Linear Second order Differential Equations with General Coefficients

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- 2nd order ODE

  (ordinary differential equation)

$$F(x, y, y', y'') = 0$$

- Linear 2nd order ODE

$$y \quad \cancel{y^2} \quad \cancel{y^3}$$
$$y' \quad \cancel{y'^2} \quad \cancel{y'^3}$$
$$y'' \quad \cancel{y''^2} \quad \cancel{y''^3}$$

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

linear 2nd order ODE

If $f(x) \neq 0$. inhomogeneous ODE

If $f(x) = 0$: homogeneous ODE

- Linear 2nd order ODE with constant coefficients :

(I) $y'' + py' + qy = f(x)$; p, q constants

To solve (I), we solve the complementary equation

(H) $y'' + py' + qy = 0 \longrightarrow y_c$

and then we find a particular solution y_p for (I)

The general solution is $y_c + y_p$ (for (I))

To find y_p:
- undetermined coefficients
- variation of parameters

General Coefficients

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Linear 2nd order ODE

$$y'' + p(x)\,y' + q(x)\,y = f(x) \qquad (I)$$

Goal: to solve (I) in general

This is not possible!

But we will learn how to solve (I) in some special cases!

$$y'' + p(x)\,y' + q(x)\,y = 0 \qquad (H)$$

If we can solve y_c for (H)

and we can find y_p for (I),

then the general solution for (I) is $y_c + y_p$

Homogeneous equations

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$$y'' + p(x)\,y' + q(x)y = 0 \quad (H)$$

Lemma. If y_1 and y_2 are solutions to (H), then $k_1 y_1 + k_2 y_2$ are solutions to (H) (k_1, k_2 constants).

[superposition of solutions]

Proof. $(k_1 y_1)'' + p(x)(k_1 y_1)' + q(x)(k_1 y_1) = 0$

$(k_2 y_2)'' + p(x)(k_2 y_2)' + q(x)(k_2 y_2) = 0$

$\Rightarrow (k_1 y_1 + k_2 y_2)'' + p(x)(k_1 y_1 + k_2 y_2)' + q(x)(k_1 y_1 + k_2 y_2) = 0$

(linearity) $\square$

$$y'' + p(x)\,y' + q(x)\,y = 0$$

- $y_1 = x^2$, $y_2 = 5x^2$ bad
- $y_1 = x^2$, $y_2 = x^2 + \sin x$ good

2 solutions y_1, y_2 are <u>linearly dependent</u> (LD) if $c_1 y_1 + c_2 y_2 = 0$ for some $(c_1, c_2) \neq (0,0)$

example $x^2, 5x^2$ are LD

because $(-5)(x^2) + (1)(5x^2) = 0$

2 solutions y_1, y_2 are <u>linearly independent</u> (LI) if there is no $(c_1, c_2) \neq (0,0)$ such that $c_1 y_1 + c_2 y_2 = 0$

example - x^2, $x^2 + \sin x$ are LI

because if $c_1 x^2 + c_2 (x^2 + \sin x) = 0$

$$\Rightarrow \quad (c_1 + c_2)\, x^2 + c_2 \sin x = 0$$

$$\Rightarrow \quad \begin{cases} c_1 + c_2 = 0 \\ c_2 = 0 \end{cases} \quad \Rightarrow \quad c_1 = c_2 = 0$$

If we have LI solutions y_1, y_2 of (H),

then $y_c = k_1 y_1 + k_2 y_2$ is a general solution of (H)

How to test for LI ?

Use Wronski determinant.

Suppose y_1, y_2 are differentiable on an open interval I

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

If y_1, y_2 are LD say $y_1 = c\, y_2$ (c constant)

$$\Rightarrow \quad W(y_1, y_2) = \begin{vmatrix} y_1 & c y_1 \\ y_1' & c y_1' \end{vmatrix} = c\, y_1 y_1' - y_1' c y_1 = 0$$

If y_1, y_2 are LD, then $W(y_1, y_2) = 0$

The converse is true !

<u>Theorem</u>: If $W(y_1, y_2) = 0$ for solutions of (H),

then y_1, y_2 are LD.

<u>Consequence</u>: y_1, y_2 are LD $\Leftrightarrow$ $W(y_1, y_2) = 0$

y_1, y_2 are LI $\Leftrightarrow$ $W(y_1, y_2) \neq 0$

<u>Abel's theorem</u>:

Suppose $p(x)$, $q(x)$ are continuous on an open interval I.

Suppose $p(x), q(x)$ are continuous on an open interval I.

Suppose y_1, y_2 are 2 solutions for (H) ; $x_0 \in I$.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(x)$$

Then $W(x) = W(x_0)\, e^{-\int_{x_0}^{x} p(t)\,dt}$

(formula for Wronski's determinant)

The theorem allows us to compute $W(x)$ even if we don't know y_1, y_2 !

In case we know y_1 but not y_2, then we can use y_1 and $W(x)$ to compute y_2 !

This is why Abel's theorem is useful !

Example. . $g(x) = \sin x$, $h(x) = \cos x$

$$W(g, h) = \begin{vmatrix} g & h \\ g' & h' \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$$

. $g(x) = x^2$, $h(x) = x^2 + \sin x$

$$W(g, h) = \begin{vmatrix} g & h \\ g' & h' \end{vmatrix}$$

$$W(g,h) = \begin{vmatrix} & \\ g' & h' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x^2 + \sin x \\ 2x & 2x + \cos x \end{vmatrix}$$

$$= x^2 (\cos x - \sin x)$$

- $g(x) = x^2 \quad , \quad h(x) = 5x^2$

$$W(g,h) = \begin{vmatrix} g & h \\ g' & h' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & 5x^2 \\ 2x & 10x \end{vmatrix} = 0$$

Theorem. If y_1, y_2 are LI solutions for (H) on I then the general solution for (H) is

$$y_c = k_1 y_1 + k_2 y_2$$

$(k_1, k_2$ constants$)$

How to find y_1, y_2? We don't know!

However, if we know y_1, then we can find y_2!

- variation of parameters
- use Wronski's determinant

Example - Homogeneous equations

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Example. solve $x^2 y'' + 2x y' - 6y = 0$.

$$y'' + \frac{2}{x} y' - \frac{6}{x^2} y = 0$$

To find y_1, we have to guess and check!

Guess : $y_1 = x^2$ is a solution

because $2 + \frac{2}{x} \cdot 2x - \frac{6}{x^2} \cdot x^2 = 0$

$y'' + \frac{2}{x} y' - \frac{6}{x^2} y = 0$, $y_1 = x^2$ is a solution

We find y_2

Variation of parameters

Put $y = u(x) \cdot y_1$ and find $u(x)$

$$\left.\begin{aligned} \frac{2}{x} \cdot \quad y' &= u' y_1 + u y_1' \\ 1 \cdot \quad y'' &= u'' y_1 + 2u' y_1' + u y_1'' \end{aligned}\right\} \text{plug in (H)}$$

$$-\frac{6}{x^2} \cdot \quad y = u \cdot y_1$$

$$0 = \frac{2}{x} \cdot u' y_1 + \frac{2}{x} u y_1' + u'' y_1 + 2u' y_1' + u y_1'' - \frac{6}{x^2} \cdot u y_1$$

add to zero

$0 = \frac{2}{x} u' y_1 + u'' y_1 + 2u' y_1'$, $y_1 = x^2$

solve for u

$$0 = 2x u' + u'' \cdot x^2 + 2u' \cdot 2x$$

$$0 = 6xu' + u''x^2$$

$$0 = 6u' + u''x$$

Put $v = u'$ $\Rightarrow$ $6v + v'x = 0$ (separable, 1st order)

$$\Rightarrow v = \frac{k}{x^6}$$

$$\Rightarrow u' = \frac{k}{x^6}$$

$$\Rightarrow u = \frac{k \cdot x^{-5}}{-5} + k'$$

$$y = u \cdot y_1 = \left(k \cdot \frac{x^{-5}}{-5} + k'\right) x^2$$ general solution

$$y'' + \frac{2}{x} y' - \frac{6}{x^2} y = 0 \quad (H), \quad y_1 = x^2 \text{ is a solution}$$

To find y_2, we can use Wronski's determinant as well

Abel's theorem $y_1 y_2' - y_1' y_2 = e^{-\int p(t)\,dt}$

$$p(t) = \frac{2}{t}.$$

$$\frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{1}{y_1^2} \cdot e^{-\int p(t)\,dt}$$

$$\left(\frac{y_2}{y_1}\right)' = \underbrace{\frac{1}{y_1^2} e^{-\int p(t)\,dt}}_{\text{we can compute}}$$

$\longrightarrow$ we can solve for $\frac{y_2}{y_1}$

$\longrightarrow$ we can get y_2

$$\left(\frac{y_2}{y_1}\right)' = \frac{1}{y_1^2} e^{-\int p(t)\,dt} \quad ; \quad p(t) = \frac{2}{t}, \quad y_1 = x^2$$

$$\left(\frac{y_2}{x^2}\right)' = \frac{1}{x^4} e^{-\int_{x_0}^{x} \frac{2}{t}\,dt}$$

$$= \frac{1}{x^4} \cdot e^{-2\ln x} = \frac{1}{x^4} \cdot x^{-2} = \frac{1}{x^6}$$

$$\left(\frac{y_2}{x^2}\right)' = \frac{1}{x^6}$$

$$\frac{y_2}{x^2} = \int \frac{1}{x^6}\,dx = \frac{-1}{5\,x^5}$$

$y_2 = \dfrac{-1}{5\,x^3}$, we can replace y_2 by $\dfrac{1}{x^3} = y_{2\,new}$

$y_1 = x^2$, $y_{2\,new} = \dfrac{1}{x^3} \Rightarrow y = k_1 x^2 + \dfrac{k_2}{x^3}$ general solution to (H)

$$y'' + p(x)\,y' + q(x)\,y = 0 \quad (H)$$

We don't know how to solve (H) in general

However, if we can find / guess y_1,

then we can find y_2 — variation of parameters / Wronski's determinant

(thanks to Abel!)

Inhomogeneous equations

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1. $y'' + p(x)\,y' + q(x)\,y = f(x)$ (I)

Canonical form (dạng chính tắc)

$y'' + p(x)\,y' + q(x)\,y = 0$ (H)

Suppose (H) has 2 solutions y_1, y_2 (LI)

To solve (I) we use variation of parameters!

Put $y = k_1 y_1 + k_2 y_2$ (k_1, k_2 are functions, not constants)

$y' = (k_1 y_1' + k_2 y_2') + \boxed{(k_1' y_1 + k_2' y_2)} = 0$

$y'' = (k_1 y_1'' + k_2 y_2'') + 2\underline{(k_1' y_1' + k_2' y_2')} + \underline{(k_1'' y_1 + k_2'' y_2)}$

$y'' + p y' + q y = f$

We want to find k_1, k_2

We first solve for k_1', k_2' and then integrate.

We assume $\boxed{k_1' y_1 + k_2' y_2 = 0}$

$\Rightarrow k_1' y_1' + k_2' y_2' + k_1'' y_1 + k_2'' y_2 = 0$

$\Rightarrow k_1' y_1' + k_2' y_2' = f(x)$

We have $\begin{cases} k_1' y_1 + k_2' y_2 = 0 \\ k_1' y_1' + k_2' y_2' = 0 \end{cases}$

$\rightsquigarrow$ solve for k_1', k_2'

Summary: we have y_1, y_2 for (H)

For (I) $y = k_1 y_1 + k_2 y_2$

with $\begin{cases} k_1' y_1 + k_2' y_2 = 0 \\ k_1' y_1' + k_2' y_2' = f(x) \end{cases}$

This is variation of parameters!

Example of solving an inhomogeneous equation

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Example $(1-x^2)y'' + 2xy' - 2y = 1 - x^2$

$$y'' + \frac{2x}{1-x^2}y' - \frac{2}{1-x^2}y = 1 \quad (I)$$

$$y'' + \frac{2x}{1-x^2}y' - \frac{2}{1-x^2}y = 0 \quad (H)$$

Suppose we know solutions $y_1 = x$, $y_2 = x^2+1$ of (H)

[Guess $y_1 = x$ is a solution, then find y_2]

To solve (I) $\quad y = k_1 x + k_2 (x^2+1)$

$$\begin{cases} k_1' x + k_2' (x^2+1) = 0 \\ k_1' \cdot 1 + k_2' \, 2x = 1 \end{cases}$$

$$\Rightarrow k_1' = -\frac{x^2+1}{x^2-1} = -1 - \frac{2}{x^2-1}$$

$$k_2' = \frac{x}{x^2-1}$$

$$\Rightarrow k_1 = -x - \ln\left|\frac{x-1}{x+1}\right| + c_1 \quad (c_1 \text{ constant})$$

$$k_2 = \frac{\ln|x^2-1|}{2} + c_2 \quad (c_2 \text{ constant})$$

$$\Rightarrow y = k_1 x + k_2 (x^2+1)$$

$$\left.\begin{aligned} y = -x\left(x + \ln\left|\frac{x-1}{x+1}\right|\right) + \frac{x^2+1}{2}\ln|x^2-1| \\ + c_1 x + c_2 (x^2+1) \end{aligned}\right\} \text{general solution to } (I)$$

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