$, we have

$$I_1 = 2a^3 \int_0^{2\pi} d\varphi \int_0^a \frac{r}{\sqrt{a^2 - r^2}} dr = 4\pi a^4$$

§2. Surface integral of the second kind

2.1 Definition

Given a function $R(x, y, z)$ which defines in an oriented surface S. Divide S into n sub-surfaces $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$. In each ΔS_i take an arbitrary point M_i. Denote by D_i the area of the orthogonal projection of ΔS_i in the plane Oxy, its sign is $(+)$ or $(-)$ if the outnormal vector at M_i make an acute or obtuse angle respectively with the positive direction of Oz. The limit, if exists, of the sum $\sum\limits_{i=1}^{n} R(M_i) D_i$ when $n \to \infty$ and $\max\limits_{1 \leq i \leq n} d(\Delta S_i) \to 0$ is called the surface integral of the second kind of the function $R(M)$ in the surface S with respect to two variables (x, y). This integral is denoted by

$$\iint_S R(x,y,z)dxdy$$

Similarly we define the surface integral of the second kind of the function $P(x,y,z)$ with respect to two variables (y,z) and of the function $Q(x,y,z)$ with respect to two variables (z,x). In general, we consider the integral

$$I = \iint_S Pdydz + Qdzdx + Rdxdy$$

2.2 Calculation formulae

1. Assume that we want to calculate

$$I_1 = \iint_S P(x,y,z)dydz,$$

   where S is a surface given by the equation $x = x(y,z); (y,z) \in D \subset Oyz$. We have

$$I_1 = \varepsilon \iint_D P(x(y,z),y,z)dydz$$

   where $\varepsilon = 1$ if the angle between the outnormal vector and the positive direction of the axis Ox is an acute angle, and $\varepsilon = -1$ if the angle between the outnormal vector and the positive direction of the axis Ox is an obtuse angle.

2. We want to calculate

$$I = \iint_S P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy$$

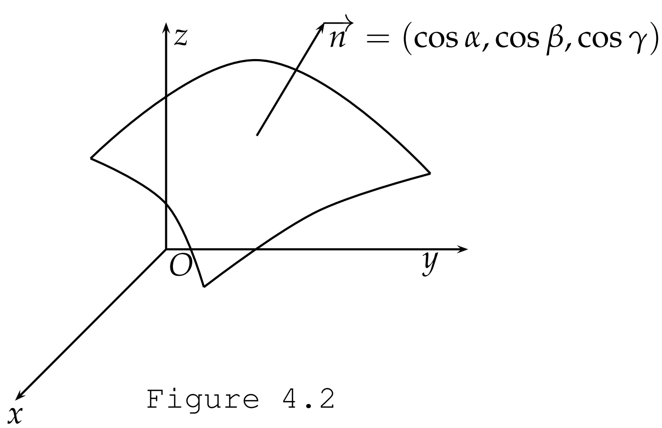

Figure 4.2

We find the unit outnormal vector $\overrightarrow{n} = (\cos\alpha, \cos\beta, \cos\gamma)$, then

$$I = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS,$$

where $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosine of vector $\overrightarrow{n}$.

3. Ostrogradsky's formula: if S is a closed surface which restricts a volume V, and P, Q, R are continuous together with their partial derivatives in V, then

$$I = \iiint_V \Big(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\Big)dxdydz$$

Example 2.1. Calculate the following surface integrals of the second kind

a) $I_2 = \iint_S ydzdx$, where S is the outside of the sphere $x^2 + y^2 + z^2 = a^2, y \le 0$.

We rewrite $y = -\sqrt{a^2 - x^2 - z^2}$.

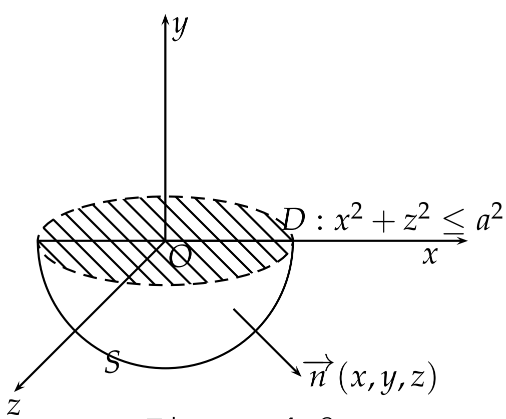

Figure 4.3

The outnormal vector makes an obtuse angle with the positive direction of the axis Oy, then $\varepsilon = -1$. Hence

$$\begin{aligned} I_2 &= \iint_{x^2+z^2\le a^2} \sqrt{a^2 - x^2 - z^2}dxdz \\ &= \int_0^{2\pi} d\varphi \int_0^a \sqrt{a^2 - r^2}rdr = \frac{2\pi a^3}{3} \end{aligned}$$

b) $I_3 = \iint_S \Big(\frac{dydz}{x} + \frac{dzdx}{y} + \frac{dxdy}{z}\Big)$, where S is the outside of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

The normal vector of the ellipsoid at the point $M(x, y, z)$ which points outwards is

$$\overrightarrow{n} = \Big(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\Big); |\overrightarrow{n}| = 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}};$$

and the unit normal vector is

$$\overrightarrow{n_0} = \frac{\overrightarrow{n}}{|\overrightarrow{n}|}$$

We use the second formula and obtain

$$I_3 = \iint\limits_S \Big(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\Big) \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} dS$$

We divide S into two pieces $S = S_1 \cup S_2$,

$$(S_1) : \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \\ z \geq 0 \end{cases} \qquad (S_2) : \begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \\ z \leq 0 \end{cases}$$

For both integrals in S_1 and S_2 we have

$$1 + (z'_x)^2 + (z'_y)^2 = 1 + c^2 \frac{\frac{x^2}{a^4} + \frac{y^2}{b^4}}{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

Hence

$$\begin{aligned} I_3 &= 2 \iint\limits_{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} \Big(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\Big) \frac{c}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} dxdy \\ &= 2 \int\limits_0^{2\pi} \int\limits_0^1 \Big(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\Big) \frac{abcr}{\sqrt{1 - r^2}} \\ &= 4\pi abc \Big(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\Big) \end{aligned}$$

In this example, although the surface is closed but we can not apply Ostrogradsky's formula because the functions $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are not continuous in the domain $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ restricted by the ellipsoid.

c) $I_4 = \iint\limits_S x^3 dydz + y^3 dzdx + z^3 dxdy$, where S is the outside of the sphere $x^2 + y^2 + z^2 = a^2$.

We can apply Ostrogradsky's theorem, then

$$I_4 = 3 \iiint\limits_{x^2 + y^2 + z^2 \leq a^2} (x^2 + y^2 + z^2) dxdydz$$

Change into spherical coordinate $x = r \sin\theta \cos\varphi, y = r \sin\theta \sin\varphi, z = r\cos\theta$, we have

$$I_4 = 3 \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} \sin\theta d\theta \int\limits_0^a r^4 dr = \frac{12\pi a^5}{5}$$

2.3 Stokes' formula

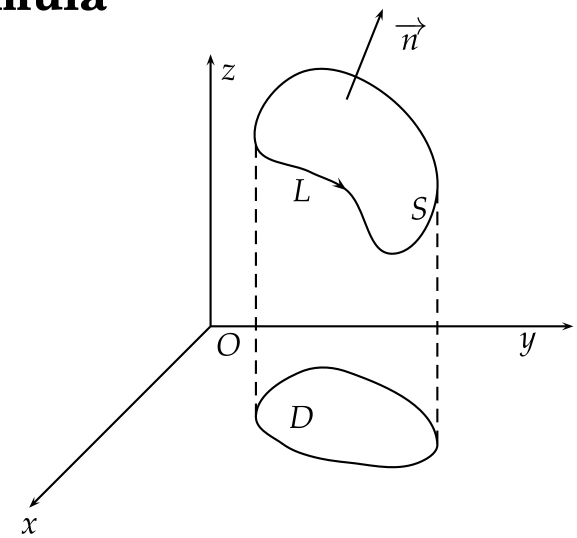

Figure 4.4

Assume that C is a closed, single, and piecewise smooth curve which restricts a piecewise smooth surface S; and P, Q, R are continuously differentiable functions in a domain containing S. We have

$$\oint_S Pdx + Qdy + Rdz = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS,$$

where $\cos\alpha, \cos\beta, \cos\gamma$ are direction cosine of the normal vector of S; and if one stands along the direction of S and follows C in its direction, then he will see S on the left.

Example 2.2. Using Stokes formula, calculate the following line integral of the second kind

a) $I_5 = \oint\limits_C (y-z)dx + (z-x)dy + (x-y)dz$, where C is the ellip $x^2 + y^2 = a^2, \dfrac{x}{a} + \dfrac{z}{h} = 1$, $(a > 0, h > 0)$, in the anticlockwise direction if one see from the positive direction of the axis Ox.

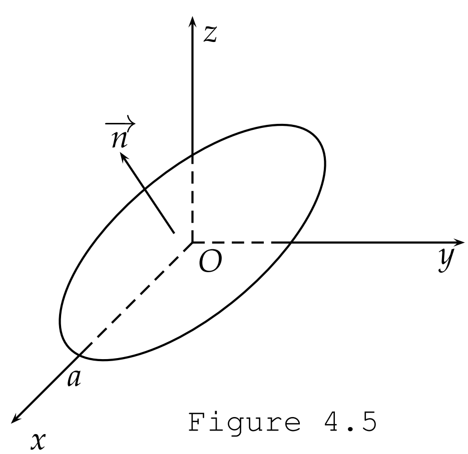

Figure 4.5

C is given, we choose S to be the surface $\frac{x}{a} + \frac{z}{h} = 1, x^2 + y^2 \leq a^2$ which is restricted by C. The outnormal vector of S makes an acute angle with the positive direction of Oz axis. Hence the normal vector is $\left(\frac{1}{a}, 0, \frac{1}{h}\right)$, then the unit outnormal vector is $\left(\frac{h}{\sqrt{a^2+h^2}}, 0, \frac{a}{\sqrt{a^2+h^2}}\right)$.

Apply Stokes' formula we have

$$I_5 = -\frac{2(a+h)}{\sqrt{a^2+h^2}} \iint\limits_S dS$$

For S, we substitute $z = h\left(1 - \frac{x}{a}\right)$ and $x^2 + y^2 \leq a^2$, then

$$I_5 = -\frac{2(a+h)}{\sqrt{a^2+h^2}} \iint\limits_{x^2+y^2 \leq a^2} \sqrt{1 + \frac{h^2}{a^2}} dxdy = -2\pi a(a+h)$$

b) $I_6 = \oint\limits_C ydx + zdy + xdz$, where C is the circle $x^2 + y^2 + z^2 = a^2, x + y + z = 0)$, in the anticlockwise direction if one see from the positive direction of the axis Ox.

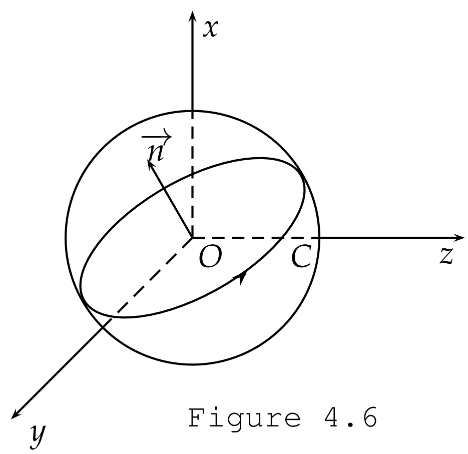

Figure 4.6

We choose S to be the surface $x^2 + y^2 + z^2 \leq a^2, x + y + z = 0$. The direction of C leads to the fact that the outnormal vector of S make an acute angle with the positive direction Ox-axis, then the unit outnormal vector is $\overrightarrow{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Apply Stokes' formula, we have

$$I_6 = \iint\limits_S -\sqrt{3} dS = -\pi\sqrt{3}a^2$$

§3. Exercises

Exercise 4.1. Calculate the following surface integral of the first kind

a) $\iint\limits_S (x + y + z) dS$, where S is the semi-sphere $x^2 + y^2 + z^2 = a^2, z \geq 0$.

b) $\iint\limits_S (x^2+y^2)dS$, where S is the boundary of the object $\sqrt{x^2+y^2} \le z \le 1$.

c) $\iint\limits_S \dfrac{dS}{(1+x+y)^2}$, where S is the surface $x+y+z=1, x \ge 0, y \ge 0, z \ge 0$.

d) $\iint\limits_S (xy+yz+zx)dS$, where S is the cone $z = \sqrt{x^2+y^2}$ intersected by the cylinder $x^2+y^2 = 2ax, (a>0)$.

Exercise 4.2. Calculate the following surface integral of the second kind

a) $\iint\limits_S z(x^2+y^2)dxdy$, where S is the semi-sphere $x^2+y^2+z^2 = 1, z \ge 0$, which points outwards.

b) $\iint\limits_S ydzdx + z^2dxdy$, where S is the inside of the ellipsoid $x^2 + \dfrac{y^2}{4} + z^2 = 1, x \ge 0, y \ge 0, z \ge 0$.

c) $\iint\limits_S x^2dydz + y^2dzdx + z^2dxdy$, where S is the outside of the sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$.

d) $\iint\limits_S (y-z)dydz + (z-x)dzdx + (x-y)dxdy$, where S is the outside of the cone $x^2 + y^2 = z^2, 0 \le z \le h$.

e) $\iint\limits_S x^2dydz + y^2dzdx + z^2dxdy$, where S is the outward boundary of the parallelepiped $0 \le x \le a, 0 \le y \le a, 0 \le z \le a$.

f) $\iint\limits_S y^2zdxdy + xzdydz + x^2ydxdz$, where S is the outside of the domain $x \ge 0, y \ge 0, x^2+y^2 \le 1, 0 \le z \le x^2+y^2$.

g) $\iint\limits_S (x-y+z)dydz + (y-z+x)dxdz + (z-x+y)dxdy$ where S is the outside of the surface $|x-y+z| + |y-z+x| + |z-x+y| = 1$.

h) $\iint\limits_S x^3dydz + y^2dzdx + zdxdy$ where S is the boundary of the cylinder $x^2+y^2 \le 1, -h \le z \le h$, ($h$ is a positive constant), which points outwards.

i) $\iint\limits_S x^3dydz + \dfrac{y^3}{2}dzdx + \dfrac{z^3}{3}dxdy$, where S is the boundary of the semi-ellipsoid $x^2 + \dfrac{y^2}{2} + \dfrac{z^2}{3} \le 1, z \ge 0$ which points outwards.

§4. Solution

Solution 4.1.

a) $z=\sqrt{a^2-x^2-y^2}; x^2+y^2\le a^2$, **then**

$$I_a=\iint\limits_{x^2+y^2\le a^2}(x+y+\sqrt{a^2-x^2-y^2})\frac{a}{\sqrt{a^2-x^2-y^2}}dxdy=\pi a^3$$

b) $z=1$: $I_1=\iint\limits_{x^2+y^2\le 1}(x^2+y^2)dxdy=\frac{\pi}{2}$

$$z=\sqrt{x^2+y^2}, \sqrt{1+(z'_x)^2+(z'_y)^2}=\sqrt{2}, I_2=\iint\limits_{x^2+y^2\le 1}(x^2+y^2)\sqrt{2}dxdy=\frac{\pi\sqrt{2}}{2}$$

$$I_b=\frac{\pi}{2}(1+\sqrt{2})$$

c) $z=1-x-y, 0\le x\le 1, 0\le y\le 1-x$

$$I_c=\int\limits_0^1 dx\int\limits_0^{1-x}\frac{\sqrt{3}}{(1+x+y)^2}dy=\sqrt{3}(\ln 2-\frac{1}{2})$$

d) $z=\sqrt{x^2+y^2}$, **then**

$$I_d=\iint\limits_{x^2+y^2\le 2ax}\left(xy+(x+y)\sqrt{x^2+y^2}\right)\sqrt{2}dxdy$$

$$=\sqrt{2}\iint\limits_{x^2+y^2\le 2ax}x\sqrt{x^2+y^2}dxdy$$

(we subtract some functions which are odd with respect to the variable y because the domain is symmetric to the axis Ox). We conclude $I_d=\frac{64a^4\sqrt{2}}{15}$

Solution 4.2.

a) $z=\sqrt{1-x^2-y^2}; (x,y): x^2+y^2\le 1; \varepsilon=1$,

$$I_a=\iint\limits_{x^2+y^2\le 1}(x^2+y^2)\sqrt{1-x^2-y^2}dxdy=\frac{4\pi}{15}$$

b) $I_1=\iint\limits_S ydzdx$; **where** $S: y=2\sqrt{1-x^2-z^2}; (x,z)\in D_1=\{x^2+z^2\le 1, x,z\ge 0\}; \varepsilon=1$

$$I_1=\iint\limits_{D_1}2\sqrt{1-x^2-z^2}dxdz=\frac{2\pi}{3}$$

$I_2=\iint\limits_S z^2dxdy$; **where** $S: z=\sqrt{1-x^2-\frac{y^2}{4}}; (x,y)\in D_2=\{x^2+\frac{y^2}{4}\le 1, x,y\ge 0\}; \varepsilon=1$

$$I_2=\iint\limits_{D_2}\sqrt{1-x^2-\frac{y^2}{4}}dxdy=\frac{\pi}{4}$$

$I_c = \dfrac{7\pi}{12}$

c) $\overrightarrow{n} = \Big(\dfrac{x-a}{R}; \dfrac{y-b}{R}; \dfrac{z-c}{R}\Big)$

$$I_c = \iint\limits_S \Big(\frac{x-a}{R}x^2 + \frac{y-b}{R}y^2 + \frac{z-c}{R}z^2\Big)dS = \frac{8\pi R^3}{3}(a+b+c)$$

$(S : z = c \pm \sqrt{R^2-(x-a)^2-(y-b)^2}; R^2 \geq (x-a)^2+(y-b)^2)$

d) $\overrightarrow{n} = \Big(\dfrac{x}{z\sqrt{2}}, \dfrac{y}{z\sqrt{2}}, \dfrac{-1}{\sqrt{2}}\Big)$ **(it makes an obtuse angle with the positive direction of Oz axis).**

$$I_d = \iint\limits_S \Big(\frac{x}{z\sqrt{2}}(y-z) + \frac{y}{z\sqrt{2}}(z-x) + \frac{-1}{\sqrt{2}}(x-y)\Big)dS = \frac{1}{\sqrt{2}}\iint\limits_S (y-x)dS,$$

$S : x^2+y^2 = z^2, 0 \leq z \leq h.$

$$I_d = \iint\limits_{x^2+y^2\leq h^2} (y-x)dxdy = 0$$

e) $I_e = 2\displaystyle\int_0^a dx \int_0^a dy \int_0^a (x+y+z)dz = 3a^4$

f) $I_f = \displaystyle\iiint\limits_V (y^2+z+x^2)dxdydz$; $V : x \geq 0, y \geq 0, x^2+y^2 \leq 1, 0 \leq z \leq x^2+y^2.$

Change into cylinderal coordinate, $x = r\cos\varphi; y = r\sin\varphi, z = z;$

$$I_f = \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 dr \int_0^{r^2} (r^2+z)rdz = \frac{\pi}{8}$$

g) $I_g = \displaystyle\iiint\limits_V 3dxdydz = 1$; $V : |x-y+z| + |y-z+x| + |z-x+y| \leq 1.$

h) $I_h = \displaystyle\iiint\limits_V (3x^2+2y+1)dxdydz$; $V : x^2+y^2 \leq 1; -h \leq z \leq h.$

$$I_h = \iiint\limits_V (3x^2+1)dxdydz = \frac{1}{2}\iiint\limits_V (3x^2+3y^2+2)dxdydz = \frac{7\pi h}{2}$$

i) $I_i = 3\displaystyle\iiint\limits_V \Big(x^2+\frac{y^2}{2}+\frac{z^2}{3}\Big)dV$, **where** $V : \begin{cases} x^2+\dfrac{y^2}{2}+\dfrac{z^2}{3} \leq 1 \\ z \geq 0 \end{cases}$

Change into spherical coordinate, $x = r\sin\theta\cos\varphi; y = r\sin\theta\sin\varphi\sqrt{2}, z = r\cos\theta\sqrt{3};$

$$I_i = 3\sqrt{6}\int_0^{2\pi} d\varphi \int_0 \frac{\pi}{2}\sin\theta d\theta \int_0^1 r^4 dr = \frac{6\sqrt{6}\pi}{5}$$

CHAPTER 5

FIELD THEORY

§1. SCALAR FIELD

A scalar field in $\mathbb{R}^3$ is a function $u : \mathbb{R}^3 \to \mathbb{R}$, $u(x, y, z) \in \mathbb{R}$.

Its gradient vector is

$$\operatorname{grad} u = \Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\Big)$$

At each point of the scalar field, gradient vector of u is the direction in which u varies fastest.

Assume that $\overrightarrow{l} = (\cos\alpha, \cos\beta, \cos\gamma)$ is a unit vector. The derivatives of u with respect to $\overrightarrow{l}$ direction is

$$\frac{\partial u}{\partial \overrightarrow{l}} = \operatorname{grad} u.\overrightarrow{l} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma$$

Example 1.1. Given a scalar field $u(x, y, z) = xy^2z^3 + 2xy - z^2$. Calculate $\operatorname{grad} u(1, 1, 1)$ and the derivatives of u with respect to the direction $\operatorname{grad} v(1, 1, 1)$ at the same point, where $v(x, y, z) = z^2 \sin(\pi xy)$.

$$\begin{aligned}
&\operatorname{grad} u = \Big(y^2z^3 + 2y, 2xyz^3 + 2x, 3xy^2z^2 - 2z\Big)\\
\Rightarrow &\operatorname{grad} u(1, 1, 1) = (3, 4, 1)\\
&\operatorname{grad} v = \Big(z^2\pi y\cos(\pi xy), z^2\pi x\cos(\pi xy), 2z\sin(\pi xy)\Big)\\
\Rightarrow &\operatorname{grad} v(1, 1, 1) = (-\pi, -\pi, 0),\ \overrightarrow{l} = \Big(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\Big)\\
&\frac{\partial u}{\partial \overrightarrow{l}}(1, 1, 1) = -\frac{1}{\sqrt{2}}.3 - \frac{1}{\sqrt{2}}.4 + 0.1 = -\frac{7}{\sqrt{2}}
\end{aligned}$$

§2. VECTOR FIELD

A vector field in $\mathbb{R}^3$ is a vector function

$$\overrightarrow{F}(x,y,z) = P(x,y,z)\overrightarrow{i} + Q(x,y,z)\overrightarrow{j} + R(x,y,z)\overrightarrow{k},$$

where $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$ are orthonormal basis of $\mathbb{R}^3$.

Divergence of this vector field is a scalar field

$$\text{div } \overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial y}$$

Rotation vector of this vector field is a vector field

$$rot\,\overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z};\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x};\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

Flow field of a vector field $\overrightarrow{F}$ through an oriented surface S is

$$\iint\limits_S Pdydz + Qdzdx + Rdxdy$$

Circulation of a vector field along a directed curve L is

$$\int\limits_L Pdx + Qdy + Rdz$$

We often use Ostrogradsky and Stokes formulae to calculate flow field and circulation of $\overrightarrow{F}$.

Example 2.1.

a) Calculate the flow field of the vector field $\overrightarrow{F} = (x,y,z)$ through the base of the cone $V: x^2 + y^2 \le z^2, 0 \le z \le h$ in the outward direction.

The flow field of $\overrightarrow{F}$ is

$$I = \iint\limits_{S^+} xdydz + ydzdx + zdxdy$$

where $S: z = h, x^2 + y^2 \le h^2$

$$I = \iint\limits_S xdydz + ydzdx + zdxdy = \iint\limits_{x^2+y^2\le h^2} hdxdy = \pi h^3$$

b) Calculate the circulation of the vector field $\overrightarrow{F} = (-y, x, 1)$ along the circle $C : x^2 + y^2 = 1, z = 0$.

The circulation is

$$J = \int_C -ydx + xdy - dz = \int_{x^2+y^2=1} -ydx + xdy$$

Apply Green's theorem we have

$$J = 2 \iint_{x^2+y^2\leq 1} 2dxdy = 2\pi$$

A vector field $\overrightarrow{F}$ is a potential vector if there exists a scalar field u such that grad $u = \overrightarrow{F}$. u is called the potential function of $\overrightarrow{F}$.

Necessary and sufficient condition for $\overrightarrow{F}$ to be a potential field is $rot\,\overrightarrow{F} = \overrightarrow{0}$. Then we find u by one of the following formulae

$$\begin{aligned}
u(x,y,z) &= \int_{(x_0,y_0,z_0)}^{(x,y,z)} Pdx + Qdy + Rdz + C \\
&= \int_{x_0}^{x} P(x,y_0,z_0)dx + \int_{y_0}^{y} Q(x,y,z_0)dy + \int_{z_0}^{z} R(x,y,z)dz + C \\
&= \int_{x_0}^{x} P(x,y,z)dx + \int_{y_0}^{y} Q(x_0,y,z)dy + \int_{z_0}^{z} R(x_0,y_0,z)dz + C \\
&\ldots
\end{aligned}$$

Example 2.2. Examine that $\overrightarrow{F} = \cos(x^2 + 2y^2 - 3z^2)(x\overrightarrow{i} + 2y\overrightarrow{j} - 3z\overrightarrow{k})$ is a potential field, and find its potential function.

It is easy to check that $rotF = \overrightarrow{0}$. We choose $x_0 = y_0 = z_0 =$, the potential function is

$$\begin{aligned}
u(x,y,z) &= \int_0^x x\cos x^2 dx + \int_0^y 2y\cos(x^2+2y^2)dy - \int_0^z 3z\cos(x^2+2y^2-3z^2)dz + C \\
&= \frac{1}{2}\sin(x^2+2y^2-3z^2) + C
\end{aligned}$$

§3. EXERCISES

Exercise 5.1. Let $u = \ln(1+\sqrt{x^2+y^2+z^2})$ be a scalar field and a point $A(1,2,-2)$ in $\mathbb{R}^3$. Calculate the derivatives of u with respect to the direction $\overrightarrow{OA}$ at A. For which direction $\overrightarrow{l}$ that $\left|\dfrac{\partial u}{\partial \overrightarrow{l}}(A)\right|$ reaches its maximum?

Exercise 5.2. Given a field $u = e^{-xyz}(x^2-y)$. Calculate the derivatives of u with respect to the direction $\operatorname{grad} u(0,1,2)$ at the point $A(1,-1,0)$.

Exercise 5.3. Find the angle between two gradient vectors of the scalar field $u = \dfrac{x}{x^2+y^2+z^2}$ at the points $A(1,2,2)$ and $B(-3,1,0)$.

Exercise 5.4. Calculate the flow field of the vector field $\overrightarrow{F} = (x,y,z)$ through the boundary of the cone $z \le 1 - \sqrt{x^2+y^2}, 0 \le z \le 1$.

Exercise 5.5. Calculate the flow field of the vector field $\overrightarrow{F} = x^2\sqrt{y^2+z^2}\,\overrightarrow{i} + x^3z\,\overrightarrow{j} + x^2y\,\overrightarrow{k}$ through the outside boundary of the domain $x^2 + \dfrac{y^2+z^2}{4} \le 1, x \ge 0$.

Exercise 5.6. Calculate the circulation of the vector field $\overrightarrow{F} = x(y+z)\,\overrightarrow{i} + y(x+z)\,\overrightarrow{j} + z(x+y)\,\overrightarrow{k}$ along the intersection curve of the sphere $x^2+y^2+z^2 = R^2, z \ge 0$ and the cylinder $x^2+y^2+y=0$.

Exercise 5.7. Prove that the following vector fields are potential fields and find their potential function

a) $\overrightarrow{F} = yz(4x^3+y^3+z^3)\,\overrightarrow{i} + zx(x^3+4y^3+z^3)\,\overrightarrow{j} + xy(x^3+y^3+4z^3)\,\overrightarrow{k}$.

b) $\overrightarrow{F} = yz(2x+y+z)\,\overrightarrow{i} + zx(x+2y+z)\,\overrightarrow{j} + xy(x+y+2z)\,\overrightarrow{k}$.

c) $\overrightarrow{F} = (x+y)\,\overrightarrow{i} + (x+z)\,\overrightarrow{j} + (y+z)\,\overrightarrow{k}$.

§4. SOLUTION

Solution 5.1. $u = \ln(1+\sqrt{x^2+y^2+z^2})$.

$$\operatorname{grad} u = \frac{1}{(1+\sqrt{x^2+y^2+z^2})\sqrt{x^2+y^2+z^2}}(x,y,z) \Rightarrow \operatorname{grad} u(1,2,-2) = \left(\frac{1}{12}, \frac{2}{12}, -\frac{2}{12}\right)$$

$$\overrightarrow{OA} = (1,2,-2) \Rightarrow \overrightarrow{l} = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

$\dfrac{\partial u}{\partial \overrightarrow{OA}} = \dfrac{1}{4}$; $\left|\dfrac{\partial u}{\partial \overrightarrow{l}}\right|$ *reaches its maximum if and only if* $\overrightarrow{l} = \operatorname{grad} u(A) = (1,2,-2)$.

Solution 5.2. $u = e^{-xyz}(x^2 - y)$

$\operatorname{grad} u = e^{-xyz}\Big(-yz(x^2-y)+2x, -xz(x^2-y)-1, -xy(x^2-y)\Big); \Rightarrow \operatorname{grad} u(A) = (2,-1,2)$

$\operatorname{grad} u(0,1,2) = (2,-1,0) \Rightarrow \overrightarrow{l} = \Big(\dfrac{2}{\sqrt{5}}, -\dfrac{1}{\sqrt{5}}, 0\Big); \dfrac{\partial u}{\partial \overrightarrow{l}} = \sqrt{5}$

Solution 5.3. $u = \dfrac{x}{x^2+y^2+z^2}$

$\operatorname{grad} u = \dfrac{1}{(x^2+y^2+z^2)^2}\Big(-x^2+y^2+z^2; -2xy; -2xz\Big)$

$\operatorname{grad} u(A) = \dfrac{1}{81}(7,-4,-4); \operatorname{grad} u(B) = \dfrac{1}{100}(-8,6,0) \Rightarrow \cos\varphi = -\dfrac{8}{9}; \varphi = \arccos\Big(-\dfrac{8}{9}\Big)$

Solution 5.4.

$$I = \iint\limits_S xdydz + ydzdx + zdxdy = \iiint\limits_V 3dxdydz$$

$V: 0 \le z \le 1 - \sqrt{x^2+y^2}; x^2+y^2 \le 1$, V is the cone whose base is a circle of radial $R = 1$ and whose altitude is $h = 1$, then $I = \pi$.

Solution 5.5.

$$I = \iint\limits_S x^2\sqrt{y^2+z^2}dydz + x^3zdzdx + x^2ydxdy = \iiint\limits_V 2x\sqrt{y^2+z^2}dxdydz$$

Set $x = r\cos\theta, y = 2r\sin\varphi\sin\theta, z = 2r\cos\varphi\sin\theta$, $|J| = 4r^2\sin\theta$

$$I = 16\int\limits_0^{\frac{\pi}{2}} \sin^2\theta\cos\theta d\theta \int\limits_0^{2\pi} d\varphi \int\limits_0^1 r^4 dr = \frac{32\pi}{15}$$

Solution 5.6. *We choose $S: x^2+y^2+z^2 = R^2; x^2+y^2+y \le 0$. The outnormal vector makes an acute angle with the positive direction of Oz axis, then $\overrightarrow{n} = \Big(\dfrac{x}{R}, \dfrac{y}{R}, \dfrac{z}{R}\Big)$.*

$\operatorname{rot}\overrightarrow{F} = \Big((z-y), (x-z), (x-y)\Big); \operatorname{rot}\overrightarrow{F}.\overrightarrow{n} = 0.$

Apply Stokes' formula, the circulation is

$$I = \int\limits_C x(y+z)dydz + y(x+z)dzdx + z(x+y)dxdy = \iint\limits_S \operatorname{rot}\overrightarrow{F}.\overrightarrow{n}dS = 0$$

Solution 5.7.

a) $\operatorname{rot}\overrightarrow{F} = 0$

$x_0 = y_0 = z_0 = 0$: $u = \displaystyle\int\limits_0^x 0dx + \int\limits_0^y 0dy + \int\limits_0^z xy(x^3+y^3+4z^3)dz = xyz(x^3+y^3+z^3) + C.$

b) $\operatorname{rot}\overrightarrow{F} = 0$

$x_0 = y_0 = z_0 = 0$: $u = xyz(x+y+z) + C.$

c) rot $\overrightarrow{F} = 0$

$$x_0 = y_0 = z_0 = 0\text{: } u = \int_0^x x dx + \int_0^x 0 dy + \int_0^z (y+z) dz = \frac{x^2}{2} + xy + yz + \frac{z^2}{2} + C.$$

CHAPTER 6

SERIES

§1. NUMBER SERIES

1.1 Definition

A number series is the expression

$$u_1 + u_2 + \ldots + u_n + \ldots =: \sum_{n=1}^{+\infty} u_n$$

The n-th partial sum of this series is denoted by

$$S_n := \sum_{k=1}^{n} u_k$$

If S_n tends to a finite value S when $n \to \infty$ then $\sum_{n=1}^{+\infty} u_n$ is said to be converged and its sum is S; if not the series $\sum_{n=1}^{+\infty} u_n$ is said to be diverged.

A series $\sum_{n=1}^{+\infty} u_n$ is absolutely convergent if $\sum_{n=1}^{+\infty} |u_n|$ is convergent. If a series is absolutely convergent then it is convergent.

A series $\sum_{n=1}^{+\infty} u_n$ is semiconvergent if it is convergent but $\sum_{n=1}^{+\infty} |u_n|$ is divergent.

Example 1.1. For $a \neq 0$, consider the sequence $\sum_{n=1}^{+\infty} aq^{n-1}$.

If $q = 1$, $S_n = na \to \infty$ when $n \to \infty$.

For $q \neq 1$, $S_n = a\dfrac{q^n - 1}{q - 1}$.

$\lim\limits_{n\to\infty} S_n$ exists if and only if $\lim\limits_{n\to\infty} q^n = 0$, or equivalently $|q| < 1$.

If $|q| < 1$, then the series $\sum\limits_{n=1}^{+\infty} aq^{n-1}$ is convergent whose sum is $\dfrac{a}{q-1}$.

If $|q| > 1$, then the series $\sum\limits_{n=1}^{+\infty} aq^{n-1}$ is divergent.

In conclusion the series $\sum\limits_{n=1}^{+\infty} aq^{n-1}$ is convergent if and only if $|q| < 1$ and its sum is $\dfrac{a}{q-1}$.

Example 1.2. Find the sum of the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{(3n-2)(3n+1)}$.

We have

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} = \sum_{k=1}^{n} \frac{1}{3}\Big(\frac{1}{3k-2} - \frac{1}{3k+1}\Big) \\ &= \frac{1}{3} - \frac{1}{3(3n+1)} \end{aligned}$$

so $\lim\limits_{n\to\infty} S_n = \dfrac{1}{3}$. This series is convergent and its sum is $\dfrac{1}{3}$.

1.2 Convergent criterion

A series $\sum\limits_{n=1}^{+\infty} u_n$ does not change its convergent property if we add or substract a finite number of terms of it. That means $\sum\limits_{n=1}^{+\infty} u_n$ and $\sum\limits_{n=n_0}^{+\infty} u_n$ have the same convergent property for a finite number n_0. Hence in the following criterion, if the statement is true for $n = 1$, one should understand that $n = n_0$ also takes effect.

1. Necessary condition

   A number series $\sum\limits_{n=1}^{+\infty} u_n$ is convergent then $\lim\limits_{n\to\infty} u_n = 0$.

   Example 1.3. $\sum\limits_{n=1}^{+\infty} n \ln\Big(1 + \dfrac{1}{n}\Big)$ is divergent because when $n \to \infty$

   $$u_n = n \ln\Big(1 + \frac{1}{n}\Big) \to 1$$

   We often use this criteria to prove **divergence of a series**.

2. Cauchy criteria

A number series $\sum\limits_{n=1}^{+\infty} u_n$ is convergent if and only if for an arbitrary number $\varepsilon > 0$, there exists an integer $N_0 > 0$ such that for all $n \geq N_0, p \geq 0$, we have

$$|u_{n+1} + u_{n+2} + \ldots + u_{n+p}| < \varepsilon$$

In our range of exercises we do not often use this criteria.

3. Comparison criterion

Assume that $\sum\limits_{n=1}^{+\infty} u_n$ and $\sum\limits_{n=1}^{+\infty} v_n$ are two positive series.

Comparison criteria 1. Furthermore assume that $u_n \leq v_n$ for all $n \geq 1$.

If $\sum\limits_{n=1}^{+\infty} v_n$ is convergent then $\sum\limits_{n=1}^{+\infty} u_n$ is convergent.

If $\sum\limits_{n=1}^{+\infty} u_n$ is divergent then $\sum\limits_{n=1}^{+\infty} v_n$ is divergent.

Comparison criteria 2. Furthermore assume that $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = k$

If $0 < k < +\infty$ then $\sum\limits_{n=1}^{+\infty} u_n$ and $\sum\limits_{n=1}^{+\infty} v_n$ have the same properties of convergence or divergence.

If $k = 0$ then $\sum\limits_{n=1}^{+\infty} u_n$ is convergent when $\sum\limits_{n=1}^{+\infty} v_n$ is convergent; and $\sum\limits_{n=1}^{+\infty} v_n$ is divergent when $\sum\limits_{n=1}^{+\infty} u_n$ is divergent.

If $k = \infty$, or equivalently $\lim\limits_{n\to\infty} \frac{v_n}{u_n} = 0$, we return to the case $k = 0$.

Note that we often choose $\frac{1}{n^s}$ to be u_n or v_n, and keep in mind that the Riemann series $\sum\limits_{n=1}^{+\infty} \frac{1}{n^s}$ is convergent if and only if $s > 1$.

Example 1.4.

a) Consider the series $\sum\limits_{n=1}^{\infty} \operatorname{arctg} \frac{\pi}{2^n}$

This is a positive series, and as $n \to \infty$, we have $\operatorname{arctg} \frac{\pi}{2^n} \sim \frac{\pi}{2^n}$, and the series $\sum\limits_{n=1}^{\infty} \frac{\pi}{2^n} = \pi \sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is convegent. Then the original series is convergent too.

b) Consider the series $\sum\limits_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n-1}}{n^\alpha}$

When $n \to \infty$: $\frac{\sqrt{n+1} - \sqrt{n-1}}{n^\alpha} = \frac{2}{(\sqrt{n+1} + \sqrt{n-1})n^\alpha} \sim \frac{1}{n^{\alpha+\frac{1}{2}}}$, then we have

If $\alpha > \dfrac{1}{2}$: the series is convergent; if $\alpha \leq \dfrac{1}{2}$, the series is divergent.

c) Consider the series $\sum\limits_{n=1}^{\infty} e^{-\sqrt{n}}$.

We know two important limits

i) $\lim\limits_{n\to\infty} \dfrac{a^n}{n^\alpha} = +\infty, (a > 1, \forall\alpha)$, or $n^\alpha \leq e^n$ when n is large enough.

ii) $\lim\limits_{n\to\infty} \dfrac{n}{\ln^\beta n} = +\infty, (\forall\beta)$, or $\ln^\beta n \leq n$ when n is large enough.

We use the first limits: $(\sqrt{n})^\alpha \leq e^{\sqrt{n}}$ when n is large enough, or equivalently, $e^{-\sqrt{n}} \leq n^{-\frac{\alpha}{2}}$, for large enough n and for all α. Choose $\alpha = 4$, and the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent; then the original series is convergent.

4. Cauchy criteria

   We calculate $l = \lim\limits_{n\to\infty} \sqrt[n]{|u_n|}$.

   If $l > 1$ then the series is divergent.

   If $l < 1$ then the series is absolutely convergent, and is convergent.

5. D'Alembert criteria

   We calculate $l = \lim\limits_{n\to\infty} \left|\dfrac{u_{n+1}}{u_n}\right|$.

   If $l > 1$ then the series is divergent.

   If $l < 1$ then the series is absolutely convergent, and is convergent.

Example 1.5.

a) $\sum\limits_{n=1}^{\infty} \dfrac{n^2+5}{3^n}$ has

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2+5}{3(n^2+5)} = \frac{1}{3} < 1$$

then this series is convergent due to D'Alembert criteria.

b) $\sum\limits_{n=1}^{\infty} \dfrac{an}{(1-a^2)^n} (0 < |a| \neq 1)$ has

$$l = \lim_{n\to\infty} \sqrt[n]{|u_n|} = \lim_{n\to\infty} \frac{\sqrt[n]{|a|n}}{|1-a^2|} = \frac{1}{|1-a^2|}$$

If $0 < |a| < \sqrt{2}$ then $l = \dfrac{1}{|1-a^2|} > 1$, this series is divergent due to Cauchy criteria.

If $|a| > \sqrt{2}$ then $l = \dfrac{1}{a^2 - 1} < 1$, this series is convergent.

If $|a| = \sqrt{2}$ then $\lim\limits_{n\to\infty} |u_n| = \lim\limits_{n\to\infty} n\sqrt{2} = +\infty$, this series is divergent due to necessary condition.

6. Integral criteria

Assume that $f(x)$ is a positive continuous function which decreases in the interval $[1, +\infty)$ and tends to 0 as $x \to +\infty$. Then the infinite integral $\int\limits_1^{+\infty} f(x)dx$ and the series $\sum\limits_{n=1}^{\infty} u_n$, where $u_n = f(n)$, have the same convergence or divergence property.

Example 1.6. Consider the convergence property of the series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n \ln^2 n}$

We define $f(x) = \dfrac{1}{x \ln^2 x}$, for $x \geq 2$. This function satisfies all the conditions in the integral criteria.

$$\int\limits_2^{\infty} \frac{dx}{x \ln^2 x} = -\frac{1}{\ln x}\Big|_2^{\infty} = \frac{1}{\ln 2}$$

This integral is convergent then the observing series is convergent.

7. Leibnitz's criteria for alternate series

An alternate series is the series $\sum\limits_{n=1}^{\infty} (-1)^n u_n$, where $u_n > 0$ for all n.

If u_n is an decreasing sequence which tends to 0 when $n \to \infty$, then the alternate series $\sum\limits_{n=1}^{\infty} (-1)^n u_n$ is convergent.

Example 1.7. We consider the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^\alpha}$, where $\alpha > 0$.

It is an alternate series which has

$$\lim_{n\to\infty} n^\alpha = +\infty \text{ for } \alpha > 0$$

and $\dfrac{1}{n^\alpha}$ is an decreasing sequence. Use Leibnitz's rule, we conclude that this series is convergent.

1.3 Exercises

Exercise 6.1. *Find the sum of the following series*

$$a)\ \sum_{n=1}^{\infty}\frac{n^2}{n!} \qquad b)\ \sum_{n=1}^{\infty}(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}) \qquad c)\ \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$d)\ \sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} \qquad e)\ \sum_{n=1}^{\infty}\operatorname{arctg}\frac{1}{1+n+n^2}$$

Exercise 6.2. *Prove that the following series are divergent*

$$a)\ \sum_{n=1}^{\infty}\frac{n^2-2n+1}{5n^2+(-1)^n\sqrt{n}} \qquad b)\ \sum_{n=1}^{\infty}\Big(\frac{n}{n+1}\Big)^{\frac{n}{2}} \qquad c)\ \sum_{n=1}^{\infty}\operatorname{arctg}\frac{2^n}{n}$$

Exercise 6.3. *Use comparison, D'Alembert, Cauchy and integral criterion, consider the convergence of the following series*

$$a)\ \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\ln\frac{n+1}{n-1} \qquad b)\ \sum_{n=1}^{\infty}(\sqrt{n^4+2n+1}-\sqrt{n^4+an}) \qquad c)\ \sum_{n=1}^{\infty}n^3e^{-n^2}$$

$$d)\ \sum_{n=1}^{\infty}\frac{n^2+5}{3^n} \qquad e)\ \sum_{n=1}^{\infty}\frac{\ln^2 2+\ln^2 3+\ldots+\ln^2 n}{n^{\alpha}} \qquad f)\ \sum_{n=1}^{\infty}\frac{a^n n!}{n^n},(a\neq e)$$

$$g)\ \sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^{2n}(n-1)!} \qquad h)\ \sum_{n=1}^{\infty}\frac{1}{5^n}\Big(1-\frac{1}{n}\Big)^{n^2} \qquad i)\ \sum_{n=1}^{\infty}\sqrt{n}\Big(\frac{n}{4n-3}\Big)^{2n}$$

$$j)\ \sum_{n=1}^{\infty}\Big(\frac{n+a}{n+b}\Big)^{n^2} \qquad k)\ \sum_{n=3}^{\infty}\frac{1}{n\ln n(\ln\ln n)^2} \qquad l)\ \sum_{n=3}^{\infty}\frac{1}{n\ln^p n},(p>0)$$

Exercise 6.4. *Use Leibnitz criteria to consider the convergent property of the following series*

$$a)\ \sum_{n=2}^{\infty}\frac{(-1)^n\ln n}{n} \qquad b)\ \sum_{n=2}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+e} \qquad c)\ \sum_{n=1}^{\infty}(-1)^n\frac{5n+3}{n^2+n}$$

Exercise 6.5. *Consider the convergent property of the following series*

$$a)\ \sum_{n=1}^{\infty}\frac{\ln n}{n^{\alpha}};(\alpha>1) \qquad b)\ \sum_{n=1}^{\infty}\frac{1}{(\ln n)^p};(p>0) \qquad c)\ \sum_{n=1}^{\infty}\frac{2n}{n+2^n}$$

$$d)\ \sum_{n=1}^{\infty}\Big(\frac{1}{n}-\ln\frac{n+1}{n}\Big) \qquad e)\ \sum_{n=1}^{\infty}\sin(\pi\sqrt{n^2+a^2});a\in\mathbb{R} \qquad f)\ \sum_{n=1}^{\infty}\sin[\pi(2+\sqrt{3})^n]$$

$$g)\ \sum_{n=3}^{\infty}\frac{1}{n^{\alpha}(\ln n)^{\beta}},(\alpha,\beta>0) \qquad h)\ \sum_{n=1}^{\infty}\Big(\cos\frac{a}{n}\Big)^{n^3};a\in\mathbb{R} \qquad i)\ \sum_{n=1}^{\infty}\frac{(n!)^2}{2^{n^2}}$$

1.4 Solution

Solution 6.1.

a) $\dfrac{n^2}{n!} = \dfrac{n}{(n-1)!} = \dfrac{1}{(n-2)!} + \dfrac{1}{(n-1)!}, n \geq 2$

$$\sum_{n=1}^{\infty} \frac{n^2}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = 2e$$

b) $a_n = \dfrac{1}{\sqrt{n+2}+\sqrt{n+1}} - \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$

$$S_n = \frac{1}{\sqrt{n+2}+\sqrt{n+1}} - \frac{1}{\sqrt{2}+1} \Rightarrow S = -\frac{1}{\sqrt{2}+1} = 1-\sqrt{2}$$

c) $\dfrac{1}{4n^2-1} = \dfrac{1}{2}\Big(\dfrac{1}{2n-1} - \dfrac{1}{2n+1}\Big)$

$$S_n = \frac{1}{2}\Big(1 - \frac{1}{2n+1}\Big) \Rightarrow S = \frac{1}{2}$$

d) $\dfrac{1}{n(n+1)(n+2)} = \dfrac{1}{2}\Big(\dfrac{1}{n(n+1)} - \dfrac{1}{(n+1)(n+2)}\Big) = \dfrac{1}{2}\Big(\dfrac{1}{n} - 2\dfrac{1}{n+1} + \dfrac{1}{n+2}\Big)$

$$S_n = \frac{1}{2}\Big(1 - \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}\Big) \Rightarrow S = \frac{1}{4}$$

e) $\operatorname{arctg}\dfrac{1}{1+n+n^2} = \operatorname{arctg}\dfrac{(n+1)-n}{1+(n+1)n} = \operatorname{arctg}(n+1) - \operatorname{arctg} n$

$$S_n = \operatorname{arctg}(n+1) - \operatorname{arctg} 1 \Rightarrow S = \frac{\pi}{4}$$

Solution 6.2. *All of these series do not satisfy necessary conditions*

a) $\lim\limits_{n\to\infty} \dfrac{n^2-2n+1}{5n^2+(-1)^n\sqrt{n}} = \dfrac{1}{5} \neq 0$

b) $\lim\limits_{n\to\infty} \Big(\dfrac{n}{n+1}\Big)^{\frac{n}{2}} = e^{-\frac{1}{2}} \neq 0$

c) $\lim\limits_{n\to\infty} \operatorname{arctg}\dfrac{2^n}{n} = \dfrac{\pi}{2} \neq 0$

Solution 6.3.

a) $\dfrac{1}{\sqrt{n}}\ln\dfrac{n+1}{n-1} \sim \dfrac{1}{\sqrt{n}}\dfrac{2}{n-1} \sim \dfrac{2}{n^{\frac{3}{2}}}$ *($n \to \infty$), the series is convergent.*

b) $\sqrt{n^4+2n+1} - \sqrt{n^4+an} = \dfrac{(2-a)n+1}{\sqrt{n^4+2n+1}+\sqrt{n^4+an}} \sim \dfrac{(2-a)n+1}{2n^2}$

If $a = 2$, $u_n \sim \dfrac{1}{2n^2}$ *the series is convergent.*

If $a \neq 2$, $u_n \sim \dfrac{a-2}{2n}$, *the series is divergent.*

c) For sufficiently large number n: $n^3e^{-n^2} \leq \dfrac{1}{n^2}$, *the series is convergent.*

d) $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{(n+1)^2+5}{3^{n+1}}\dfrac{3^n}{n^2+5} = \dfrac{1}{3} < 1$, *the series is convergent.*

e) $\alpha > 2$: $a_n \leq \dfrac{\ln^2 n}{n^{\alpha-1}} \leq \dfrac{1}{n^{1+\varepsilon}}$; $(0 < \varepsilon < \alpha - 2)$, *the series is convergent.*

$\alpha \leq 2$: $a_n \geq \dfrac{\ln^2 2}{n^{\alpha-1}}$, *the series is divergent.*

f) $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{a}{\left(1+\frac{1}{n}\right)^n} = \dfrac{a}{e}$. *The series is convergent if* $a < e$, *is divergent if* $a > e$.

g) $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{(2n+1)!!}{2^{2(n+1)}n!} \dfrac{2^{2n}(n-1)!}{(2n-1)!!} = \dfrac{1}{2} < 1$, *the series is convergent.*

h) $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \dfrac{1}{5} \lim\limits_{n\to\infty} \left(1 - \dfrac{1}{n}\right)^n = \dfrac{1}{5e} < 1$, *the series is convergent.*

i) $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{n} \left(\dfrac{n}{4n-3}\right)^2 = \dfrac{1}{16} < 1$, *the series is convergent.*

j) $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \left(\dfrac{n+a}{n+b}\right)^n = e^{a-b}$.

If $a > b$ *then* $e^{a-b} > 1$ *and the series is divergent.*

If $a < b$ *then* $e^{a-b} < 1$ *and the series is convergent.*

If $a = b$, $a_n = 1$ *does not satisfy the necessary condition, the series is divergent.*

k) $f(x) = \dfrac{1}{x \ln x (\ln\ln x)^2}, x \geq 3$

$\displaystyle\int_3^\infty f(x)dx = -\frac{1}{\ln\ln x}\Big|_3^\infty < +\infty$, *the series is convergent.*

l) $f(x) = \dfrac{1}{x \ln^p x}, x \geq 2$

$$\int_2^\infty f(x)dx = \begin{cases} \ln\ln x\Big|_2^\infty & \text{if } p = 1 \\ \dfrac{(\ln x)^{1-p}}{1-p}\Big|_2^\infty & \text{if } p \neq 1 \end{cases}$$

The series is convergent if $p > 1$, *is divergent if* $0 < p \leq 1$.

Solution 6.4.

a) $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$; $a_n = \dfrac{\ln n}{n}$ *is decreasing as* $n \to \infty$ *because*

$$f(x) = \frac{\ln x}{x}; f'(x) = \frac{1 - \ln x}{x^2} < 0, \forall x \geq 3$$

The series is convergent.

b) $\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{n+e} = 0$; $a_n = \dfrac{\sqrt{n}}{n+e}$ *is decreasing as* $n \to \infty$ *because*

$$f(x) = \frac{\sqrt{x}}{x+e}; f'(x) = \frac{e-x}{2\sqrt{x}(x+e)^2} < 0, \forall x \geq 3$$

The series is convergent.

c) $\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{5n+3}{n^2+n} = 2\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n+1} + 3\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}$, *this series is convergent because both series in the righthand-side are convergent.*

Solution 6.5.

a) We choose $0 < \varepsilon < \alpha - 1$, *when* n *is large enough* $\dfrac{\ln n}{n^\alpha} \le \dfrac{1}{n^{\alpha-\varepsilon}}$, $\alpha - \varepsilon > 1$ *so the series is convergent.*

b) For an arbitrary $1 > \varepsilon > 0$, *we have for a large enough number* n: $\dfrac{1}{(\ln n)^p} \ge \dfrac{1}{n^\varepsilon}$, *the series is divergent.*

c) $\dfrac{2n}{n+2^n} \sim \dfrac{2n}{2^n}$ *as* $n \to \infty$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2(n+1)}{2^{n+1}}\frac{2^n}{2n} = \frac{1}{2} < 1, \text{ the series is convergent.}$$

d) $\ln(1+x) = x - \dfrac{x^2}{2} + o(x^2)$ *as* $x \to 0$, *so*

$$n - \ln\Big(1+\frac{1}{n}\Big) \sim \frac{1}{2n^2}, \text{ as } n \to \infty$$

the series is convergent.

e) $\sin(\pi\sqrt{n^2+a^2}) = (-1)^n\sin(\pi\sqrt{n^2+a^2} - n\pi) = (-1)^n\sin\dfrac{a^2\pi}{\sqrt{n^2+a^2}+n}$

$0 < \dfrac{a^2\pi}{\sqrt{n^2+a^2}+n} < \pi, \forall n$, *when* n *is large enough* $\Big\{\sin\dfrac{a^2\pi}{\sqrt{n^2+a^2}+n}\Big\}$ *is a positive sequence which converges to* 0. *The original series is convergent.*

f) $\{S_n\}, S_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$ *satisfy* $S_{n+2} = 4S_{n+1} - S_n$, *for all* $n \ge 0$.

By induction prove that S_n *is divisible by* 4, *then it is even for all* n.

Hence $\sin[\pi(2+\sqrt{3})^n] = -\sin[\pi(2-\sqrt{3})^n] \sim -\pi(2-\sqrt{3})^n$ *as* $n \to \infty$.

$\sum\limits_{n=0}^{\infty}\pi(2-\sqrt{3})^n$ *is convergent because* $0 < \pi(2-\sqrt{3}) < 1$, *the original series is convergent.*

g) $\alpha > 1$: $\dfrac{1}{n^\alpha(\ln n)^\beta} \le \dfrac{1}{n^{\alpha-\varepsilon}}$ *where* $0 < \varepsilon < \alpha - 1$, *the series is convergent.*

$0 < \alpha < 1$: $\dfrac{1}{n^\alpha(\ln n)^\beta} \ge \dfrac{1}{n^{\alpha+\varepsilon}}$ *where* $0 < \varepsilon < 1 - \alpha$, *the series is divergent.*

$\alpha = 1$, *see (1.3.l).*

Summary the series is convergent if and only if $\alpha > 1$ *or* $\alpha = 1, \beta > 1$; *and is divergent if* $0 < \alpha < 1$ *or* $\alpha = 1, 0 < \beta \le 1$.

h) $\lim\limits_{n\to\infty}\sqrt[n]{a_n} = \lim\limits_{n\to\infty}\Big(\cos\dfrac{a}{n}\Big)^{n^2} = e^{-\frac{a^2}{2}} < 1$, *the series is convergent.*

i) $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\dfrac{(n+1)^2}{2^{2n+1}} = 0$, *the series is convergent.*

§2. Function series

2.1 Function sequence

Assume that $f_1, f_2, \ldots, f_n, \ldots$ is a sequence of functions defined in a set $X \subset \mathbb{R}$. $x_0 \in X$ is called a convergent point of the above sequence if $\{f_n(x_0)\}$ is a convergent sequence in $\mathbb{R}$.

A sequence $\{f_n\}$ is called uniformly convergent in a set X to a function f, denoted by $f_n \overset{X}{\rightrightarrows} f$, if for an arbitrary positive number $\varepsilon > 0$, there exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$|f_n(x) - f(x)| < \varepsilon, \forall x \in X$$

The number n_0 depends only on ε, does not depend on x.

Example 2.1. The sequence $f_n(x) = \dfrac{x^n}{n}$ is uniformly convergent in $[0,1]$ to the function $f(x) = 0$ because

$$|f_n(x) - f(x)| = \frac{x^n}{n} \leq \frac{1}{n}$$

for all $x \in [0,1]$. Then for an arbitrary $\varepsilon > 0$ we can choose $n_0 = \left[\dfrac{1}{\varepsilon}\right] + 1$.

Proposition: $f_n \overset{X}{\rightrightarrows} f$ if and only if $\lim\limits_{n\to\infty} \max\limits_{x\in X} |f_n(x) - f(x)| = 0$.

2.2 Function series

Definition

A function series is $\sum\limits_{n=1}^{\infty} u_n(x)$, where $u_n(x), n \geq 1$, are functions defined in a set $X \subset \mathbb{R}$. Denote by $S_n(x)$ the $n-$th partial sum of the above function series.

The function series $\sum\limits_{n=1}^{\infty} u_n(x)$ is called to converge at a point x_0 if the sequence $\{S_n(x_0)\}$ converges, is called to converge in a set X if $\{S_n(x)\}$ converges for every point $x \in X$. The set of all convergent points of $\sum\limits_{n=1}^{\infty} u_n(x)$ is called the domain of convergence. The limits S of the sequence $\{S_n\}$ is called the sum of the function series.

Example 2.2. We consider the function series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^x}$. It is convergent if and only if $x > 1$. Then the domain of convergence of this series is $(1, +\infty)$.

The function series $\sum\limits_{n=1}^{\infty} u_n(x)$ is called to converge uniformly to a function S in a set X if the sequence $\{S_n\}$ converges uniformly to S in X.

Convergence criterion

To prove uniform convergence, we often use the following criterion.

1. $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent in X to a function $S(x)$ if and only if

$$\lim_{n\to\infty} \max_{x\in X} |S_n(x) - S(x)| = 0$$

2. Weierstrass' criteria: If for all $x \in X$, we have

$$|u_n(x)| \leq a_n, \forall n \geq 1$$

and the number series $\sum_{n=1}^{\infty} a_n$ is convergent, then the function series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly in X.

Example 2.3.

i) The function series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2+x^2}$ converge uniformly in $\mathbb{R}$ due to Weierstrass' criteria. Indeed,

$$\left|\frac{\cos nx}{n^2+x^2}\right| \leq \frac{1}{n^2+x^2} \leq \frac{1}{n^2}, \forall x \in \mathbb{R}$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

ii) Consider the function series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+x^2}$.

For each $x \in \mathbb{R}$, the corresponding number series is convergent due to Leibnitz's criteria. Denote by $S(x), x \in \mathbb{R}$ the sum of the number series. For all $x \in \mathbb{R}$, we have

$$|S(x) - S_n(x)| \leq \frac{1}{x^2+n+1} \leq \frac{1}{n+1}$$

then

$$0 \leq \lim_{n\to\infty} \max_{x\in X} |S_n(x) - S(x)| \leq \lim_{n\to\infty} \frac{1}{n+1} = 0$$

Hence $\lim_{n\to\infty} \max_{x\in X} |S_n(x) - S(x)| = 0$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+x^2}$ converges uniformly in $\mathbb{R}$.

Properties of uniformly convergent function series

Given a function series $\sum_{n=1}^{\infty} u_n$.

If $\{u_n\}, n \in \mathbb{N}$, are continuous functions in the interval $[a,b]$ and $\sum_{n=1}^{\infty} u_n$ converges uniformly in $[a,b]$ to $S(x)$, then $S(x)$ is continuous in $[a,b]$, then is integrable in this interval and

$$\int_a^b S(x)dx = \int_a^b \Big[\sum_{n=1}^{\infty} u_n(x)\Big]dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)dx$$

If $\{u_n\}, n \in \mathbb{N}$, are continuous functions together with their derivatives in the interval (a,b) and $\sum\limits_{n=1}^{\infty} u_n$ is convergent to $S(x)$, $\sum\limits_{n=1}^{\infty} u'_n$ converges uniformly in (a,b), then $S(x)$ is differentiable in (a,b) and

$$S'(x) = \Big[\sum_{n=1}^{\infty} u_n(x)\Big]' = \sum_{n=1}^{\infty} u'_n(x)$$

2.3 Power series

Power series, radial of convergence, domain of convergence

A power series is a function series of the following form

$$\sum_{n=1}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

Radial of convergence of a power series is a number such that $\sum\limits_{n=1}^{\infty} a_n x^n$ is absolutely convergent when $|x| < R$ and is divergent when $|x| > R$.

If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \rho$ (or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \rho$), then the radial of convergence is determined by

$$R = \begin{cases} \dfrac{1}{\rho} & \text{if } 0 < \rho < \infty \\ 0 & \text{if } \rho = \infty \\ \infty & \text{if } \rho = 0 \end{cases}$$

The domain of convergence contains $(-R, R)$, together with the end points $x = R$ or $x = -R$ if the power series converges at $x = \pm R$ respectively.

Properties of a power series

The power series $\sum\limits_{n=1}^{\infty} u_n$ converges uniformly in every closed interval $[a,b] \subset (-R, R)$.

The sum of the power series $\sum\limits_{n=1}^{\infty} u_n$ is continuous in its domain of convergence. We can integrate or differentiate each terms of this series:

$$\Big(\sum_{n=1}^{\infty} u_n\Big)' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$\int_a^b \Big(\sum_{n=1}^{\infty} u_n\Big) dx = \sum_{n=1}^{\infty} \int_a^b a_n x^n dx$$

for all closed intervals $[a,b] \subset (-R, R)$.

If $\sum\limits_{n=1}^{\infty} u_n(x)$ also converges at $x = \pm R$, then

$$\sum_{n=1}^{\infty} u_n(\pm R) = \lim_{x \to \pm R} S(x)$$

Example 2.4. Consider the series $\sum\limits_{n=1}^{\infty} \frac{x^n}{n}$

$R = \lim\limits_{n \to \infty} \sqrt[n]{n} = 1$, then radial of convergence is $R = 1$.

$x = 1$, $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent.

$x = -1$, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent.

So the domain of convergence is $[-1, 1)$.

We can calculte the sum $S(x)$ of the observing series. In $(-1, 1)$, $S(x)$ is differentiable and

$$S'(x) = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

Hence

$$S(x) = S(0) + \int_0^x S'(t)dt = \int_0^x \frac{dt}{1-t} = -ln(1-x)$$

Because the series also converges at $x = -1$, then

$$-ln2 = S(-1) = \lim_{x \to -1} S(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

As in the above example, we often use differentiation and integration to find the sum of a power series. We differentiate or integrate a series to obtain a new series, which we can use some fundamental series expansions to calculate this. Here are some fundamental series.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots (|x| < 1)$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 + \ldots + (-1)^n x^n + \ldots (|x| < 1)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots (x \in \mathbb{R})$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots ; (x \in \mathbb{R})$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots (x \in \mathbb{R})$$

Example 2.5. Find the sum of the series $1+\sqrt{2}+\frac{3}{2}+\ldots+\frac{n+1}{(\sqrt{2})^n}+\ldots$

We consider the function series $\sum\limits_{n=0}^{\infty}(n+1)x^n =: f(x)$.

Radial of convergence is $R=1$, then for $x\in(-1,1)$, we can integrate each term of this series in the interval $[0,x]$:

$$\begin{aligned}\int_0^x f(t)dt &= \int_0^x\Big(\sum_{n=0}^{\infty}(n+1)t^n\Big)dt = \sum_{n=0}^{\infty}\int_0^x(n+1)t^n dt\\ &= \sum_{n=0}^{\infty}x^{n+1} = \frac{x}{1-x}, x\in(-1,1)\end{aligned}$$

Hence $f(x)=\Big(\frac{x}{1-x}\Big)'=\frac{1}{(1-x)^2}$. The finding sum is

$$1+\sqrt{2}+\frac{3}{2}+\ldots+\frac{n+1}{(\sqrt{2})^n}+\ldots=f\Big(\frac{1}{\sqrt{2}}\Big)=2(3+2\sqrt{2})$$

2.4 Exercises

Exercise 6.6. *Find the domain of convergence of the following series*

$a)\ \sum\limits_{n=1}^{\infty}\frac{\ln^n(x+\frac{1}{n})}{\sqrt{x-e}}$ $\quad b)\ \sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{1+n^2x}$ $\quad c)\ \sum\limits_{n=0}^{\infty}\frac{(n+x)^n}{n^{x+n}}$

$d)\ \sum\limits_{n=1}^{\infty}\frac{\cos nx}{2^{nx}}$ $\quad e)\ \sum\limits_{n=1}^{\infty}\frac{x^n}{1+x^{2n}}$ $\quad f)\ \sum\limits_{n=1}^{\infty}\mathrm{tg}^n(x+\frac{1}{n})$

Exercise 6.7. *Examine the uniform convergence of the following function sequences and function series*

a) $f_n(x)=x^n-x^{n+1}$ *in the interval* $[0,1]$.

b) $f_n(x)=\sin\frac{x}{n}$ *in the interval* $[0,1]$ *and in* $\mathbb{R}$.

c) $\sum\limits_{n=1}^{\infty}(1-x)x^n$ *in the interval* $[0,1]$.

d) $\sum\limits_{n=1}^{\infty}\ln\Big(1+\frac{x^2}{n^2\ln n}\Big)$ *in the interval* $[-a,a],(a>0)$.

e) $\sum\limits_{n=1}^{\infty}2^n\sin\frac{x}{3^n}$ *in the interval* $[-a,a],(a>0)$.

f) $\sum\limits_{n=1}^{\infty}a^n\Big(\frac{2x+1}{x+2}\Big)^n$ *in* $[-1,1]$, *(*$|a|<1$*).*

Exercise 6.8. *Find the convergent radial and domain of convergence of the following exponential series*

$$a)\ \sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} \qquad b)\ \sum_{n=1}^{\infty} x^n \ln(n+1) \qquad c)\ \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} x^n$$

$$d)\ \sum_{n=1}^{\infty} \frac{2^n.n!}{(2n)!} x^{2n} \qquad e)\ \sum_{n=1}^{\infty} (-2)^n \frac{x^{3n+1}}{n+1} \qquad f)\ \sum_{n=1}^{\infty} \frac{2^{n-1}.x^{n-1}}{(2n-1)^2\sqrt{3^{n-1}}}$$

Exercise 6.9. *Find the sum of the following series*

$$a)\ \sum_{n=1}^{\infty} \frac{x^{n+1}}{(2n)!!} \qquad b)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{2n-1} \qquad c)\ \sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1}$$

$$d)\ \sum_{n=1}^{\infty} n^2 x^{n-1} \qquad e)\ \sum_{n=1}^{\infty} n(n+2)x^n \qquad f)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

2.5 Solution

Solution 6.6.

a) *Domain of determination: $x > e$. $\sqrt[n]{a_n} = \ln(x + \frac{1}{n}) \to \ln x > 1$ as $n \to \infty$ then the series is divergent at $x > e$. The domain of convergence is $\emptyset$.*

b) *$x = 0$, $|a_n| = 1$, the series is divergent.*

$x \neq 0$, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{1+n^2x} = \dfrac{1}{x} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{\frac{1}{x}+n^2}$.

For each x, $\{\dfrac{1}{\frac{1}{x}+n^2}\}$ is a positive decreasing sequence when n is large enough which tends to 0, so the series is convergent. The domain of convergence is $\mathbb{R}^*$.

c) *$a_n = \left(\dfrac{n+x}{n}\right)^n \dfrac{1}{n^x} \sim e^x \dfrac{1}{n^x}$, so the series is convergent iff $x > 1$. Domain of convergence is $(1, +\infty)$.*

d) *$x > 0$: $\left|\dfrac{\cos nx}{2^{nx}}\right| \leq \dfrac{1}{2^{nx}}$, the series is $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{2^x}\right)^n$ is convergent because $2^x > 1$.*

$x \leq 0$, if the series is convergent at x then necessary condition leads to

$$\lim_{n\to\infty} \frac{\cos nx}{2^{nx}} = 0 \Rightarrow \lim_{n\to\infty} \cos nx = 0,$$

this is impossible. Domain of convergence is $(0, +\infty)$.

e) *$|x| > 1$: $|a_n| = \dfrac{|x|^n}{1+x^{2n}} \sim \left(\dfrac{1}{|x|}\right)^n$ as $n \to \infty$; $\dfrac{1}{|x|} < 1$ so the series is convergent.*

$|x| < 1$: $|a_n| = \dfrac{|x|^n}{1+x^{2n}} \sim |x|^n$ as $n \to \infty$; $|x| < 1$ so the series is convergent.

$|x| = 1$, $|a_n| = \dfrac{1}{2} \nrightarrow 0$, the series is divergent. Domain of convergence is $\mathbb{R}\backslash\{\pm 1\}$.

f) $\sqrt[n]{|a_n|} = \operatorname{tg}\left(x + \frac{1}{n}\right) \to \operatorname{tg} x$ *as* $n \to \infty$.

If $\operatorname{tg} x < 1 \Leftrightarrow -\frac{\pi}{4} + k\pi < x < \frac{\pi}{4} + k\pi$, *the series is convergent.*

If $\operatorname{tg} x = 1 \Leftrightarrow x = \pm\frac{\pi}{4} + k\pi$: $a_n \to e^{\pm 2} \neq 0$ *as* $n \to \infty$, *the series is divergent.*

If $\operatorname{tg} x > 1$, *the series is divergent.*

Domain of convergence: $\left(-\frac{\pi}{4} + k\pi, \frac{\pi}{4} + k\pi\right); (k \in \mathbb{Z})$.

Solution 6.7.

a) $f(x) = 0, \forall x \in [0,1]$; $|f_n(x) - f(x)| = x^n(1-x) \leq \frac{n^n}{(n+1)^{n+1}} \leq \frac{1}{n+1} \to 0$ *as* $n \to \infty$. $f_n \overset{[0,1]}{\rightrightarrows} f$.

b) $f(x) = 0, \forall x \in \mathbb{R}$; $|f_n(x) - f(x)| = \left|\sin\frac{x}{n}\right|$

For all $x \in [0,1]$: $\left|\sin\frac{x}{n}\right| \leq \left|\frac{x}{n}\right| \leq \frac{1}{n} \to 0$ *as* $n \to \infty$. $f_n \overset{[0,1]}{\rightrightarrows} f$.

For $x \in \mathbb{R}$: $\max\limits_{x\in\mathbb{R}}\left|\sin\frac{x}{n}\right| \geq \left|\sin\frac{n}{n}\right| = \sin 1 \nrightarrow 0$ *as* $n \to \infty$. f_n *does not converge uniformly to* f *in* $[0,1]$.

c) $S_n(x) = x - x^{n+1} \to x$ *if* $0 \leq x < 1$, *and* $S_n(1) \to 0$ *as* $n \to \infty$. *The function* $f(x) = 0$ *if* $x = 1$; $f(x) = x$ *if* $0 \leq x < 1$ *is not continuous in* $[0,1]$ *then* S_n *does not converge uniformly. The series does not converge uniformly too.*

d) $\ln(1+x) \leq x, \forall x \geq 0$; *then*

$$\ln\left(1 + \frac{x^2}{n\ln^2 n}\right) \leq \frac{x^2}{n\ln^2 n} \leq \frac{a^2}{n\ln^2 n}; \forall x \in [-a,a]$$

$a^2 \sum\limits_{n=1}^{\infty} \frac{1}{n\ln^2 n}$ *is convergent then use Weierstrass' criteria, the series converges uniformly in* $[-a,a]$.

e) $\left|2^n \sin\frac{x}{3^n}\right| \leq a\left(\frac{2}{3}\right)^n$; $\sum\limits_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ *is convergent then use Weierstrass' criteria, the series converges uniformly in* $[-a,a]$.

f) $\frac{2x+1}{x+2} \in [-1,1]$ *for all* $x \in [-1,1]$, *then* $\left|a^n\left(\frac{2x+1}{x+2}\right)^n\right| \leq |a|^n$, *the series* $\sum\limits_{n=1}^{\infty}|a|^n$ *is convergent then the function series converges uniformly in* $[-1,1]$.

Solution 6.8.

a) $R = 1$, *domain of convergence is* $[-1,1)$.

b) $R = 1$, *domain of convergence is* $(-1,1)$.

c) $R = \frac{1}{e}$, *domain of convergence is* $\left(-\frac{1}{e}, \frac{1}{e}\right)$.

d) $R = +\infty$, *domain of convergence is* $(-\infty, +\infty)$.

e) $R = \dfrac{1}{\sqrt[3]{2}}$, *domain of convergence is* $\left(-\dfrac{1}{\sqrt[3]{2}}, \dfrac{1}{\sqrt[3]{2}}\right]$.

f) $R = \dfrac{\sqrt{3}}{2}$, *domain of convergence is* $\left[-\dfrac{\sqrt{3}}{2}, \dfrac{\sqrt{3}}{2}\right]$.

Solution 6.9.

a) $S(x) = x(e^{\frac{x}{2}} - 1), \forall x \in \mathbb{R}$.

b) $R = 1$, $\forall x \in (-1, 1)$: $S(x) = x \operatorname{arctg} x$.

c) $R = 1$, $\forall x \in (-1, 1)$: $S(x) = \dfrac{1}{4} \ln \dfrac{1+x}{1-x} + \dfrac{1}{2} \operatorname{arctg} x$.

d) $R = 1$, $\forall x \in (-1, 1)$: $S(x) = \dfrac{1+x}{(1-x)^3}$

e) $R = 1$, $\forall x \in (-1, 1)$: $S(x) = \dfrac{x^2 - 3x}{(x-1)^3}$

f) $S(x) = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n+1)(2n-1)} = \dfrac{1}{2} - \dfrac{x^2+1}{2x} \operatorname{arctg} x$

$\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{4n^2 - 1} = \lim\limits_{x \to 1} S(x) = \dfrac{1}{2} - \dfrac{\pi}{4}$

§3. Fourier series

3.1 Decomposition theorem

Assume that $f(x)$ is a 2π periodic funtion and integrable in the closed interval $[-\pi, \pi]$. Its Fourier series is a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

whose coefficients are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx; \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, n \geq 1$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, n \geq 1$$

Theorem 6.1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a 2π periodic function that satisfies one of the following conditions:*

i) f is piecewise continuous function and its derivatives is piecewise continuous;

ii) f is piecewise monotonous and is bounded.

Then, the Fourier series of $f(x)$ converges at every points and its sum $S(x)$ coincides with $f(x)$ at every continuous points of f. At discontinuous point c of $f(x)$, we have

$$S(c) = \frac{f(c+0) + f(c-0)}{2}$$

If $f(x)$ is an odd function then $a_n = 0, \forall n \geq 0$,

$$b_n = \frac{2}{\pi} \int\limits_0^\pi f(x) \sin nx dx, n \geq 1$$

If $f(x)$ is an even function then $b_n = 0, \forall n \geq 1$,

$$a_0 = \frac{2}{\pi} \int\limits_0^\pi f(x) dx; \quad a_n = \frac{2}{\pi} \int\limits_0^\pi f(x) \cos nx dx, n \geq 1$$

Example 3.1. Find the Fourier series of the 2π periodic function $f(x)$, $f(x) = x$ for $x \in (-\pi, \pi)$.

$f(x)$ is bounded and is an increasing function in every intervals $(-\pi + 2k\pi, \pi + 2k\pi)$, then it can be decomposed into Fourier series. We calculate the coefficients.

Because $f(x) = x$ in $(-\pi, \pi)$ is an odd function then

$$\begin{aligned} a_n &= \frac{1}{\pi} \int\limits_{-\pi}^\pi f(x) \cos nx dx = 0, n \geq 0 \\ b_n &= \frac{2}{\pi} \int\limits_0^\pi x \sin nx dx = \frac{2}{\pi} \Big[-\frac{x}{n} \cos nx \Big|_0^\pi + \int\limits_0^\pi \frac{\cos nx}{n} dx \Big] \\ &= (-1)^{n+1} \frac{2}{n}; (n \geq 1) \end{aligned}$$

Hence for $x \neq (2n+1)\pi$,

$$f(x) = 2 \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} \sin nx$$

Theorem 6.2. *If $f(x)$ is $2l$ periodic function which also satisfies one of the conditions mentioned in the above theorem in the interval $[-l, l]$, then $f(x)$ can be decomposed into Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}$$

whose coefficients are

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx, n \geq 1$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx, n \geq 1$$

Example 3.2. Find the Fourier series of the $2l$ periodic function $f(x)$ that $f(x) = x$ in $(a, a+2l)$.

$f(x)$ is bounded and is an increasing function in every intervals $(a+2kl, a+2(k+1)l)$, then it can be decomposed into Fourier series. We calculate the coefficients

Because the integral of a periodic function in every interval whose length is equal to the periodic is the same then

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{1}{l}\int_{a}^{a+2l} f(x)dx = \frac{1}{l}\int_{a}^{a+2l} xdx = 2(a+l)$$

$$\begin{aligned}
a_n &= \frac{1}{l}\int_{a}^{a+2l} x\cos\frac{n\pi x}{l}dx = \frac{1}{l}\Big(\frac{xl}{n\pi}\sin\frac{n\pi x}{l}\Big|_a^{a+2l} - \frac{l}{n\pi}\int_a^{a+2l}\sin\frac{n\pi x}{l}dx\Big)\\
&= \frac{1}{l}\Big(\frac{2l^2}{n\pi}\sin\frac{\pi na}{l} + \frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\Big|_a^{a+2l}\Big)\\
&= \frac{2l}{n\pi}\sin\frac{\pi na}{l}, n \geq 1\\
b_n &= \frac{1}{l}\int_{a}^{a+2l} x\sin\frac{n\pi x}{l}dx = \frac{1}{l}\Big(\frac{-xl}{n\pi}\cos\frac{n\pi x}{l}\Big|_a^{a+2l} + \frac{l}{n\pi}\int_a^{a+2l}\cos\frac{n\pi x}{l}dx\Big)\\
&= \frac{-2l}{n\pi}\cos\frac{\pi na}{l}, n \geq 1
\end{aligned}$$

Hence for $x \neq a + 2nl$,

$$\begin{aligned}
f(x) =&(a+l) + \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\Big(\sin\frac{n\pi a}{l}\cos\frac{n\pi x}{l} - \cos\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\Big)\\
=&a + l + \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{l}(a-x)
\end{aligned}$$

Now we consider a function which satisfies one of the conditions in the first theorem in a closed $[a, b]$. To expand $f(x)$ into a Fourier series, we construct a periodic function $g(x)$ whose periodic is larger or equal $b - a$ such that

$$g(x) = f(x), \forall x \in [a, b]$$

If $g(x)$ can be expanded into Fourier series then its sum coincides with $g(x)$, and also $f(x)$, at every continuous points in $[a, b]$. If $g(x)$ is an odd function then its Fourier series is sum of sine functions. If $g(x)$ is an even function then its Fourier series is sum of cosine functions.

Example 3.3. Find the expansion of the function $f(x) = x$ for $0 < x < 2$ into Fourier series of cosine functions and into Fourier series of sine functions.

To expand $f(x)$ into Fourier series of cosine functions we construct an even function $g(x)$, which is 4 periodic function and $g(x) = x$ for $0 < x < 2$, $l = 2$. $g(x)$ is even then $b_n = 0, n \geq 1$

$$\begin{aligned}
a_0 &= \int_0^2 x dx = 2; \\
a_n &= \int_0^2 x \cos \frac{n\pi x}{2} dx = \frac{2x}{n\pi} \sin \frac{n\pi x}{2}\Big|_0^2 - \frac{2}{n\pi} \int_0^2 \sin \frac{n\pi x}{2} dx \\
&= \frac{4}{n^2\pi^2}(\cos n\pi - 1) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{8}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Hence for $0 < x < 2$,

$$f(x) = 1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos \frac{(2n+1)\pi x}{2}$$

To expand $f(x)$ into Fourier series of sine functions we construct an odd function $h(x)$, which is 4 periodic function and $h(x) = x$ for $0 < x < 2$, $l = 2$. $h(x)$ is odd then $a_n = 0, n \geq 0$

$$\begin{aligned}
b_n &= \int_0^2 x \sin \frac{n\pi x}{2} dx = -\frac{2x}{n\pi} \cos \frac{n\pi x}{2}\Big|_0^2 + \frac{2}{n\pi} \int_0^2 \cos \frac{n\pi x}{2} dx \\
&= -\frac{4}{n\pi} \cos n\pi = 4\frac{(-1)^{n+1}}{n\pi}
\end{aligned}$$

Hence for $0 < x < 2$,

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{2}$$

3.2 Exercises

Exercise 6.10. Find the Fourier expansion of the following functions

a) $f(x)$ is a periodic function with $T = 2\pi$ and $f(x) = |x|$ in the interval $[-\pi, \pi]$.

b) $f(x)$ is a periodic function with $T = 2\pi$, and $f(x) = \dfrac{\pi - x}{2}$ in the interval $(0, 2\pi)$.

c) $f(x)$ is a periodic function with $T = 2\pi$ and $f(x) = \sin ax$ in the interval $(-\pi, \pi)$, $a \neq \mathbb{Z}$.

Exercise 6.11. Decompose the following functions into Fourier series of cosine and sine functions

a) $f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq h \\ 1 & \text{if } h < x \leq \pi \end{cases}$ in the interval $[0, \pi]$.

b) $f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x < 2 \\ 3 - x & \text{if } 2 \leq x \leq 3 \end{cases}$ in the interval $(0, 3)$.

3.3 Solution

Solution 6.10.

a) $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi} \sum\limits_{n=0}^{\infty} \dfrac{\cos(2n+1)x}{(2n+1)^2}, \forall x \in [-\pi, \pi].$

b) $f(x) = \sum\limits_{n=1}^{\infty} \dfrac{\sin nx}{n}, \forall x \in (0, 2\pi).$

c) $f(x) = \dfrac{2 \sin a\pi}{\pi} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n n}{a^2 - n^2} \sin nx, \forall x \in (-\pi, \pi).$

Solution 6.11.

a) $f(x) = \dfrac{2}{\pi} \sum\limits_{n=1}^{\infty} \dfrac{\cos(nh) + (-1)^{n+1}}{n} \sin nx$ *and* $f(x) = \dfrac{\pi - h}{\pi} - \dfrac{2}{\pi} \sum\limits_{n=1}^{\infty} \dfrac{\sin(nh)}{n} \cos nx$

b) $f(x) = \dfrac{2}{3} + \dfrac{3}{\pi} \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2} \left(\cos \dfrac{2n\pi}{3} - 1 \right) \cos \dfrac{2n\pi x}{3}$ *and*

$$f(x) = \frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sin \frac{n\pi}{3} + \sin \frac{2n\pi}{3} \right) \sin \frac{2n\pi x}{3}$$