Linear versus First-order

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What is a linear differential equation ?

it may have high derivatives $y^{(n)}$ but with exponent at most 1

. example . it may have y'', but it has no $(y'')^2$, no $(y'')^3$

. $a(x)\, y'' + b(x)\, y' + c(x)\, y + d(x) = 0$: linear DE

. $a(x)(y'')^2 + a_1(x)\, y'' + b(x) = 0$: nonlinear DE

. $b(x)\, y' + c(x)\, y^2 + d(x) = 0$: nonlinear DE

Linearity / nonlinearity and first-order are different conditions

We will study <u>linear</u> <u>first-order</u> differential equation

Such an equation has the form

$$y' + a(x)\, y = b(x)$$

Linear First-order Differential Equations - a,b are constants

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$$y' + a(x)\,y = b(x)$$

<u>Case 1</u> $\quad a(x) = a\,,\quad b(x) = b$

$$y' + ay = b$$

Smart idea: multiply both sides by e^{ax}

$$\underbrace{e^{ax}y' + ae^{ax}y}_{\left(e^{ax}y\right)'} = e^{ax}b$$

$$d\left(e^{ax}y\right) = e^{ax}b\,dx$$

$$e^{ax}y = \int e^{ax}b\,dx = \frac{b}{a}e^{ax} + k$$

The solution is $y = \dfrac{b}{a} + ke^{-ax}$

Linear First-order Differential Equations - a is constant

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$$y' + ay = b(x)$$

Smart idea: multiply by e^{ax} (call it an <u>integrating factor</u>) (thừa số tích phân)

$$\underbrace{e^{ax} y' + a e^{ax} y}_{(y e^{ax})'} = b(x) e^{ax}$$

$$y e^{ax} = \underbrace{\int b(x) e^{ax}\, dx}_{\text{compute this!}}$$

<u>Example</u>: solve $y' + 2y = x$

$$\underbrace{e^{2x} y' + 2e^{2x} y}_{(e^{2x} y)'} = x e^{2x} \quad (\text{integrating factor: } e^{2x})$$

$$e^{2x} y = \underbrace{\int x e^{2x}}_{\text{integrate by parts (IBP)}} = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + k$$

$$y = \frac{x}{2} - \frac{1}{4} + k e^{-2x} \quad (k : \text{constant})$$

Linear First-order Differential Equations - a(x), b(x) are functions

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$$y' + a(x)\,y = b(x) \quad (*)$$

Theorem. (Existence and Uniqueness of Solution)

Assume that $a(x)$, $b(x)$ are continuous on an open interval (r, s)

(IVP) initial value problem
$$\begin{cases} y' + a(x)\,y = b(x) \\ y(x_0) = y_0 \end{cases}$$

Then this IVP has a unique solution on (r, s)

$$y' + a(x)\,y = b(x) \quad (I) \quad \text{`inhomogeneous equation'}$$

Look at another equation

$$y' + a(x)\,y = 0 \quad (H) \quad \text{`homogeneous equation'}$$

phương trình thuần nhất

We will solve (H) first, then solve (I)

$$y' + a(x)\,y = 0 \quad (H)$$

$$\int \frac{1}{y}\,dy = \int -a(x)\,dx$$

$\exp(T) = e^T$

exponential

$$\ln|y| = -\int a(x)\,dx$$

$$y = K\cdot\exp\left(-\int a(x)\,dx\right) = e^{-\int a(x)dx}$$

(K constant)

This is a solution for (H) $y' + a(x)\,y = 0$

- For (I) $y' + a(x)\,y = b(x)$

Smart idea: Put $y = K(x) \cdot \exp\left(-\int a(x)\,dx\right)$

let $K(x)$ be a function

and solve for $K(x)$

'Method of variation of constants'

Example of variations of constants

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<u>Example</u>: solve $y' - 2xy = 2xe^{x^2}$

This is a linear first-order DE

Consider (H) $y' - 2xy = 0$ now it is separable

$$\int \frac{1}{y}\, dy = \int 2x\, dx$$

$$\ln|y| = x^2 + k$$

$$y = K e^{x^2} \quad (K \text{ constant})$$

This is a solution for (H)

Consider (I) $y' - 2xy = 2xe^{x^2}$

look at $y = K(x)\, e^{x^2}$, plug in (I):

$$K'(x) e^{x^2} + \cancel{K(x) \cdot 2x e^{x^2}} - \cancel{2x \cdot K(x) e^{x^2}} = 2xe^{x^2}$$

$$K'(x) \cancel{e^{x^2}} = 2x \cancel{e^{x^2}}$$

$$K(x) = x^2 + l \quad (l \text{ constant})$$

so $y = K(x)\, e^{x^2} = (x^2 + l)\, e^{x^2}$

Check that this is a solution for (I) $y' - 2xy = 2xe^{x^2}$.

Bernoulli's equations

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Bernoulli's equations : $y' + p(x)\,y = q(x)\,y^{\alpha}$

- If $\alpha = 0, 1$: linear first-order DE
- If $\alpha \neq 0, 1$: nonlinear first-order DE

$$y' + p(x)\,y = q(x)\,y^{\alpha}$$

Idea : multiply by $y^{-\alpha}$ $(\alpha \neq 0, 1)$ not an integrating factor !

$$\underbrace{y^{-\alpha}\,y' + p(x)\,y^{1-\alpha}}_{\frac{1}{1-\alpha}\cdot\frac{dz}{dx} + p(x)\,z} = q(x)$$

$$\frac{1}{1-\alpha}\cdot\frac{dz}{dx} + p(x)\,z = q(x)$$. linear first-order DE in terms of z

Put $z = y^{1-\alpha}$, take $\frac{dz}{dx} = (1-\alpha)\,y^{-\alpha}\cdot y'$

$$\frac{1}{1-\alpha}\cdot\frac{dz}{dx} = y^{-\alpha}\,y'$$

Example of Bernoulli's equation

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Example. solve $y' + xy = x^3 y^3$

$y' + p(x)\, y = q(x)\, y^{\alpha}$ $(\alpha = 3)$

This is Bernoulli's equation

Multiply by $y^{-\alpha} = y^{-3}$

$$\boxed{y^{-3} y' + x y^{-2} = x^3}$$

Put $z = y^{-2}$

so $\dfrac{dz}{dx} = -2\, y^{-3}\, y'$. Therefore

$$-\frac{1}{2} \cdot \frac{dz}{dx} + xz = x^3$$

$z' - 2xz = -2x^3$: linear first-order DE

Use variation of constants/parameters.

$$z = x^2 + 1 + K \cdot e^{x^2} = y^{-2}$$

so the solution for $y' + xy = x^3 y^3$ is

$$y^2\left(x^2 + 1 + K e^{x^2}\right) = 1 \quad (K \text{ constant})$$

Exact differential equation

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Differential equation in the form of a total differential

What is a total differential ?

$u(x, y)$

$$du = u_x \, dx + u_y \, dy$$ total differential of u

Suppose we have a DE

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \qquad (*)$$

where $P\,dx + Q\,dy = du$ is a total differential

we call $(*)$ an exact differential equation

How to solve $(*)$?

We find u so that $du = P\,dx + Q\,dy$

Formulas for u satisfying $du = P\,dx + Q\,dy$: choose (x_0, y_0)

① $$u(x,y) = \int_{x_0}^{x} P(\cdot\,, y_0) + \int_{y_0}^{y} Q(x, \cdot)$$

$$= \int_{x_0}^{x} P(u, y_0)\,du + \int_{y_0}^{y} Q(x, v)\,dv$$

② $$u(x,y) = \int_{x_0}^{x} P(\cdot\,, y) + \int_{y_0}^{y} Q(x_0, \cdot)$$

(2) $u(x,y) = \int_{x_0} P(\cdot, y) \quad \int_{y_0} Q(\ ,\ )$

$$= \int_{x_0}^{x} P(u, y)\, du + \int_{y_0}^{y} Q(x_0, v)\, dv$$

Example of exact differential equation

Example solve $(e^x + y + \sin y) + y'(e^y + x + x\cos y) = 0$

This is

$$\underbrace{(e^x + y + \sin y)}_{P}\, dx + \underbrace{(e^y + x + x\cos y)}_{Q}\, dy = 0$$

$$P_y = 1 + \cos y \qquad \Rightarrow \quad P_y = Q_x$$

$$Q_x = 1 + \cos y$$

$\Rightarrow$ $P\,dx + Q\,dy = du$ is a total differential

We have an exact differential equation

Choose $(x_0, y_0) = (0, 0)$

$$u(x,y) = \int_0^x P(u, 0)\, du + \int_0^y Q(x, v)\, dv + k$$

$$P = e^x + y + \sin y \qquad Q = e^y + x + x\cos y$$

$$u(x,y) = \underbrace{\int_0^x e^u\, du} + \int_0^y (e^v + x + x\cos v)\, dv$$

$$= e^x - 1 + \left[e^v + xv + x\sin v\right]_0^y$$

$$= e^x - 1 + (e^y + xy + x\sin y) - (1)$$

$$= (e^x + e^y + xy + x\sin y) - 2$$

$$d\,u(x,y) = 0$$

the solution is $u(x,y) = k$

$$e^x + e^y + xy + x\sin y = k' \quad (k' \text{ constant})$$

Non-exact differential equation

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Suppose we have

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \quad (*)$$

but $P\,dx + Q\,dy \neq du$ is not a total differential

$$(P_y \neq Q_x)$$

We call $(*)$ a non-exact DE

Suppose we can find $I(x,y)$ so that

$I P\,dx + I Q\,dy = du$ is a total differential,

then we have an exact DE and we can solve it

This $I(x,y)$ is called an integrating factor

(thừa số tích phân)

Two cases

Case 1: If $\dfrac{P_y - Q_x}{Q} = g(x)$ is a function of x,

then $I(x,y) = I(x) = \exp\left(\int g(x)\,dx\right)$

$$= e^{\int \frac{P_y - Q_x}{Q}\,dx}$$

Case 2: If $\dfrac{P_y - Q_x}{-P} = h(y)$ is a function of y,

then $I(x,y) = I(y) = \exp\left(\int h(y)\,dy\right)$

then $I(x,y) = I(y) = \exp\left(\int h(y)\, dy\right)$

$$= e^{\int \frac{P_y - Q_x}{-P} dy}$$

Other cases: we don't learn in this course!

Example of non-exact equation

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Example. solve $(3xy - y^2) + (x^2 - xy)\, y' = 0$

$$\underbrace{(3xy - y^2)}_{P}\, dx + \underbrace{(x^2 - xy)}_{Q}\, dy = 0$$

$$P_y = 3x - 2y$$

$$Q_x = 2x - y$$

$P_y \neq Q_x$, $\quad P_y - Q_x = x - y \neq 0$

$P\,dx + Q\,dy$ is not a total differential

We have a non-exact DE

$$\frac{P_y - Q_x}{Q} = \frac{x - y}{x^2 - xy} = \frac{1}{x} \text{ is a function of } x$$

integrating factor $I(x) = \exp\left(\int \frac{1}{x}\, dx\right)$

$$= \exp(\ln|x|) = |x|$$

we can choose $I(x) = x$ to multiply in

$$\underbrace{(3x^2y - y^2x)}_{P_1}\, dx + \underbrace{(x^3 - x^2y)}_{Q_1}\, dy = 0$$

$$P_{1,y} = 3x^2 - 2xy$$

$$Q_{1,x} = 3x^2 - 2xy$$

so $P_{1y} = Q_{1x}$ exact DE

so we find $u(x,y)$ with $du = P_1 dx + Q_1 dy$ as before

we find $u(x,y) = x^3 y - \frac{x^2 y^2}{2}$

(check $u_x = 3x^2 y - xy = P_1$

$u_y = x^3 - x^2 = Q_1$)

so the solution is $u(x,y) = k$

$x^3 y - \frac{x^2 y^2}{2} = k$ (k constant)

Summary

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- linear first-order differential equation

$$y' + a(x)\, y = b(x)$$

  - a, b constants
  - a is constant
  - a, b functions : variation of constants / parameters

- Bernoulli's equation

$$y' + p(x)\, y = q(x)\, y^{\alpha} \quad (\alpha \neq 0, 1)$$

multiply by $y^{-\alpha}$ to make it linear

- exact / non-exact } differential equation