# Chapter 3
# PAIRS OF RANDOM VARIABLES

NGUYỄN THỊ THU THỦY [1]

SCHOOL OF APPLIED MATHEMATICS AND INFORMATICS
HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY

HANOI – 2021

[1] Email: thuy.nguyenthithu2@hust.edu.vn

# Introduction

**This chapter analyzes experiments that produce two random variables, $X$ and $Y$.**

1. $F_{X,Y}(x, y)$, the joint cumulative distribution function of two random variables.
2. $P_{X,Y}(x, y)$, the joint probability mass function for two discrete random variables.
3. $f_{X,Y}(x, y)$, the joint probability density function of two continuous random variables.
4. Functions of two random variables.
5. Conditional probability and independence.

# Introduction

**This chapter analyzes experiments that produce two random variables, $X$ and $Y$.**

1. $F_{X,Y}(x,y)$, the joint cumulative distribution function of two random variables.
2. $P_{X,Y}(x,y)$, the joint probability mass function for two discrete random variables.
3. $f_{X,Y}(x,y)$, the joint probability density function of two continuous random variables.
4. Functions of two random variables.
5. Conditional probability and independence.

# Introduction

**This chapter analyzes experiments that produce two random variables, $X$ and $Y$.**

1. $F_{X,Y}(x, y)$, the joint cumulative distribution function of two random variables.
2. $P_{X,Y}(x, y)$, the joint probability mass function for two discrete random variables.
3. $f_{X,Y}(x, y)$, the joint probability density function of two continuous random variables.
4. Functions of two random variables.
5. Conditional probability and independence.

# Introduction

**This chapter analyzes experiments that produce two random variables, $X$ and $Y$.**

1. $F_{X,Y}(x, y)$, the joint cumulative distribution function of two random variables.
2. $P_{X,Y}(x, y)$, the joint probability mass function for two discrete random variables.
3. $f_{X,Y}(x, y)$, the joint probability density function of two continuous random variables.
4. Functions of two random variables.
5. Conditional probability and independence.

# Introduction

**This chapter analyzes experiments that produce two random variables, $X$ and $Y$.**

1. $F_{X,Y}(x, y)$, the joint cumulative distribution function of two random variables.
2. $P_{X,Y}(x, y)$, the joint probability mass function for two discrete random variables.
3. $f_{X,Y}(x, y)$, the joint probability density function of two continuous random variables.
4. Functions of two random variables.
5. Conditional probability and independence.

# Introduction

**This chapter analyzes experiments that produce two random variables, $X$ and $Y$.**

1. $F_{X,Y}(x, y)$, the joint cumulative distribution function of two random variables.
2. $P_{X,Y}(x, y)$, the joint probability mass function for two discrete random variables.
3. $f_{X,Y}(x, y)$, the joint probability density function of two continuous random variables.
4. Functions of two random variables.
5. Conditional probability and independence.

# 3.1.1 Definitions

**Definition 1 (Joint Cumulative Distribution Function – Joint CDF)**

The joint cumulative distribution function of random variables $X$ and $Y$ is

$$\boxed{F_{X,Y}(x,y) = P[X < x, Y < y]} \quad (3.1)$$

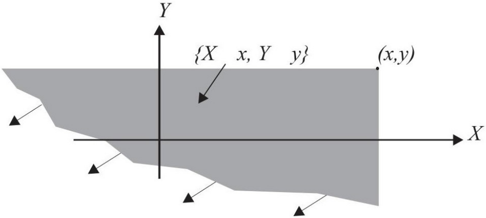


**Figure:** The area of the $(X,Y)$ plane corresponding to the joint CDF $F_{X,Y}(x,y)$

# 3.1.1 Definitions

**Definition 1 (Joint Cumulative Distribution Function – Joint CDF)**

The joint cumulative distribution function of random variables $X$ and $Y$ is

$$\boxed{F_{X,Y}(x,y) = P[X < x, Y < y]} \tag{3.1}$$

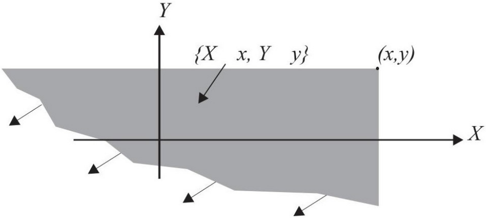


**Figure:** The area of the $(X,Y)$ plane corresponding to the joint CDF $F_{X,Y}(x,y)$

# 3.1.2 Properties

**Note**

The joint CDF has properties that are direct consequences of the definition. For example, we note that the event $\{X < x\}$ suggests that $Y$ can have any value so long as the condition on $X$ is met. This corresponds to the joint event $\{X < x, Y < \infty\}$. Therefore,

$$\boxed{F_X(x) = P[X < x] = P[X < x, Y < \infty] = \lim_{y \to \infty} F_{X,Y}(x, y) = F_{X,Y}(x, \infty)} \tag{3.2}$$

# 3.1.2 Properties

**Theorem 1**

For any pair of random variables, $X, Y$,

(a) $0 \le F_{X,Y}(x, y) \le 1$,

(b) $F_X(x) = F_{X,Y}(x, \infty)$,

(c) $F_Y(y) = F_{X,Y}(\infty, y)$,

(d) $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$,

(e) If $x < x_1$ and $y < y_1$, then $F_{X,Y}(x, y) \le F_{X,Y}(x_1, y_1)$,

(f) $F_{X,Y}(\infty, \infty) = 1$,

(g) If $x_1 < x_2$, $y_1 < y_2$, then

$$P[x_1 \le X < x_2, y_1 \le Y < y_2] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1).$$

# 3.2.1 Definitions

**Definition 2 (Joint Probability Mass Function – Joint PMF)**

The joint probability mass function of discrete random variables $X$ and $Y$ is

$$\boxed{P_{X,Y}(x,y) = P[X = x, Y = y]} \tag{3.3}$$

**Note**

Corresponding to $S_X$, the range of a single discrete random variable, we use the notation $S_{X,Y}$ to denote the set of possible values of the pair $(X,Y)$. That is,

$$S_{X,Y} = \{(x,y) \mid P_{X,Y}(x,y) > 0\}. \tag{3.4}$$

# 3.2.1 Definitions

**Definition 2 (Joint Probability Mass Function – Joint PMF)**

The joint probability mass function of discrete random variables $X$ and $Y$ is

$$\boxed{P_{X,Y}(x,y) = P[X = x, Y = y]} \tag{3.3}$$

**Note**

Corresponding to $S_X$, the range of a single discrete random variable, we use the notation $S_{X,Y}$ to denote the set of possible values of the pair $(X, Y)$. That is,

$$S_{X,Y} = \{(x,y) \mid P_{X,Y}(x,y) > 0\}. \tag{3.4}$$

# 3.2.1 Definitions

**Definition 3 (Joint Probability Distribution)**

| $X \backslash Y$ | $y_1$ | $\dots$ | $y_j$ | $\dots$ | $y_n$ | $\sum_j$ |
|---|---|---|---|---|---|---|
| $x_1$ | $p_{11}$ | $\dots$ | $p_{1j}$ | $\dots$ | $p_{1n}$ | $P[X=x_1]$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_i$ | $p_{i1}$ | $\dots$ | $p_{ij}$ | $\dots$ | $p_{in}$ | $P[X=x_i]$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_m$ | $p_{m1}$ | $\dots$ | $p_{mj}$ | $\dots$ | $p_{mn}$ | $P[X=x_m]$ |
| $\sum_i$ | $P[Y=y_1]$ | $\dots$ | $P[Y=y_j]$ | $\dots$ | $P[Y=y_n]$ | $\sum_i \sum_j = 1$ |

# 3.2.1 Definitions

**Theorem 2**

(a) $0 \leq p_{ij} \leq 1$ for all $i = 1, 2, \dots, m,\ j = 1, 2, \dots, n.$

(b) $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1.$

# 3.2.1 Definitions

**Remark**

From (3.1) and Definition 3, the joint cumulative distribution function of discrete random variables $X$ and $Y$ is

$$\boxed{F_{XY}(x, y) = \sum_{x_i < x} \sum_{y_j < y} p_{ij}} \tag{3.5}$$

# 3.2.1 Definitions

**Example 1**

Test two integrated circuits one after the other. On each test, the possible outcomes are $a$ (accept) and $r$ (reject). Assume that all circuits are acceptable with probability 0.9 and that the outcomes of successive tests are independent. Count the number of acceptable circuits $X$ and count the number of successful tests $Y$ before you observe the first reject. (If both tests are successful, let $Y = 2$.) Find the joint PMF of $X$ and $Y$.

# 3.2.1 Definitions

**Example 1 Solution**

The sample space of the experiment is

$$S = \{aa, ar, ra, rr\}.$$

We compute

$$P[aa] = 0.81, \quad P[ar] = P[ra] = 0.09, \quad P[rr] = 0.01.$$

Each outcome specifies a pair of values $X$ and $Y$. Let $g(s)$ be the function that transforms each outcome $s$ in the sample space $S$ into the pair of random variables $(X, Y)$. Then

$$g(aa) = (2, 2), \quad g(ar) = (1, 1), \quad g(ra) = (1, 0), \quad g(rr) = (0, 0).$$

# 3.2.1 Definitions

**Example 1 Solution (continuous)**

For each pair of values $x, y$, $P_{X,Y}(x,y)$ is the sum of the probabilities of the outcomes for which $X = x$ and $Y = y$. For example, $P_{X,Y}(1,1) = P[ar]$. The joint PMF can be given as a set of labeled points in the $x, y$ plane where each point is a possible value (probability $> 0$) of the pair $(x, y)$, or as a simple list:

$$P_{X,Y}(x,y) = \begin{cases} 0.81, & x = 2, y = 2, \\ 0.09, & x = 1, y = 1, \\ 0.09, & x = 1, y = 0, \\ 0.01, & x = 0, y = 0, \\ 0, & \text{otherwise.} \end{cases}$$

# 3.2.1 Definitions

**Example 1 Solution (continuous)**

A second representation of $P_{X,Y}(x,y)$ is the matrix:

| $P_{X,Y}(x,y)$ | $y=0$ | $y=1$ | $y=2$ |
|---|---|---|---|
| $x=0$ | 0.01 | 0 | 0 |
| $x=1$ | 0.09 | 0.09 | 0 |
| $x=2$ | 0 | 0 | 0.81 |

**Note**

Note that all of the probabilities add up to 1. Using the notation of random variables, we write this as

$$\boxed{\sum_{x\in S_X}\sum_{y\in S_Y} P_{X,Y}(x,y)=1}$$

# 3.2.1 Definitions

**Example 1 Solution (continuous)**

A second representation of $P_{X,Y}(x,y)$ is the matrix:

| $P_{X,Y}(x,y)$ | $y=0$ | $y=1$ | $y=2$ |
|---|---|---|---|
| $x=0$ | 0.01 | 0 | 0 |
| $x=1$ | 0.09 | 0.09 | 0 |
| $x=2$ | 0 | 0 | 0.81 |

**Note**

Note that all of the probabilities add up to 1. Using the notation of random variables, we write this as

$$\boxed{\sum_{x\in S_X}\sum_{y\in S_Y} P_{X,Y}(x,y) = 1}$$

# 3.2.2 Properties

**Note**

We represent an event $B$ as a region in the $(X, Y)$ plane. Figure 2 shows two examples of events.

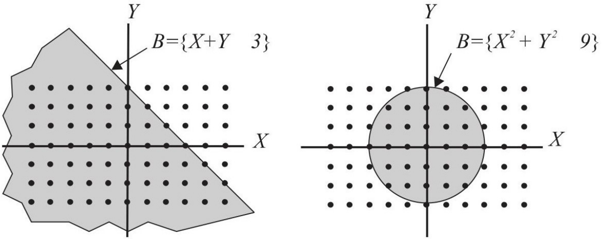


**Figure:** Subset $B$ of the $(X, Y)$ plane. Point $(X, Y) \in S_{X,Y}$ are marked by bullets

# 3.2.2 Properties

**Note**

We represent an event $B$ as a region in the $(X, Y)$ plane. Figure 2 shows two examples of events.

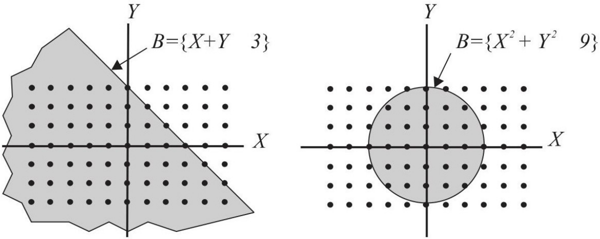


**Figure:** Subset $B$ of the $(X, Y)$ plane. Point $(X, Y) \in S_{X,Y}$ are marked by bullets

# 3.2.2 Properties

**Theorem 3**

For discrete random variables $X$ and $Y$ and any set $B$ in the $(X, Y)$ plane, the probability of the event $\{(X, Y) \in B\}$ is

$$\boxed{P[B] = \sum_{(x,y) \in B} P_{X,Y}(x, y)} \tag{3.6}$$

# 3.2.2 Properties

**Example 2**

Continuing Example 1, find the probability of the event $B$ that $X$, the number of acceptable circuits, equals $Y$, the number of tests before observing the first failure.

# 3.2.2 Properties

**Example 2 Solution**

Mathematically, $B$ is the event $\{X = Y\}$. The elements of $B$ with nonzero probability are

$$B \cap S_{X,Y} = \{(0,0),(1,1),(2,2)\}.$$

Therefore,

$$\begin{aligned} P[B] &= P_{X,Y}(0,0) + P_{X,Y}(1,1) + P_{X,Y}(2,2) \\ &= 0.01 + 0.09 + 0.81 = 0.91. \end{aligned}$$

# 3.2.3 Marginal PMF

**Theorem 4**

For discrete random variables $X$ and $Y$ with joint PMF $P_{X,Y}(x,y)$,

$$\boxed{P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x,y), \quad P_Y(y) = \sum_{x \in S_X} P_{X,Y}(x,y)} \tag{3.7}$$

# 3.2.3 Marginal PMF

**Remark. From Definition 3,**

**(a)** Marginal CD of $X$:

| $X$ | $x_1$ | $x_2$ | $\dots$ | $x_m$ |
|---|---|---|---|---|
| $p$ | $P[X = x_1]$ | $P[X = x_2]$ | $\dots$ | $P[X = x_m]$ |

**(b)** Marginal CD of $Y$:

| $Y$ | $y_1$ | $y_2$ | $\dots$ | $y_n$ |
|---|---|---|---|---|
| $p$ | $P[Y = y_1]$ | $P[Y = y_2]$ | $\dots$ | $P[Y = y_n]$ |

# 3.2.3 Marginal PMF

**Example 3**

In Example 1, we found the joint PMF of $X$ and $Y$ to be

| $P_{X,Y}(x,y)$ | $y=0$ | $y=1$ | $y=2$ |
|---|---|---|---|
| $x=0$ | 0.01 | 0 | 0 |
| $x=1$ | 0.09 | 0.09 | 0 |
| $x=2$ | 0 | 0 | 0.81 |

Find the marginal PMFs for the random variables $X$ and $Y$.

# 3.2.3 Marginal PMF

**Example 3 Solution**

To find $P_X(x)$, we note that both $X$ and $Y$ have range $\{0, 1, 2\}$. Theorem 3 gives

$$P_X(0) = \sum_{y=0}^{2} P_{X,Y}(0, y) = 0.01, \quad P_X(1) = \sum_{y=0}^{2} P_{X,Y}(1, y) = 0.18$$

$$P_X(2) = \sum_{y=0}^{2} P_{X,Y}(2, y) = 0.81, \quad P_X(x) = 0, \quad x \neq 0, 1, 2.$$

# 3.2.3 Marginal PMF

**Example 3 Solution (continuous)**

For the PMF of $Y$, we obtain

$$P_Y(0) = \sum_{x=0}^{2} P_{X,Y}(x,0) = 0.10, \quad P_Y(1) = \sum_{x=0}^{2} P_{X,Y}(x,1) = 0.09$$

$$P_Y(2) = \sum_{x=0}^{2} P_{X,Y}(x,2) = 0.81, \quad P_Y(y) = 0, \quad y \neq 0, 1, 2.$$

# 3.2.3 Marginal PMF

## Example 3 Solution (continuous)

We display $P_X(x)$ and $P_Y(y)$ by rewriting the matrix in Example 1 and placing the row sums and column sums in the margins.

| $P_{X,Y}(x,y)$ | $y=0$ | $y=1$ | $y=2$ | $P_X(x)$ |
|---|---|---|---|---|
| $x=0$ | 0.01 | 0 | 0 | 0.01 |
| $x=1$ | 0.09 | 0.09 | 0 | 0.18 |
| $x=2$ | 0 | 0 | 0.81 | 0.81 |
| $P_Y(y)$ | 0.10 | 0.09 | 0.81 | |

$$P_X(x)=\begin{cases}0.01, & x=0,\\ 0.18, & x=1,\\ 0.81, & x=2,\\ 0, & \text{otherwise.}\end{cases} \qquad P_Y(y)=\begin{cases}0.1, & y=0,\\ 0.09, & y=1,\\ 0.81, & y=2,\\ 0, & \text{otherwise.}\end{cases}$$

# 4.3.1 Definitions

**Definition 4 (Joint Probability Density Function – Joint PDF)**

The joint PDF of the continuous random variables $X$ and $Y$ is a function $f_{X,Y}(x,y)$ with the property

$$\boxed{F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)dvdu} \tag{3.8}$$

**Theorem 5**

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}.$$

# 4.3.1 Definitions

**Definition 4 (Joint Probability Density Function – Joint PDF)**

The joint PDF of the continuous random variables $X$ and $Y$ is a function $f_{X,Y}(x,y)$ with the property

$$\boxed{F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)dvdu} \tag{3.8}$$

**Theorem 5**

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}.$$

# 3.3.2 Properties

**Theorem 6**

A joint PDF $f_{X,Y}(x,y)$ has the following properties corresponding to first and second axioms of probability:

(a) $f_{X,Y}(x,y) \geq 0$ for all $(x,y)$,

(b) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy = 1.$

**Theorem 7**

The probability that the continuous random variables $(X,Y)$ are in $A$ is

$$P[A] = \iint_A f_{X,Y}(x,y)dxdy.$$

# 3.3.2 Properties

**Theorem 6**

A joint PDF $f_{X,Y}(x,y)$ has the following properties corresponding to first and second axioms of probability:

(a) $f_{X,Y}(x,y) \geq 0$ for all $(x,y)$,

(b) $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy = 1.$

**Theorem 7**

The probability that the continuous random variables $(X,Y)$ are in $A$ is

$$P[A] = \iint_A f_{X,Y}(x,y)dxdy.$$

# 3.3.2 Properties

**Example 4**

Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c, & 0 \le x \le 5,\ 0 \le y \le 3 \\ 0, & \text{otherwise.} \end{cases}$$

Find the constant $c$ and $P[A] = P[2 \le X < 3,\ 1 \le Y < 3]$.

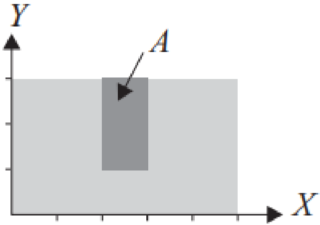


**Figure:** Example 4

# 3.3.2 Properties

**Example 4 Solution**

- The large rectangle in the diagram is the area of nonzero probability. Theorem 6 states that the integral of the joint PDF over this rectangle is 1:

$$1 = \int_0^5 \int_0^3 c dy dx = 15c.$$

Therefore, $c = 1/15$.

- The small dark rectangle in the diagram is the event $A = \{2 \leq X < 3,\ 1 \leq Y < 3\}$. $P[A]$ is the integral of the PDF over this rectangle, which is

$$P[A] = \int_2^3 \int_1^3 \frac{1}{15} dx dy = \frac{2}{15}.$$

# 4.3.3 Marginal PDF

**Theorem 8**

If $X$ and $Y$ are random variables with joint PDF $f_{X,Y}(x,y)$,

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx.$$

**Example 5**

The joint PDF of $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} \frac{5y}{4}, & -1 \le x \le 1, \; x^2 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

# 4.3.3 Marginal PDF

**Theorem 8**

If $X$ and $Y$ are random variables with joint PDF $f_{X,Y}(x,y)$,

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx.$$

**Example 5**

The joint PDF of $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{5y}{4}, & -1 \le x \le 1,\ x^2 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

# 3.3.2 Properties

**Example 5 Solution**

Set $D = \{(x, y) \in \mathbb{R}^2 : -1 \le x \le 1,\ x^2 \le y \le 1\}$.

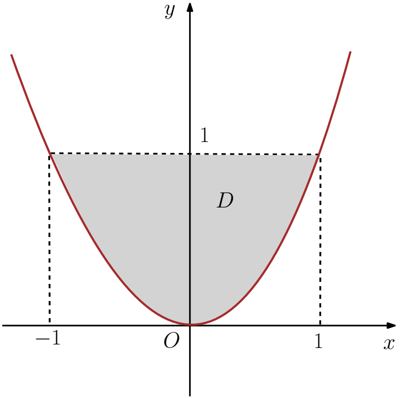


**Figure:** Example 5

# 3.3.3 Marginal PDF

**Example 5 Solution (continuous)**

We use Theorem 8 to find the marginal PDF $f_X(x)$.

- When $x < -1$ or when $x > 1$, $f_{X,Y}(x,y) = 0$, and therefore $f_X(x) = 0$.
- For $-1 \le x \le 1$,

$$f_X(x) = \int_{x^2}^{1} \frac{5y}{4} dy = \frac{5(1-x^4)}{8}.$$

The complete expression for the marginal PDF of $X$ is

$$f_X(x) = \begin{cases} \frac{5(1-x^4)}{8}, & -1 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

# 3.3.3 Marginal PDF

**Example 5 Solution (continuous)**

For the marginal PDF of $Y$,

- We note that for $y < 0$ or $y > 1$, $f_Y(y) = 0$.
- For $0 \leq y \leq 1$, we integrate over the horizontal bar marked $Y = y$. The boundaries of the bar are $x = -\sqrt{y}$ and $x = \sqrt{y}$. Therefore, for $0 \leq y \leq 1$,

$$f_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{5y}{4} dx = \frac{5y}{4} x \Big|_{x=-\sqrt{y}}^{x=\sqrt{y}} = \frac{5y^{3/2}}{2}.$$

The complete marginal PDF of $Y$ is

$$f_Y(y) = \begin{cases} \dfrac{5y^{3/2}}{2}, & 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

# 3.3.3 Marginal PDF

**Example 6**

The joint PDF of $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} kx, & \text{if } 0 < y < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

**(a)** Find constant $k$.

**(b)** Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

# 3.3.2 Properties

**Example 6 Solution**

Set $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1; 0 < y < x\}$.

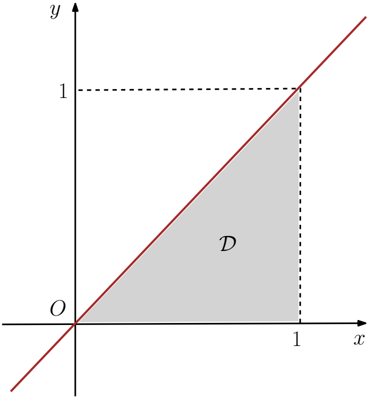


**Figure:** Example 6

# 3.3.3 Marginal PDF

**Example 6 Solution**

(a) From Theorem 7, $k \geq 0$ and

$$1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)dxdy = \int_0^1 dx \int_0^x kxdy = k\int_0^1 x^2dx = \frac{k}{3}.$$

Hence $k = 3$ and $f_{X,Y}(x,y) = \begin{cases} 3x, & \text{if } (x,y) \in \mathcal{D} \\ 0, & \text{otherwise.} \end{cases}$

# 3.3.3 Marginal PDF

**Example 6 Solution (continuous)**

(b) From Theorem 7,

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = \begin{cases} \int_0^x 3x dy, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx = \begin{cases} \int_y^1 3x dx, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{3}{2} - \frac{3}{2}y^2, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

# 3.4.1 Expected Values

**Theorem 9**

For random variables $X$ and $Y$, the expected value of $W = g(X,Y)$ is

$$\text{Discrete: } E[W] = \sum_{x\in S_X}\sum_{y\in S_Y} g(x,y)P_{X,Y}(x,y),$$

$$\text{or } E[W] = \sum_{i}\sum_{j} g(x_i,y_j)P[X=x_i, Y=y_j],$$

$$\text{Continuous: } E[W] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f_{X,Y}(x,y)dxdy.$$

# 3.4.1 Expected Values

**Theorem 10**

$$E[g_1(X,Y) + \ldots + g_n(X,Y)] = E[g_1(X,Y)] + \ldots + E[g_n(X,Y)].$$

**Theorem 11**

For any two random variables $X$ and $Y$,

$$E[X+Y] = E[X] + E[Y].$$

**Theorem 12**

The variance of the sum of two random variables is

$$Var[X+Y] = Var[X] + Var[Y] + 2E[(X-\mu_X)(Y-\mu_Y)].$$

# 3.4.1 Expected Values

**Theorem 10**

$$E[g_1(X,Y) + \ldots + g_n(X,Y)] = E[g_1(X,Y)] + \ldots + E[g_n(X,Y)].$$

**Theorem 11**

For any two random variables $X$ and $Y$,

$$E[X+Y] = E[X] + E[Y].$$

**Theorem 12**

The variance of the sum of two random variables is

$$Var[X+Y] = Var[X] + Var[Y] + 2E[(X-\mu_X)(Y-\mu_Y)].$$

# 3.4.1 Expected Values

**Theorem 10**

$$E[g_1(X,Y) + \ldots + g_n(X,Y)] = E[g_1(X,Y)] + \ldots + E[g_n(X,Y)].$$

**Theorem 11**

For any two random variables $X$ and $Y$,

$$E[X+Y] = E[X] + E[Y].$$

**Theorem 12**

The variance of the sum of two random variables is

$$Var[X+Y] = Var[X] + Var[Y] + 2E[(X-\mu_X)(Y-\mu_Y)].$$

# 3.4.2 Covariance. Covariance Matrix. Correlation Coefficient

**Definition 5 (Covariance)**

The covariance of two random variables $X$ and $Y$ is

$$Cov[X, Y] = E[(X - \mu_X)(Y - \mu_Y)].$$

**Remark**

From the properites of the expected value,

$$\boxed{Cov(X, Y) = E[XY] - E[X].E[Y]} \tag{3.9}$$

where $E[XY]$ is

$$E[XY] = \begin{cases} \sum_i \sum_j x_i y_j p_{ij}, & \text{(Discrete Random Variables)} \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf(x, y)dxdy, & \text{(Continuous Random Variables).} \end{cases}$$

# 3.4.2 Covariance. Covariance Matrix. Correlation Coefficient

**Definition 5 (Covariance)**

The covariance of two random variables $X$ and $Y$ is

$$Cov[X, Y] = E[(X - \mu_X)(Y - \mu_Y)].$$

**Remark**

From the properites of the expected value,

$$\boxed{Cov(X, Y) = E[XY] - E[X].E[Y]} \tag{3.9}$$

where $E[XY]$ is

$$E[XY] = \begin{cases} \sum\limits_i \sum\limits_j x_i y_j p_{ij}, & \text{(Discrete Random Variables)} \\ \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} xyf(x, y)dxdy, & \text{(Continuous Random Variables).} \end{cases}$$

# 3.4.2 Covariance. Covariance Matrix. Correlation Coefficient

**Theorem 13**

**(a)** $Cov(X,Y) = E[XY] - E[X].E[Y] = r_{X,Y} - \mu_X \mu_Y.$

**(b)** $Var[X+Y] = Var[X] + Var[Y] + 2Cov[X,Y].$

**(c)** $Var[X] = Cov[X,X],\ Var[Y] = Cov[Y,Y].$

# 3.4.2 Covariance. Covariance Matrix. Correlation Coefficient

**Definition 6 (Covariance Matrix)**

The covariance matrix of two random variables $X$ and $Y$ is

$$\Gamma = \begin{bmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(Y,X) & Cov(Y,Y) \end{bmatrix} = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix}$$

# 3.4.2 Covariance. Covariance Matrix. Correlation Coefficient

**Definition 7 (Correlation)**

The correlation of $X$ and $Y$ is $r_{X,Y} = E[XY]$, where

$$E[XY] = \begin{cases} \sum\limits_{x \in S_X} \sum\limits_{y \in S_Y} xyP_{X,Y}(x,y), & \text{(Discrete Random Variables)} \\ \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} xyf_{X,Y}(x,y)dxdy, & \text{(Continuous Random Variables).} \end{cases}$$

# 3.4.2 Covariance. Covariance Matrix. Correlation Coefficient

**Example 7**

For the integrated circuits tests in Example 1, we found in Example 3 that the probability model for $X$ and $Y$ is given by the following matrix.

| $P_{X,Y}(x,y)$ | $y=0$ | $y=1$ | $y=2$ | $P_X(x)$ |
|---|---|---|---|---|
| $x=0$ | 0.01 | 0 | 0 | 0.01 |
| $x=1$ | 0.09 | 0.09 | 0 | 0.18 |
| $x=2$ | 0 | 0 | 0.81 | 0.81 |
| $P_Y(y)$ | 0.10 | 0.09 | 0.81 | |

Find $r_{X,Y}$ and $Cov[X,Y]$.

# 3.4.2 Covariance. Covariance Matrix. Correlation Coefficient

**Example 7 Solution**

By Definition 5,

$$r_{X,Y} = E[XY] = \sum_{x=0}^{2}\sum_{y=0}^{2} xyP_{X,Y}(x,y) = 1 \times 1 \times 0.09 + 2 \times 2 \times 0.81 = 3.33.$$

To use Theorem 13(a) to find the covariance, we find

$$E[X] = (0)(0.01) + (1)(0.18) + (2)(0.81) = 1.80$$
$$E[Y] = (0)(0.10) + (1)(0.09) + (2)(0.81) = 1.71.$$

Therefore, by Theorem 13(a), $Cov[X,Y] = 3.33 - 1.80 \times 1.71 = 0.252$.

# 3.4.2 Covariance. Covariance Matrix. Correlation Coefficient

**Example 8**

The joint CD of $X$ and $Y$ is

| $X \backslash Y$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $-1$ | $4/15$ | $1/15$ | $4/15$ |
| $0$ | $1/15$ | $2/15$ | $1/15$ |
| $1$ | $0$ | $2/15$ | $0$ |

**(a)** Find $E(X)$, $E(Y)$, $Cov(X,Y)$.

**(b)** Find the CDs of $X$ and $Y$.

# 3.4.2 Covariance. Covariance Matrix. Correlation Coefficient

**Example 8 Solution**

(a) We have

$$E[X] = (-1) \times \frac{9}{15} + 0 \times \frac{4}{15} + 1 \times \frac{2}{15} = -\frac{7}{15}.$$

$$E[Y] = (-1) \times \frac{5}{15} + 0 \times \frac{5}{15} + 1 \times \frac{5}{15} = 0.$$

$$E[XY] = (-1) \times (-1) \times \frac{4}{15} + (-1) \times (1) \times \frac{4}{15} + 1 \times (-1) \times 0 + 1 \times 1 \times 0 = 0.$$

Hence $Cov[X, Y] = E[XY] - E[X] \times E[Y] = 0.$

# 3.4.2 Covariance. Covariance Matrix. Correlation Coefficient

**Example 8 Solution (continuous)**

(b) The CDs of $X$ and $Y$ are

| $X$ | -1 | 0 | 1 |
|---|---|---|---|
| $P$ | $^9/_{15}$ | $^4/_{15}$ | $^5/_{15}$ |

| $Y$ | -1 | 0 | 1 |
|---|---|---|---|
| $P$ | $^5/_{15}$ | $^5/_{15}$ | $^5/_{15}$ |

# 3.4.2 Covariance. Covariance Matrix. Correlation Coefficient

**Definition 8 (Orthogonal Random Variables)**

Random variables $X$ and $Y$ are orthogonal if $r_{X,Y} = 0$.

**Definition 9 (Uncorrelated Random Variables)**

Random variables $X$ and $Y$ are uncorrelated if $Cov[X, Y] = 0$.

# 3.4.2 Covariance. Covariance Matrix. Correlation Coefficient

**Definition 8 (Orthogonal Random Variables)**

Random variables $X$ and $Y$ are orthogonal if $r_{X,Y} = 0$.

**Definition 9 (Uncorrelated Random Variables)**

Random variables $X$ and $Y$ are uncorrelated if $Cov[X, Y] = 0$.

# 3.4.3 Correlation Coefficient

**Definition 10 (Correlation Coefficient)**

The correlation coefficient of two random variables $X$ and $Y$ is

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y}.$$

**Theorem 14**

$$-1 \le \rho_{X,Y} \le 1.$$

**Theorem 15**

If $X$ and $Y$ are random variables such that $Y = aX + b$

$$\rho_{X,Y} = \begin{cases} -1, & a < 0, \\ 0, & a = 0, \end{cases}$$

# 3.4.3 Correlation Coefficient

**Definition 10 (Correlation Coefficient)**

The correlation coefficient of two random variables $X$ and $Y$ is

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y}.$$

**Theorem 14**

$$-1 \leq \rho_{X,Y} \leq 1.$$

**Theorem 15**

If $X$ and $Y$ are random variables such that $Y = aX + b$

$$\rho_{X,Y} = \begin{cases} -1, & a < 0, \\ 0, & a = 0, \end{cases}$$

# 3.4.3 Correlation Coefficient

**Definition 10 (Correlation Coefficient)**

The correlation coefficient of two random variables $X$ and $Y$ is

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y}.$$

**Theorem 14**

$$-1 \leq \rho_{X,Y} \leq 1.$$

**Theorem 15**

If $X$ and $Y$ are random variables such that $Y = aX + b$

$$\rho_{X,Y} = \begin{cases} -1, & a < 0, \\ 0, & a = 0, \end{cases}$$

# 3.5.1 Conditional Joint PMF

**Definition 11 (Conditional Joint PMF)**

For discrete random variables $X$ and $Y$ and an event, $B$ with $P[B] > 0$, the conditional joint PMF of $X$ and $Y$ given $B$ is

$$P_{X,Y|B}(x, y) = P[X = x, Y = y|B].$$

**Theorem 16**

For any event $B$, a region of the $X, Y$ plane with $P[B] > 0$,

$$P_{X,Y|B}(x, y) = \begin{cases} \dfrac{P_{X,Y}(x, y)}{P[B]}, & (x, y) \in B, \\ 0, & \text{otherwise.} \end{cases}$$

# 3.5.1 Conditional Joint PMF

**Definition 11 (Conditional Joint PMF)**

For discrete random variables $X$ and $Y$ and an event, $B$ with $P[B] > 0$, the conditional joint PMF of $X$ and $Y$ given $B$ is

$$P_{X,Y|B}(x, y) = P[X = x, Y = y|B].$$

**Theorem 16**

For any event $B$, a region of the $X, Y$ plane with $P[B] > 0$,

$$P_{X,Y|B}(x, y) = \begin{cases} \dfrac{P_{X,Y}(x, y)}{P[B]}, & (x, y) \in B, \\ 0, & \text{otherwise.} \end{cases}$$

# 3.5.2 Conditional Joint PDF

**Definition 12 (Conditional Joint PDF)**

*Given an event $B$ with $P[B] > 0$, the conditional joint probability density function of $X$ and $Y$ is*

$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P[B]}, & (x,y) \in B, \\ 0, & \text{otherwise.} \end{cases}$$

# 3.5.2 Conditional Joint PDF

**Example 9**

$X$ and $Y$ are random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{15}, & 0 \le x \le 5,\ 0 \le y \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional PDF of $X$ and $Y$ given the event $B = \{X + Y \ge 4\}$.

# 3.5.2 Conditional Joint PDF

**Example 9 Solution**

We calculate $P[B]$ by integrating $f_{X,Y}(x,y)$ over the region $B$.

$$P[B] = \int_0^3 \int_{4-y}^5 \frac{1}{15} dx dy = \frac{1}{15} \int_0^3 (1+y) dy = \frac{1}{2}.$$

Definition 11 leads to the conditional joint PDF

$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{2}{15}, & 0 \le x \le 5,\ 0 \le y \le 3,\ x+y \ge 4 \\ 0, & \text{otherwise.} \end{cases}$$

# 3.5.2 Conditional Joint PDF

**Example 9 Solution**

We calculate $P[B]$ by integrating $f_{X,Y}(x,y)$ over the region $B$.

$$P[B] = \int_0^3 \int_{4-y}^5 \frac{1}{15} dxdy = \frac{1}{15} \int_0^3 (1+y)dy = \frac{1}{2}.$$

Definition 11 leads to the conditional joint PDF

$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{2}{15}, & 0 \le x \le 5,\ 0 \le y \le 3,\ x+y \ge 4 \\ 0, & \text{otherwise.} \end{cases}$$

# 3.5.3 Conditional Expected Value

**Theorem 17 (Conditional Expected Value)**

For random variables $X$ and $Y$ and an event $B$ of nonzero probability, the conditional expected value of $W = g(X, Y)$ given $B$ is

$$\text{Discrete: } E[W|B] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X,Y|B}(x, y),$$

$$\text{Continuous: } E[W|B] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) f_{X,Y|B}(x, y) dx dy.$$

# 3.5.3 Conditional Expected Value

**Example 10**

Continuing Example 9, find the conditional expected value of $W = XY$ given the event $B = \{X + Y \geq 4\}$.

**Solution**

From Theorem 17,

$$E[XY|B] = \int_0^3 \int_{4-y}^5 \frac{2}{15} xydxdy = \frac{1}{15} \int_0^3 x^2 \Big|_{4-y}^5 ydy = \frac{1}{15} \int_0^3 (9y + 8y^2 - y^3)dy$$
$$= \frac{123}{20}.$$

# 3.5.3 Conditional Expected Value

**Example 10**

Continuing Example 9, find the conditional expected value of $W = XY$ given the event $B = \{X + Y \geq 4\}$.

**Solution**

From Theorem 17,

$$E[XY|B] = \int_0^3 \int_{4-y}^5 \frac{2}{15} xydxdy = \frac{1}{15} \int_0^3 x^2\Big|_{4-y}^5 ydy = \frac{1}{15} \int_0^3 (9y + 8y^2 - y^3)dy$$
$$= \frac{123}{20}.$$

# 3.5.4 Conditional Variance

**Definition 13 (Conditional Variance)**

The conditional variance of the random variable $W = g(X, Y)$ is

$$Var[W|B] = E([(W - E[W|B])^2|B].$$

**Theorem 18**

$$Var[W|B] = E[W^2|B] - (E[W|B])^2.$$

# 3.5.4 Conditional Variance

**Definition 13 (Conditional Variance)**

The conditional variance of the random variable $W = g(X, Y)$ is

$$Var[W|B] = E([(W - E[W|B])^2|B].$$

**Theorem 18**

$$Var[W|B] = E[W^2|B] - (E[W|B])^2.$$

# 3.6.1 Conditional PMF

**Definition 14 (Conditional PMF)**

For any event $Y = y$ such that $P_Y(y) > 0$, the conditional PMF of $X$ given $Y = y$ is

$$P_{X|Y}(x|y) = P[X = x|Y = y].$$

**Theorem 19**

For random variables $X$ and $Y$ with joint PMF $P_{X,Y}(x, y)$, and $x$ and $y$ such that $P_X(x) > 0$ and $P_Y(y) > 0$,

$$P_{X,Y}(x, y) = P_{X|Y}(x|y)P_Y(y) = P_{Y|X}(y|x)P_X(x).$$

# 3.6.1 Conditional PMF

**Definition 14 (Conditional PMF)**

For any event $Y = y$ such that $P_Y(y) > 0$, the conditional PMF of $X$ given $Y = y$ is

$$P_{X|Y}(x|y) = P[X = x|Y = y].$$

**Theorem 19**

For random variables $X$ and $Y$ with joint PMF $P_{X,Y}(x, y)$, and $x$ and $y$ such that $P_X(x) > 0$ and $P_Y(y) > 0$,

$$P_{X,Y}(x, y) = P_{X|Y}(x|y)P_Y(y) = P_{Y|X}(y|x)P_X(x).$$

# 3.6.2 Conditional Expected Value of a Function

**Theorem 20 (Conditional Expected Value of a Function)**

$X$ and $Y$ are discrete random variables. For any $y \in S_Y$, the conditional expected value of $g(X, Y)$ given $Y = y$ is

$$E[g(X,Y)|Y=y] = \sum_{x \in S_X} g(x,y) P_{X|Y}(x|y).$$

**Note**

The conditional expected value of $X$ given $Y = y$ is a special case of Theorem 20:

$$\boxed{E[X|Y=y] = \sum_{x \in S_X} x P_{X|Y}(x|y)} \tag{3.10}$$

# 3.6.2 Conditional Expected Value of a Function

**Theorem 20 (Conditional Expected Value of a Function)**

$X$ and $Y$ are discrete random variables. For any $y \in S_Y$, the conditional expected value of $g(X, Y)$ given $Y = y$ is

$$E[g(X,Y)|Y=y] = \sum_{x \in S_X} g(x,y) P_{X|Y}(x|y).$$

**Note**

The conditional expected value of $X$ given $Y = y$ is a special case of Theorem 20:

$$\boxed{E[X|Y=y] = \sum_{x \in S_X} x P_{X|Y}(x|y)} \tag{3.10}$$

# 3.6.3 Conditional PDF

**Definition 15 (Conditional PDF)**

For $y$ such that $f_Y(y) > 0$, the conditional PDF of $X$ given $\{Y = y\}$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

**Note**

Definition 15 implies

$$\boxed{f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}} \qquad (3.11)$$

# 3.6.3 Conditional PDF

**Definition 15 (Conditional PDF)**

For $y$ such that $f_Y(y) > 0$, the conditional PDF of $X$ given $\{Y = y\}$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

**Note**

Definition 15 implies

$$\boxed{f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}} \qquad (3.11)$$

# 3.6.3 Conditional PDF

**Example 11**

Returning to Example 4, random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 < y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $0 < x < 1$, find the conditional PDF $f_{Y|X}(y|x)$. For $0 < y < 1$, find the conditional PDF $f_{X|Y}(x|y)$.

# 3.6.3 Conditional PDF

**Example 11 Solution**

For $0 < x < 1$, Theorem 9 implies

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = \int_0^x 2dy = 2x.$$

The conditional PDF of $Y$ given $X$ is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} 1/x, & 0 < y < x, \\ 0, & \text{otherwise.} \end{cases}$$

# 3.6.3 Conditional PDF

**Example 11 Solution**

For $0 < y < 1$, Theorem 9 implies

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx = \int_{y}^{1} 2dx = 2(1-y).$$

Furthermore, Equation (3.11) implies

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 1/(1-y), & y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

# 3.6.3 Conditional PDF

**Theorem 21**

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x) = f_{X|Y}(x|y)f_Y(y).$$

# 3.6.4 Conditional Expected Value of a Function

**Definition 16 (Conditional Expected Value of a Function)**

For continuous random variables $X$ and $Y$ and any $y$ such that $f_Y(y) > 0$, the conditional expected value of $g(X, Y)$ given $Y = y$ is

$$E[g(X,Y)|Y=y] = \int_{-\infty}^{+\infty} g(x,y) f_{X|Y}(x|y)\,dx.$$

**Note**

The conditional expected value of $X$ given $Y = y$ is a special case of Definition 16:

$$\boxed{E[X|Y=y] = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y)\,dx} \qquad (4.12)$$

# 3.6.4 Conditional Expected Value of a Function

**Definition 16 (Conditional Expected Value of a Function)**

For continuous random variables $X$ and $Y$ and any $y$ such that $f_Y(y) > 0$, the conditional expected value of $g(X, Y)$ given $Y = y$ is

$$E[g(X,Y)|Y=y] = \int_{-\infty}^{+\infty} g(x,y) f_{X|Y}(x|y)\,dx.$$

**Note**

The conditional expected value of $X$ given $Y = y$ is a special case of Definition 16:

$$\boxed{E[X|Y=y] = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y)\,dx} \tag{4.12}$$

# 3.6.5 Conditional Expected Value

**Definition 17 (Conditional Expected Value)**

The conditional expected value $E[X|Y]$ is a function of random variable $Y$ such that if $Y = y$ then $E[X|Y] = E[X|Y = y]$.

**Example 12**

For random variables $X$ and $Y$ in Example 4.5, we found in Example 11 that the conditional PDF of $X$ given $Y$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 1/(1-y), & y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional expected values $E[X|Y = y]$ and $E[X|Y]$.

# 3.6.5 Conditional Expected Value

**Definition 17 (Conditional Expected Value)**

The conditional expected value $E[X|Y]$ is a function of random variable $Y$ such that if $Y = y$ then $E[X|Y] = E[X|Y = y]$.

**Example 12**

For random variables $X$ and $Y$ in Example 4.5, we found in Example 11 that the conditional PDF of $X$ given $Y$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 1/(1-y), & y < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the conditional expected values $E[X|Y = y]$ and $E[X|Y]$.

# 3.6.5 Conditional Expected Value

**Example 12 Solution**

Given the conditional PDF $f_{X|Y}(x|y)$, we perform the integration

$$E[X|Y=y] = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y)dx$$

$$= \int_{y}^{1} \frac{1}{1-y} x dx = \frac{x^2}{2(1-y)}\Big|_{x=y}^{x=1} = \frac{1+y}{2}.$$

Since $E[X|Y=y] = (1+y)/2$, $E[X|Y] = (1+Y)/2$.

# 3.7.1 Definitions

**Definition 18 (Independent Random Variables)**

Random variables $X$ and $Y$ are independent if and only if

$$\text{Discrete: } P_{X,Y}(x,y) = P_X(x)P_Y(y),$$
$$\text{Continuous: } f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

**Note**

Theorem 19 implies that if $X$ and $Y$ are independent discrete random variables, then

$$\boxed{P_{X|Y}(x|y) = P_X(x), \quad P_{Y|X}(y|x) = P_Y(y)} \tag{4.13}$$

Theorem 21 implies that if $X$ and $Y$ are independent continuous random variables, then

$$\boxed{f_{X|Y}(x|y) = f_X(x), \quad f_{Y|X}(y|x) = f_Y(y)} \tag{4.14}$$

# 3.7.1 Definitions

**Definition 18 (Independent Random Variables)**

Random variables $X$ and $Y$ are independent if and only if

$$\text{Discrete: } P_{X,Y}(x,y) = P_X(x)P_Y(y),$$
$$\text{Continuous: } f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

**Note**

Theorem 19 implies that if $X$ and $Y$ are independent discrete random variables, then

$$\boxed{P_{X|Y}(x|y) = P_X(x), \quad P_{Y|X}(y|x) = P_Y(y)} \tag{4.13}$$

Theorem 21 implies that if $X$ and $Y$ are independent continuous random variables, then

$$\boxed{f_{X|Y}(x|y) = f_X(x), \quad f_{Y|X}(y|x) = f_Y(y)} \tag{4.14}$$

# 3.7.1 Definitions

**Example 13**

Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Are $X$ and $Y$ independent?

**Example 13 Solution**

The marginal PDFs of $X$ and $Y$ are

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise} \end{cases}, \quad f_Y(y) = \begin{cases} 2y, & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily verified that $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ for all pairs $(x,y)$ and so we conclude that $X$ and $Y$ are independent.

# 3.7.1 Definitions

**Example 13**

Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Are $X$ and $Y$ independent?

**Example 13 Solution**

The marginal PDFs of $X$ and $Y$ are

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise} \end{cases}, \quad f_Y(y) = \begin{cases} 2y, & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily verified that $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ for all pairs $(x,y)$ and so we conclude that $X$ and $Y$ are independent.

# 3.7.2 Properties

**Theorem 22**

For independent random variables $X$ and $Y$,

(a) $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$,

(b) $r_{X,Y} = E[XY] = E[X]E[Y]$,

(c) $Cov[X, Y] = \rho_{X,Y} = 0$,

(d) $Var[X + Y] = Var[X] + Var[Y]$,

(e) $E[X|Y = y] = E[X]$ for all $y \in S_Y$,

(f) $E[Y|X = x] = E[Y]$ for all $x \in S_X$.

# 3.7.2 Properties

**Example 14**

Random variables $X$ and $Y$ have a joint PMF given by the following matrix

| $P_{X,Y}(x,y)$ | $y=-1$ | $y=0$ | $y=1$ |
|---|---|---|---|
| $x=-1$ | 0 | 0.25 | 0 |
| $x=1$ | 0.25 | 0.25 | 0.25 |

Are $X$ and $Y$ independent? Are $X$ and $Y$ uncorrelated?

# 3.7.2 Properties

**Example 14 Solution**

- For the marginal PMFs, we have $P_X(-1) = 0.25$ and $P_Y(-1) = 0.25$. Thus

$$P_X(-1)P_Y(-1) = 0.0625 \neq P_{X,Y}(-1,-1) = 0,$$

and we conclude that $X$ and $Y$ are not independent.

- To find $Cov[X,Y]$, we calculate

$$E[X] = 0.5, \quad E[Y] = 0, \quad E[XY] = 0.$$

Therefore, Theorem 14(a) implies

$$Cov[X,Y] = E[XY] - E[X]E[Y] = \rho_{X,Y} = 0,$$

and by definition $X$ and $Y$ are uncorrelated.