Week 4

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Question 1. Suppose X is continuous with density

$$f_X(x) = \begin{cases} cx^2 & 0 \le x < 9 \\ 0 & \text{ow} \end{cases}$$

Write an expression for the value of c that makes X a valid PDF, and set up expressions (integrals) for its mean and variance. Also, ~~find the~~ CDF of X, FX.

+ Conditions for valid PDF: $\int_{-\infty}^{+\infty} f(x)\, dx = 1$

$$\Leftrightarrow \frac{cx^3}{3}\Big|_0^9 = 1$$

$$\Leftrightarrow \boxed{c = \frac{1}{243}}$$

+ Mean: $E[X] = \int_{-\infty}^{+\infty} x f_X(t)\, dt \quad \int_0^9 t \cdot ct^2 dt$

$$= \boxed{\frac{1}{243}\int_0^9 t^3\, dt}$$

+ Variance: We have $E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(t)\, dt$

$$= \frac{1}{243}\int_0^9 t^4\, dt$$

$$\Rightarrow \boxed{Var(X) = \left(\frac{1}{243}\int_0^9 t^3\, dt\right)^2 - \frac{1}{243}\int_0^9 t^4\, dt}$$

+ $\frac{dF_X(x)}{dx} = f_X(x) \Rightarrow \boxed{F_X(x)} = \begin{cases} 0 & , x < 0 \\ \frac{c}{3}x^3 & , 0 \le x < 9 \\ 1 & , x \ge 9 \end{cases}$

Question 2.

a. Suppose the current (in Amperes) flowing through a 1-ohm resistor is a $Uniform(a,b)$ random variable I for $a,b>0$. The power dissipated by this resistor is $X=I$^2. What is the expected power dissipated by the resistor?

b. Continuing with the previous example, suppose that the current Iinstead follows an $Exponential(\lambda)$ distribution. What is the expected power dissipated by the resistor?

a. The CDF of the current flowing through the resistor:

$$F_X(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \le x \le b \\ 1 & , x > b \end{cases}$$

$\Rightarrow$ For the power dissipated by the resistor is:

$$F_X(x) = \begin{cases} 0 & , x < a^2 \\ \frac{\sqrt{x}-a}{b-a} & , a^2 \le x \le b^2 \\ 1 & , x > b^2 \end{cases}$$

$$\Rightarrow f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} \frac{1}{2(b-a)\sqrt{x}} & , a^2 \le x \le b^2 \end{cases}$$

$$\rightarrow E[X] = \int_{a^2}^{b^2} x \cdot \frac{1}{2(b-a)\sqrt{x}} \, dx = \frac{b^3 - a^3}{3(b-a)}$$

0 ow

b, $E[I^2] = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \boxed{\frac{2}{\lambda^2}}$

Question 3: A flea of negligible size is trapped in a large, spherical, inflated beach ball with radius r. At this moment, it is equally likely to be at any point within the ball. Let X be the distance of the flea from the centre of the ball.
a. Find the range of X, ΩX.
b. Find the cumulative distribution function $FX(x)=P(X\le x)$.
c. Find the probability density function $fX(x)$.
d. Find an integral for $E[X]$.

a. $\Omega_X = [0, r]$

b. $P(X \le x) = \frac{\text{Volume of the sphere of radius } x}{\text{Volume of the sphere of radius } r} = \boxed{\frac{x^3}{r^3}}$

c. $f_X(x) = \frac{dF_X(x)}{dx} = \frac{3x^2}{r^3}, \quad 0 < x < r$

$$\boxed{f_X(x)} = \begin{cases} \frac{3x^2}{r^3} & , 0 < x < r \\ 0 & \text{ow} \end{cases}$$

d. $E[X] = \int_0^r x f(x) dx = 3\int_0^r \frac{x^3}{r^3} dx = \boxed{\frac{3}{4} \cdot r}$

Question 4: Suppose that you are s minutes early for an appointment, then you incur the cost cs, and if you are s minutes late, then you incur the cost ks. Suppose also the travel time from where you presently are to the location of your appointment is a continuous random variable having probability density function f. Determine the time at which you should depart if you want to minimize your expected cost.

Suppose we leave t minutes before the appointment, then:

$$\text{Cost}: \quad C_t(x) = \begin{cases} c(t-x) & \text{if } X \le t \\ k(X-t) & \text{if } X > t \end{cases}$$

where X is the travel time

$$\text{Then} \quad E[C_t(x)] = \int_{-\infty}^{\infty} C_t(x) f(x) dx$$
$$= \int_0^t c(t-x) f(x) dx + \int_t^{+\infty} k(x-t) f(x) dx$$
$$= ct \cdot \int_0^t f(x) dx - c\int_0^t x f(x) dx$$
$$+ kt \int_t^{+\infty} x f(x) dx - kt \int_t^{+\infty} f(x) dx$$

We have: $\frac{d E[C_t(x)]}{dx} = c \cdot t \cdot f(t) - c \cdot F(t) - ct \cdot f(t)$
$- kt \cdot f(t) + kt f(t)$
$- k[1 - F(t)]$
$= \boxed{(k+c) F(t) - k}$

Question 5: You are waiting for a bus to take home from VinUni. You can either take the A-line, B-line, and C-line. The distribution of the waiting time in minutes for each is the following:
- A-line: $A\sim Exp(\lambda=0.1)$
- B-line: $B\sim \overline{Unif}(0,20)$

- C-line: has range $(1,\infty)$ and density function $fC(x)=1/(x^2)$

Assume the three bus arrival times are independent. You take the first bus that arrives.

- Find the CDFs of A,B, and C.
- What is ~~the pr~~obability you wait more than 5 minutes for a bus?
- What is the probability you wait ~~more than 30 minute~~s for a bus?
- What is the expected amount of ~~time you will wait fo~~r a bus?

+ CDF:

A: $F_A(x) = \begin{cases} \boxed{1-e^{-(0.1)x}} & , x \geq 0 \\ 0 & \text{ow} \end{cases}$

B: $F_B(x) = \begin{cases} 0 & , x < 0 \\ \boxed{\frac{x}{20}} & , 0 \leq x \leq 20 \\ 1 & , x > 20 \end{cases}$

C: $F_C(x) = \int_1^x \frac{1}{t^2}\, dt = \boxed{1 - \frac{1}{x}}$

$\Rightarrow F_C(x) = \begin{cases} 1-\frac{1}{x} & x > 1 \\ 0 & \text{ow} \end{cases}$

+ Let $D = \min\{A, B, C\}$ be the time until the first bus arrives.

then:

$$P(D>5) = P(A>5) \cdot P(B>5) \cdot P(C>5)$$
$$= (1-F_A(x)) \cdot (1-F_B(x)) \cdot (1-F_C(x))$$
$$= (e^{-0.5}) \cdot \left(\frac{15}{20}\right) \cdot \left(\frac{1}{5}\right)$$
$$= \boxed{\frac{3}{20} e^{-0.5}}$$

+ Similarly: $P(D>30) = P(A>30) \cdot P(B>30) \cdot P(C>30)$

$$= \boxed{0}$$

+ $E[D] = \min\{E(A), E(B), E(C)\}$

$$= \min\left\{\frac{1}{0.1}, \frac{10}{2}, \int_1^\infty x \frac{1}{x^2}\, dx\right\}$$
$$= \min\{10, 10, +\infty\}$$
$$= \boxed{10}$$