Chapter 2: Random Variables and Probability Distributions

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Nội dung

Random Variables

Introduction

In this chapter and for most of the remainder of the course,

1. We will examine probability models that assign numbers to the outcomes in the sample space.
2. When we observe one of these numbers, we refer to the observation as a random variable.
3. The name of a random variable is always a capital letter, for example, X.
4. The set of possible values of X is the range of X. Denote by S_X.

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A probability model always begins with an experiment. Each random variable is related directly to this experiment. There are three types of relationships.

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2.1.1 Random Variable

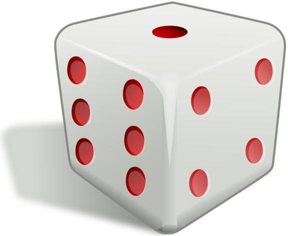
Definition 2.1 (Random Variable)

A variable X is a **random variable** if the value that it assumes, corresponding to the outcome of an experiment, is random event.

2.1.1 Random Variable

Example 2.1 (Some random variables)

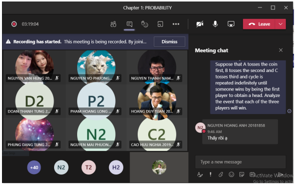
(1) X, the **number of dots** on the side facing upwards when we roll a six-sided die.



2.1.1 Random Variable

Example 2.1

(2) Y, the **number of students** of asleep in the next probability lecture.



2.1.1 Random Variable

Example 2.1

- Z, the **number of phone calls** you answer in the next hour.

2.1.1 Random Variable

Example 2.1

- V, the **number of minutes** you wait until you next answer the phone.

2.1.1 Random Variable

Example 2.1

- W, the **distance** from the center of the beer to the target point when you shoot a bullet into a beer.

2.1.1 Random Variable

Discrete random and Continuous random variable

1. Random variables X, Y, and Z are **discrete random** variables. The possible values of these random variables form a **countable set**.

   $S_X = \{1, 2, \ldots, 6\}$; $S_Y = \{1, 2, \ldots, n\}$; $S_Z = \{1, 2, \ldots, n, \ldots\}$.

2. The random variable V and W can be any nonnegative real number. It is a **continuous random** variable.

   $S_V = (0; a)$, $a > 0$; $S_W = [0, r]$, $r > 0$ is a radius of beer.

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2.1.2 Discrete Random Variable

Definition 2.2 (Discrete sample space)

If a sample space contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers, it is called a **discrete sample space**.

2.1.2 Discrete Random Variable

Definition 2.3 (Discrete Random Variable)

X is a **discrete random variable** if the range of X is a countable set

$$S_X = \{x_1, x_2, \ldots\}.$$

Definition 2.4 (Finite Random Variable)

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Example 2.2 (Some examples of discrete random variable)

1. X is outcome of rolling one die. X is the finite random variable.

$$S_X = \{1, 2, \ldots, 6\}.$$

2. Shoot continuously on a beer until you hit the target. Call Y the number of bullets used to shoot.

$$S_Y = \{1, 2, \ldots\}.$$

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If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a **continuous sample space**.

Definition 2.6 (Continuous random variable)

When a random variable can take on values on a continuous scale, it is called a **continuous random variable**.

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2.2.1 Probability Function

Definition 3.1 (Probability/Probability mass function)

The set of ordered pairs $(x, P_X(x))$ is a probability function, or probability mass function of the discrete random variable X if, for each possible outcome x,

1. $P_X(x) = P[X = x]$.
2. $P_X(x) \geq 0$ **for all** x.
3. $\sum_{x \in S_X} P_X(x) = 1$.

Remark 3.1

1. $[X = x]$ is an event consisting of all outcomes s of the underlying experiment for which $X(s) = x$.
2. $P_X(x)$ is a function ranging over all real numbers x.
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Example 3.1

Suppose we observe three calls at a telephone switch where voice calls (V) and data calls (D) are equally likely. Let X denote the number of voice calls, Y the number of data calls, and let $R = XY$. The sample space of the experiment and the corresponding values of the random variables X, Y, and R are

Outcomes	DDD	DDV	DVD	DVV	VDD	VDV	VVD	VVV
P[.]	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
random variable X	0	1	1	2	1	2	2	3
random variable Y	3	2	2	1	2	1	1	0
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What is the PMF of R?

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2.2.1 Probability Function

Example 3.1 Solution

- We see that $R = 0$ if either outcome, DDD or VVV, occurs so that

$$P[R = 0] = P[DDD] + P[VVV] = \frac{1}{4}.$$

- For the other six outcomes of the experiment $R = 2$ so that $P[R = 2] = 6/8$.
- The PMF of R is

$$P_R[r] = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0, & \text{otherwise.} \end{cases}$$

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2.2.2 Probability Distribution

Definition 3.2 (Probability distribution)

The **probability distribution** for a discrete random variable X is a formula, table, or graph that gives the possible values of X, and the probability associated with each value of X.

$X = x_i$	x_1	x_2	$\ldots$	x_n
$P[X = x_i]$	p_1	p_2	$\ldots$	p_n

(2.1)

where $p_i = P[X = x_i]$, $i = 1, 2, \ldots, n$.

Note

$$\sum_{i=1}^{n} P[X = x_i] = 1.$$

2.2.2 Probability Distribution

Example 3.2

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

2.2.2 Probability Distribution

Example 3.2 Solution

1. Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school.
2. X can only take the numbers 0, 1, and 2.
3. $P_X(0) = P[X = 0] = \dfrac{C_3^0 C_{17}^2}{C_{20}^2} = \dfrac{68}{95}$;

   $P_X(1) = P[X = 1] = \dfrac{51}{190}$;

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4. Thus, the probability distribution of X is

$X = x_i$	0	1	2
$P[X = x_i]$	$\frac{68}{95}$	$\frac{51}{190}$	$\frac{3}{190}$

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3. $$P_X(0) = P[X = 0] = \frac{C_3^0 C_{17}^2}{C_{20}^2} = \frac{68}{95};$$
   $$P_X(1) = P[X = 1] = \frac{51}{190};$$
   $$P_X(2) = P[X = 2] = \frac{3}{190}.$$
4. Thus, the probability distribution of X is

$X = x_i$	0	1	2
$P[X = x_i]$	$\frac{68}{95}$	$\frac{51}{190}$	$\frac{3}{190}$

2.2.2 Probability Distribution

Theorem 3.1

For a discrete random variable X with PMF $P_X(x)$ and range S_X. If $B \subset S_X$, the probability that X is in the set B is

$$\boxed{P[B] = \sum_{x \in B} P_X(x)} \tag{2.2}$$

Proof

Since the events $\{X = x\}$ and $\{X = y\}$ are disjoint when $x \neq y$, B can be written as the union of disjoint events $B = \cup_{x \in B}\{X = x\}$. Thus,

$$P[B] = \sum_{x \in B} P[X = x] = \sum_{x \in B} P_X(x).$$

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2.2.2 Probability Distribution

Practice Test 1

A midterm test has 4 multiple choice questions with four choices with one correct answer each. Assume that you answer all and you will get $(+5)$ points for 1 question correct, (-2) points for 1 question wrong. Let X is number of points that you get. (a) Find the probability mass function of X. (b) Find the probability distribution of X.

2.2.3 Cumulative Distribution Function

Definition 3.3 (Cumulative distribution function)

The cumulative distribution function (CDF) $F_X(x)$ of a discrete random variable X with probability distribution $P_X(x)$ is

$$\boxed{F_X(x) = P[X < x] = \sum_{t<x} P_X(t), \quad \text{for } -\infty < x < \infty} \tag{2.3}$$

2.2.3 Cumulative Distribution Function

Note

If X is a discrete random variable with probability distribution is (2.1), then the CDF is

$$F_X(x) = \begin{cases} 0, & x \le x_1, \\ p_1, & x_1 < x \le x_2, \\ p_1 + p_2, & x_2 < x \le x_3, \\ \dots & \\ 1, & x > x_n. \end{cases} \tag{2.3.1}$$

2.2.3 Cumulative Distribution Function

Figure of CDF

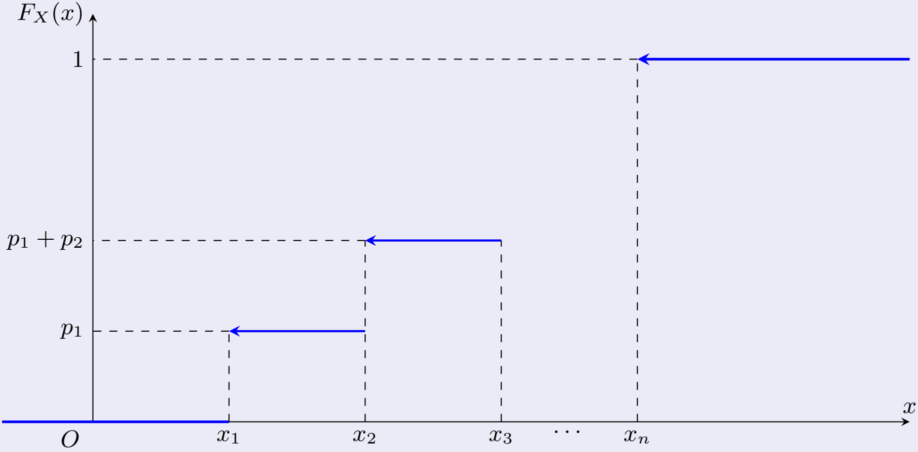

Figure: Figure of the CDF (2.3.1)

2.2.3 Cumulative Distribution Function

Theorem 3.2

(a) $0 \le F_X(x) \le 1$.

(b) For all $x_1 < x_2$, $F_X(x_1) \le F_X(x_2)$ and $\lim_{x\to a^-} F_X(x) = F_X(a)$ for all $a \in \mathbb{R}$.

(c) $F_X(-\infty) = 0$ and $F_X(+\infty) = 1$.

(d) $F_X(b) - F_X(a) = P[a \le X < b]$.

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2.2.3 Cumulative Distribution Function

Theorem 3.2 Proof

(a) $F_X(x) = P[X < x]$.

(b) For all $x_1 < x_2$, $P[X < x_2] = P[X < x_1] + P[x_1 \leq X < x_2]$.

(c) $F_X(-\infty) = P[X < -\infty] = P[\emptyset]$ and $F_X(+\infty) = P[X < \infty] = P[S]$.

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2.2.3 Cumulative Distribution Function

Example 3.3

In Example 3.1, we found that random variable R has PMF

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find and sketch the CDF of random variable R.

Example 3.3 Solution

From the PMF $P_R(r)$, random variable R has CDF

$$F_R(r) = P[R < r] = \begin{cases} 0, & r \leq 0, \\ 1/4, & 0 < r \leq 2, \\ 1, & r > 2. \end{cases}$$

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Nội dung

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Definition 4.1 (Cumulative distribution function)

The cumulative distribution function (CDF) of random variable X is

$$\boxed{F_X(x) = P[X < x], \quad x \in \mathbb{R}} \tag{2.4}$$

Definition 4.2 (Continuous random variable)

X is a continuous random variable if the CDF $F_X(x)$ is a continuous function.

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2.3.2 Probability Density Function

Definition 4.3 (Probability density function)

The probability density function (PDF) of a continuous random variable X is

$$\boxed{f_X(x) = \frac{dF_X(x)}{dx}} \tag{2.5}$$

Theorem 4.1 (Properties of the PDF)

For a continuous random variable X with PDF $f_X(x)$,

(a) $f_X(x) \geq 0$ for all x;

(b) $\int\limits_{-\infty}^{+\infty} f_X(x)dx = 1$;

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2.3.2 Probability Density Function

Proof

(a) $F_X(x)$ is a nondecreasing function of x;

(b) Follows from $F_X(-\infty) = 0$ and $F_X(+\infty) = 1$;

(c) Follows directly from the definition of $f_X(x)$ and the fact that $F_X(-\infty) = 0$.

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2.3.2 Probability Density Function

Theorem 4.2

$$\boxed{P[a \leq X < b] = \int\limits_a^b f_X(x)dx} \tag{2.6}$$

Proof

$$\begin{aligned} P[a \leq X < b] &= P[X < b] - P[X < a] = F_X(b) - F_X(a) \\ &= \int\limits_{-\infty}^{b} f_X(x)dx - \int\limits_{-\infty}^{a} f_X(x)dx \\ &= \int\limits_a^b f_X(x)dx. \end{aligned}$$

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2.3.2 Probability Density Function

The probability that X will fall into a particular interval say, from a to b is equal to the area under the curve between the two points a and b. This is the shaded area in Figure 2.

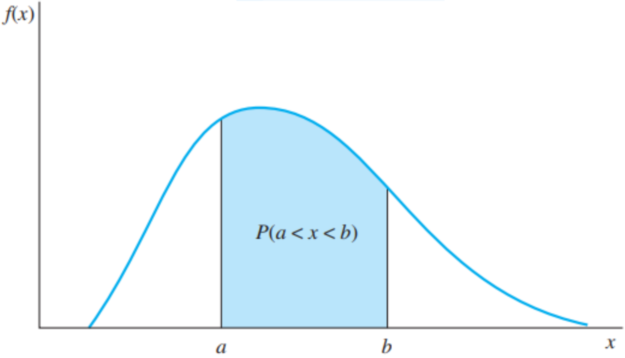

Figure: 2 The probability distribution $f_X(x)$; $P[a < X < b]$ is equal to the shaded area under the curve

2.3.2 Probability Density Function

Remark 4.1

$P[X = a] = 0$ for continuous random variables.

- Follows directly from $P[a \le X < b] = F_X(b) - F_X(a)$ and $F_X(x)$ is a continuous function.
- $P[X \ge a] = P[X > a]$ and $P[X \le a] = P[X < a]$.
- This is not true in general for discrete random variables.

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2.3.2 Probability Density Function

Example 4.2

The continuous random variable X has PDF $f_X(x) = ae^{-|x|}$, $(-\infty < x < +\infty)$. Define the random variable Y by $Y = X^2$.

(a) What is a?

(b) What is the CDF $F_X(x)$?

(c) What is the CDF $F_Y(x)$?

(d) Find $P[0 < X < \ln 3]$?

2.3.2 Probability Density Function

Example 4.2 Solution

(a) It follows from $f_X(x) \geq 0, \forall x$ and $\int_{-\infty}^{+\infty} f_X(x)dx = 1$ that

- $a \geq 0$ and
- $1 = \int_{-\infty}^{0} ae^{x}dx + \int_{0}^{+\infty} ae^{-x}dx = 2a.$
- Hence, $a = \frac{1}{2}.$

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2.3.2 Probability Density Function

Example 4.2 Solution

(b) What is the CDF $F_X(x)$?

- Since $F_X(x) = \int_{-\infty}^{x} f_X(u)du$, $f_X(u) = \frac{1}{2}e^{-|u|}$, $-\infty < u < +\infty$,

- If $x \le 0$, $F_X(x) = \int_{-\infty}^{x} \frac{1}{2}e^{u}du = \frac{1}{2}e^{x}$.

- If $x > 0$, $F_X(x) = \int_{-\infty}^{0} \frac{1}{2}e^{u}du + \int_{0}^{x} \frac{1}{2}e^{-u}du = \frac{1}{2} - \frac{1}{2}e^{-x} + \frac{1}{2} = 1 - \frac{1}{2}e^{-x}$.

- Hence,

$$F_X(x) = \begin{cases} \frac{1}{2}e^{x}, & \text{if } \quad x \le 0, \\ 1 - \frac{1}{2}e^{-x}, & \text{if } \quad x > 0. \end{cases}$$

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Example 4.2 Solution

(d) Find $P[0 < X < \ln 3]$?

- $P[0 < X < \ln 3] = F_X(\ln 3) - F_X(0) = ?$
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2.3.1. Cumulative Distribution Function

Practice Test 3

The cumulative distribution function of the continuous random variable X is $F(x) = a + b \arctan x$, $(-\infty < x < +\infty)$. (a) What are a and b? (b) What is $P[-1 < X < 1]$?

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Nội dung

2.4.1 Expected/Mean of a Random Variable

Example 5.1

For one quiz, 10 students have the following grades (on a scale of 0 to 10):

$$9, 5, 10, 8, 4, 7, 5, 5, 8, 7$$

Find the mean, the median, and the mode.

Example 5.1 Solution

$$4, \quad 5, \quad 5, \quad 5, \quad 7, \quad 7, \quad 8, \quad 8, \quad 9, \quad 10$$

- The sum of the ten grades is 68. The mean value is $68/10 = 6,8$.
- The median is 7 since there are four scores below 7 and four scores above 7.
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2.4.1 Expected/Mean of a Random Variable

Example 5.1

For one quiz, 10 students have the following grades (on a scale of 0 to 10):

$$9, 5, 10, 8, 4, 7, 5, 5, 8, 7$$

Find the mean, the median, and the mode.

Example 5.1 Solution

$$4, \quad 5, \quad 5, \quad 5, \quad 7, \quad 7, \quad 8, \quad 8, \quad 9, \quad 10$$

- The sum of the ten grades is 68. The mean value is $68/10 = 6,8$.
- The median is 7 since there are four scores below 7 and four scores above 7.
- The mode is 5 since that score occurs more often than any other. It occurs three times.

(a) Mode and Median

Definition 5.1 (Mode)

A mode of random variable X is a number x_{mod} satisfying

$$\boxed{P_X(x_{mod}) \geq P_X(x) \quad \text{for all} \quad x} \tag{2.7}$$

Definition 5.2 (Median)

A median, x_{med}, of random variable X is a number that satisfies

$$\boxed{P[X < x_{med}] = P[X \geq x_{med}]} \tag{2.8}$$

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Example 5.2

The probability density function of the continuous random variable X is

$$f_X(x) = \begin{cases} \frac{3}{4}x(2-x), & 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

What is x_{mod}? What is x_{med}?

Example 5.2 Solution

Applying Theorem 4.1(c),

$$F_X(x) = \begin{cases} 0, & x \le 0, \\ \frac{3}{4}\left(x^2 - \frac{x^3}{3}\right), & 0 < x \le 2, \\ 1, & x > 2. \end{cases}$$

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Example 5.2 Solution

- x_{med} is a solution of the equation $F_X(x) = \frac{1}{2}$, or $x^3 - 3x^2 + 2 = 0$ with $0 < x \leq 2$. Hence $x_{med} = 1$.
- Taking the derivative of the PDF $f_X(x)$,

$$g(x) := f'_X(x) = \begin{cases} 0, & x \leq 0, \\ \frac{3}{2}(1-x), & 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

We can see the function $g(x)$ reaches maximum at $x = 1$, so $x_{mod} = 1$.

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(b) Expected value/Mean value

Definition 5.3 (Expected value/Mean value)

Let X be a random variable with probability distribution $P_X(x)$, or $f_X(x)$. The mean value, or expected value, of X is

$$\boxed{\mu_X = E[X] = \sum_{x \in S_X} x P_X(x) \quad \text{if } X \text{ is discrete}} \tag{2.9}$$

and

$$\boxed{\mu_X = E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx \quad \text{if } X \text{ is continuous}} \tag{2.10}$$

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Example 5.3

Random variable R in Example 3.1 has PMF

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0. & \text{otherwise.} \end{cases}$$

What is $E[R]$?

Example 5.3 Solution

$$E[R] = \mu_R = (0)P_R(0) + (2)P_R(2) = (0)\left(\frac{1}{4}\right) + (2)\left(\frac{3}{4}\right) = \frac{3}{2}.$$

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Definition 5.4

Continuous random variable X has the PDF

$$f_X(x) = \begin{cases} 1, & 0 \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the expected value of X.

Example 5.4 Solution

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx = \int_0^1 x dx = 1/2.$$

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2.4.2 Functions of a Random Variable

Definition 5.5 (Mathematical function)

Each sample value y of a derived random variable Y is a mathematical function $g(x)$ of a sample value x of another random variable X. We adopt the notation $Y = g(X)$ to describe the relationship of the two random variables.

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Example 5.4

The random variable X is the number of pages in a facsimile transmission. Based on experience, you have a probability model $P_X(x)$ for the number of pages in each fax you send. The phone company offers you a new charging plan for faxes: \$0.10 for the first page, \$0.09 for the second page, etc., down to \$0.06 for the fifth page. For all faxes between 6 and 10 pages, the phone company will charge \$0.50 per fax. (It will not accept faxes longer than ten pages.)

A function $Y = g(X)$ for the charge in cents for sending one fax is

$$Y = g(X) = \begin{cases} 10.5X - 0.5X^2, & 1 \le X \le 5, \\ 50, & 6 \le X \le 10. \end{cases}$$

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For a discrete random variable X, the PMF of $Y = g(X)$ is

$$\boxed{P_Y(y) = \sum_{x:g(x)=y} P_X(x)} \tag{2.11}$$

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Theorem 5.2

Let X be a random variable with probability distribution $P_X(x)$, or $f_X(x)$. The expected value of the random variable $Y = g(X)$ is

$$\boxed{\mu_Y = E[g(X)] = \sum_{x \in S_X} g(x)P_X(x) \quad \text{if } X \text{ is discrete}} \tag{2.12}$$

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$$\boxed{\mu_Y = E[g(X)] = \int_{-\infty}^{+\infty} g(x)f_X(x)dx \quad \text{if } X \text{ is continuous}} \tag{2.13}$$

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Example 5.5

In Example 5.4, suppose all your faxes contain $1, 2, 3$, or 4 pages with equal probability. Find the PMF and expected value of Y, the charge for a fax.

2.4.2 Functions of a Random Variable

Example 5.5 Solution

- The number of pages X has PMF

$$P_X(x) = \begin{cases} 1/4, & x = 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

- The charge for the fax, Y, has range $S_Y = \{10, 19, 27, 34\}$ corresponding to $S_X = \{1, 2, 3, 4\}$. Here each value of Y results in a unique value of X. Hence,

$$P_Y(y) = \begin{cases} 1/4, & x = 10, 19, 27, 34, \\ 0, & \text{otherwise.} \end{cases}$$

- The expected fax bill is $E[Y] = \dfrac{1}{4}(10 + 19 + 27 + 34) = 22.5$ cents.

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2.4.2 Functions of a Random Variable

Example 5.6

In Example 5.5,

$$P_X(x) = \begin{cases} 1/4, & x = 1, 2, 3, 4, \\ 0, & \text{otherwise} \end{cases}$$

and

$$Y = g(X) = \begin{cases} 10.5X - 0.5X^2, & 1 \le X \le 5, \\ 50, & 6 \le X \le 10. \end{cases}$$

What is $E[Y]$?

2.4.2 Functions of a Random Variable

Example 5.6 Solution

Applying Theorem 5.2 we have

$$\begin{aligned} E[Y] &= \sum_{x=1}^{4} P_X(x)g(x) \\ &= \frac{1}{4}[(10.5)(1) - (0.5)(1)^2] + \frac{1}{4}[(10.5)(2) - (0.5)(2)^2] \\ &\quad + \frac{1}{4}[(10.5)(3) - (0.5)(3)^2] + \frac{1}{4}[(10.5)(4) - (0.5)(4)^2] \\ &= \frac{1}{4}[10 + 19 + 27 + 34] = 22.5 \text{ cents.} \end{aligned}$$

2.4.2 Functions of a Random Variable

Example 5.6 Solution

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Example 5.6 Solution

Applying Theorem 5.2 we have

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\begin{aligned}
E[Y] &= \sum_{x=1}^{4} P_X(x)g(x) \\
&= \frac{1}{4}[(10.5)(1) - (0.5)(1)^2] + \frac{1}{4}[(10.5)(2) - (0.5)(2)^2] \\
&\quad + \frac{1}{4}[(10.5)(3) - (0.5)(3)^2] + \frac{1}{4}[(10.5)(4) - (0.5)(4)^2] \\
&= \frac{1}{4}[10 + 19 + 27 + 34] = 22.5 \text{ cents.}
\end{aligned}
$$

2.4.2 Functions of a Random Variable

Example 5.6 Solution

Applying Theorem 5.2 we have

$$\begin{aligned}
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&= \frac{1}{4}[10 + 19 + 27 + 34] = 22.5 \text{ cents.}
\end{aligned}$$

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2.4.2 Functions of a Random Variable

Theorem 5.3

For any random variable X,

(i) $E[X - \mu_X] = 0$.

(ii) $E[aX + b] = aE[X] + b$.

Corollary 5.1

(i) Setting $a = 0$, we see that $E[b] = b$.

(ii) Setting $b = 0$, we see that $E[aX] = aE[X]$.

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2.4.2 Functions of a Random Variable

Example 5.7

Recall that in Examples 3.1 and 5.3, we found that R has PMF

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and expected value $E[R] = 3/2$. What is the expected value of $V = g(R) = 4R + 7$?

2.4.2 Functions of a Random Variable

Example 5.7 Solution

- From Theorem 5.3(ii),

$$E[V] = E[g(R)] = 4E[R] + 7 = 4 \times \frac{3}{2} + 7 = 13.$$

- We can verify this result by applying Theorem 5.2. Using the PMF $P_R(r)$ given in Example 3.1, we can write

$$E[V] = g(0)P_R(0) + g(2)P_R(2) = 7 \times \frac{1}{4} + 15 \times \frac{3}{4} = 13.$$

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2.4.2 Functions of a Random Variable

Theorem 5.4

The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,

$$\boxed{E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]} \tag{2.14}$$

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2.4.2 Functions of a Random Variable

Example 5.8

Let X be a random variable with probability distribution as follows:

$X = x_i$	0	1	2	3
$P[X = x_i]$	1/3	1/2	0	1/6

Find the expected value of $Y = (X - 1)^2$.

2.4.2 Functions of a Random Variable

Example 5.8 Solution

- Applying Theorem 5.4 to the function $Y = (X-1)^2$, we can write

$$E[(X-1)^2] = E[X^2] - 2E[X] + E[1].$$

- From Corollary 5.1, $E[1] = 1$,
- and by direct computation $E[X] = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{2}\right) + (2)(0) + (3)\left(\frac{1}{6}\right) = 1$
- and $E[X^2] = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{2}\right) + (4)(0) + (9)\left(\frac{1}{6}\right) = 2.$
- Hence, $E[(X-1)^2] = 2 - (2)(1) + 1 = 1.$

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2.4.3 Variance and Standard Deviation

Definition 5.6 (Variance)

The variance of random variable X is

$$Var[X] = E\big[(X - \mu_X)^2\big]. \tag{2.15}$$

Definition 5.7 (Standard deviation)

The standard deviation of random variable X is

$$\sigma_X = \sqrt{Var[X]}. \tag{2.16}$$

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2.4.3 Variance and Standard Deviation

Remark 5.1

Because $(X - \mu_X)^2$ is a function of X, $Var[X]$ can be computed according to Theorem 5.2.

$$\sigma_X^2 = Var[X] = \sum_{x \in S_X} (x - \mu_X)^2 P_X(x) \quad \text{if } X \text{ is discrete,} \tag{2.18}$$

and

$$\sigma_X^2 = Var[X] = \int_{-\infty}^{+\infty} (x - \mu_X)^2 f_X(x) dx \quad \text{if } X \text{ is continuous.} \tag{2.19}$$

2.4.3 Variance and Standard Deviation

Theorem 5.5

$Var[X] = E[X^2] - \mu_X^2$ where

$$E[X^2] = \sum_{x \in S_X} x^2 P_X(x) \quad \text{if } X \text{ is discrete,} \tag{2.20}$$

and

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)dx \quad \text{if } X \text{ is continuous.} \tag{2.21}$$

2.4.3 Variance and Standard Deviation

Example 5.9

In Example 3.1, we found that random variable R has PMF

$$P_R(r) = \begin{cases} 1/4, & r = 0, \\ 3/4, & r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

In Example 5.3, we calculated $E[R] = \mu_R = 3/2$. What is the variance of R?

2.4.3 Variance and Standard Deviation

Example 5.9 Solution

We present three ways to compute $Var[R]$.

1. Define $W = (R - \mu_R)^2 = (R - 3/2)^2$. The PMF of W is

$$P_W(w) = \begin{cases} 1/4, & w = (0 - 3/2)^2 = 9/4, \\ 3/4, & w = (2 - 3/2)^2 = 1/4, \\ 0 & \text{otherwise.} \end{cases}$$

   Then $Var[R] = E[W] = (1/4)(9/4) + (3/4)(1/4) = 3/4$.

2. Recall that Theorem 5.2 produces the same result without requiring the derivation of $P_W(w)$.

$$\begin{aligned} Var[R] &= E[(R - \mu_R)^2] \\ &= (0 - 3/2)^2 P_R(0) + (2 - 3/2)^2 P_R(2) = 3/4. \end{aligned}$$

3. To apply Theorem 5.5, we find that $E[R^2] = 0^2 P_R(0) + 2^2 P_R(2) = 3$, and

$$Var[R] = E[R^2] - \mu_R^2 = 3 - (3/2)^2 = 3/4.$$

2.4.3 Variance and Standard Deviation

Theorem 5.6

$Var[aX + b] = a^2 Var[X]$.

Theorem 5.7

Let X be a random variable with probability distribution $P_X(x)$, or $f_X(x)$. The variance of the random variable $Y = g(X)$ is

$$\sigma_Y^2 = E[g(X) - \mu_{g(X)}]^2 = \sum_{x \in S_X} [g(x) - \mu_{g(X)}]^2 P_X(x) \quad \text{if } X \text{ is discrete,}$$

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2.4.3 Variance and Standard Deviation

Example 5.10

Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution

$X = x_i$	0	1	2	3
$P[X = x_i]$	1/4	1/8	1/2	1/8

Example 5.10 Solution

- According to Theorem 5.2,

$$\mu_{2X+3} = E[2X+3] = \sum_{x=0}^{3}(2x+3)P_X(x) = 6.$$

- Using Theorem 5.7, we have

$$\sigma^2_{2X+3} = E[(2X+3-6)^2] = E[(4X^2-12X+9)] = \sum_{x=0}^{3}(4x^2-12x+9)P_X(x) = 4.$$

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2.4.3 Variance and Standard Deviation

Example 5.11

Let X be a random variable with density function

$$f_X(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find the expected value and the variance of $g(X) = 4X + 3$.

Example 5.11 Solution

- By Theorem 5.2 we have
$$E[4X+3] = \int_{-1}^{2} \frac{(4x+3)x^2}{3} dx = \frac{1}{3}\int_{-1}^{2}(4x^3+3x^2)dx = 8.$$
- Using Theorem 5.7,

$$\sigma^2_{4X+3} = E[(4X+3) - 8^2] = E[(4X-5)^2]$$
$$= \int^{2} (4x-5)^2 \frac{x^2}{3} dx = \frac{51}{5}.$$

2.4.3 Variance and Standard Deviation

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- By Theorem 5.2 we have
$E[4X+3] = \int_{-1}^{2} \frac{(4x+3)x^2}{3} dx = \frac{1}{3} \int_{-1}^{2} (4x^3 + 3x^2) dx = 8.$
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2.4.3 Variance and Standard Deviation

Example 5.12

Let X be a random variable with probability distribution as follows:

$X = x_i$	0	1	2	3
$P[X = x_i]$	1/3	1/2	0	1/6

Find the expected value of $Y = (X-1)^2$.

2.4.3 Variance and Standard Deviation

Example 5.12 Solution

- Applying Theorem 5.7 to the function $Y = (X-1)^2$, we can write
$$E[(X-1)^2] = E[X^2] - 2E[X] + E[1].$$
- $E[1] = 1$, and by direct computation,
$$E[X] = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{2}\right) + (2)(0) + (3)\left(\frac{1}{6}\right) = 1 \quad \text{and}$$
$$E[X^2] = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{2}\right) + (4)(0) + (9)\left(\frac{1}{6}\right) = 2.$$
- Hence,
$$E[(X-1)^2] = 2 - (2)(1) + 1 = 1.$$

Nội dung

Important Probability Distributions

Some Discrete Probability Distributions

1. Binomial Distribution
2. Discrete Uniform Distribution
3. Poisson Distribution

Some Continuous Probability Distributions

1. Continuous Uniform Distribution
2. Normal Distribution
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(a) Binomial Distribution $\mathcal{B}(n, p)$

Definition 6.1 (Bernoulli Random Variable)

X is a Bernoulli random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} 1-p, & x = 0, \\ p, & x = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where the parameter p is in the range $0 < p < 1$.

Theorem 6.1

The mean and variance of the Bernoulli random variable X are

$$\mu = E[X] = p \quad \text{and} \quad \sigma^2 = Var[X] = p(1-p).$$

(a) Binomial Distribution $\mathcal{B}(n, p)$

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$$\mu = E[X] = p \quad \text{and} \quad \sigma^2 = Var[X] = p(1-p).$$

(a) Binomial Distribution $\mathcal{B}(n,p)$

Definition 6.1 (Bernoulli Random Variable)

X is a Bernoulli random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} 1-p, & x=0, \\ p, & x=1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

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Remark 6.1 (Bernoulli Process)

Strictly speaking, the Bernoulli process must possess the following properties:

1. The experiment consists of repeated trials.
2. Each trial results in an outcome that may be classified as a success or a failure.
3. The probability of success, denoted by p, remains constant from trial to trial.
4. The repeated trials are independent.

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Definition 6.2 (Binomial random variable)

X is a binomial random variable if the probability distribution of X has the form

X	0	1	$\ldots$	k	$\ldots$	n
p	$C_n^0 p^0 q^n$	$C_n^1 p^1 q^{n-1}$	$\ldots$	$C_n^k p^k q^{n-k}$	$\ldots$	$C_n^n p^n q^0$

(3.2)

where $p_X(k) = P(X = k) = C_n^k p^k q^{n-k}$.

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X is a binomial random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} C_n^x p^x (1-p)^{n-x}, & \text{for } x = 0, 1, 2, \ldots, n, \\ 0, & \text{o.w.} \end{cases} \tag{3.3}$$

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The probability distribution of this discrete random variable is called the binomial distribution, and is denoted by $\mathcal{B}(n,p)$ (or $X \sim \mathcal{B}(n,p)$).

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The mean and variance of the binomial random variable X are

$$\mu = E[X] = np \quad \text{and} \quad \sigma^2 = Var[X] = npq, \quad q = 1-p. \tag{3.4}$$

Example 6.1

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) at least 10 survive, (b) from 3 to 8 survive, and (c) exactly 5 survive?

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- Let X be the number of people who survive.
- $X \sim \mathcal{B}(15, 0.4)$ and the PMF of X is

$$P_X(x) = C_{15}^{x}(0.4)^x(0.6)^{15-x}, \quad x = 0, 1, \ldots, 15.$$

Hence,

(a) $P[X \geq 10] = \sum_{x=10}^{15} P_X(x)$
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(b) Discrete Uniform Distribution

Definition 6.5 (Discrete Uniform Random Variable)

X is a discrete uniform random variable if the probability distribution of X has the form

X	1	2	...	n
p	$\frac{1}{n}$	$\frac{1}{n}$	...	$\frac{1}{n}$

(3.5)

Theorem 6.3

For the discrete uniform random variable X of Definition 6.5

$$\boxed{E(X) = \frac{n+1}{2}, \quad V(X) = \frac{n^2-1}{12}} \tag{3.6}$$

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(c) Poisson Distribution

Definition 6.6 (Poisson Random Variable)

X is a Poisson random variable if the probability distribution of X has the form

X	0	1	...	k	...	n	...
p	$\frac{\lambda^0}{0!}e^{-\lambda}$	$\frac{\lambda^1}{1!}e^{-\lambda}$	...	$\frac{\lambda^k}{k!}e^{-\lambda}$	...	$\frac{\lambda^n}{n!}e^{-\lambda}$	...

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where $P(X = k) = \dfrac{\lambda^k}{k!}e^{-\lambda}$, $\lambda > 0$ is a constant.

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X is a Poisson random variable if the PMF of X has the form

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X is a Poisson random variable if the PMF of X has the form

$$P_X(x)=\begin{cases}\dfrac{\lambda^x e^{-\lambda}}{x!}, & x=0,1,2,\dots\\ 0, & \text{otherwise},\end{cases} \tag{3.8}$$

where the parameter λ is in the range $\lambda > 0$.

(c) Poisson Distribution

Definition 6.6 (Poisson Random Variable)

X is a Poisson random variable if the probability distribution of X has the form

X	0	1	$\ldots$	k	$\ldots$	n	$\ldots$
p	$\frac{\lambda^0}{0!}e^{-\lambda}$	$\frac{\lambda^1}{1!}e^{-\lambda}$	$\ldots$	$\frac{\lambda^k}{k!}e^{-\lambda}$	$\ldots$	$\frac{\lambda^n}{n!}e^{-\lambda}$	$\ldots$

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Probability distribution of Poisson random variable is called the Poisson distribution, and is denoted by $\mathcal{P}(\lambda)$.

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Both the mean and the variance of the Poisson distribution $\mathcal{P}(\lambda)$ are λ.

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Remark 6.2

Here are some examples of experiments for which the random variable X can be modeled by the Poisson random variable:

1. The number of calls received by a switchboard during a given period of time.
2. The number of customer arrivals at a checkout counter during a given minute.
3. The number of machine breakdowns during a given day.
4. The number of traffic accidents at a given intersection during a given time period.

In each example, X represents the number of events that occur in a period of time or space during which an average of λ such events can be expected to occur. The only assumptions needed when one uses the Poisson distribution to model experiments such as these are that the counts or events occur randomly and independently of one another.

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Example 6.2

The number of hits at a Web site in any time interval is a Poisson random variable. A particular site has on average $\alpha = 2$ hits per second.

(a) What is the probability that there are no hits in an interval of 0.25 seconds?

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(c) Poisson Distribution

Example 6.2 Solution

(a) In an interval of 0.25 seconds, the number of hits H is a Poisson random variable with $\lambda = \alpha T = (2\text{hits}/s) \times (0.25s) = 0.5$ hits.

- The PMF of H is

$$P_H(h) = \begin{cases} \dfrac{(0.5)^h \times e^{-5}}{h!}, & h = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

- The probability of no hits is

$$P[H = 0] = P_H(0) = \frac{(0.5)^0 \times e^{-0.5}}{0!} = 0.607.$$

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Example 6.2 Solution

(b) In an interval of 1 second, $\lambda = \alpha T = (2\text{hits}/s) \times (1s) = 2$ hits.

- Letting J denote the number of hits in one second, the PMF of J is

$$P_J(j) = \begin{cases} \dfrac{(2)^j \times e^{-2}}{j!}, & j = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

- To find the probability of no more than two hits, we note that $\{J \le 2\} = \{J = 0\} \cup \{J = 1\} \cup \{J = 2\}$ is the union of three mutually exclusive events. Therefore,

$$\begin{aligned} P[J \le 2] &= P[J = 0] + P[J = 1] + P[J = 2] \\ &= P_J(0) + P_J(1) + P_J(2) \\ &= e^{-2} + \frac{2^1 \times e^{-2}}{1!} + \frac{2^2 \times e^{-2}}{2!} = 0.677. \end{aligned}$$

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(d) Approximation of Binomial Distribution by a Poisson Distribution

Theorem 6.5

Let X be a binomial random variable with probability distribution $\mathcal{B}(n, p)$. When $n \to \infty$, $p \to 0$, and $np \to \mu$ as $n \to \infty$ remains constant,

$$\mathcal{B}(n, p) \to \mathcal{P}(\lambda) \quad \text{as} \quad n \to \infty.$$

Remark 6.3

The Poisson distribution provides a simple, easy-to-compute, and accurate approximation to binomial probabilities when n is large and $\lambda = np$ is small, preferably with $np < 7$.

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Example 6.3

Suppose a life insurance company insures the lives of 5000 men aged 42. If actuarial studies show the probability that any 42-year-old man will die in a given year to be 0.001, find the exact probability that the company will have to pay $X = 4$ claims during a given year.

(d) Approximation of Binomial Distribution by a Poisson Distribution

Example 6.3 Solution

- The exact probability is given by the binomial distribution as

$$P[X = 4] = C^4_{5000}(0.001)^4(1 - 0.001)^{5000-4} = \frac{5000!}{4!4996!}(0.001)^4(0.999)^{4996}$$

  for which binomial tables are not available.

- To compute $P[X = 4]$ without the aid of a computer would be very time-consuming, but the Poisson distribution can be used to provide a good approximation to $P[X = 4]$.
- Computing $\lambda = np = (5000)(0.001) = 5$ and substituting into the formula for the Poisson probability distribution, we have

$$P[X = 4] \simeq \frac{5^4}{4!}e^{-5} = \frac{(625)(0.006738)}{24} = 0.175.$$

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Example 6.4

In a certain industrial facility, accidents occur infrequently. It is known that the probability of an accident on any given day is 0.005 and accidents are independent of each other.

(a) What is the probability that in any given period of 400 days there will be an accident on one day?

(b) What is the probability that there are at most three days with an accident?

(d) Approximation of Binomial Distribution by a Poisson Distribution

Example 6.4 Solution

Let X be a binomial random variable with $n = 400$ and $p = 0.005$. Thus, $np = 2$. Using the Poisson approximation,

(a) $P[X = 1] = e^{-1}2^1 = 0.271$ and

(b) $P[X \leq 3] = \sum_{x=0}^{3} \dfrac{e^{-2}2^x}{x!} = 0.857.$

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(a) Continuous Uniform Distribution

Definition 6.9 (Uniform random variable)

X is a uniform $\mathcal{U}[a,b]$ random variable if the PDF of X is

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & x \in [a,b], \\ 0, & x \notin [a,b]. \end{cases} \tag{3.9}$$

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Theorem 6.6

If X is a uniform $\mathcal{U}[a,b]$ random variable,

(a) The CDF of X is

$$F_X(x) = \begin{cases} 0, & x \leq a, \\ \dfrac{x-a}{b-a}, & a < x \leq b, \\ 1, & x > b. \end{cases} \tag{3.10}$$

(b) The expected value of X is $\mu = E[X] = \dfrac{a+b}{2}$.

(c) The variance of X is $\sigma^2 = Var[X] = \dfrac{(b-a)^2}{12}$.

(a) Continuous Uniform Distribution

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(a) Continuous Uniform Distribution

Example 6.5

Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval $[0, 4]$.

(a) What is the probability density function?

(b) What is the probability that any given conference lasts at least 3 hours?

(a) Continuous Uniform Distribution

Example 6.5 Solution

(a) The appropriate density function for the uniformly distributed random variable X in this situation is

$$f_X(x) = \begin{cases} \frac{1}{4}, & 0 \le x \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $P[X \ge 3] = \int_3^4 \frac{1}{4} dx = \frac{1}{4}.$

(a) Continuous Uniform Distribution

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(b) $P[X \ge 3] = \int_3^4 \frac{1}{4} dx = \frac{1}{4}.$

(b) Exponential Distribution

Definition 6.10 (Exponential random variable)

X is an exponential $exp(\lambda)$ random variable if the PDF of X is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{3.11}$$

where the parameter $\lambda > 0$.

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(b) Exponential Distribution

Example 6.6

The probability that a telephone call lasts no more than t minutes is often modeled as an exponential CDF.

$$F_T(t) = \begin{cases} 1 - e^{-t/3}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

What is the PDF of the duration in minutes of a telephone conversation? What is the probability that a conversation will last between 2 and 4 minutes?

(b) Exponential Distribution

Example 6.6 Solution

- We find the PDF of T by taking the derivative of the CDF:

$$f_T(t) = \frac{dF_T(t)}{dt} = \begin{cases} \frac{1}{3}e^{-t/3}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

- Therefore, observing Definition 3.8, we recognize that T is an exponential $(\lambda = 1/3)$ random variable.
- The probability that a call lasts between 2 and 4 minutes is

$$P[2 \leq T \leq 4] = F_T(4) - F_T(2) = e^{-2/3} - e^{-4/3} = 0.250.$$

(b) Exponential Distribution

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(b) Exponential Distribution

Example 6.7

In Example 6.6, what is $E[T]$, the expected duration of a telephone call? What are the variance and standard deviation of T? What is the probability that a call duration is within ± 1 standard deviation of the expected call duration?

(b) Exponential Distribution

Example 6.7 Solution

- Using the PDF $f_T(t)$ in Example 3.6, we calculate the expected duration of a call:

$$E[T] = \int_{-\infty}^{+\infty} t f_T(t) dt = \int_{0}^{+\infty} t \frac{1}{3} e^{-t/3} dt.$$

- Integration by parts yields

$$E[T] = -te^{-t/3}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-t/3} dt = 3 \text{ minutes}.$$

- To calculate the variance, we begin with the second moment of T:

$$E[T^2] = \int_{-\infty}^{+\infty} t^2 f_T(t) dt = \int_0^{+\infty} t^2 \frac{1}{3} e^{-t/3} dt.$$

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(b) Exponential Distribution

Example 6.7 Solution

- Again integrating by parts, we have

$$E[T^2] = -t^2e^{-t/3}\Big|_0^{+\infty} + \int_0^{+\infty} 2te^{-t/3}dt = 2\int_0^{+\infty} te^{-t/3}dt.$$

- With the knowledge that $E[T] = 3$, we observe that $\int_0^{+\infty} te^{-t/3}dt = 3E[T] = 9.$ Thus $E[T^2] = 6E[T] = 18$ and

$$Var[T] = E[T^2] - (E[T])^2 = 18 - 3^2 = 9.$$

- The standard deviation is $\sigma_T = \sqrt{Var[T]} = 3$ minutes. The probability that the call duration is within ± 1 standard deviation of the expected value is

$$P[0 \le T \le 6] = F_T(6) - F_T(0) = 1 - e^{-2} = 0.865.$$

(b) Exponential Distribution

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(b) Exponential Distribution

Theorem 6.7

If X is an exponential $exp(\lambda)$ random variable,

(a) $F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$

(b) $\mu = E[X] = 1/\lambda$.

(c) $\sigma^2 = Var[X] = 1/\lambda^2$.

(b) Exponential Distribution

Example 6.8

Phone company A charges \$0.15 per minute for telephone calls. For any fraction of a minute at the end of a call, they charge for a full minute. Phone company B also charges \$0.15 per minute. However, Phone company B calculates its charge based on the exact duration of a call. If T, the duration of a call in minutes, is an exponential $(\lambda = 1/3)$ random variable, what is the PDF of T? What is the expected value of T? What are the expected revenues per call $E[R_A]$ and $E[R_B]$ for companies A and B?

(b) Exponential Distribution

Example 6.8 Solution

- Because T is an exponential $(\lambda = 1/3)$ random variable,

$$f_T(t) = \begin{cases} \frac{1}{3}e^{-\frac{1}{3}t}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

- We have in Theorem 6.7 (and in Example 6.6),

$$E[T] = \int_{-\infty}^{+\infty} t f_T(t) dt = \frac{1}{\lambda} = 3 \quad \text{minutes per call.}$$

- Therefore, for phone company B, which charges for the exact duration of a call,

$$E[R_B] = 0.15 \times E[T] = 0.45 \quad \text{dolars per call.}$$

(b) Exponential Distribution

Example 6.8 Solution

- Because T is an exponential ($\lambda = 1/3$) random variable,

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(b) Exponential Distribution

Example 6.8 Solution

- Company A, by contrast, collects \$0.15$[T]$ for a call of duration T minutes ($T = 1.2$, $[T] = 2\dots$).
- Put $K = [T]$. The expected revenue for company A is $E[R_A] = 0.15 \times E[K]$.
- $P[K = k] = P[k-1 < X \le k] = F_X(k) - F_X(k-1) = (e^{-\lambda})^{k-1}(1-e^{-\lambda})$.
- $E[K] = \sum\limits_{k=1}^{\infty} kP[K=k] = \sum\limits_{k=1}^{\infty} k(1-p)^{k-1}p = \frac{1}{p}$, where $p = 1-e^{-\lambda}$.
- Hence,

$$E[R_A] = \frac{0.15}{p} = \frac{0.15}{0.2834} = (0.15)\times(3.5285) = 0.5292 \quad \text{dolars per call.}$$

(b) Exponential Distribution

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(c1) Normal Distribution $\mathcal{N}(\mu, \sigma)$

Definition 6.11 (Uniform random variable)

The most important continuous probability distribution in the entire field of statistics is the normal distribution. Its graph, called the normal curve, is the bell-shaped curve of Figure 3, which approximately describes many phenomena that occur in nature, industry, and research.

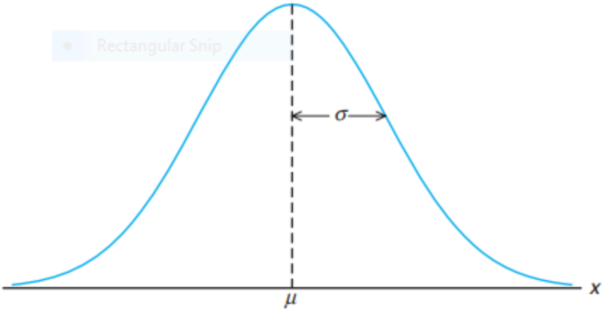

Figure: Figure 2 The normal curve

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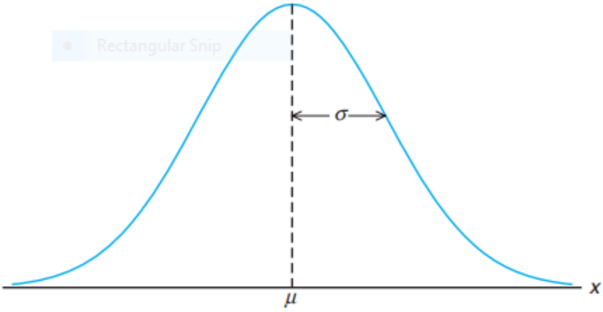

Figure: Figure 2 The normal curve

(c1) Normal Distribution $\mathcal{N}(\mu, \sigma)$

Introduction

The normal distribution is often referred to as the **Gaussian distribution**, in honor of Karl Friedrich Gauss (1777–1855), who also derived its equation from a study of errors in repeated measurements of the same quantity.

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A continuous random variable X having the bell-shaped distribution of Figure 3 is called a **normal random variable**. The mathematical equation for the probability distribution of the normal variable depends on the two parameters μ and σ, its mean and standard deviation, respectively. Hence, we denote the normal distribution by $\mathcal{N}(\mu, \sigma)$.

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(c1) Normal Distribution $\mathcal{N}(\mu, \sigma)$

Definition 6.12 (Density)

The PDF of the normal random variable X, with mean μ and variance σ^2, is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty, \tag{3.11}$$

where $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$.

Note

Once μ and σ are specified, the normal curve is completely determined. For example, if $\mu = 50$ and $\sigma = 5$, then the ordinates $\mathcal{N}(50, 5)$ can be computed for various values of x and the curve drawn.

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Figure 4

In Figure 4, we have sketched two normal curves having the same standard deviation but different means. The two curves are identical in form but are centered at different positions along the horizontal axis.

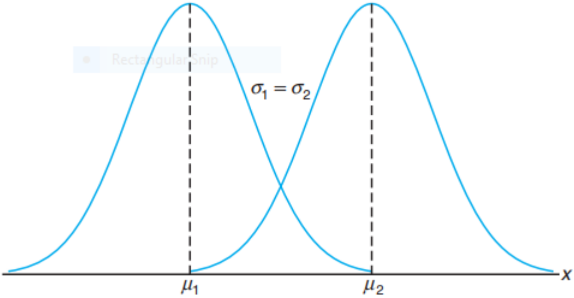

Figure: Figure 3.3: Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$

(c1) Normal Distribution $\mathcal{N}(\mu, \sigma)$

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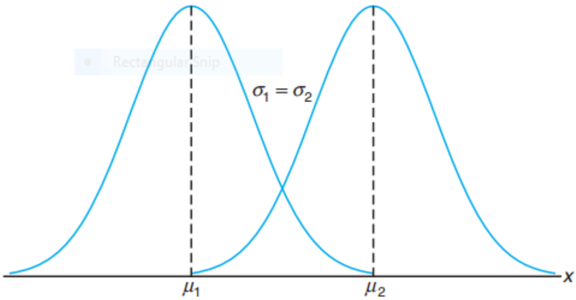

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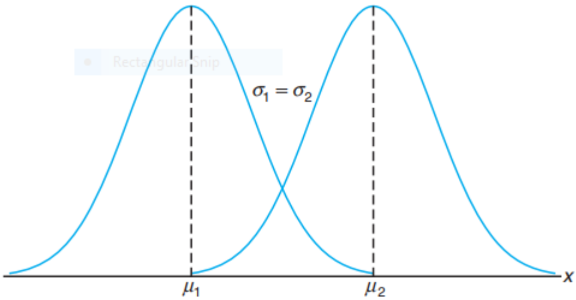

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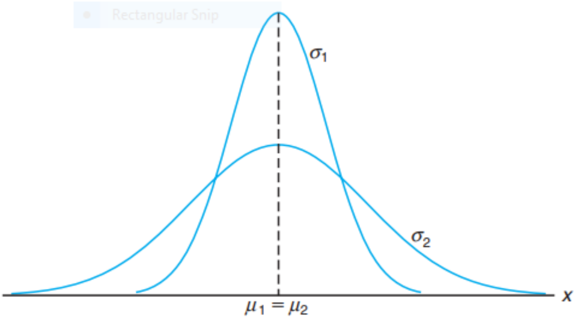
(c1) Normal Distribution $\mathcal{N}(\mu, \sigma)$

Figure 5

In Figure 5, we have sketched two normal curves with the same mean but different standard deviations.



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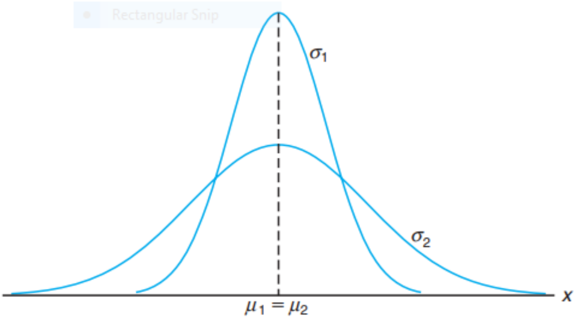

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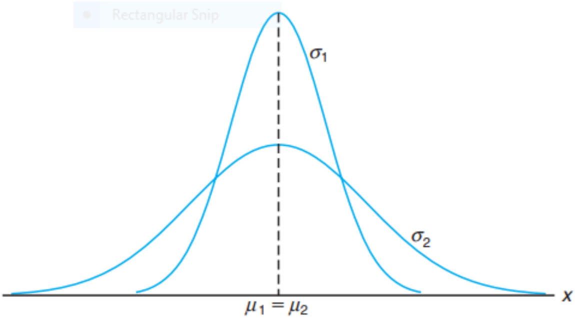

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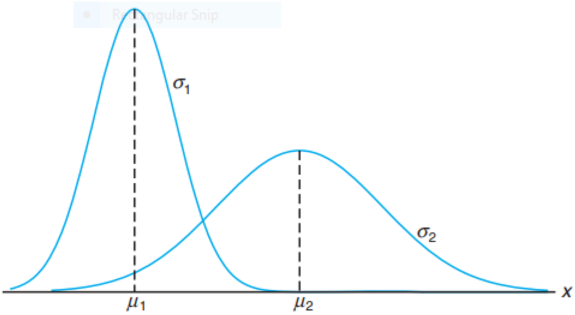
(c1) Normal Distribution $\mathcal{N}(\mu, \sigma)$

Figure 6

Figure 6 shows two normal curves having different means and different standard deviations.



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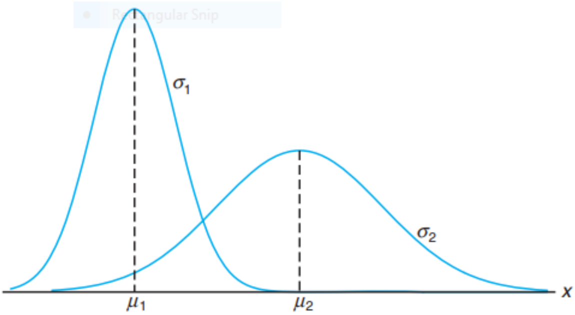

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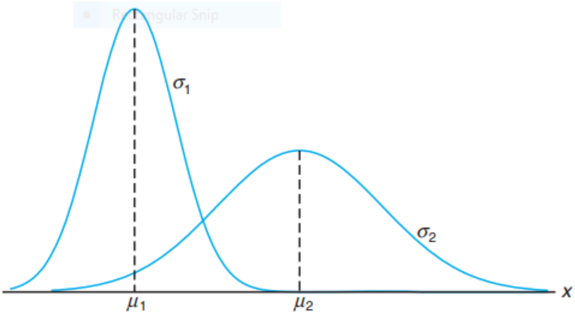

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(c1) Normal Distribution $\mathcal{N}(\mu, \sigma)$

Theorem 6.8

The mean and variance of the normal random variable are μ and σ^2, respectively. Hence, the standard deviation is σ.

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(c1) Normal Distribution $\mathcal{N}(\mu, \sigma)$

Proof

- To evaluate the mean, we first calculate

$$E[X-\mu] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu) e^{-\dfrac{(x-\mu)^2}{2\sigma^2}} dx.$$

- Setting $z = (x-\mu)/\sigma$ and $dx = \sigma dz$, we obtain

$$E[X-\mu] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\dfrac{z^2}{2}} dz = 0,$$

since the integrand above is an odd function of z.

- Hence,

$$E[X] = \mu.$$

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(c2) Standard Normal Distribution. Area Under the Normal Curve

Introduction

Put

$$Z = \frac{X-\mu}{\sigma}.$$

If X is a normal random variable with mean μ and variance σ^2 then Z is seen to be a normal random variable with mean 0 and variance 1.

Definition 6.13 (Standard normal distribution)

The distribution of a normal random variable with mean 0 and variance 1 is called a standard normal distribution $\mathcal{N}(0,1)$.

PDF

The PDF of the standard normal random variable Z is

$$\varphi_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

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If X is a normal random variable $\mathcal{N}(\mu, \sigma)$, the CDF of X is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

The probability that X is in the interval (x_1, x_2) is

$$P[x_1 < X < x_2] = \Phi\left(\frac{x_2-\mu}{\sigma}\right) - \Phi\left(\frac{x_1-\mu}{\sigma}\right).$$

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$$P[x_1 < X < x_2] = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

is represented by the area of the shaded region.

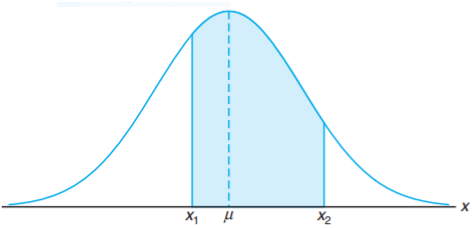

Figure: 7 $P[x_1 < X < x_2]$ = area of the shaded region

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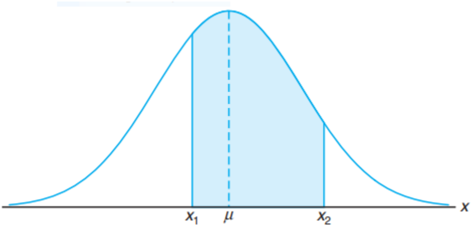

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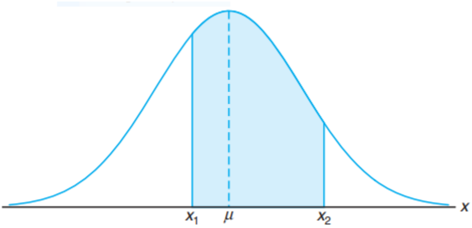

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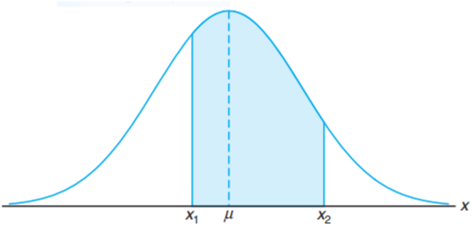

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Remark 6.4

(a) In using Theorem 6.9, we transform values of a norm random variable, X, to equivalent values of the standard normal random variable, Z. For a sample value x of the random variable X, the corresponding sample value of Z is

$$z = \frac{x-\mu}{\sigma} \quad \text{or equivalently,} \quad x = \mu + z\sigma. \tag{3.12}$$

The original and transformed distributions are illustrated in Figure 8. Since all the values of X falling between x_1 and x_2 have corresponding z values between z_1 and z_2, the area under the X-curve between the ordinates $x = x_1$ and $x = x_2$ in Figure 8 equals the area under the Z-curve between the transformed ordinates $z = z_1$ and $z = z_2$.

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(c2) Standard Normal Distribution. Area Under the Normal Curve

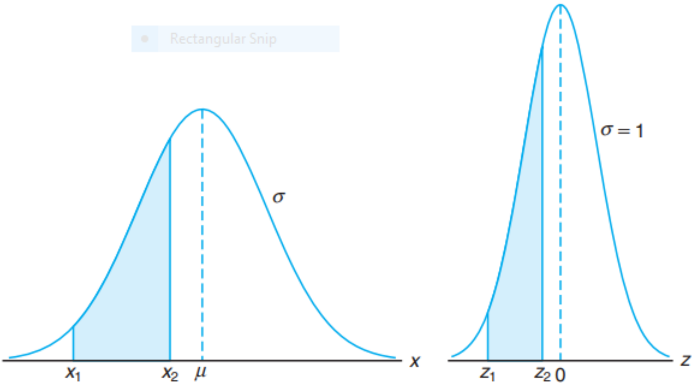

Figure: 8 The original and transformed normal distributions

(c2) Standard Normal Distribution. Area Under the Normal Curve

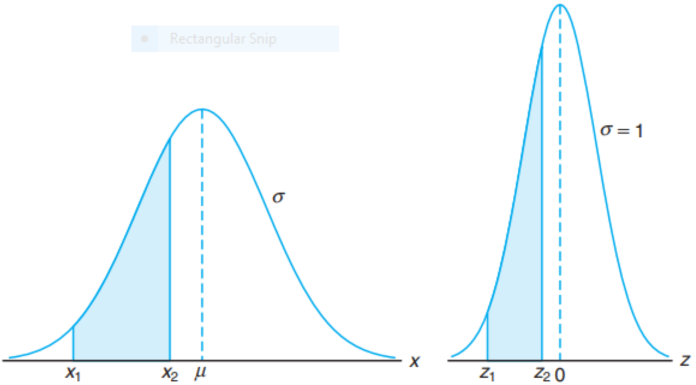

Figure: 8 The original and transformed normal distributions

(c2) Standard Normal Distribution. Area Under the Normal Curve

Remark 6.5

(b) The probability distribution for Z, shown in Figure 9, is called the standardized normal distribution because its mean is 0 and its standard deviation is 1. Values of Z on the left side of the curve are negative, while values on the right side are positive. The area under the standard normal curve to the left of a specified value of Z say, z_0 is the probability $P[Z \leq z_0]$. This cumulative area is recorded in Table 3.1 and is shown as the shaded area in Figure 9.

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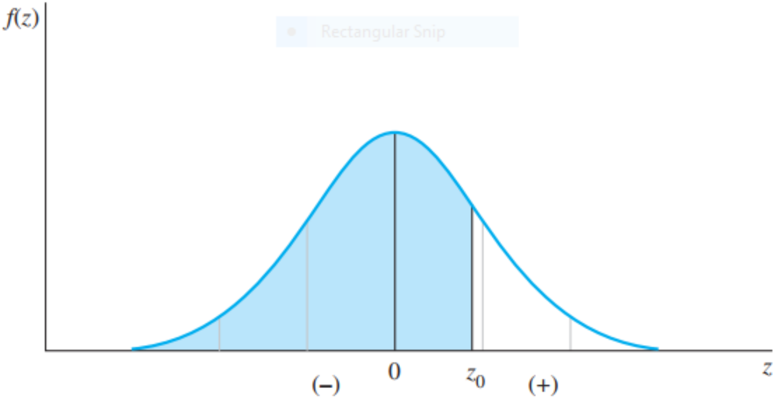

Figure: 9 Standardized normal distribution

(c2) Standard Normal Distribution. Area Under the Normal Curve

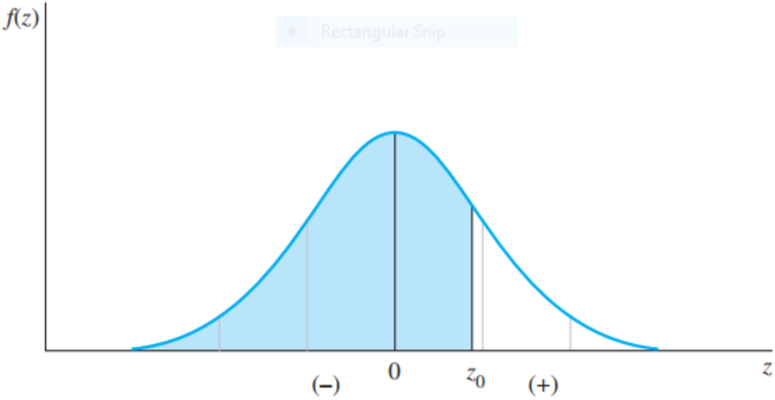

Figure: 9 Standardized normal distribution

(c2) Standard Normal Distribution. Area Under the Normal Curve

Example 6.9

Suppose your score on a test is $x = 46$, a sample value of the Gaussian $(61, 10)$ random variable. Express your test score as a sample value of the standard normal random variable, Z.

Example 6.9 Solution

Equation (3.12) indicates that $z = (46 - 61)/10 = -1.5$. Therefore your score is 1.5 standard deviations less than the expected value.

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Remark 6.6

To find probabilities of norm random variables, we use the values of (z) presented in Table 2.1. Note that this table contains entries only for $z \geq 0$. For negative values of z, we apply the following property of $\Phi(z)$.

Theorem 6.10

$$\Phi(-z) = 1 - \Phi(z).$$

Remark 6.7

Figure 10 displays the symmetry properties of $\Phi(z)$. Both graphs contain the standard normal PDF. In Figure 10(a), the shaded area under the PDF is $\Phi(z)$. Since the area under the PDF equals 1, the unshaded area the PDF is $1 - \Phi(z)$. In Figure 10(b), the shaded area on the right is $1 - \Phi(z)$ and the shaded area on the left is $\Phi(-z)$. This graph demonstrates that $\Phi(-z) = 1 - \Phi(z)$.

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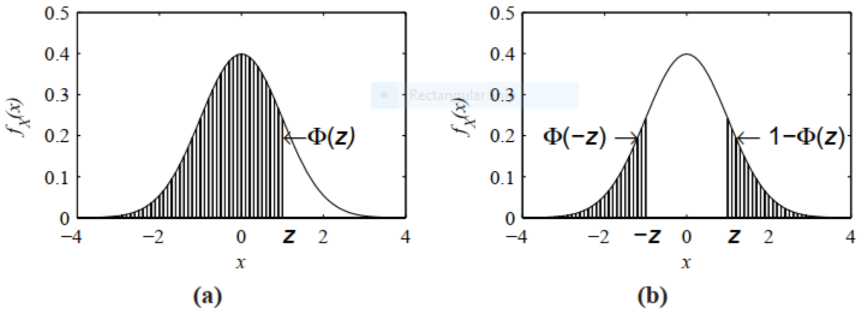

Figure: 10 Standardized normal distribution

(c2) Standard Normal Distribution. Area Under the Normal Curve

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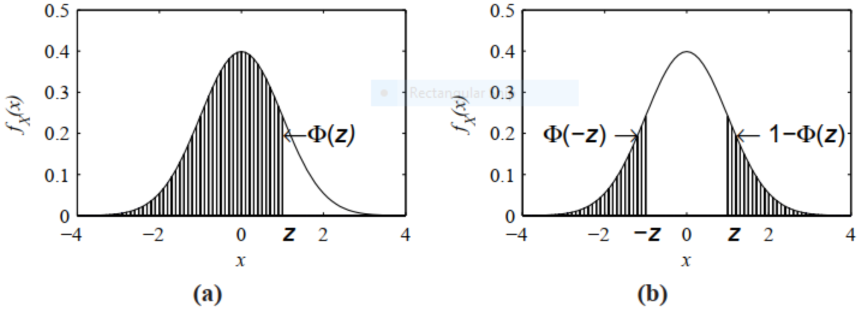

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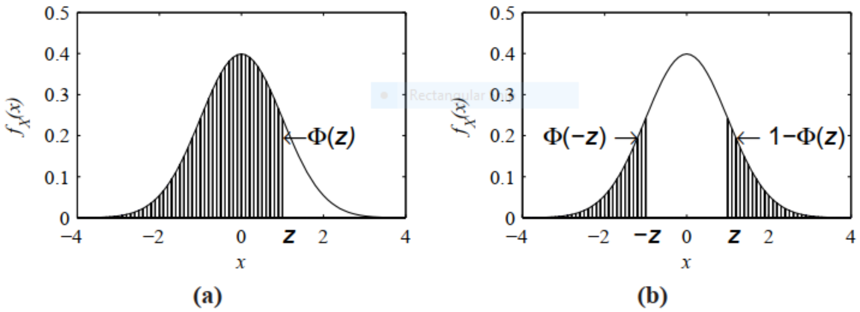

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Example 6.10

Find the probability that a normally distributed random variable will fall within these ranges:

(a) One standard deviation of its mean.

(b) Two standard deviations of its mean.

Example 6.10 Solution

(a) Since the standard normal random variable z measures the distance from the mean in units of standard deviations, you need to find

$$P[-1 \leq Z \leq 1] = \Phi(1) - \Phi(-1) = \Phi(1) - [1 - \Phi(1)] = (2)(0.84134) - 1 = 0.68268.$$

(b) As in part (a), $P[-2 \leq Z \leq 2] = 2\Phi(2) - 1 = (2)(0.97725) - 1 = 0.9545.$

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Table 2.1: The values of $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt$

x	0	1	2	3	4	5	6	7	8	9
0,0	0,50000	50399	50798	51197	51595	51994	52392	52790	53188	53586
0,1	53983	54380	54776	55172	55567	55962	56356	56749	57142	57535
0,2	57926	58317	58706	59095	59483	59871	60257	60642	61026	61409
0,3	61791	62172	62556	62930	63307	63683	64058	64431	64803	65173
0,4	65542	65910	66276	66640	67003	67364	67724	68082	68439	68739
0,5	69146	69447	69847	70194	70544	70884	71226	71566	71904	72240
0,6	72575	72907	73237	73565	73891	74215	74537	74857	75175	75490
0,7	75804	76115	76424	76730	77035	77337	77637	77935	78230	78524
0,8	78814	79103	79389	79673	79955	80234	80511	80785	81057	81327
0,9	81594	81859	82121	82381	82639	82894	83147	83398	83646	83891
1,0	84134	84375	84614	84850	85083	85314	85543	85769	85993	86214
1,1	86433	86650	86864	87076	87286	87493	87698	87900	88100	88298
1,2	88493	88686	88877	89065	89251	89435	89617	89796	89973	90147
1,3	90320	90490	90658	90824	90988	91149	91309	91466	91621	91774
1,4	91924	92073	92220	92364	92507	92647	92786	92922	93056	93189
1,5	93319	93448	93574	93699	93822	93943	94062	94179	94295	94408
1,6	94520	94630	94738	94845	94950	95053	95154	95254	95352	95449
1,7	95543	95637	95728	95818	95907	95994	96080	96164	96246	96327
1,8	96407	96485	96562	96638	96712	96784	96856	96926	96995	97062
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2,3	98928	98956	98983	99010	99036	99061	99086	99111	99134	99158
2,4	99180	99202	99224	99245	99266	99285	99305	99324	99343	99361
2,5	99379	99396	99413	99430	99446	99261	99477	99492	99506	99520
2,6	99534	99547	99560	99573	99585	99598	99609	99621	99632	99643
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3,0	0,99865	3,1	99903	3,2	99931	3,3	99952	3,4	99966	
3,5	99977	3,6	99984	3,7	99989	3,8	99993	3,9	99995	
4,0	999968									
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(c2) Standard Normal Distribution. Area Under the Normal Curve

Example 6.11

If X is the norm random variable $\mathcal{N}(61, 10)$, what is $P[X \leq 46]$?

Example 6.11 Solution

- Applying Theorem 6.9, Theorem 6.10, we have

$$P[X \leq 46] = P(-\infty < X \leq 46) = \Phi\left(\frac{46-61}{10}\right) - \Phi(-\infty) = \Phi(-1.5)$$

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Example 6.12

If X is a Gaussian random variable with $\mu = 61$ and $\sigma = 10$, what is $P[51 < X \leq 71]$?

Example 6.12 Solution

$$P[51 < X \leq 71] = \Phi\left(\frac{71-61}{10}\right) - \Phi\left(\frac{51-61}{10}\right)$$
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If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of

$$Z = \frac{X - np}{\sqrt{npq}},$$

as $n \to \infty$, is the standard normal distribution $\mathcal{N}(0, 1)$.

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Note

To illustrate the normal approximation to the binomial distribution,

- we first draw the histogram for $\mathcal{B}(15, 0.4)$ and
- then superimpose the particular normal curve having the same mean and variance as the binomial variable X.
- Hence, we draw a normal curve with $\mu = np = (15)(0.4) = 6$ and $\sigma^2 = npq = (15)(0.4)(0.6) = 3.6$.

The histogram of $\mathcal{B}(15, 0.4)$ and the corresponding superimposed normal curve, which is completely determined by its mean and variance, are illustrated in Figure 11.

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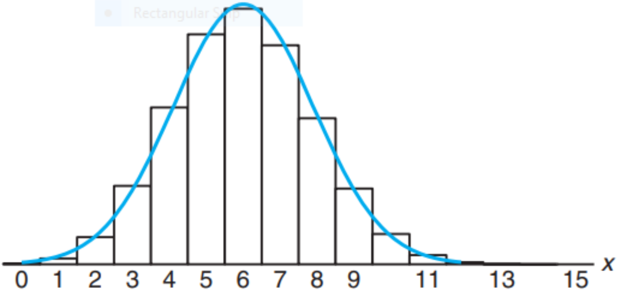

Figure: 11 Normal approximation of $\mathcal{B}(15, 0.4)$

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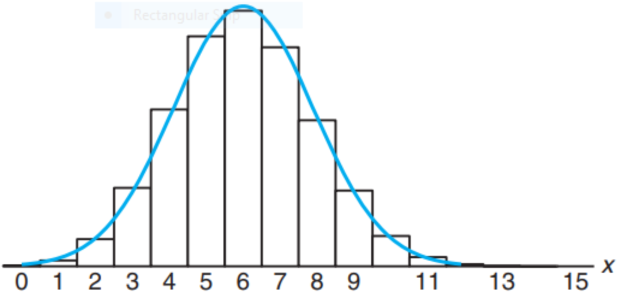

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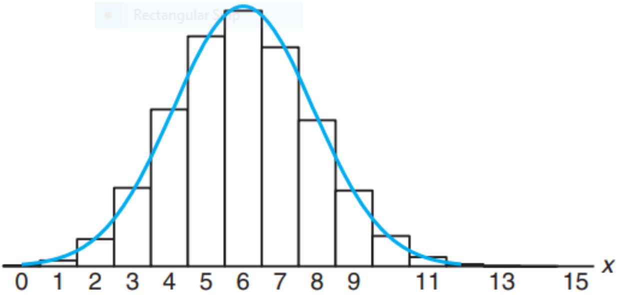

Figure: 11 Normal approximation of $\mathcal{B}(15, 0.4)$

(c3) Normal Approximation to the Binomial Distribution

Definition 6.15 (Normal Approximation to the Binomial Distribution)

Let X be a binomial random variable with n trials and probability p of success. The probability distribution of X is approximated using a normal curve with

$$\mu = np \quad \text{and} \quad \sigma = \sqrt{npq},$$

and

$$P[x_1 < X < x_2] = \sum_{k=x_1}^{x_2} C_n^k (p)^k (1-p)^{n-k} \simeq \Phi\left(\frac{x_2 + 0.5 - \mu}{\sigma}\right) - \Phi\left(\frac{x_1 - 0.5 - \mu}{\sigma}\right) \tag{3.15}$$

and the approximation will be good if np and $n(1-p)$ are greater than or equal to 5.

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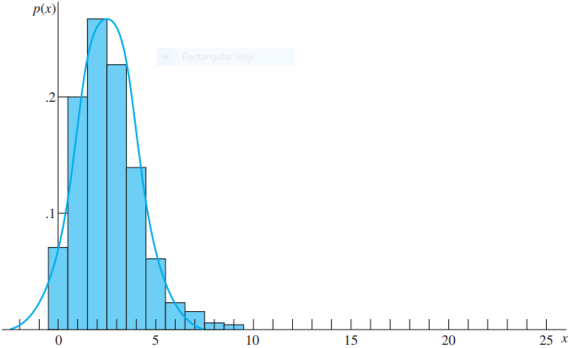

Figure: 12 The binomial probability distribution and the approximating normal distribution for $n = 25$ and $p = 0.1$

(c3) Normal Approximation to the Binomial Distribution

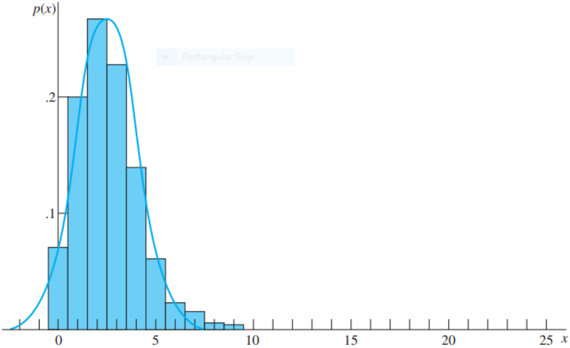

Figure: 12 The binomial probability distribution and the approximating normal distribution for $n = 25$ and $p = 0.1$

(c3) Normal Approximation to the Binomial Distribution

Example 6.13

Use the normal curve to approximate the probability that $X = 8$, 9, or 10 for a binomial random variable with $n = 25$ and $p = 0.5$. Compare this approximation to the exact binomial probability.

Example 6.13 Solution

- You can find the exact binomial probability for this example because there are cumulative binomial tables for $n = 25$,

  $P[X = 8] + P[X = 9] + P[X = 10] = \left(C_{25}^{8} + C_{25}^{9} + C_{25}^{10}\right)(0.5)^{25} \simeq 0.190535.$
- To use the normal approximation, first find the appropriate mean and standard deviation for the normal curve: $\mu = np = 12.5$, $\sigma = \sqrt{npq} = 2.5$. It follows from (3.15) that

$$P[8 \leq X \leq 10] = \Phi(-0.8) - \Phi(-2) = 0.18911.$$

- You can compare the approximation, 0.18911, to the actual probability, 0.190535. They are quite close!

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Remark 6.8

The normal approximation to the binomial probabilities will be adequate if both $np > 5$ and $n(1-p) > 5$.

Example 6.14

The reliability of an electrical fuse is the probability that a fuse, chosen at random from production, will function under its designed conditions. A random sample of 1000 fuses was tested and $X = 27$ defectives were observed. Calculate the approximate probability of observing 27 or more defectives, assuming that the fuse reliability is 0.98.

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Example 6.14 Solution

- The probability of observing a defective when a single fuse is tested is $p = 0.02$, given that the fuse reliability is 0.98.
- Then $\mu = np = 20$, $\sigma = \sqrt{npq} = 4.43$.
- The probability of 27 or more defective fuses, given $n = 1000$, is

$$P[X \geq 27] = P[X = 27] + P[X = 28] + \cdots + P[X = 1000].$$

- It is appropriate to use the normal approximation to the binomial probability because $np = 20$ and $nq = 980$ are both greater than 5.
- So

$$\begin{aligned} P[27 \leq X \leq 1000] &= \Phi\Big(\frac{1000 + 0.5 - 20}{4.43}\Big) - \Phi\Big(\frac{27 - 0.5 - 20}{4.43}\Big) \\ &= 1 - \Phi(1.47) = 1 - 0.92922 = 0.07078. \end{aligned}$$

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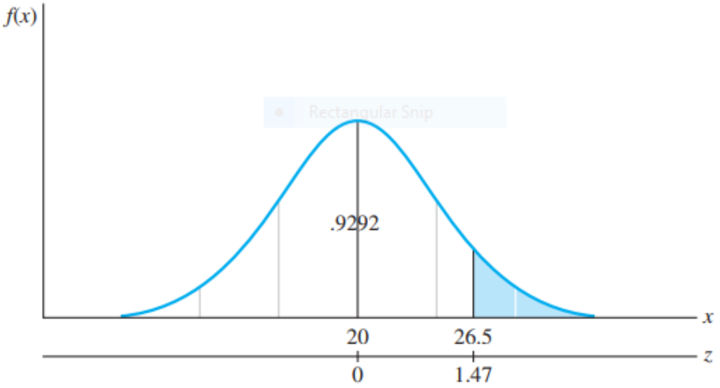

Figure: 13 Normal approximation to the binomial for Example 6.14