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FUNDAMENTALS OF OPTIMIZATION

Integer Linear Programming

CONTENT

- Relaxation and Bound
- Branch and Bound
- Cutting plan
- Gomory Cut
- Branch and Cut

Relaxation and Bound

- Given an Integer Program (IP)

$$z = \max\{cx: x \in X \subseteq \mathbb{Z}^n\}$$

- Find decreasing sequence of upper bounds

$$\bar{z}_1 > \bar{z}_2 > \dots > \bar{z}_s \geq z$$

- Find increasing sequence of lower bounds

$$\underline{z}_1 < \underline{z}_2 < \dots < \underline{z}_t \leq z$$

- Algorithm stop when $\bar{z}_s - \underline{z}_t \leq \varepsilon$

Relaxation and Bound

- Primal bounds
 - Every feasible solution $x^* \in X$ provides a lower bound of the maximization problem: $\underline{z} = cx^* \leq z$
 - Example: in TSP, every close tour is a upper bound of the objective function (as TSP is a minimization problem)

Relaxation and Bound

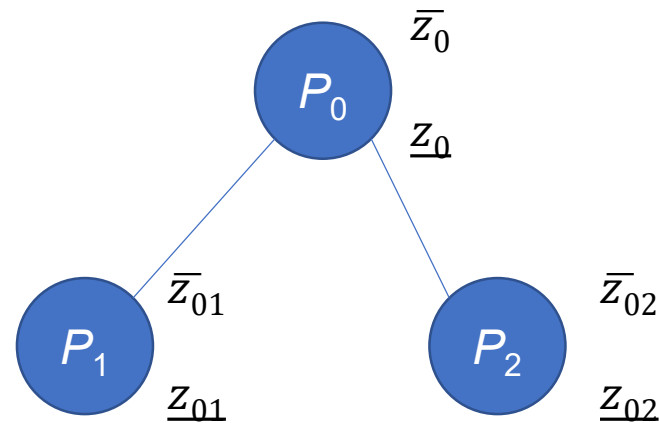
- Dual bounds
 - Finding upper bounds for a maximization problem (or lower bounds for a minimization problem) gives dual bounds of the objective
- **Definition** A problem (RP) $z^R = \max\{f(x): x \in T \subseteq R^n\}$ is a relaxation of (IP) $z = \max\{cx: x \in X \subseteq Z^n\}$ if:
 - $X \subseteq T$
 - $f(x) \geq cx, \forall x \in X$
- **Proposition** RP is a relaxation of IP , $z^R \geq z$

Relaxation and Bound

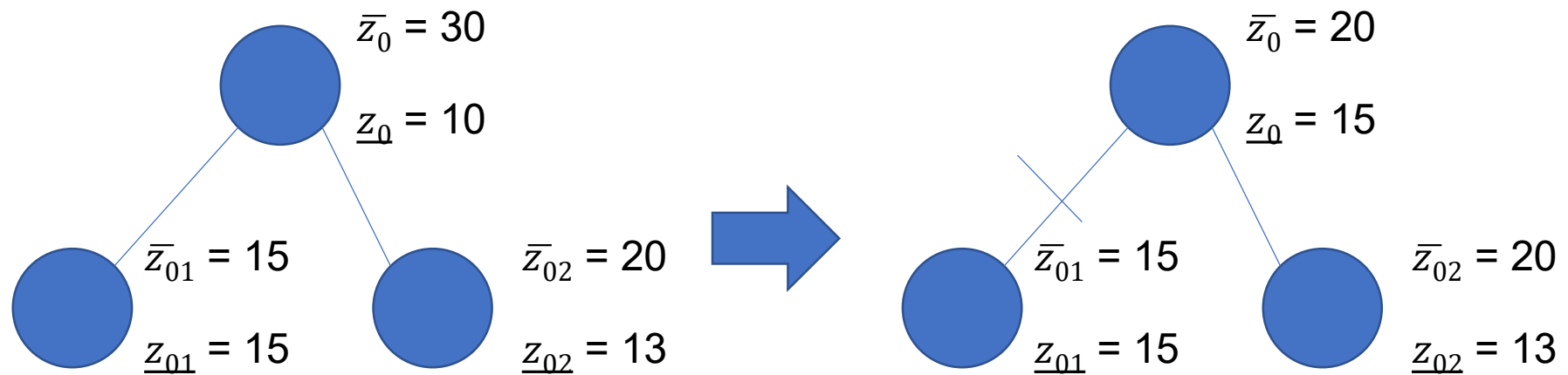
- Linear Relaxation
 - $Z^{LP} = \max\{cx: x \in P\}$ with $P = \{x \in R^n: Ax \leq b\}$ is a linear relaxation program of the $IP \max\{cx: x \in P \cap Z^n\}$

Branch and Bound

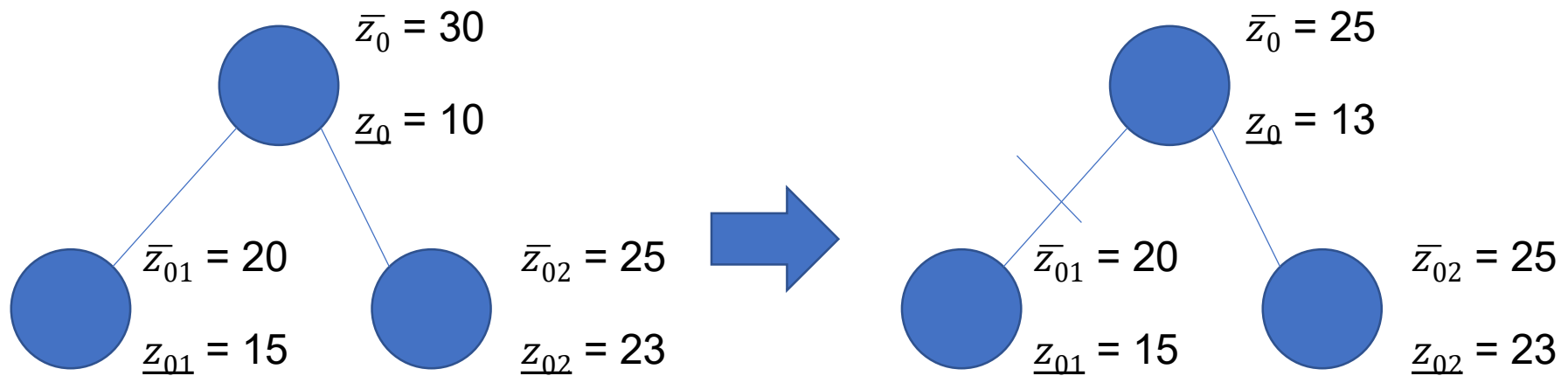
- Feasible region of P_0 is divided into feasible regions of P_1 and P_2 : $X(P_0) = X(P_1) \cup X(P_2)$



Branch and Bound



Branch and Bound



LP-based Branch and Bound

```
Initial problem  $S$  with formulation  $P$  on a list  $L$ 
Incumbent  $x^*$  is initialized with primal bound  $\underline{z} = -INF$ 
while  $L$  not empty do{
  Select a problem  $S^i$  with formulation  $P^i$  from  $L$ 
  Solve LP relaxation over  $P^i$  got dual bound  $\bar{z}^i$  and solution  $x^i(LP)$ 
  if  $\bar{z}^i \leq \underline{z}$  then continue; // prune by dual bound
  if  $x^i(LP)$  integer then{
     $\underline{z} = \bar{z}^i$ 
     $x^* = x^i(LP)$ 
  }else{
    select a component  $x_i$  of  $x^i(LP)$  whose value  $\lambda_i$  is fractional
     $P_1^i = P^i \cup (x_i \leq \lfloor \lambda_i \rfloor)$ ,  $P_2^i = P^i \cup (x_i \geq \lceil \lambda_i \rceil)$ 
    add  $P_1^i$  and  $P_2^i$  to  $L$ 
  }
}
```

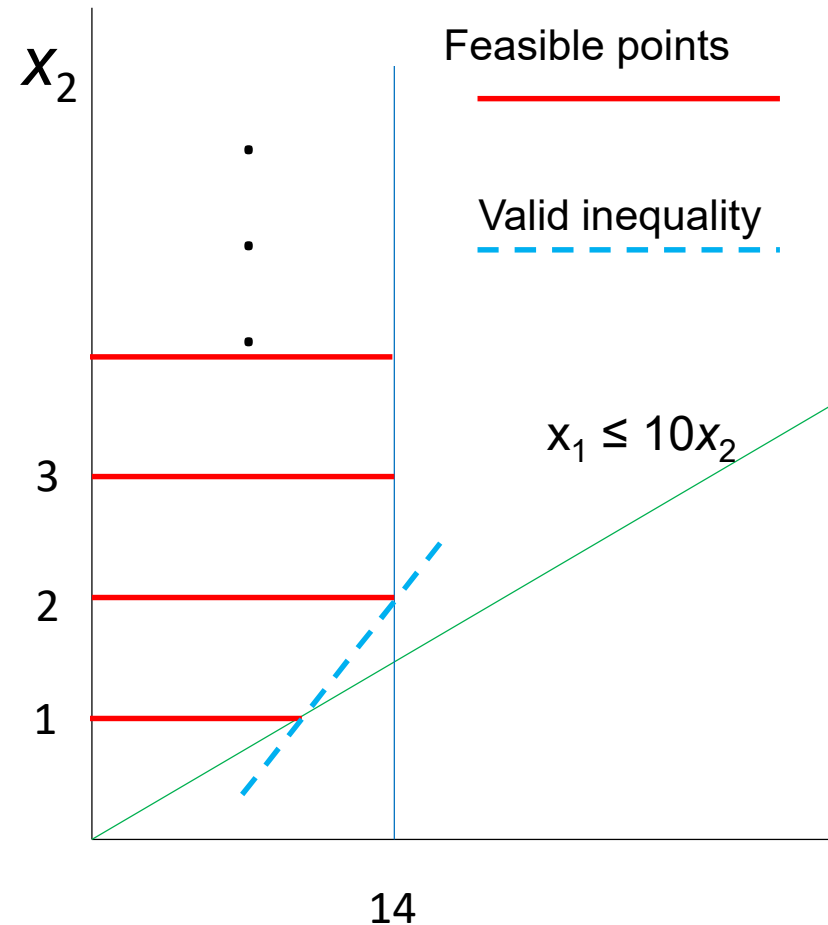
Return x^*

Cutting Plane

- Given a MIP $\max\{cz: z \in X\}$
- Inequality $\pi z \leq \pi_0$ is called a valid inequality if $\pi z \leq \pi_0$ is true for all $z \in X$
- Finding valid inequalities allows us to narrow the search space, transform MIP to corresponding LP in which an optimal solution to LP is an optimal solution to the original MIP

Cutting Plane

- Example, consider a MIP with $X = \{(x_1, x_2): x_1 \leq 10x_2, 0 \leq x_1 \leq 14, x_2 \in \mathbb{Z}_+^1\}$
- Red lines represent X
- $x_1 \leq 6 + 4x_2$ is a valid inequality (dashed line)



Example Integer Rounding

- Consider feasible region $X = P \cap \mathbb{Z}^3$ where $P = \{x \in \mathbb{R}_+^3 : 5x_1 + 9x_2 + 13x_3 \geq 19\}$
- From $5x_1 + 9x_2 + 13x_3 \geq 19$ we have $x_1 + \frac{9}{5}x_2 + \frac{13}{5}x_3 \geq \frac{19}{5}$
 $\rightarrow x_1 + 2x_2 + 3x_3 \geq \frac{19}{5}$
- As x_1, x_2, x_3 are integers, so we have
$$x_1 + 2x_2 + 3x_3 \geq \lceil \frac{19}{5} \rceil = 4 \text{ (this is a valid inequality for } X)$$

Gomory Cut

- (IP) $\max \{cx: Ax = b, x \geq 0 \text{ and integer}\}$
- Solve corresponding linear programming relaxation (LP) $\max \{cx: Ax = b, x \geq 0\}$
- Suppose with an optimal basis, the LP is rewritten in the form

$$\overline{a_{00}} + \sum_{j \in J_N} \overline{a_{0j}} x_j \rightarrow \max$$

$$x_{B_u} + \sum_{j \in J_N} \overline{a_{uj}} x_j = \overline{a_{u0}}, u = 1, 2, \dots, m$$

$$x \geq 0 \text{ and integer}$$

with $\overline{a_{0j}} \leq 0$ (as these coefficients corresponds to a maximizer), and $\overline{a_{u0}} \geq 0$

Gomory Cut

- If the basic optimal solution x^* is not integer, then there exists some row u with $\overline{a_{u0}}$ is not integer

→ Create a Gomory cut $x_{B_u} + \sum_{j \in J_N} \lfloor \overline{a_{uj}} \rfloor x_j \leq \lfloor \overline{a_{u0}} \rfloor$ (1)

→ Rewriting this inequality (as x_{B_u} is integer)

$$\sum_{j \in J_N} (\overline{a_{uj}} - \lfloor \overline{a_{uj}} \rfloor) x_j \geq \overline{a_{u0}} - \lfloor \overline{a_{u0}} \rfloor$$

or

$$\sum_{j \in J_N} f_{u,j} x_j \geq f_{u,0} \quad (2)$$

with $f_{u,j} = \overline{a_{uj}} - \lfloor \overline{a_{uj}} \rfloor$ and $f_{u,0} = \overline{a_{u0}} - \lfloor \overline{a_{u0}} \rfloor$

- As $0 \leq f_{u,j} < 1$ and $0 < f_{u,0} < 1$ and $x_j^* = 0, \forall j \in J_N \rightarrow (2)$ cuts off x^* .

Gomory Cut

- Difference between the left-hand side (LHS) and right-hand side (RHS) of (1) is integral (as x is integral) \rightarrow the difference between LHS and RHS of (2) is also integral
- \rightarrow rewrite (2) in the form $s = \sum_{j \in J_N} f_{u,j} x_j - f_{u,0}$ where the slack variable s is nonnegative integer

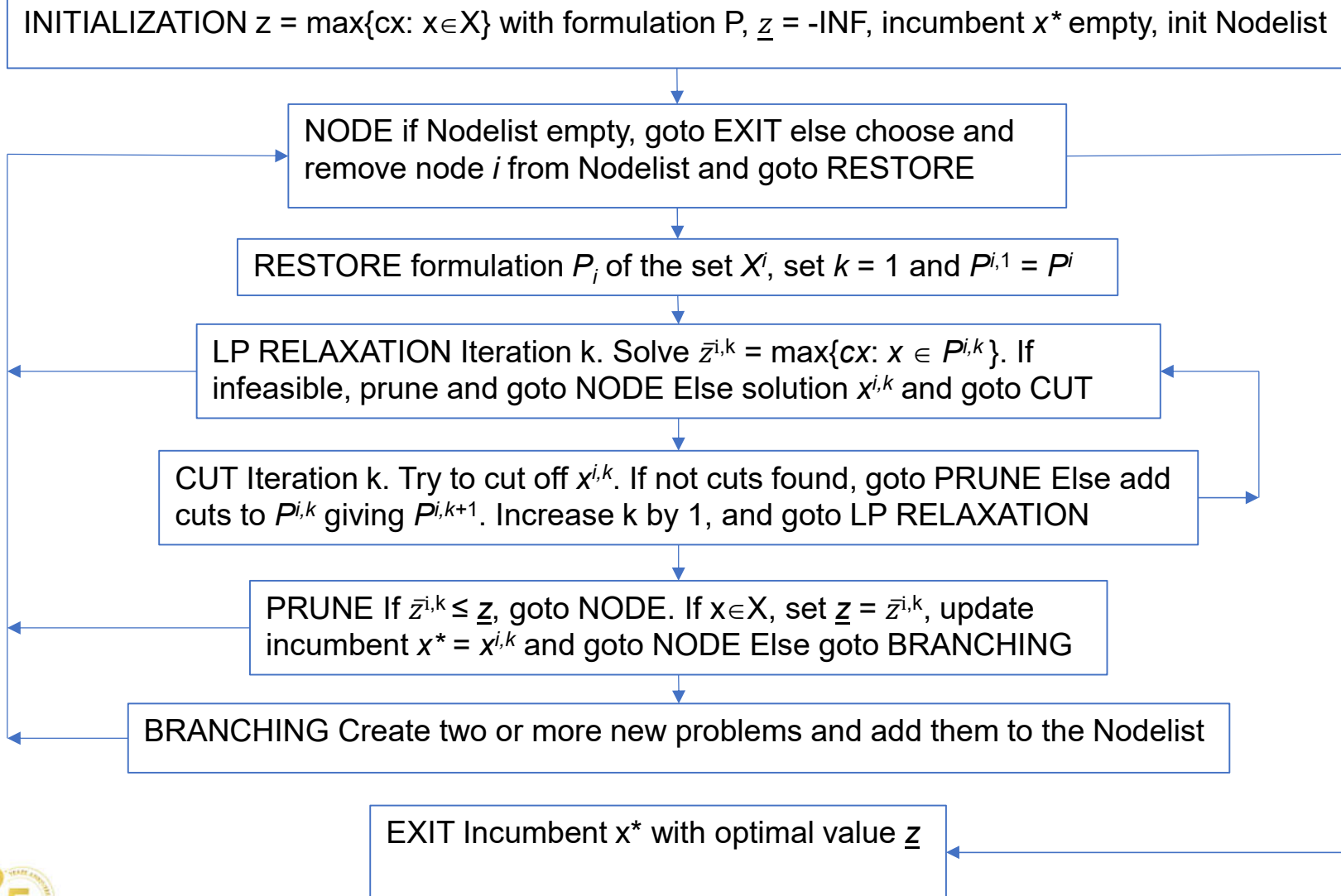
Gomory Cut

Consider the Integer Program (IP)

$$\begin{aligned} f(x_1, x_2, x_3, x_4, x_5) &= x_1 + x_2 \rightarrow \max \\ 2x_1 + x_2 + x_3 &= 8 \\ 3x_1 + 4x_2 + x_4 &= 24 \\ x_1 - x_2 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \text{ and integer} \end{aligned}$$

Solve ?

Branch and Cut [Wolsey, 98]





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