Second-order differential equations

Tuesday, November 9, 2021 7:18 AM

A second-order differential equation has the form

$$F\left(x, y, y', y''\right) = 0$$

We may have an ordinary differential equation without initial conditions or we may have an initial value problem (IVP).

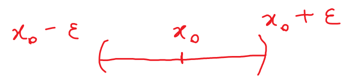
<u>Cauchy's problem:</u>

$$\begin{cases} y'' = f(x, y, y') \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$$

<u>Theorem:</u> We assume that $f(x, y, y')$, $\dfrac{\partial f}{\partial y}(x, y, y')$, $\dfrac{\partial f}{\partial y'}(x, y, y')$ are continuous on a domain D in $\mathbb{R}^3$; suppose $(x_0, y_0, y_1) \in D$.

Then there is an open neighborhood around x_0 such that there exists a unique solution to Cauchy's problem in this neighborhood

"local solution"


$x_0 - \varepsilon$
x_0
$x_0 + \varepsilon$

Linear second-order differential equations

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$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

This is a linear second-order differential equation.

$p(x)$, $q(x)$: coefficients

If $f(x) = 0$: homogeneous equation

If $f(x) \neq 0$: inhomogeneous equation

For homogeneous equation

$$y'' + p(x)\, y' + q(x)\, y = 0$$

If y_1, y_2 are solutions: $\begin{cases} y_1'' + p(x) y_1' + q(x) y_1 = 0 \\ y_2'' + p(x) y_2' + q(x) y_2 = 0 \end{cases}$

$$\Rightarrow (y_1 + y_2)'' + p(x)(y_1 + y_2)' + q(x)(y_1 + y_2) = 0$$

$\Rightarrow$ $y_1 + y_2$ is a solution

More generally, $k_1 y_1 + k_2 y_2$ is a solution (k_1, k_2 constants)

<u>Example</u> $y'' + y = 0$

- $y_1 = \cos x$, $y_2 = \sin x$
- $y = k_1 \cos x + k_2 \sin x$ is a general solution.

How about inhomogeneous equation.

$$y'' + p(x)\,y' + q(x)\,y = f(x) \qquad (I)$$

We want to solve (I)

$$y'' + p(x)\,y' + q(x)\,y = 0 \qquad (H)$$

the complementary equation of (I)

(phương trình bổ trợ)

Suppose y_p is a solution of (I) and y_c of (H)

then $y_c + y_p$ is a solution of (I).

Linear 2nd order ODE, homogeneous, constant coefficients

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$$y'' + p(x)\,y' + q(x)\,y = 0$$

$p(x)$, $q(x)$ · coefficients

We assume that $p(x)$, $q(x)$ are constants

We have

$$\boxed{\begin{array}{c} ay'' + by' + cy = 0 \\ \text{linear, 2nd order, constant coefficients} \end{array}}$$

Auxiliary quadratic equation :

$$ar^2 + br + c = 0$$

Auxiliary quadratic equation	General solution to DE
2 distinct real zeros r_1, r_2	$y = k_1 e^{r_1 x} + k_2 e^{r_2 x}$
a double real zero $r = r_1 = r_2$	$y = k_2 e^{rx} + k_2 x e^{rx}$
2 complex zeros $\alpha \pm i\beta$	$y = k_1 e^{\alpha x} \cos \beta x + k_2 e^{\alpha x} \sin \beta x$

We can check that these are solutions very easily by just plugging in the ODE.

Example. $y'' + y = 0$

Auxiliary equation $r^2 + 1 = 0$

$r = \pm i$

General solution : $y = k_1 \cos x + k_2 \sin x$

Example. solve $4y'' + 12y' + 9y = 0$

Example. solve $4y'' + 12y' + 9y = 0$

Auxiliary equation $4r^2 + 12r + 9 = 0$

$$(2r+3)^2 = 0$$

$$r = -3/2$$

$$\langle e^{rx},\ x \cdot e^{rx} \rangle$$

General solution $y = k_1 e^{rx} + k_2 x e^{rx}$.

Linear 2nd order ODE, inhomogeneous, constant coefficients

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2 methods:
- undetermined coefficients
- variation of parameters

$$y'' + py' + qy = f(x) \qquad (I)$$

The above 2 methods are used to find a particular solution y_p to (I).

Method of Undetermined Coefficients

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$$y'' + py' + qy = f(x)$$

(1) If $f(x) = e^{kx} \cdot P(x)$, where $P(x)$ is a polynomial of degree n, then we try to find a solution

$$y_p = e^{kx} \cdot Q(x)$$ where $Q(x)$ is a polynomial of degree n.

In some cases, we may have to find

$$y_p = e^{kx} \, x \, Q(x) \quad \text{or} \quad e^{kx} \cdot x^2 Q(x)$$

(2) If $f(x) = e^{kx} \, P(x) \cdot \cos(mx)$

or $f(x) = e^{kx} \cdot P(x) \cdot \sin(mx)$

with $P(x)$ a polynomial of degree n,

then we try $y_p = e^{kx} \cdot Q(x) \cdot \cos(mx) + e^{kx} \, R(x) \cdot \sin(mx)$

with $Q(x)$, $R(x)$ polynomials of degree n

Sometimes we may have to multiply $Q(x)$ and $R(x)$ by either x or x^2

Examples

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Example: solve $y'' + y' - 2y = x^2 \quad (I)$

Here $f(x) = x^2 = e^{0x} \cdot x^2$

$y'' + y' - 2y = 0 \quad (H)$

Auxiliary equation $r^2 + r - 2 = 0$

$(r-1)(r+2) = 0$

$r = 1, -2$

General solution to (H) $\quad y_c = k_1 e^{x} + k_2 e^{-2x}$

$(I) \quad y'' + y' - 2y = x^2$

we find $\quad y_p = e^{0x} \underbrace{Q(x)}_{\text{a polynomial of degree 2}}$

$y_p = Ax^2 + Bx + C$

A, B, C : undetermined coefficients

$y_p' = 2Ax + B$

$y_p'' = 2A$

$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$

$\Rightarrow A = -\frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = -\frac{3}{4}$

$y_p = -\frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}$

So the general solution to (I) is

$y = y_c + y_p = k_1 e^{x} + k_2 e^{-2x} + \left(-\frac{x^2}{2} - \frac{x}{2} - \frac{3}{4}\right)$

Example. solve $y'' - 4y' + 13y = \underline{e^{2x} \cos 3x} \quad (I)$

Complementary equation.

$$y'' - 4y' + 13y = 0 \quad (H)$$

Auxiliary equation $r^2 - 4r + 13 = 0$

$$r = 2 \pm 3i$$

General solution to (H) $\quad y_c = k_1 e^{2x} \cos 3x + k_2 e^{2x} \sin 3x$

Note that $f(x) = \underline{e^{2x} \cos 3x}$ is a solution to (H)

For (I) $\quad y'' - 4y' + 13y = e^{2x} \cos 3x$,

we find $\quad y_p = x \left(e^{2x} \cos 3x \cdot \underbrace{Q(x)}_{\substack{\text{polynomial} \\ \text{of degree } 0}} + e^{2x} \sin 3x \cdot \underbrace{R(x)}_{\substack{\text{polynomial} \\ \text{of degree } 0}} \right)$

(If we had a double zero, we multiply by x^2.)

So we have $\quad y_p = x \left(Q \cdot e^{2x} \cos 3x + R \cdot e^{2x} \sin 3x \right)$

(Q, R constants)

is a solution of $y'' - 4y' + 13y = e^{2x} \cos 3x$

Then we find y_p', y_p'', plug in to compute Q, R

Method of Variation of Parameters

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$$y'' + p y' + q y = f(x) \qquad (I)$$

Complementary equation

$$y'' + py' + qy = 0 \qquad (H)$$

with general solution

$$y_c = k_1 \cdot y_1(x) + k_2 \, y_2(x)$$

$(k_1, k_2$ constants$)$

We will find a solution y_p to (I) of the form

$$y_p = u_1(x) \, y_1(x) + u_2(x) \, y_2(x)$$

$(u_1(x), u_2(x)$ functions$)$

Example of Variation of Parameters

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Example: solve $y'' + y = \tan x \quad \left(0 < x < \frac{\pi}{2}\right) \quad (I)$

Complementary equation $y'' + y = 0 \quad (H)$

$$y_c = k_1 \cos x + k_2 \sin x$$

We find a solution y_p of (I)

$$y_p = u_1(x) \cos x + u_2(x) \sin x$$

$$y_p'' + y_p = \tan x$$

$$y_p' = \underbrace{\left(u_1' \cos x + u_2' \sin x\right)}_{\text{we set this to be zero}} + \left(u_1(-\sin x) + u_2 \cos x\right)$$

$$u_1' \cos x + u_2' \sin x = 0$$

Then $y_p' = -u_1 \sin x + u_2 \cos x$

$$y_p'' = \left(-u_1' \sin x + u_2' \cos x\right) + \left(-u_1 \cos x - u_2 \sin x\right)$$

$$y_p = u_1 \cos x + u_2 \sin x$$

$$y_p'' + y_p = \tan x,$$

so $\begin{cases} -u_1' \sin x + u_2' \cos x = \tan x \\ u_1' \cos x + u_2' \sin x = 0 \end{cases}$

Solve for u_1', u_2'

Integrate to find u_1, u_2

$$u_2' = \sin x, \quad u_1' = -\frac{\sin^2 x}{\cos x}$$

$$u_2 = -\cos x \;, \quad u_1 = \sin x - \ln\left(\frac{1}{\cos x} + \tan x\right)$$

$$y_p = u_1 \cos x + u_2 \sin x \quad \text{solution of (I)}$$

General solution to (I) is $y = y_c + y_p$

2nd order ODE: without y and y'

Tuesday, November 9, 2021 8:58 AM

$$F(x, y, y', y'') = 0$$

If there is no y and y'.

$$F(x, y'') = 0$$

Case 1: $y'' = g(x)$

Integrate to find y'

Integrate to find y

Case 2 $x = h(y'')$

Idea. convert 2nd order ODE to 1st order ODE

Put $p = y'$, so $p' = y''$

We have $x = h(p')$

this is first-order!

2nd order ODE: without y

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$$F(x, y, y', y'') = 0$$

If there is no y

$$F(x, y', y'') = 0$$

We convert it to first-order!

Put $p = y'$, so $p' = y''$

$$F(x, p, p') = 0$$

This is first-order ODE of $p = p(x)$

2nd order ODE: without x

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$$F(x, y, y', y'') = 0$$

If there is no x

$$F(y, y', y'') = 0$$

We want to make it first-order!

Put $p = y'$, so $p'(x) = y''$.

$$y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \underbrace{\frac{dy}{dx}}_{= y' = p} = \frac{dp}{dy} \cdot y' = \frac{dp}{dy} \cdot p$$

$$F\left(y, p, \frac{dp}{dy} \cdot p\right) = 0$$

This is first-order! Think of p as a function of y, and you have a relation of $y, p, \frac{dp}{dy}$.