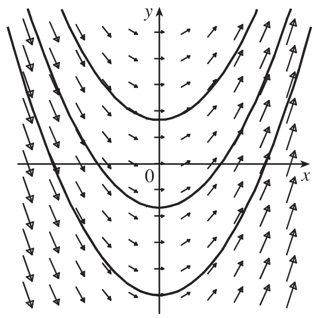
$dy/dt = -y \quad \Rightarrow \quad dy/y = -dt \quad \Rightarrow \quad \ln|y| = -t + K \quad \Rightarrow \quad y = \pm e^{-t+K} = Be^{-t}$ for some constant B. Therefore $xy = Ae^t Be^{-t} = AB =$ constant. If the flow line passes through $(1,1)$ then $(1)(1) =$ constant $= 1 \quad \Rightarrow \quad xy = 1 \quad \Rightarrow$ $y = 1/x$, $x > 0$.

36. (a) We sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\,\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be parabolas.



(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point (x, y) is $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = \mathbf{i} + x\,\mathbf{j} \quad \Rightarrow \quad \dfrac{dx}{dt} = 1, \ \dfrac{dy}{dt} = x$. Thus $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{x}{1} = x$.

(c) From part (b), $dy/dx = x$. Integrating, we have $y = \frac{1}{2}x^2 + c$. Since the particle starts at the origin, we know $(0, 0)$ is on the curve, so $0 = 0 + c \quad \Rightarrow \quad c = 0$ and the path the particle follows is $y = \frac{1}{2}x^2$.

16.2 Line Integrals

1. $x = t^3$ and $y = t$, $0 \le t \le 2$, so by Formula 3

$$\int_C y^3\,ds = \int_0^2 t^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^2 t^3\,\sqrt{(3t^2)^2 + (1)^2}\,dt = \int_0^2 t^3\,\sqrt{9t^4 + 1}\,dt$$
$$= \tfrac{1}{36} \cdot \tfrac{2}{3}\left(9t^4 + 1\right)^{3/2}\Big]_0^2 = \tfrac{1}{54}(145^{3/2} - 1) \text{ or } \tfrac{1}{54}\left(145\sqrt{145} - 1\right)$$

2. $$\int_C xy\,ds = \int_0^1 (t^2)(2t)\sqrt{(2t)^2 + (2)^2}\,dt = \int_0^1 2t^3\sqrt{4t^2 + 4}\,dt = \int_0^1 4t^3\sqrt{t^2 + 1}\,dt \quad \left[\begin{array}{l}\text{Substitute } u = t^2 + 1 \quad \Rightarrow \\ t^2 = u - 1,\ du = 2t\,dt\end{array}\right]$$
$$= \int_1^2 2(u - 1)\sqrt{u}\,du = 2\int_1^2 (u^{3/2} - u^{1/2})\,du = 2\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^2$$
$$= 2\left(\tfrac{8}{5}\sqrt{2} - \tfrac{4}{3}\sqrt{2} - \tfrac{2}{5} + \tfrac{2}{3}\right) = \tfrac{8}{15}\left(\sqrt{2} + 1\right)$$

3. Parametric equations for C are $x = 4\cos t,\ \ y = 4\sin t,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt = \int_{-\pi/2}^{\pi/2} 4^5\cos t\,\sin^4 t\,\sqrt{16(\sin^2 t + \cos^2 t)}\,dt$$
$$= 4^5\int_{-\pi/2}^{\pi/2} (\sin^4 t\cos t)(4)\,dt = (4)^6\left[\tfrac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = \tfrac{2\cdot 4^6}{5} = 1638.4$$

4. Parametric equations for C are $x = 4t,\ \ y = 3 + 3t,\ \ 0 \le t \le 1$. Then

$$\int_C x\sin y\,ds = \int_0^1 (4t)\sin(3 + 3t)\sqrt{4^2 + 3^2}\,dt = 20\int_0^1 t\sin(3 + 3t)dt$$

Integrating by parts with $u = t \Rightarrow du = dt, dv = \sin(3+3t)dt \Rightarrow v = -\frac{1}{3}\cos(3+3t)$ gives

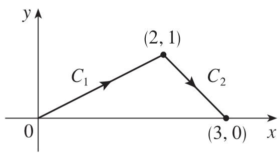
$$\int_C x \sin y\, ds = 20\left[-\tfrac{1}{3}t\cos(3+3t) + \tfrac{1}{9}\sin(3+3t)\right]_0^1 = 20\left[-\tfrac{1}{3}\cos 6 + \tfrac{1}{9}\sin 6 + 0 - \tfrac{1}{9}\sin 3\right]$$
$$= \tfrac{20}{9}(\sin 6 - 3\cos 6 - \sin 3)$$

5. If we choose x as the parameter, parametric equations for C are $x = x$, $y = \sqrt{x}$ for $1 \le x \le 4$ and

$$\int_C \left(x^2y^3 - \sqrt{x}\right) dy = \int_1^4 \left[x^2 \cdot (\sqrt{x}\,)^3 - \sqrt{x}\right] \frac{1}{2\sqrt{x}}\, dx = \tfrac{1}{2}\int_1^4 \left(x^3 - 1\right) dx$$
$$= \tfrac{1}{2}\left[\tfrac{1}{4}x^4 - x\right]_1^4 = \tfrac{1}{2}\left(64 - 4 - \tfrac{1}{4} + 1\right) = \tfrac{243}{8}$$

6. Choosing y as the parameter, we have $x = y^3$, $y = y$, $-1 \le y \le 1$. Then

$$\int_C e^x\, dx = \int_{-1}^1 e^{y^3} \cdot 3y^2\, dy = e^{y^3}\Big]_{-1}^1 = e^1 - e^{-1} = e - \tfrac{1}{e}.$$

7. 

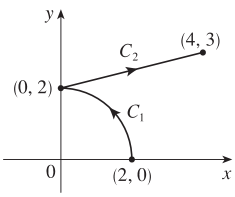
$C = C_1 + C_2$

On C_1: $x = x, y = \frac{1}{2}x \Rightarrow dy = \frac{1}{2}\,dx$, $0 \le x \le 2$.

On C_2: $x = x, y = 3 - x \Rightarrow dy = -dx$, $2 \le x \le 3$.

Then

$$\int_C (x+2y)\,dx + x^2\,dy = \int_{C_1} (x+2y)\,dx + x^2\,dy + \int_{C_2} (x+2y)\,dx + x^2\,dy$$
$$= \int_0^2 \left[x + 2\left(\tfrac{1}{2}x\right) + x^2\left(\tfrac{1}{2}\right)\right] dx + \int_2^3 \left[x + 2(3-x) + x^2(-1)\right] dx$$
$$= \int_0^2 \left(2x + \tfrac{1}{2}x^2\right) dx + \int_2^3 \left(6 - x - x^2\right) dx$$
$$= \left[x^2 + \tfrac{1}{6}x^3\right]_0^2 + \left[6x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_2^3 = \tfrac{16}{3} - 0 + \tfrac{9}{2} - \tfrac{22}{3} = \tfrac{5}{2}$$

8. 

$C = C_1 + C_2$

On C_1: $x = 2\cos t \Rightarrow dx = -2\sin t\,dt$, $y = 2\sin t \Rightarrow$
$dy = 2\cos t\ dt$, $0 \le t \le \frac{\pi}{2}$.

On C_2: $x = 4t \Rightarrow dx = 4\,dt, y = 2 + t \Rightarrow$
$dy = dt$, $0 \le t \le 1$.

Then

$$\int_C x^2\,dx + y^2\,dy = \int_{C_1} x^2\,dx + y^2\,dy + \int_{C_2} x^2\,dx + y^2\,dy$$
$$= \int_0^{\pi/2} (2\cos t)^2(-2\sin t\,dt) + (2\sin t)^2(2\cos t\,dt) + \int_0^1 (4t)^2(4\,dt) + (2+t)^2\,dt$$
$$= 8\int_0^{\pi/2} (-\cos^2 t\sin t + \sin^2 t\cos t)\,dt + \int_0^1 (65t^2 + 4t + 4)\,dt$$
$$= 8\left[\tfrac{1}{3}\cos^3 t + \tfrac{1}{3}\sin^3 t\right]_0^{\pi/2} + \left[\tfrac{65}{3}t^3 + 2t^2 + 4t\right]_0^1 = 8\left(\tfrac{1}{3} - \tfrac{1}{3}\right) + \tfrac{65}{3} + 2 + 4 = \tfrac{83}{3}$$

9. $x = 2\sin t,\ \ y = t,\ \ z = -2\cos t,\ \ 0 \le t \le \pi$. Then by Formula 9,

$$\begin{aligned}\int_C xyz\,ds &= \int_0^\pi (2\sin t)(t)(-2\cos t)\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 + \left(\tfrac{dz}{dt}\right)^2}\,dt \\ &= \int_0^\pi -4t\sin t\,\cos t\,\sqrt{(2\cos t)^2 + (1)^2 + (2\sin t)^2}\,dt = \int_0^\pi -2t\sin 2t\,\sqrt{4(\cos^2 t + \sin^2 t) + 1}\,dt \\ &= -2\sqrt{5}\int_0^\pi t\sin 2t\,dt = -2\sqrt{5}\left[-\tfrac{1}{2}t\cos 2t + \tfrac{1}{4}\sin 2t\right]_0^\pi \qquad \left[\begin{array}{l}\text{integrate by parts with} \\ u = t,\ dv = \sin 2t\,dt\end{array}\right] \\ &= -2\sqrt{5}\left(-\tfrac{\pi}{2} - 0\right) = \sqrt{5}\,\pi\end{aligned}$$

10. Parametric equations for C are $x = -1 + 2t,\ y = 5 + t,\ z = 4t,\ 0 \le t \le 1$. Then

$$\begin{aligned}\int_C xyz^2\,ds &= \int_0^1 (-1+2t)(5+t)(4t)^2\sqrt{2^2 + 1^2 + 4^2}\,dt = \sqrt{21}\int_0^1 (32t^4 + 144t^3 - 80t^2)\,dt \\ &= \sqrt{21}\left[32\cdot\frac{t^5}{5} + 144\cdot\frac{t^4}{4} - 80\cdot\frac{t^3}{3}\right]_0^1 = \sqrt{21}\left(\tfrac{32}{5} + 36 - \tfrac{80}{3}\right) = \tfrac{236}{15}\sqrt{21}\end{aligned}$$

11. Parametric equations for C are $x = t,\ \ y = 2t,\ \ z = 3t,\ 0 \le t \le 1$. Then

$$\int_C xe^{yz}\,ds = \int_0^1 te^{(2t)(3t)}\sqrt{1^2 + 2^2 + 3^2}\,dt = \sqrt{14}\int_0^1 te^{6t^2}dt = \sqrt{14}\left[\tfrac{1}{12}e^{6t^2}\right]_0^1 = \tfrac{\sqrt{14}}{12}(e^6 - 1).$$

12. $\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} = \sqrt{1^2 + (-2\sin 2t)^2 + (2\cos 2t)^2} = \sqrt{1 + 4(\sin^2 2t + \cos^2 2t)} = \sqrt{5}$. Then

$$\begin{aligned}\int_C (x^2 + y^2 + z^2)\,ds &= \int_0^{2\pi} (t^2 + \cos^2 2t + \sin^2 2t)\,\sqrt{5}\,dt = \sqrt{5}\int_0^{2\pi} (t^2 + 1)\,dt \\ &= \sqrt{5}\left[\tfrac{1}{3}t^3 + t\right]_0^{2\pi} = \sqrt{5}\left[\tfrac{1}{3}(8\pi^3) + 2\pi\right] = \sqrt{5}\left(\tfrac{8}{3}\pi^3 + 2\pi\right)\end{aligned}$$

13. $\int_C xye^{yz}\,dy = \int_0^1 (t)(t^2)e^{(t^2)(t^3)}\cdot 2t\,dt = \int_0^1 2t^4e^{t^5}\,dt = \tfrac{2}{5}e^{t^5}\Big]_0^1 = \tfrac{2}{5}(e^1 - e^0) = \tfrac{2}{5}(e - 1)$

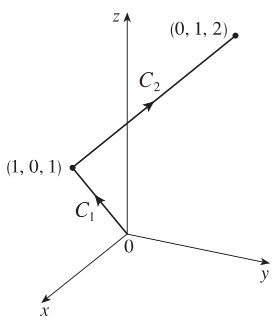
14. $\int_C y\,dx + z\,dy + x\,dz = \int_1^4 t\cdot\tfrac{1}{2}t^{-1/2}\,dt + t^2\cdot dt + \sqrt{t}\cdot 2t\,dt = \int_1^4 \left(\tfrac{1}{2}t^{1/2} + t^2 + 2t^{3/2}\right)dt$

$$= \left[\tfrac{1}{3}t^{3/2} + \tfrac{1}{3}t^3 + \tfrac{4}{5}t^{5/2}\right]_1^4 = \tfrac{8}{3} + \tfrac{64}{3} + \tfrac{128}{5} - \tfrac{1}{3} - \tfrac{1}{3} - \tfrac{4}{5} = \tfrac{722}{15}$$

15. Parametric equations for C are $x = 1 + 3t,\ \ y = t,\ \ z = 2t,\ 0 \le t \le 1$. Then

$$\begin{aligned}\int_C z^2\,dx + x^2\,dy + y^2\,dz &= \int_0^1 (2t)^2\cdot 3\,dt + (1+3t)^2\,dt + t^2\cdot 2\,dt = \int_0^1 \left(23t^2 + 6t + 1\right)dt \\ &= \left[\tfrac{23}{3}t^3 + 3t^2 + t\right]_0^1 = \tfrac{23}{3} + 3 + 1 = \tfrac{35}{3}\end{aligned}$$

16.



On C_1: $x = t \ \Rightarrow\ dx = dt,\ y = 0 \ \Rightarrow$
$dy = 0\,dt,\ z = t \ \Rightarrow\ dz = dt,\ 0 \le t \le 1.$

On C_2: $x = 1 - t \ \Rightarrow\ dx = -dt,\ y = t \ \Rightarrow$
$dy = dt,\ z = 1 + t \ \Rightarrow\ dz = dt,\ 0 \le t \le 1.$

Then

$$\int_C (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$$
$$= \int_{C_1} (y+z)\,dx + (x+z)\,dy + (x+y)\,dz + \int_{C_2} (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$$
$$= \int_0^1 (0+t)\,dt + (t+t)\cdot 0\,dt + (t+0)\,dt + \int_0^1 (t+1+t)(-dt) + (1-t+1+t)\,dt + (1-t+t)\,dt$$
$$= \int_0^1 2t\,dt + \int_0^1 (-2t+2)\,dt = \left[t^2\right]_0^1 + \left[-t^2+2t\right]_0^1 = 1+1=2$$

17. (a) Along the line $x=-3$, the vectors of $\mathbf{F}$ have positive y-components, so since the path goes upward, the integrand $\mathbf{F}\cdot\mathbf{T}$ is always positive. Therefore $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds$ is positive.

(b) All of the (nonzero) field vectors along the circle with radius 3 are pointed in the clockwise direction, that is, opposite the direction to the path. So $\mathbf{F}\cdot\mathbf{T}$ is negative, and therefore $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds$ is negative.

18. Vectors starting on C_1 point in roughly the same direction as C_1, so the tangential component $\mathbf{F}\cdot\mathbf{T}$ is positive. Then $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds$ is positive. On the other hand, no vectors starting on C_2 point in the same direction as C_2, while some vectors point in roughly the opposite direction, so we would expect $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds$ to be negative.

19. $\mathbf{r}(t) = 11t^4\,\mathbf{i} + t^3\,\mathbf{j}$, so $\mathbf{F}(\mathbf{r}(t)) = (11t^4)(t^3)\,\mathbf{i} + 3(t^3)^2\,\mathbf{j} = 11t^7\,\mathbf{i} + 3t^6\,\mathbf{j}$ and $\mathbf{r}'(t) = 44t^3\,\mathbf{i} + 3t^2\,\mathbf{j}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 (11t^7\cdot 44t^3 + 3t^6\cdot 3t^2)\,dt = \int_0^1 (484t^{10} + 9t^8)\,dt = \left[44t^{11} + t^9\right]_0^1 = 45.$$

20. $\mathbf{F}(\mathbf{r}(t)) = (t^2+t^3)\,\mathbf{i} + (t^3-t^2)\,\mathbf{j} + (t^2)^2\mathbf{k} = (t^2+t^3)\,\mathbf{i} + (t^3-t^2)\,\mathbf{j} + t^4\,\mathbf{k}$, $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + 2t\,\mathbf{k}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 (2t^3 + 2t^4 + 3t^5 - 3t^4 + 2t^5)\,dt = \int_0^1 (5t^5 - t^4 + 2t^3)\,dt$$
$$= \left[\tfrac{5}{6}t^6 - \tfrac{1}{5}t^5 + \tfrac{1}{2}t^4\right]_0^1 = \tfrac{5}{6} - \tfrac{1}{5} + \tfrac{1}{2} = \tfrac{17}{15}.$$

21. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle \sin t^3, \cos(-t^2), t^4\right\rangle \cdot \left\langle 3t^2, -2t, 1\right\rangle dt$

$$= \int_0^1 (3t^2\sin t^3 - 2t\cos t^2 + t^4)\,dt = \left[-\cos t^3 - \sin t^2 + \tfrac{1}{5}t^5\right]_0^1 = \tfrac{6}{5} - \cos 1 - \sin 1$$

22. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \langle \cos t, \sin t, \cos t\sin t\rangle \cdot \langle -\sin t, \cos t, 1\rangle\,dt = \int_0^\pi \sin t\cos t\,dt = \tfrac{1}{2}\sin^2 t\Big]_0^\pi = 0$

23. $\mathbf{F}(\mathbf{r}(t)) = (e^t)\left(e^{-t^2}\right)\mathbf{i} + \sin\left(e^{-t^2}\right)\mathbf{j} = e^{t-t^2}\,\mathbf{i} + \sin\left(e^{-t^2}\right)\mathbf{j}$, $\mathbf{r}'(t) = e^t\,\mathbf{i} - 2te^{-t^2}\,\mathbf{j}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^2 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_1^2 \left[e^{t-t^2}e^t + \sin\left(e^{-t^2}\right)\cdot\left(-2te^{-t^2}\right)\right]dt$$
$$= \int_1^2 \left[e^{2t-t^2} - 2te^{-t^2}\sin\left(e^{-t^2}\right)\right]dt \approx 1.9633$$

24. $\mathbf{F}(\mathbf{r}(t)) = (\sin t)\sin(\sin 5t)\,\mathbf{i} + (\sin 5t)\sin(\cos t)\,\mathbf{j} + (\cos t)\sin(\sin t)\,\mathbf{k}$, $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 5\cos 5t\,\mathbf{k}$.

Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$$
$$= \int_0^\pi [-\sin^2 t\,\sin(\sin 5t) + \cos t\,\sin 5t\,\sin(\cos t) + 5\cos t\,\cos 5t\,\sin(\sin t)]\,dt \approx -0.1363$$

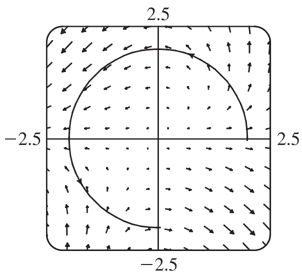
25. $x = t^2,\ \ y = t^3,\ \ z = t^4$ so by Formula 9,

$$\begin{aligned}\int_C x\sin(y+z)\,ds &= \int_0^5 (t^2)\sin(t^3+t^4)\sqrt{(2t)^2+(3t^2)^2+(4t^3)^2}\,dt\\ &= \int_0^5 t^2\sin(t^3+t^4)\sqrt{4t^2+9t^4+16t^6}\,dt \approx 15.0074\end{aligned}$$

26. $\int_C ze^{-xy}\,ds = \int_0^1 (e^{-t})e^{-t\cdot t^2}\sqrt{(1)^2+(2t)^2+(-e^{-t})^2}\,dt = \int_0^1 e^{-t-t^3}\sqrt{1+4t^2+e^{-2t}}\,dt \approx 0.8208$

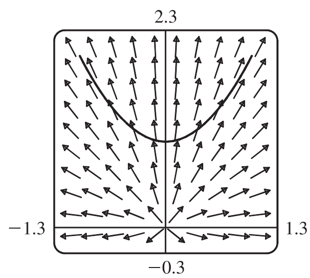
27. We graph $\mathbf{F}(x,y) = (x-y)\,\mathbf{i} + xy\,\mathbf{j}$ and the curve C. We see that most of the vectors starting on C point in roughly the same direction as C, so for these portions of C the tangential component $\mathbf{F}\cdot\mathbf{T}$ is positive. Although some vectors in the third quadrant which start on C point in roughly the opposite direction, and hence give negative tangential components, it seems reasonable that the effect of these portions of C is outweighed by the positive tangential components. Thus, we would expect $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ to be positive.

To verify, we evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. The curve C can be represented by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\ \ 0\le t\le \frac{3\pi}{2}$, so $\mathbf{F}(\mathbf{r}(t)) = (2\cos t - 2\sin t)\,\mathbf{i} + 4\cos t\sin t\,\mathbf{j}$ and $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$. Then



$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{3\pi/2} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt\\ &= \int_0^{3\pi/2}[-2\sin t(2\cos t - 2\sin t) + 2\cos t(4\cos t\sin t)]\,dt\\ &= 4\int_0^{3\pi/2}(\sin^2 t - \sin t\cos t + 2\sin t\cos^2 t)\,dt\\ &= 3\pi + \tfrac{2}{3} \qquad \text{[using a CAS]}\end{aligned}$$

28. We graph $\mathbf{F}(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\,\mathbf{j}$ and the curve C. In the first quadrant, each vector starting on C points in roughly the same direction as C, so the tangential component $\mathbf{F}\cdot\mathbf{T}$ is positive. In the second quadrant, each vector starting on C points in roughly the direction opposite to C, so $\mathbf{F}\cdot\mathbf{T}$ is negative. Here, it appears that the tangential components in the first and second quadrants counteract each other, so it seems reasonable to guess that $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is zero. To verify, we evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. The curve C can be represented by



$\mathbf{r}(t) = t\,\mathbf{i} + (1+t^2)\,\mathbf{j},\ -1\le t\le 1$, so $\mathbf{F}(\mathbf{r}(t)) = \dfrac{t}{\sqrt{t^2+(1+t^2)^2}}\,\mathbf{i} + \dfrac{1+t^2}{\sqrt{t^2+(1+t^2)^2}}\,\mathbf{j}$ and $\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j}$. Then

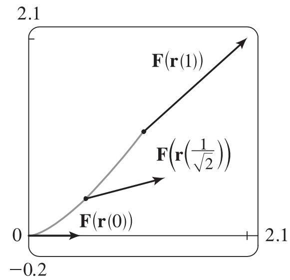
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt &= \int_{-1}^1\left(\frac{t}{\sqrt{t^2+(1+t^2)^2}} + \frac{2t(1+t^2)}{\sqrt{t^2+(1+t^2)^2}}\right)dt\\ &= \int_{-1}^1 \frac{t(3+2t^2)}{\sqrt{t^4+3t^2+1}}\,dt = 0 \qquad \text{[since the integrand is an odd function]}\end{aligned}$$

29. (a) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle e^{t^2-1}, t^5\right\rangle \cdot \langle 2t, 3t^2\rangle\,dt = \int_0^1 \left(2te^{t^2-1} + 3t^7\right)dt = \left[e^{t^2-1} + \frac{3}{8}t^8\right]_0^1 = \frac{11}{8} - 1/e$

(b) $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{F}(\mathbf{r}(0)) = \langle e^{-1}, 0\rangle$;

$\mathbf{r}\left(\frac{1}{\sqrt{2}}\right) = \left\langle \frac{1}{2}, \frac{1}{2\sqrt{2}}\right\rangle$, $\mathbf{F}\left(\mathbf{r}\left(\frac{1}{\sqrt{2}}\right)\right) = \left\langle e^{-1/2}, \frac{1}{4\sqrt{2}}\right\rangle$;

$\mathbf{r}(1) = \langle 1,1\rangle$, $\mathbf{F}(\mathbf{r}(1)) = \langle 1,1\rangle$.



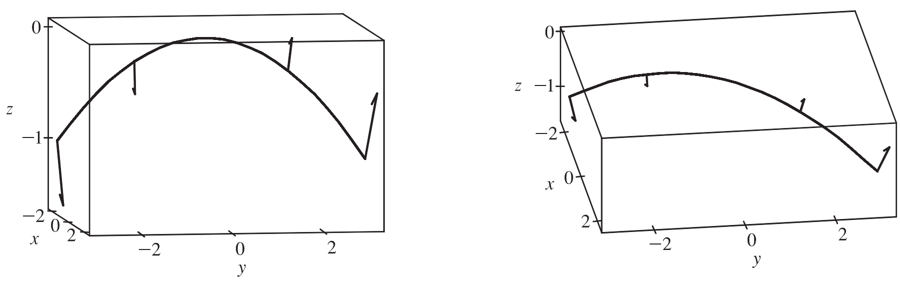
In order to generate the graph with Maple, we use the `line` command in the `plottools` package to define each of the vectors. For example,

```
v1:=line([0,0],[exp(-1),0]):
```

generates the vector from the vector field at the point $(0,0)$ (but without an arrowhead) and gives it the name `v1`. To show everything on the same screen, we use the `display` command. In Mathematica, we use `ListPlot` (with the `PlotJoined -> True` option) to generate the vectors, and then `Show` to show everything on the same screen.

30. (a) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^1 \langle 2t, t^2, 3t\rangle \cdot \langle 2, 3, -2t\rangle\,dt = \int_{-1}^1 (4t + 3t^2 - 6t^2)\,dt = \left[2t^2 - t^3\right]_{-1}^1 = -2$

(b) Now $\mathbf{F}(\mathbf{r}(t)) = \langle 2t, t^2, 3t\rangle$, so $\mathbf{F}(\mathbf{r}(-1)) = \langle -2, 1, -3\rangle$, $\mathbf{F}\left(\mathbf{r}\left(-\frac{1}{2}\right)\right) = \left\langle -1, \frac{1}{4}, -\frac{3}{2}\right\rangle$, $\mathbf{F}\left(\mathbf{r}\left(\frac{1}{2}\right)\right) = \left\langle 1, \frac{1}{4}, \frac{3}{2}\right\rangle$, and $\mathbf{F}(\mathbf{r}(1)) = \langle 2,1,3\rangle$.



31. $x = e^{-t}\cos 4t$, $y = e^{-t}\sin 4t$, $z = e^{-t}$, $0 \le t \le 2\pi$.

Then $\dfrac{dx}{dt} = e^{-t}(-\sin 4t)(4) - e^{-t}\cos 4t = -e^{-t}(4\sin 4t + \cos 4t)$,

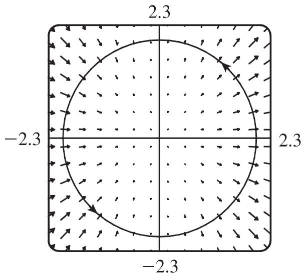
$\dfrac{dy}{dt} = e^{-t}(\cos 4t)(4) - e^{-t}\sin 4t = -e^{-t}(-4\cos 4t + \sin 4t)$, and $\dfrac{dz}{dt} = -e^{-t}$, so

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(-e^{-t})^2[(4\sin 4t + \cos 4t)^2 + (-4\cos 4t + \sin 4t)^2 + 1]}$$
$$= e^{-t}\sqrt{16(\sin^2 4t + \cos^2 4t) + \sin^2 4t + \cos^2 4t + 1} = 3\sqrt{2}\,e^{-t}$$

Therefore
$$\int_C x^3y^2z\,ds = \int_0^{2\pi} (e^{-t}\cos 4t)^3(e^{-t}\sin 4t)^2(e^{-t})\,(3\sqrt{2}\,e^{-t})\,dt$$
$$= \int_0^{2\pi} 3\sqrt{2}\,e^{-7t}\cos^3 4t\sin^2 4t\,dt = \tfrac{172{,}704}{5{,}632{,}705}\sqrt{2}\,(1 - e^{-14\pi})$$

32. (a) We parametrize the circle C as $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$. So $\mathbf{F}(\mathbf{r}(t)) = \left\langle 4\cos^2 t, 4\cos t\sin t\right\rangle$, $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, and $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} (-8\cos^2 t\sin t + 8\cos^2 t\sin t)\,dt = 0$.

(b)



From the graph, we see that all of the vectors in the field are perpendicular to the path. This indicates that the field does no work on the particle, since the field never pulls the particle in the direction in which it is going. In other words, at any point along C, $\mathbf{F}\cdot\mathbf{T}=0$, and so certainly $\int_C \mathbf{F}\cdot d\mathbf{r}=0$.

33. We use the parametrization $x=2\cos t$, $y=2\sin t$, $-\frac{\pi}{2}\le t\le\frac{\pi}{2}$. Then

$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt=\sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt=2\,dt$, so $m=\int_C k\,ds=2k\int_{-\pi/2}^{\pi/2}dt=2k(\pi)$,

$\overline{x}=\frac{1}{2\pi k}\int_C xk\,ds=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\cos t)2\,dt=\frac{1}{2\pi}\big[4\sin t\big]_{-\pi/2}^{\pi/2}=\frac{4}{\pi}$, $\overline{y}=\frac{1}{2\pi k}\int_C yk\,ds=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(2\sin t)2\,dt=0$.

Hence $(\overline{x},\overline{y})=\left(\frac{4}{\pi},0\right)$.

34. We use the parametrization $x=a\cos t$, $y=a\sin t$, $0\le t\le\frac{\pi}{2}$. Then

$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt=\sqrt{(-a\sin t)^2+(a\cos t)^2}\,dt=a\,dt$, so

$m=\int_C\rho(x,y)\,ds=\int_C kxy\,ds=\int_0^{\pi/2}k(a\cos t)(a\sin t)\,a\,dt=ka^3\int_0^{\pi/2}\cos t\sin t\,dt=ka^3\left[\frac{1}{2}\sin^2 t\right]_0^{\pi/2}=\frac{1}{2}ka^3$,

$$\overline{x}=\frac{1}{ka^3/2}\int_C x(kxy)\,ds=\frac{2}{ka^3}\int_0^{\pi/2}k(a\cos t)^2(a\sin t)a\,dt=\frac{2}{ka^3}\cdot ka^4\int_0^{\pi/2}\cos^2 t\,\sin t\,dt$$

$=2a\left[-\frac{1}{3}\cos^3 t\right]_0^{\pi/2}=2a\left(0+\frac{1}{3}\right)=\frac{2}{3}a$, and

$$\overline{y}=\frac{1}{ka^3/2}\int_C y(kxy)\,ds=\frac{2}{ka^3}\int_0^{\pi/2}k(a\cos t)(a\sin t)^2a\,dt=\frac{2}{ka^3}\cdot ka^4\int_0^{\pi/2}\sin^2 t\,\cos t\,dt$$

$=2a\left[\frac{1}{3}\sin^3 t\right]_0^{\pi/2}=2a\left(\frac{1}{3}-0\right)=\frac{2}{3}a$.

Therefore the mass is $\frac{1}{2}ka^3$ and the center of mass is $(\overline{x},\overline{y})=\left(\frac{2}{3}a,\frac{2}{3}a\right)$.

35. (a) $\overline{x}=\dfrac{1}{m}\displaystyle\int_C x\rho(x,y,z)\,ds$, $\overline{y}=\dfrac{1}{m}\displaystyle\int_C y\rho(x,y,z)\,ds$, $\overline{z}=\dfrac{1}{m}\displaystyle\int_C z\rho(x,y,z)\,ds$ where $m=\int_C\rho(x,y,z)\,ds$.

(b) $m=\int_C k\,ds=k\int_0^{2\pi}\sqrt{4\sin^2 t+4\cos^2 t+9}\,dt=k\sqrt{13}\int_0^{2\pi}dt=2\pi k\sqrt{13}$,

$$\overline{x}=\frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi}2k\sqrt{13}\,\sin t\,dt=0,\ \overline{y}=\frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi}2k\sqrt{13}\,\cos t\,dt=0,$$

$$\overline{z}=\frac{1}{2\pi k\sqrt{13}}\int_0^{2\pi}\left(k\sqrt{13}\right)(3t)\,dt=\frac{3}{2\pi}\left(2\pi^2\right)=3\pi.$$ Hence $(\overline{x},\overline{y},\overline{z})=(0,0,3\pi)$.

36. $m=\int_C(x^2+y^2+z^2)\,ds=\int_0^{2\pi}(t^2+1)\sqrt{(1)^2+(-\sin t)^2+(\cos t)^2}\,dt=\int_0^{2\pi}(t^2+1)\sqrt{2}\,dt=\sqrt{2}\left(\frac{8}{3}\pi^3+2\pi\right)$,

$$\overline{x}=\frac{1}{\sqrt{2}\left(\frac{8}{3}\pi^3+2\pi\right)}\int_0^{2\pi}\sqrt{2}\,(t^3+t)\,dt=\frac{4\pi^4+2\pi^2}{\frac{8}{3}\pi^3+2\pi}=\frac{3\pi\left(2\pi^2+1\right)}{4\pi^2+3},$$

$$\overline{y} = \frac{3}{2\sqrt{2}\,\pi(4\pi^2+3)}\int_0^{2\pi}\left(\sqrt{2}\cos t\right)(t^2+1)\,dt = 0, \text{ and}$$

$$\overline{z} = \frac{3}{2\sqrt{2}\,\pi(4\pi^2+3)}\int_0^{2\pi}\left(\sqrt{2}\sin t\right)(t^2+1)\,dt = 0. \text{ Hence } (\overline{x},\overline{y},\overline{z}) = \left(\frac{3\pi(2\pi^2+1)}{4\pi^2+3}, 0, 0\right).$$

37. From Example 3, $\rho(x,y) = k(1-y)$, $x = \cos t$, $y = \sin t$, and $ds = dt$, $0 \le t \le \pi \quad \Rightarrow$

$$I_x = \int_C y^2\rho(x,y)\,ds = \int_0^\pi \sin^2 t\,[k(1-\sin t)]\,dt = k\int_0^\pi (\sin^2 t - \sin^3 t)\,dt$$

$$= \tfrac{1}{2}k\int_0^\pi (1-\cos 2t)\,dt - k\int_0^\pi (1-\cos^2 t)\sin t\,dt \qquad \left[\begin{array}{c}\text{Let } u = \cos t,\ du = -\sin t\,dt \\ \text{in the second integral}\end{array}\right]$$

$$= k\left[\tfrac{\pi}{2} + \int_1^{-1}(1-u^2)\,du\right] = k\left(\tfrac{\pi}{2} - \tfrac{4}{3}\right)$$

$$I_y = \int_C x^2\rho(x,y)\,ds = k\int_0^\pi \cos^2 t\,(1-\sin t)\,dt = \tfrac{k}{2}\int_0^\pi (1+\cos 2t)\,dt - k\int_0^\pi \cos^2 t\sin t\,dt$$

$= k\left(\frac{\pi}{2} - \frac{2}{3}\right)$, using the same substitution as above.

38. The wire is given as $x = 2\sin t$, $y = 2\cos t$, $z = 3t$, $0 \le t \le 2\pi$ with $\rho(x,y,z) = k$. Then

$ds = \sqrt{(2\cos t)^2 + (-2\sin t)^2 + 3^2}\,dt = \sqrt{4(\cos^2 t + \sin^2 t) + 9}\,dt = \sqrt{13}\,dt$ and

$$I_x = \int_C (y^2+z^2)\rho(x,y,z)\,ds = \int_0^{2\pi}(4\cos^2 t + 9t^2)(k)\sqrt{13}\,dt = \sqrt{13}\,k\left[4\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right) + 3t^3\right]_0^{2\pi}$$

$$= \sqrt{13}\,k(4\pi + 24\pi^3) = 4\sqrt{13}\,\pi k(1+6\pi^2)$$

$$I_y = \int_C (x^2+z^2)\rho(x,y,z)\,ds = \int_0^{2\pi}\left(4\sin^2 t + 9t^2\right)(k)\sqrt{13}\,dt = \sqrt{13}\,k\left[4\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right) + 3t^3\right]_0^{2\pi}$$

$$= \sqrt{13}\,k(4\pi + 24\pi^3) = 4\sqrt{13}\,\pi k(1+6\pi^2)$$

$$I_z = \int_C (x^2+y^2)\rho(x,y,z)\,ds = \int_0^{2\pi}(4\sin^2 t + 4\cos^2 t)(k)\sqrt{13}\,dt = 4\sqrt{13}\,k\int_0^{2\pi}dt = 8\pi\sqrt{13}\,k$$

39. $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\langle t - \sin t, 3 - \cos t\rangle \cdot \langle 1 - \cos t, \sin t\rangle\,dt$

$$= \int_0^{2\pi}(t - t\cos t - \sin t + \sin t\cos t + 3\sin t - \sin t\cos t)\,dt$$

$$= \int_0^{2\pi}(t - t\cos t + 2\sin t)\,dt = \left[\tfrac{1}{2}t^2 - (t\sin t + \cos t) - 2\cos t\right]_0^{2\pi} \qquad \left[\begin{array}{c}\text{integrate by parts} \\ \text{in the second term}\end{array}\right]$$

$$= 2\pi^2$$

40. Choosing y as the parameter, the curve C is parametrized by $x = y^2+1$, $y = y$, $0 \le y \le 1$. Then

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle \left(y^2+1\right)^2, ye^{y^2+1}\right\rangle \cdot \langle 2y, 1\rangle\,dy = \int_0^1 \left[2y\left(y^2+1\right)^2 + ye^{y^2+1}\right]dy$$

$$= \left[\tfrac{1}{3}\left(y^2+1\right)^3 + \tfrac{1}{2}e^{y^2+1}\right]_0^1 = \tfrac{8}{3} + \tfrac{1}{2}e^2 - \tfrac{1}{3} - \tfrac{1}{2}e = \tfrac{1}{2}e^2 - \tfrac{1}{2}e + \tfrac{7}{3}$$

41. $\mathbf{r}(t) = \langle 2t, t, 1-t\rangle$, $0 \le t \le 1$.

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle 2t - t^2, t - (1-t)^2, 1 - t - (2t)^2\right\rangle \cdot \langle 2, 1, -1\rangle\,dt$$

$$= \int_0^1 (4t - 2t^2 + t - 1 + 2t - t^2 - 1 + t + 4t^2)\,dt = \int_0^1 (t^2 + 8t - 2)\,dt = \left[\tfrac{1}{3}t^3 + 4t^2 - 2t\right]_0^1 = \tfrac{7}{3}$$

42. $\mathbf{r}(t) = 2\,\mathbf{i} + t\,\mathbf{j} + 5t\,\mathbf{k}$, $0 \le t \le 1$. Therefore

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \frac{K\langle 2, t, 5t\rangle}{(4+26t^2)^{3/2}} \cdot \langle 0, 1, 5\rangle\,dt = K\int_0^1 \frac{26t}{(4+26t^2)^{3/2}}\,dt = K\left[-(4+26t^2)^{-1/2}\right]_0^1 = K\left(\tfrac{1}{2} - \tfrac{1}{\sqrt{30}}\right).$$

43. (a) $\mathbf{r}(t) = at^2\,\mathbf{i} + bt^3\,\mathbf{j} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = 2at\,\mathbf{i} + 3bt^2\,\mathbf{j} \quad \Rightarrow \quad \mathbf{a}(t) = \mathbf{v}'(t) = 2a\,\mathbf{i} + 6bt\,\mathbf{j}$, and force is mass times acceleration: $\mathbf{F}(t) = m\,\mathbf{a}(t) = 2ma\,\mathbf{i} + 6mbt\,\mathbf{j}$.

(b) $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2ma\,\mathbf{i} + 6mbt\,\mathbf{j}) \cdot (2at\,\mathbf{i} + 3bt^2\,\mathbf{j})\,dt = \int_0^1 (4ma^2t + 18mb^2t^3)\,dt$

$$= \left[2ma^2t^2 + \tfrac{9}{2}mb^2t^4\right]_0^1 = 2ma^2 + \tfrac{9}{2}mb^2$$

44. $\mathbf{r}(t) = a\sin t\,\mathbf{i} + b\cos t\,\mathbf{j} + ct\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = a\cos t\,\mathbf{i} - b\sin t\,\mathbf{j} + c\,\mathbf{k} \quad \Rightarrow \quad \mathbf{a}(t) = \mathbf{v}'(t) = -a\sin t\,\mathbf{i} - b\cos t\,\mathbf{j}$ and $\mathbf{F}(t) = m\,\mathbf{a}(t) = -ma\sin t\,\mathbf{i} - mb\cos t\,\mathbf{j}$. Thus

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (-ma\sin t\,\mathbf{i} - mb\cos t\,\mathbf{j}) \cdot (a\cos t\,\mathbf{i} - b\sin t\,\mathbf{j} + c\,\mathbf{k})\,dt$$
$$= \int_0^{\pi/2} (-ma^2\sin t\cos t + mb^2\sin t\cos t)\,dt = m(b^2 - a^2)\left[\tfrac{1}{2}\sin^2 t\right]_0^{\pi/2} = \tfrac{1}{2}m(b^2 - a^2)$$

45. Let $\mathbf{F} = 185\,\mathbf{k}$. To parametrize the staircase, let $x = 20\cos t,\ \ y = 20\sin t,\ \ z = \frac{90}{6\pi}t = \frac{15}{\pi}t,\ \ 0 \le t \le 6\pi \ \ \Rightarrow$ $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{6\pi} \langle 0, 0, 185\rangle \cdot \left\langle -20\sin t, 20\cos t, \frac{15}{\pi}\right\rangle dt = (185)\frac{15}{\pi}\int_0^{6\pi} dt = (185)(90) \approx 1.67 \times 10^4$ ft-lb

46. This time m is a function of t: $m = 185 - \frac{9}{6\pi}t = 185 - \frac{3}{2\pi}t$. So let $\mathbf{F} = \left(185 - \frac{3}{2\pi}t\right)\mathbf{k}$. To parametrize the staircase, let $x = 20\cos t,\ \ y = 20\sin t,\ \ z = \frac{90}{6\pi}t = \frac{15}{\pi}t,\ \ 0 \le t \le 6\pi$. Therefore

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{6\pi} \left\langle 0, 0, 185 - \tfrac{3}{2\pi}t\right\rangle \cdot \left\langle -20\sin t, 20\cos t, \tfrac{15}{\pi}\right\rangle dt = \tfrac{15}{\pi}\int_0^{6\pi} \left(185 - \tfrac{3}{2\pi}t\right) dt$$
$$= \tfrac{15}{\pi}\left[185t - \tfrac{3}{4\pi}t^2\right]_0^{6\pi} = 90\left(185 - \tfrac{9}{2}\right) \approx 1.62 \times 10^4 \text{ ft-lb}$$

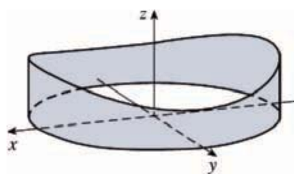
47. (a) $\mathbf{r}(t) = \langle \cos t, \sin t\rangle,\ \ 0 \le t \le 2\pi$, and let $\mathbf{F} = \langle a, b\rangle$. Then

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle a, b\rangle \cdot \langle -\sin t, \cos t\rangle\ dt = \int_0^{2\pi} (-a\sin t + b\cos t)\,dt = \left[a\cos t + b\sin t\right]_0^{2\pi}$$
$$= a + 0 - a + 0 = 0$$

(b) Yes. $\mathbf{F}(x, y) = k\,\mathbf{x} = \langle kx, ky\rangle$ and

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \langle k\cos t, k\sin t\rangle \cdot \langle -\sin t, \cos t\rangle\,dt = \int_0^{2\pi} (-k\sin t\ \cos t + k\sin t\ \cos t)\,dt = \int_0^{2\pi} 0\,dt = 0.$$

48. Consider the base of the fence in the xy-plane, centered at the origin, with the height given by $z = h(x, y)$. The fence can be graphed using the parametric equations $x = 10\cos u,\ \ y = 10\sin u$,



$$z = v\left[4 + 0.01((10\cos u)^2 - (10\sin u)^2)\right] = v(4 + \cos^2 u - \sin^2 u)$$
$$= v(4 + \cos 2u),\ 0 \le u \le 2\pi,\ 0 \le v \le 1.$$

The area of the fence is $\int_C h(x, y)\,ds$ where C, the base of the fence, is given by $x = 10\cos t,\ \ y = 10\sin t,\ \ 0 \le t \le 2\pi$. Then

$$\int_C h(x, y)\,ds = \int_0^{2\pi} \left[4 + 0.01((10\cos t)^2 - (10\sin t)^2)\right]\sqrt{(-10\sin t)^2 + (10\cos t)^2}\,dt$$
$$= \int_0^{2\pi} (4 + \cos 2t)\sqrt{100}\,dt = 10\left[4t + \tfrac{1}{2}\sin 2t\right]_0^{2\pi} = 10(8\pi) = 80\pi \text{ m}^2$$

If we paint both sides of the fence, the total surface area to cover is 160π m^2, and since 1 L of paint covers 100 m^2, we require $\frac{160\pi}{100} = 1.6\pi \approx 5.03$ L of paint.

49. Let $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then

$$\begin{aligned}
\int_C \mathbf{v}\cdot d\mathbf{r} &= \int_a^b \langle v_1, v_2, v_3\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\,dt = \int_a^b [v_1\,x'(t) + v_2\,y'(t) + v_3\,z'(t)]\,dt \\
&= \big[v_1\,x(t) + v_2\,y(t) + v_3\,z(t)\big]_a^b = [v_1\,x(b) + v_2\,y(b) + v_3\,z(b)] - [v_1\,x(a) + v_2\,y(a) + v_3\,z(a)] \\
&= v_1\,[x(b) - x(a)] + v_2\,[y(b) - y(a)] + v_3\,[z(b) - z(a)] \\
&= \langle v_1, v_2, v_3\rangle \cdot \langle x(b) - x(a), y(b) - y(a), z(b) - z(a)\rangle \\
&= \langle v_1, v_2, v_3\rangle \cdot [\langle x(b), y(b), z(b)\rangle - \langle x(a), y(a), z(a)\rangle] = \mathbf{v}\cdot[\mathbf{r}(b) - \mathbf{r}(a)]
\end{aligned}$$

50. If $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ then

$$\begin{aligned}
\int_C \mathbf{r}\cdot d\mathbf{r} &= \int_a^b \langle x(t), y(t), z(t)\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\,dt = \int_a^b [x(t)\,x'(t) + y(t)\,y'(t) + z(t)\,z'(t)]\,dt \\
&= \left[\tfrac{1}{2}[x(t)]^2 + \tfrac{1}{2}[y(t)]^2 + \tfrac{1}{2}[z(t)]^2\right]_a^b \\
&= \tfrac{1}{2}\left\{\left([x(b)]^2 + [y(b)]^2 + [z(b)]^2\right) - \left([x(a)]^2 + [y(a)]^2 + [z(a)]^2\right)\right\} \\
&= \tfrac{1}{2}\left[|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2\right]
\end{aligned}$$

51. The work done in moving the object is $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$. We can approximate this integral by dividing C into 7 segments of equal length $\Delta s = 2$ and approximating $\mathbf{F}\cdot\mathbf{T}$, that is, the tangential component of force, at a point (x_i^*, y_i^*) on each segment. Since C is composed of straight line segments, $\mathbf{F}\cdot\mathbf{T}$ is the scalar projection of each force vector onto C. If we choose (x_i^*, y_i^*) to be the point on the segment closest to the origin, then the work done is

$\int_C \mathbf{F}\cdot\mathbf{T}\,ds \approx \sum\limits_{i=1}^{7} [\mathbf{F}(x_i^*, y_i^*)\cdot\mathbf{T}(x_i^*, y_i^*)]\,\Delta s = [2 + 2 + 2 + 2 + 1 + 1 + 1](2) = 22$. Thus, we estimate the work done to be approximately 22 J.

52. Use the orientation pictured in the figure. Then since $\mathbf{B}$ is tangent to any circle that lies in the plane perpendicular to the wire, $\mathbf{B} = |\mathbf{B}|\,\mathbf{T}$ where $\mathbf{T}$ is the unit tangent to the circle C: $x = r\cos\theta,\;\; y = r\sin\theta$. Thus $\mathbf{B} = |\mathbf{B}|\,\langle -\sin\theta, \cos\theta\rangle$. Then $\int_C \mathbf{B}\cdot d\mathbf{r} = \int_0^{2\pi} |\mathbf{B}|\,\langle -\sin\theta, \cos\theta\rangle \cdot \langle -r\sin\theta, r\cos\theta\rangle\,d\theta = \int_0^{2\pi} |\mathbf{B}|\,r\,d\theta = 2\pi r\,|\mathbf{B}|$. (Note that $|\mathbf{B}|$ here is the magnitude of the field at a distance r from the wire's center.) But by Ampere's Law $\int_C \mathbf{B}\cdot d\mathbf{r} = \mu_0 I$. Hence $|\mathbf{B}| = \mu_0 I/(2\pi r)$.

16.3 The Fundamental Theorem for Line Integrals

1. C appears to be a smooth curve, and since ∇f is continuous, we know f is differentiable. Then Theorem 2 says that the value of $\int_C \nabla f\cdot d\mathbf{r}$ is simply the difference of the values of f at the terminal and initial points of C. From the graph, this is $50 - 10 = 40$.

2. C is represented by the vector function $\mathbf{r}(t) = (t^2 + 1)\,\mathbf{i} + (t^3 + t)\,\mathbf{j}$, $0 \le t \le 1$, so $\mathbf{r}'(t) = 2t\,\mathbf{i} + (3t^2 + 1)\,\mathbf{j}$. Since $3t^2 + 1 \ne 0$, we have $\mathbf{r}'(t) \ne \mathbf{0}$, thus C is a smooth curve. ∇f is continuous, and hence f is differentiable, so by Theorem 2 we have $\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(2, 2) - f(1, 0) = 9 - 3 = 6$.

3. $\partial(2x - 3y)/\partial y = -3 = \partial(-3x + 4y - 8)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so by Theorem 6 $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f = \mathbf{F}$, that is, $f_x(x, y) = 2x - 3y$ and $f_y(x, y) = -3x + 4y - 8$. But $f_x(x, y) = 2x - 3y$ implies $f(x, y) = x^2 - 3xy + g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x, y) = -3x + g'(y)$. Thus $-3x + 4y - 8 = -3x + g'(y)$ so $g'(y) = 4y - 8$ and $g(y) = 2y^2 - 8y + K$ where K is a constant. Hence $f(x, y) = x^2 - 3xy + 2y^2 - 8y + K$ is a potential function for $\mathbf{F}$.

4. $\partial(e^x \sin y)/\partial y = e^x \cos y = \partial(e^x \cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = e^x \sin y$ implies $f(x, y) = e^x \sin y + g(y)$ and $f_y(x, y) = e^x \cos y + g'(y)$. But $f_y(x, y) = e^x \cos y$ so $g'(y) = 0 \quad \Rightarrow \quad g(y) = K$. Then $f(x, y) = e^x \sin y + K$ is a potential function for $\mathbf{F}$.

5. $\partial(e^x \cos y)/\partial y = -e^x \sin y$, $\partial(e^x \sin y)/\partial x = e^x \sin y$. Since these are not equal, $\mathbf{F}$ is not conservative.

6. $\partial(3x^2 - 2y^2)/\partial y = -4y$, $\partial(4xy + 3)/\partial x = 4y$. Since these are not equal, $\mathbf{F}$ is not conservative.

7. $\partial(ye^x + \sin y)/\partial y = e^x + \cos y = \partial(e^x + x\cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = ye^x + \sin y$ implies $f(x, y) = ye^x + x\sin y + g(y)$ and $f_y(x, y) = e^x + x\cos y + g'(y)$. But $f_y(x, y) = e^x + x\cos y$ so $g(y) = K$ and $f(x, y) = ye^x + x\sin y + K$ is a potential function for $\mathbf{F}$.

8. $\partial(2xy + y^{-2})/\partial y = 2x - 2y^{-3} = \partial(x^2 - 2xy^{-3})/\partial x$ and the domain of $\mathbf{F}$ is $\{(x, y) \mid y > 0\}$ which is open and simply-connected. Hence $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = 2xy + y^{-2}$ implies $f(x, y) = x^2 y + xy^{-2} + g(y)$ and $f_y(x, y) = x^2 - 2xy^{-3} + g'(y)$. But $f_y(x, y) = x^2 - 2xy^{-3}$ so $g'(y) = 0 \quad \Rightarrow \quad g(y) = K$. Then $f(x, y) = x^2 y + xy^{-2} + K$ is a potential function for $\mathbf{F}$.

9. $\partial(\ln y + 2xy^3)/\partial y = 1/y + 6xy^2 = \partial(3x^2 y^2 + x/y)/\partial x$ and the domain of $\mathbf{F}$ is $\{(x, y) \mid y > 0\}$ which is open and simply connected. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = \ln y + 2xy^3$ implies $f(x, y) = x\ln y + x^2 y^3 + g(y)$ and $f_y(x, y) = x/y + 3x^2 y^2 + g'(y)$. But $f_y(x, y) = 3x^2 y^2 + x/y$ so $g'(y) = 0 \quad \Rightarrow \quad g(y) = K$ and $f(x, y) = x\ln y + x^2 y^3 + K$ is a potential function for $\mathbf{F}$.

10. $$\frac{\partial(xy\cosh xy + \sinh xy)}{\partial y} = x^2 y \sinh xy + x\cosh xy + x\cosh xy = x^2 y\sinh xy + 2x\cosh xy = \frac{\partial(x^2\cosh xy)}{\partial x}$$

and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Thus $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = xy\cosh xy + \sinh xy$ implies $f(x, y) = x\sinh xy + g(y) \quad \Rightarrow \quad f_y(x, y) = x^2\cosh xy + g'(y)$. But $f_y(x, y) = x^2 \cosh xy$ so $g(y) = K$ and $f(x, y) = x\sinh xy + K$ is a potential function for $\mathbf{F}$.

11. (a) $\mathbf{F}$ has continuous first-order partial derivatives and $\dfrac{\partial}{\partial y}\, 2xy = 2x = \dfrac{\partial}{\partial x}\,(x^2)$ on $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative by Theorem 6. Then we know that the line integral of $\mathbf{F}$ is independent of path; in particular, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of C. Since all three curves have the same initial and terminal points, $\int_C \mathbf{F} \cdot d\mathbf{r}$ will have the same value for each curve.

(b) We first find a potential function f, so that $\nabla f = \mathbf{F}$. We know $f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$. Integrating $f_x(x,y)$ with respect to x, we have $f(x,y) = x^2y + g(y)$. Differentiating both sides with respect to y gives $f_y(x,y) = x^2 + g'(y)$, so we must have $x^2 + g'(y) = x^2 \quad \Rightarrow \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = K$, a constant. Thus $f(x,y) = x^2y + K$. All three curves start at $(1,2)$ and end at $(3,2)$, so by Theorem 2, $\int_C \mathbf{F} \cdot d\mathbf{r} = f(3,2) - f(1,2) = 18 - 2 = 16$ for each curve.

12. (a) $f_x(x,y) = x^2$ implies $f(x,y) = \frac{1}{3}x^3 + g(y)$ and $f_y(x,y) = 0 + g'(y)$. But $f_y(x,y) = y^2$ so $g'(y) = y^2 \quad \Rightarrow \quad g(y) = \frac{1}{3}y^3 + K$. We can take $K = 0$, so $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$.

(b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(2,8) - f(-1,2) = \left(\frac{8}{3} + \frac{512}{3}\right) - \left(-\frac{1}{3} + \frac{8}{3}\right) = 171$.

13. (a) $f_x(x,y) = xy^2$ implies $f(x,y) = \frac{1}{2}x^2y^2 + g(y)$ and $f_y(x,y) = x^2y + g'(y)$. But $f_y(x,y) = x^2y$ so $g'(y) = 0 \quad \Rightarrow$ $g(y) = K$, a constant. We can take $K = 0$, so $f(x,y) = \frac{1}{2}x^2y^2$.

(b) The initial point of C is $\mathbf{r}(0) = (0,1)$ and the terminal point is $\mathbf{r}(1) = (2,1)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(2,1) - f(0,1) = 2 - 0 = 2$.

14. (a) $f_y(x,y) = x^2e^{xy}$ implies $f(x,y) = xe^{xy} + g(x) \quad \Rightarrow \quad f_x(x,y) = xye^{xy} + e^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$. But $f_x(x,y) = (1+xy)e^{xy}$ so $g'(x) = 0 \quad \Rightarrow \quad g(x) = K$. We can take $K = 0$, so $f(x,y) = xe^{xy}$.

(b) The initial point of C is $\mathbf{r}(0) = (1,0)$ and the terminal point is $\mathbf{r}(\pi/2) = (0,2)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(0,2) - f(1,0) = 0 - e^0 = -1$.

15. (a) $f_x(x,y,z) = yz$ implies $f(x,y,z) = xyz + g(y,z)$ and so $f_y(x,y,z) = xz + g_y(y,z)$. But $f_y(x,y,z) = xz$ so $g_y(y,z) = 0 \quad \Rightarrow \quad g(y,z) = h(z)$. Thus $f(x,y,z) = xyz + h(z)$ and $f_z(x,y,z) = xy + h'(z)$. But $f_z(x,y,z) = xy + 2z$, so $h'(z) = 2z \quad \Rightarrow \quad h(z) = z^2 + K$. Hence $f(x,y,z) = xyz + z^2$ (taking $K = 0$).

(b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(4,6,3) - f(1,0,-2) = 81 - 4 = 77$.

16. (a) $f_x(x,y,z) = y^2z + 2xz^2$ implies $f(x,y,z) = xy^2z + x^2z^2 + g(y,z)$ and so $f_y(x,y,z) = 2xyz + g_y(y,z)$. But $f_y(x,y,z) = 2xyz$ so $g_y(y,z) = 0 \quad \Rightarrow \quad g(y,z) = h(z)$. Thus $f(x,y,z) = xy^2z + x^2z^2 + h(z)$ and $f_z(x,y,z) = xy^2 + 2x^2z + h'(z)$. But $f_z(x,y,z) = xy^2 + 2x^2z$, so $h'(z) = 0 \quad \Rightarrow \quad h(z) = K$. Hence $f(x,y,z) = xy^2z + x^2z^2$ (taking $K = 0$).

(b) $t = 0$ corresponds to the point $(0,1,0)$ and $t = 1$ corresponds to $(1,2,1)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,2,1) - f(0,1,0) = 5 - 0 = 5$.

17. (a) $f_x(x,y,z) = yze^{xz}$ implies $f(x,y,z) = ye^{xz} + g(y,z)$ and so $f_y(x,y,z) = e^{xz} + g_y(y,z)$. But $f_y(x,y,z) = e^{xz}$ so $g_y(y,z) = 0 \quad \Rightarrow \quad g(y,z) = h(z)$. Thus $f(x,y,z) = ye^{xz} + h(z)$ and $f_z(x,y,z) = xye^{xz} + h'(z)$. But $f_z(x,y,z) = xye^{xz}$, so $h'(z) = 0 \quad \Rightarrow \quad h(z) = K$. Hence $f(x,y,z) = ye^{xz}$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 1,-1,0 \rangle$, $\mathbf{r}(2) = \langle 5,3,0 \rangle$ so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(5,3,0) - f(1,-1,0) = 3e^0 + e^0 = 4$.

18. (a) $f_x(x,y,z) = \sin y$ implies $f(x,y,z) = x\sin y + g(y,z)$ and so $f_y(x,y,z) = x\cos y + g_y(y,z)$. But $f_y(x,y,z) = x\cos y + \cos z$ so $g_y(y,z) = \cos z \;\Rightarrow\; g(y,z) = y\cos z + h(z)$. Thus $f(x,y,z) = x\sin y + y\cos z + h(z)$ and $f_z(x,y,z) = -y\sin z + h'(z)$. But $f_z(x,y,z) = -y\sin z$, so $h'(z) = 0 \;\Rightarrow\; h(z) = K$. Hence $f(x,y,z) = x\sin y + y\cos z$ (taking $K = 0$).

(b) $\mathbf{r}(0) = \langle 0,0,0\rangle$, $\mathbf{r}(\pi/2) = \langle 1,\pi/2,\pi\rangle$ so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,\pi/2,\pi) - f(0,0,0) = 1 - \frac{\pi}{2} - 0 = 1 - \frac{\pi}{2}$.

19. The functions $2xe^{-y}$ and $2y - x^2e^{-y}$ have continuous first-order derivatives on $\mathbb{R}^2$ and $\dfrac{\partial}{\partial y}\left(2xe^{-y}\right) = -2xe^{-y} = \dfrac{\partial}{\partial x}\left(2y - x^2e^{-y}\right)$, so $\mathbf{F}(x,y) = 2xe^{-y}\,\mathbf{i} + \left(2y - x^2e^{-y}\right)\mathbf{j}$ is a conservative vector field by Theorem 6 and hence the line integral is independent of path. Thus a potential function f exists, and $f_x(x,y) = 2xe^{-y}$ implies $f(x,y) = x^2e^{-y} + g(y)$ and $f_y(x,y) = -x^2e^{-y} + g'(y)$. But $f_y(x,y) = 2y - x^2e^{-y}$ so $g'(y) = 2y \;\Rightarrow\; g(y) = y^2 + K$. We can take $K = 0$, so $f(x,y) = x^2e^{-y} + y^2$. Then $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy = f(2,1) - f(1,0) = 4e^{-1} + 1 - 1 = 4/e$.

20. The functions $\sin y$ and $x\cos y - \sin y$ have continuous first-order derivatives on $\mathbb{R}^2$ and $\dfrac{\partial}{\partial y}(\sin y) = \cos y = \dfrac{\partial}{\partial x}(x\cos y - \sin y)$, so $\mathbf{F}(x,y) = \sin y\,\mathbf{i} + (x\cos y - \sin y)\,\mathbf{j}$ is a conservative vector field by Theorem 6 and hence the line integral is independent of path. Thus a potential function f exists, and $f_x(x,y) = \sin y$ implies $f(x,y) = x\sin y + g(y)$ and $f_y(x,y) = x\cos y + g'(y)$. But $f_y(x,y) = x\cos y - \sin y$ so $g'(y) = -\sin y \;\Rightarrow\; g(y) = \cos y + K$. We can take $K = 0$, so $f(x,y) = x\sin y + \cos y$. Then $\int_C \sin y\,dx + (x\cos y - \sin y)\,dy = f(1,\pi) - f(2,0) = -1 - 1 = -2$.

21. If $\mathbf{F}$ is conservative, then $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path. This means that the work done along all piecewise-smooth curves that have the described initial and terminal points is the same. Your reply: It doesn't matter which curve is chosen.

22. The curves C_1 and C_2 connect the same two points but $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} \neq \int_{C_2}\mathbf{F}\cdot d\mathbf{r}$. Thus $\mathbf{F}$ is not independent of path, and therefore is not conservative.

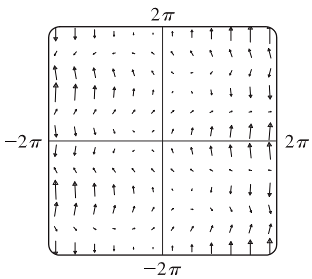
23. $\mathbf{F}(x,y) = 2y^{3/2}\,\mathbf{i} + 3x\sqrt{y}\,\mathbf{j}$, $W = \int_C \mathbf{F}\cdot d\mathbf{r}$. Since $\partial(2y^{3/2})/\partial y = 3\sqrt{y} = \partial(3x\sqrt{y}\,)/\partial x$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x(x,y) = 2y^{3/2} \;\Rightarrow\; f(x,y) = 2xy^{3/2} + g(y) \;\Rightarrow\; f_y(x,y) = 3xy^{1/2} + g'(y)$. But $f_y(x,y) = 3x\sqrt{y}$ so $g'(y) = 0$ or $g(y) = K$. We can take $K = 0 \;\Rightarrow\; f(x,y) = 2xy^{3/2}$. Thus $W = \int_C \mathbf{F}\cdot d\mathbf{r} = f(2,4) - f(1,1) = 2(2)(8) - 2(1) = 30$.

24. $\mathbf{F}(x,y) = e^{-y}\,\mathbf{i} - xe^{-y}\,\mathbf{j}$, $W = \int_C \mathbf{F}\cdot d\mathbf{r}$. Since $\dfrac{\partial}{\partial y}\left(e^{-y}\right) = -e^{-y} = \dfrac{\partial}{\partial x}\left(-xe^{-y}\right)$, there exists a function f such that $\nabla f = \mathbf{F}$. In fact, $f_x = e^{-y} \;\Rightarrow\; f(x,y) = xe^{-y} + g(y) \;\Rightarrow\; f_y = -xe^{-y} + g'(y) \;\Rightarrow\; g'(y) = 0$, so we can take $f(x,y) = xe^{-y}$ as a potential function for $\mathbf{F}$. Thus $W = \int_C \mathbf{F}\cdot d\mathbf{r} = f(2,0) - f(0,1) = 2 - 0 = 2$.

25. We know that if the vector field (call it $\mathbf{F}$) is conservative, then around any closed path C, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. But take C to be a circle centered at the origin, oriented counterclockwise. All of the field vectors that start on C are roughly in the direction of motion along C, so the integral around C will be positive. Therefore the field is not conservative.

26. If a vector field $\mathbf{F}$ is conservative, then around any closed path C, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. For any closed path we draw in the field, it appears that some vectors on the curve point in approximately the same direction as the curve and a similar number point in roughly the opposite direction. (Some appear perpendicular to the curve as well.) Therefore it is plausible that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C which means $\mathbf{F}$ is conservative.

27.



From the graph, it appears that $\mathbf{F}$ is conservative, since around all closed paths, the number and size of the field vectors pointing in directions similar to that of the path seem to be roughly the same as the number and size of the vectors pointing in the opposite direction. To check, we calculate $\dfrac{\partial}{\partial y}(\sin y) = \cos y = \dfrac{\partial}{\partial x}(1 + x\cos y)$. Thus $\mathbf{F}$ is conservative, by Theorem 6.

28. $\nabla f(x, y) = \cos(x - 2y)\,\mathbf{i} - 2\cos(x - 2y)\,\mathbf{j}$

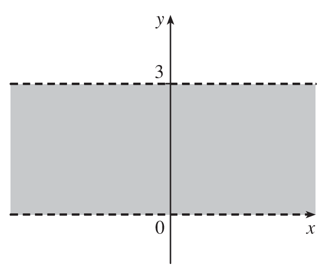
(a) We use Theorem 2: $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C_1 starts at $t = a$ and ends at $t = b$. So because $f(0, 0) = \sin 0 = 0$ and $f(\pi, \pi) = \sin(\pi - 2\pi) = 0$, one possible curve C_1 is the straight line from $(0, 0)$ to (π, π); that is, $\mathbf{r}(t) = \pi t\,\mathbf{i} + \pi t\,\mathbf{j}$, $0 \le t \le 1$.

(b) From (a), $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So because $f(0, 0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}, 0\right) = 1$, one possible curve C_2 is $\mathbf{r}(t) = \frac{\pi}{2} t\,\mathbf{i}$, $0 \le t \le 1$, the straight line from $(0, 0)$ to $\left(\frac{\pi}{2}, 0\right)$.

29. Since $\mathbf{F}$ is conservative, there exists a function f such that $\mathbf{F} = \nabla f$, that is, $P = f_x$, $Q = f_y$, and $R = f_z$. Since P, Q, and R have continuous first order partial derivatives, Clairaut's Theorem says that $\partial P/\partial y = f_{xy} = f_{yx} = \partial Q/\partial x$, $\partial P/\partial z = f_{xz} = f_{zx} = \partial R/\partial x$, and $\partial Q/\partial z = f_{yz} = f_{zy} = \partial R/\partial y$.

30. Here $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + xyz\,\mathbf{k}$. Then using the notation of Exercise 29 , $\partial P/\partial z = 0$ while $\partial R/\partial x = yz$. Since these aren't equal, $\mathbf{F}$ is not conservative. Thus by Theorem 4, the line integral of $\mathbf{F}$ is not independent of path.

31. $D = \{(x, y) \mid 0 < y < 3\}$ consists of those points between, but not on, the horizontal lines $y = 0$ and $y = 3$.

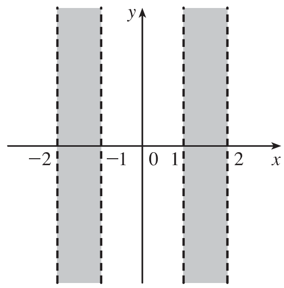


(a) Since D does not include any of its boundary points, it is open. More formally, at any point in D there is a disk centered at that point that lies entirely in D.

(b) Any two points chosen in D can always be joined by a path that lies entirely in D, so D is connected. (D consists of just one "piece.")

(c) D is connected and it has no holes, so it's simply-connected. (Every simple closed curve in D encloses only points that are in D.)

32. $D = \{(x, y) \mid 1 < |x| < 2\}$ consists of those points between, but not on, the vertical lines $x = 1$ and $x = 2$, together with the points between the vertical lines $x = -1$ and $x = -2$.

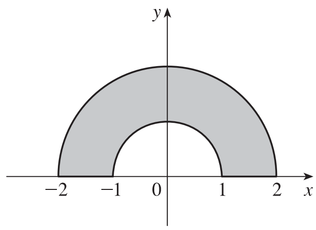


(a) The region does not include any of its boundary points, so it is open.

(b) D consists of two separate pieces, so it is not connected. [For instance, both the points $(-1.5, 0)$ and $(1.5, 0)$ lie in D but they cannot be joined by a path that lies entirely in D.]

(c) Because D is not connected, it's not simply-connected.

33. $D = \left\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4,\ y \geq 0\right\}$ is the semiannular region in the upper half-plane between circles centered at the origin of radii 1 and 2 (including all boundary points).

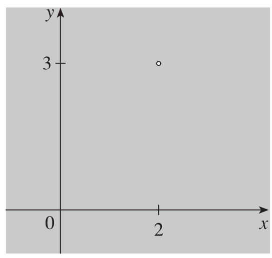


(a) D includes boundary points, so it is not open. [Note that at any boundary point, $(1, 0)$ for instance, any disk centered there cannot lie entirely in D.]

(b) The region consists of one piece, so it's connected.

(c) D is connected and has no holes, so it's simply-connected.

34. $D = \{(x, y) \mid (x, y) \neq (2, 3)\}$ consists of all points in the xy-plane except for $(2, 3)$.



(a) D has only one boundary point, namely $(2, 3)$, which is not included, so the region is open.

(b) D is connected, as it consists of only one piece.

(c) D is not simply-connected, as it has a hole at $(2, 3)$. Thus any simple closed curve that encloses $(2, 3)$ lies in D but includes a point that is not in D.

35. (a) $P = -\dfrac{y}{x^2 + y^2}$, $\dfrac{\partial P}{\partial y} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$ and $Q = \dfrac{x}{x^2 + y^2}$, $\dfrac{\partial Q}{\partial x} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$. Thus $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) C_1: $x = \cos t$, $y = \sin t$, $0 \leq t \leq \pi$, C_2: $x = \cos t$, $y = \sin t$, $t = 2\pi$ to $t = \pi$. Then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \frac{(-\sin t)(-\sin t) + (\cos t)(\cos t)}{\cos^2 t + \sin^2 t}\, dt = \int_0^\pi dt = \pi \text{ and } \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^{\pi} dt = -\pi$$

Since these aren't equal, the line integral of $\mathbf{F}$ isn't independent of path. (Or notice that $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi$ where C_3 is the circle $x^2 + y^2 = 1$, and apply the contrapositive of Theorem 3.) This doesn't contradict Theorem 6, since the domain of $\mathbf{F}$, which is $\mathbb{R}^2$ except the origin, isn't simply-connected.

36. (a) Here $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Then $f(\mathbf{r}) = -c/|\mathbf{r}|$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$. (See the discussion of gradient fields in Section 16.1.) Hence $\mathbf{F}$ is conservative and its line integral is independent of path. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = f(P_2) - f(P_1) = -\frac{c}{(x_2^2 + y_2^2 + z_2^2)^{1/2}} + \frac{c}{(x_1^2 + y_1^2 + z_1^2)^{1/2}} = c\left(\frac{1}{d_1} - \frac{1}{d_2}\right).$$

(b) In this case, $c = -(mMG) \quad \Rightarrow$

$$W = -mMG\left(\frac{1}{1.52 \times 10^{11}} - \frac{1}{1.47 \times 10^{11}}\right)$$
$$= -(5.97 \times 10^{24})(1.99 \times 10^{30})(6.67 \times 10^{-11})(-2.2377 \times 10^{-13}) \approx 1.77 \times 10^{32} \text{ J}$$

(c) In this case, $c = \epsilon q Q \quad \Rightarrow$

$$W = \epsilon q Q\left(\frac{1}{10^{-12}} - \frac{1}{5 \times 10^{-13}}\right) = \left(8.985 \times 10^9\right)(1)\left(-1.6 \times 10^{-19}\right)\left(-10^{12}\right) \approx 1400 \text{ J.}$$

16.4 Green's Theorem

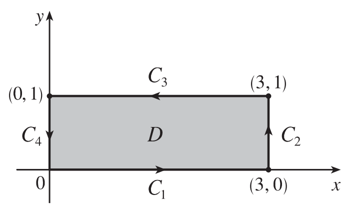
1. (a) Parametric equations for C are $x = 2\cos t,\ \ y = 2\sin t,\ \ 0 \le t \le 2\pi$. Then

$$\oint_C (x - y)\,dx + (x + y)\,dy = \int_0^{2\pi} [(2\cos t - 2\sin t)(-2\sin t) + (2\cos t + 2\sin t)(2\cos t)]\,dt$$
$$= \int_0^{2\pi} (4\sin^2 t + 4\cos^2 t)\,dt = \int_0^{2\pi} 4\,dt = 4t\Big]_0^{2\pi} = 8\pi$$

(b) Note that C as given in part (a) is a positively oriented, smooth, simple closed curve. Then by Green's Theorem,

$$\oint_C (x - y)\,dx + (x + y)\,dy = \iint_D \left[\frac{\partial}{\partial x}(x + y) - \frac{\partial}{\partial y}(x - y)\right] dA = \iint_D [1 - (-1)]\,dA = 2\iint_D dA$$
$$= 2A(D) = 2\pi(2)^2 = 8\pi$$

2. (a)



C_1: $x = t \quad \Rightarrow \quad dx = dt,\ \ y = 0 \quad \Rightarrow \quad dy = 0\,dt,\ \ 0 \le t \le 3.$

C_2: $x = 3 \quad \Rightarrow \quad dx = 0\,dt,\ \ y = t \quad \Rightarrow \quad dy = dt,\ \ 0 \le t \le 1.$

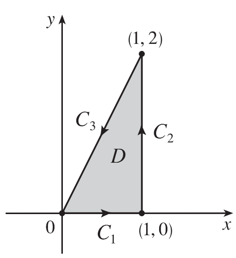
C_3: $x = 3 - t \quad \Rightarrow \quad dx = -dt,\ \ y = 1 \quad \Rightarrow \quad dy = 0\,dt,\ \ 0 \le t \le 3.$

C_4: $x = 0 \quad \Rightarrow \quad dx = 0\,dt,\ \ y = 1 - t \quad \Rightarrow \quad dy = -dt,\ \ 0 \le t \le 1$

Thus $$\oint_C xy\,dx + x^2\,dy = \oint_{C_1 + C_2 + C_3 + C_4} xy\,dx + x^2\,dy = \int_0^3 0\,dt + \int_0^1 9\,dt + \int_0^3 (3 - t)(-1)\,dt + \int_0^1 0\,dt$$
$$= \big[9t\big]_0^1 + \left[\tfrac{1}{2}t^2 - 3t\right]_0^3 = 9 + \tfrac{9}{2} - 9 = \tfrac{9}{2}$$

(b) $\oint_C xy\,dx + x^2\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^3 \int_0^1 (2x - x)\,dy\,dx = \int_0^3 x\,dx \int_0^1 dy = \left[\frac{1}{2}x^2\right]_0^3 \cdot 1 = \frac{9}{2}$

3. (a)



C_1: $x = t \quad \Rightarrow \quad dx = dt,\ \ y = 0 \quad \Rightarrow \quad dy = 0\,dt,\ \ 0 \le t \le 1.$

C_2: $x = 1 \quad \Rightarrow \quad dx = 0\,dt,\ \ y = t \quad \Rightarrow \quad dy = dt,\ \ 0 \le t \le 2.$

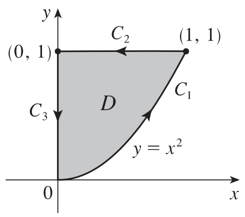
C_3: $x = 1 - t \quad \Rightarrow \quad dx = -dt,\ \ y = 2 - 2t \quad \Rightarrow \quad dy = -2\,dt,\ \ 0 \le t \le 1.$

Thus
$$\oint_C xy\,dx + x^2y^3\,dy = \oint_{C_1+C_2+C_3} xy\,dx + x^2y^3\,dy$$
$$= \int_0^1 0\,dt + \int_0^2 t^3\,dt + \int_0^1 \left[-(1-t)(2-2t) - 2(1-t)^2(2-2t)^3\right] dt$$
$$= 0 + \left[\tfrac{1}{4}t^4\right]_0^2 + \left[\tfrac{2}{3}(1-t)^3 + \tfrac{8}{3}(1-t)^6\right]_0^1 = 4 - \tfrac{10}{3} = \tfrac{2}{3}$$

(b) $\oint_C xy\,dx + x^2y^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(xy)\right] dA = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$

$$= \int_0^1 \left[\tfrac{1}{2}xy^4 - xy\right]_{y=0}^{y=2x} dx = \int_0^1 (8x^5 - 2x^2)\,dx = \tfrac{4}{3} - \tfrac{2}{3} = \tfrac{2}{3}$$

4\. (a)



C_1: $x = t \;\Rightarrow\; dx = dt,\; y = t^2 \;\Rightarrow\; dy = 2t\,dt,\; 0 \le t \le 1$

C_2: $x = 1 - t \;\Rightarrow\; dx = -dt,\; y = 1 \;\Rightarrow\; dy = 0\,dt,\; 0 \le t \le 1$

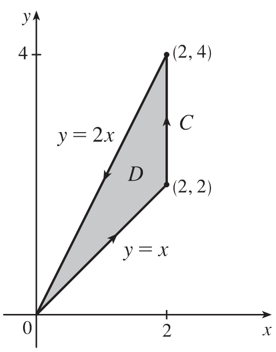
C_3: $x = 0 \;\Rightarrow\; dx = 0\,dt,\; y = 1 - t \;\Rightarrow\; dy = -dt,\; 0 \le t \le 1$

Thus
$$\oint_C x^2y^2\,dx + xy\,dy = \oint_{C_1+C_2+C_3} x^2y^2\,dx + xy\,dy$$
$$= \int_0^1 \left[t^2(t^2)^2\,dt + t(t^2)(2t\,dt)\right] + \int_0^1 \left[(1-t)^2(1)^2(-dt) + (1-t)(1)(0\,dt)\right]$$
$$+ \int_0^1 \left[(0)^2(1-t)^2(0\,dt) + (0)(1-t)(-dt)\right]$$
$$= \int_0^1 (t^6 + 2t^4)\,dt + \int_0^1 (-1 + 2t - t^2)\,dt + \int_0^1 0\,dt$$
$$= \left[\tfrac{1}{7}t^7 + \tfrac{2}{5}t^5\right]_0^1 + \left[-t + t^2 - \tfrac{1}{3}t^3\right]_0^1 + 0 = \left(\tfrac{1}{7} + \tfrac{2}{5}\right) + \left(-1 + 1 - \tfrac{1}{3}\right) = \tfrac{22}{105}$$

(b) $\oint_C x^2y^2\,dx + xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2y^2)\right] dA = \int_0^1 \int_{x^2}^1 (y - 2x^2y)\,dy\,dx$

$$= \int_0^1 \left[\tfrac{1}{2}y^2 - x^2y^2\right]_{y=x^2}^{y=1} dx = \int_0^1 \left(\tfrac{1}{2} - x^2 - \tfrac{1}{2}x^4 + x^6\right) dx$$
$$= \left[\tfrac{1}{2}x - \tfrac{1}{3}x^3 - \tfrac{1}{10}x^5 + \tfrac{1}{7}x^7\right]_0^1 = \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{10} + \tfrac{1}{7} = \tfrac{22}{105}$$

5\.



The region D enclosed by C is given by $\{(x, y) \mid 0 \le x \le 2, x \le y \le 2x\}$, so

$$\int_C xy^2\,dx + 2x^2y\,dy = \iint_D \left[\frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(xy^2)\right] dA$$
$$= \int_0^2 \int_x^{2x} (4xy - 2xy)\,dy\,dx$$
$$= \int_0^2 \left[xy^2\right]_{y=x}^{y=2x} dx$$
$$= \int_0^2 3x^3\,dx = \tfrac{3}{4}x^4\Big]_0^2 = 12$$

6\. The region D enclosed by C is $[0, 5] \times [0, 2]$, so

$$\int_C \cos y\,dx + x^2 \sin y\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2 \sin y) - \frac{\partial}{\partial y}(\cos y)\right] dA = \int_0^5 \int_0^2 [2x \sin y - (-\sin y)]\,dy\,dx$$
$$= \int_0^5 (2x + 1)\,dx \int_0^2 \sin y\,dy = \left[x^2 + x\right]_0^5 \left[-\cos y\right]_0^2 = 30(1 - \cos 2)$$

7. $\int_C \left(y + e^{\sqrt{x}}\right) dx + (2x + \cos y^2)\, dy = \iint_D \left[\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}\left(y + e^{\sqrt{x}}\right)\right] dA$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (2-1)\, dx\, dy = \int_0^1 (y^{1/2} - y^2)\, dy = \tfrac{1}{3}$$

8. $\int_C y^4\, dx + 2xy^3\, dy = \iint_D \left[\frac{\partial}{\partial x}(2xy^3) - \frac{\partial}{\partial y}(y^4)\right] dA = \iint_D (2y^3 - 4y^3)\, dA$

$$= -2 \iint_D y^3\, dA = 0$$

because $f(x,y) = y^3$ is an odd function with respect to y and D is symmetric about the x-axis.

9. $\int_C y^3\, dx - x^3\, dy = \iint_D \left[\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3)\right] dA = \iint_D (-3x^2 - 3y^2)\, dA = \int_0^{2\pi} \int_0^2 (-3r^2)\, r\, dr\, d\theta$

$$= -3 \int_0^{2\pi} d\theta \int_0^2 r^3\, dr = -3(2\pi)(4) = -24\pi$$

10. $\int_C (1 - y^3)\, dx + (x^3 + e^{y^2})\, dy = \iint_D \left[\frac{\partial}{\partial x}(x^3 + e^{y^2}) - \frac{\partial}{\partial y}(1 - y^3)\right] dA = \iint_D (3x^2 + 3y^2)\, dA$

$$= \int_0^{2\pi} \int_2^3 (3r^2)\, r\, dr\, d\theta = 3 \int_0^{2\pi} d\theta \int_2^3 r^3\, dr$$

$$= 3\big[\theta\big]_0^{2\pi} \left[\tfrac{1}{4} r^4\right]_2^3 = 3(2\pi) \cdot \tfrac{1}{4}(81 - 16) = \tfrac{195}{2}\pi$$

11. $\mathbf{F}(x,y) = \langle y\cos x - xy\sin x, xy + x\cos x\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 2, 0 \le y \le 4 - 2x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} (y\cos x - xy\sin x)\, dx + (xy + x\cos x)\, dy = -\iint_D \left[\frac{\partial}{\partial x}(xy + x\cos x) - \frac{\partial}{\partial y}(y\cos x - xy\sin x)\right] dA$$

$$= -\iint_D (y - x\sin x + \cos x - \cos x + x\sin x)\, dA = -\int_0^2 \int_0^{4-2x} y\, dy\, dx$$

$$= -\int_0^2 \left[\tfrac{1}{2} y^2\right]_{y=0}^{y=4-2x} dx = -\int_0^2 \tfrac{1}{2}(4-2x)^2\, dx = -\int_0^2 (8 - 8x + 2x^2)\, dx = -\left[8x - 4x^2 + \tfrac{2}{3}x^3\right]_0^2$$

$$= -\left(16 - 16 + \tfrac{16}{3} - 0\right) = -\tfrac{16}{3}$$

12. $\mathbf{F}(x,y) = \left\langle e^{-x} + y^2, e^{-y} + x^2\right\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid -\pi/2 \le x \le \pi/2, 0 \le y \le \cos x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \left(e^{-x} + y^2\right) dx + \left(e^{-y} + x^2\right) dy = -\iint_D \left[\frac{\partial}{\partial x}\left(e^{-y} + x^2\right) - \frac{\partial}{\partial y}\left(e^{-x} + y^2\right)\right] dA$$

$$= -\int_{-\pi/2}^{\pi/2} \int_0^{\cos x} (2x - 2y)\, dy\, dx = -\int_{-\pi/2}^{\pi/2} \left[2xy - y^2\right]_{y=0}^{y=\cos x} dx$$

$$= -\int_{-\pi/2}^{\pi/2} (2x\cos x - \cos^2 x)\, dx = -\int_{-\pi/2}^{\pi/2} \left(2x\cos x - \tfrac{1}{2}(1 + \cos 2x)\right) dx$$

$$= -\left[2x\sin x + 2\cos x - \tfrac{1}{2}\left(x + \tfrac{1}{2}\sin 2x\right)\right]_{-\pi/2}^{\pi/2} \qquad \text{[integrate by parts in the first term]}$$

$$= -\left(\pi - \tfrac{1}{4}\pi - \pi - \tfrac{1}{4}\pi\right) = \tfrac{1}{2}\pi$$

13. $\mathbf{F}(x,y) = \langle y - \cos y, x\sin y\rangle$ and the region D enclosed by C is the disk with radius 2 centered at $(3, -4)$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} (y - \cos y)\, dx + (x\sin y)\, dy = -\iint_D \left[\frac{\partial}{\partial x}(x\sin y) - \frac{\partial}{\partial y}(y - \cos y)\right] dA$$

$$= -\iint_D (\sin y - 1 - \sin y)\, dA = \iint_D dA = \text{area of } D = \pi(2)^2 = 4\pi$$

14. $\mathbf{F}(x,y) = \left\langle \sqrt{x^2+1}, \tan^{-1} x \right\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 1, x \le y \le 1\}$.

C is oriented positively, so

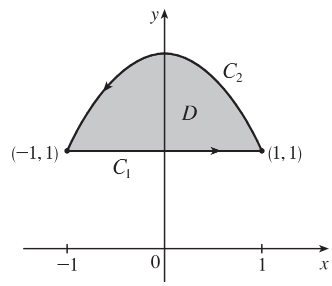
$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \sqrt{x^2+1}\,dx + \tan^{-1} x\,dy &= \iint_D \left[\frac{\partial}{\partial x}\left(\tan^{-1} x\right) - \frac{\partial}{\partial y}\left(\sqrt{x^2+1}\right)\right] dA \\
&= \int_0^1 \int_x^1 \left(\frac{1}{1+x^2} - 0\right) dy\,dx = \int_0^1 \frac{1}{1+x^2}\,[y]_{y=x}^{y=1}\,dx = \int_0^1 \frac{1}{1+x^2}(1-x)\,dx \\
&= \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right) dx = \left[\tan^{-1} x - \frac{1}{2}\ln(1+x^2)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2
\end{aligned}$$

15. Here $C = C_1 + C_2$ where

C_1 can be parametrized as $x = t,\ \ y = 1,\ \ -1 \le t \le 1$, and

C_2 is given by $x = -t,\ \ y = 2 - t^2,\ \ -1 \le t \le 1$.

Then the line integral is



$$\begin{aligned}
\oint_{C_1+C_2} y^2 e^x\,dx + x^2 e^y\,dy &= \int_{-1}^1 [1 \cdot e^t + t^2 e \cdot 0]\,dt \\
&\quad + \int_{-1}^1 [(2-t^2)^2 e^{-t}(-1) + (-t)^2 e^{2-t^2}(-2t)]\,dt \\
&= \int_{-1}^1 [e^t - (2-t^2)^2 e^{-t} - 2t^3 e^{2-t^2}]\,dt = -8e + 48e^{-1}
\end{aligned}$$

according to a CAS. The double integral is

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{-1}^1 \int_1^{2-x^2} (2xe^y - 2ye^x)\,dy\,dx = -8e + 48e^{-1},$$ verifying Green's Theorem in this case.

16. We can parametrize C as $x = \cos\theta,\ \ y = 2\sin\theta,\ \ 0 \le \theta \le 2\pi$. Then the line integral is

$$\begin{aligned}
\oint_C P\,dx + Q\,dy &= \int_0^{2\pi} \left[2\cos\theta - (\cos\theta)^3(2\sin\theta)^5\right](-\sin\theta)\,d\theta + \int_0^{2\pi} (\cos\theta)^3(2\sin\theta)^8 \cdot 2\cos\theta\,d\theta \\
&= \int_0^{2\pi} [-2\cos\theta\sin\theta + 32\cos^3\theta\sin^6\theta + 512\cos^4\theta\sin^8\theta]\,d\theta = 7\pi,
\end{aligned}$$

according to a CAS. The double integral is $\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{-1}^1 \int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} (3x^2y^8 + 5x^3y^4)\,dy\,dx = 7\pi.$

17. By Green's Theorem, $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x(x+y)\,dx + xy^2\,dy = \iint_D (y^2 - x)\,dA$ where C is the path described in the question and D is the triangle bounded by C. So

$$\begin{aligned}
W &= \int_0^1 \int_0^{1-x} (y^2 - x)\,dy\,dx = \int_0^1 \left[\tfrac{1}{3}y^3 - xy\right]_{y=0}^{y=1-x} dx = \int_0^1 \left(\tfrac{1}{3}(1-x)^3 - x(1-x)\right) dx \\
&= \left[-\tfrac{1}{12}(1-x)^4 - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3\right]_0^1 = \left(-\tfrac{1}{2} + \tfrac{1}{3}\right) - \left(-\tfrac{1}{12}\right) = -\tfrac{1}{12}
\end{aligned}$$

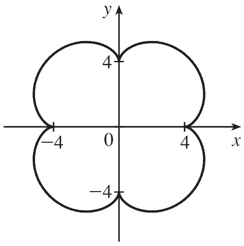
18. By Green's Theorem, $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x\,dx + (x^3 + 3xy^2)\,dy = \iint_D (3x^2 + 3y^2 - 0)\,dA$, where D is the semicircular region bounded by C. Converting to polar coordinates, we have $W = 3\int_0^2 \int_0^\pi r^2 \cdot r\,d\theta\,dr = 3\pi\left[\frac{1}{4}r^4\right]_0^2 = 12\pi$.

19. Let C_1 be the arch of the cycloid from $(0,0)$ to $(2\pi, 0)$, which corresponds to $0 \le t \le 2\pi$, and let C_2 be the segment from $(2\pi, 0)$ to $(0,0)$, so C_2 is given by $x = 2\pi - t$, $y = 0$, $0 \le t \le 2\pi$. Then $C = C_1 \cup C_2$ is traversed clockwise, so $-C$ is

oriented positively. Thus $-C$ encloses the area under one arch of the cycloid and from (5) we have

$$A = -\oint_{-C} y\,dx = \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt + \int_0^{2\pi} 0\,(-dt)$$
$$= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt + 0 = \left[t - 2\sin t + \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi$$

20.



$$A = \oint_C x\,dy = \int_0^{2\pi} (5\cos t - \cos 5t)(5\cos t - 5\cos 5t)\,dt$$
$$= \int_0^{2\pi} (25\cos^2 t - 30\cos t\cos 5t + 5\cos^2 5t)\,dt$$
$$= \left[25\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right) - 30\left(\tfrac{1}{8}\sin 4t + \tfrac{1}{12}\sin 6t\right) + 5\left(\tfrac{1}{2}t + \tfrac{1}{20}\sin 10t\right)\right]_0^{2\pi}$$

[Use Formula 80 in the Table of Integrals]

$$= 30\pi$$

21. (a) Using Equation 16.2.8, we write parametric equations of the line segment as $x = (1-t)x_1 + tx_2$, $y = (1-t)y_1 + ty_2$, $0 \le t \le 1$. Then $dx = (x_2 - x_1)\,dt$ and $dy = (y_2 - y_1)\,dt$, so

$$\int_C x\,dy - y\,dx = \int_0^1 [(1-t)x_1 + tx_2](y_2 - y_1)\,dt + [(1-t)y_1 + ty_2](x_2 - x_1)\,dt$$
$$= \int_0^1 (x_1(y_2 - y_1) - y_1(x_2 - x_1) + t[(y_2 - y_1)(x_2 - x_1) - (x_2 - x_1)(y_2 - y_1)])\,dt$$
$$= \int_0^1 (x_1 y_2 - x_2 y_1)\,dt = x_1 y_2 - x_2 y_1$$

(b) We apply Green's Theorem to the path $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i is the line segment that joins (x_i, y_i) to (x_{i+1}, y_{i+1}) for $i = 1, 2, \ldots, n-1$, and C_n is the line segment that joins (x_n, y_n) to (x_1, y_1). From (5), $\frac{1}{2}\int_C x\,dy - y\,dx = \iint_D dA$, where D is the polygon bounded by C. Therefore

$$\text{area of polygon} = A(D) = \iint_D dA = \tfrac{1}{2}\int_C x\,dy - y\,dx$$
$$= \tfrac{1}{2}\left(\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \cdots + \int_{C_{n-1}} x\,dy - y\,dx + \int_{C_n} x\,dy - y\,dx\right)$$

To evaluate these integrals we use the formula from (a) to get

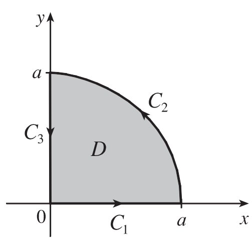
$A(D) = \frac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)]$.

(c) $A = \frac{1}{2}[(0\cdot 1 - 2\cdot 0) + (2\cdot 3 - 1\cdot 1) + (1\cdot 2 - 0\cdot 3) + (0\cdot 1 - (-1)\cdot 2) + (-1\cdot 0 - 0\cdot 1)]$

$= \frac{1}{2}(0 + 5 + 2 + 2) = \frac{9}{2}$

22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dA = \frac{1}{A}\iint_D x\,dA = \overline{x}$ and $-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_D (-2y)\,dA = \frac{1}{A}\iint_D y\,dA = \overline{y}$.

23. We orient the quarter-circular region as shown in the figure.



$A = \frac{1}{4}\pi a^2$ so $\overline{x} = \dfrac{1}{\pi a^2/2}\displaystyle\oint_C x^2\,dy$ and $\overline{y} = -\dfrac{1}{\pi a^2/2}\displaystyle\oint_C y^2\,dx$.

Here $C = C_1 + C_2 + C_3$ where C_1: $x = t,\ \ y = 0,\ \ 0 \le t \le a$;

C_2: $x = a\cos t,\ \ y = a\sin t,\ \ 0 \le t \le \frac{\pi}{2}$; and

C_3: $x = 0$, $y = a - t$, $0 \le t \le a$. Then

$\oint_C x^2\,dy = \int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy = \int_0^a 0\,dt + \int_0^{\pi/2}(a\cos t)^2(a\cos t)\,dt + \int_0^a 0\,dt$

$= \int_0^{\pi/2} a^3\cos^3 t\,dt = a^3\int_0^{\pi/2}(1-\sin^2 t)\cos t\,dt = a^3\left[\sin t - \frac{1}{3}\sin^3 t\right]_0^{\pi/2} = \frac{2}{3}a^3$

so $\overline{x} = \dfrac{1}{\pi a^2/2}\displaystyle\oint_C x^2\,dy = \dfrac{4a}{3\pi}$.

$\oint_C y^2 dx = \int_{C_1} y^2\,dx + \int_{C_2} y^2\,dx + \int_{C_3} y^2\,dx = \int_0^a 0\,dt + \int_0^{\pi/2}(a\sin t)^2(-a\sin t)\,dt + \int_0^a 0\,dt$

$= \int_0^{\pi/2}(-a^3\sin^3 t)\,dt = -a^3\int_0^{\pi/2}(1-\cos^2 t)\sin t\,dt = -a^3\left[\frac{1}{3}\cos^3 t - \cos t\right]_0^{\pi/2} = -\frac{2}{3}a^3$,

so $\overline{y} = -\dfrac{1}{\pi a^2/2}\displaystyle\oint_C y^2 dx = \dfrac{4a}{3\pi}$. Thus $(\overline{x},\overline{y}) = \left(\dfrac{4a}{3\pi}, \dfrac{4a}{3\pi}\right)$.

24. Here $A = \frac{1}{2}ab$ and $C = C_1 + C_2 + C_3$, where C_1: $x = x$, $y = 0$, $0 \le x \le a$;

C_2: $x = a$, $y = y$, $0 \le y \le b$; and C_3: $x = x$, $y = \frac{b}{a}x$, $x = a$ to $x = 0$. Then

$\oint_C x^2\,dy = \int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy = 0 + \int_0^b a^2\,dy + \int_a^0 (x^2)\left(\frac{b}{a}\,dx\right)$

$= a^2 b + \frac{b}{a}\left[\frac{1}{3}x^3\right]_a^0 = a^2 b - \frac{1}{3}a^2 b = \frac{2}{3}a^2 b.$

Similarly, $\oint_C y^2\,dx = \int_{C_1} y^2\,dx + \int_{C_2} y^2\,dx + \int_{C_3} y^2\,dx = 0 + 0 + \int_a^0 \left(\frac{b}{a}x\right)^2\,dx = \frac{b^2}{a^2}\cdot\frac{1}{3}x^3\Big]_a^0 = -\frac{1}{3}ab^2$. Thus

$\overline{x} = \frac{1}{2A}\oint_C x^2\,dy = \frac{1}{ab}\cdot\frac{2}{3}a^2 b = \frac{2}{3}a$ and $\overline{y} = -\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{ab}\left(-\frac{1}{3}ab^2\right) = \frac{1}{3}b$, so $(\overline{x},\overline{y}) = \left(\frac{2}{3}a, \frac{1}{3}b\right)$.

25. By Green's Theorem, $-\frac{1}{3}\rho\oint_C y^3\,dx = -\frac{1}{3}\rho\iint_D(-3y^2)\,dA = \iint_D y^2\rho\,dA = I_x$ and

$\frac{1}{3}\rho\oint_C x^3\,dy = \frac{1}{3}\rho\iint_D(3x^2)\,dA = \iint_D x^2\rho\,dA = I_y$.

26. By symmetry the moments of inertia about any two diameters are equal. Centering the disk at the origin, the moment of inertia about a diameter equals

$$I_y = \tfrac{1}{3}\rho\oint_C x^3\,dy = \tfrac{1}{3}\rho\int_0^{2\pi}(a^4\cos^4 t)\,dt = \tfrac{1}{3}a^4\rho\int_0^{2\pi}\left[\tfrac{3}{8} + \tfrac{1}{2}\cos 2t + \tfrac{1}{8}\cos 4t\right]dt = \tfrac{1}{3}a^4\rho\cdot\tfrac{3(2\pi)}{8} = \tfrac{1}{4}\pi a^4\rho$$

27. As in Example 5, let C' be a counterclockwise-oriented circle with center the origin and radius a, where a is chosen to be small enough so that C' lies inside C, and D the region bounded by C and C'. Here

$P = \dfrac{2xy}{(x^2+y^2)^2} \quad\Rightarrow\quad \dfrac{\partial P}{\partial y} = \dfrac{2x(x^2+y^2)^2 - 2xy\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4} = \dfrac{2x^3-6xy^2}{(x^2+y^2)^3}$ and

$Q = \dfrac{y^2-x^2}{(x^2+y^2)^2} \quad\Rightarrow\quad \dfrac{\partial Q}{\partial x} = \dfrac{-2x(x^2+y^2)^2 - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} = \dfrac{2x^3-6xy^2}{(x^2+y^2)^3}$. Thus, as in the example,

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \iint_D 0\,dA = 0$$

and $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C'} \mathbf{F}\cdot d\mathbf{r}$. We parametrize C' as $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$. Then

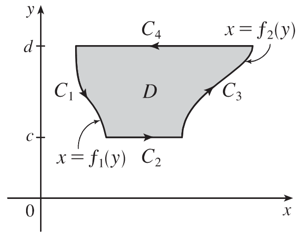
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C'}\mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}\frac{2\,(a\cos t)\,(a\sin t)\,\mathbf{i} + \left(a^2\sin^2 t - a^2\cos^2 t\right)\mathbf{j}}{\left(a^2\cos^2 t + a^2\sin^2 t\right)^2}\cdot\Big(-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}\Big)\,dt \\ &= \frac{1}{a}\int_0^{2\pi}\left(-\cos t\sin^2 t - \cos^3 t\right)dt = \frac{1}{a}\int_0^{2\pi}\left(-\cos t\sin^2 t - \cos t\left(1-\sin^2 t\right)\right)dt \\ &= -\frac{1}{a}\int_0^{2\pi}\cos t\,dt = -\frac{1}{a}\sin t\Big]_0^{2\pi} = 0\end{aligned}$$

28. P and Q have continuous partial derivatives on $\mathbb{R}^2$, so by Green's Theorem we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (3-1)\, dA = 2 \iint_D dA = 2 \cdot A(D) = 2 \cdot 6 = 12$$

29. Since C is a simple closed path which doesn't pass through or enclose the origin, there exists an open region that doesn't contain the origin but does contain D. Thus $P = -y/(x^2+y^2)$ and $Q = x/(x^2+y^2)$ have continuous partial derivatives on this open region containing D and we can apply Green's Theorem. But by Exercise 16.3.35(a), $\partial P/\partial y = \partial Q/\partial x$, so $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 0\, dA = 0$.

30. We express D as a type II region: $D = \{(x,y) \mid f_1(y) \le x \le f_2(y), c \le y \le d\}$ where f_1 and f_2 are continuous functions. Then $\displaystyle\iint_D \frac{\partial Q}{\partial x}\, dA = \int_c^d \int_{f_1(y)}^{f_2(y)} \frac{\partial Q}{\partial x}\, dx\, dy = \int_c^d \left[Q(f_2(y), y) - Q(f_1(y), y) \right] dy$ by the Fundamental Theorem of Calculus. But referring to the figure, $\displaystyle\oint_C Q\, dy = \oint_{C_1 + C_2 + C_3 + C_4} Q\, dy$.



Then $\int_{C_1} Q\, dy = \int_d^c Q(f_1(y), y)\, dy$, $\int_{C_2} Q\, dy = \int_{C_4} Q\, dy = 0$, and $\int_{C_3} Q\, dy = \int_c^d Q(f_2(y), y)\, dy$. Hence $\oint_C Q\, dy = \int_c^d \left[Q(f_2(y), y) - Q(f_1(y), y) \right] dy = \iint_D (\partial Q/\partial x)\, dA$.

31. Using the first part of (5), we have that $\iint_R dx\, dy = A(R) = \int_{\partial R} x\, dy$. But $x = g(u,v)$, and $dy = \dfrac{\partial h}{\partial u}\, du + \dfrac{\partial h}{\partial v}\, dv$, and we orient ∂S by taking the positive direction to be that which corresponds, under the mapping, to the positive direction along ∂R, so

$$\begin{aligned} \int_{\partial R} x\, dy &= \int_{\partial S} g(u,v) \left(\frac{\partial h}{\partial u}\, du + \frac{\partial h}{\partial v}\, dv \right) = \int_{\partial S} g(u,v)\, \frac{\partial h}{\partial u}\, du + g(u,v)\, \frac{\partial h}{\partial v}\, dv \\ &= \pm \iint_S \left[\tfrac{\partial}{\partial u} \left(g(u,v)\, \tfrac{\partial h}{\partial v} \right) - \tfrac{\partial}{\partial v} \left(g(u,v)\, \tfrac{\partial h}{\partial u} \right) \right] dA \qquad \text{[using Green's Theorem in the } uv\text{-plane]} \\ &= \pm \iint_S \left(\tfrac{\partial g}{\partial u} \tfrac{\partial h}{\partial v} + g(u,v)\, \tfrac{\partial^2 h}{\partial u\, \partial v} - \tfrac{\partial g}{\partial v} \tfrac{\partial h}{\partial u} - g(u,v)\, \tfrac{\partial^2 h}{\partial v\, \partial u} \right) dA \qquad \text{[using the Chain Rule]} \\ &= \pm \iint_S \left(\tfrac{\partial x}{\partial u} \tfrac{\partial y}{\partial v} - \tfrac{\partial x}{\partial v} \tfrac{\partial y}{\partial u} \right) dA \quad \text{[by the equality of mixed partials]} \quad = \pm \iint_S \tfrac{\partial(x,y)}{\partial(u,v)}\, du\, dv \end{aligned}$$

The sign is chosen to be positive if the orientation that we gave to ∂S corresponds to the usual positive orientation, and it is negative otherwise. In either case, since $A(R)$ is positive, the sign chosen must be the same as the sign of $\dfrac{\partial(x,y)}{\partial(u,v)}$.

Therefore $\displaystyle A(R) = \iint_R dx\, dy = \iint_S \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$.

16.5 Curl and Divergence

1. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+yz & y+xz & z+xy \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(z+xy) - \frac{\partial}{\partial z}(y+xz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(z+xy) - \frac{\partial}{\partial z}(x+yz)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(y+xz) - \frac{\partial}{\partial y}(x+yz)\right]\mathbf{k}$$

$$= (x-x)\,\mathbf{i} - (y-y)\,\mathbf{j} + (z-z)\,\mathbf{k} = \mathbf{0}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(x+yz) + \dfrac{\partial}{\partial y}(y+xz) + \dfrac{\partial}{\partial z}(z+xy) = 1+1+1 = 3$

2. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2z^3 & x^3yz^2 & x^2y^3z \end{vmatrix} = (3x^2y^2z - 2x^3yz)\,\mathbf{i} - (2xy^3z - 3xy^2z^2)\,\mathbf{j} + (3x^2yz^2 - 2xyz^3)\,\mathbf{k}$

$$= x^2yz(3y-2x)\,\mathbf{i} + xy^2z(3z-2y)\,\mathbf{j} + xyz^2(3x-2z)\,\mathbf{k}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(xy^2z^3) + \dfrac{\partial}{\partial y}(x^3yz^2) + \dfrac{\partial}{\partial z}(x^2y^3z) = y^2z^3 + x^3z^2 + x^2y^3$

3. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xye^z & 0 & yze^x \end{vmatrix} = (ze^x - 0)\,\mathbf{i} - (yze^x - xye^z)\,\mathbf{j} + (0 - xe^z)\,\mathbf{k}$

$$= ze^x\,\mathbf{i} + (xye^z - yze^x)\,\mathbf{j} - xe^z\,\mathbf{k}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(xye^z) + \dfrac{\partial}{\partial y}(0) + \dfrac{\partial}{\partial z}(yze^x) = ye^z + 0 + ye^x = y(e^z + e^x)$

4. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin yz & \sin zx & \sin xy \end{vmatrix}$

$$= (x\cos xy - x\cos zx)\,\mathbf{i} - (y\cos xy - y\cos yz)\,\mathbf{j} + (z\cos zx - z\cos yz)\,\mathbf{k}$$
$$= x(\cos xy - \cos zx)\,\mathbf{i} + y(\cos yz - \cos xy)\,\mathbf{j} + z(\cos zx - \cos yz)\,\mathbf{k}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(\sin yz) + \dfrac{\partial}{\partial y}(\sin zx) + \dfrac{\partial}{\partial z}(\sin xy) = 0+0+0 = 0$

5. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}}\,[(-yz+yz)\,\mathbf{i} - (-xz+xz)\,\mathbf{j} + (-xy+xy)\,\mathbf{k}] = \mathbf{0}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{\sqrt{x^2+y^2+z^2}}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{\sqrt{x^2+y^2+z^2}}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$

$$= \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2-y^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2-z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

6. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & e^{xy}\sin z & y\tan^{-1}(x/z) \end{vmatrix}$

$$= \left[\tan^{-1}(x/z) - e^{xy}\cos z\right]\mathbf{i} - \left(y\cdot\frac{1}{1+(x/z)^2}\cdot\frac{1}{z} - 0\right)\mathbf{j} + (ye^{xy}\sin z - 0)\,\mathbf{k}$$

$$= \left[\tan^{-1}(x/z) - e^{xy}\cos z\right]\mathbf{i} - \frac{yz}{x^2+z^2}\,\mathbf{j} + ye^{xy}\sin z\,\mathbf{k}$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(0) + \dfrac{\partial}{\partial y}(e^{xy}\sin z) + \dfrac{\partial}{\partial z}\left[y\tan^{-1}(x/z)\right]$

$$= 0 + xe^{xy}\sin z + y\cdot\frac{1}{1+(x/z)^2}\left(-\frac{x}{z^2}\right) = xe^{xy}\sin z - \frac{xy}{x^2+z^2}$$

7. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x\sin y & e^y\sin z & e^z\sin x \end{vmatrix} = (0 - e^y\cos z)\,\mathbf{i} - (e^z\cos x - 0)\,\mathbf{j} + (0 - e^x\cos y)\,\mathbf{k}$

$$= \langle -e^y\cos z, -e^z\cos x, -e^x\cos y\rangle$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}(e^x\sin y) + \dfrac{\partial}{\partial y}(e^y\sin z) + \dfrac{\partial}{\partial z}(e^z\sin x) = e^x\sin y + e^y\sin z + e^z\sin x$

8. (a) $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^{-1} & yz^{-1} & zx^{-1} \end{vmatrix} = (0 + yz^{-2})\,\mathbf{i} - (-zx^{-2} - 0)\,\mathbf{j} + (0 + xy^{-2})\,\mathbf{k}$

$$= \langle y/z^2, z/x^2, x/y^2\rangle$$

(b) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{y}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{z}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{z}{x}\right) = \dfrac{1}{y} + \dfrac{1}{z} + \dfrac{1}{x}$

9. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, the x-component of each vector of $\mathbf{F}$ is 0, so $P = 0$, hence $\dfrac{\partial P}{\partial x} = \dfrac{\partial P}{\partial y} = \dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = 0$. Q decreases as y increases, so $\dfrac{\partial Q}{\partial y} < 0$, but Q doesn't change in the x- or z-directions, so $\dfrac{\partial Q}{\partial x} = \dfrac{\partial Q}{\partial z} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 0 + \dfrac{\partial Q}{\partial y} + 0 < 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$

10. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, P and Q don't vary in the z-direction, so $\dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = \dfrac{\partial P}{\partial z} = \dfrac{\partial Q}{\partial z} = 0$. As x increases, the x-component of each vector of $\mathbf{F}$ increases while the y-component remains constant, so $\dfrac{\partial P}{\partial x} > 0$ and $\dfrac{\partial Q}{\partial x} = 0$. Similarly, as y increases, the y-component of each vector increases while the x-component remains constant, so $\dfrac{\partial Q}{\partial y} > 0$ and $\dfrac{\partial P}{\partial y} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + 0 > 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$

11. If the vector field is $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, then we know $R = 0$. In addition, the y-component of each vector of $\mathbf{F}$ is 0, so $Q = 0$, hence $\dfrac{\partial Q}{\partial x} = \dfrac{\partial Q}{\partial y} = \dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial x} = \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = 0$. P increases as y increases, so $\dfrac{\partial P}{\partial y} > 0$, but P doesn't change in the x- or z-directions, so $\dfrac{\partial P}{\partial x} = \dfrac{\partial P}{\partial z} = 0$.

(a) $\operatorname{div}\mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 0 + 0 + 0 = 0$

(b) $\operatorname{curl}\mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + \left(0 - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = -\dfrac{\partial P}{\partial y}\mathbf{k}$

Since $\dfrac{\partial P}{\partial y} > 0$, $-\dfrac{\partial P}{\partial y}\mathbf{k}$ is a vector pointing in the negative z-direction.

12. (a) $\operatorname{curl} f = \nabla \times f$ is meaningless because f is a scalar field.

(b) $\operatorname{grad} f$ is a vector field.

(c) $\operatorname{div}\mathbf{F}$ is a scalar field.

(d) $\operatorname{curl}(\operatorname{grad} f)$ is a vector field.

(e) $\operatorname{grad}\mathbf{F}$ is meaningless because $\mathbf{F}$ is not a scalar field.

(f) $\operatorname{grad}(\operatorname{div}\mathbf{F})$ is a vector field.

(g) $\operatorname{div}(\operatorname{grad} f)$ is a scalar field.

(h) $\operatorname{grad}(\operatorname{div} f)$ is meaningless because f is a scalar field.

(i) $\operatorname{curl}(\operatorname{curl}\mathbf{F})$ is a vector field.

(j) $\operatorname{div}(\operatorname{div}\mathbf{F})$ is meaningless because $\operatorname{div}\mathbf{F}$ is a scalar field.

(k) $(\operatorname{grad} f) \times (\operatorname{div}\mathbf{F})$ is meaningless because $\operatorname{div}\mathbf{F}$ is a scalar field.

(l) $\operatorname{div}(\operatorname{curl}(\operatorname{grad} f))$ is a scalar field.

13. $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = (6xyz^2 - 6xyz^2)\,\mathbf{i} - (3y^2z^2 - 3y^2z^2)\,\mathbf{j} + (2yz^3 - 2yz^3)\,\mathbf{k} = \mathbf{0}$

and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ with component functions which have continuous partial derivatives, so by Theorem 4, $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_x(x,y,z) = y^2z^3$ implies $f(x,y,z) = xy^2z^3 + g(y,z)$ and $f_y(x,y,z) = 2xyz^3 + g_y(y,z)$. But $f_y(x,y,z) = 2xyz^3$, so $g(y,z) = h(z)$ and $f(x,y,z) = xy^2z^3 + h(z)$. Thus $f_z(x,y,z) = 3xy^2z^2 + h'(z)$ but $f_z(x,y,z) = 3xy^2z^2$ so $h(z) = K$, a constant. Hence a potential function for $\mathbf{F}$ is $f(x,y,z) = xy^2z^3 + K$.

14. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz^2 & x^2yz^2 & x^2y^2z \end{vmatrix} = (2x^2yz - 2x^2yz)\,\mathbf{i} - (2xy^2z - 2xyz)\,\mathbf{j} + (2xyz^2 - xz^2)\,\mathbf{k} \neq \mathbf{0}$,

so $\mathbf{F}$ is not conservative.

15. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3xy^2z^2 & 2x^2yz^3 & 3x^2y^2z^2 \end{vmatrix}$

$$= (6x^2yz^2 - 6x^2yz^2)\,\mathbf{i} - (6xy^2z^2 - 6xy^2z)\,\mathbf{j} + (4xyz^3 - 6xyz^2)\,\mathbf{k}$$
$$= 6xy^2z(1-z)\,\mathbf{j} + 2xyz^2(2z-3)\,\mathbf{k} \neq \mathbf{0}$$

so $\mathbf{F}$ is not conservative.

16. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & \sin z & y\cos z \end{vmatrix} = (\cos z - \cos z)\,\mathbf{i} - (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$, $\mathbf{F}$ is defined on all of $\mathbb{R}^3$,

and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = 1$ implies $f(x,y,z) = x + g(y,z)$ and $f_y(x,y,z) = g_y(y,z)$. But $f_y(x,y,z) = \sin z$, so $g(y,z) = y\sin z + h(z)$ and $f(x,y,z) = x + y\sin z + h(z)$. Thus $f_z(x,y,z) = y\cos z + h'(z)$ but $f_z(x,y,z) = y\cos z$ so $h(z) = K$ and $f(x,y,z) = x + y\sin z + K$.

17. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{yz} & xze^{yz} & xye^{yz} \end{vmatrix}$

$$= [xyze^{yz} + xe^{yz} - (xyze^{yz} + xe^{yz})]\,\mathbf{i} - (ye^{yz} - ye^{yz})\,\mathbf{j} + (ze^{yz} - ze^{yz})\,\mathbf{k} = \mathbf{0}$$

$\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = e^{yz}$ implies $f(x,y,z) = xe^{yz} + g(y,z) \Rightarrow$ $f_y(x,y,z) = xze^{yz} + g_y(y,z)$. But $f_y(x,y,z) = xze^{yz}$, so $g(y,z) = h(z)$ and $f(x,y,z) = xe^{yz} + h(z)$. Thus $f_z(x,y,z) = xye^{yz} + h'(z)$ but $f_z(x,y,z) = xye^{yz}$ so $h(z) = K$ and a potential function for $\mathbf{F}$ is $f(x,y,z) = xe^{yz} + K$.

18. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x\sin yz & ze^x\cos yz & ye^x\cos yz \end{vmatrix}$

$$= [-yze^x\sin yz + e^x\cos yz - (-yze^x\sin yz + e^x\cos yz)]\,\mathbf{i} - (ye^x\cos yz - ye^x\cos yz)\,\mathbf{j}$$
$$+ (ze^x\cos yz - ze^x\cos yz)\,\mathbf{k} = \mathbf{0}$$

$\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = e^x\sin yz$ implies $f(x,y,z) = e^x\sin yz + g(y,z) \Rightarrow$

$f_y(x,y,z) = ze^x \cos yz + g_y(y,z)$. But $f_y(x,y,z) = ze^x \cos yz$, so $g(y,z) = h(z)$ and $f(x,y,z) = e^x \sin yz + h(z)$. Thus $f_z(x,y,z) = ye^x \cos yz + h'(z)$ but $f_z(x,y,z) = ye^x \cos yz$ so $h(z) = K$ and a potential function for $\mathbf{F}$ is $f(x,y,z) = e^x \sin yz + K$.

19. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = \dfrac{\partial}{\partial x}(x \sin y) + \dfrac{\partial}{\partial y}(\cos y) + \dfrac{\partial}{\partial z}(z - xy) = \sin y - \sin y + 1 \neq 0$, which contradicts Theorem 11.

20. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = yz - 2yz + 2yz = yz \neq 0$ which contradicts Theorem 11.

21. $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$. Hence $\mathbf{F} = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$ is irrotational.

22. $\operatorname{div}\mathbf{F} = \dfrac{\partial(f(y,z))}{\partial x} + \dfrac{\partial(g(x,z))}{\partial y} + \dfrac{\partial(h(x,y))}{\partial z} = 0$ so $\mathbf{F}$ is incompressible.

For Exercises 23–29, let $\mathbf{F}(x,y,z) = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G}(x,y,z) = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$.

23.
$$\begin{aligned}
\operatorname{div}(\mathbf{F}+\mathbf{G}) &= \operatorname{div}\langle P_1+P_2, Q_1+Q_2, R_1+R_2\rangle = \frac{\partial(P_1+P_2)}{\partial x} + \frac{\partial(Q_1+Q_2)}{\partial y} + \frac{\partial(R_1+R_2)}{\partial z} \\
&= \frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_1}{\partial z} + \frac{\partial R_2}{\partial z} = \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right) \\
&= \operatorname{div}\langle P_1, Q_1, R_1\rangle + \operatorname{div}\langle P_2, Q_2, R_2\rangle = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}
\end{aligned}$$

24.
$$\begin{aligned}
\operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G} &= \left[\left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)\mathbf{k}\right] \\
&\quad + \left[\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\mathbf{k}\right] \\
&= \left[\frac{\partial(R_1+R_2)}{\partial y} - \frac{\partial(Q_1+Q_2)}{\partial z}\right]\mathbf{i} + \left[\frac{\partial(P_1+P_2)}{\partial z} - \frac{\partial(R_1+R_2)}{\partial x}\right]\mathbf{j} \\
&\quad + \left[\frac{\partial(Q_1+Q_2)}{\partial x} - \frac{\partial(P_1+P_2)}{\partial y}\right]\mathbf{k} = \operatorname{curl}(\mathbf{F}+\mathbf{G})
\end{aligned}$$

25.
$$\begin{aligned}
\operatorname{div}(f\mathbf{F}) &= \operatorname{div}(f\,\langle P_1, Q_1, R_1\rangle) = \operatorname{div}\langle fP_1, fQ_1, fR_1\rangle = \frac{\partial(fP_1)}{\partial x} + \frac{\partial(fQ_1)}{\partial y} + \frac{\partial(fR_1)}{\partial z} \\
&= \left(f\,\frac{\partial P_1}{\partial x} + P_1\,\frac{\partial f}{\partial x}\right) + \left(f\,\frac{\partial Q_1}{\partial y} + Q_1\,\frac{\partial f}{\partial y}\right) + \left(f\,\frac{\partial R_1}{\partial z} + R_1\,\frac{\partial f}{\partial z}\right) \\
&= f\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \langle P_1, Q_1, R_1\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f
\end{aligned}$$

26. $$\operatorname{curl}(f\mathbf{F}) = \left[\frac{\partial(fR_1)}{\partial y} - \frac{\partial(fQ_1)}{\partial z}\right]\mathbf{i} + \left[\frac{\partial(fP_1)}{\partial z} - \frac{\partial(fR_1)}{\partial x}\right]\mathbf{j} + \left[\frac{\partial(fQ_1)}{\partial x} - \frac{\partial(fP_1)}{\partial y}\right]\mathbf{k}$$

$$= \left[f\,\frac{\partial R_1}{\partial y} + R_1\,\frac{\partial f}{\partial y} - f\,\frac{\partial Q_1}{\partial z} - Q_1\,\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[f\,\frac{\partial P_1}{\partial z} + P_1\,\frac{\partial f}{\partial z} - f\,\frac{\partial R_1}{\partial x} - R_1\,\frac{\partial f}{\partial x}\right]\mathbf{j}$$

$$+ \left[f\,\frac{\partial Q_1}{\partial x} + Q_1\,\frac{\partial f}{\partial x} - f\,\frac{\partial P_1}{\partial y} - P_1\,\frac{\partial f}{\partial y}\right]\mathbf{k}$$

$$= f\left[\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right]\mathbf{i} + f\left[\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right]\mathbf{j} + f\left[\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right]\mathbf{k}$$

$$+ \left[R_1\frac{\partial f}{\partial y} - Q_1\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[P_1\frac{\partial f}{\partial z} - R_1\frac{\partial f}{\partial x}\right]\mathbf{j} + \left[Q_1\frac{\partial f}{\partial x} - P_1\frac{\partial f}{\partial y}\right]\mathbf{k}$$

$$= f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}$$

27. $$\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \nabla\cdot(\mathbf{F}\times\mathbf{G}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{vmatrix} = \frac{\partial}{\partial x}\begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} - \frac{\partial}{\partial y}\begin{vmatrix} P_1 & R_1 \\ P_2 & R_2 \end{vmatrix} + \frac{\partial}{\partial z}\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix}$$

$$= \left[Q_1\,\frac{\partial R_2}{\partial x} + R_2\,\frac{\partial Q_1}{\partial x} - Q_2\,\frac{\partial R_1}{\partial x} - R_1\,\frac{\partial Q_2}{\partial x}\right] - \left[P_1\,\frac{\partial R_2}{\partial y} + R_2\,\frac{\partial P_1}{\partial y} - P_2\,\frac{\partial R_1}{\partial y} - R_1\,\frac{\partial P_2}{\partial y}\right]$$

$$+ \left[P_1\,\frac{\partial Q_2}{\partial z} + Q_2\,\frac{\partial P_1}{\partial z} - P_2\,\frac{\partial Q_1}{\partial z} - Q_1\,\frac{\partial P_2}{\partial z}\right]$$

$$= \left[P_2\left(\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right) + Q_2\left(\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right) + R_2\left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right)\right]$$

$$- \left[P_1\left(\frac{\partial R_2}{\partial y} - \frac{\partial Q_2}{\partial z}\right) + Q_1\left(\frac{\partial P_2}{\partial z} - \frac{\partial R_2}{\partial x}\right) + R_1\left(\frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y}\right)\right]$$

$$= \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$$

28. $\operatorname{div}(\nabla f\times\nabla g) = \nabla g\cdot\operatorname{curl}(\nabla f) - \nabla f\cdot\operatorname{curl}(\nabla g)$ [by Exercise 27] $= 0$ [by Theorem 3]

29. $$\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \nabla\times(\nabla\times\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial R_1/\partial y - \partial Q_1/\partial z & \partial P_1/\partial z - \partial R_1/\partial x & \partial Q_1/\partial x - \partial P_1/\partial y \end{vmatrix}$$

$$= \left(\frac{\partial^2 Q_1}{\partial y\partial x} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2} + \frac{\partial^2 R_1}{\partial z\partial x}\right)\mathbf{i} + \left(\frac{\partial^2 R_1}{\partial z\partial y} - \frac{\partial^2 Q_1}{\partial z^2} - \frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial x\partial y}\right)\mathbf{j}$$

$$+ \left(\frac{\partial^2 P_1}{\partial x\partial z} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial y\partial z}\right)\mathbf{k}$$

Now let's consider $\operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F}$ and compare with the above.
(Note that $\nabla^2\mathbf{F}$ is defined on page 1119 [ET 1095].)

[continued]

$$\operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F} = \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial y \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$

$$- \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial y^2} + \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j}\right.$$

$$\left. + \left(\frac{\partial^2 R_1}{\partial x^2} + \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$

$$= \left(\frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 R_1}{\partial y \partial z} - \frac{\partial^2 Q_1}{\partial x^2} - \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j}$$

$$+ \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_2}{\partial y^2}\right)\mathbf{k}$$

Then applying Clairaut's Theorem to reverse the order of differentiation in the second partial derivatives as needed and comparing, we have $\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F}$ as desired.

30. (a) $\nabla \cdot \mathbf{r} = \left(\frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}\right) \cdot (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = 1 + 1 + 1 = 3$

(b) $\nabla \cdot (r\mathbf{r}) = \nabla \cdot \sqrt{x^2+y^2+z^2}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

$$= \left(\frac{x^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right) + \left(\frac{y^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right) + \left(\frac{z^2}{\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2}\right)$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\,(4x^2+4y^2+4z^2) = 4\sqrt{x^2+y^2+z^2} = 4r$$

Another method:
By Exercise 25, $\nabla \cdot (r\mathbf{r}) = \operatorname{div}(r\mathbf{r}) = r\operatorname{div}\mathbf{r} + \mathbf{r} \cdot \nabla r = 3r + \mathbf{r} \cdot \dfrac{\mathbf{r}}{r}$ [see Exercise 31(a) below] $= 4r$.

(c) $\nabla^2 r^3 = \nabla^2\left(x^2+y^2+z^2\right)^{3/2}$

$$= \tfrac{\partial}{\partial x}\left[\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}(2x)\right] + \tfrac{\partial}{\partial y}\left[\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}(2y)\right] + \tfrac{\partial}{\partial z}\left[\tfrac{3}{2}(x^2+y^2+z^2)^{1/2}(2z)\right]$$

$$= 3\left[\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)(x) + (x^2+y^2+z^2)^{1/2}\right]$$

$$+ 3\left[\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)(y) + (x^2+y^2+z^2)^{1/2}\right]$$

$$+ 3\left[\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)(z) + (x^2+y^2+z^2)^{1/2}\right]$$

$$= 3(x^2+y^2+z^2)^{-1/2}(4x^2+4y^2+4z^2) = 12(x^2+y^2+z^2)^{1/2} = 12r$$

Another method: $\frac{\partial}{\partial x}\,(x^2+y^2+z^2)^{3/2} = 3x\sqrt{x^2+y^2+z^2} \;\Rightarrow\; \nabla r^3 = 3r(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = 3r\,\mathbf{r}$,
so $\nabla^2 r^3 = \nabla \cdot \nabla r^3 = \nabla \cdot (3r\,\mathbf{r}) = 3(4r) = 12r$ by part (b).

31. (a) $\nabla r = \nabla\sqrt{x^2+y^2+z^2} = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \dfrac{\mathbf{r}}{r}$

(b) $\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left[\dfrac{\partial}{\partial y}(z) - \dfrac{\partial}{\partial z}(y)\right]\mathbf{i} + \left[\dfrac{\partial}{\partial z}(x) - \dfrac{\partial}{\partial x}(z)\right]\mathbf{j} + \left[\dfrac{\partial}{\partial x}(y) - \dfrac{\partial}{\partial y}(x)\right]\mathbf{k} = \mathbf{0}$

(c) $\nabla\left(\dfrac{1}{r}\right) = \nabla\left(\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right)$

$$= \frac{-\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2x)}{x^2+y^2+z^2}\,\mathbf{i} - \frac{\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2y)}{x^2+y^2+z^2}\,\mathbf{j} - \frac{\dfrac{1}{2\sqrt{x^2+y^2+z^2}}(2z)}{x^2+y^2+z^2}\,\mathbf{k}$$

$$= -\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

(d) $\nabla \ln r = \nabla \ln(x^2+y^2+z^2)^{1/2} = \frac{1}{2}\nabla \ln(x^2+y^2+z^2)$

$$= \frac{x}{x^2+y^2+z^2}\,\mathbf{i} + \frac{y}{x^2+y^2+z^2}\,\mathbf{j} + \frac{z}{x^2+y^2+z^2}\,\mathbf{k} = \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}$$

32. $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \quad \Rightarrow \quad r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$, so

$$\mathbf{F} = \frac{\mathbf{r}}{r^p} = \frac{x}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{p/2}}\,\mathbf{k}$$

Then $\dfrac{\partial}{\partial x}\dfrac{x}{(x^2+y^2+z^2)^{p/2}} = \dfrac{(x^2+y^2+z^2)-px^2}{(x^2+y^2+z^2)^{1+p/2}} = \dfrac{r^2-px^2}{r^{p+2}}$. Similarly,

$\dfrac{\partial}{\partial y}\dfrac{y}{(x^2+y^2+z^2)^{p/2}} = \dfrac{r^2-py^2}{r^{p+2}}$ and $\dfrac{\partial}{\partial z}\dfrac{z}{(x^2+y^2+z^2)^{p/2}} = \dfrac{r^2-pz^2}{r^{p+2}}$. Thus

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{r^2-px^2}{r^{p+2}} + \frac{r^2-py^2}{r^{p+2}} + \frac{r^2-pz^2}{r^{p+2}} = \frac{3r^2-px^2-py^2-pz^2}{r^{p+2}}$$

$$= \frac{3r^2-p(x^2+y^2+z^2)}{r^{p+2}} = \frac{3r^2-pr^2}{r^{p+2}} = \frac{3-p}{r^p}$$

Consequently, if $p=3$ we have $\operatorname{div}\mathbf{F}=0$.

33. By (13), $\oint_C f(\nabla g)\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}(f\nabla g)\,dA = \iint_D [f\operatorname{div}(\nabla g) + \nabla g\cdot\nabla f]\,dA$ by Exercise 25. But $\operatorname{div}(\nabla g) = \nabla^2 g$.

Hence $\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA$.

34. By Exercise 33, $\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla g\cdot\nabla f\,dA$ and

$\iint_D g\nabla^2 f\,dA = \oint_C g(\nabla f)\cdot\mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$. Hence

$\iint_D \left(f\nabla^2 g - g\nabla^2 f\right)dA = \oint_C [f(\nabla g)\cdot\mathbf{n} - g(\nabla f)\cdot\mathbf{n}]\,ds + \iint_D (\nabla f\cdot\nabla g - \nabla g\cdot\nabla f)\,dA = \oint_C [f\nabla g - g\nabla f]\cdot\mathbf{n}\,ds.$

35. Let $f(x,y)=1$. Then $\nabla f = \mathbf{0}$ and Green's first identity (see Exercise 33) says

$\iint_D \nabla^2 g\,dA = \oint_C (\nabla g)\cdot\mathbf{n}\,ds - \iint_D \mathbf{0}\cdot\nabla g\,dA \quad \Rightarrow \quad \iint_D \nabla^2 g\,dA = \oint_C \nabla g\cdot\mathbf{n}\,ds$. But g is harmonic on D, so

$\nabla^2 g = 0 \quad \Rightarrow \quad \oint_C \nabla g\cdot\mathbf{n}\,ds = 0$ and $\oint_C D_{\mathbf{n}} g\,ds = \oint_C (\nabla g\cdot\mathbf{n})\,ds = 0$.

36. Let $g = f$. Then Green's first identity (see Exercise 33) says $\iint_D f\,\nabla^2 f\,dA = \oint_C (f)(\nabla f)\cdot \mathbf{n}\,ds - \iint_D \nabla f \cdot \nabla f\,dA$.

But f is harmonic, so $\nabla^2 f = 0$, and $\nabla f \cdot \nabla f = |\nabla f|^2$, so we have $0 = \oint_C (f)\,(\nabla f)\cdot \mathbf{n}\,ds - \iint_D |\nabla f|^2\,dA \quad\Rightarrow$

$\iint_D |\nabla f|^2\,dA = \oint_C (f)\,(\nabla f)\cdot \mathbf{n}\,ds = 0$ since $f(x,y) = 0$ on C.

37. (a) We know that $\omega = v/d$, and from the diagram $\sin\theta = d/r \quad\Rightarrow\quad v = d\omega = (\sin\theta)r\omega = |\mathbf{w}\times\mathbf{r}|$. But $\mathbf{v}$ is perpendicular to both $\mathbf{w}$ and $\mathbf{r}$, so that $\mathbf{v} = \mathbf{w}\times\mathbf{r}$.

(b) From (a), $\mathbf{v} = \mathbf{w}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (0\cdot z - \omega y)\,\mathbf{i} + (\omega x - 0\cdot z)\,\mathbf{j} + (0\cdot y - x\cdot 0)\,\mathbf{k} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$

(c) $\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\omega y & \omega x & 0 \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(\omega x)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(-\omega y) - \frac{\partial}{\partial x}(0)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right]\mathbf{k}$$

$$= [\omega - (-\omega)]\,\mathbf{k} = 2\omega\,\mathbf{k} = 2\mathbf{w}$$

38. Let $\mathbf{H} = \langle h_1, h_2, h_3\rangle$ and $\mathbf{E} = \langle E_1, E_2, E_3\rangle$.

(a) $\nabla\times(\nabla\times\mathbf{E}) = \nabla\times(\operatorname{curl}\mathbf{E}) = \nabla\times\left(-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right) = -\frac{1}{c}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial h_1/\partial t & \partial h_2/\partial t & \partial h_3/\partial t \end{vmatrix}$

$$= -\frac{1}{c}\left[\left(\frac{\partial^2 h_3}{\partial y\,\partial t} - \frac{\partial^2 h_2}{\partial z\,\partial t}\right)\mathbf{i} + \left(\frac{\partial^2 h_1}{\partial z\,\partial t} - \frac{\partial^2 h_3}{\partial x\,\partial t}\right)\mathbf{j} + \left(\frac{\partial^2 h_2}{\partial x\,\partial t} - \frac{\partial^2 h_1}{\partial y\,\partial t}\right)\mathbf{k}\right]$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)\mathbf{k}\right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$= -\frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl}\mathbf{H} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$$

(b) $\nabla\times(\nabla\times\mathbf{H}) = \nabla\times(\operatorname{curl}\mathbf{H}) = \nabla\times\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right) = \frac{1}{c}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial E_1/\partial t & \partial E_2/\partial t & \partial E_3/\partial t \end{vmatrix}$

$$= \frac{1}{c}\left[\left(\frac{\partial^2 E_3}{\partial y\,\partial t} - \frac{\partial^2 E_2}{\partial z\,\partial t}\right)\mathbf{i} + \left(\frac{\partial^2 E_1}{\partial z\,\partial t} - \frac{\partial^2 E_3}{\partial x\,\partial t}\right)\mathbf{j} + \left(\frac{\partial^2 E_2}{\partial x\,\partial t} - \frac{\partial^2 E_1}{\partial y\,\partial t}\right)\mathbf{k}\right]$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right)\mathbf{k}\right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$= \frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl}\mathbf{E} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}$$

(c) Using Exercise 29, we have that $\operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{grad}\operatorname{div}\mathbf{E} - \nabla^2\mathbf{E} \quad \Rightarrow$

$$\nabla^2\mathbf{E} = \operatorname{grad}\operatorname{div}\mathbf{E} - \operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{grad} 0 + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \quad \text{[from part (a)]} \quad = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

(d) As in part (c), $\nabla^2\mathbf{H} = \operatorname{grad}\operatorname{div}\mathbf{H} - \operatorname{curl}\operatorname{curl}\mathbf{H} = \operatorname{grad} 0 + \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2}$ [using part (b)] $= \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2}$.

39. For any continuous function f on $\mathbb{R}^3$, define a vector field $\mathbf{G}(x,y,z) = \langle g(x,y,z), 0, 0\rangle$ where $g(x,y,z) = \int_0^x f(t,y,z)\,dt$. Then $\operatorname{div}\mathbf{G} = \dfrac{\partial}{\partial x}(g(x,y,z)) + \dfrac{\partial}{\partial y}(0) + \dfrac{\partial}{\partial z}(0) = \dfrac{\partial}{\partial x}\int_0^x f(t,y,z)\,dt = f(x,y,z)$ by the Fundamental Theorem of Calculus. Thus every continuous function f on $\mathbb{R}^3$ is the divergence of some vector field.

16.6 Parametric Surfaces and Their Areas

1. $P(7,10,4)$ lies on the parametric surface $\mathbf{r}(u,v) = \langle 2u+3v, 1+5u-v, 2+u+v\rangle$ if and only if there are values for u and v where $2u+3v=7$, $1+5u-v=10$, and $2+u+v=4$. But solving the first two equations simultaneously gives $u=2$, $v=1$ and these values do not satisfy the third equation, so P does not lie on the surface.

   $Q(5,22,5)$ lies on the surface if $2u+3v=5$, $1+5u-v=22$, and $2+u+v=5$ for some values of u and v. Solving the first two equations simultaneously gives $u=4$, $v=-1$ and these values satisfy the third equation, so Q lies on the surface.

2. $P(3,-1,5)$ lies on the parametric surface $\mathbf{r}(u,v) = \left\langle u+v, u^2-v, u+v^2\right\rangle$ if and only if there are values for u and v where $u+v=3$, $u^2-v=-1$, and $u+v^2=5$. From the first equation we have $v=3-u$ and substituting into the second equation gives $u^2-3+u=-1 \quad\Leftrightarrow\quad u^2+u-2=0 \quad\Leftrightarrow\quad (u+2)(u-1)=0$, so $u=-2 \quad\Rightarrow$ $v=5 \quad \text{or} \quad u=1 \quad\Rightarrow\quad v=2$. The third equation is satisified by $u=1$, $v=2$ so P does lie on the surface.

   $Q(-1,3,4)$ lies on $\mathbf{r}(u,v)$ if and only if $u+v=-1$, $u^2-v=3$, and $u+v^2=4$, but substituting the first equation into the second gives $u=-2, v=1 \quad \text{or} \quad u=1, v=-2$, and neither of these pairs satisfies the third equation. Thus, Q does not lie on the surface.

3. $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (3-v)\,\mathbf{j} + (1+4u+5v)\,\mathbf{k} = \langle 0,3,1\rangle + u\,\langle 1,0,4\rangle + v\,\langle 1,-1,5\rangle$. From Example 3, we recognize this as a vector equation of a plane through the point $(0,3,1)$ and containing vectors $\mathbf{a} = \langle 1,0,4\rangle$ and $\mathbf{b} = \langle 1,-1,5\rangle$. If we wish to find a more conventional equation for the plane, a normal vector to the plane is $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 4 \\ 1 & -1 & 5 \end{vmatrix} = 4\,\mathbf{i} - \mathbf{j} - \mathbf{k}$ and an equation of the plane is $4(x-0)-(y-3)-(z-1)=0$ or $4x-y-z=-4$.

4. $\mathbf{r}(u,v) = 2\sin u\,\mathbf{i} + 3\cos u\,\mathbf{j} + v\,\mathbf{k}$, so the corresponding parametric equations for the surface are $x = 2\sin u$, $y = 3\cos u$, $z = v$. For any point (x, y, z) on the surface, we have $(x/2)^2 + (y/3)^2 = \sin^2 u + \cos^2 u = 1$, so cross-sections parallel to the yz-plane are all ellipses. Since $z = v$ with $0 \le v \le 2$, the surface is the portion of the elliptical cylinder $x^2/4 + y^2/9 = 1$ for $0 \le z \le 2$.

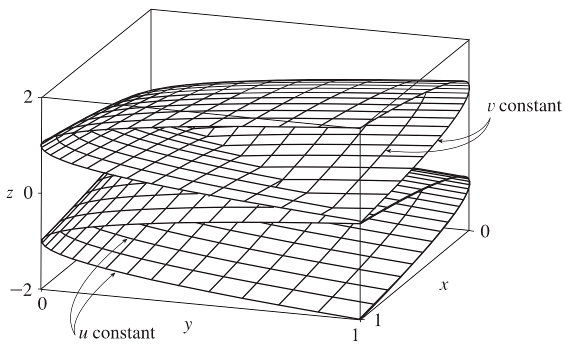
5. $\mathbf{r}(s,t) = \langle s, t, t^2 - s^2 \rangle$, so the corresponding parametric equations for the surface are $x = s$, $y = t$, $z = t^2 - s^2$. For any point (x, y, z) on the surface, we have $z = y^2 - x^2$. With no restrictions on the parameters, the surface is $z = y^2 - x^2$, which we recognize as a hyperbolic paraboloid.

6. $\mathbf{r}(s,t) = s\sin 2t\,\mathbf{i} + s^2\,\mathbf{j} + s\cos 2t\,\mathbf{k}$, so the corresponding parametric equations for the surface are $x = s\sin 2t$, $y = s^2$, $z = s\cos 2t$. For any point (x, y, z) on the surface, we have $x^2 + z^2 = s^2\sin^2 2t + s^2\cos^2 2t = s^2 = y$. Since no restrictions are placed on the parameters, the surface is $y = x^2 + z^2$, which we recognize as a circular paraboloid whose axis is the y-axis.

7. $\mathbf{r}(u,v) = \langle u^2, v^2, u + v \rangle$, $-1 \le u \le 1$, $-1 \le v \le 1$.

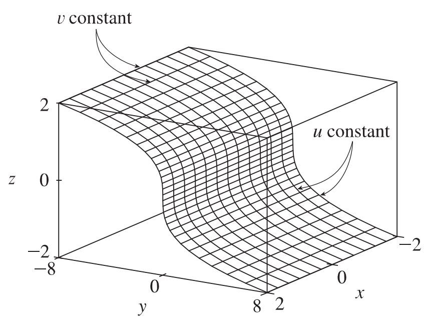
The surface has parametric equations $x = u^2$, $y = v^2$, $z = u + v$, $-1 \le u \le 1$, $-1 \le v \le 1$.

In Maple, the surface can be graphed by entering `plot3d([u^2,v^2,u+v],u=-1..1,v=-1..1);`. In Mathematica we use the `ParametricPlot3D` command. If we keep u constant at u_0, $x = u_0^2$, a constant, so the corresponding grid curves must be the curves parallel to the yz-plane. If v is constant, we have $y = v_0^2$, a constant, so these grid curves are the curves parallel to the xz-plane.



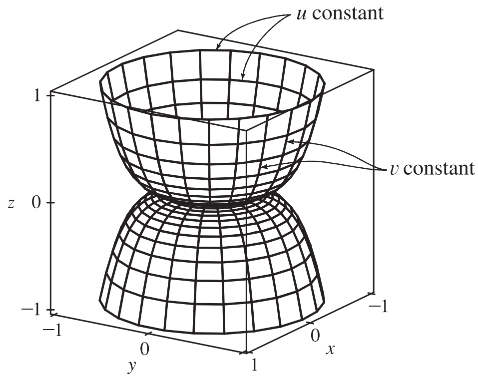
8. $\mathbf{r}(u,v) = \langle u, v^3, -v \rangle$, $-2 \le u \le 2$, $-2 \le v \le 2$.

The surface has parametric equations $x = u$, $y = v^3$, $z = -v$, $-2 \le u \le 2$, $-2 \le v \le 2$. If $u = u_0$ is constant, $x = u_0 =$ constant, so the corresponding grid curves are the curves parallel to the yz-plane. If $v = v_0$ is constant, $y = v_0^3 =$ constant, so the corresponding grid curves are the curves parallel to the xz-plane.



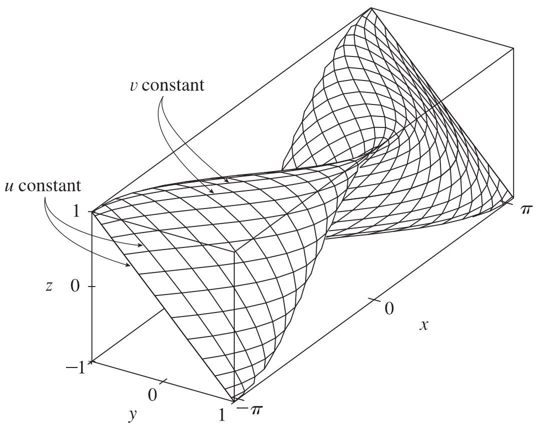
9. $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, u^5\rangle$.

The surface has parametric equations $x = u\cos v$, $y = u\sin v$, $z = u^5$, $-1 \leq u \leq 1$, $0 \leq v \leq 2\pi$. Note that if $u = u_0$ is constant then $z = u_0^5$ is constant and $x = u_0 \cos v$, $y = u_0 \sin v$ describe a circle in x, y of radius $|u_0|$, so the corresponding grid curves are circles parallel to the xy-plane. If $v = v_0$, a constant, the parametric equations become $x = u\cos v_0$, $y = u\sin v_0$, $z = u^5$. Then $y = (\tan v_0)x$, so these are the grid curves we see that lie in vertical planes $y = kx$ through the z-axis.



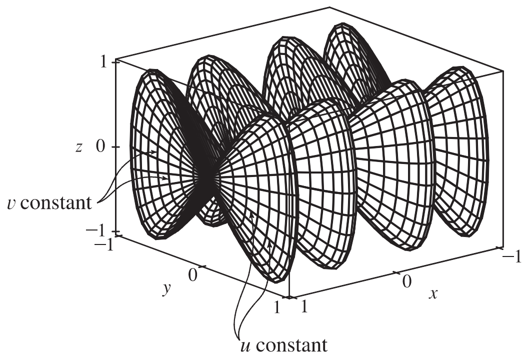
10. $\mathbf{r}(u,v) = \langle u, \sin(u+v), \sin v\rangle$, $-\pi \leq u \leq \pi$, $-\pi \leq v \leq \pi$.

The surface has parametric equations $x = u$, $y = \sin(u+v)$, $z = \sin v$, $-\pi \leq u \leq \pi$, $-\pi \leq v \leq \pi$. If $u = u_0$ is constant, $x = u_0 =$ constant, so the corresponding grid curves are the curves parallel to the yz-plane. If $v = v_0$ is constant, $z = \sin v_0 =$ constant, so the corresponding grid curves are the curves parallel to the xy-plane.



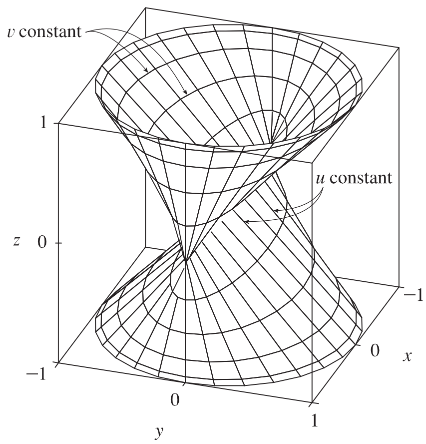
11. $x = \sin v$, $y = \cos u \sin 4v$, $z = \sin 2u \sin 4v$, $0 \leq u \leq 2\pi$, $-\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$.

Note that if $v = v_0$ is constant, then $x = \sin v_0$ is constant, so the corresponding grid curves must be parallel to the yz-plane. These are the vertically oriented grid curves we see, each shaped like a "figure-eight." When $u = u_0$ is held constant, the parametric equations become $x = \sin v$, $y = \cos u_0 \sin 4v$, $z = \sin 2u_0 \sin 4v$. Since z is a constant multiple of y, the corresponding grid curves are the curves contained in planes $z = ky$ that pass through the x-axis.



12. $x = \sin u,\ y = \cos u\ \sin v,\ z = \sin v,\ 0 \le u \le 2\pi,\ \ 0 \le v \le 2\pi$.

If $u = u_0$ is constant, then $x = \sin u_0 =$ constant, so the corresponding grid curves are the curves parallel to the yz-plane. If $v = v_0$ is constant, then $z = \sin v_0 =$ constant, so the corresponding grid curves are the curves parallel to the xy-plane.



13. $\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + v\, \mathbf{k}$. The parametric equations for the surface are $x = u \cos v$, $y = u \sin v$, $z = v$. We look at the grid curves first; if we fix v, then x and y parametrize a straight line in the plane $z = v$ which intersects the z-axis. If u is held constant, the projection onto the xy-plane is circular; with $z = v$, each grid curve is a helix. The surface is a spiraling ramp, graph IV.

14. $\mathbf{r}(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + \sin u\, \mathbf{k}$. The corresponding parametric equations for the surface are $x = u \cos v$, $y = u \sin v$, $z = \sin u$, $-\pi \le u \le \pi$. If $u = u_0$ is held constant, then $x = u_0 \cos v$, $y = u_0 \sin v$ so each grid curve is a circle of radius $|u_0|$ in the horizontal plane $z = \sin u_0$. If $v = v_0$ is constant, then $x = u \cos v_0$, $y = u \sin v_0 \quad \Rightarrow \quad y = (\tan v_0)x$, so the grid curves lie in vertical planes $y = kx$ through the z-axis. In fact, since x and y are constant multiples of u and $z = \sin u$, each of these traces is a sine wave. The surface is graph I.

15. $\mathbf{r}(u, v) = \sin v\, \mathbf{i} + \cos u\ \sin 2v\, \mathbf{j} + \sin u\ \sin 2v\, \mathbf{k}$. Parametric equations for the surface are $x = \sin v$, $y = \cos u\ \sin 2v$, $z = \sin u\ \sin 2v$. If $v = v_0$ is fixed, then $x = \sin v_0$ is constant, and $y = (\sin 2v_0) \cos u$ and $z = (\sin 2v_0) \sin u$ describe a circle of radius $|\sin 2v_0|$, so each corresponding grid curve is a circle contained in the vertical plane $x = \sin v_0$ parallel to the yz-plane. The only possible surface is graph II. The grid curves we see running lengthwise along the surface correspond to holding u constant, in which case $y = (\cos u_0) \sin 2v$, $z = (\sin u_0) \sin 2v \quad \Rightarrow \quad z = (\tan u_0)y$, so each grid curve lies in a plane $z = ky$ that includes the x-axis.

16. $x = (1 - u)(3 + \cos v) \cos 4\pi u$, $y = (1 - u)(3 + \cos v) \sin 4\pi u$, $z = 3u + (1 - u) \sin v$. These equations correspond to graph V: when $u = 0$, then $x = 3 + \cos v$, $y = 0$, and $z = \sin v$, which are equations of a circle with radius 1 in the xz-plane centered at $(3, 0, 0)$. When $u = \frac{1}{2}$, then $x = \frac{3}{2} + \frac{1}{2} \cos v$, $y = 0$, and $z = \frac{3}{2} + \frac{1}{2} \sin v$, which are equations of a circle with radius $\frac{1}{2}$ in the xz-plane centered at $\left(\frac{3}{2}, 0, \frac{3}{2}\right)$. When $u = 1$, then $x = y = 0$ and $z = 3$, giving the topmost point shown in the graph. This suggests that the grid curves with u constant are the vertically oriented circles visible on the surface. The spiralling grid curves correspond to keeping v constant.

17. $x = \cos^3 u \, \cos^3 v$, $y = \sin^3 u \, \cos^3 v$, $z = \sin^3 v$. If $v = v_0$ is held constant then $z = \sin^3 v_0$ is constant, so the corresponding grid curve lies in a horizontal plane. Several of the graphs exhibit horizontal grid curves, but the curves for this surface are neither circles nor straight lines, so graph III is the only possibility. (In fact, the horizontal grid curves here are members of the family $x = a\cos^3 u$, $y = a\sin^3 u$ and are called astroids.) The vertical grid curves we see on the surface correspond to $u = u_0$ held constant, as then we have $x = \cos^3 u_0 \, \cos^3 v$, $y = \sin^3 u_0 \, \cos^3 v$ so the corresponding grid curve lies in the vertical plane $y = (\tan^3 u_0)x$ through the z-axis.

18. $x = (1 - |u|)\cos v$, $y = (1 - |u|)\sin v$, $z = u$. Then $x^2 + y^2 = (1 - |u|)^2 \cos^2 v + (1 - |u|)^2 \sin^2 v = (1 - |u|)^2$, so if u is held constant, each grid curve is a circle of radius $(1 - |u|)$ in the horizontal plane $z = u$. The graph then must be graph VI. If v is held constant, so $v = v_0$, we have $x = (1 - |u|)\cos v_0$ and $y = (1 - |u|)\sin v_0$. Then $y = (\tan v_0)x$, so the grid curves we see running vertically along the surface in the planes $y = kx$ correspond to keeping v constant.

19. From Example 3, parametric equations for the plane through the point $(0, 0, 0)$ that contains the vectors $\mathbf{a} = \langle 1, -1, 0\rangle$ and $\mathbf{b} = \langle 0, 1, -1\rangle$ are $x = 0 + u(1) + v(0) = u, \ y = 0 + u(-1) + v(1) = v - u, \ z = 0 + u(0) + v(-1) = -v$.

20. From Example 3, parametric equations for the plane through the point $(0, -1, 5)$ that contains the vectors $\mathbf{a} = \langle 2, 1, 4\rangle$ and $\mathbf{b} = \langle -3, 2, 5\rangle$ are $x = 0 + u(2) + v(-3) = 2u - 3v, \ y = -1 + u(1) + v(2) = -1 + u + 2v,$ $z = 5 + u(4) + v(5) = 5 + 4u + 5v$.

21. Solving the equation for x gives $x^2 = 1 + y^2 + \frac{1}{4}z^2 \quad\Rightarrow\quad x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$. (We choose the positive root since we want the part of the hyperboloid that corresponds to $x \geq 0$.) If we let y and z be the parameters, parametric equations are $y = y$, $z = z, \ x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$.

22. Solving the equation for y gives $y^2 = \frac{1}{2}(1 - x^2 - 3z^2) \quad\Rightarrow\quad y = -\sqrt{\frac{1}{2}(1 - x^2 - 3z^2)}$ (since we want the part of the ellipsoid that corresponds to $y \leq 0$). If we let x and z be the parameters, parametric equations are $x = x, \ z = z,$ $y = -\sqrt{\frac{1}{2}(1 - x^2 - 3z^2)}$.

Alternate solution: The equation can be rewritten as $x^2 + \dfrac{y^2}{\left(1/\sqrt{2}\right)^2} + \dfrac{z^2}{(1/\sqrt{3})^2} = 1$, and if we let $x = u\cos v$ and $z = \dfrac{1}{\sqrt{3}}\, u \sin v$, then $y = -\sqrt{\frac{1}{2}(1 - x^2 - 3z^2)} = -\sqrt{\frac{1}{2}(1 - u^2\cos^2 v - u^2 \sin^2 v)} = -\sqrt{\frac{1}{2}(1 - u^2)}$, where $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

Second alternate solution: We can adapt the formulas for converting from spherical to rectangular coordinates as follows. We let $x = \sin\phi\cos\theta, \ y = \dfrac{1}{\sqrt{2}}\sin\phi\sin\theta, \ z = \dfrac{1}{\sqrt{3}}\cos\phi$; the surface is generated for $0 \leq \phi \leq \pi, \ \pi \leq \theta \leq 2\pi$.

23. Since the cone intersects the sphere in the circle $x^2 + y^2 = 2$, $z = \sqrt{2}$ and we want the portion of the sphere above this, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $x^2 + y^2 \le 2$.

Alternate solution: Using spherical coordinates, $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$ where $0 \le \phi \le \frac{\pi}{4}$ and $0 \le \theta \le 2\pi$.

24. In spherical coordinates, parametric equations are $x = 4\sin\phi\cos\theta$, $y = 4\sin\phi\sin\theta$, $z = 4\cos\phi$. The intersection of the sphere with the plane $z = 2$ corresponds to $z = 4\cos\phi = 2 \quad\Rightarrow\quad \cos\phi = \frac{1}{2} \quad\Rightarrow\quad \phi = \frac{\pi}{3}$. By symmetry, the intersection of the sphere with the plane $z = -2$ corresponds to $\phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Thus the surface is described by $0 \le \theta \le 2\pi$, $\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$.

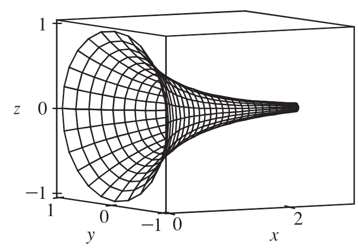
25. Parametric equations are $x = x$, $y = 4\cos\theta$, $z = 4\sin\theta$, $0 \le x \le 5$, $0 \le \theta \le 2\pi$.

26. Using x and y as the parameters, $x = x$, $y = y$, $z = x + 3$ where $0 \le x^2 + y^2 \le 1$. Also, since the plane intersects the cylinder in an ellipse, the surface is a planar ellipse in the plane $z = x + 3$. Thus, parametrizing with respect to s and θ, we have $x = s\cos\theta$, $y = s\sin\theta$, $z = 3 + s\cos\theta$ where $0 \le s \le 1$ and $0 \le \theta \le 2\pi$.

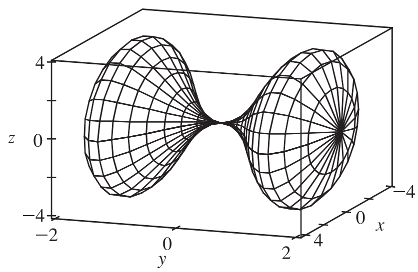
27. The surface appears to be a portion of a circular cylinder of radius 3 with axis the x-axis. An equation of the cylinder is $y^2 + z^2 = 9$, and we can impose the restrictions $0 \le x \le 5$, $y \le 0$ to obtain the portion shown. To graph the surface on a CAS, we can use parametric equations $x = u$, $y = 3\cos v$, $z = 3\sin v$ with the parameter domain $0 \le u \le 5$, $\frac{\pi}{2} \le v \le \frac{3\pi}{2}$. Alternatively, we can regard x and z as parameters. Then parametric equations are $x = x$, $z = z$, $y = -\sqrt{9 - z^2}$, where $0 \le x \le 5$ and $-3 \le z \le 3$.

28. The surface appears to be a portion of a sphere of radius 1 centered at the origin. In spherical coordinates, the sphere has equation $\rho = 1$, and imposing the restrictions $\frac{\pi}{2} \le \theta \le 2\pi$, $\frac{\pi}{4} \le \phi \le \pi$ will give only the portion of the sphere shown. Thus, to graph the surface on a CAS we can either use spherical coordinates with the stated restrictions, or we can use parametric equations: $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$, $\frac{\pi}{2} \le \theta \le 2\pi$, $\frac{\pi}{4} \le \phi \le \pi$.

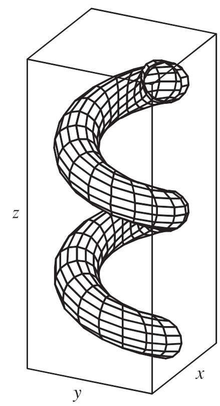
29. Using Equations 3, we have the parametrization $x = x$, $\;y = e^{-x}\cos\theta$, $z = e^{-x}\sin\theta$, $\;0 \le x \le 3$, $\;0 \le \theta \le 2\pi$.



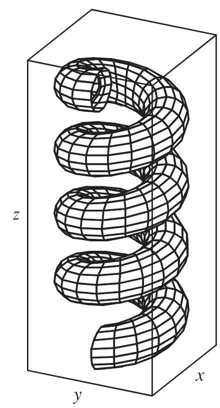
30. Letting θ be the angle of rotation about the y-axis, we have the parametrization $x = (4y^2 - y^4)\cos\theta$, $\;y = y$, $\;z = (4y^2 - y^4)\sin\theta$, $-2 \le y \le 2$, $\;0 \le \theta \le 2\pi$.



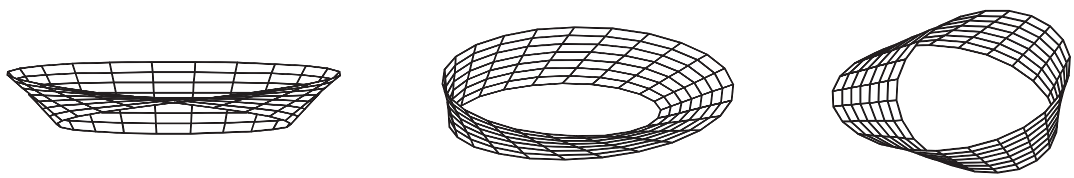
31. (a) Replacing $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$ gives parametric equations $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = u + \cos v$. From the graph, it appears that the direction of the spiral is reversed. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\sin u$, $y = (2 + \sin v)\cos u$, $z = 0$, draws a circle in the clockwise direction for each value of v. The original equations, on the other hand, give circular projections drawn in the counterclockwise direction. The equation for z is identical in both surfaces, so as z increases, these grid curves spiral up in opposite directions for the two surfaces.



(b) Replacing $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$ gives parametric equations $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = u + \cos v$. From the graph, it appears that the number of coils in the surface doubles within the same parametric domain. We can verify this observation by noting that the projection of the spiral grid curves onto the xy-plane, given by $x = (2 + \sin v)\cos 2u$, $y = (2 + \sin v)\sin 2u$, $z = 0$ (where v is constant), complete circular revolutions for $0 \le u \le \pi$ while the original surface requires $0 \le u \le 2\pi$ for a complete revolution. Thus, the new surface winds around twice as fast as the original surface, and since the equation for z is identical in both surfaces, we observe twice as many circular coils in the same z-interval.



32. First we graph the surface as viewed from the front, then from two additional viewpoints.



The surface appears as a twisted sheet, and is unusual because it has only one side. (The Möbius strip is discussed in more detail in Section 16.7.)

33. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + 3u^2\,\mathbf{j} + (u - v)\,\mathbf{k}$.

$\mathbf{r}_u = \mathbf{i} + 6u\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_v = \mathbf{i} - \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6u\,\mathbf{i} + 2\,\mathbf{j} - 6u\,\mathbf{k}$. Since the point $(2, 3, 0)$ corresponds to $u = 1$, $v = 1$, a normal vector to the surface at $(2, 3, 0)$ is $-6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}$, and an equation of the tangent plane is $-6x + 2y - 6z = -6$ or $3x - y + 3z = 3$.

34. $\mathbf{r}(u, v) = (u^2 + 1)\,\mathbf{i} + (v^3 + 1)\,\mathbf{j} + (u + v)\,\mathbf{k}$.

$\mathbf{r}_u = 2u\,\mathbf{i} + \mathbf{k}$ and $\mathbf{r}_v = 3v^2\,\mathbf{j} + \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -3v^2\,\mathbf{i} - 2u\,\mathbf{j} + 6uv^2\,\mathbf{k}$. Since the point $(5, 2, 3)$ corresponds to $u = 2$, $v = 1$, a normal vector to the surface at $(5, 2, 3)$ is $-3\,\mathbf{i} - 4\,\mathbf{j} + 12\,\mathbf{k}$, and an equation of the tangent plane is $-3(x - 5) - 4(y - 2) + 12(z - 3) = 0$ or $3x + 4y - 12z = -13$.

35. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}\left(1,\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$.

$\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$ and $\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + \mathbf{k}$, so a normal vector to the surface at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ is

$\mathbf{r}_u\left(1,\frac{\pi}{3}\right) \times \mathbf{r}_v\left(1,\frac{\pi}{3}\right) = \left(\frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}\right) \times \left(-\frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j} + \mathbf{k}\right) = \frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{2}\,\mathbf{j} + \mathbf{k}$. Thus an equation of the tangent plane at

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ is $\frac{\sqrt{3}}{2}\left(x - \frac{1}{2}\right) - \frac{1}{2}\left(y - \frac{\sqrt{3}}{2}\right) + 1\left(z - \frac{\pi}{3}\right) = 0$ or $\frac{\sqrt{3}}{2}x - \frac{1}{2}y + z = \frac{\pi}{3}$.

36. $\mathbf{r}(u,v) = \sin u\,\mathbf{i} + \cos u\,\sin v\,\mathbf{j} + \sin v\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}\left(\frac{\pi}{6},\frac{\pi}{6}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$.

$\mathbf{r}_u = \cos u\,\mathbf{i} - \sin u\sin v\,\mathbf{j}$ and $\mathbf{r}_v = \cos u\cos v\,\mathbf{j} + \cos v\,\mathbf{k}$, so a normal vector to the surface at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$ is

$\mathbf{r}_u\left(\frac{\pi}{6},\frac{\pi}{6}\right) \times \mathbf{r}_v\left(\frac{\pi}{6},\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{4}\,\mathbf{j}\right) \times \left(\frac{3}{4}\,\mathbf{j} + \frac{\sqrt{3}}{2}\,\mathbf{k}\right) = -\frac{\sqrt{3}}{8}\,\mathbf{i} - \frac{3}{4}\,\mathbf{j} + \frac{3\sqrt{3}}{8}\,\mathbf{k}$.

Thus an equation of the tangent plane at $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$ is $-\frac{\sqrt{3}}{8}\left(x - \frac{1}{2}\right) - \frac{3}{4}\left(y - \frac{\sqrt{3}}{4}\right) + \frac{3\sqrt{3}}{8}\left(z - \frac{1}{2}\right) = 0$ or

$\sqrt{3}\,x + 6y - 3\sqrt{3}\,z = \frac{\sqrt{3}}{2}$ or $2x + 4\sqrt{3}\,y - 6z = 1$.

37. $\mathbf{r}(u,v) = u^2\,\mathbf{i} + 2u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}(1,0) = (1,0,1)$.

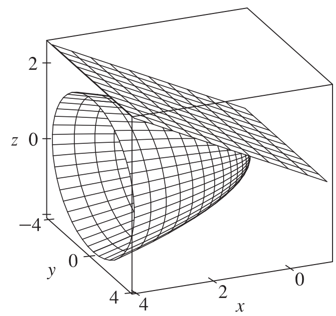
$\mathbf{r}_u = 2u\,\mathbf{i} + 2\sin v\,\mathbf{j} + \cos v\,\mathbf{k}$ and $\mathbf{r}_v = 2u\cos v\,\mathbf{j} - u\sin v\,\mathbf{k}$,

so a normal vector to the surface at the point $(1,0,1)$ is

$\mathbf{r}_u(1,0) \times \mathbf{r}_v(1,0) = (2\,\mathbf{i} + \mathbf{k}) \times (2\,\mathbf{j}) = -2\,\mathbf{i} + 4\,\mathbf{k}$.

Thus an equation of the tangent plane at $(1,0,1)$ is

$-2(x-1) + 0(y-0) + 4(z-1) = 0$ or $-x + 2z = 1$.



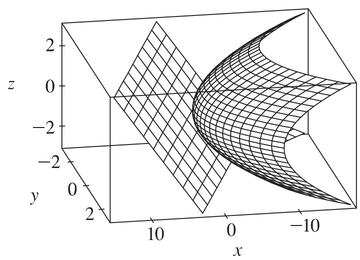
38. $\mathbf{r}(u,v) = (1 - u^2 - v^2)\,\mathbf{i} - v\,\mathbf{j} - u\,\mathbf{k}$.

$\mathbf{r}_u = -2u\,\mathbf{i} - \mathbf{k}$ and $\mathbf{r}_v = -2v\,\mathbf{i} - \mathbf{j}$. Since the point $(-1,-1,-1)$

corresponds to $u = 1$, $v = 1$, a normal vector to the surface at

$(-1,-1,-1)$ is

$\mathbf{r}_u(1,1) \times \mathbf{r}_v(1,1) = (-2\,\mathbf{i} - \mathbf{k}) \times (-2\,\mathbf{i} - \mathbf{j}) = -\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$.



Thus an equation of the tangent plane is $-1(x+1) + 2(y+1) + 2(z+1) = 0$ or $-x + 2y + 2z = -3$.

39. The surface S is given by $z = f(x,y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x + 2y = 6$, so D is the triangular region given by $\left\{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3 - \frac{3}{2}x\right\}$. By Formula 9, the surface area of S is

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA \\ &= \iint_D \sqrt{1 + (-3)^2 + (-2)^2}\,dA = \sqrt{14}\iint_D dA = \sqrt{14}\,A(D) = \sqrt{14}\left(\tfrac{1}{2}\cdot 2\cdot 3\right) = 3\sqrt{14}. \end{aligned}$$

40. $\mathbf{r}_u = \langle 1, -3, 1 \rangle$, $\mathbf{r}_v = \langle 1, 0, -1 \rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle 3, 2, 3 \rangle$. Then by Definition 6,

$$A(S) = \iint_D |\,\mathbf{r}_u \times \mathbf{r}_v\,|\, dA = \int_0^2 \int_{-1}^1 |\,\langle 3, 2, 3 \rangle\,|\, dv\, du = \sqrt{22} \int_0^2 du \int_{-1}^1 dv = \sqrt{22}\,(2)(2) = 4\sqrt{22}$$

41. Here we can write $z = f(x, y) = \frac{1}{3} - \frac{1}{3}x - \frac{2}{3}y$ and D is the disk $x^2 + y^2 \leq 3$, so by Formula 9 the area of the surface is

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + \left(-\tfrac{1}{3}\right)^2 + \left(-\tfrac{2}{3}\right)^2}\, dA = \tfrac{\sqrt{14}}{3} \iint_D dA \\ &= \tfrac{\sqrt{14}}{3}\, A(D) = \tfrac{\sqrt{14}}{3} \cdot \pi\left(\sqrt{3}\right)^2 = \sqrt{14}\,\pi \end{aligned}$$

42. $z = f(x, y) = \sqrt{x^2 + y^2} \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = \dfrac{1}{2}\left(x^2 + y^2\right)^{-1/2} \cdot 2x = \dfrac{x}{\sqrt{x^2 + y^2}}$, $\dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2}}$, and

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{1 + \frac{x^2 + y^2}{x^2 + y^2}} = \sqrt{2}$$

Here D is given by $\left\{(x, y) \mid 0 \leq x \leq 1, x^2 \leq y \leq x\right\}$, so by Formula 9, the surface area of S is

$$A(S) = \iint_D \sqrt{2}\, dA = \int_0^1 \int_{x^2}^x \sqrt{2}\, dy\, dx = \sqrt{2} \int_0^1 \left(x - x^2\right) dx = \sqrt{2}\, \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \sqrt{2}\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \tfrac{\sqrt{2}}{6}$$

43. $z = f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + \left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2}\, dA = \int_0^1 \int_0^1 \sqrt{1 + x + y}\, dy\, dx \\ &= \int_0^1 \left[\tfrac{2}{3}(x + y + 1)^{3/2}\right]_{y=0}^{y=1} dx = \tfrac{2}{3} \int_0^1 \left[(x + 2)^{3/2} - (x + 1)^{3/2}\right] dx \\ &= \tfrac{2}{3}\left[\tfrac{2}{5}(x + 2)^{5/2} - \tfrac{2}{5}(x + 1)^{5/2}\right]_0^1 = \tfrac{4}{15}(3^{5/2} - 2^{5/2} - 2^{5/2} + 1) = \tfrac{4}{15}(3^{5/2} - 2^{7/2} + 1) \end{aligned}$$

44. $z = f(x, y) = 1 + 3x + 2y^2$ with $0 \leq x \leq 2y$, $0 \leq y \leq 1$. Thus, by Formula 9,

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + 3^2 + (4y)^2}\, dA = \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\, dx\, dy = \int_0^1 2y\, \sqrt{10 + 16y^2}\, dy \\ &= \tfrac{1}{16} \cdot \tfrac{2}{3}(10 + 16y^2)^{3/2}\Big]_0^1 = \tfrac{1}{24}(26^{3/2} - 10^{3/2}) \end{aligned}$$

45. $z = f(x, y) = xy$ with $x^2 + y^2 \leq 1$, so $f_x = y$, $f_y = x \quad \Rightarrow$

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + y^2 + x^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\tfrac{1}{3}\left(r^2 + 1\right)^{3/2}\right]_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi} \tfrac{1}{3}\left(2\sqrt{2} - 1\right) d\theta = \tfrac{2\pi}{3}\left(2\sqrt{2} - 1\right) \end{aligned}$$

46. A parametric representation of the surface is $x = y^2 + z^2$, $y = y$, $z = z$ with $0 \leq y^2 + z^2 \leq 9$.

Hence $\mathbf{r}_y \times \mathbf{r}_z = (2y\,\mathbf{i} + \mathbf{j}) \times (2z\,\mathbf{i} + \mathbf{k}) = \mathbf{i} - 2y\,\mathbf{j} - 2z\,\mathbf{k}$.

Note: In general, if $x = f(y, z)$ then $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i} - \dfrac{\partial f}{\partial y}\,\mathbf{j} - \dfrac{\partial f}{\partial z}\,\mathbf{k}$, and $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\, dA$. Then

$$\begin{aligned} A(S) &= \iint\limits_{0 \leq y^2 + z^2 \leq 9} \sqrt{1 + 4y^2 + 4z^2}\, dA = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^3 r\, \sqrt{1 + 4r^2}\, dr = 2\pi\left[\tfrac{1}{12}(1 + 4r^2)^{3/2}\right]_0^3 = \tfrac{\pi}{6}\left(37\sqrt{37} - 1\right) \end{aligned}$$

47. A parametric representation of the surface is $x = x$, $y = 4x + z^2$, $z = z$ with $0 \le x \le 1$, $0 \le z \le 1$.

Hence $\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i} + 4\,\mathbf{j}) \times (2z\,\mathbf{j} + \mathbf{k}) = 4\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$.

Note: In general, if $y = f(x, z)$ then $\mathbf{r}_x \times \mathbf{r}_z = \dfrac{\partial f}{\partial x}\,\mathbf{i} - \mathbf{j} + \dfrac{\partial f}{\partial z}\,\mathbf{k}$ and $A\,(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}\,dA$. Then

$$\begin{aligned} A(S) &= \textstyle\int_0^1\int_0^1 \sqrt{17 + 4z^2}\,dx\,dz = \int_0^1 \sqrt{17 + 4z^2}\,dz \\ &= \textstyle\frac{1}{2}\left(z\sqrt{17 + 4z^2} + \frac{17}{2}\ln\left|2z + \sqrt{4z^2 + 17}\,\right|\right)\Big]_0^1 = \frac{\sqrt{21}}{2} + \frac{17}{4}\left[\ln\left(2 + \sqrt{21}\,\right) - \ln\sqrt{17}\,\right] \end{aligned}$$

48. $\mathbf{r}_u = \langle \cos v, \sin v, 0\rangle$, $\mathbf{r}_v = \langle -u\sin v, u\cos v, 1\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$. Then

$$\begin{aligned} A(S) &= \textstyle\int_0^\pi\int_0^1 \sqrt{1 + u^2}\,du\,dv = \int_0^\pi dv\,\int_0^1 \sqrt{1 + u^2}\,du \\ &= \textstyle\pi\left[\frac{u}{2}\sqrt{u^2 + 1} + \frac{1}{2}\ln\left|u + \sqrt{u^2 + 1}\,\right|\right]_0^1 = \frac{\pi}{2}\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\,\right)\right] \end{aligned}$$

49. $\mathbf{r}_u = \langle 2u, v, 0\rangle$, $\mathbf{r}_v = \langle 0, u, v\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \left\langle v^2, -2uv, 2u^2\right\rangle$. Then

$$\begin{aligned} A(S) &= \textstyle\iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA = \int_0^1\int_0^2 \sqrt{v^4 + 4u^2v^2 + 4u^4}\,dv\,du = \int_0^1\int_0^2 \sqrt{(v^2 + 2u^2)^2}\,dv\,du \\ &= \textstyle\int_0^1\int_0^2 (v^2 + 2u^2)\,dv\,du = \int_0^1 \left[\frac{1}{3}v^3 + 2u^2 v\right]_{v=0}^{v=2} du = \int_0^1 \left(\frac{8}{3} + 4u^2\right) du = \left[\frac{8}{3}u + \frac{4}{3}u^3\right]_0^1 = 4 \end{aligned}$$

50. The cylinder encloses separate portions of the sphere in the upper and lower halves. The top half of the sphere is $z = f(x, y) = \sqrt{b^2 - x^2 - y^2}$ and D is given by $\left\{(x, y) \mid x^2 + y^2 \le a^2\right\}$. By Formula 9, the surface area of the upper enclosed portion is

$$\begin{aligned} A &= \iint_D \sqrt{1 + \left(\frac{-x}{\sqrt{b^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{b^2 - x^2 - y^2}}\right)^2}\,dA = \iint_D \sqrt{1 + \frac{x^2 + y^2}{b^2 - x^2 - y^2}}\,dA \\ &= \iint_D \sqrt{\frac{b^2}{b^2 - x^2 - y^2}}\,dA = \int_0^{2\pi}\int_0^a \frac{b}{\sqrt{b^2 - r^2}}\,r\,dr\,d\theta = b\int_0^{2\pi} d\theta\,\int_0^a \frac{r}{\sqrt{b^2 - r^2}}\,dr \\ &= b\left[\theta\right]_0^{2\pi}\,\left[-\sqrt{b^2 - r^2}\,\right]_0^a = 2\pi b\left(-\sqrt{b^2 - a^2} + \sqrt{b^2 - 0}\,\right) = 2\pi b\left(b - \sqrt{b^2 - a^2}\,\right) \end{aligned}$$

The lower portion of the sphere enclosed by the cylinder has identical shape, so the total area is $2A = 4\pi b\left(b - \sqrt{b^2 - a^2}\right)$.

51. From Equation 9 we have $A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA$. But if $|f_x| \le 1$ and $|f_y| \le 1$ then $0 \le (f_x)^2 \le 1$, $0 \le (f_y)^2 \le 1 \quad\Rightarrow\quad 1 \le 1 + (f_x)^2 + (f_y)^2 \le 3 \quad\Rightarrow\quad 1 \le \sqrt{1 + (f_x)^2 + (f_y)^2} \le \sqrt{3}$. By Property 15.3.11, $\iint_D 1\,dA \le \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA \le \iint_D \sqrt{3}\,dA \quad\Rightarrow\quad A(D) \le A(S) \le \sqrt{3}\,A(D) \quad\Rightarrow$ $\pi R^2 \le A(S) \le \sqrt{3}\pi R^2$.

52. $z = f(x, y) = \cos(x^2 + y^2)$ with $x^2 + y^2 \le 1$.

$$\begin{aligned} A(S) &= \textstyle\iint_D \sqrt{1 + (-2x\sin(x^2 + y^2))^2 + (-2y\sin(x^2 + y^2))^2}\,dA \\ &= \textstyle\iint_D \sqrt{1 + 4x^2\sin^2(x^2 + y^2) + 4y^2\sin^2(x^2 + y^2)}\,dA = \iint_D \sqrt{1 + 4(x^2 + y^2)\sin^2(x^2 + y^2)}\,dA \\ &= \textstyle\int_0^{2\pi}\int_0^1 \sqrt{1 + 4r^2\sin^2(r^2)}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta\,\int_0^1 r\sqrt{1 + 4r^2\sin^2(r^2)}\,dr \\ &= \textstyle 2\pi\int_0^1 r\sqrt{1 + 4r^2\sin^2(r^2)}\,dr \approx 4.1073 \end{aligned}$$

53. $z = f(x,y) = e^{-x^2-y^2}$ with $x^2+y^2 \le 4$.

$$A(S) = \iint_D \sqrt{1 + \left(-2xe^{-x^2-y^2}\right)^2 + \left(-2ye^{-x^2-y^2}\right)^2}\, dA = \iint_D \sqrt{1 + 4(x^2+y^2)e^{-2(x^2+y^2)}}\, dA$$
$$= \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2e^{-2r^2}}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 r\sqrt{1+4r^2e^{-2r^2}}\, dr = 2\pi\int_0^2 r\sqrt{1+4r^2e^{-2r^2}}\, dr \approx 13.9783$$

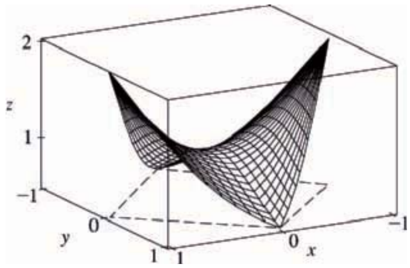
54. Let $f(x,y) = \dfrac{1+x^2}{1+y^2}$. Then $f_x = \dfrac{2x}{1+y^2}$,

$$f_y = (1+x^2)\left[-\frac{2y}{(1+y^2)^2}\right] = -\frac{2y(1+x^2)}{(1+y^2)^2}.$$

We use a CAS to estimate

$\int_{-1}^{1}\int_{-(1-|x|)}^{1-|x|} \sqrt{1+f_x^2+f_y^2}\, dy\, dx \approx 2.6959.$

In order to graph only the part of the surface above the square, we use $-(1-|x|) \le y \le 1-|x|$ as the y-range in our plot command.



55. (a) $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \int_0^6\int_0^4 \sqrt{1 + \frac{4x^2+4y^2}{(1+x^2+y^2)^4}}\, dy\, dx.$

Using the Midpoint Rule with $f(x,y) = \sqrt{1 + \dfrac{4x^2+4y^2}{(1+x^2+y^2)^4}}$, $m=3$, $n=2$ we have

$$A(S) \approx \sum_{i=1}^{3}\sum_{j=1}^{2} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A = 4\left[f(1,1)+f(1,3)+f(3,1)+f(3,3)+f(5,1)+f(5,3)\right] \approx 24.2055$$

(b) Using a CAS we have $A(S) = \displaystyle\int_0^6\int_0^4 \sqrt{1+\frac{4x^2+4y^2}{(1+x^2+y^2)^4}}\, dy\, dx \approx 24.2476$. This agrees with the estimate in part (a) to the first decimal place.

56. $\mathbf{r}(u,v) = \langle \cos^3 u\, \cos^3 v, \sin^3 u\, \cos^3 v, \sin^3 v\rangle$, so $\mathbf{r}_u = \langle -3\cos^2 u\, \sin u \cos^3 v, 3\sin^2 u\, \cos u \cos^3 v, 0\rangle$,

$\mathbf{r}_v = \langle -3\cos^3 u\, \cos^2 v\, \sin v, -3\sin^3 u\, \cos^2 v\, \sin v, 3\sin^2 v\, \cos v\rangle$, and

$\mathbf{r}_u \times \mathbf{r}_v = \langle 9\cos u\, \sin^2 u\, \cos^4 v\, \sin^2 v, 9\cos^2 u\, \sin u\, \cos^4 v\, \sin^2 v, 9\cos^2 u\, \sin^2 u\, \cos^5 v\, \sin v\rangle$. Then

$$\begin{aligned}|\mathbf{r}_u\times\mathbf{r}_v| &= 9\sqrt{\cos^2 u\, \sin^4 u\, \cos^8 v\, \sin^4 v + \cos^4 u\, \sin^2 u\, \cos^8 v\, \sin^4 v + \cos^4 u\, \sin^4 u\, \cos^{10} v\, \sin^2 v} \\ &= 9\sqrt{\cos^2 u\, \sin^2 u\, \cos^8 v\, \sin^2 v\,(\sin^2 v + \cos^2 u\, \sin^2 u\, \cos^2 v)} \\ &= 9\cos^4 v\, |\cos u\, \sin u\, \sin v|\sqrt{\sin^2 v + \cos^2 u\, \sin^2 u\, \cos^2 v}\end{aligned}$$

Using a CAS, we have $A(S) = \int_0^{\pi}\int_0^{2\pi} 9\cos^4 v\, |\cos u\, \sin u\, \sin v|\sqrt{\sin^2 v + \cos^2 u\, \sin^2 u\, \cos^2 v}\, dv\, du \approx 4.4506.$

57. $z = 1 + 2x + 3y + 4y^2$, so

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \int_1^4\int_0^1 \sqrt{1+4+(3+8y)^2}\, dy\, dx = \int_1^4\int_0^1 \sqrt{14+48y+64y^2}\, dy\, dx.$$

Using a CAS, we have

$\int_1^4\int_0^1 \sqrt{14+48y+64y^2}\, dy\, dx = \frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\left(11\sqrt{5}+3\sqrt{14}\sqrt{5}\right) - \frac{15}{16}\ln\left(3\sqrt{5}+\sqrt{14}\sqrt{5}\right)$

or $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\frac{11\sqrt{5}+3\sqrt{70}}{3\sqrt{5}+\sqrt{70}}$.

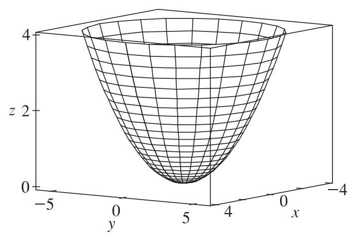
58. (a) $\mathbf{r}_u = a\cos v\,\mathbf{i} + b\sin v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = -au\sin v\,\mathbf{i} + bu\cos v\,\mathbf{j} + 0\,\mathbf{k}$, and

$\mathbf{r}_u \times \mathbf{r}_v = -2bu^2\cos v\,\mathbf{i} - 2au^2\sin v\,\mathbf{j} + abu\,\mathbf{k}$.

$A(S) = \int_0^{2\pi}\int_0^2 |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \int_0^{2\pi}\int_0^2 \sqrt{4b^2u^4\cos^2 v + 4a^2u^4\sin^2 v + a^2b^2u^2}\,du\,dv$

(b) $x^2 = a^2u^2\cos^2 v$, $y^2 = b^2u^2\sin^2 v$, $z = u^2 \;\Rightarrow\; x^2/a^2 + y^2/b^2 = u^2 = z$ which is an elliptic paraboloid. To find D, notice that $0 \le u \le 2 \;\Rightarrow\; 0 \le z \le 4 \;\Rightarrow\; 0 \le x^2/a^2 + y^2/b^2 \le 4$. Therefore, using Formula 9, we have

$$A(S) = \int_{-2a}^{2a}\int_{-b\sqrt{4-(x^2/a^2)}}^{b\sqrt{4-(x^2/a^2)}} \sqrt{1 + (2x/a^2)^2 + (2y/b^2)^2}\,dy\,dx.$$

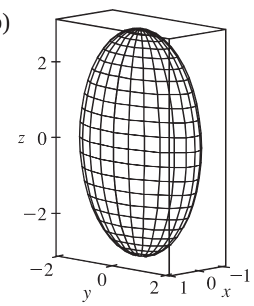
(c) 

(d) We substitute $a = 2$, $b = 3$ in the integral in part (a) to get $A(S) = \int_0^{2\pi}\int_0^2 2u\sqrt{9u^2\cos^2 v + 4u^2\sin^2 v + 9}\,du\,dv$. We use a CAS to estimate the integral accurate to four decimal places. To speed up the calculation, we can set `Digits:=7;` (in Maple) or use the approximation command `N` (in Mathematica). We find that $A(S) \approx 115.6596$.

59. (a) $x = a\sin u\cos v$, $y = b\sin u\sin v$, $z = c\cos u \;\Rightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (\sin u\cos v)^2 + (\sin u\sin v)^2 + (\cos u)^2$$
$$= \sin^2 u + \cos^2 u = 1$$

and since the ranges of u and v are sufficient to generate the entire graph, the parametric equations represent an ellipsoid.

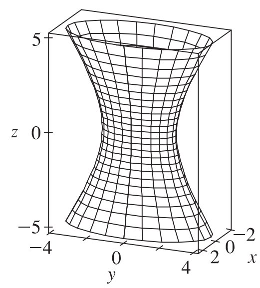
(b) 

(c) From the parametric equations (with $a = 1$, $b = 2$, and $c = 3$), we calculate $\mathbf{r}_u = \cos u\cos v\,\mathbf{i} + 2\cos u\sin v\,\mathbf{j} - 3\sin u\,\mathbf{k}$ and $\mathbf{r}_v = -\sin u\sin v\,\mathbf{i} + 2\sin u\cos v\,\mathbf{j}$. So $\mathbf{r}_u \times \mathbf{r}_v = 6\sin^2 u\cos v\,\mathbf{i} + 3\sin^2 u\sin v\,\mathbf{j} + 2\sin u\cos u\,\mathbf{k}$, and the surface area is given by $A(S) = \int_0^{2\pi}\int_0^{\pi} |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \int_0^{2\pi}\int_0^{\pi} \sqrt{36\sin^4 u\cos^2 v + 9\sin^4 u\sin^2 v + 4\cos^2 u\sin^2 u}\,du\,dv$

60. (a) $x = a\cosh u\,\cos v$, $y = b\cosh u\,\sin v$, $z = c\sinh u \;\Rightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \cosh^2 u\,\cos^2 v + \cosh^2 u\,\sin^2 v - \sinh^2 u$$
$$= \cosh^2 u - \sinh^2 u = 1$$

and the parametric equations represent a hyperboloid of one sheet.

(b) 

(c) $\mathbf{r}_u = \sinh u\,\cos v\,\mathbf{i} + 2\sinh u\,\sin v\,\mathbf{j} + 3\cosh u\,\mathbf{k}$ and $\mathbf{r}_v = -\cosh u\,\sin v\,\mathbf{i} + 2\cosh u\,\cos v\,\mathbf{j}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6\cosh^2 u\,\cos v\,\mathbf{i} - 3\cosh^2 u\,\sin v\,\mathbf{j} + 2\cosh u\,\sinh u\,\mathbf{k}$. We integrate between $u = \sinh^{-1}(-1) = -\ln\left(1+\sqrt{2}\right)$ and $u = \sinh^{-1} 1 = \ln\left(1+\sqrt{2}\right)$, since then z varies between

-3 and 3, as desired. So the surface area is

$$\begin{aligned} A(S) &= \int_0^{2\pi} \int_{-\ln\left(1+\sqrt{2}\right)}^{\ln\left(1+\sqrt{2}\right)} |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv \\ &= \int_0^{2\pi} \int_{-\ln\left(1+\sqrt{2}\right)}^{\ln\left(1+\sqrt{2}\right)} \sqrt{36\cosh^4 u\,\cos^2 v + 9\cosh^4 u\,\sin^2 v + 4\cosh^2 u\,\sinh^2 u}\,du\,dv \end{aligned}$$

61. To find the region D: $z = x^2 + y^2$ implies $z + z^2 = 4z$ or $z^2 - 3z = 0$. Thus $z = 0$ or $z = 3$ are the planes where the surfaces intersect. But $x^2 + y^2 + z^2 = 4z$ implies $x^2 + y^2 + (z-2)^2 = 4$, so $z = 3$ intersects the upper hemisphere. Thus $(z-2)^2 = 4 - x^2 - y^2$ or $z = 2 + \sqrt{4 - x^2 - y^2}$. Therefore D is the region inside the circle $x^2 + y^2 + (3-2)^2 = 4$, that is, $D = \left\{(x,y) \mid x^2 + y^2 \le 3\right\}$.

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + [(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2}\,dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + \frac{r^2}{4 - r^2}}\,r\,dr\,d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r\,dr}{\sqrt{4 - r^2}}\,d\theta = \int_0^{2\pi} \left[-2(4 - r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}}\,d\theta \\ &= \int_0^{2\pi} (-2 + 4)\,d\theta = 2\theta\Big]_0^{2\pi} = 4\pi \end{aligned}$$

62. We first find the area of the face of the surface that intersects the positive y-axis. A parametric representation of the surface is $x = x,\ \ y = \sqrt{1 - z^2},\ \ z = z$ with $x^2 + z^2 \le 1$. Then $\mathbf{r}(x,z) = \left\langle x, \sqrt{1 - z^2}, z\right\rangle \ \Rightarrow\ \mathbf{r}_x = \langle 1, 0, 0\rangle$, $\mathbf{r}_z = \left\langle 0, -z/\sqrt{1 - z^2}, 1\right\rangle$ and $\mathbf{r}_x \times \mathbf{r}_z = \left\langle 0, -1, -z/\sqrt{1 - z^2}\,\right\rangle \ \Rightarrow\ |\,\mathbf{r}_x \times \mathbf{r}_z\,| = \sqrt{1 + \dfrac{z^2}{1 - z^2}} = \dfrac{1}{\sqrt{1 - z^2}}$.

$$A(S) = \iint\limits_{x^2 + z^2 \le 1} |\,\mathbf{r}_x \times \mathbf{r}_z\,|\,dA = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1 - z^2}}\,dx\,dz = 4\int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1 - z^2}}\,dx\,dz \quad \left[\begin{array}{c}\text{by the symmetry}\\ \text{of the surface}\end{array}\right]$$

This integral is improper [when $z = 1$], so

$$A(S) = \lim_{t \to 1^-} 4\int_0^t \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1 - z^2}}\,dx\,dz = \lim_{t \to 1^-} 4\int_0^t \frac{\sqrt{1 - z^2}}{\sqrt{1 - z^2}}\,dz = \lim_{t \to 1^-} 4\int_0^t dz = \lim_{t \to 1^-} 4t = 4$$

Since the complete surface consists of four congruent faces, the total surface area is $4(4) = 16$.

Alternate solution: The face of the surface that intersects the positive y-axis can also be parametrized as $\mathbf{r}(x,\theta) = \langle x, \cos\theta, \sin\theta\rangle$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $x^2 + z^2 \le 1 \ \Leftrightarrow\ x^2 + \sin^2\theta \le 1 \ \Leftrightarrow$ $-\sqrt{1 - \sin^2\theta} \le x \le \sqrt{1 - \sin^2\theta} \ \Leftrightarrow\ -\cos\theta \le x \le \cos\theta$. Then $\mathbf{r}_x = \langle 1, 0, 0\rangle$, $\mathbf{r}_\theta = \langle 0, -\sin\theta, \cos\theta\rangle$ and $\mathbf{r}_x \times \mathbf{r}_\theta = \langle 0, -\cos\theta, -\sin\theta\rangle \ \Rightarrow\ |\,\mathbf{r}_x \times \mathbf{r}_\theta\,| = 1$, so

$A(S) = \int_{-\pi/2}^{\pi/2} \int_{-\cos\theta}^{\cos\theta} 1\,dx\,d\theta = \int_{-\pi/2}^{\pi/2} 2\cos\theta\,d\theta = 2\sin\theta\Big]_{-\pi/2}^{\pi/2} = 4$. Again, the area of the complete surface is $4(4) = 16$.

63. Let $A(S_1)$ be the surface area of that portion of the surface which lies above the plane $z = 0$. Then $A(S) = 2A(S_1)$. Following Example 10, a parametric representation of S_1 is $x = a\sin\phi\cos\theta$, $y = a\sin\phi\sin\theta$, $z = a\cos\phi$ and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2\sin\phi$. For D, $0 \le \phi \le \frac{\pi}{2}$ and for each fixed ϕ, $\left(x - \frac{1}{2}a\right)^2 + y^2 \le \left(\frac{1}{2}a\right)^2$ or $\left[a\sin\phi\cos\theta - \frac{1}{2}a\right]^2 + a^2\sin^2\phi\sin^2\theta \le (a/2)^2$ implies $a^2\sin^2\phi - a^2\sin\phi\cos\theta \le 0$ or $\sin\phi\,(\sin\phi - \cos\theta) \le 0$. But $0 \le \phi \le \frac{\pi}{2}$, so $\cos\theta \ge \sin\phi$ or $\sin\left(\frac{\pi}{2} + \theta\right) \ge \sin\phi$ or $\phi - \frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \phi$. Hence $D = \left\{(\phi, \theta) \mid 0 \le \phi \le \frac{\pi}{2}, \phi - \frac{\pi}{2} \le \theta \le \frac{\pi}{2} - \phi\right\}$. Then

$$\begin{aligned} A(S_1) &= \int_0^{\pi/2}\int_{\phi-(\pi/2)}^{(\pi/2)-\phi} a^2\sin\phi\,d\theta\,d\phi = a^2\int_0^{\pi/2}(\pi - 2\phi)\sin\phi\,d\phi \\ &= a^2\left[(-\pi\cos\phi) - 2(-\phi\cos\phi + \sin\phi)\right]_0^{\pi/2} = a^2(\pi - 2) \end{aligned}$$

Thus $A(S) = 2a^2(\pi - 2)$.

Alternate solution: Working on S_1 we could parametrize the portion of the sphere by $x = x$, $y = y$, $z = \sqrt{a^2 - x^2 - y^2}$.

Then $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2}{a^2 - x^2 - y^2} + \dfrac{y^2}{a^2 - x^2 - y^2}} = \dfrac{a}{\sqrt{a^2 - x^2 - y^2}}$ and

$$\begin{aligned} A(S_1) &= \iint\limits_{0 \le (x-(a/2))^2 + y^2 \le (a/2)^2} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \frac{a}{\sqrt{a^2 - r^2}}\,r\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} -a(a^2 - r^2)^{1/2}\Big]_{r=0}^{r=a\cos\theta}\,d\theta = \int_{-\pi/2}^{\pi/2} a^2\left[1 - (1 - \cos^2\theta)^{1/2}\right]d\theta \\ &= \int_{-\pi/2}^{\pi/2} a^2(1 - |\sin\theta|)\,d\theta = 2a^2\int_0^{\pi/2}(1 - \sin\theta)\,d\theta = 2a^2\left(\tfrac{\pi}{2} - 1\right) \end{aligned}$$

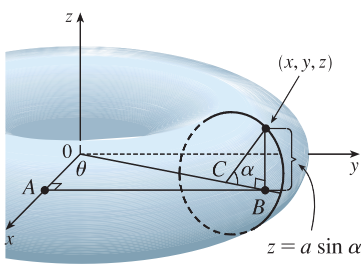
Thus $A(S) = 4a^2\left(\frac{\pi}{2} - 1\right) = 2a^2(\pi - 2)$.

Notes:

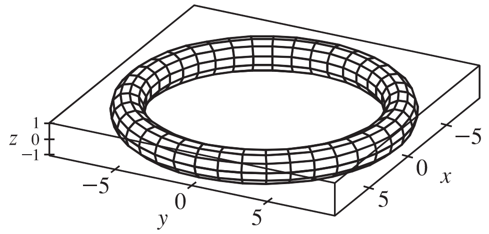
(1) Perhaps working in spherical coordinates is the most obvious approach here. However, you must be careful in setting up D.

(2) In the alternate solution, you can avoid having to use $|\sin\theta|$ by working in the first octant and then multiplying by 4. However, if you set up S_1 as above and arrived at $A(S_1) = a^2\pi$, you now see your error.

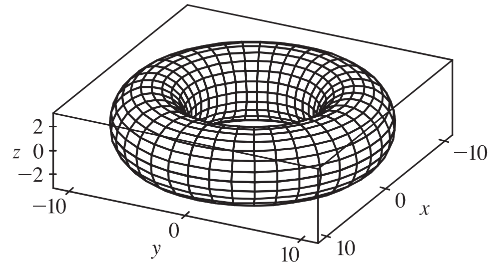
64. (a) Here $z = a\sin\alpha$, $y = |AB|$, and $x = |OA|$. But $|OB| = |OC| + |CB| = b + a\cos\alpha$ and $\sin\theta = \dfrac{|AB|}{|OB|}$ so that $y = |OB|\sin\theta = (b + a\cos\alpha)\sin\theta$. Similarly $\cos\theta = \dfrac{|OA|}{|OB|}$ so $x = (b + a\cos\alpha)\cos\theta$. Hence a parametric representation for the torus is $x = b\cos\theta + a\cos\alpha\,\cos\theta$, $y = b\sin\theta + a\cos\alpha\,\sin\theta$, $z = a\sin\alpha$, where $0 \le \alpha \le 2\pi$, $0 \le \theta \le 2\pi$.



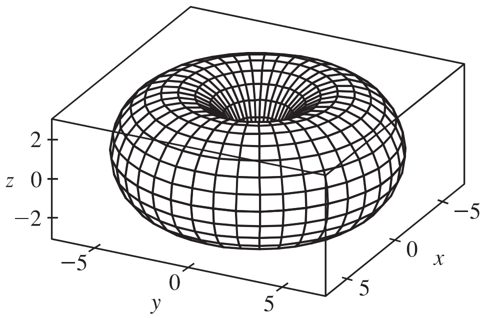
(b)



$a = 1, b = 8$



$a = 3, b = 8$



$a = 3, b = 4$

(c) $x = b\cos\theta + a\cos\alpha\,\cos\theta$, $y = b\sin\theta + a\cos\alpha\,\sin\theta$, $z = a\sin\alpha$, so $\mathbf{r}_\alpha = \langle -a\sin\alpha\,\cos\theta, -a\sin\alpha\,\sin\theta, a\cos\alpha\rangle$, $\mathbf{r}_\theta = \langle -(b + a\cos\alpha)\sin\theta, (b + a\cos\alpha)\cos\theta, 0\rangle$ and

$$\begin{aligned}\mathbf{r}_\alpha \times \mathbf{r}_\theta &= \left(-ab\cos\alpha\,\cos\theta - a^2\cos\alpha\,\cos^2\theta\right)\mathbf{i} + \left(-ab\sin\alpha\,\cos\theta - a^2\sin\alpha\,\cos^2\theta\right)\mathbf{j}\\ &\quad + \left(-ab\cos^2\alpha\,\sin\theta - a^2\cos^2\alpha\,\sin\theta\,\cos\theta - ab\sin^2\alpha\,\sin\theta - a^2\sin^2\alpha\,\sin\theta\,\cos\theta\right)\mathbf{k}\\ &= -a\,(b + a\cos\alpha)\left[(\cos\theta\,\cos\alpha)\,\mathbf{i} + (\sin\theta\,\cos\alpha)\,\mathbf{j} + (\sin\alpha)\,\mathbf{k}\right]\end{aligned}$$

Then $|\mathbf{r}_\alpha \times \mathbf{r}_\theta| = a(b + a\cos\alpha)\sqrt{\cos^2\theta\,\cos^2\alpha + \sin^2\theta\,\cos^2\alpha + \sin^2\alpha} = a(b + a\cos\alpha)$.

Note: $b > a$, $-1 \le \cos\alpha \le 1$ so $|b + a\cos\alpha| = b + a\cos\alpha$. Hence

$A\,(S) = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)\,d\alpha\,d\theta = 2\pi\left[ab\alpha + a^2\sin\alpha\right]_0^{2\pi} = 4\pi^2 ab.$

16.7 Surface Integrals

1. The faces of the box in the planes $x = 0$ and $x = 2$ have surface area 24 and centers $(0, 2, 3)$, $(2, 2, 3)$. The faces in $y = 0$ and $y = 4$ have surface area 12 and centers $(1, 0, 3)$, $(1, 4, 3)$, and the faces in $z = 0$ and $z = 6$ have area 8 and centers $(1, 2, 0)$, $(1, 2, 6)$. For each face we take the point P_{ij}^* to be the center of the face and $f(x, y, z) = e^{-0.1(x+y+z)}$, so by Definition 1,

$$\begin{aligned}\iint_S f(x,y,z)\,dS &\approx [f(0,2,3)](24) + [f(2,2,3)](24) + [f(1,0,3)](12)\\ &\quad + [f(1,4,3)](12) + [f(1,2,0)](8) + [f(1,2,6)](8)\\ &= 24(e^{-0.5} + e^{-0.7}) + 12(e^{-0.4} + e^{-0.8}) + 8(e^{-0.3} + e^{-0.9}) \approx 49.09\end{aligned}$$

2. Each quarter-cylinder has surface area $\frac{1}{4}[2\pi(1)(2)] = \pi$, and the top and bottom disks have surface area $\pi(1)^2 = \pi$. We can take $(0,0,1)$ as a sample point in the top disk, $(0,0,-1)$ in the bottom disk, and $(\pm1,0,0)$, $(0,\pm1,0)$ in the four quarter-cylinders. Then $\iint_S f(x,y,z)\,dS$ can be approximated by the Riemann sum

$$f(1,0,0)(\pi) + f(-1,0,0)(\pi) + f(0,1,0)\,(\pi) + f(0,-1,0)(\pi) + f(0,0,1)(\pi) + f(0,0,-1)(\pi)$$
$$= (2+2+3+3+4+4)\pi = 18\pi \approx 56.5.$$

3. We can use the xz- and yz-planes to divide H into four patches of equal size, each with surface area equal to $\frac{1}{8}$ the surface area of a sphere with radius $\sqrt{50}$, so $\Delta S = \frac{1}{8}(4)\pi\left(\sqrt{50}\right)^2 = 25\pi$. Then $(\pm3,\pm4,5)$ are sample points in the four patches, and using a Riemann sum as in Definition 1, we have

$$\iint_H f(x,y,z)\,dS \approx f(3,4,5)\,\Delta S + f(3,-4,5)\,\Delta S + f(-3,4,5)\,\Delta S + f(-3,-4,5)\,\Delta S$$
$$= (7+8+9+12)(25\pi) = 900\pi \approx 2827$$

4. On the surface, $f(x,y,z) = g\left(\sqrt{x^2+y^2+z^2}\right) = g(2) = -5$. So since the area of a sphere is $4\pi r^2$,

$\iint_S f(x,y,z)\,dS = \iint_S g(2)\,dS = -5\iint_S dS = -5[4\pi(2)^2] = -80\pi.$

5. $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (u-v)\,\mathbf{j} + (1+2u+v)\,\mathbf{k}$, $0 \le u \le 2$, $0 \le v \le 1$ and

$\mathbf{r}_u \times \mathbf{r}_v = (\mathbf{i}+\mathbf{j}+2\,\mathbf{k}) \times (\mathbf{i}-\mathbf{j}+\mathbf{k}) = 3\,\mathbf{i}+\mathbf{j}-2\,\mathbf{k} \quad\Rightarrow\quad |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{3^2+1^2+(-2)^2} = \sqrt{14}$. Then by Formula 2,

$$\iint_S (x+y+z)\,dS = \iint_D (u+v+u-v+1+2u+v)\;|\mathbf{r}_u \times \mathbf{r}_v|\;dA = \int_0^1\int_0^2 (4u+v+1)\cdot\sqrt{14}\,du\,dv$$
$$= \sqrt{14}\int_0^1 \left[2u^2+uv+u\right]_{u=0}^{u=2} dv = \sqrt{14}\int_0^1 (2v+10)\,dv = \sqrt{14}\left[v^2+10v\right]_0^1 = 11\sqrt{14}$$

6. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi/2$ and

$\mathbf{r}_u \times \mathbf{r}_v = (\cos v\,\mathbf{i} + \sin v\,\mathbf{j} + \mathbf{k}) \times (-u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}) = -u\cos v\,\mathbf{i} - u\sin v\,\mathbf{j} + u\,\mathbf{k} \quad\Rightarrow$

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2u^2} = \sqrt{2}\,u$ [since $u \ge 0$]. Then by Formula 2,

$$\iint_S xyz\,dS = \iint_D (u\cos v)(u\sin v)(u)\;|\mathbf{r}_u \times \mathbf{r}_v|\;dA = \int_0^1\int_0^{\pi/2} (u^3\sin v\cos v)\cdot\sqrt{2}\,u\,dv\,du$$
$$= \sqrt{2}\int_0^1 u^4\,du\int_0^{\pi/2}\sin v\cos v\,dv = \sqrt{2}\left[\tfrac{1}{5}u^5\right]_0^1\left[\tfrac{1}{2}\sin^2 v\right]_0^{\pi/2} = \sqrt{2}\cdot\tfrac{1}{5}\cdot\tfrac{1}{2} = \tfrac{1}{10}\sqrt{2}$$

7. $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$ and

$\mathbf{r}_u \times \mathbf{r}_v = \langle\cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle = \langle\sin v, -\cos v, u\rangle \quad\Rightarrow$

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{u^2+1}$. Then

$$\iint_S y\,dS = \iint_D (u\sin v)\;|\mathbf{r}_u \times \mathbf{r}_v|\;dA = \int_0^1\int_0^\pi (u\sin v)\cdot\sqrt{u^2+1}\,dv\,du = \int_0^1 u\sqrt{u^2+1}\,du\int_0^\pi \sin v\,dv$$
$$= \left[\tfrac{1}{3}(u^2+1)^{3/2}\right]_0^1\;\left[-\cos v\right]_0^\pi = \tfrac{1}{3}(2^{3/2}-1)\cdot 2 = \tfrac{2}{3}(2\sqrt{2}-1)$$

8. $\mathbf{r}(u,v) = \left\langle 2uv, u^2-v^2, u^2+v^2\right\rangle$, $u^2+v^2 \le 1$ and

$\mathbf{r}_u \times \mathbf{r}_v = \langle 2v, 2u, 2u\rangle \times \langle 2u, -2v, 2v\rangle = \left\langle 8uv, 4u^2-4v^2, -4u^2-4v^2\right\rangle$, so

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{(8uv)^2 + (4u^2-4v^2)^2 + (-4u^2-4v^2)^2} = \sqrt{64u^2v^2 + 32u^4 + 32v^4}$$
$$= \sqrt{32(u^2+v^2)^2} = 4\sqrt{2}(u^2+v^2)$$

Then

$$\begin{aligned}\iint_S (x^2+y^2)\,dS &= \iint_D \left[(2uv)^2+(u^2-v^2)^2\right]\,|\mathbf{r}_u\times\mathbf{r}_v|\,dA = \iint_D (4u^2v^2+u^4-2u^2v^2+v^4)\cdot 4\sqrt{2}(u^2+v^2)\,dA\\ &= 4\sqrt{2}\iint_D (u^4+2u^2v^2+v^4)\,(u^2+v^2)\,dA = 4\sqrt{2}\iint_D (u^2+v^2)^3\,dA = 4\sqrt{2}\int_0^{2\pi}\int_0^1 (r^2)^3\,r\,dr\,d\theta\\ &= 4\sqrt{2}\int_0^{2\pi} d\theta\int_0^1 r^7\,dr = 4\sqrt{2}\,\left[\theta\right]_0^{2\pi}\,\left[\tfrac{1}{8}r^8\right]_0^1 = 4\sqrt{2}\cdot 2\pi\cdot\tfrac{1}{8} = \sqrt{2}\,\pi\end{aligned}$$

9. $z = 1+2x+3y$ so $\dfrac{\partial z}{\partial x} = 2$ and $\dfrac{\partial z}{\partial y} = 3$. Then by Formula 4,

$$\begin{aligned}\iint_S x^2yz\,dS &= \iint_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA = \int_0^3\int_0^2 x^2y(1+2x+3y)\,\sqrt{4+9+1}\,dy\,dx\\ &= \sqrt{14}\int_0^3\int_0^2 (x^2y+2x^3y+3x^2y^2)\,dy\,dx = \sqrt{14}\int_0^3\left[\tfrac{1}{2}x^2y^2+x^3y^2+x^2y^3\right]_{y=0}^{y=2}dx\\ &= \sqrt{14}\int_0^3 (10x^2+4x^3)\,dx = \sqrt{14}\left[\tfrac{10}{3}x^3+x^4\right]_0^3 = 171\sqrt{14}\end{aligned}$$

10. S is the part of the plane $z = 4-2x-2y$ over the region $D = \{(x,y)\mid 0\le x\le 2, 0\le y\le 2-x\}$. Thus

$$\begin{aligned}\iint_S xz\,dS &= \iint_D x(4-2x-2y)\sqrt{(-2)^2+(-2)^2+1}\,dA = 3\int_0^2\int_0^{2-x}\left(4x-2x^2-2xy\right)dy\,dx\\ &= 3\int_0^2\left[4xy-2x^2y-xy^2\right]_{y=0}^{y=2-x}dx = 3\int_0^2\left[4x(2-x)-2x^2(2-x)-x(2-x)^2\right]dx\\ &= 3\int_0^2\left(x^3-4x^2+4x\right)dx = 3\left[\tfrac{1}{4}x^4-\tfrac{4}{3}x^3+2x^2\right]_0^2 = 3\left(4-\tfrac{32}{3}+8\right) = 4\end{aligned}$$

11. An equation of the plane through the points $(1,0,0)$, $(0,-2,0)$, and $(0,0,4)$ is $4x-2y+z=4$, so S is the region in the plane $z = 4-4x+2y$ over $D = \{(x,y)\mid 0\le x\le 1, 2x-2\le y\le 0\}$. Thus by Formula 4,

$$\begin{aligned}\iint_S x\,dS &= \iint_D x\sqrt{(-4)^2+(2)^2+1}\,dA = \sqrt{21}\int_0^1\int_{2x-2}^0 x\,dy\,dx = \sqrt{21}\int_0^1\left[xy\right]_{y=2x-2}^{y=0}dx\\ &= \sqrt{21}\int_0^1(-2x^2+2x)\,dx = \sqrt{21}\left[-\tfrac{2}{3}x^3+x^2\right]_0^1 = \sqrt{21}\left(-\tfrac{2}{3}+1\right) = \tfrac{\sqrt{21}}{3}\end{aligned}$$

12. $z = \frac{2}{3}(x^{3/2}+y^{3/2})$ and

$$\begin{aligned}\iint_S y\,dS &= \iint_D y\sqrt{\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2+1}\,dA = \int_0^1\int_0^1 y\sqrt{x+y+1}\,dx\,dy\\ &= \int_0^1 y\left[\tfrac{2}{3}(x+y+1)^{3/2}\right]_{x=0}^{x=1}dy = \int_0^1 \tfrac{2}{3}y\left[(y+2)^{3/2}-(y+1)^{3/2}\right]dy\end{aligned}$$

Substituting $u = y+2$ in the first term and $t = y+1$ in the second, we have

$$\begin{aligned}\iint_S y\,dS &= \tfrac{2}{3}\int_2^3 (u-2)u^{3/2}\,du - \tfrac{2}{3}\int_1^2 (t-1)t^{3/2}\,dt = \tfrac{2}{3}\left[\tfrac{2}{7}u^{7/2}-\tfrac{4}{5}u^{5/2}\right]_2^3 - \tfrac{2}{3}\left[\tfrac{2}{7}t^{7/2}-\tfrac{2}{5}t^{5/2}\right]_1^2\\ &= \tfrac{2}{3}\left[\tfrac{2}{7}(3^{7/2}-2^{7/2})-\tfrac{4}{5}(3^{5/2}-2^{5/2})-\tfrac{2}{7}(2^{7/2}-1)+\tfrac{2}{5}(2^{5/2}-1)\right]\\ &= \tfrac{2}{3}\left(\tfrac{18}{35}\sqrt{3}+\tfrac{8}{35}\sqrt{2}-\tfrac{4}{35}\right) = \tfrac{4}{105}\left(9\sqrt{3}+4\sqrt{2}-2\right)\end{aligned}$$

13. S is the portion of the cone $z^2 = x^2+y^2$ for $1\le z\le 3$, or equivalently, S is the part of the surface $z = \sqrt{x^2+y^2}$ over the region $D = \left\{(x,y)\mid 1\le x^2+y^2\le 9\right\}$. Thus

$$\iint_S x^2 z^2\,dS = \iint_D x^2(x^2+y^2)\sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2+1}\,dA$$

$$= \iint_D x^2(x^2+y^2)\sqrt{\frac{x^2+y^2}{x^2+y^2}+1}\,dA = \iint_D \sqrt{2}\,x^2(x^2+y^2)\,dA = \sqrt{2}\int_0^{2\pi}\int_1^3 (r\cos\theta)^2(r^2)\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\cos^2\theta\,d\theta\int_1^3 r^5\,dr = \sqrt{2}\left[\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{6}r^6\right]_1^3 = \sqrt{2}\,(\pi)\cdot\tfrac{1}{6}(3^6-1) = \frac{364\sqrt{2}}{3}\,\pi$$

14. Using y and z as parameters, we have $\mathbf{r}(y,z) = (y+2z^2)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $0 \le y \le 1$, $0 \le z \le 1$.

Then $\mathbf{r}_y \times \mathbf{r}_z = (\mathbf{i}+\mathbf{j}) \times (4z\,\mathbf{i}+\mathbf{k}) = \mathbf{i} - \mathbf{j} - 4z\,\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = \sqrt{2+16z^2}$. Thus

$$\iint_S z\,dS = \int_0^1\int_0^1 z\sqrt{2+16z^2}\,dy\,dz = \int_0^1 z\sqrt{2+16z^2}\,dz = \left[\tfrac{1}{32}\cdot\tfrac{2}{3}(2+16z^2)^{3/2}\right]_0^1 = \tfrac{1}{48}(18^{3/2}-2^{3/2}) = \tfrac{13}{12}\sqrt{2}.$$

15. Using x and z as parameters, we have $\mathbf{r}(x,z) = x\,\mathbf{i} + (x^2+z^2)\,\mathbf{j} + z\,\mathbf{k}$, $x^2+z^2 \le 4$. Then

$\mathbf{r}_x \times \mathbf{r}_z = (\mathbf{i}+2x\,\mathbf{j}) \times (2z\,\mathbf{j}+\mathbf{k}) = 2x\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$ and $|\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+1+4z^2} = \sqrt{1+4(x^2+z^2)}$. Thus

$$\iint_S y\,dS = \iint_{x^2+z^2\le 4}(x^2+z^2)\sqrt{1+4(x^2+z^2)}\,dA = \int_0^{2\pi}\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr$$

$$= 2\pi\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr \qquad \left[\text{let } u = 1+4r^2 \;\Rightarrow\; r^2 = \tfrac{1}{4}(u-1) \text{ and } \tfrac{1}{8}du = r\,dr\right]$$

$$= 2\pi\int_1^{17}\tfrac{1}{4}(u-1)\sqrt{u}\cdot\tfrac{1}{8}du = \tfrac{1}{16}\pi\int_1^{17}(u^{3/2}-u^{1/2})\,du$$

$$= \tfrac{1}{16}\pi\left[\tfrac{2}{5}u^{5/2}-\tfrac{2}{3}u^{3/2}\right]_1^{17} = \tfrac{1}{16}\pi\left[\tfrac{2}{5}(17)^{5/2}-\tfrac{2}{3}(17)^{3/2}-\tfrac{2}{5}+\tfrac{2}{3}\right] = \frac{\pi}{60}(391\sqrt{17}+1)$$

16. The sphere intersects the cylinder in the circle $x^2+y^2=1$, $z=\sqrt{3}$, so S is the portion of the sphere where $z \ge \sqrt{3}$.

Using spherical coordinates to parametrize the sphere we have $\mathbf{r}(\phi,\theta) = 2\sin\phi\,\cos\theta\,\mathbf{i} + 2\sin\phi\,\sin\theta\,\mathbf{j} + 2\cos\phi\,\mathbf{k}$, and

$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 4\sin\phi$ (see Example 16.6.10). The portion where $z \ge \sqrt{3}$ corresponds to $0 \le \phi \le \frac{\pi}{6}$, $0 \le \theta \le 2\pi$ so

$$\iint_S y^2\,dS = \int_0^{2\pi}\int_0^{\pi/6}(2\sin\phi\sin\theta)^2(4\sin\phi)\,d\phi\,d\theta = 16\int_0^{2\pi}\sin^2\theta\,d\theta\int_0^{\pi/6}\sin^3\phi\,d\phi$$

$$= 16\left[\tfrac{1}{2}\theta-\tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{3}\cos^3\phi-\cos\phi\right]_0^{\pi/6} = 16(\pi)\left(\tfrac{\sqrt{3}}{8}-\tfrac{\sqrt{3}}{2}-\tfrac{1}{3}+1\right) = \left(\tfrac{32}{3}-6\sqrt{3}\right)\pi$$

17. Using spherical coordinates and Example 16.6.10 we have $\mathbf{r}(\phi,\theta) = 2\sin\phi\cos\theta\,\mathbf{i} + 2\sin\phi\sin\theta\,\mathbf{j} + 2\cos\phi\,\mathbf{k}$ and

$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 4\sin\phi$. Then $\iint_S(x^2z+y^2z)\,dS = \int_0^{2\pi}\int_0^{\pi/2}(4\sin^2\phi)(2\cos\phi)(4\sin\phi)\,d\phi\,d\theta = 16\pi\sin^4\phi\big]_0^{\pi/2} = 16\pi.$

18. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x+y=5$; and the back, S_3, in the plane $x=0$.

On S_1: the surface is given by $\mathbf{r}(u,v) = u\,\mathbf{i} + 3\cos v\,\mathbf{j} + 3\sin v\,\mathbf{k}$, $0 \le v \le 2\pi$, and $0 \le x \le 5-y \;\Rightarrow$ $0 \le u \le 5-3\cos v$. Then $\mathbf{r}_u \times \mathbf{r}_v = -3\cos v\,\mathbf{j} - 3\sin v\,\mathbf{k}$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{9\cos^2 v+9\sin^2 v} = 3$, so

$$\iint_{S_1} xz\,dS = \int_0^{2\pi}\int_0^{5-3\cos v} u(3\sin v)(3)\,du\,dv = 9\int_0^{2\pi}\left[\tfrac{1}{2}u^2\right]_{u=0}^{u=5-3\cos v}\sin v\,dv$$

$$= \tfrac{9}{2}\int_0^{2\pi}(5-3\cos v)^2\sin v\,dv = \tfrac{9}{2}\left[\tfrac{1}{9}(5-3\cos v)^3\right]_0^{2\pi} = 0.$$

On S_2: $\mathbf{r}(y,z) = (5-y)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $|\mathbf{r}_y \times \mathbf{r}_z| = |\mathbf{i}+\mathbf{j}| = \sqrt{2}$, where $y^2+z^2 \le 9$ and

$$\iint_{S_2} xz\,dS = \iint_{y^2+z^2\le 9} (5-y)z\sqrt{2}\,dA = \sqrt{2}\int_0^{2\pi}\int_0^3 (5-r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$$
$$= \sqrt{2}\int_0^{2\pi}\int_0^3 (5r^2 - r^3\cos\theta)(\sin\theta)\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\left[\tfrac{5}{3}r^3 - \tfrac{1}{4}r^4\cos\theta\right]_{r=0}^{r=3}\sin\theta\,d\theta$$
$$= \sqrt{2}\int_0^{2\pi}\left(45 - \tfrac{81}{4}\cos\theta\right)\sin\theta\,d\theta = \sqrt{2}\left(\tfrac{4}{81}\right)\cdot\tfrac{1}{2}\left(45 - \tfrac{81}{4}\cos\theta\right)^2\Big]_0^{2\pi} = 0$$

On S_3: $x=0$ so $\iint_{S_3} xz\,dS = 0$. Hence $\iint_S xz\,dS = 0+0+0 = 0$.

19. S is given by $\mathbf{r}(u,v) = u\,\mathbf{i} + \cos v\,\mathbf{j} + \sin v\,\mathbf{k}$, $0 \le u \le 3$, $0 \le v \le \pi/2$. Then

$\mathbf{r}_u \times \mathbf{r}_v = \mathbf{i} \times (-\sin v\,\mathbf{j} + \cos v\,\mathbf{k}) = -\cos v\,\mathbf{j} - \sin v\,\mathbf{k}$ and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\cos^2 v + \sin^2 v} = 1$, so

$$\iint_S (z + x^2y)\,dS = \int_0^{\pi/2}\int_0^3 (\sin v + u^2\cos v)(1)\,du\,dv = \int_0^{\pi/2}(3\sin v + 9\cos v)\,dv$$
$$= [-3\cos v + 9\sin v]_0^{\pi/2} = 0 + 9 + 3 - 0 = 12$$

20. Let S_1 be the lateral surface, S_2 the top disk, and S_3 the bottom disk.

On S_1: $\mathbf{r}(\theta,z) = 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 2$, $|\mathbf{r}_\theta \times \mathbf{r}_z| = 3$,

$\iint_{S_1}(x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^2 (9+z^2)\,3\,dz\,d\theta = 2\pi(54+8) = 124\pi$.

On S_2: $\mathbf{r}(\theta,r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2\,\mathbf{k}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$,

$\iint_{S_2}(x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2+4)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4} + 18\right) = \frac{153}{2}\pi$.

On S_3: $\mathbf{r}(\theta,r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$,

$\iint_{S_3}(x^2+y^2+z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2+0)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4}\right) = \frac{81}{2}\pi$.

Hence $\iint_S \left(x^2+y^2+z^2\right)dS = 124\pi + \frac{153}{2}\pi + \frac{81}{2}\pi = 241\pi$.

21. From Exercise 5, $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (u-v)\,\mathbf{j} + (1+2u+v)\,\mathbf{k}$, $0 \le u \le 2$, $0 \le v \le 1$, and $\mathbf{r}_u \times \mathbf{r}_v = 3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$.

Then

$$\mathbf{F}(\mathbf{r}(u,v)) = (1+2u+v)e^{(u+v)(u-v)}\,\mathbf{i} - 3(1+2u+v)e^{(u+v)(u-v)}\,\mathbf{j} + (u+v)(u-v)\,\mathbf{k}$$
$$= (1+2u+v)e^{u^2-v^2}\,\mathbf{i} - 3(1+2u+v)e^{u^2-v^2}\,\mathbf{j} + (u^2-v^2)\,\mathbf{k}$$

Because the z-component of $\mathbf{r}_u \times \mathbf{r}_v$ is negative we use $-(\mathbf{r}_u \times \mathbf{r}_v)$ in Formula 9 for the upward orientation:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(-(\mathbf{r}_u \times \mathbf{r}_v))\,dA = \int_0^1\int_0^2\left[-3(1+2u+v)e^{u^2-v^2} + 3(1+2u+v)e^{u^2-v^2} + 2(u^2-v^2)\right]du\,dv$$
$$= \int_0^1\int_0^2 2(u^2-v^2)\,du\,dv = 2\int_0^1\left[\tfrac{1}{3}u^3 - uv^2\right]_{u=0}^{u=2}dv = 2\int_0^1\left(\tfrac{8}{3} - 2v^2\right)dv$$
$$= 2\left[\tfrac{8}{3}v - \tfrac{2}{3}v^3\right]_0^1 = 2\left(\tfrac{8}{3} - \tfrac{2}{3}\right) = 4$$

22. $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$ and

$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle = \langle \sin v, -\cos v, u\rangle$. Here $\mathbf{F}(\mathbf{r}(u,v)) = v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}$ and, by Formula 9,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA = \int_0^1\int_0^\pi (v\sin v - u\sin v\cos v + u^2\cos v)\,dv\,du$$
$$= \int_0^1 \left[\sin v - v\cos v - \tfrac{1}{2}u\sin^2 v + u^2\sin v\right]_{v=0}^{v=\pi} du = \int_0^1 \pi\,du = \pi u\big]_0^1 = \pi$$

23. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, $z = g(x,y) = 4 - x^2 - y^2$, and D is the square $[0,1]\times[0,1]$, so by Equation 10

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D [-xy(-2x) - yz(-2y) + zx]\,dA = \int_0^1\int_0^1 [2x^2y + 2y^2(4-x^2-y^2) + x(4-x^2-y^2)]\,dy\,dx$$
$$= \int_0^1 \left(\tfrac{1}{3}x^2 + \tfrac{11}{3}x - x^3 + \tfrac{34}{15}\right)dx = \tfrac{713}{180}$$

24. $\mathbf{F}(x,y,z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$, $z = g(x,y) = \sqrt{x^2+y^2}$, and D is the annular region $\{(x,y) \mid 1 \le x^2+y^2 \le 9\}$. Since S has downward orientation, we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D \left[-(-x)\left(\frac{x}{\sqrt{x^2+y^2}}\right) - (-y)\left(\frac{y}{\sqrt{x^2+y^2}}\right) + z^3\right] dA$$
$$= -\iint_D \left[\frac{x^2+y^2}{\sqrt{x^2+y^2}} + \left(\sqrt{x^2+y^2}\right)^3\right] dA = -\int_0^{2\pi}\int_1^3 \left(\frac{r^2}{r} + r^3\right) r\,dr\,d\theta$$
$$= -\int_0^{2\pi} d\theta \int_1^3 (r^2+r^4)\,dr = -\left[\theta\right]_0^{2\pi}\left[\tfrac{1}{3}r^3 + \tfrac{1}{5}r^5\right]_1^3$$
$$= -2\pi\left(9 + \tfrac{243}{5} - \tfrac{1}{3} - \tfrac{1}{5}\right) = -\tfrac{1712}{15}\pi$$

25. $\mathbf{F}(x,y,z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$, $z = g(x,y) = \sqrt{4-x^2-y^2}$ and D is the quarter disk $\left\{(x,y)\,\middle|\, 0\le x\le 2, 0\le y\le \sqrt{4-x^2}\right\}$. S has downward orientation, so by Formula 10,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D \left[-x\cdot\tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2x) - (-z)\cdot\tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2y) + y\right] dA$$
$$= -\iint_D \left(\frac{x^2}{\sqrt{4-x^2-y^2}} - \sqrt{4-x^2-y^2}\cdot\frac{y}{\sqrt{4-x^2-y^2}} + y\right) dA$$
$$= -\iint_D x^2(4-(x^2+y^2))^{-1/2}\,dA = -\int_0^{\pi/2}\int_0^2 (r\cos\theta)^2(4-r^2)^{-1/2}\,r\,dr\,d\theta$$
$$= -\int_0^{\pi/2}\cos^2\theta\,d\theta \int_0^2 r^3(4-r^2)^{-1/2}\,dr \qquad \left[\text{let } u = 4-r^2 \;\Rightarrow\; r^2 = 4-u \text{ and } -\tfrac{1}{2}\,du = r\,dr\right]$$
$$= -\int_0^{\pi/2}\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right)d\theta \int_4^0 -\tfrac{1}{2}(4-u)(u)^{-1/2}\,du$$
$$= -\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}\left(-\tfrac{1}{2}\right)\left[8\sqrt{u} - \tfrac{2}{3}u^{3/2}\right]_4^0 = -\tfrac{\pi}{4}\left(-\tfrac{1}{2}\right)\left(-16 + \tfrac{16}{3}\right) = -\tfrac{4}{3}\pi$$

26. $\mathbf{F}(x,y,z) = xz\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$

Using spherical coordinates, S is given by $x = 5\sin\phi\cos\theta$, $y = 5\sin\phi\sin\theta$, $z = 5\cos\phi$, $0\le\theta\le\pi$, $0\le\phi\le\pi$. $\mathbf{F}(\mathbf{r}(\phi,\theta)) = (5\sin\phi\cos\theta)(5\cos\phi)\,\mathbf{i} + (5\sin\phi\cos\theta)\,\mathbf{j} + (5\sin\phi\sin\theta)\,\mathbf{k}$ and $\mathbf{r}_\phi\times\mathbf{r}_\theta = 25\sin^2\phi\cos\theta\,\mathbf{i} + 25\sin^2\phi\sin\theta\,\mathbf{j} + 25\cos\phi\sin\phi\,\mathbf{k}$, so

$$\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta) = 625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\cos\theta\sin\theta + 125\sin^2\phi\cos\phi\sin\theta$$

Then

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \left[\mathbf{F}(\mathbf{r}(\phi,\theta))\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)\right] dA \\ &= \int_0^\pi\int_0^\pi (625\sin^3\phi\cos\phi\cos^2\theta + 125\sin^3\phi\cos\theta\sin\theta + 125\sin^2\phi\cos\phi\sin\theta)\,d\theta\,d\phi \\ &= 125\int_0^\pi \left[5\sin^3\phi\cos\phi\left(\tfrac12\theta+\tfrac14\sin 2\theta\right)+\sin^3\phi\left(\tfrac12\sin^2\theta\right)+\sin^2\phi\cos\phi\,(-\cos\theta)\right]_{\theta=0}^{\theta=\pi} d\phi \\ &= 125\int_0^\pi \left(\tfrac52\pi\sin^3\phi\cos\phi + 2\sin^2\phi\cos\phi\right)d\phi = 125\left[\tfrac52\pi\cdot\tfrac14\sin^4\phi + 2\cdot\tfrac13\sin^3\phi\right]_0^\pi = 0\end{aligned}$$

27. Let S_1 be the paraboloid $y = x^2+z^2$, $0\le y\le 1$ and S_2 the disk $x^2+z^2\le 1$, $y=1$. Since S is a closed surface, we use the outward orientation.

On S_1: $\mathbf{F}(\mathbf{r}(x,z)) = (x^2+z^2)\,\mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_x\times\mathbf{r}_z = 2x\,\mathbf{i}-\mathbf{j}+2z\,\mathbf{k}$ (since the $\mathbf{j}$-component must be negative on S_1). Then

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \iint_{x^2+z^2\le 1}\left[-(x^2+z^2)-2z^2\right]dA = -\int_0^{2\pi}\int_0^1 (r^2+2r^2\sin^2\theta)\,r\,dr\,d\theta \\ &= -\int_0^{2\pi}\int_0^1 r^3(1+2\sin^2\theta)\,dr\,d\theta = -\int_0^{2\pi}(1+1-\cos 2\theta)\,d\theta\int_0^1 r^3\,dr \\ &= -\left[2\theta-\tfrac12\sin 2\theta\right]_0^{2\pi}\left[\tfrac14 r^4\right]_0^1 = -4\pi\cdot\tfrac14 = -\pi\end{aligned}$$

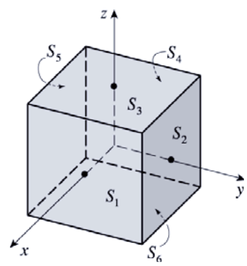
On S_2: $\mathbf{F}(\mathbf{r}(x,z)) = \mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_z\times\mathbf{r}_x = \mathbf{j}$. Then $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1}(1)\,dA = \pi$.

Hence $\iint_S \mathbf{F}\cdot d\mathbf{S} = -\pi+\pi = 0$.

28. $\mathbf{F}(x,y,z) = xy\,\mathbf{i}+4x^2\,\mathbf{j}+yz\,\mathbf{k}$, $z = g(x,y) = xe^y$, and D is the square $[0,1]\times[0,1]$, so by Equation 10

$$\begin{aligned}\iint_S\mathbf{F}\cdot d\mathbf{S} &= \iint_D[-xy(e^y)-4x^2(xe^y)+yz]dA = \int_0^1\int_0^1(-xye^y-4x^3e^y+xye^y)\,dy\,dx \\ &= \int_0^1\left[-4x^3e^y\right]_{y=0}^{y=1}dx = (e-1)\int_0^1(-4x^3)\,dx = 1-e\end{aligned}$$

29. Here S consists of the six faces of the cube as labeled in the figure. On S_1:



$\mathbf{F} = \mathbf{i}+2y\,\mathbf{j}+3z\,\mathbf{k}$, $\mathbf{r}_y\times\mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 dy\,dz = 4$;

S_2: $\mathbf{F} = x\,\mathbf{i}+2\,\mathbf{j}+3z\,\mathbf{k}$, $\mathbf{r}_z\times\mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 2\,dx\,dz = 8$;

S_3: $\mathbf{F} = x\,\mathbf{i}+2y\,\mathbf{j}+3\,\mathbf{k}$, $\mathbf{r}_x\times\mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 3\,dx\,dy = 12$;

S_4: $\mathbf{F} = -\mathbf{i}+2y\,\mathbf{j}+3z\,\mathbf{k}$, $\mathbf{r}_z\times\mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4}\mathbf{F}\cdot d\mathbf{S} = 4$;

S_5: $\mathbf{F} = x\,\mathbf{i}-2\,\mathbf{j}+3z\,\mathbf{k}$, $\mathbf{r}_x\times\mathbf{r}_z = -\mathbf{j}$ and $\iint_{S_5}\mathbf{F}\cdot d\mathbf{S} = 8$;

S_6: $\mathbf{F} = x\,\mathbf{i}+2y\,\mathbf{j}-3\,\mathbf{k}$, $\mathbf{r}_y\times\mathbf{r}_x = -\mathbf{k}$ and $\iint_{S_6}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 3\,dx\,dy = 12$.

Hence $\iint_S\mathbf{F}\cdot d\mathbf{S} = \sum_{i=1}^{6}\iint_{S_i}\mathbf{F}\cdot d\mathbf{S} = 48$.

30. Here S consists of three surfaces: S_1, the lateral surface of the cylinder; S_2, the front formed by the plane $x+y=2$; and the back, S_3, in the plane $y=0$.

On S_1: $\mathbf{F}(\mathbf{r}(\theta,y)) = \sin\theta\,\mathbf{i}+y\,\mathbf{j}+5\,\mathbf{k}$ and $\mathbf{r}_\theta\times\mathbf{r}_y = \sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{k}$ $\Rightarrow$

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{2-\sin\theta}(\sin^2\theta+5\cos\theta)\,dy\,d\theta \\ &= \int_0^{2\pi}(2\sin^2\theta+10\cos\theta-\sin^3\theta-5\sin\theta\,\cos\theta)\,d\theta = 2\pi\end{aligned}$$

On S_2: $\mathbf{F}(\mathbf{r}(x,z)) = x\,\mathbf{i} + (2-x)\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{i} + \mathbf{j}$.

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+z^2 \le 1} [x + (2-x)]\, dA = 2\pi$$

On S_3: $\mathbf{F}(\mathbf{r}(x,z)) = x\,\mathbf{i} + 5\,\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ so $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = 0$. Hence $\iint_S \mathbf{F} \cdot d\mathbf{S} = 4\pi$.

31. Here S consists of four surfaces: S_1, the top surface (a portion of the circular cylinder $y^2 + z^2 = 1$); S_2, the bottom surface (a portion of the xy-plane); S_3, the front half-disk in the plane $x = 2$, and S_4, the back half-disk in the plane $x = 0$.

On S_1: The surface is $z = \sqrt{1-y^2}$ for $0 \le x \le 2$, $-1 \le y \le 1$ with upward orientation, so

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_{-1}^1 \left[-x^2\,(0) - y^2 \left(-\frac{y}{\sqrt{1-y^2}} \right) + z^2 \right] dy\,dx = \int_0^2 \int_{-1}^1 \left(\frac{y^3}{\sqrt{1-y^2}} + 1 - y^2 \right) dy\,dx$$

$$= \int_0^2 \left[-\sqrt{1-y^2} + \tfrac{1}{3}(1-y^2)^{3/2} + y - \tfrac{1}{3}y^3 \right]_{y=-1}^{y=1} dx = \int_0^2 \tfrac{4}{3}\,dx = \tfrac{8}{3}$$

On S_2: The surface is $z = 0$ with downward orientation, so

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_{-1}^1 \left(-z^2\right) dy\,dx = \int_0^2 \int_{-1}^1 (0)\,dy\,dx = 0$$

On S_3: The surface is $x = 2$ for $-1 \le y \le 1$, $0 \le z \le \sqrt{1-y^2}$, oriented in the positive x-direction. Regarding y and z as parameters, we have $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and

$$\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} x^2\,dz\,dy = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} 4\,dz\,dy = 4A\,(S_3) = 2\pi$$

On S_4: The surface is $x = 0$ for $-1 \le y \le 1$, $0 \le z \le \sqrt{1-y^2}$, oriented in the negative x-direction. Regarding y and z as parameters, we use $-\left(\mathbf{r}_y \times \mathbf{r}_z\right) = -\mathbf{i}$ and

$$\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} x^2\,dz\,dy = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} (0)\;dz\,dy = 0$$

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{8}{3} + 0 + 2\pi + 0 = 2\pi + \frac{8}{3}$.

32. Here S consists of four surfaces: S_1, the triangular face with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$; S_2, the face of the tetrahedron in the xy-plane; S_3, the face in the xz-plane; and S_4, the face in the yz-plane.

On S_1: The face is the portion of the plane $z = 1 - x - y$ for $0 \le x \le 1$, $0 \le y \le 1 - x$ with upward orientation, so

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-x} [-y\,(-1) - (z-y)\,(-1) + x]\,dy\,dx = \int_0^1 \int_0^{1-x} (z+x)\,dy\,dx = \int_0^1 \int_0^{1-x} (1-y)\,dy\,dx$$

$$= \int_0^1 \left[y - \tfrac{1}{2}y^2 \right]_{y=0}^{y=1-x} dx = \tfrac{1}{2} \int_0^1 \left(1-x^2\right) dx = \tfrac{1}{2} \left[x - \tfrac{1}{3}x^3 \right]_0^1 = \tfrac{1}{3}$$

On S_2: The surface is $z = 0$ with downward orientation, so

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-x} (-x)\,dy\,dx = -\int_0^1 x\,(1-x)\,dx = -\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \right]_0^1 = -\tfrac{1}{6}$$

On S_3: The surface is $y = 0$ for $0 \le x \le 1$, $0 \le z \le 1 - x$, oriented in the negative y-direction. Regarding x and z as parameters, we have $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ and

$$\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-x} -(z-y)\,dz\,dx = -\int_0^1 \int_0^{1-x} z\,dz\,dx = -\int_0^1 \left[\tfrac{1}{2}z^2 \right]_{z=0}^{z=1-x} dx$$

$$= -\tfrac{1}{2} \int_0^1 (1-x)^2\,dx = \tfrac{1}{6} \left[(1-x)^3 \right]_0^1 = -\tfrac{1}{6}$$

On S_4: The surface is $x = 0$ for $0 \le y \le 1$, $0 \le z \le 1 - y$, oriented in the negative x-direction. Regarding y and z as parameters, we have $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ so we use $-(\mathbf{r}_y \times \mathbf{r}_z) = -\mathbf{i}$ and

$$\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{1-y} (-y)\,dz\,dy = -\int_0^1 y\,(1-y)\,dy = -\left[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right]_0^1 = -\tfrac{1}{6}$$

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} = -\frac{1}{6}$.

33. $z = xe^y \;\Rightarrow\; \partial z/\partial x = e^y$, $\partial z/\partial y = xe^y$, so by Formula 4, a CAS gives $\iint_S (x^2 + y^2 + z^2)\,dS = \int_0^1 \int_0^1 (x^2 + y^2 + x^2 e^{2y})\sqrt{e^{2y} + x^2 e^{2y} + 1}\,dx\,dy \approx 4.5822$.

34. $z = xy \;\Rightarrow\; \partial z/\partial x = y$, $\partial z/\partial y = x$, so by Formula 4, a CAS gives

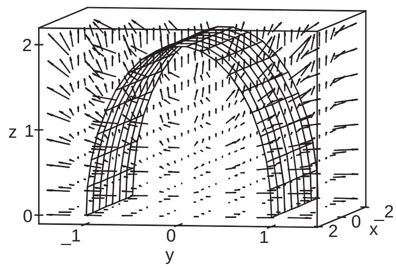
$$\begin{aligned}\iint_S x^2 yz\,dS &= \int_0^1 \int_0^1 x^2 y(xy)\sqrt{y^2 + x^2 + 1}\,dx\,dy \\ &= \tfrac{1}{60}\sqrt{3} - \tfrac{1}{12}\ln\!\left(1 + \sqrt{3}\right) - \tfrac{1}{192}\ln\!\left(\sqrt{2} + 1\right) + \tfrac{317}{2880}\sqrt{2} + \tfrac{1}{24}\ln 2\end{aligned}$$

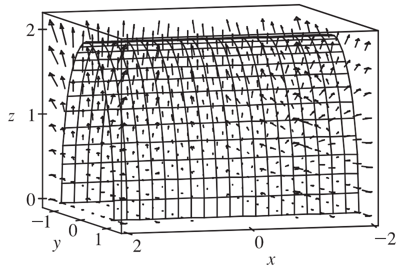
35. We use Formula 4 with $z = 3 - 2x^2 - y^2 \;\Rightarrow\; \partial z/\partial x = -4x$, $\partial z/\partial y = -2y$. The boundaries of the region $3 - 2x^2 - y^2 \ge 0$ are $-\sqrt{\frac{3}{2}} \le x \le \sqrt{\frac{3}{2}}$ and $-\sqrt{3 - 2x^2} \le y \le \sqrt{3 - 2x^2}$, so we use a CAS (with precision reduced to seven or fewer digits; otherwise the calculation may take a long time) to calculate

$$\iint_S x^2 y^2 z^2\,dS = \int_{-\sqrt{3/2}}^{\sqrt{3/2}} \int_{-\sqrt{3-2x^2}}^{\sqrt{3-2x^2}} x^2 y^2 (3 - 2x^2 - y^2)^2 \sqrt{16x^2 + 4y^2 + 1}\,dy\,dx \approx 3.4895$$

36. The flux of $\mathbf{F}$ across S is given by $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS$. Now on S, $z = g(x, y) = 2\sqrt{1 - y^2}$, so $\partial g/\partial x = 0$ and $\partial g/\partial y = -2y(1 - y^2)^{-1/2}$. Therefore, by (10),

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_{-2}^{2} \int_{-1}^{1} \left(-x^2 y\left[-2y(1 - y^2)^{-1/2}\right] + \left[2\sqrt{1 - y^2}\right]^2 e^{x/5}\right) dy\,dx = \tfrac{1}{3}(16\pi + 80e^{2/5} - 80e^{-2/5})$$





37. If S is given by $y = h(x, z)$, then S is also the level surface $f(x, y, z) = y - h(x, z) = 0$.

$\mathbf{n} = \dfrac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|} = \dfrac{-h_x\,\mathbf{i} + \mathbf{j} - h_z\,\mathbf{k}}{\sqrt{h_x^2 + 1 + h_z^2}}$, and $-\mathbf{n}$ is the unit normal that points to the left. Now we proceed as in the derivation of (10), using Formula 4 to evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iint_D (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \frac{\dfrac{\partial h}{\partial x}\mathbf{i} - \mathbf{j} + \dfrac{\partial h}{\partial z}\mathbf{k}}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2 + 1 + \left(\dfrac{\partial h}{\partial z}\right)^2}} \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + 1 + \left(\frac{\partial h}{\partial z}\right)^2}\,dA$$

where D is the projection of S onto the xz-plane. Therefore $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(P\,\frac{\partial h}{\partial x} - Q + R\,\frac{\partial h}{\partial z}\right) dA$.

38. If S is given by $x = k(y, z)$, then S is also the level surface $f(x, y, z) = x - k(y, z) = 0$.

$\mathbf{n} = \dfrac{\nabla f(x, y, z)}{|\nabla f(x, y, z)|} = \dfrac{\mathbf{i} - k_y\,\mathbf{j} - k_z\,\mathbf{k}}{\sqrt{1 + k_y^2 + k_z^2}}$, and since the x-component is positive this is the unit normal that points forward.

Now we proceed as in the derivation of (10), using Formula 4 for

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \frac{\mathbf{i} - \dfrac{\partial k}{\partial y}\mathbf{j} - \dfrac{\partial k}{\partial z}\mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial k}{\partial y}\right)^2 + \left(\dfrac{\partial k}{\partial z}\right)^2}} \sqrt{1 + \left(\frac{\partial k}{\partial y}\right)^2 + \left(\frac{\partial k}{\partial z}\right)^2}\, dA$$

where D is the projection of S onto the yz-plane. Therefore $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(P - Q\,\frac{\partial k}{\partial y} - R\,\frac{\partial k}{\partial z}\right) dA.$

39. $m = \iint_S K\, dS = K \cdot 4\pi\left(\frac{1}{2}a^2\right) = 2\pi a^2 K$; by symmetry $M_{xz} = M_{yz} = 0$, and

$M_{xy} = \iint_S zK\, dS = K\int_0^{2\pi}\int_0^{\pi/2}(a\cos\phi)(a^2\sin\phi)\, d\phi\, d\theta = 2\pi K a^3\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \pi K a^3.$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{1}{2}a\right)$.

40. S is given by $\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + \sqrt{x^2 + y^2}\,\mathbf{k}$, $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \dfrac{x^2 + y^2}{x^2 + y^2}} = \sqrt{2}$ so

$$m = \iint_S \left(10 - \sqrt{x^2 + y^2}\right) dS = \iint\limits_{1 \le x^2 + y^2 \le 16} \left(10 - \sqrt{x^2 + y^2}\right)\sqrt{2}\, dA$$
$$= \int_0^{2\pi}\int_1^4 \sqrt{2}\,(10 - r)\, r\, dr\, d\theta = 2\pi\sqrt{2}\left[5r^2 - \tfrac{1}{3}r^3\right]_1^4 = 108\sqrt{2}\,\pi$$

41. (a) $I_z = \iint_S (x^2 + y^2)\rho(x, y, z)\, dS$

(b) $I_z = \iint_S (x^2 + y^2)\left(10 - \sqrt{x^2 + y^2}\right) dS = \displaystyle\iint\limits_{1 \le x^2 + y^2 \le 16} (x^2 + y^2)\left(10 - \sqrt{x^2 + y^2}\right)\sqrt{2}\, dA$

$= \int_0^{2\pi}\int_1^4 \sqrt{2}\,(10r^3 - r^4)\, dr\, d\theta = 2\sqrt{2}\,\pi\left(\frac{4329}{10}\right) = \frac{4329}{5}\sqrt{2}\,\pi$

42. Using spherical coordinates to parametrize the sphere we have $\mathbf{r}(\phi, \theta) = 5\sin\phi\cos\theta\,\mathbf{i} + 5\sin\phi\sin\theta\,\mathbf{j} + 5\cos\phi\,\mathbf{k}$, and $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 25\sin\phi$ (see Example 16.6.10). S is the portion of the sphere where $z \ge 4$, so $0 \le \phi \le \tan^{-1}(3/4)$ and $0 \le \theta \le 2\pi$.

(a) $m = \iint_S \rho(x, y, z) dS = \int_0^{2\pi}\int_0^{\tan^{-1}(3/4)} k(25\sin\phi)\, d\phi\, d\theta = 25k\int_0^{2\pi} d\theta\int_0^{\tan^{-1}(3/4)} \sin\phi\, d\phi$

$= 25k(2\pi)\left[-\cos\left(\tan^{-1}\frac{3}{4}\right) + 1\right] = 50\pi k\left(-\frac{4}{5} + 1\right) = 10\pi k.$

Because S has constant density, $\overline{x} = \overline{y} = 0$ by symmetry, and

$\overline{z} = \frac{1}{m}\iint_S z\rho(x, y, z) dS = \frac{1}{10\pi k}\int_0^{2\pi}\int_0^{\tan^{-1}(3/4)} k(5\cos\phi)(25\sin\phi)\, d\phi\, d\theta$

$= \frac{1}{10\pi k}(125k)\int_0^{2\pi} d\theta\int_0^{\tan^{-1}(3/4)} \sin\phi\cos\phi\, d\phi = \frac{1}{10\pi k}(125k)(2\pi)\left[\frac{1}{2}\sin^2\phi\right]_0^{\tan^{-1}(3/4)} = 25 \cdot \frac{1}{2}\left(\frac{3}{5}\right)^2 = \frac{9}{2},$

so the center of mass is $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{9}{2}\right)$.

(b) $I_z = \iint_S (x^2 + y^2)\rho(x, y, z) dS = \int_0^{2\pi}\int_0^{\tan^{-1}(3/4)} k(25\sin^2\phi)(25\sin\phi)\, d\phi\, d\theta$

$= 625k\int_0^{2\pi} d\theta\int_0^{\tan^{-1}(3/4)} \sin^3\phi\, d\phi = 625k(2\pi)\left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\tan^{-1}(3/4)}$

$= 1250\pi k\left[\frac{1}{3}\left(\frac{4}{5}\right)^3 - \frac{4}{5} - \frac{1}{3} + 1\right] = 1250\pi k\left(\frac{14}{375}\right) = \frac{140}{3}\pi k$

43. The rate of flow through the cylinder is the flux $\iint_S \rho\mathbf{v}\cdot\mathbf{n}\,dS = \iint_S \rho\mathbf{v}\cdot\, d\mathbf{S}$. We use the parametric representation $\mathbf{r}(u,v) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j} + v\,\mathbf{k}$ for S, where $0 \le u \le 2\pi$, $0 \le v \le 1$, so $\mathbf{r}_u = -2\sin u\,\mathbf{i} + 2\cos u\,\mathbf{j}$, $\mathbf{r}_v = \mathbf{k}$, and the outward orientation is given by $\mathbf{r}_u \times \mathbf{r}_v = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j}$. Then

$$\begin{aligned}\iint_S \rho\mathbf{v}\cdot\, d\mathbf{S} &= \rho\int_0^{2\pi}\int_0^1 \left(v\,\mathbf{i} + 4\sin^2 u\,\mathbf{j} + 4\cos^2 u\,\mathbf{k}\right)\cdot(2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j})\,dv\,du \\ &= \rho\int_0^{2\pi}\int_0^1 \left(2v\cos u + 8\sin^3 u\right)dv\,du = \rho\int_0^{2\pi}\left(\cos u + 8\sin^3 u\right)du \\ &= \rho\left[\sin u + 8\left(-\tfrac{1}{3}\right)(2+\sin^2 u)\cos u\right]_0^{2\pi} = 0\text{ kg/s}\end{aligned}$$

44. A parametric representation for the hemisphere S is $\mathbf{r}(\phi,\theta) = 3\sin\phi\,\cos\theta\,\mathbf{i} + 3\sin\phi\,\sin\theta\,\mathbf{j} + 3\cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi = 3\cos\phi\,\cos\theta\,\mathbf{i} + 3\cos\phi\,\sin\theta\,\mathbf{j} - 3\sin\phi\,\mathbf{k}$, $\mathbf{r}_\theta = -3\sin\phi\sin\theta\,\mathbf{i} + 3\sin\phi\cos\theta\,\mathbf{j}$, and the outward orientation is given by $\mathbf{r}_\phi \times \mathbf{r}_\theta = 9\sin^2\phi\,\cos\theta\,\mathbf{i} + 9\sin^2\phi\,\sin\theta\,\mathbf{j} + 9\sin\phi\,\cos\phi\,\mathbf{k}$. The rate of flow through S is

$$\begin{aligned}\iint_S \rho\mathbf{v}\cdot\, d\mathbf{S} &= \rho\int_0^{\pi/2}\int_0^{2\pi}(3\sin\phi\,\sin\theta\,\mathbf{i} + 3\sin\phi\,\cos\theta\,\mathbf{j})\cdot\left(9\sin^2\phi\,\cos\theta\,\mathbf{i} + 9\sin^2\phi\sin\theta\,\mathbf{j} + 9\sin\phi\,\cos\phi\,\mathbf{k}\right)d\theta\,d\phi \\ &= 27\rho\int_0^{\pi/2}\int_0^{2\pi}\left(\sin^3\phi\,\sin\theta\,\cos\theta + \sin^3\phi\,\sin\theta\,\cos\theta\right)d\theta\,d\phi = 54\rho\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^{2\pi}\sin\theta\,\cos\theta\,d\theta \\ &= 54\rho\left[-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi/2}\left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi} = 0\text{ kg/s}\end{aligned}$$

45. S consists of the hemisphere S_1 given by $z = \sqrt{a^2 - x^2 - y^2}$ and the disk S_2 given by $0 \le x^2 + y^2 \le a^2$, $z = 0$.

On S_1: $\mathbf{E} = a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + 2a\cos\phi\,\mathbf{k}$,

$\mathbf{T}_\phi \times \mathbf{T}_\theta = a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}$. Thus

$$\begin{aligned}\iint_{S_1}\mathbf{E}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2}(a^3\sin^3\phi + 2a^3\sin\phi\,\cos^2\phi)\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2}(a^3\sin\phi + a^3\sin\phi\,\cos^2\phi)\,d\phi\,d\theta = (2\pi)a^3\left(1+\tfrac{1}{3}\right) = \tfrac{8}{3}\pi a^3\end{aligned}$$

On S_2: $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\mathbf{r}_y \times \mathbf{r}_x = -\mathbf{k}$ so $\iint_{S_2}\mathbf{E}\cdot d\mathbf{S} = 0$. Hence the total charge is $q = \varepsilon_0\iint_S \mathbf{E}\cdot d\mathbf{S} = \frac{8}{3}\pi a^3\varepsilon_0$.

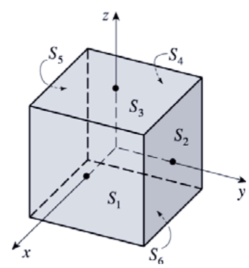
46. Referring to the figure, on

S_1: $\mathbf{E} = \mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1}\mathbf{E}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 dy\,dz = 4$;

S_2: $\mathbf{E} = x\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2}\mathbf{E}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 dx\,dz = 4$;

S_3: $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3}\mathbf{E}\cdot d\mathbf{S} = \int_{-1}^1\int_{-1}^1 dx\,dy = 4$;

S_4: $\mathbf{E} = -\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4}\mathbf{E}\cdot d\mathbf{S} = 4$.



Similarly $\iint_{S_5}\mathbf{E}\cdot d\mathbf{S} = \iint_{S_6}\mathbf{E}\cdot d\mathbf{S} = 4$. Hence $q = \varepsilon_0\iint_S \mathbf{E}\cdot d\mathbf{S} = \varepsilon_0\sum\limits_{i=1}^{6}\iint_{S_i}\mathbf{E}\cdot d\mathbf{S} = 24\varepsilon_0$.

47. $K\nabla u = 6.5(4y\,\mathbf{j} + 4z\,\mathbf{k})$. S is given by $\mathbf{r}(x,\theta) = x\,\mathbf{i} + \sqrt{6}\,\cos\theta\,\mathbf{j} + \sqrt{6}\,\sin\theta\,\mathbf{k}$ and since we want the inward heat flow, we use $\mathbf{r}_x \times \mathbf{r}_\theta = -\sqrt{6}\,\cos\theta\,\mathbf{j} - \sqrt{6}\,\sin\theta\,\mathbf{k}$. Then the rate of heat flow inward is given by

$\iint_S (-K\,\nabla u)\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^4 -(6.5)(-24)\,dx\,d\theta = (2\pi)(156)(4) = 1248\pi$.

48. $u(x,y,z) = c/\sqrt{x^2+y^2+z^2}$,

$$\mathbf{F} = -K\,\nabla u = -K\left[-\frac{cx}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{i} - \frac{cy}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{j} - \frac{cz}{(x^2+y^2+z^2)^{3/2}}\,\mathbf{k}\right]$$
$$= \frac{cK}{(x^2+y^2+z^2)^{3/2}}\,(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})$$

and the outward unit normal is $\mathbf{n} = \dfrac{1}{a}\,(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})$.

Thus $\mathbf{F}\cdot\mathbf{n} = \dfrac{cK}{a(x^2+y^2+z^2)^{3/2}}\,(x^2+y^2+z^2)$, but on S, $x^2+y^2+z^2=a^2$ so $\mathbf{F}\cdot\mathbf{n} = \dfrac{cK}{a^2}$. Hence the rate of heat flow across S is $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \frac{cK}{a^2}\iint_S dS = \frac{cK}{a^2}\,(4\pi a^2) = 4\pi Kc$.

49. Let S be a sphere of radius a centered at the origin. Then $|\mathbf{r}| = a$ and $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3 = \left(c/a^3\right)(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})$. A parametric representation for S is $\mathbf{r}(\phi,\theta) = a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}$, $0\le\phi\le\pi$, $0\le\theta\le 2\pi$. Then $\mathbf{r}_\phi = a\cos\phi\,\cos\theta\,\mathbf{i} + a\cos\phi\,\sin\theta\,\mathbf{j} - a\sin\phi\,\mathbf{k}$, $\mathbf{r}_\theta = -a\sin\phi\,\sin\theta\,\mathbf{i} + a\sin\phi\,\cos\theta\,\mathbf{j}$, and the outward orientation is given by $\mathbf{r}_\phi\times\mathbf{r}_\theta = a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}$. The flux of $\mathbf{F}$ across S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^\pi\int_0^{2\pi}\frac{c}{a^3}\,(a\sin\phi\,\cos\theta\,\mathbf{i} + a\sin\phi\,\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k})$$
$$\cdot\left(a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}\right)d\theta\,d\phi$$
$$= \frac{c}{a^3}\int_0^\pi\int_0^{2\pi} a^3\left(\sin^3\phi + \sin\phi\,\cos^2\phi\right)d\theta\,d\phi = c\int_0^\pi\int_0^{2\pi}\sin\phi\,d\theta\,d\phi = 4\pi c$$

Thus the flux does not depend on the radius a.

16.8 Stokes' Theorem

1. Both H and P are oriented piecewise-smooth surfaces that are bounded by the simple, closed, smooth curve $x^2+y^2=4$, $z=0$ (which we can take to be oriented positively for both surfaces). Then H and P satisfy the hypotheses of Stokes' Theorem, so by (3) we know $\iint_H \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_P \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ (where C is the boundary curve).

2. The boundary curve C is the circle $x^2+y^2=9$, $z=0$ oriented in the counterclockwise direction when viewed from above. A vector equation of C is $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$, $0\le t\le 2\pi$, so $\mathbf{r}'(t) = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$ and $\mathbf{F}(\mathbf{r}(t)) = 2(3\sin t)(\cos 0)\,\mathbf{i} + e^{3\cos t}(\sin 0)\,\mathbf{j} + (3\cos t)e^{3\sin t}\,\mathbf{k} = 6\sin t\,\mathbf{i} + (3\cos t)e^{3\sin t}\,\mathbf{k}$. Then, by Stokes' Theorem,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}(-18\sin^2 t + 0 + 0)\,dt = -18\left[\tfrac12 t - \tfrac14\sin 2t\right]_0^{2\pi} = -18\pi.$$

3. The paraboloid $z = x^2+y^2$ intersects the cylinder $x^2+y^2=4$ in the circle $x^2+y^2=4$, $z=4$. This boundary curve C should be oriented in the counterclockwise direction when viewed from above, so a vector equation of C is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 4\,\mathbf{k}$, $0\le t\le 2\pi$. Then $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$,

$$\mathbf{F}(\mathbf{r}(t)) = (4\cos^2 t)(16)\,\mathbf{i} + (4\sin^2 t)(16)\,\mathbf{j} + (2\cos t)(2\sin t)(4)\,\mathbf{k} = 64\cos^2 t\,\mathbf{i} + 64\sin^2 t\,\mathbf{j} + 16\sin t\,\cos t\,\mathbf{k},$$

and by Stokes' Theorem,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} (-128\cos^2 t\,\sin t + 128\sin^2 t\,\cos t + 0)\,dt$$
$$= 128\left[\tfrac{1}{3}\cos^3 t + \tfrac{1}{3}\sin^3 t\right]_0^{2\pi} = 0$$

4. The boundary curve C is the circle $y^2+z^2=4$, $x=2$ which should be oriented in the counterclockwise direction when viewed from the front, so a vector equation of C is $\mathbf{r}(t) = 2\,\mathbf{i} + 2\cos t\,\mathbf{j} + 2\sin t\,\mathbf{k}$, $0\le t\le 2\pi$. Then $\mathbf{F}(\mathbf{r}(t)) = \tan^{-1}(32\cos t\sin^2 t)\,\mathbf{i} + 8\cos t\,\mathbf{j} + 16\sin^2 t\,\mathbf{k}$, $\mathbf{r}'(t) = -2\sin t\,\mathbf{j} + 2\cos t\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -16\sin t\cos t + 32\sin^2 t\cos t$. Thus

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}(-16\sin t\cos t + 32\sin^2 t\cos t)\,dt$$
$$= \left[-8\sin^2 t + \tfrac{32}{3}\sin^3 t\right]_0^{2\pi} = 0$$

5. C is the square in the plane $z=-1$. Rather than evaluating a line integral around C we can use Equation 3: $\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_2}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube. $\operatorname{curl}\mathbf{F} = x^2 z\,\mathbf{i} + (xy - 2xyz)\,\mathbf{j} + (y - xz)\,\mathbf{k}$. For S_2, we choose $\mathbf{n}=\mathbf{k}$ so that C has the same orientation for both surfaces. Then $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = y - xz = x + y$ on S_2, where $z=-1$. Thus $\iint_{S_2}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1}(x+y)\,dx\,dy = 0$ so $\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0$.

6. The boundary curve C is the circle $x^2+z^2=1$, $y=0$ which should be oriented in the counterclockwise direction when viewed from the right, so a vector equation of C is $\mathbf{r}(t) = \cos(-t)\,\mathbf{i} + \sin(-t)\,\mathbf{k} = \cos t\,\mathbf{i} - \sin t\,\mathbf{k}$, $0\le t\le 2\pi$. Then $\mathbf{F}(\mathbf{r}(t)) = \mathbf{i} + e^{-\cos t\sin t}\,\mathbf{j} - \cos^2 t\,\sin t\,\mathbf{k}$, $\mathbf{r}'(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -\sin t + \cos^3 t\,\sin t$. Thus

$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}(-\sin t + \cos^3 t\,\sin t)\,dt$$
$$= \left[\cos t - \tfrac{1}{4}\cos^4 t\right]_0^{2\pi} = 0$$

7. $\operatorname{curl}\mathbf{F} = -2z\,\mathbf{i} - 2x\,\mathbf{j} - 2y\,\mathbf{k}$ and we take the surface S to be the planar region enclosed by C, so S is the portion of the plane $x+y+z=1$ over $D = \{(x,y) \mid 0\le x\le 1, 0\le y\le 1-x\}$. Since C is oriented counterclockwise, we orient S upward. Using Equation 16.7.10, we have $z = g(x,y) = 1-x-y$, $P=-2z$, $Q=-2x$, $R=-2y$, and

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D [-(-2z)(-1) - (-2x)(-1) + (-2y)]\,dA$$
$$= \int_0^1\int_0^{1-x}(-2)\,dy\,dx = -2\int_0^1(1-x)\,dx = -1$$

8. $\operatorname{curl}\mathbf{F} = (x-y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}$ and S is the portion of the plane $3x+2y+z=1$ over $D = \left\{(x,y) \mid 0\le x\le\tfrac{1}{3}, 0\le y\le\tfrac{1}{2}(1-3x)\right\}$. We orient S upward and use Equation 16.7.10 with $z = g(x,y) = 1-3x-2y$:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D[-(x-y)(-3) - (-y)(-2) + 1]\,dA = \int_0^{1/3}\int_0^{(1-3x)/2}(1+3x-5y)\,dy\,dx$$
$$= \int_0^{1/3}\left[(1+3x)y - \tfrac{5}{2}y^2\right]_{y=0}^{y=(1-3x)/2}dx = \int_0^{1/3}\left[\tfrac{1}{2}(1+3x)(1-3x) - \tfrac{5}{2}\cdot\tfrac{1}{4}(1-3x)^2\right]dx$$
$$= \int_0^{1/3}\left(-\tfrac{81}{8}x^2 + \tfrac{15}{4}x - \tfrac{1}{8}\right)dx = \left[-\tfrac{27}{8}x^3 + \tfrac{15}{8}x^2 - \tfrac{1}{8}x\right]_0^{1/3} = -\tfrac{1}{8} + \tfrac{5}{24} - \tfrac{1}{24} = \tfrac{1}{24}$$

9. $\operatorname{curl}\mathbf{F} = (xe^{xy} - 2x)\,\mathbf{i} - (ye^{xy} - y)\,\mathbf{j} + (2z - z)\,\mathbf{k}$ and we take S to be the disk $x^2 + y^2 \le 16$, $z = 5$. Since C is oriented counterclockwise (from above), we orient S upward. Then $\mathbf{n} = \mathbf{k}$ and $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = 2z - z$ on S, where $z = 5$. Thus

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_S (2z - z)\;dS = \iint_S (10 - 5)\,dS = 5(\text{area of } S) = 5(\pi\cdot 4^2) = 80\pi$$

10. The curve of intersection is an ellipse in the plane $z = 5 - x$. $\operatorname{curl}\mathbf{F} = \mathbf{i} - x\,\mathbf{k}$ and we take the surface S to be the planar region enclosed by C with upward orientation, so

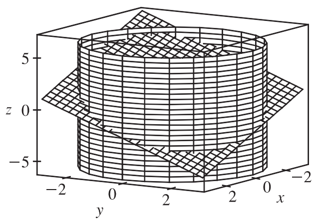
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+y^2\le 9} [-1\,(-1) - 0 + (-x)]\,dA = \int_0^{2\pi}\int_0^3 (1 - r\cos\theta)\;r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 \left(r - r^2\cos\theta\right) dr\,d\theta = \int_0^{2\pi}\left(\tfrac{9}{2} - 9\cos\theta\right) d\theta = \left[\tfrac{9}{2}\theta - 9\sin\theta\right]_0^{2\pi} = 9\pi$$

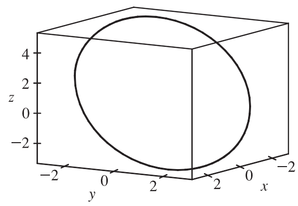
11. (a) The curve of intersection is an ellipse in the plane $x + y + z = 1$ with unit normal $\mathbf{n} = \frac{1}{\sqrt{3}}\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $\operatorname{curl}\mathbf{F} = x^2\,\mathbf{j} + y^2\,\mathbf{k}$, and $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = \frac{1}{\sqrt{3}}(x^2 + y^2)$. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \tfrac{1}{\sqrt{3}}\left(x^2 + y^2\right) dS = \iint_{x^2+y^2\le 9}\left(x^2 + y^2\right) dx\,dy = \int_0^{2\pi}\int_0^3 r^3\,dr\,d\theta = 2\pi\left(\tfrac{81}{4}\right) = \tfrac{81\pi}{2}$$

(b)



(c) One possible parametrization is $x = 3\cos t$, $y = 3\sin t$, $z = 1 - 3\cos t - 3\sin t$, $0 \le t \le 2\pi$.

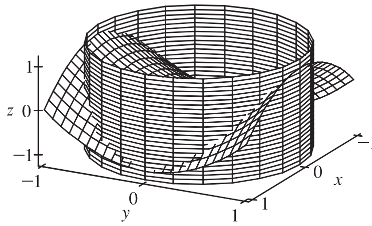


12. (a) S is the part of the surface $z = y^2 - x^2$ that lies above the unit disk D. $\operatorname{curl}\mathbf{F} = x\,\mathbf{i} - y\,\mathbf{j} + (x^2 - x^2)\,\mathbf{k} = x\,\mathbf{i} - y\,\mathbf{j}$. Using Equation 16.7.10 with $g(x, y) = y^2 - x^2$, $P = x$, $Q = -y$, we have

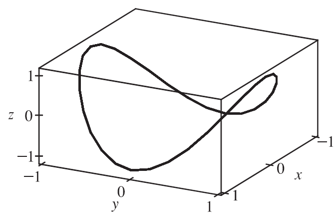
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D[-x(-2x) - (-y)(2y)]\,dA = 2\iint_D (x^2 + y^2)\,dA$$

$$= 2\int_0^{2\pi}\int_0^1 r^2 r\,dr\,d\theta = 2(2\pi)\left[\tfrac{1}{4}r^4\right]_0^1 = \pi$$

(b)



(c) One possible set of parametric equations is $x = \cos t$, $y = \sin t$, $z = \sin^2 t - \cos^2 t$, $0 \le t \le 2\pi$.



13. The boundary curve C is the circle $x^2 + y^2 = 16$, $z = 4$ oriented in the clockwise direction as viewed from above (since S is oriented downward). We can parametrize C by $\mathbf{r}(t) = 4\cos t\,\mathbf{i} - 4\sin t\,\mathbf{j} + 4\,\mathbf{k}$, $0 \le t \le 2\pi$, and then $\mathbf{r}'(t) = -4\sin t\,\mathbf{i} - 4\cos t\,\mathbf{j}$. Thus $\mathbf{F}(\mathbf{r}(t)) = 4\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} - 2\,\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -16\sin^2 t - 16\cos^2 t = -16$, and

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}(-16)\,dt = -16\,(2\pi) = -32\pi$$

Now curl $\mathbf{F} = 2\,\mathbf{k}$, and the projection D of S on the xy-plane is the disk $x^2 + y^2 \leq 16$, so by Equation 16.7.10 with $z = g(x, y) = \sqrt{x^2 + y^2}$ [and multiplying by -1 for the downward orientation] we have

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = -\iint_D (-0 - 0 + 2)\,dA = -2 \cdot A(D) = -2 \cdot \pi(4^2) = -32\pi$$

14. The paraboloid intersects the plane $z = 1$ when $1 = 5 - x^2 - y^2 \quad \Leftrightarrow \quad x^2 + y^2 = 4$, so the boundary curve C is the circle $x^2 + y^2 = 4$, $z = 1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \leq t \leq 2\pi$, and then $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$. Thus $\mathbf{F}(\mathbf{r}(t)) = -4\sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 6\cos t\,\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 8\sin^2 t + 4\sin t\cos t$, and

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (8\sin^2 t + 4\sin t\cos t)\,dt = 8\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right) + 2\sin^2 t\Big]_0^{2\pi} = 8\pi$$

Now curl $\mathbf{F} = (-3 - 2y)\,\mathbf{j} + 2z\,\mathbf{k}$, and the projection D of S on the xy-plane is the disk $x^2 + y^2 \leq 4$, so by Equation 16.7.10 with $z = g(x, y) = 5 - x^2 - y^2$ we have

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_D [-0 - (-3 - 2y)(-2y) + 2z]\,dA = \iint_D [-6y - 4y^2 + 2(5 - x^2 - y^2)]\,dA \\ &= \int_0^{2\pi}\int_0^2 \left[-6r\sin\theta - 4r^2\sin^2\theta + 2(5 - r^2)\right] r\,dr\,d\theta = \int_0^{2\pi} \left[-2r^3\sin\theta - r^4\sin^2\theta + 5r^2 - \tfrac{1}{2}r^4\right]_{r=0}^{r=2} d\theta \\ &= \int_0^{2\pi} \left(-16\sin\theta - 16\sin^2\theta + 20 - 8\right) d\theta = 16\cos\theta - 16\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) + 12\theta\Big]_0^{2\pi} = 8\pi\end{aligned}$$

15. The boundary curve C is the circle $x^2 + z^2 = 1$, $y = 0$ oriented in the counterclockwise direction as viewed from the positive y-axis. Then C can be described by $\mathbf{r}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{k}$, $0 \leq t \leq 2\pi$, and $\mathbf{r}'(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{k}$. Thus $\mathbf{F}(\mathbf{r}(t)) = -\sin t\,\mathbf{j} + \cos t\,\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -\cos^2 t$, and $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\cos^2 t)\,dt = -\frac{1}{2}t - \frac{1}{4}\sin 2t\Big]_0^{2\pi} = -\pi$.
Now curl $\mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$, and S can be parametrized (see Example 16.6.10) by $\mathbf{r}(\phi, \theta) = \sin\phi\,\cos\theta\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \pi$. Then $\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\,\cos\theta\,\mathbf{i} + \sin^2\phi\,\sin\theta\,\mathbf{j} + \sin\phi\,\cos\phi\,\mathbf{k}$ and

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint\limits_{x^2+z^2\leq 1} \operatorname{curl} \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\,dA = \int_0^\pi \int_0^\pi (-\sin^2\phi\,\cos\theta - \sin^2\phi\,\sin\theta - \sin\phi\,\cos\phi)\,d\theta\,d\phi \\ &= \int_0^\pi (-2\sin^2\phi - \pi\sin\phi\,\cos\phi)\,d\phi = \left[\tfrac{1}{2}\sin 2\phi - \phi - \tfrac{\pi}{2}\sin^2\phi\right]_0^\pi = -\pi\end{aligned}$$

16. Let S be the surface in the plane $x + y + z = 1$ with upward orientation enclosed by C. Then an upward unit normal vector for S is $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$. Orient C in the counterclockwise direction, as viewed from above. $\int_C z\,dx - 2x\,dy + 3y\,dz$ is equivalent to $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F}(x, y, z) = z\,\mathbf{i} - 2x\,\mathbf{j} + 3y\,\mathbf{k}$, and the components of $\mathbf{F}$ are polynomials, which have continuous partial derivatives throughout $\mathbb{R}^3$. We have curl $\mathbf{F} = 3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$, so by Stokes' Theorem,

$$\begin{aligned}\int_C z\,dx - 2x\,dy + 3y\,dz &= \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,dS = \iint_S (3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}) \cdot \tfrac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})\,dS \\ &= \tfrac{2}{\sqrt{3}} \iint_S dS = \tfrac{2}{\sqrt{3}}\,(\text{surface area of } S)\end{aligned}$$

Thus the value of $\int_C z\,dx - 2x\,dy + 3y\,dz$ is always $\frac{2}{\sqrt{3}}$ times the area of the region enclosed by C, regardless of its shape or location. [Notice that because $\mathbf{n}$ is normal to a plane, it is constant. But curl $\mathbf{F}$ is also constant, so the dot product curl $\mathbf{F} \cdot \mathbf{n}$ is constant and we could have simply argued that $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,dS$ is a constant multple of $\iint_S dS$, the surface area of S.]

17. It is easier to use Stokes' Theorem than to compute the work directly. Let S be the planar region enclosed by the path of the particle, so S is the portion of the plane $z = \frac{1}{2}y$ for $0 \le x \le 1$, $0 \le y \le 2$, with upward orientation.

curl $\mathbf{F} = 8y\,\mathbf{i} + 2z\,\mathbf{j} + 2y\,\mathbf{k}$ and

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D \left[-8y\,(0) - 2z\left(\tfrac{1}{2}\right) + 2y\right] dA = \int_0^1 \int_0^2 \left(2y - \tfrac{1}{2}y\right) dy\,dx$$
$$= \int_0^1 \int_0^2 \tfrac{3}{2}y\,dy\,dx = \int_0^1 \left[\tfrac{3}{4}y^2\right]_{y=0}^{y=2} dx = \int_0^1 3\,dx = 3$$

18. $\int_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz = \int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = (y + \sin x)\,\mathbf{i} + (z^2 + \cos y)\,\mathbf{j} + x^3\,\mathbf{k} \quad \Rightarrow$ curl $\mathbf{F} = -2z\,\mathbf{i} - 3x^2\,\mathbf{j} - \mathbf{k}$. Since $\sin 2t = 2\sin t\,\cos t$, C lies on the surface $z = 2xy$. Let S be the part of this surface that is bounded by C. Then the projection of S onto the xy-plane is the unit disk D $[x^2 + y^2 \le 1]$. C is traversed clockwise (when viewed from above) so S is oriented downward. Using Equation 16.7.10 with $g(x, y) = 2xy$, $P = -2z = -2(2xy) = -4xy$, $Q = -3x^2$, $R = -1$ and multiplying by -1 for the downward orientation, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = -\iint_D \left[-(-4xy)(2y) - (-3x^2)(2x) - 1\right] dA$$
$$= -\iint_D (8xy^2 + 6x^3 - 1)\,dA = -\int_0^{2\pi} \int_0^1 (8r^3 \cos\theta \sin^2\theta + 6r^3 \cos^3\theta - 1)\,r\,dr\,d\theta$$
$$= -\int_0^{2\pi} \left(\tfrac{8}{5}\cos\theta \sin^2\theta + \tfrac{6}{5}\cos^3\theta - \tfrac{1}{2}\right) d\theta = -\left[\tfrac{8}{15}\sin^3\theta + \tfrac{6}{5}\left(\sin\theta - \tfrac{1}{3}\sin^3\theta\right) - \tfrac{1}{2}\theta\right]_0^{2\pi} = \pi$$

19. Assume S is centered at the origin with radius a and let H_1 and H_2 be the upper and lower hemispheres, respectively, of S. Then $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{H_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} + \iint_{H_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$ by Stokes' Theorem. But C_1 is the circle $x^2 + y^2 = a^2$ oriented in the counterclockwise direction while C_2 is the same circle oriented in the clockwise direction. Hence $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ so $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ as desired.

20. (a) By Exercise 16.5.26, $\operatorname{curl}(f\nabla g) = f \operatorname{curl}(\nabla g) + \nabla f \times \nabla g = \nabla f \times \nabla g$ since $\operatorname{curl}(\nabla g) = \mathbf{0}$. Hence by Stokes' Theorem $\int_C (f\nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$.

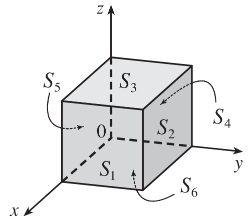
(b) As in (a), $\operatorname{curl}(f\nabla f) = \nabla f \times \nabla f = \mathbf{0}$, so by Stokes' Theorem, $\int_C (f\nabla f) \cdot d\mathbf{r} = \iint_S [\operatorname{curl}(f\nabla f)] \cdot d\mathbf{S} = 0$.

(c) As in part (a),

$$\operatorname{curl}(f\nabla g + g\nabla f) = \operatorname{curl}(f\nabla g) + \operatorname{curl}(g\nabla f) \qquad \text{[by Exercise 16.5.24]}$$
$$= (\nabla f \times \nabla g) + (\nabla g \times \nabla f) = \mathbf{0} \qquad [\text{since } \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})]$$

Hence by Stokes' Theorem, $\int_C (f\nabla g + g\nabla f) \cdot d\mathbf{r} = \iint_S \operatorname{curl}(f\nabla g + g\nabla f) \cdot d\mathbf{S} = 0$.

16.9 The Divergence Theorem



1. div $\mathbf{F} = 3 + x + 2x = 3 + 3x$, so

$\iiint_E \operatorname{div} \mathbf{F}\, dV = \int_0^1 \int_0^1 \int_0^1 (3x + 3)\, dx\, dy\, dz = \frac{9}{2}$ (notice the triple integral is three times the volume of the cube plus three times $\overline{x}$).

To compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, on

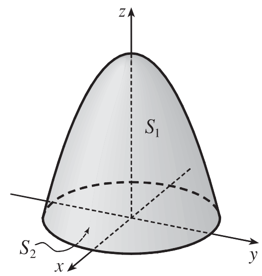
S_1: $\mathbf{n} = \mathbf{i}$, $\mathbf{F} = 3\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$, and $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} 3\, dS = 3$;

S_2: $\mathbf{F} = 3x\,\mathbf{i} + x\,\mathbf{j} + 2xz\,\mathbf{k}$, $\mathbf{n} = \mathbf{j}$ and $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} x\, dS = \frac{1}{2}$;

S_3: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2x\,\mathbf{k}$, $\mathbf{n} = \mathbf{k}$ and $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_3} 2x\, dS = 1$;

S_4: $\mathbf{F} = \mathbf{0}$, $\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} = 0$; S_5: $\mathbf{F} = 3x\,\mathbf{i} + 2x\,\mathbf{k}$, $\mathbf{n} = -\mathbf{j}$ and $\iint_{S_5} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_5} 0\, dS = 0$;

S_6: $\mathbf{F} = 3x\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_6} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_6} 0\, dS = 0$. Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{9}{2}$.



2. div $\mathbf{F} = 2x + x + 1 = 3x + 1$ so

$$\begin{aligned}
\iiint_E \operatorname{div} \mathbf{F}\, dV &= \iiint_E (3x + 1)\, dV = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (3r\cos\theta + 1)\, r\, dz\, dr\, d\theta \\
&= \int_0^2 \int_0^{2\pi} r(3r\cos\theta + 1)(4 - r^2)\, d\theta\, dr \\
&= \int_0^{2\pi} r(4 - r^2)\big[3r\sin\theta + \theta\big]_{\theta=0}^{\theta=2\pi}\, dr \\
&= 2\pi \int_0^2 (4r - r^3)\, dr = 2\pi\big[2r^2 - \tfrac{1}{4}r^4\big]_0^2 \\
&= 2\pi(8 - 4) = 8\pi
\end{aligned}$$

On S_1: The surface is $z = 4 - x^2 - y^2$, $x^2 + y^2 \le 4$, with upward orientation, and $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j} + (4 - x^2 - y^2)\,\mathbf{k}$. Then

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D [-(x^2)(-2x) - (xy)(-2y) + (4 - x^2 - y^2)]\, dA \\
&= \iint_D \left[2x(x^2 + y^2) + 4 - (x^2 + y^2)\right]\, dA = \int_0^{2\pi} \int_0^2 (2r\cos\theta \cdot r^2 + 4 - r^2)\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \left[\tfrac{2}{5}r^5\cos\theta + 2r^2 - \tfrac{1}{4}r^4\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} \left(\tfrac{64}{5}\cos\theta + 4\right) d\theta = \left[\tfrac{64}{5}\sin\theta + 4\theta\right]_0^{2\pi} = 8\pi
\end{aligned}$$

On S_2: The surface is $z = 0$ with downward orientation, so $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} 0\, dS = 0$.

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 8\pi$.

3. div $\mathbf{F} = 0 + 1 + 0 = 1$, so $\iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E 1\, dV = V(E) = \frac{4}{3}\pi \cdot 4^3 = \frac{256}{3}\pi$. S is a sphere of radius 4 centered at the origin which can be parametrized by $\mathbf{r}(\phi, \theta) = \langle 4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi\rangle$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$ (similar to Example 16.6.10). Then

$$\begin{aligned}
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \langle 4\cos\phi\cos\theta, 4\cos\phi\sin\theta, -4\sin\phi\rangle \times \langle -4\sin\phi\sin\theta, 4\sin\phi\cos\theta, 0\rangle \\
&= \left\langle 16\sin^2\phi\cos\theta, 16\sin^2\phi\sin\theta, 16\cos\phi\sin\phi\right\rangle
\end{aligned}$$

and $\mathbf{F}(\mathbf{r}(\phi, \theta)) = \langle 4\cos\phi, 4\sin\phi\sin\theta, 4\sin\phi\cos\theta\rangle$. Thus

$\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 64\cos\phi\sin^2\phi\cos\theta + 64\sin^3\phi\sin^2\theta + 64\cos\phi\sin^2\phi\cos\theta = 128\cos\phi\sin^2\phi\cos\theta + 64\sin^3\phi\sin^2\theta$

and

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA = \int_0^{2\pi} \int_0^{\pi} (128\cos\phi\sin^2\phi\cos\theta + 64\sin^3\phi\sin^2\theta)\, d\phi\, d\theta \\
&= \int_0^{2\pi} \left[\tfrac{128}{3}\sin^3\phi\cos\theta + 64\left(-\tfrac{1}{3}(2 + \sin^2\phi)\cos\phi\right)\sin^2\theta\right]_{\phi=0}^{\phi=\pi} d\theta \\
&= \int_0^{2\pi} \tfrac{256}{3}\sin^2\theta\, d\theta = \tfrac{256}{3}\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{256}{3}\pi
\end{aligned}$$

4. $\operatorname{div}\mathbf{F} = 2x - 1 + 1 = 2x$, so

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \iint_{y^2+z^2\le 9}\left[\int_0^2 2x\,dx\right]dA = \iint_{y^2+z^2\le 9} 4\,dA = 4(\text{area of circle}) = 4(\pi\cdot 3^2) = 36\pi$$

Let S_1 be the front of the cylinder (in the plane $x=2$), S_2 the back (in the yz-plane), and S_3 the lateral surface of the cylinder.

S_1 is the disk $x=2$, $y^2+z^2\le 9$. A unit normal vector is $\mathbf{n} = \langle 1,0,0\rangle$ and $\mathbf{F} = \langle 4,-y,z\rangle$ on S_1, so

$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} 4\,dS = 4(\text{surface area of } S_1) = 4(\pi\cdot 3^2) = 36\pi$. S_2 is the disk $x=0$, $y^2+z^2\le 9$.

Here $\mathbf{n} = \langle -1,0,0\rangle$ and $\mathbf{F} = \langle 0,-y,z\rangle$, so $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_2} 0\,dS = 0$.

S_3 can be parametrized by $\mathbf{r}(x,\theta) = \langle x, 3\cos\theta, 3\sin\theta\rangle$, $0\le x\le 2$, $0\le\theta\le 2\pi$. Then

$\mathbf{r}_x\times\mathbf{r}_\theta = \langle 1,0,0\rangle\times\langle 0,-3\sin\theta,3\cos\theta\rangle = \langle 0,-3\cos\theta,-3\sin\theta\rangle$. For the outward (positive) orientation we use

$-(\mathbf{r}_x\times\mathbf{r}_\theta)$ and $\mathbf{F}(\mathbf{r}(x,\theta)) = \left\langle x^2,-3\cos\theta,3\sin\theta\right\rangle$, so

$$\begin{aligned}\iint_{S_3}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(-(\mathbf{r}_x\times\mathbf{r}_\theta))\,dA = \int_0^2\int_0^{2\pi}(0 - 9\cos^2\theta + 9\sin^2\theta)\,d\theta\,dx\\ &= -9\int_0^2 dx\int_0^{2\pi}\cos 2\theta\,d\theta = -9\,(2)\left[\tfrac12\sin 2\theta\right]_0^{2\pi} = 0\end{aligned}$$

Thus $\iint_S \mathbf{F}\cdot d\mathbf{S} = 36\pi + 0 + 0 = 36\pi$.

5. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(xye^z) + \frac{\partial}{\partial y}(xy^2z^3) + \frac{\partial}{\partial z}(-ye^z) = ye^z + 2xyz^3 - ye^z = 2xyz^3$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^3\int_0^2\int_0^1 2xyz^3\,dz\,dy\,dx = 2\int_0^3 x\,dx\int_0^2 y\,dy\int_0^1 z^3\,dz\\ &= 2\left[\tfrac12 x^2\right]_0^3\left[\tfrac12 y^2\right]_0^2\left[\tfrac14 z^4\right]_0^1 = 2\left(\tfrac92\right)(2)\left(\tfrac14\right) = \tfrac92\end{aligned}$$

6. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2) = 2xyz + 2xyz + 2xyz = 6xyz$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^a\int_0^b\int_0^c 6xyz\,dz\,dy\,dx = 6\int_0^a x\,dx\int_0^b y\,dy\int_0^c z\,dz\\ &= 6\left[\tfrac12 x^2\right]_0^a\left[\tfrac12 y^2\right]_0^b\left[\tfrac12 z^2\right]_0^c = 6\left(\tfrac12 a^2\right)\left(\tfrac12 b^2\right)\left(\tfrac12 c^2\right) = \tfrac34 a^2b^2c^2\end{aligned}$$

7. $\operatorname{div}\mathbf{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E (3y^2+3z^2)\,dV = \int_0^{2\pi}\int_0^1\int_{-1}^2 (3r^2\cos^2\theta + 3r^2\sin^2\theta)\,r\,dx\,dr\,d\theta\\ &= 3\int_0^{2\pi} d\theta\int_0^1 r^3\,dr\int_{-1}^2 dx = 3(2\pi)\left(\tfrac14\right)(3) = \tfrac{9\pi}{2}\end{aligned}$$

8. $\operatorname{div}\mathbf{F} = 3x^2 + 3y^2 + 3z^2$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 3(x^2+y^2+z^2)\,dV = \int_0^\pi\int_0^{2\pi}\int_0^2 3\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = 3\int_0^\pi \sin\phi\,d\phi\int_0^{2\pi} d\theta\int_0^2 \rho^4\,d\rho\\ &= 3\left[-\cos\phi\right]_0^\pi\left[\theta\right]_0^{2\pi}\left[\tfrac15\rho^5\right]_0^2 = 3\,(2)\,(2\pi)\left(\tfrac{32}{5}\right) = \tfrac{384}{5}\pi\end{aligned}$$

9. $\operatorname{div}\mathbf{F} = 2x\sin y - x\sin y - x\sin y = 0$, so by the Divergence Theorem, $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 0\,dV = 0$.

10. The tetrahedron has vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ and is described by

$E = \left\{(x,y,z) \mid 0 \le x \le a, 0 \le y \le b\left(1-\frac{x}{a}\right), 0 \le z \le c\left(1-\frac{x}{a}-\frac{y}{b}\right)\right\}$. Here we have $\operatorname{div}\mathbf{F} = 0+1+x = x+1$, so

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E (x+1)\,dV = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} (x+1)\,dz\,dy\,dx \\
&= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} (x+1)\left[c\left(1-\frac{x}{a}-\frac{y}{b}\right)\right] dy\,dx = c\int_0^a (x+1)\left[\left(1-\frac{x}{a}\right)y - \frac{1}{2b}y^2\right]_{y=0}^{y=b\left(1-\frac{x}{a}\right)} dx \\
&= c\int_0^a (x+1)\left[\left(1-\frac{x}{a}\right)\cdot b\left(1-\frac{x}{a}\right) - \frac{1}{2b}\cdot b^2\left(1-\frac{x}{a}\right)^2\right] dx = \tfrac{1}{2}bc\int_0^a (x+1)\left(1-\frac{x}{a}\right)^2 dx \\
&= \tfrac{1}{2}bc\int_0^a \left(\tfrac{1}{a^2}x^3 + \tfrac{1}{a^2}x^2 - \tfrac{2}{a}x^2 + x - \tfrac{2}{a}x + 1\right) dx \\
&= \tfrac{1}{2}bc\left[\tfrac{1}{4a^2}x^4 + \tfrac{1}{3a^2}x^3 - \tfrac{2}{3a}x^3 + \tfrac{1}{2}x^2 - \tfrac{1}{a}x^2 + x\right]_0^a \\
&= \tfrac{1}{2}bc\left(\tfrac{1}{4}a^2 + \tfrac{1}{3}a - \tfrac{2}{3}a^2 + \tfrac{1}{2}a^2 - a + a\right) = \tfrac{1}{2}bc\left(\tfrac{1}{12}a^2 + \tfrac{1}{3}a\right) = \tfrac{1}{24}abc(a+4)
\end{aligned}$$

11. $\operatorname{div}\mathbf{F} = y^2 + 0 + x^2 = x^2 + y^2$ so

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^3(4-r^2)\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^2 (4r^3 - r^5)\,dr = 2\pi\left[r^4 - \tfrac{1}{6}r^6\right]_0^2 = \tfrac{32}{3}\pi
\end{aligned}$$

12. $\operatorname{div}\mathbf{F} = 4x^3 + 4xy^2$ so

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 4x(x^2+y^2)\,dV = \int_0^{2\pi}\int_0^1\int_0^{r\cos\theta+2} (4r^3\cos\theta)\,r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 (4r^5\cos^2\theta + 8r^4\cos\theta)\,dr\,d\theta = \int_0^{2\pi}\left(\tfrac{2}{3}\cos^2\theta + \tfrac{8}{5}\cos\theta\right)d\theta = \tfrac{2}{3}\pi
\end{aligned}$$

13. $\mathbf{F}(x,y,z) = x\sqrt{x^2+y^2+z^2}\,\mathbf{i} + y\sqrt{x^2+y^2+z^2}\,\mathbf{j} + z\sqrt{x^2+y^2+z^2}\,\mathbf{k}$, so

$$\begin{aligned}
\operatorname{div}\mathbf{F} &= x\cdot\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x) + (x^2+y^2+z^2)^{1/2} + y\cdot\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y) + (x^2+y^2+z^2)^{1/2} \\
&\qquad + z\cdot\tfrac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z) + (x^2+y^2+z^2)^{1/2} \\
&= (x^2+y^2+z^2)^{-1/2}\left[x^2 + (x^2+y^2+z^2) + y^2 + (x^2+y^2+z^2) + z^2 + (x^2+y^2+z^2)\right] \\
&= \frac{4(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} = 4\sqrt{x^2+y^2+z^2}.
\end{aligned}$$

Then

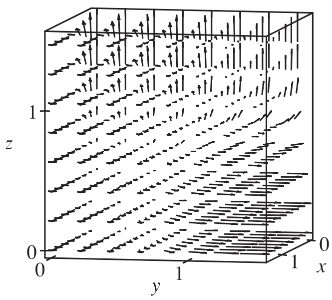
$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 4\sqrt{x^2+y^2+z^2}\,dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 4\sqrt{\rho^2}\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&= \int_0^{\pi/2}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^1 4\rho^3\,d\rho = \left[-\cos\phi\right]_0^{\pi/2}\left[\theta\right]_0^{2\pi}\left[\rho^4\right]_0^1 = (1)\,(2\pi)\,(1) = 2\pi
\end{aligned}$$

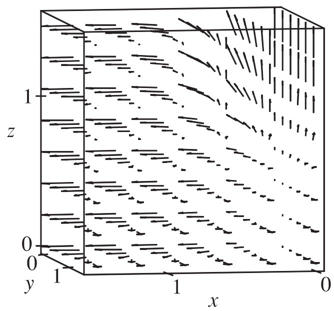
14. $\mathbf{F}(x,y,z) = x(x^2+y^2+z^2)\,\mathbf{i} + y(x^2+y^2+z^2)\,\mathbf{j} + z(x^2+y^2+z^2)\,\mathbf{k}$, so

$\operatorname{div}\mathbf{F} = x\cdot 2x + (x^2+y^2+z^2) + y\cdot 2y + (x^2+y^2+z^2) + z\cdot 2z + (x^2+y^2+z^2) = 5(x^2+y^2+z^2)$. Then

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 5(x^2+y^2+z^2)\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^R 5\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&= 5\int_0^{\pi}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^R \rho^4\,d\rho = 5\left[-\cos\phi\right]_0^{\pi}\left[\theta\right]_0^{2\pi}\left[\tfrac{1}{5}\rho^5\right]_0^R = 5\,(2)\,(2\pi)\left(\tfrac{1}{5}R^5\right) = 4\pi R^5
\end{aligned}$$

15. $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \sqrt{3-x^2}\,dV = \int_{-1}^1\int_{-1}^1\int_0^{2-x^4-y^4} \sqrt{3-x^2}\,dz\,dy\,dx = \frac{341}{60}\sqrt{2} + \frac{81}{20}\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

16. 



By the Divergence Theorem, the flux of $\mathbf{F}$ across the surface of the cube is

$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\left[\cos x\,\cos^2 y + 3\sin^2 y\,\cos y\,\cos^4 z + 5\sin^4 z\,\cos z\,\cos^6 x\right] dz\,dy\,dx = \frac{19}{64}\pi^2.$

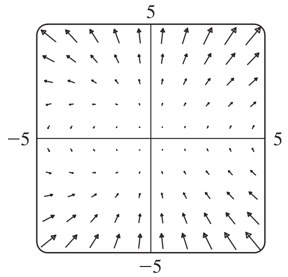
17. For S_1 we have $\mathbf{n} = -\mathbf{k}$, so $\mathbf{F}\cdot\mathbf{n} = \mathbf{F}\cdot(-\mathbf{k}) = -x^2z - y^2 = -y^2$ (since $z = 0$ on S_1). So if D is the unit disk, we get $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_D(-y^2)\,dA = -\int_0^{2\pi}\int_0^1 r^2\left(\sin^2\theta\right) r\,dr\,d\theta = -\frac{1}{4}\pi$. Now since S_2 is closed, we can use the Divergence Theorem. Since $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}\left(z^2x\right) + \frac{\partial}{\partial y}\left(\frac{1}{3}y^3 + \tan z\right) + \frac{\partial}{\partial z}\left(x^2z + y^2\right) = z^2 + y^2 + x^2$, we use spherical coordinates to get $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2}{5}\pi$. Finally $\iint_S\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} - \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \frac{2}{5}\pi - \left(-\frac{1}{4}\pi\right) = \frac{13}{20}\pi$.

18. As in the hint to Exercise 17, we create a closed surface $S_2 = S\cup S_1$, where S is the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$, and S_1 is the disk $x^2 + y^2 = 1$ on the plane $z = 1$ oriented downward, and we then apply the Divergence Theorem. Since the disk S_1 is oriented downward, its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and $\mathbf{F}\cdot(-\mathbf{k}) = -z = -1$ on S_1. So $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1}(-1)\,dS = -A(S_1) = -\pi$. Let E be the region bounded by S_2. Then $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E 1\,dV = \int_0^1\int_0^{2\pi}\int_1^{2-r^2} r\,dz\,d\theta\,dr = \int_0^1\int_0^{2\pi}(r - r^3)\,d\theta\,dr = (2\pi)\frac{1}{4} = \frac{\pi}{2}$. Thus the flux of $\mathbf{F}$ across S is $\iint_S\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2}\mathbf{F}\cdot d\mathbf{S} - \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}$.

19. The vectors that end near P_1 are longer than the vectors that start near P_1, so the net flow is inward near P_1 and $\operatorname{div}\mathbf{F}(P_1)$ is negative. The vectors that end near P_2 are shorter than the vectors that start near P_2, so the net flow is outward near P_2 and $\operatorname{div}\mathbf{F}(P_2)$ is positive.

20. (a) The vectors that end near P_1 are shorter than the vectors that start near P_1, so the net flow is outward and P_1 is a source. The vectors that end near P_2 are longer than the vectors that start near P_2, so the net flow is inward and P_2 is a sink.

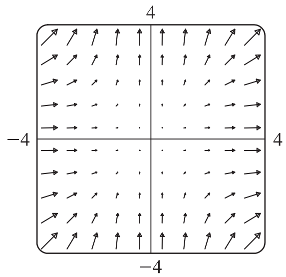
(b) $\mathbf{F}(x,y) = \left\langle x, y^2\right\rangle \Rightarrow \operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = 1 + 2y$. The y-value at P_1 is positive, so $\operatorname{div}\mathbf{F} = 1 + 2y$ is positive, thus P_1 is a source. At P_2, $y < -1$, so $\operatorname{div}\mathbf{F} = 1 + 2y$ is negative, and P_2 is a sink.

21. 

From the graph it appears that for points above the x-axis, vectors starting near a particular point are longer than vectors ending there, so divergence is positive. The opposite is true at points below the x-axis, where divergence is negative.

$\mathbf{F}(x,y) = \langle xy, x + y^2 \rangle \quad \Rightarrow \quad \operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(x + y^2\right) = y + 2y = 3y.$

Thus $\operatorname{div} \mathbf{F} > 0$ for $y > 0$, and $\operatorname{div} \mathbf{F} < 0$ for $y < 0$.

22. 

From the graph it appears that for points above the line $y = -x$, vectors starting near a particular point are longer than vectors ending there, so divergence is positive. The opposite is true at points below the line $y = -x$, where divergence is negative.

$\mathbf{F}(x,y) = \langle x^2, y^2 \rangle \quad \Rightarrow \quad \operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) = 2x + 2y$. Then $\operatorname{div} \mathbf{F} > 0$ for $2x + 2y > 0 \quad \Rightarrow \quad y > -x$, and $\operatorname{div} \mathbf{F} < 0$ for $y < -x$.

23. Since $\dfrac{\mathbf{x}}{|\mathbf{x}|^3} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \dfrac{(x^2+y^2+z^2) - 3x^2}{(x^2+y^2+z^2)^{5/2}}$ with similar expressions for $\dfrac{\partial}{\partial y}\left(\dfrac{y}{(x^2+y^2+z^2)^{3/2}}\right)$ and $\dfrac{\partial}{\partial z}\left(\dfrac{z}{(x^2+y^2+z^2)^{3/2}}\right)$, we have

$\operatorname{div}\left(\dfrac{\mathbf{x}}{|\mathbf{x}|^3}\right) = \dfrac{3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$, except at $(0,0,0)$ where it is undefined.

24. We first need to find $\mathbf{F}$ so that $\iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iint_S (2x + 2y + z^2)\,dS$, so $\mathbf{F} \cdot \mathbf{n} = 2x + 2y + z^2$. But for S,

$\mathbf{n} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Thus $\mathbf{F} = 2\,\mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ and $\operatorname{div} \mathbf{F} = 1$.

If $B = \left\{(x,y,z) \mid x^2 + y^2 + z^2 \le 1\right\}$, then $\iint_S (2x + 2y + z^2)\,dS = \iiint_B dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$.

25. $\iint_S \mathbf{a} \cdot \mathbf{n}\,dS = \iiint_E \operatorname{div} \mathbf{a}\,dV = 0$ since $\operatorname{div} \mathbf{a} = 0$.

26. $\frac{1}{3}\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3}\iiint_E \operatorname{div} \mathbf{F}\,dV = \frac{1}{3}\iiint_E 3\,dV = V(E)$

27. $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F})\,dV = 0$ by Theorem 16.5.11.

28. $\iint_S D_{\mathbf{n}} f\,dS = \iint_S (\nabla f \cdot \mathbf{n})\,dS = \iiint_E \operatorname{div}(\nabla f)\,dV = \iiint_E \nabla^2 f\,dV$

29. $\iint_S (f\nabla g) \cdot \mathbf{n}\,dS = \iiint_E \operatorname{div}(f\nabla g)\,dV = \iiint_E (f\nabla^2 g + \nabla g \cdot \nabla f)\,dV$ by Exercise 16.5.25.

30. $\iint_S (f\nabla g - g\nabla f) \cdot \mathbf{n}\,dS = \iiint_E \left[(f\nabla^2 g + \nabla g \cdot \nabla f) - (g\nabla^2 f + \nabla g \cdot \nabla f)\right] dV$ [by Exercise 29].

But $\nabla g \cdot \nabla f = \nabla f \cdot \nabla g$, so that $\iint_S (f\nabla g - g\nabla f) \cdot \mathbf{n}\,dS = \iiint_E (f\nabla^2 g - g\nabla^2 f)\,dV$.

31. If $\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$ is an arbitrary constant vector, we define $\mathbf{F} = f\mathbf{c} = fc_1\,\mathbf{i} + fc_2\,\mathbf{j} + fc_3\,\mathbf{k}$. Then

$\operatorname{div}\mathbf{F} = \operatorname{div} f\mathbf{c} = \dfrac{\partial f}{\partial x}\,c_1 + \dfrac{\partial f}{\partial y}\,c_2 + \dfrac{\partial f}{\partial z}\,c_3 = \nabla f \cdot \mathbf{c}$ and the Divergence Theorem says $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV \;\;\Rightarrow$

$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_E \nabla f\cdot\mathbf{c}\,dV$. In particular, if $\mathbf{c} = \mathbf{i}$ then $\iint_S f\mathbf{i}\cdot\mathbf{n}\,dS = \iiint_E \nabla f\cdot\mathbf{i}\,dV \;\;\Rightarrow$

$\displaystyle\iint_S fn_1\,dS = \iiint_E \frac{\partial f}{\partial x}\,dV$ (where $\mathbf{n} = n_1\,\mathbf{i} + n_2\,\mathbf{j} + n_3\,\mathbf{k}$). Similarly, if $\mathbf{c} = \mathbf{j}$ we have $\displaystyle\iint_S fn_2\,dS = \iiint_E \frac{\partial f}{\partial y}\,dV$,

and $\mathbf{c} = \mathbf{k}$ gives $\displaystyle\iint_S fn_3\,dS = \iiint_E \frac{\partial f}{\partial z}\,dV$. Then

$$\begin{aligned}\textstyle\iint_S f\mathbf{n}\,dS &= \left(\textstyle\iint_S fn_1\,dS\right)\mathbf{i} + \left(\textstyle\iint_S fn_2\,dS\right)\mathbf{j} + \left(\textstyle\iint_S fn_3\,dS\right)\mathbf{k}\\ &= \left(\iiint_E \frac{\partial f}{\partial x}\,dV\right)\mathbf{i} + \left(\iiint_E \frac{\partial f}{\partial y}\,dV\right)\mathbf{j} + \left(\iiint_E \frac{\partial f}{\partial z}\,dV\right)\mathbf{k} = \iiint_E \left(\frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}\right)dV\\ &= \textstyle\iiint_E \nabla f\,dV \quad \text{as desired.}\end{aligned}$$

32. By Exercise 31, $\iint_S p\mathbf{n}\,dS = \iiint_E \nabla p\,dV$, so

$\mathbf{F} = -\iint_S p\mathbf{n}\,dS = -\iiint_E \nabla p\,dV = -\iiint_E \nabla(\rho g z)\,dV = -\iiint_E (\rho g\,\mathbf{k})\,dV = -\rho g\left(\iiint_E dV\right)\mathbf{k} = -\rho g V(E)\,\mathbf{k}$.

But the weight of the displaced liquid is volume $\times$ density $\times\, g = \rho g V(E)$, thus $\mathbf{F} = -W\mathbf{k}$ as desired.

16 Review

CONCEPT CHECK

1. See Definitions 1 and 2 in Section 16.1. A vector field can represent, for example, the wind velocity at any location in space, the speed and direction of the ocean current at any location, or the force vectors of Earth's gravitational field at a location in space.

2. (a) A conservative vector field $\mathbf{F}$ is a vector field which is the gradient of some scalar function f.

   (b) The function f in part (a) is called a potential function for $\mathbf{F}$, that is, $\mathbf{F} = \nabla f$.

3. (a) See Definition 16.2.2.

   (b) We normally evaluate the line integral using Formula 16.2.3.

   (c) The mass is $m = \int_C \rho(x,y)\;ds$, and the center of mass is $(\overline{x},\overline{y})$ where $\overline{x} = \frac{1}{m}\int_C x\rho(x,y)\;ds$, $\overline{y} = \frac{1}{m}\int_C y\rho(x,y)\;ds$.

   (d) See (5) and (6) in Section 16.2 for plane curves; we have similar definitions when C is a space curve [see the equation preceding (10) in Section 16.2].

   (e) For plane curves, see Equations 16.2.7. We have similar results for space curves [see the equation preceding (10) in Section 16.2].

4. (a) See Definition 16.2.13.

   (b) If $\mathbf{F}$ is a force field, $\int_C \mathbf{F}\cdot d\mathbf{r}$ represents the work done by $\mathbf{F}$ in moving a particle along the curve C.

   (c) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$

5. See Theorem 16.3.2.

6. (a) $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if the line integral has the same value for any two curves that have the same initial and terminal points.

   (b) See Theorem 16.3.4.

7. See the statement of Green's Theorem on page 1108 [ET 1084].

8. See Equations 16.4.5.

9. (a) $\operatorname{curl} \mathbf{F} = \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial P}{\partial z} - \dfrac{\partial R}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{F}$

   (b) $\operatorname{div} \mathbf{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$

   (c) For $\operatorname{curl} \mathbf{F}$, see the discussion accompanying Figure 1 on page 1118 [ET 1094] as well as Figure 6 and the accompanying discussion on page 1150 [ET 1126]. For $\operatorname{div} \mathbf{F}$, see the discussion following Example 5 on page 1119 [ET 1095] as well as the discussion preceding (8) on page 1157 [ET 1133].

10. See Theorem 16.3.6; see Theorem 16.5.4.

11. (a) See (1) and (2) and the accompanying discussion in Section 16.6; See Figure 4 and the accompanying discussion on page 1124 [ET 1100].

    (b) See Definition 16.6.6.

    (c) See Equation 16.6.9.

12. (a) See (1) in Section 16.7.

    (b) We normally evaluate the surface integral using Formula 16.7.2.

    (c) See Formula 16.7.4.

    (d) The mass is $m = \iint_S \rho(x, y, z)\, dS$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \frac{1}{m} \iint_S x\rho(x, y, z)\, dS$, $\overline{y} = \frac{1}{m} \iint_S y\rho(x, y, z)\, dS$, $\overline{z} = \frac{1}{m} \iint_S z\rho(x, y, z)\, dS$.

13. (a) See Figures 6 and 7 and the accompanying discussion in Section 16.7. A Möbius strip is a nonorientable surface; see Figures 4 and 5 and the accompanying discussion on page 1139 [ET 1115].

    (b) See Definition 16.7.8.

    (c) See Formula 16.7.9.

    (d) See Formula 16.7.10.

14. See the statement of Stokes' Theorem on page 1146 [ET 1122].

15. See the statement of the Divergence Theorem on page 1153 [ET 1129].

16. In each theorem, we have an integral of a "derivative" over a region on the left side, while the right side involves the values of the original function only on the boundary of the region.

TRUE-FALSE QUIZ

1. False; $\operatorname{div}\mathbf{F}$ is a scalar field.
2. True. (See Definition 16.5.1.)
3. True, by Theorem 16.5.3 and the fact that $\operatorname{div}\mathbf{0} = 0$.
4. True, by Theorem 16.3.2.
5. False. See Exercise 16.3.35. (But the assertion is true if D is simply-connected; see Theorem 16.3.6.)
6. False. See the discussion accompanying Figure 8 on page 1092 [ET 1068].
7. False. For example, $\operatorname{div}(y\,\mathbf{i}) = 0 = \operatorname{div}(x\,\mathbf{j})$ but $y\,\mathbf{i} \neq x\,\mathbf{j}$.
8. True. Line integrals of conservative vector fields are independent of path, and by Theorem 16.3.3, work $= \int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for any closed path C.
9. True. See Exercise 16.5.24.
10. False. $\mathbf{F}\cdot\mathbf{G}$ is a scalar field, so $\operatorname{curl}(\mathbf{F}\cdot\mathbf{G})$ has no meaning.
11. True. Apply the Divergence Theorem and use the fact that $\operatorname{div}\mathbf{F} = 0$.
12. False by Theorem 16.5.11, because if it were true, then $\operatorname{div}\operatorname{curl}\mathbf{F} = 3 \neq 0$.

EXERCISES

1. (a) Vectors starting on C point in roughly the direction opposite to C, so the tangential component $\mathbf{F}\cdot\mathbf{T}$ is negative. Thus $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is negative.

   (b) The vectors that end near P are shorter than the vectors that start near P, so the net flow is outward near P and $\operatorname{div}\mathbf{F}(P)$ is positive.

2. We can parametrize C by $x = x$, $y = x^2$, $0 \leq x \leq 1$ so

$$\int_C x\,ds = \int_0^1 x\sqrt{1+(2x)^2}\,dx = \tfrac{1}{12}(1+4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{12}\big(5\sqrt{5}-1\big).$$

3. $$\int_C yz\cos x\,ds = \int_0^\pi (3\cos t)(3\sin t)\cos t\sqrt{(1)^2+(-3\sin t)^2+(3\cos t)^2}\,dt = \int_0^\pi (9\cos^2 t\,\sin t)\sqrt{10}\,dt$$
$$= 9\sqrt{10}\left(-\tfrac{1}{3}\cos^3 t\right)\Big]_0^\pi = -3\sqrt{10}\,(-2) = 6\sqrt{10}$$

4. $x = 3\cos t \quad\Rightarrow\quad dx = -3\sin t\,dt$, $y = 2\sin t \quad\Rightarrow\quad dy = 2\cos t\,dt$, $0 \leq t \leq 2\pi$, so

$$\begin{aligned}\int_C y\,dx + \left(x+y^2\right)dy &= \int_0^{2\pi}\left[(2\sin t)(-3\sin t) + (3\cos t + 4\sin^2 t)(2\cos t)\right]dt \\ &= \int_0^{2\pi}(-6\sin^2 t + 6\cos^2 t + 8\sin^2 t\,\cos t)\,dt = \int_0^{2\pi}\left[6(\cos^2 t - \sin^2 t) + 8\sin^2 t\,\cos t\right]dt \\ &= \int_0^{2\pi}(6\cos 2t + 8\sin^2 t\,\cos t)\,dt = 3\sin 2t + \tfrac{8}{3}\sin^3 t\Big]_0^{2\pi} = 0\end{aligned}$$

Or: Notice that $\frac{\partial}{\partial y}(y) = 1 = \frac{\partial}{\partial x}\left(x+y^2\right)$, so $\mathbf{F}(x,y) = \left\langle y, x+y^2\right\rangle$ is a conservative vector field. Since C is a closed curve, $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C y\,dx + (x+y^2)\,dy = 0$.

5. $\int_C y^3\,dx + x^2\,dy = \int_{-1}^{1}\left[y^3(-2y) + (1-y^2)^2\right]dy = \int_{-1}^{1}(-y^4 - 2y^2 + 1)\,dy$

$$= \left[-\tfrac{1}{5}y^5 - \tfrac{2}{3}y^3 + y\right]_{-1}^{1} = -\tfrac{1}{5} - \tfrac{2}{3} + 1 - \tfrac{1}{5} - \tfrac{2}{3} + 1 = \tfrac{4}{15}$$

6. $\int_C \sqrt{xy}\,dx + e^y\,dy + xz\,dz = \int_0^1\left(\sqrt{t^4\cdot t^2}\cdot 4t^3 + e^{t^2}\cdot 2t + t^4\cdot t^3\cdot 3t^2\right)dt = \int_0^1(4t^6 + 2te^{t^2} + 3t^9)\,dt$

$$= \left[\tfrac{4}{7}t^7 + e^{t^2} + \tfrac{3}{10}t^{10}\right]_0^1 = e - \tfrac{9}{70}$$

7. C: $x = 1 + 2t \;\Rightarrow\; dx = 2\,dt,\ y = 4t \;\Rightarrow\; dy = 4\,dt,\ z = -1 + 3t \;\Rightarrow\; dz = 3\,dt,\ 0 \le t \le 1.$

$$\int_C xy\,dx + y^2\,dy + yz\,dz = \int_0^1[(1+2t)(4t)(2) + (4t)^2(4) + (4t)(-1+3t)(3)]\,dt$$

$$= \int_0^1(116t^2 - 4t)\,dt = \left[\tfrac{116}{3}t^3 - 2t^2\right]_0^1 = \tfrac{116}{3} - 2 = \tfrac{110}{3}$$

8. $\mathbf{F}(\mathbf{r}(t)) = (\sin t)(1+t)\,\mathbf{i} + (\sin^2 t)\,\mathbf{j}$, $\mathbf{r}'(t) = \cos t\,\mathbf{i} + \mathbf{j}$ and

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi((1+t)\sin t\cos t + \sin^2 t)\,dt = \int_0^\pi\left(\tfrac{1}{2}(1+t)\sin 2t + \sin^2 t\right)dt$$

$$= \left[\tfrac{1}{2}\left((1+t)\left(-\tfrac{1}{2}\cos 2t\right) + \tfrac{1}{4}\sin 2t\right) + \tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right]_0^\pi = \tfrac{\pi}{4}$$

9. $\mathbf{F}(\mathbf{r}(t)) = e^{-t}\,\mathbf{i} + t^2(-t)\,\mathbf{j} + (t^2 + t^3)\,\mathbf{k}$, $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} - \mathbf{k}$ and

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1(2te^{-t} - 3t^5 - (t^2 + t^3))\,dt = \left[-2te^{-t} - 2e^{-t} - \tfrac{1}{2}t^6 - \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4\right]_0^1 = \tfrac{11}{12} - \tfrac{4}{e}.$$

10. (a) C: $x = 3 - 3t,\ y = \tfrac{\pi}{2}t,\ z = 3t,\ 0 \le t \le 1.$ Then

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1\left[3t\,\mathbf{i} + (3-3t)\,\mathbf{j} + \tfrac{\pi}{2}t\,\mathbf{k}\right]\cdot\left[-3\,\mathbf{i} + \tfrac{\pi}{2}\,\mathbf{j} + 3\,\mathbf{k}\right]dt = \int_0^1\left[-9t + \tfrac{3\pi}{2}\right]dt = \tfrac{1}{2}(3\pi - 9).$$

(b) $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2}(3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + t\,\mathbf{k})\cdot(-3\sin t\,\mathbf{i} + \mathbf{j} + 3\cos t\,\mathbf{k})\,dt$

$$= \int_0^{\pi/2}(-9\sin^2 t + 3\cos t + 3t\cos t)\,dt = \left[-\tfrac{9}{2}(t - \sin t\cos t) + 3\sin t + 3(t\sin t + \cos t)\right]_0^{\pi/2}$$

$$= -\tfrac{9\pi}{4} + 3 + \tfrac{3\pi}{2} - 3 = -\tfrac{3\pi}{4}$$

11. $\frac{\partial}{\partial y}\left[(1+xy)e^{xy}\right] = 2xe^{xy} + x^2ye^{xy} = \frac{\partial}{\partial x}\left[e^y + x^2e^{xy}\right]$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_y(x,y) = e^y + x^2e^{xy}$ implies $f(x,y) = e^y + xe^{xy} + g(x)$ and then $f_x(x,y) = xye^{xy} + e^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$. But $f_x(x,y) = (1+xy)e^{xy}$, so $g'(x) = 0 \;\Rightarrow\; g(x) = K$. Thus $f(x,y) = e^y + xe^{xy} + K$ is a potential function for $\mathbf{F}$.

12. $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, its components have continuous partial derivatives, and $\operatorname{curl}\mathbf{F} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k} = \mathbf{0}$, so $\mathbf{F}$ is conservative by Theorem 16.5.4. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = \sin y$ implies $f(x,y,z) = x\sin y + g(y,z)$ and then $f_y(x,y,z) = x\cos y + g_y(y,z)$. But $f_y(x,y,z) = x\cos y$, so $g_y(y,z) = 0 \;\Rightarrow\; g(y,z) = h(z)$. Then $f(x,y,z) = x\sin y + h(z)$ implies $f_z(x,y,z) = h'(z)$. But $f_z(x,y,z) = -\sin z$, so $h(z) = \cos z + K$. Thus a potential function for $\mathbf{F}$ is $f(x,y,z) = x\sin y + \cos z + K$.

13. Since $\frac{\partial}{\partial y}\left(4x^3y^2 - 2xy^3\right) = 8x^3y - 6xy^2 = \frac{\partial}{\partial x}\left(2x^4y - 3x^2y^2 + 4y^3\right)$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$, $\mathbf{F}$ is conservative.

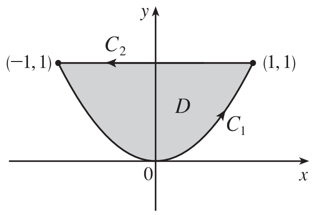
Furthermore $f(x, y) = x^4y^2 - x^2y^3 + y^4$ is a potential function for $\mathbf{F}$. $t = 0$ corresponds to the point $(0, 1)$ and $t = 1$ corresponds to $(1, 1)$, so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(0, 1) = 1 - 1 = 0$.

14. Here curl $\mathbf{F} = \mathbf{0}$, the domain of $\mathbf{F}$ is $\mathbb{R}^3$, and the components of $\mathbf{F}$ have continuous partial derivatives, so $\mathbf{F}$ is conservative.

Furthermore $f(x, y, z) = xe^y + ye^z$ is a potential function for $\mathbf{F}$. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 0, 3) - f(0, 2, 0) = 4 - 2 = 2$.

15. C_1: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$, $-1 \le t \le 1$;

C_2: $\mathbf{r}(t) = -t\,\mathbf{i} + \mathbf{j}$, $-1 \le t \le 1$.



Then

$$\int_C xy^2\,dx - x^2y\,dy = \int_{-1}^{1}(t^5 - 2t^5)\,dt + \int_{-1}^{1} t\,dt$$
$$= \left[-\tfrac{1}{6}t^6\right]_{-1}^{1} + \left[\tfrac{1}{2}t^2\right]_{-1}^{1} = 0$$

Using Green's Theorem, we have

$$\int_C xy^2\,dx - x^2y\,dy = \iint_D \left[\frac{\partial}{\partial x}\left(-x^2y\right) - \frac{\partial}{\partial y}\left(xy^2\right)\right] dA = \iint_D (-2xy - 2xy)\,dA = \int_{-1}^{1}\int_{x^2}^{1} -4xy\,dy\,dx$$
$$= \int_{-1}^{1}\left[-2xy^2\right]_{y=x^2}^{y=1}\,dx = \int_{-1}^{1}\left(2x^5 - 2x\right)dx = \left[\tfrac{1}{3}x^6 - x^2\right]_{-1}^{1} = 0$$

16. $\int_C \sqrt{1 + x^3}\,dx + 2xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}\left(\sqrt{1 + x^3}\right)\right] dA = \int_0^1\int_0^{3x}(2y - 0)\,dy\,dx = \int_0^1 9x^2\,dx = \left.3x^3\right]_0^1 = 3$

17. $\int_C x^2y\,dx - xy^2\,dy = \iint\limits_{x^2 + y^2 \le 4} \left[\frac{\partial}{\partial x}\left(-xy^2\right) - \frac{\partial}{\partial y}\left(x^2y\right)\right] dA = \iint\limits_{x^2 + y^2 \le 4} \left(-y^2 - x^2\right) dA = -\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = -8\pi$

18. $\operatorname{curl}\mathbf{F} = (0 - e^{-y}\cos z)\,\mathbf{i} - (e^{-z}\cos x - 0)\,\mathbf{j} + (0 - e^{-x}\cos y)\,\mathbf{k} = -e^{-y}\cos z\,\mathbf{i} - e^{-z}\cos x\,\mathbf{j} - e^{-x}\cos y\,\mathbf{k}$,

$\operatorname{div}\mathbf{F} = -e^{-x}\sin y - e^{-y}\sin z - e^{-z}\sin x$

19. If we assume there is such a vector field $\mathbf{G}$, then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = 2 + 3z - 2xz$. But $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ for all vector fields $\mathbf{F}$.

Thus such a $\mathbf{G}$ cannot exist.

20. Let $\mathbf{F} = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G} = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$ be vector fields whose first partials exist and are continuous. Then

$$\mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} = \left[P_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{i} + Q_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{j} + R_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]$$
$$- \left[P_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{i} + Q_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{j}\right.$$
$$\left. + R_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]$$

and

$$(\mathbf{G}\cdot\nabla)\,\mathbf{F}-(\mathbf{F}\cdot\nabla)\,\mathbf{G}=\left[\left(P_2\,\frac{\partial P_1}{\partial x}+Q_2\,\frac{\partial P_1}{\partial y}+R_2\,\frac{\partial P_1}{\partial z}\right)\mathbf{i}+\left(P_2\,\frac{\partial Q_1}{\partial x}+Q_2\,\frac{\partial Q_1}{\partial y}+R_2\,\frac{\partial Q_1}{\partial z}\right)\mathbf{j}\right.$$
$$\left.+\left(P_2\,\frac{\partial R_1}{\partial x}+Q_2\,\frac{\partial R_1}{\partial y}+R_2\,\frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]$$
$$-\left[\left(P_1\,\frac{\partial P_2}{\partial x}+Q_1\,\frac{\partial P_2}{\partial y}+R_1\,\frac{\partial P_2}{\partial z}\right)\mathbf{i}+\left(P_1\,\frac{\partial Q_2}{\partial x}+Q_1\,\frac{\partial Q_2}{\partial y}+R_1\,\frac{\partial Q_2}{\partial z}\right)\mathbf{j}\right.$$
$$\left.+\left(P_1\,\frac{\partial R_2}{\partial x}+Q_1\,\frac{\partial R_2}{\partial y}+R_1\,\frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]$$

Hence

$$\mathbf{F}\operatorname{div}\mathbf{G}-\mathbf{G}\operatorname{div}\mathbf{F}+(\mathbf{G}\cdot\nabla)\,\mathbf{F}-(\mathbf{F}\cdot\nabla)\,\mathbf{G}$$
$$=\left[\left(P_1\,\frac{\partial Q_2}{\partial y}+Q_2\,\frac{\partial P_1}{\partial x}\right)-\left(P_2\,\frac{\partial Q_1}{\partial y}+Q_1\,\frac{\partial P_2}{\partial y}\right)\right.$$
$$\left.-\left(P_2\,\frac{\partial R_1}{\partial z}+R_1\,\frac{\partial P_2}{\partial z}\right)+\left(P_1\,\frac{\partial R_2}{\partial z}+R_2\,\frac{\partial P_1}{\partial z}\right)\right]\mathbf{i}$$
$$+\left[\left(Q_1\,\frac{\partial R_2}{\partial z}+R_2\,\frac{\partial Q_1}{\partial z}\right)-\left(Q_2\,\frac{\partial R_1}{\partial z}+R_1\,\frac{\partial Q_2}{\partial z}\right)\right.$$
$$\left.-\left(P_1\,\frac{\partial Q_2}{\partial x}+Q_2\,\frac{\partial P_1}{\partial x}\right)+\left(P_2\,\frac{\partial Q_1}{\partial x}+Q_1\,\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j}$$
$$+\left[\left(P_2\,\frac{\partial R_1}{\partial x}+R_1\,\frac{\partial P_2}{\partial x}\right)-\left(P_1\,\frac{\partial R_2}{\partial x}+R_2\,\frac{\partial P_1}{\partial x}\right)\right.$$
$$\left.-\left(Q_1\,\frac{\partial R_2}{\partial y}+R_2\,\frac{\partial Q_1}{\partial y}\right)+\left(Q_2\,\frac{\partial R_1}{\partial y}+R_1\,\frac{\partial Q_2}{\partial y}\right)\right]\mathbf{k}$$
$$=\left[\frac{\partial}{\partial y}\,(P_1Q_2-P_2Q_1)-\frac{\partial}{\partial z}\,(P_2R_1-P_1R_2)\right]\mathbf{i}$$
$$+\left[\frac{\partial}{\partial z}\,(Q_1R_2-Q_2R_1)-\frac{\partial}{\partial x}\,(P_1Q_2-P_2Q_1)\right]\mathbf{j}$$
$$+\left[\frac{\partial}{\partial x}\,(P_2R_1-P_1R_2)-\frac{\partial}{\partial y}\,(Q_1R_2-Q_2R_1)\right]\mathbf{k}$$
$$=\operatorname{curl}\,(\mathbf{F}\times\mathbf{G})$$

21. For any piecewise-smooth simple closed plane curve C bounding a region D, we can apply Green's Theorem to $\mathbf{F}(x,y)=f(x)\,\mathbf{i}+g(y)\,\mathbf{j}$ to get $\int_C f(x)\,dx+g(y)\,dy=\iint_D\left[\frac{\partial}{\partial x}\,g(y)-\frac{\partial}{\partial y}\,f(x)\right]dA=\iint_D 0\,dA=0$.

22. $\nabla^2(fg) = \dfrac{\partial^2(fg)}{\partial x^2} + \dfrac{\partial^2(fg)}{\partial y^2} + \dfrac{\partial^2(fg)}{\partial z^2}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}\right) \quad \text{[Product Rule]}$$

$$= \frac{\partial^2 f}{\partial x^2}g + 2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}g + 2\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}$$

$$+ f\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}g + 2\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\frac{\partial^2 g}{\partial z^2} \quad \text{[Product Rule]}$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + 2\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle \cdot \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right\rangle$$

$$= f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$$

Another method: Using the rules in Exercises 14.6.37(b) and 16.5.25, we have

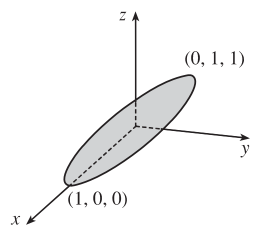
$$\nabla^2(fg) = \nabla\cdot\nabla(fg) = \nabla\cdot(g\,\nabla f + f\,\nabla g) = \nabla g\cdot\nabla f + g\nabla\cdot\nabla f + \nabla f\cdot\nabla g + f\nabla\cdot\nabla g$$
$$= g\,\nabla^2 f + f\,\nabla^2 g + 2\nabla f\cdot\nabla g$$

23. $\nabla^2 f = 0$ means that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. Now if $\mathbf{F} = f_y\,\mathbf{i} - f_x\,\mathbf{j}$ and C is any closed path in D, then applying Green's Theorem, we get

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C f_y\,dx - f_x\,dy = \iint_D \left[\tfrac{\partial}{\partial x}(-f_x) - \tfrac{\partial}{\partial y}(f_y)\right] dA$$
$$= -\iint_D (f_{xx} + f_{yy})\,dA = -\iint_D 0\,dA = 0$$

Therefore the line integral is independent of path, by Theorem 16.3.3.

24. (a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so C lies on the circular cylinder $x^2 + y^2 = 1$. But also $y = z$, so C lies on the plane $y = z$. Thus C is the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$.



(b) Apply Stokes' Theorem, $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$:

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{2y} & 2x^2e^{2y} + 2y\cot z & -y^2\csc^2 z \end{vmatrix} = \left\langle -2y\csc^2 z - (-2y\csc^2 z), 0, 4xe^{2y} - 4xe^{2y}\right\rangle = \mathbf{0}$$

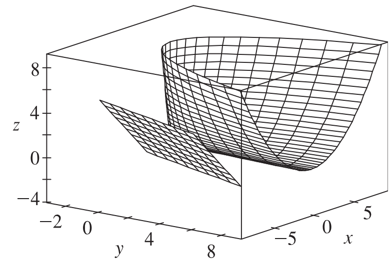
Therefore $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0$.

25. $z = f(x,y) = x^2 + 2y$ with $0 \le x \le 1$, $0 \le y \le 2x$. Thus

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4}\,dA = \int_0^1\int_0^{2x} \sqrt{5 + 4x^2}\,dy\,dx = \int_0^1 2x\sqrt{5 + 4x^2}\,dx = \tfrac{1}{6}(5 + 4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{6}\left(27 - 5\sqrt{5}\right).$$

26. (a) $\mathbf{r}_u = -v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = 2v\,\mathbf{i} - u\,\mathbf{j}$ and $\mathbf{r}_u \times \mathbf{r}_v = 2u^2\,\mathbf{i} + 4uv\,\mathbf{j} + 2v^2\,\mathbf{k}$. Since the point $(4, -2, 1)$ corresponds to $u = 1$, $v = 2$ (or $u = -1$, $v = -2$ but $\mathbf{r}_u \times \mathbf{r}_v$ is the same for both), a normal vector to the surface at $(4, -2, 1)$ is $2\,\mathbf{i} + 8\,\mathbf{j} + 8\,\mathbf{k}$ and an equation of the tangent plane is $2x + 8y + 8z = 0$ or $x + 4y + 4z = 0$.

(b)



(c) By Definition 16.6.6, the area of S is given by

$A(S) = \int_0^3 \int_{-3}^3 \sqrt{(2u^2)^2 + (4uv)^2 + (2v^2)^2}\,dv\,du = 2\int_0^3 \int_{-3}^3 \sqrt{u^4 + 4u^2v^2 + v^4}\,dv\,du.$

(d) By Equation 16.7.9, the surface integral is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^3 \int_{-3}^3 \left\langle \frac{(u^2)^2}{1+(v^2)^2}, \frac{(v^2)^2}{1+(-uv)^2}, \frac{(-uv)^2}{1+(u^2)^2} \right\rangle \cdot \langle 2u^2, 4uv, 2v^2\rangle\,dv\,du$$

$$= \int_0^3 \int_{-3}^3 \left(\frac{2u^6}{1+v^4} + \frac{4uv^5}{1+u^2v^2} + \frac{2u^2v^4}{1+u^4} \right) dv\,du \approx 1524.0190$$

27. $z = f(x, y) = x^2 + y^2$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (using upward orientation). Then

$$\iint_S z\,dS = \iint_{x^2+y^2 \le 4} (x^2 + y^2)\sqrt{4x^2 + 4y^2 + 1}\,dA$$

$$= \int_0^{2\pi} \int_0^2 r^3 \sqrt{1 + 4r^2}\,dr\,d\theta = \tfrac{1}{60}\pi\left(391\sqrt{17} + 1\right)$$

(Substitute $u = 1 + 4r^2$ and use tables.)

28. $z = f(x, y) = 4 + x + y$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then

$$\iint_S (x^2 z + y^2 z)\,dS = \iint_{x^2+y^2 \le 4} (x^2 + y^2)(4 + x + y)\sqrt{3}\,dA$$

$$= \int_0^2 \int_0^{2\pi} \sqrt{3}\,r^3(4 + r\cos\theta + r\sin\theta)\,d\theta\,dr = \int_0^2 8\pi\sqrt{3}\,r^3\,dr = 32\pi\sqrt{3}$$

29. Since the sphere bounds a simple solid region, the Divergence Theorem applies and

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E (z - 2)\,dV = \iiint_E z\,dV - 2\iiint_E dV$$

$$= 0 \begin{bmatrix}\text{odd function in } z \\ \text{and } E \text{ is symmetric}\end{bmatrix} - 2\cdot V(E) = -2\cdot\tfrac{4}{3}\pi(2)^3 = -\tfrac{64}{3}\pi$$

Alternate solution: $\mathbf{F}(\mathbf{r}(\phi, \theta)) = 4\sin\phi\,\cos\theta\,\cos\phi\,\mathbf{i} - 4\sin\phi\,\sin\theta\,\mathbf{j} + 6\sin\phi\,\cos\theta\,\mathbf{k}$,

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\,\cos\theta\,\mathbf{i} + 4\sin^2\phi\,\sin\theta\,\mathbf{j} + 4\sin\phi\,\cos\phi\,\mathbf{k}$, and

$\mathbf{F}\cdot(\mathbf{r}_\phi \times \mathbf{r}_\theta) = 16\sin^3\phi\,\cos^2\theta\,\cos\phi - 16\sin^3\phi\,\sin^2\theta + 24\sin^2\phi\,\cos\phi\,\cos\theta$. Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} (16\sin^3\phi\,\cos\phi\,\cos^2\theta - 16\sin^3\phi\,\sin^2\theta + 24\sin^2\phi\,\cos\phi\,\cos\theta)\,d\phi\,d\theta$$

$$= \int_0^{2\pi} \tfrac{4}{3}(-16\sin^2\theta)\,d\theta = -\tfrac{64}{3}\pi$$

30. $z = f(x, y) = x^2 + y^2$, $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (because of upward orientation) and $\mathbf{F}(\mathbf{r}(x, y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -2x^3 - 2xy^2 + x^2 + y^2$. Then

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint\limits_{x^2+y^2\le 1} (-2x^3 - 2xy^2 + x^2 + y^2)\,dA \\ &= \int_0^1 \int_0^{2\pi} (-2r^3 \cos^3\theta - 2r^3 \cos\theta\, \sin^2\theta + r^2)\,r\,dr\,d\theta = \int_0^1 r^3(2\pi)\,dr = \tfrac{\pi}{2}\end{aligned}$$

31. Since $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\iint_S (\operatorname{curl}\mathbf{F}) \cdot d\mathbf{S} = 0$. We parametrize C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$ and

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\cos^2 t\, \sin t + \sin^2 t\, \cos t)\,dt = \frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t\Big]_0^{2\pi} = 0.$

32. $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$ where C: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 8\cos^2 t\, \sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + e^{4\cos t \sin t}\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -16\cos^2 t \sin^2 t + 4\sin t \cos t$. Thus

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-16\cos^2 t\, \sin^2 t + 4\sin t\, \cos t)\,dt = \left[-16\left(-\frac{1}{4}\sin t\, \cos^3 t + \frac{1}{16}\sin 2t + \frac{1}{8}t\right) + 2\sin^2 t\right]_0^{2\pi} = -4\pi.$

33. The surface is given by $x + y + z = 1$ or $z = 1 - x - y$, $0 \le x \le 1$, $0 \le y \le 1 - x$ and $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Then

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_D (-y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\,dA = \iint_D (-1)\,dA = -(\text{area of } D) = -\frac{1}{2}.$

34. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + z^2)\,dV = \int_0^{2\pi} \int_0^1 \int_0^2 (3r^2 + 3z^2)\,r\,dz\,dr\,d\theta = 2\pi \int_0^1 (6r^3 + 8r)\,dr = 11\pi$

35. $\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint\limits_{x^2+y^2+z^2\le 1} 3\,dV = 3(\text{volume of sphere}) = 4\pi$. Then

$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \sin^3\phi\, \cos^2\theta + \sin^3\phi\, \sin^2\theta + \sin\phi\, \cos^2\phi = \sin\phi$ and

$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi} \sin\phi\,d\phi\,d\theta = (2\pi)(2) = 4\pi.$

36. Here we must use Equation 16.9.7 since $\mathbf{F}$ is not defined at the origin. Let S_1 be the sphere of radius 1 with center at the origin and outer unit normal $\mathbf{n}_1$. Let S_2 be the surface of the ellipsoid with outer unit normal $\mathbf{n}_2$ and let E be the solid region between S_1 and S_2. Then the outward flux of $\mathbf{F}$ through the ellipsoid is given by

$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\,dS = -\iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\,dS + \iiint_E \operatorname{div}\mathbf{F}\,dV$. But $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$, so

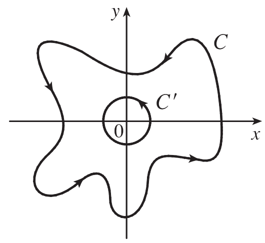
$\operatorname{div}\mathbf{F} = \nabla \cdot \left(|\mathbf{r}|^{-3}\,\mathbf{r}\right) = |\mathbf{r}|^{-3}(\nabla \cdot \mathbf{r}) + \mathbf{r} \cdot \left(\nabla\,|\mathbf{r}|^{-3}\right) = |\mathbf{r}|^{-3}\,(3) + \mathbf{r} \cdot \left(-3\,|\mathbf{r}|^{-4}\right)\left(\mathbf{r}\,|\mathbf{r}|^{-1}\right) = 0$. [Here we have used Exercises 16.5.30(a) and 16.5.31(a).] And $\mathbf{F} \cdot \mathbf{n}_1 = \dfrac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \dfrac{\mathbf{r}}{|\mathbf{r}|} = |\mathbf{r}|^{-2} = 1$ on S_1.

Thus $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\,dS = \iint_{S_1} dS + \iiint_E 0\,dV = (\text{surface area of the unit sphere}) = 4\pi(1)^2 = 4\pi.$

37. Because $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x, y, z) = 3x^2yz - 3y$ implies $f(x, y, z) = x^3yz - 3xy + g(y, z) \quad\Rightarrow\quad f_y(x, y, z) = x^3z - 3x + g_y(y, z)$. But $f_y(x, y, z) = x^3z - 3x$, so $g(y, z) = h(z)$ and $f(x, y, z) = x^3yz - 3xy + h(z)$. Then $f_z(x, y, z) = x^3y + h'(z)$ but $f_z(x, y, z) = x^3y + 2z$, so $h(z) = z^2 + K$ and a potential function for $\mathbf{F}$ is $f(x, y, z) = x^3yz - 3xy + z^2$. Hence

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(0, 3, 0) - f(0, 0, 2) = 0 - 4 = -4.$

38. Let C' be the circle with center at the origin and radius a as in the figure. Let D be the region bounded by C and C'. Then D's positively oriented boundary is $C \cup (-C')$. Hence by Green's Theorem



$$\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0, \text{ so}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot dr = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^{2\pi} \left[\frac{2a^3 \cos^3 t + 2a^3 \cos t \, \sin^2 t - 2a \sin t}{a^2} (-a \sin t) + \frac{2a^3 \sin^3 t + 2a^3 \cos^2 t \, \sin t + 2a \cos t}{a^2} (a \cos t) \right] dt$$

$$= \int_0^{2\pi} \frac{2a^2}{a^2}\, dt = 4\pi$$

39. By the Divergence Theorem, $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV = 3(\text{volume of } E) = 3(8 - 1) = 21$.

40. The stated conditions allow us to use the Divergence Theorem. Hence $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F})\, dV = 0$ since $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.

41. Let $\mathbf{F} = \mathbf{a} \times \mathbf{r} = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle = \langle a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x \rangle$. Then $\operatorname{curl} \mathbf{F} = \langle 2a_1, 2a_2, 2a_3 \rangle = 2\mathbf{a}$, and $\iint_S 2\mathbf{a} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\mathbf{a} \times \mathbf{r}) \cdot d\mathbf{r}$ by Stokes' Theorem.

PROBLEMS PLUS

1. Let S_1 be the portion of $\Omega(S)$ between $S(a)$ and S, and let ∂S_1 be its boundary. Also let S_L be the lateral surface of S_1 [that is, the surface of S_1 except S and $S(a)$]. Applying the Divergence Theorem we have $\iint_{\partial S_1} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iiint_{S_1} \nabla\cdot\frac{\mathbf{r}}{r^3}\,dV$.

But
$$\nabla\cdot\frac{\mathbf{r}}{r^3} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right\rangle$$
$$= \frac{(x^2+y^2+z^2-3x^2)+(x^2+y^2+z^2-3y^2)+(x^2+y^2+z^2-3z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$$

$\Rightarrow \quad \iint_{\partial S_1} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iiint_{S_1} 0\,dV = 0$. On the other hand, notice that for the surfaces of ∂S_1 other than $S(a)$ and S, $\mathbf{r}\cdot\mathbf{n} = 0 \quad\Rightarrow$

$$0 = \iint_{\partial S_1} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS + \iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS + \iint_{S_L} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS + \iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS \quad\Rightarrow$$

$\iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = -\iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$. Notice that on $S(a)$, $r = a \quad\Rightarrow\quad \mathbf{n} = -\frac{\mathbf{r}}{r} = -\frac{\mathbf{r}}{a}$ and $\mathbf{r}\cdot\mathbf{r} = r^2 = a^2$, so that $-\iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS = \iint_{S(a)} \frac{\mathbf{r}\cdot\mathbf{r}}{a^4}\,dS = \iint_{S(a)} \frac{a^2}{a^4}\,dS = \frac{1}{a^2}\iint_{S(a)} dS = \frac{\text{area of } S\,(a)}{a^2} = |\Omega(S)|$.

Therefore $|\Omega(S)| = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,dS$.

2. By Green's Theorem

$$\int_C (y^3 - y)\,dx - 2x^3\,dy = \iint_D \left[\frac{\partial(-2x^3)}{\partial x} - \frac{\partial(y^3-y)}{\partial y}\right] dA = \iint_D (1 - 6x^2 - 3y^2)\,dA$$

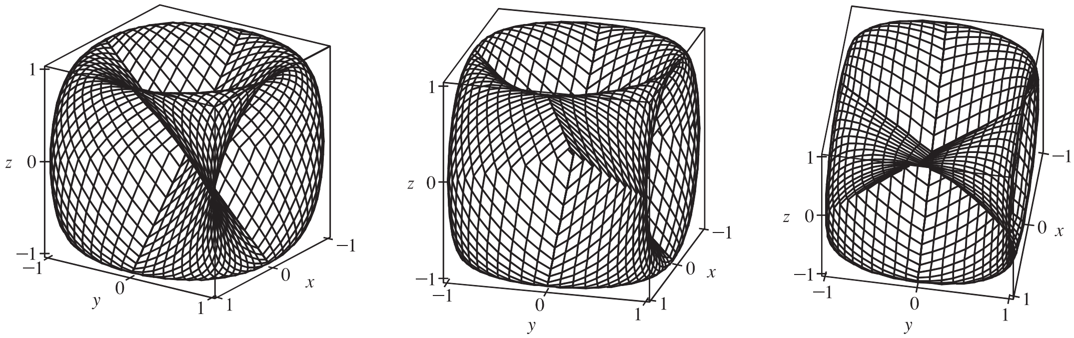
Notice that for $6x^2 + 3y^2 > 1$, the integrand is negative. The integral has maximum value if it is evaluated only in the region where the integrand is positive, which is within the ellipse $6x^2 + 3y^2 = 1$. So the simple closed curve that gives a maximum value for the line integral is the ellipse $6x^2 + 3y^2 = 1$.

3. The given line integral $\frac{1}{2}\int_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz$ can be expressed as $\int_C \mathbf{F}\cdot d\mathbf{r}$ if we define the vector field $\mathbf{F}$ by $\mathbf{F}(x,y,z) = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k} = \frac{1}{2}(bz - cy)\,\mathbf{i} + \frac{1}{2}(cx - az)\,\mathbf{j} + \frac{1}{2}(ay - bx)\,\mathbf{k}$. Then define S to be the planar interior of C, so S is an oriented, smooth surface. Stokes' Theorem says $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$. Now

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$
$$= \left(\tfrac{1}{2}a + \tfrac{1}{2}a\right)\mathbf{i} + \left(\tfrac{1}{2}b + \tfrac{1}{2}b\right)\mathbf{j} + \left(\tfrac{1}{2}c + \tfrac{1}{2}c\right)\mathbf{k} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k} = \mathbf{n}$$

so $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{n} = |\mathbf{n}|^2 = 1$, hence $\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_S dS$ which is simply the surface area of S. Thus, $\int_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}\int_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz$ is the plane area enclosed by C.

4. The surface given by $x = \sin u$, $y = \sin v$, $z = \sin(u + v)$ is difficult to visualize, so we first graph the surface from three different points of view.

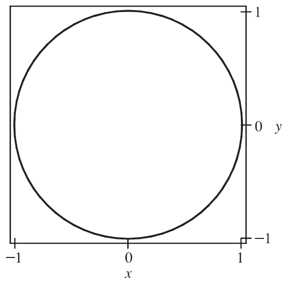


The trace in the horizontal plane $z = 0$ is given by $z = \sin(u + v) = 0 \quad \Rightarrow \quad u + v = k\pi$ [k an integer]. Then we can write $v = k\pi - u$, and the trace is given by the parametric equations $x = \sin u$, $y = \sin v = \sin(k\pi - u) = \sin k\pi \, \cos u - \cos k\pi \, \sin u = \pm \sin u$, and since $\sin u = x$, the trace consists of the two lines $y = \pm x$.

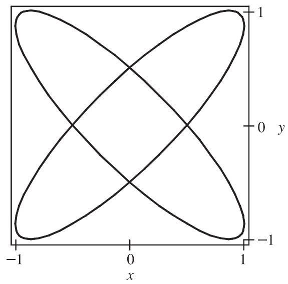
If $z = 1$, $z = \sin(u + v) = 1 \quad \Rightarrow \quad u + v = \frac{\pi}{2} + 2k\pi$. So $v = \left(\frac{\pi}{2} + 2k\pi\right) - u$ and the trace in $z = 1$ is given by the parametric equations $x = \sin u$, $y = \sin v = \sin\left(\left(\frac{\pi}{2} + 2k\pi\right) - u\right) = \sin\left(\frac{\pi}{2} + 2k\pi\right)\cos u - \cos\left(\frac{\pi}{2} + 2k\pi\right)\sin u = \cos u$. This curve is equivalent to $x^2 + y^2 = 1$, $z = 1$, a circle of radius 1. Similarly, in $z = -1$ we have $z = \sin(u + v) = -1 \quad \Rightarrow$ $u + v = \frac{3\pi}{2} + 2k\pi \quad \Rightarrow \quad v = \left(\frac{3\pi}{2} + 2k\pi\right) - u$, so the trace is given by the parametric equations $x = \sin u$, $y = \sin v = \sin\left(\left(\frac{3\pi}{2} + 2k\pi\right) - u\right) = \sin\left(\frac{3\pi}{2} + 2k\pi\right)\cos u - \cos\left(\frac{3\pi}{2} + 2k\pi\right)\sin u = -\cos u$, which again is a circle, $x^2 + y^2 = 1$, $z = -1$.

If $z = \frac{1}{2}$, $z = \sin(u + v) = \frac{1}{2} \quad \Rightarrow \quad u + v = \alpha + 2k\pi$ where $\alpha = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. Then $v = (\alpha + 2k\pi) - u$ and the trace in $z = \frac{1}{2}$ is given by the parametric equations $x = \sin u$, $y = \sin v = \sin[(\alpha + 2k\pi) - u] = \sin(\alpha + 2k\pi)\, \cos u - \cos(\alpha + 2k\pi)\, \sin u = \frac{1}{2}\cos u \pm \frac{\sqrt{3}}{2}\sin u$. In rectangular coordinates, $x = \sin u$ so $y = \frac{1}{2}\cos u \pm \frac{\sqrt{3}}{2}x \quad \Rightarrow \quad y \pm \frac{\sqrt{3}}{2}x = \frac{1}{2}\cos u \quad \Rightarrow \quad 2y \pm \sqrt{3}\,x = \cos u$. But then $x^2 + \left(2y \pm \sqrt{3}\,x\right)^2 = \sin^2 u + \cos^2 u = 1 \quad \Rightarrow \quad x^2 + 4y^2 \pm 4\sqrt{3}\,xy + 3x^2 = 1 \quad \Rightarrow \quad 4x^2 \pm 4\sqrt{3}\,xy + 4y^2 = 1$, which may be recognized as a conic section. In particular, each equation is an ellipse rotated $\pm 45^\circ$ from the standard orientation (see the following graph). The trace in $z = -\frac{1}{2}$ is similar: $z = \sin(u + v) = -\frac{1}{2} \quad \Rightarrow \quad u + v = \beta + 2k\pi$ where $\beta = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. Then $v = (\beta + 2k\pi) - u$ and the trace is given by the parametric equations $x = \sin u$, $y = \sin v = \sin[(\beta + 2k\pi) - u] = \sin(\beta + 2k\pi)\, \cos u - \cos(\beta + 2k\pi)\, \sin u = -\frac{1}{2}\cos u \pm \frac{\sqrt{3}}{2}\sin u$. If we convert to rectangular coordinates, we arrive at the same pair of equations, $4x^2 \pm 4\sqrt{3}\,xy + 4y^2 = 1$, so the trace is identical to the trace in $z = \frac{1}{2}$.

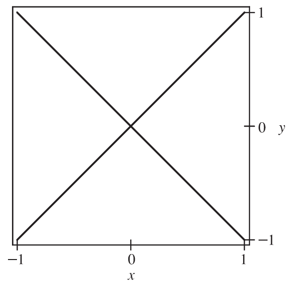
Graphing each of these, we have the following 5 traces.



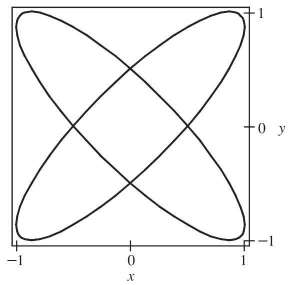
$z = -1$



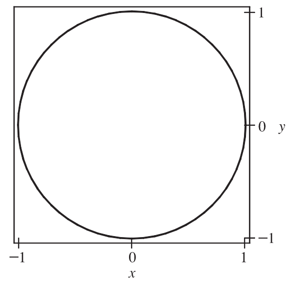
$z = -\frac{1}{2}$



$z = 0$



$z = \frac{1}{2}$



$z = 1$

Visualizing these traces on the surface reveals that horizontal cross sections are pairs of intersecting ellipses whose major axes are perpendicular to each other. At the bottom of the surface, $z = -1$, the ellipses coincide as circles of radius 1. As we move up the surface, the ellipses become narrower until at $z = 0$ they collapse into line segments, after which the process is reversed, and the ellipses widen to again coincide as circles at $z = 1$.

5. $(\mathbf{F} \cdot \nabla)\,\mathbf{G} = \left(P_1 \dfrac{\partial}{\partial x} + Q_1 \dfrac{\partial}{\partial y} + R_1 \dfrac{\partial}{\partial z}\right)(P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k})$

$$= \left(P_1 \frac{\partial P_2}{\partial x} + Q_1 \frac{\partial P_2}{\partial y} + R_1 \frac{\partial P_2}{\partial z}\right)\mathbf{i} + \left(P_1 \frac{\partial Q_2}{\partial x} + Q_1 \frac{\partial Q_2}{\partial y} + R_1 \frac{\partial Q_2}{\partial z}\right)\mathbf{j} + \left(P_1 \frac{\partial R_2}{\partial x} + Q_1 \frac{\partial R_2}{\partial y} + R_1 \frac{\partial R_2}{\partial z}\right)\mathbf{k}$$

$$= (\mathbf{F} \cdot \nabla P_2)\,\mathbf{i} + (\mathbf{F} \cdot \nabla Q_2)\,\mathbf{j} + (\mathbf{F} \cdot \nabla R_2)\,\mathbf{k}.$$

Similarly, $(\mathbf{G} \cdot \nabla)\,\mathbf{F} = (\mathbf{G} \cdot \nabla P_1)\,\mathbf{i} + (\mathbf{G} \cdot \nabla Q_1)\,\mathbf{j} + (\mathbf{G} \cdot \nabla R_1)\,\mathbf{k}$. Then

$$\mathbf{F} \times \operatorname{curl} \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_1 & Q_1 & R_1 \\ \partial R_2/\partial y - \partial Q_2/\partial z & \partial P_2/\partial z - \partial R_2/\partial x & \partial Q_2/\partial x - \partial P_2/\partial y \end{vmatrix}$$

$$= \left(Q_1 \frac{\partial Q_2}{\partial x} - Q_1 \frac{\partial P_2}{\partial y} - R_1 \frac{\partial P_2}{\partial z} + R_1 \frac{\partial R_2}{\partial x}\right)\mathbf{i} + \left(R_1 \frac{\partial R_2}{\partial y} - R_1 \frac{\partial Q_2}{\partial z} - P_1 \frac{\partial Q_2}{\partial x} + P_1 \frac{\partial P_2}{\partial y}\right)\mathbf{j} + \left(P_1 \frac{\partial P_2}{\partial z} - P_1 \frac{\partial R_2}{\partial x} - Q_1 \frac{\partial R_2}{\partial y} + Q_1 \frac{\partial Q_2}{\partial z}\right)\mathbf{k}$$

[continued]

and

$$\mathbf{G}\times\operatorname{curl}\mathbf{F} = \left(Q_2\frac{\partial Q_1}{\partial x} - Q_2\frac{\partial P_1}{\partial y} - R_2\frac{\partial P_1}{\partial z} + R_2\frac{\partial R_1}{\partial x}\right)\mathbf{i} + \left(R_2\frac{\partial R_1}{\partial y} - R_2\frac{\partial Q_1}{\partial z} - P_2\frac{\partial Q_1}{\partial x} + P_2\frac{\partial P_1}{\partial y}\right)\mathbf{j}$$
$$+ \left(P_2\frac{\partial P_1}{\partial z} - P_2\frac{\partial R_1}{\partial x} - Q_2\frac{\partial R_1}{\partial y} + Q_2\frac{\partial Q_1}{\partial z}\right)\mathbf{k}.$$

Then

$$(\mathbf{F}\cdot\nabla)\,\mathbf{G} + \mathbf{F}\times\operatorname{curl}\mathbf{G} = \left(P_1\frac{\partial P_2}{\partial x} + Q_1\frac{\partial Q_2}{\partial x} + R_1\frac{\partial R_2}{\partial x}\right)\mathbf{i} + \left(P_1\frac{\partial P_2}{\partial y} + Q_1\frac{\partial Q_2}{\partial y} + R_1\frac{\partial R_2}{\partial y}\right)\mathbf{j}$$
$$+ \left(P_1\frac{\partial P_2}{\partial z} + Q_1\frac{\partial Q_2}{\partial z} + R_1\frac{\partial R_2}{\partial z}\right)\mathbf{k}$$

and

$$(\mathbf{G}\cdot\nabla)\,\mathbf{F} + \mathbf{G}\times\operatorname{curl}\mathbf{F} = \left(P_2\frac{\partial P_1}{\partial x} + Q_2\frac{\partial Q_1}{\partial x} + R_2\frac{\partial R_1}{\partial x}\right)\mathbf{i} + \left(P_2\frac{\partial P_1}{\partial y} + Q_2\frac{\partial Q_1}{\partial y} + R_2\frac{\partial R_1}{\partial y}\right)\mathbf{j}$$
$$+ \left(P_2\frac{\partial P_1}{\partial z} + Q_2\frac{\partial Q_1}{\partial z} + R_2\frac{\partial R_1}{\partial z}\right)\mathbf{k}.$$

Hence

$$\begin{aligned}
&(\mathbf{F}\cdot\nabla)\,\mathbf{G} + \mathbf{F}\times\operatorname{curl}\mathbf{G} + (\mathbf{G}\cdot\nabla)\,\mathbf{F} + \mathbf{G}\times\operatorname{curl}\mathbf{F}\\
&\quad= \left[\left(P_1\frac{\partial P_2}{\partial x} + P_2\frac{\partial P_1}{\partial x}\right) + \left(Q_1\frac{\partial Q_2}{\partial x} + Q_2\frac{\partial Q_1}{\partial y}\right) + \left(R_1\frac{\partial R_2}{\partial x} + R_2\frac{\partial R_1}{\partial x}\right)\right]\mathbf{i}\\
&\qquad + \left[\left(P_1\frac{\partial P_2}{\partial y} + P_2\frac{\partial P_1}{\partial y}\right) + \left(Q_1\frac{\partial Q_2}{\partial y} + Q_2\frac{\partial Q_1}{\partial y}\right) + \left(R_1\frac{\partial R_2}{\partial y} + R_2\frac{\partial R_1}{\partial y}\right)\right]\mathbf{j}\\
&\qquad + \left[\left(P_1\frac{\partial P_2}{\partial z} + P_2\frac{\partial P_1}{\partial z}\right) + \left(Q_1\frac{\partial Q_2}{\partial z} + Q_2\frac{\partial Q_1}{\partial z}\right) + \left(R_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial R_1}{\partial z}\right)\right]\mathbf{k}\\
&\quad= \nabla(P_1P_2 + Q_1Q_2 + R_1R_2) = \nabla(\mathbf{F}\cdot\mathbf{G}).
\end{aligned}$$

6. (a) First we place the piston on coordinate axes so the top of the cylinder is at the origin and $x(t) \geq 0$ is the distance from the top of the cylinder to the piston at time t. Let C_1 be the curve traced out by the piston during one four-stroke cycle, so C_1 is given by $\mathbf{r}(t) = x(t)\,\mathbf{i}$, $a \leq t \leq b$. (Thus, the curve lies on the positive x-axis and reverses direction several times.) The force on the piston is $AP(t)\,\mathbf{i}$, where A is the area of the top of the piston and $P(t)$ is the pressure in the cylinder at time t. As in Section 16.2, the work done on the piston is $\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_a^b AP(t)\,\mathbf{i}\cdot x'(t)\,\mathbf{i}\,dt = \int_a^b AP(t)\,x'(t)\;dt$. Here, the volume of the cylinder at time t is $V(t) = Ax(t) \;\;\Rightarrow\;\; V'(t) = Ax'(t) \;\;\Rightarrow\;\; \int_a^b AP(t)\,x'(t)\;dt = \int_a^b P(t)\,V'(t)\;dt$. Since the curve C in the PV-plane corresponds to the values of P and V at time t, $a \leq t \leq b$, we have

$$W = \int_a^b AP(t)\,x'(t)\;dt = \int_a^b P(t)\,V'(t)\;dt = \int_C P\,dV$$

Another method: If we divide the time interval $[a, b]$ into n subintervals of equal length Δt, the amount of work done on the piston in the ith time interval is approximately $AP(t_i)[x(t_i) - x(t_{i-1})]$. Thus we estimate the total work done during one cycle to be $\sum_{i=1}^{n} AP(t_i)[x(t_i) - x(t_{i-1})]$. If we allow $n \to \infty$, we have

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} AP(t_i)[x(t_i) - x(t_{i-1})] = \lim_{n\to\infty} \sum_{i=1}^{n} P(t_i)[Ax(t_i) - Ax(t_{i-1})] = \lim_{n\to\infty} \sum_{i=1}^{n} P(t_i)[V(t_i) - V(t_{i-1})]$$
$$= \int_C P\,dV$$

(b) Let C_L be the lower loop of the curve C and C_U the upper loop. Then $C = C_L \cup C_U$. C_L is positively oriented, so from Formula 16.4.5 we know the area of the lower loop in the PV-plane is given by $-\oint_{C_L} P\,dV$. C_U is negatively oriented, so the area of the upper loop is given by $-\left(-\oint_{C_U} P\,dV\right) = \oint_{C_U} P\,dV$. From part (a),

$$W = \int_C P\,dV = \int_{C_L \cup C_U} P\,dV = \oint_{C_L} P\,dV + \oint_{C_U} P\,dV = \oint_{C_U} P\,dV - \left(-\oint_{C_L} P\,dV\right),$$

the difference of the areas enclosed by the two loops of C.