ĐẠI HỌC BÁCH KHOA

25 YEARS ANNIVERSARY SOICT

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FUNDAMENTALS OF OPTIMIZATION

Mathematical foundation

Optimization problems

- Maximize or minimize some function relative to some set (range of choices)
- The function represents the quality of the choice, indicating which is the “best”
- Example
  - A shipper need to find the shortest route to deliver packages to customers 1, 2, …, N

Notations

- $x \in R^n$: vector of decision variables x_j, $j = 1, 2, \ldots, n$
- $f : R^n \rightarrow R$ is the objective function (**dom** $f = R^n$)
- $g_i : R^n \rightarrow R$ is the constraint function defining restriction on x, $i = 1, 2, \ldots, m$

minimize $f(x)$ over $x = (x_1, x_2, \ldots, x_n) \in X \subset R^n$ satisfying a property P:

$g_i(x) \leq b_i$, $i = 1, 2, \ldots, s$
$g_i(x) = d_i$, $i = s + 1, 2, \ldots, m$

Convex sets

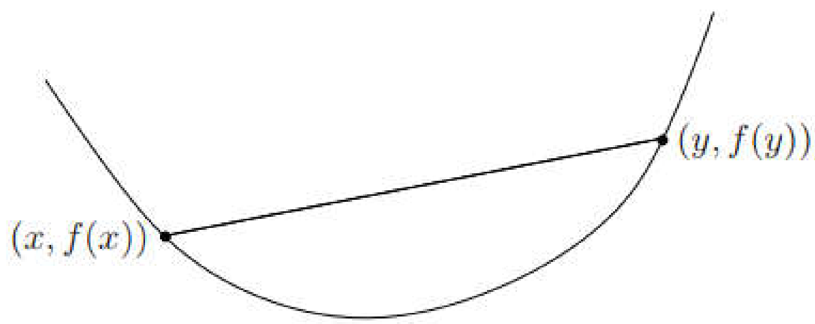
- S is called a convex set if: $\forall$ $u_1, u_2, \ldots, u_k$ in S, $\forall$ non-negative numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = 1$, then $\sum_{i=1}^{k} u_i\lambda_i$ is in S

Convex set

Non Convex set

Convex functions

- Linear function: $f(x) = Ax$
- Affine function: $f(x) = Ax + b$
- Convex function
  - f is called convex if $\forall x_1, x_2$ and $\forall \lambda \in [0,1]$:
    $$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$
  - f is called strictly convex if $\forall x_1 \neq x_2$ and $\forall \lambda \in (0,1)$:
    $$f(\lambda x_1 + (1 - \lambda)x_2) < \lambda f(x_1) + (1 - \lambda)f(x_2)$$



Convex functions

- Example: $f(x) = 2x + 3$
  - $f(\lambda x_1 + (1-\lambda)x_2) = 2(\lambda x_1 + (1-\lambda)x_2) + 3$

    $= (2\lambda x_1 + 3\lambda) + (2(1-\lambda)x_2) + (1-\lambda)3)$

    $= \lambda f(x_1) + (1-\lambda)f(x_2)$

Convex functions

- Examples
  - $f(x) = x^2$
  - $f(x) = x^n$, n is a constant is convex on x > 0
    - Is convex if n $\geq$ 1 or n $\leq$ 0
    - Is concave is 0 $\leq$ n $\leq$ 1
  - $f(x) = e^{ax}$, a is a constant
  - $f(x) = x\ln x$

Basis

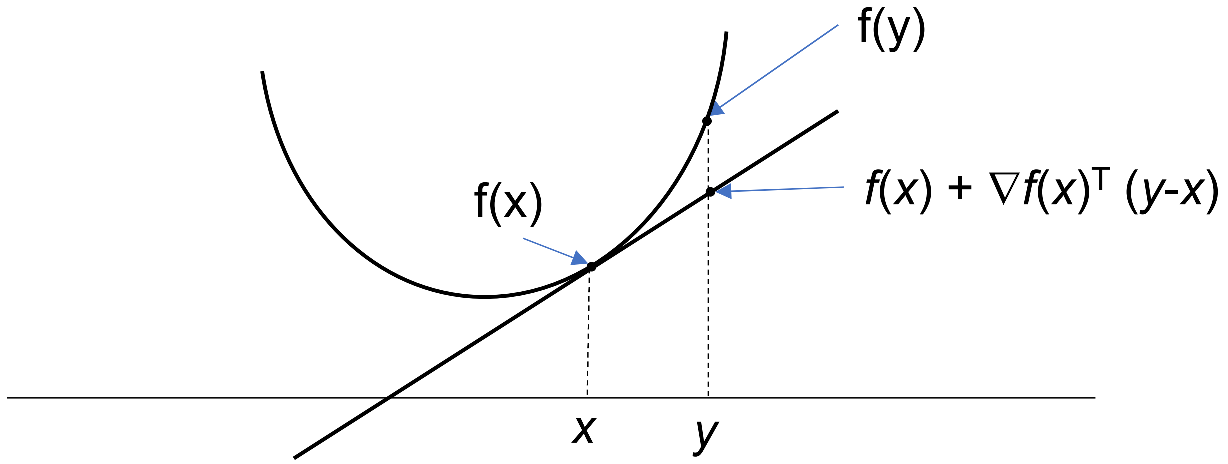
- $f(x_1, x_2, \ldots, x_n)$ is a multivariable function

$$\nabla f(x) \ \ (\text{or } f'(x)) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \frac{\partial f}{\partial x_2}(x) \\ \ldots \\ \frac{\partial f}{\partial xn}(x) \end{pmatrix}$$

$$\nabla^2 f(x) \ \ (\text{or } f''(x)) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)$$ called Hessian matrix

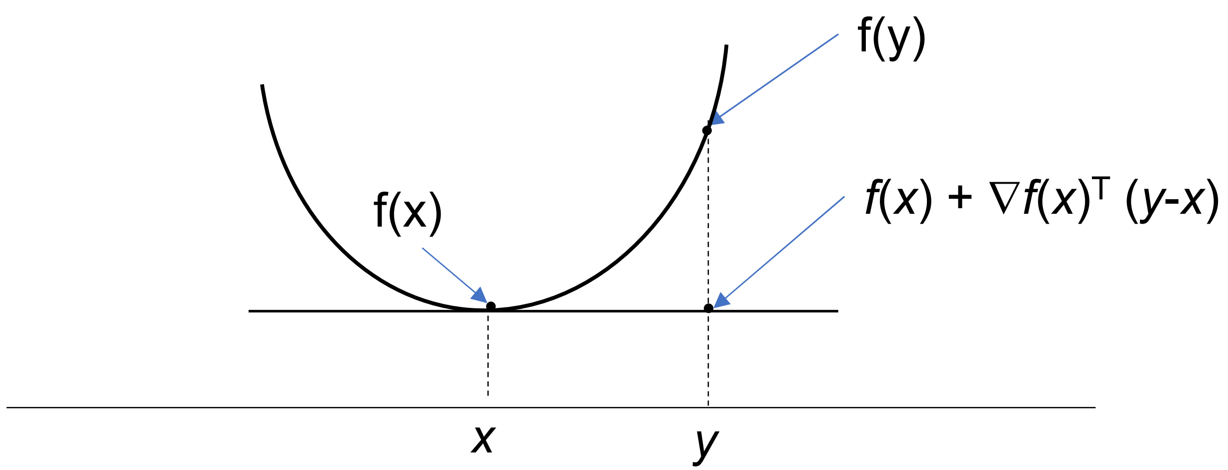
Convex functions

- First-order condition
  - Suppose f is differentiable (i.e., its gradient exists at all points in **dom** f, which is open). f is convex if and only if **dom** f is convex and $f(x) + \nabla f(x)^{T} (y-x) \leq f(y), \forall x,y \in$ **dom** f



Basis

- If $\nabla f(x) = 0$, then $f(y) \geq f(x)$, $\forall y \in$ **dom** f → x is global minimizer of the function f



Norms

- Norm: A real-valued function $f(x)$ on R^n is called a norm, if
  - $f(x) \geq 0$
  - $\lambda f(x) = f(\lambda x)$
  - $f(x + y) \leq f(x) + f(y)$ (triangle inequality)
- Examples
  - $||x||_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{1/p}$
  - $||x||_1 = (|x_1| + |x_2| + \ldots + |x_n|)$
  - $||x||_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$

Taylor approximation

- Single variable Taylor series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- First-order Taylor approximation

$$f(x + h) \approx f(x) + h^T \nabla(x)$$

- Second-order Taylor approximation

$$f(x+h) \approx f(x) + h^T \nabla f(x) + \frac{1}{2} h^T \nabla^2 f(x) h$$

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Thank you for your attentions!

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