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Calculus 2 Exercises

Chapter 1

VECTORS AND THE GEOMETRY OF SPACE

Reference: James Stewart. *Calculus*, sixth edition. Thomson, USA 2008.

1.1 Three-dimensional coordinate systems

1. Find the lengths of the sides of the triangle PQR. Is it a right triangle? Is it an isosceles triangle?

a) $P(3;-2;-3)$, $Q(7;0;1)$, $R(1;2;1)$.

b) $P(2;-1;0)$, $Q(4;1;1)$, $R(4;-5;4)$.

2. Find an equation of the sphere with center $(1;-4;3)$ and radius 5. Describe its intersection with each of the coordinate planes.

3. Find an equation of the sphere that passes through the origin and whose center is $(1;2;3)$.

4. Find an equation of a sphere if one of its diameters has end points $(2;1;4)$ and $(4;3;10)$.

5. Find an equation of the largest sphere with center $(5,4,9)$ that is contained in the first octant.

6. Write inequalities to describe the following regions

a) The region consisting of all points between (but not on) the spheres of radius r and R centered at the origin, where $r < R$.

b) The solid upper hemisphere of the sphere of radius 2 centered at the origin.

7. Consider the points P such that the distance from P to $A(-1;5;3)$ is twice the distance from P to $B(6;2;-2)$. Show that the set of all such points is a sphere, and find its center and radius.

8. Find an equation of the set of all points equidistant from the points $A(-1;5;3)$ and $B(6;2;-2)$. Describe the set.

1.2 Vectors

9. Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2;4)$.

10. Find the unit vectors that are parallel to the tangent line to the curve $y = 2\sin x$ at the point $(\pi/6;1)$.

11. Find the unit vectors that are perpendicular to the tangent line to the curve $y = 2\sin x$ at the point $(\pi/6;1)$.

12. Let C be the point on the line segment AB that is twice as far from B as it is from A. If $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$, show that $\mathbf{c} = \mathbf{\frac{2}{3}a} + \mathbf{\frac{1}{3}b}$.

1.3 The dot product

13. Determine whether the given vectors are orthogonal, parallel, or neither

a) $a = (-5;3;7), \quad b = (6;-8;2)$

b) $a = (4;6), \quad b = (-3;2)$

c) $a = -i + 2j + 5k, \quad b = 3i + 4j - k$

d) $u = (a, b, c), \quad v = (-b;a;0)$

14. For what values of b are the vectors $(-6;b;2)$ and $(b;b^2;b)$ orthogonal?

15. Find two unit vectors that make an angle of 60^o with $v = (3;4)$.

16. If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$, find the third direction angle γ.

17. Find the angle between a diagonal of a cube and one of its edges.

18. Find the angle between a diagonal of a cube and a diagonal of one of its faces.

1.4 The cross product

19. Find the area of the parallelogram with vertices $A(-2;1)$, $B(0;4)$, $C(4;2)$, and $D(2;-1)$.

20. Find the area of the parallelogram with vertices $K(1;2;3)$, $L(1;3;6)$, $M(3;8;6)$ and $N(3;7;3)$.

21. Find the volume of the parallelepiped determined by the vectors a, b, and c.

a) $a = (6;3;-1), \quad b = (0;1;2), \quad c = (4;-2;5)$.

b) $a = i + j - k, \quad b = i - j + k, \quad c = -i + j + k$.

22. Let $v = 5j$ and let u be a vector with length 3 that starts at the origin and rotates in the xy-plane. Find the maximum and minimum values of the length of the vector $u \times v$. In what direction does $u \times v$ point?

1.5 Equations of lines and planes

23. Determine whether each statement is true or false.

a) Two lines parallel to a third line are parallel.

b) Two lines perpendicular to a third line are parallel.

c) Two planes parallel to a third plane are parallel.

d) Two planes perpendicular to a third plane are parallel.

e) Two lines parallel to a plane are parallel.

f) Two lines perpendicular to a plane are parallel.

g) Two planes parallel to a line are parallel.

h) Two planes perpendicular to a line are parallel.

i) Two planes either intersect or are parallel.

j) Two lines either intersect or are parallel.

k) A plane and a line either intersect or are parallel.

24. Find a vector equation and parametric equations for the line.

a) The line through the point $(6; -5; 2)$ and parallel to the vector $(1; 3; -2/3)$.

b) The line through the point $(0; 14; -10)$ and parallel to the line $x = -1 + 2t; y = 6 - 3t; z = 3 + 9t$.

c) The line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$.

25. Find parametric equations and symmetric equations for the line of intersection of the plane $x + y + z = 1$ and $x + z = 0$.

26. Find a vector equation for the line segment from $(2; -1; 4)$ to $(4; 6; 1)$.

27. Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

a) $L_1 : x = -6t, y = 1 + 9t, z = -3t;\quad L_2 : x = 1 + 2s, y = 4 - 3s, z = s$.

b) $L_1 : \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3};\quad L_2 : \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$.

28. Find an equation of the plane.

a) The plane through the point $(6; 3; 2)$ and perpendicular to the vector $(-2; 1; 5)$

b) The plane through the point $(-2; 8; 10)$ and perpendicular to the line $x = 1 + t, y = 2t, z = 4 - 3t$.

c) The plane that contains the line $x = 3+2t, y = t, z = 8-t$ and is parallel to the plane $2x + 4y + 8z = 17$.

29. Find the cosine of the angle between the planes $x + y + z = 0$ and $x + 2y + 3z = 1$.

30. Find parametric equations for the line through the point $(0; 1; 2)$ that is perpendicular to the line $x = 1+t, y = 1-t, z = 2t$, and intersects this line.

31. Find the distance between the skew lines with parametric equations $x = 1 + t, y = 1 + 6t, z = 2t$ and $x = 1 + 2s, y = 5 + 15s, z = -2 + 6s$.

1.6 Quadric surfaces

32. Find an equation for the surface obtained by rotating the parabola $y = x^2$ about the y-axis.

33. Find an equation for the surface consisting of all points that are equidistant from the point $(-1; 0; 0)$ and the plane $x = 1$. Identify the surface.

Chapter 2

VECTOR FUNCTIONS

Reference: James Stewart. *Calculus*, sixth edition. Thomson, USA 2008.

2.1 Vector functions

34. Find the domain of the vector function.

a) $r(t) = (\sqrt{4-t^2}, e^{-3t}, \ln(t+1))$

b) $r(t) = \frac{t-2}{t+2}i + \sin t j + \ln(9-t^2)k$

35. Find the limit

a) $\lim\limits_{t\to 0}(\frac{e^t-1}{t}, \frac{\sqrt{1+t}-1}{t}, \frac{3}{t+1})$

b) $\lim\limits_{t\to\infty}(\arctan t, e^{-2t}, \frac{\ln t}{t+1})$

36. Find a vector function that represents the curve of intersection of the two surfaces.

a) The cylinder $x^2+y^2=4$ and the surface $z=xy$.

b) The paraboloid $z=4x^2+y^2$ and the parabolic cylinder $y=x^2$.

37. Suppose u and v are vector functions that possess limits as $t\to a$ and let c be a constant. Prove the following properties of limits.

a) $\lim\limits_{t\to a}[u(t)+v(t)] = \lim\limits_{t\to a} u(t) + \lim\limits_{t\to a} v(t)$

b) $\lim\limits_{t\to a} cu(t) = c\lim\limits_{t\to a} u(t)$

c) $\lim\limits_{t\to a}[u(t).v(t)] = \lim\limits_{t\to a} u(t). \lim\limits_{t\to a} v(t)$

d) $\lim_{t\to a}[u(t)\times v(t)] = \lim_{t\to a} u(t) \times \lim_{t\to a} v(t)$

38. Find the derivative of the vector function.

a) $r(t) = (t\sin t, t^3, t\cos 2t)$.

b) $r(t) = \arcsin t i + \sqrt{1-t^2} j + k$

c) $r(t) = e^{t^2} i - \sin^2 t j + \ln(1+3t)$

39. Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point. Illustrate by graphing both the curve and the tangent line on a common screen.

a) $x = t, y = e^{-t}, z = 2t - t^2; (0; 1; 0)$

b) $x = 2\cos t, y = 2\sin t, z = 4\cos 2t; (\sqrt{3}, 1, 2)$

c) $x = t\cos t, y = t, z = t\sin t; (-\pi, \pi, 0)$

40. Find the point of intersection of the tangent lines to the curve $r(t) = (\sin \pi t, 2\sin \pi t, \cos \pi t)$ at the points where $t = 0$ and $t = 0.5$

41. Evaluate the integral

a) $\int_0^{\pi/2} (3\sin^2 t \cos t\, i + 3\sin t \cos^2 t\, j + 2\sin t \cos t\, k)dt$

b) $\int_1^2 (t^2\, i + t\sqrt{t-1}\, j + t\sin \pi t\, k)dt$

c) $\int (e^t\, i + 2t\, j + \ln t\, k)dt$

d) $\int (\cos \pi t\, i + \sin \pi t\, j + t^2\, k)dt$

42. If a curve has the property that the position vector $r(t)$ is always perpendicular to the tangent vector $r'(t)$, show that the curve lies on a sphere with center the origin.

2.2 Arc length and curvature

43. Find the length of the curve.

a) $r(t) = (2\sin t, 5t, 2\cos t), \quad -10 \le t \le 10$

b) $r(t) = (2t, t^2, \frac{1}{3}t^3), \quad 0 \le t \le 1$

c) $r(t) = \cos t\, i + \sin t\, j + \ln\cos t\, k, \quad 0 \le t \le \pi/4$

44. Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point $(6; 18; 36)$.

45. Suppose you start at the point $(0; 0; 3)$ and move 5 units along the curve $x = 3\sin t, y = 4t, z = 3\cos t$ in the positive direction. Where are you now?

46. Reparametrize the curve

$$r(t) = \left(\frac{2}{t^2+1} - 1\right) i + \frac{2t}{t^2+1} j$$

with respect to arc length measured from the point $(1; 0)$ in the direction of increasing . Express the reparametrization in its simplest form. What can you conclude about the curve?

47. Find the curvature

a) $r(t) = t^2\, i + t\, k$

b) $r(t) = t\, i + t\, j + (1 + t^2)\, k$

c) $r(t) = 3t\, i + 4\sin t\, j + 4\cos t\, k$

d) $x = e^t \cos t, y = e^t \sin t$

e) $x = t^3 + 1, y = t^2 + 1$

48. Find the curvature of $r(r) = (e^t \cos t, e^t \sin t, t)$ at the point (1, 0, 0).

49. Find the curvature of $r(r) = (t, t^2, t^3)$ at the point (1, 1, 1).

50. Find the curvature

a) $y = 2x - x^2$, b) $y = \cos x$, c) $y = 4x^{5/2}$.

51. At what point does the curve have maximum curvature? What happens to the curvature as $x \to \infty$?

a) $y = \ln x$, b) $y = e^x$.

52. Find an equation of a parabola that has curvature 4 at the origin.

Chapter 3

Multiple Integrals

3.1 Double Integrals

3.1.1 Double Integrals in Cartesian coordinate

53. *Evaluate*

a) $\iint\limits_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} x\sin(x+y)dxdy,$

b) $\iint\limits_{[0,2]\times[1,2]} (x-3y^2)dxdy,$

c) $\iint\limits_{[1,2]\times[0,\pi]} y\sin(xy)dxdy,$

d) $\iint\limits_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} \sin(x-y)dxdy,$

e) $\iint\limits_{[0,2]\times[1,2]} (y+xy^{-2})dxdy,$

f) $\iint\limits_{[0,1]\times[-3,2]} \frac{xy^2}{x^2+1}dxdy,$

g) $\iint\limits_{[0,1]\times[0,1]} \frac{1+x^2}{1+y^2}dxdy,$

h) $\iint\limits_{[0,\frac{\pi}{6}]\times[0,\frac{\pi}{3}]} x\sin(x+y)dxdy,$

i) $\iint\limits_{[0,1]\times[0,1]} \frac{x}{1+xy}dxdy,$

j) $\iint\limits_{[0,2]\times[0,3]} ye^{-xy}dxdy,$

k) $\iint\limits_{[1,3]\times[1,2]} \frac{1}{1+x+y}dxdy.$

54. *Evaluate*

a) $\iint\limits_{D} x^2(y-x)\,dxdy$ *where* D *is the region bounded by* $y=x^2$ *and* $x=y^2$.

b) $\iint\limits_{D} |x+y|dxdy, D:=\{(x,y)\in\mathbb{R}^2\,||x\le 1|\,,|y|\le 1\}$

c) $\iint\limits_{D} \sqrt{|y-x^2|}dxdy, D:=\{(x,y)\in\mathbb{R}^2\,||x|\le 1, 0\le y\le 1\}$

d) $\iint\limits_{[0,1]\times[0,1]} \frac{ydxdy}{(1+x^2+y^2)^{\frac{3}{2}}}$

e) $\iint\limits_D \frac{x^2}{y^2}dxdy$, *where D is bounded by the lines $x=2, y=x$ and the hyperbola $xy=1$.*

f) $\iint\limits_D \frac{y}{1+x^5}dxdy$, *where* $D=\{(x,y)|0\le x\le 1, 0\le y\le x^2\}$,

g) $\iint\limits_D y^2e^{xy}dxdy$, *where* $D=\{(x,y)|0\le y\le 4, 0\le x\le y\}$,

h) $\iint\limits_D x\sqrt{y^2-x^2}dxdy$,*where* $D=\{(x,y)|0\le y\le 1, 0\le x\le y\}$,

i) $\iint\limits_D (x+y)dxdy$, *where D is bounded by $y=\sqrt{x}$ and $y=x^2$,*

j) $\iint\limits_D y^3dxdy$, *where D is the triangle region with vertices $(0,2),(1,1)$ and $(3,2)$,*

k) $\iint\limits_D xy^2dxdy$, *where D is enclosed by $x=0$ and $x=\sqrt{1-y^2}$.*

Change the order of integration

55. *Change the order of integration*

a) $\int\limits_{-1}^{1} dx \int\limits_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)\,dy.$

b) $\int\limits_{0}^{1} dy \int\limits_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\,dx.$

c) $\int\limits_{0}^{2} dx \int\limits_{\sqrt{2x-x^2}}^{\sqrt{2x}} f(x,y)\,dx.$

d) $\int\limits_{0}^{4} dx \int\limits_{0}^{\sqrt{x}} f(x,y)dy,$

e) $\int\limits_{0}^{1} dx \int\limits_{4x}^{4} f(x,y)dy,$

f) $\int\limits_{0}^{3} dy \int\limits_{-\sqrt{9-y^2}}^{9-y^2} f(x,y)dx,$

g) $\int\limits_{0}^{3} dy \int\limits_{0}^{\sqrt{9-y}} f(x,y)dx,$

h) $\int\limits_{0}^{2} dx \int\limits_{0}^{\ln x} f(x,y)dy,$

i) $\int\limits_{0}^{1} dx \int\limits_{\arctan x}^{\frac{\pi}{4}} f(x,y)dy,$

j) $\int\limits_{0}^{\sqrt{2}} dy \int\limits_{0}^{y} f(x,y)\,dx + \int\limits_{\sqrt{2}}^{2} dy \int\limits_{0}^{\sqrt{4-y^2}} f(x,y)\,dx.$

56. *Evaluate the integral by reversing the order of integration*

a) $\int\limits_{0}^{1} dy \int\limits_{3y}^{3} e^{x^2}dx,$

b) $\int\limits_{0}^{\sqrt{\pi}} dy \int\limits_{y}^{\sqrt{\pi}} \cos(x^2)dx,$

c) $\int\limits_{0}^{4} dx \int\limits_{\sqrt{x}}^{2} \frac{1}{y^3+1}dy,$

d) $\int\limits_{0}^{1} dx \int\limits_{x}^{1} e^{\frac{x}{y}}dy,$

e) $\int\limits_{0}^{1} dy \int\limits_{\arcsin y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}dx,$

f) $\int\limits_{0}^{8} dy \int\limits_{\sqrt[3]{y}}^{2} e^{x^4}dx.$

Change of variables

57. *Evaluate* $I = \iint\limits_D (4x^2 - 2y^2)\,dxdy$, *where* $D : \begin{cases} 1 \le xy \le 4 \\ x \le y \le 4x. \end{cases}$

58. *Evaluate*

$$I = \iint\limits_D \frac{x^2 \sin xy}{y} dxdy,$$

where D is bounded by parabolas

$$x^2 = ay, x^2 = by, y^2 = px, y^2 = qx, \ (0 < a < b, 0 < p < q).$$

59. *Evaluate* $I = \iint\limits_D xydxdy$, *where D is bounded by the curves*

$$y = ax^3, y = bx^3, y^2 = px, y^2 = qx, \ (0 < b < a, 0 < p < q).$$

Hint: Change of variables $u = \frac{x^3}{y}, v = \frac{y^2}{x}$.

60. *Prove that*

$$\int\limits_0^1 dx \int\limits_0^{1-x} e^{\frac{y}{x+y}} dy = \frac{e-1}{2}.$$

Hint: Change of variables $u = x + y, v = y$.

61. *Find the area of the domain bounded by* $xy = 4, xy = 8, xy^3 = 5, xy^3 = 15$.

Hint: Change of variables $u = xy, v = xy^3$, $(S = 2\ln 3)$.

62. *Find the area of the domain bounded by* $y^2 = x, y^2 = 8x, x^2 = y, x^2 = 8y$.

Hint: Change of variables $u = \frac{y^2}{x}, v = \frac{x^2}{y}$, $(S = \frac{279\pi}{2})$.

63. *Hint: Change of variables* $y = x^3, y = 4x^3, x = y^3, x = 4y^3$.

64. *Prove that*

$$\iint\limits_{x+y\le 1, x\ge 0, y\ge 0} \cos\left(\frac{x-y}{x+y}\right) dxdy = \frac{\sin 1}{2}.$$

Hint: Change of variables $u = x - y, v = x + y$.

65. *Evaluate*

$$I = \iint\limits_D \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}}\right) dxdy,$$

where D is bounded by the axes and the parabola $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$.

Double Integrals in polar coordinate

66. *Express the double integral* $I = \iint\limits_D f(x,y)\,dxdy$ *in terms of polar coordinates, where* D *is given by* $x^2+y^2 \geq 4x, x^2+y^2 \leq 8x, y \geq x, y \leq \sqrt{3}x$.

67. *Evaluate* $\iint\limits_D xy^2dxdy$ *where* D *is bounded by* $\begin{cases} x^2+(y-1)^2 = 1 \\ x^2+y^2-4y = 0. \end{cases}$

68. *Evaluate*

a) $\iint\limits_D |x+y|dxdy,$ b) $\iint\limits_D |x-y|dxdy,$

where $D : x^2+y^2 \leq 1$.

69. *Evaluate* $\iint\limits_D \frac{dxdy}{(x^2+y^2)^2}$, *where* $D : \begin{cases} 4y \leq x^2+y^2 \leq 8y \\ x \leq y \leq x\sqrt{3}. \end{cases}$

70. *Evaluate* $\iint\limits_D \frac{xy}{x^2+y^2}dxdy$, *where* $D : \begin{cases} x^2+y^2 \leq 12, x^2+y^2 \geq 2x \\ x^2+y^2 \geq 2\sqrt{3}y, x \geq 0, y \geq 0. \end{cases}$

71. *Evaluate* $\iint\limits_D (x+y)dxdy$, *where* D *is the region that lies to the left of the* y*-axis, between the circles* $x^2+y^2 = 1$ *and* $x^2+y^2 = 4$.

72. *Evaluate* $\iint\limits_D \cos(x^2+y^2)dxdy$, *where* D *is the region that lies above the* x*-axis within the circle* $x^2+y^2 = 9$.

Evaluate $\iint\limits_D \sqrt{4-x^2-y^2}dxdy$, *where* $D = \{(x,y)|x^2+y^2 \leq 4, x \geq 0\}$.

73. *Evaluate* $\iint\limits_D ye^x dxdy$, *where* D *is the region in the first quadrant enclosed by the circle* $x^2+y^2 = 25$.

74. *Evaluate* $\iint\limits_D \arctan\frac{y}{x}dxdy$, *where* $D = \{(x,y)|1 \leq x^2+y^2 \leq 4, 0 \leq y \leq x\}$.

75. *Evaluate* $\iint\limits_D xdxdy$, *where* D *is the region in the first quadrant that lies between the circles* $x^2+y^2 = 4$ *and* $x^2+y^2 = 2x$.

3.1.2 Applications of Double Integrals

76. *Compute the area of the domain* D *bounded by* *a)* $\begin{cases} y = 2^x, y = 2^{-x}, \\ y = 4. \end{cases}$

b) $\begin{cases} y^2 = x, y^2 = 2x \\ x^2 = y, x^2 = 2y. \end{cases}$

c) $\begin{cases} y = 0, y^2 = 4ax \\ x + y = 3a, \ (a > 0). \end{cases}$

d) $\begin{cases} x^2 + y^2 = 2x, x^2 + y^2 = 4x \\ x = y, y = 0. \end{cases}$

e) $r = 1, r = \frac{2}{\sqrt{3}} \cos\varphi.$

f) $\left(x^2 + y^2\right)^2 = 2a^2xy \ (a > 0).$

g) $x^3+y^3 = axy \ (a > 0)$ *(Descartes leaf)*

h) $r = a\,(1 + \cos\varphi) \ (a > 0)$ *(Cardioids)*

77. *Compute the volume of the object given by*

a) $\begin{cases} 3x + y \geq 1, y \geq 0 \\ 3x + 2y \leq 2, \\ 0 \leq z \leq 1 - x - y. \end{cases}$

b) $\begin{cases} 0 \leq z \leq 1 - x^2 - y^2, \\ x \leq y \leq x\sqrt{3}. \end{cases}$

78. *Compute the volume of the object bounded by the surfaces*

a) $\begin{cases} z = 4 - x^2 - y^2 \\ 2z = 2 + x^2 + y^2 \end{cases}$ *b)* $\begin{cases} z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}, z = 0 \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{2x}{a} \end{cases}$ *c)* $\begin{cases} az = x^2 + y^2 \\ z = \sqrt{x^2 + y^2}. \end{cases}$

79. *Find the area of the part of the paraboloid $x = y^2 + z^2$ that satisfies $x \leq 1$.*

3.1.3 Triple Integrals

Triple Integrals in Cartesian coordinate

80. *Evaluate*

a) $\iiint\limits_V (x^2 + y^2)\,dxdydz$*, where V is bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the cone $x^2 + y^2 - z^2 = 0$.*

b) $\iiint\limits_E y\,dxdydz$*, where E is bounded by the planes $x = 0, y = 0, z = 0$ and $2x+2y+z = 4$.*

c) $\iiint\limits_E x^2e^y dxdydz$, *where* E *is bounded by the parabolic cylinder* $z = 1 - y^2$ *and the planes* $z = 0, x = 1$ *and* $x = -1$.

d) $\iiint\limits_E xydxdydz$, *where* E *is bounded by the parabolic cylinder* $y = x^2$ *and* $x = y^2$ *and the planes* $z = 0$ *and* $z = x + y$.

e) $\iiint\limits_E xyzdxdydz$, *where* E *is the solid tetrahedron with vertices* $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ *and* $(0, 0, 1)$.

f) $\iiint\limits_E xdxdydz$, *where* E *is the bounded by the paraboloid* $x = 4y^2 + 4z^2$ *and the plane* $x = 4$.

g) $\iiint\limits_E zdxdydz$, *where* E *is the bounded by the cylinder* $y^2 + z^2 = 9$ *and the planes* $x = 0, y = 3x$ *and* $z = 0$ *in the first octant.*

Change of variables

81. *Evaluate*

a) $\iiint\limits_V (x + y + z)dxdydz$, *where* V *is bounded by* $\begin{cases} x + y + z = \pm 3 \\ x + 2y - z = \pm 1 \\ x + 4y + z = \pm 2 \end{cases}$.

b) $\iiint\limits_V (3x^2 + 2y + z)dxdydz$, *where* $V : |x - y| \le 1, |y - z| \le 1, |z + x| \le 1$.

c) $\iiint\limits_V dxdydz$, *where* $V : |x - y| + |x + 3y| + |x + y + z| \le 1$.

Triple Integrals in Cylindrical Coordinates

82. *Evaluate* $\iiint\limits_V (x^2 + y^2)\, dxdydz$, *where* $V : \begin{cases} x^2 + y^2 \le 1 \\ 1 \le z \le 2 \end{cases}$

83. *Evaluate* $\iiint\limits_V z\sqrt{x^2 + y^2}dxdydz$, *where:*

a) V *is bounded by:* $x^2 + y^2 = 2x$ *and* $z = 0, z = a \ (a > 0)$.

b) V *is a half of the sphere* $x^2 + y^2 + z^2 \le a^2, z \ge 0 \ (a > 0)$

84. *Evaluate* $I = \iiint\limits_V \sqrt{x^2 + y^2}dxdydz$ *where* V *is bounded by:* $\begin{cases} x^2 + y^2 = z^2 \\ z = 1. \end{cases}$

85. *Evaluate* $\iiint\limits_V \frac{dxdydz}{\sqrt{x^2+y^2+(z-2)^2}}$, *where* $V : \begin{cases} x^2 + y^2 \le 1 \\ |z| \le 1. \end{cases}$

Triple Integrals in Spherical Coordinates

86. *Evaluate* $\iiint\limits_V (x^2+y^2+z^2)\,dxdydz$, *where* $V : \begin{cases} 1 \le x^2+y^2+z^2 \le 4 \\ x^2+y^2 \le z^2. \end{cases}$

87. *Evaluate* $\iiint\limits_V \sqrt{x^2+y^2+z^2}dxdydz$, *where* $V :\ x^2+y^2+z^2 \le z$.

88. *Evaluate* $\iiint\limits_V z\sqrt{x^2+y^2}dxdydz$, *where* V *is a half of the ellipsoid* $\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2} \le 1, z \ge 0, (a, b > 0)$.

89. *Evaluate* $\iiint\limits_V \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dxdydz$, *where* $V : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1, (a, b, c > 0)$.

90. *Evaluate* $\iiint\limits_V \sqrt{z-x^2-y^2-z^2}dxdydz$, *where* $V :\ x^2+y^2+z^2 \le z$.

91. *Evaluate* $\iiint\limits_V (4z-x^2-y^2-z^2)dxdydz$, *where* V *is the sphere* $x^2+y^2+z^2 \le 4z$.

92. *Evaluate* $\iiint\limits_V xzdxdydz$, *where* V *is the domain* $x^2+y^2+z^2-2x-2y-2z \le -2$.

93. *Evaluate*

$$I = \iiint\limits_V \frac{dxdydz}{(1+x+y+z)^3},$$

where V *is bounded by* $x=0, y=0, z=0$ *and* $x+y+z=1$.

94. *Evaluate*

$$\iiint\limits_V zdxdydz,$$

where V *is a half of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2} \le 1, (z \ge 0).$$

95. *Evaluate*

a) $I_1 = \iiint\limits_B \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$, *where* B *is the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$.

b) $I_2 = \iiint\limits_C zdxdydz$, *where* C *is the domain bounded by the cone* $z^2 = \frac{h^2}{R^2}(x^2+y^2)$ *and the plane* $z = h$.

c) $I_3 = \iiint\limits_D z^2dxdydz$, *where* D *is bounded by the sphere* $x^2+y^2+z^2 \le R^2$ *and the sphere* $x^2+y^2+z^2 \le 2Rz$.

d) $I_4 = \iiint\limits_V (x+y+z)^2dxdydz$, *where* V *is bounded by the paraboloid* $x^2+y^2 \le 2az$ *and the sphere* $x^2+y^2+z^2 \le 3a^2$.

96. *Find the volume of the object bounded by the planes* $Oxy, x = 0, x = a, y = 0, y = b$, *and the paraboloid elliptic*

$$z = \frac{x^2}{2p} + \frac{y^2}{2y}, \ (p > 0, q > 0).$$

97. *Evaluate*

$$I = \iiint\limits_V \sqrt{x^2 + y^2 + z^2} dxdydz,$$

where V *is the domain bounded by* $x^2 + y^2 + z^2 = z$.

98. *Evaluate*

$$I = \iiint\limits_V zdxdydz,$$

where V *is the domain bounded by the surfaces* $z = x^2 + y^2$ *and* $x^2 + y^2 + z^2 = 6$.

99. *Evaluate*

$$I = \iiint\limits_V \frac{xyz}{x^2 + y^2} dxdydz,$$

where V *is the domain bounded by the surface* $(x^2 + y^2 + z^2)^2 = a^2xy$ *and the plane* $z = 0$.

Chapter 4

Line Integrals

4.1 Line Integrals of scalar Fields

100. *Evaluate*

a) $\int\limits_C (x-y)\,ds$, *where* C *is the circle* $x^2+y^2=2x$.

b) $\int\limits_C y^2 ds$, *where* C *is the curve* $\begin{cases} x = a\,(t-\sin t) \\ y = a\,(1-\cos t) \end{cases}, 0 \le t \le 2\pi, a > 0.$

c) $\int\limits_C \sqrt{x^2+y^2}ds$, *where* C *is the curve* $\begin{cases} x = (\cos t + t\sin t) \\ y = (\sin t - t\cos t) \end{cases}, 0 \le t \le 2\pi.$

d) $\int\limits_C (x+y)ds$, *where* C *is the circle* $x^2+y^2=2y$.

e) $\int\limits_L xyds$, *where* L *is the part of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, x \ge 0, y \ge 0$.

f) $I = \int\limits_L |y|ds$, *where* L *is the Cardioid curve* $r = a(1+\cos\varphi)$ $(a > 0)$.

g) $I = \int\limits_L |y|ds$, *where* L *is the Lemniscate curve* $(x^2+y^2)^2 = a^2(x^2-y^2)$.

4.2 Line Integrals of vector Fields

101. *Evaluate* $\int\limits_{ABCA} 2\,(x^2+y^2)\,dx + x\,(4y+3)\,dy$, *where* $ABCA$ *is the quadrangular curve,* $A(0,0), B(1,1), C(0,2)$.

102. *Evaluate* $\int\limits_{ABCDA} \frac{dx+dy}{|x|+|y|}$, *where* $ABCDA$ *is the triangular curve,* $A(1,0), B(0,1), C(-1,0), D(0,-$

Green's Theorem

103. *Evaluate the integral* $\int\limits_C (xy+x+y)\,dx + (xy+x-y)\,dy$, *where* C *is the positively oriented circle* $x^2+y^2=R^2$ *by*

i) computing it directly and

ii) Green's Theorem, then compare the results,

104. *Evaluate the following integrals, where* C *is a half the circle* $x^2+y^2=2x$, *traced from* $O(0,0)$ *to* $A(2,0)$.

a) $\int\limits_C (xy+x+y)\,dx + (xy+x-y)\,dy$

b) $\int\limits_C x^2\left(y+\frac{x}{4}\right)dy - y^2\left(x+\frac{y}{4}\right)dx.$

c) $\int\limits_C (xy+e^x\sin x+x+y)\,dx - (xy-e^{-y}+x-\sin y)\,dy.$

105. *Evaluate* $\oint\limits_{OABO} e^x\left[(1-\cos y)\,dx - (y-\sin y)\,dy\right]$, *where* $OABO$ *is the triangle,* $O(0,0), A(1,1),$

Applications of Line Integrals

106. *Find the area of the domain bounded by an arch of the cycloid* $\begin{cases} x=a(\theta-\sin\theta) \\ y=a(1-\cos\theta) \end{cases}$ *and* Ox $(a>0)$.

Independence of Path

107. *Evaluate* $\int\limits_{(-2,1)}^{(3,0)} (x^4+4xy^3)\,dx + (6x^2y^2-5y^4)\,dy.$

108. *Evaluate* $\int\limits_{(1,\pi)}^{(2,2\pi)} \left(1-\frac{y^2}{x^2}\cos\frac{y}{x}\right)dx + \left(\sin\frac{y}{x}+\frac{y}{x}\cos\frac{y}{x}\right)dy.$

Chapter 5

Surface Integrals

5.1 Surface Integrals of scalar Fields

109. *Evaluate* $\iint\limits_S \left(z + 2x + \frac{4y}{3}\right) dS$*, where* $S = \left\{(x, y, z) \,|\, \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1, x, y, z \geq 0\right\}$.

110. *Evaluate* $\iint\limits_S (x^2 + y^2)\, dS$*, where* $S = \{(x, y, z) \,|\, z = x^2 + y^2, 0 \leq z \leq 1\}$.

111. *Evaluate* $\iint\limits_S x^2y^2z dS$*, where* S *is the part of the cone* $z = \sqrt{x^2 + y^2}$ *lies below the plane* $z = 1$.

112. *Evaluate* $\iint\limits_S \dfrac{dS}{(2 + x + y + z)^2}$*, where* S *is the boundary of the triangular pyramid* $x + y + z \leq 1, x \geq 0, y \geq 0, z \geq 0$.

5.2 Surface Integrals of vector Fields

113. *Evaluate* $\iint\limits_S z\,(x^2 + y^2)\, dxdy$*, where* S *is a half of the sphere* $x^2 + y^2 + z^2 = 1, z \geq 0$*, with the outward normal vector.*

114. *Evaluate* $\iint\limits_S ydxdz + z^2dxdy$*, where* S *is the surface* $x^2 + \frac{y^2}{4} + z^2 = 1, x \geq 0, y \geq 0, z \geq 0$*, and is oriented downward.*

115. *Evaluate* $\iint\limits_S x^2y^2zdxdy$*, where* S *is the surface* $x^2 + y^2 + z^2 = R^2, z \leq 0$ *and is oriented upward.*

The Divergence Theorem

116. *Evaluate the following integrals, where* S *is the surface* $x^2 + y^2 + z^2 = a^2$ *with outward orientation.*

a. $\iint\limits_S xdydz + ydzdx + zdxdy$ *b.* $\iint\limits_S x^3dydz + y^3dzdx + z^3dxdy.$

117. *Evaluate* $\iint\limits_S y^2zdxdy + xzdydz + x^2ydxdz$*, where* S *is the boundary of the domain* $x \geq 0, y \geq 0, x^2 + y^2 \leq 1, 0 \leq z \leq x^2 + y^2$ *which is outward oriented.*

118. *Evaluate* $\iint\limits_S xdydz + ydzdx + zdxdy$*, where* S *the boundary of the domain* $(z-1)^2 \leq x^2 + y^2, a \leq z \leq 1, a > 0$ *which is outward oriented.*

Stokes' Theorem

119. *Use Stokes' Theorem to evaluate* $\int\limits_C F \cdot dr = \int\limits_C Pdx + Qdy + Rdz$*. In each case* C *is oriented counterclockwise as viewed from above.*

1. $F(x, y, z) = (x + y^2)\mathbf{i} + (y + z^2)\mathbf{j} + (z + x^2)\mathbf{k}$, C *is the triangle with vertices* $(1, 0, 0), (0, 1, 0)$ *and* $(0, 0, 1)$.
2. $F(x, y, z) = \mathbf{i} + (x + yz)\mathbf{j} + (xy - \sqrt{z})\mathbf{k}$, C *is the boundary of the part of the plane* $3x + 2y + z = 1$ *in the first octant.*
3. $F(x, y, z) = yz\mathbf{i} + 2xz\mathbf{j} + e^{xy}\mathbf{k}$, C *is the circle* $x^2 + y^2 = 16, z = 5$.
4. $F(x, y, z) = xy\mathbf{i} + 2z\mathbf{j} + 3y\mathbf{k}$, C *is the curve of intersection of the plane* $x + z = 5$ *and the cylinder* $x^2 + y^2 = 9$.