Probability and Statistics – Problem Set 6

Question 1.

Suppose X and Y have joint PDF $f_{X,Y}(x,y) = c(x^2 + xy)$ on $[0,1] \times [0,1]$.

a) Find c ad the joint CDF $F_{X,Y}(x,y)$

b) Find the marginal CDFs F_X, F_Y and the marginal PDF f_X, f_Y.

c) Find $E[X]$ and $Var(X)$

d) Find the covariance and correlation of X and Y.

Question 2:

Let X be the roll of a fair $3 - sided$ die. We then flip a fair coin X times independently; Let Y be the number of heads.

a) What are Ω_X and Ω_Y? What is $\Omega_{X,Y}$? What is $X's$ marginal distribution?

b) What is $p_{X,Y}(x,y)$?

c) What is $p_Y(y)$?

d) Are X and Y independent?

Question 3:

Suppose (X, Y, Z) are jointly distributed with density function

$$f_{X,Y,Z}(x,y,z) = \begin{cases} ce^{-12x}e^{-13y}, & \text{x, y} > 0 \text{ and } 0 < \text{z} < 47 \\ 0, & \text{otherwise} \end{cases}$$

a) Set up an appropriate triple integral with the order $dxdydz$ for the value of c.

b) Computing the marginal PDF $f_X(x)$. What distribution is the random variable X?

c) Computing the marginal PDF $f_Z(z)$

d) Are X, Y, Z mutually independent?

e) Write an expression for $E\left[\log\left(\frac{1}{Z^{X+Y}}\right)\right]$

Question 4:

Suppose $X \sim Bin(n,p)$ and $Y \sim Bin(m,p)$ are independent and let $Z = X + Y$. What is the conditional PMF $P(X = k|Z = z)$?

Question 5:

The covariance matrix of a random vector $Z = (Z_1, Z_2, \ldots, Z_n)$ is defined to the $n \times n$ matrix Σ such that $\Sigma_{ij} = Cov(X_i, X_j)$.

a) Let $X_1, X_2, \ldots, X_4$ be independent and identically distributed random variables with mean μ and variance σ^2. What is the 4×4 covariance matrix of X?

b) Define $Y = (X_1 + X_2, X_2 + X_3, X_3 + X_4)$. What is the 3×3 covariance matrix of Y.

Question 6:

Suppose we throw 12 balls independently and uniformly into 7 bins. For $i = 1,2, \ldots, 7$, let X_i be the indicator random variable of whether bin i is empty.

Let $X = (X_1, X_2, \ldots, X_7)$ be the random vector of indicators.

a) What is the covariance matrix of X?

b) Let $Y = \sum_{i=1}^{7} X_i$ be the number of empty bins. What is $Var(Y)$?