Introduction

Tuesday, October 26, 2021 7:25 AM

What is a differential equation? (phương trình vi phân)

- Find a function $y(x)$ satisfying $y'(x) = y(x)$.

  Guess: $y(x) = Ke^x$ (K constant)

  Check: $y'(x) = Ke^x = y(x)$ ✓

  $y(x) = Ke^x$ is called a solution to $\left[y'(x) = y(x) \right]$

- Find a function $y(x)$ satisfying $y''(x) = -y(x)$.

  Guess: $y_1(x) = \sin x$, $y_2(x) = \cos x$

  Check: $y_1 = \sin x$ $\qquad$ $y_2 = \cos x$

  $y_1' = \cos x$ $\qquad$ $y_2' = -\sin x$

  $y_1'' = -\sin x = -y_1(x)$ $\qquad$ $y_2'' = -\cos x = -y_2(x)$

- Observations:

  ① In the identity/equation,
  on both side we have x (independent variable)
  $y = y(x)$,
  and derivatives $y', y'', y^{(3)}, \dots, y^{(n)}$.

  ② we need to find a solution
  which is a function

- Find a function $y(x)$ satisfying $\begin{cases} y'(x) = y(x) \\ y(0) = 2021 \end{cases}$ (initial value condition) (giá trị ban đầu)

  $y(x) = Ke^x$ satisfies $y' = y$

  INITIAL VALUE PROBLEM

$$y(0) = 2021$$

$$\Rightarrow 2021 = K \cdot e^0 \Rightarrow K = 2021.$$

So $y(x) = 2021 \cdot e^x$ (unique solution)

A differential equation has the form

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

Order of a differential equation.

If a differential equation involves $y^{(n)}$ but not $y^{(k)}$ $(k > n)$, then we say the differential equation has order n.

- $y' = y$ first-order differential equation
- $y'' = -y$ second-order differential equation

We will start with first-order differential equations.

Cauchy's differential equation: (Initial Value Problem IVP)

$$\begin{cases} y' = f(x, y) & (\text{first-order differential equation}) \\ y(x_0) = y_0 \end{cases}$$

- $\begin{cases} y' = y \\ y'(0) = 2021 \end{cases}$ $\qquad \begin{cases} y' = e^x \cdot y \\ y'(0) = 2021 \end{cases}$

Existence and Uniqueness of Solution

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Theorem: Consider the differential equation

$$y' = f(x, y)$$

Suppose that $\left.\begin{matrix} f(x,y) \\ f_y(x,y) \end{matrix}\right\}$ are continuous in a domain D;

$(x_0, y_0) \in D$.

Then there is an open neighborhood $(x_0 - \varepsilon, x_0 + \varepsilon)$ of x_0 such that there exists a unique function $y(x)$ defined on $(x_0 - \varepsilon, x_0 + \varepsilon)$ satisfying $y' = f(x, y)$ and $y(x_0) = y_0$

'local solution'

The theorem is useful when we have a method to find a solution to the IVP.

It is not easy to find such a solution!

We will learn many methods to solve an IVP / a first-order differential equation.

Separable First-order Differential Equations

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Separable equation

A separable equation is a differential equation reduced to the form

$$f(x)\,dx = g(y)\,dy.$$

- example $y' = y$

$$\frac{dy}{dx} = y$$

$$\frac{dy}{y} = dx \qquad \text{separable equation}$$

Solving $f(x)\,dx = g(y)\,dy$:

$$\int f(x)\,dx = \int g(y)\,dy$$

$$\underbrace{F(x) = G(y) + k}_{\text{solution of } f(x)\,dx = g(y)\,dy} \qquad (k \text{ constant})$$

Examples of separable equations

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Example 1 $\quad (1+x)\,y\,dx + (1-y)\,x\,dy = 0$

move x to one side, move y to the other side

$$(1+x)\,y\,dx = (y-1)\,x\,dy$$

$$\frac{1+x}{x}\,dx = \frac{y-1}{y}\,dy$$

$$\left(1+\frac{1}{x}\right)dx = \left(1-\frac{1}{y}\right)dy$$

$$\int\left(1+\frac{1}{x}\right)dx = \left(1-\frac{1}{y}\right)dy$$

$$x + \ln|x| = y - \ln|y| + k$$

$$\boxed{\ln|xy| + x - y = k} \quad (k \text{ constant})$$

this is a solution when $x \neq 0$, $y \neq 0$

Example 2.

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{y^2-1}{x} \\ y(1) = 2 \end{cases} \qquad \text{IVP}$$

$$\frac{dy}{dx} = \frac{y^2-1}{x}$$

'move x to one side, move y to the other side'

$$\frac{dy}{dx} = \frac{y^2 - 1}{x}$$

Move x ... move y to the other side,

$$\int \frac{dy}{y^2 - 1} = \int \frac{dx}{x}$$

$$\frac{1}{2} \ln \left| \frac{y-1}{y+1} \right| = \ln|x| + k$$

$$\ln \left| \frac{y-1}{y+1} \right| = \ln|x|^2 + 2k \quad (k \text{ constant})$$

$$\left| \frac{y-1}{y+1} \right| = |x|^2 \; K \quad (K = e^{2k} > 0 \text{ constant})$$

$$\frac{y-1}{y+1} = K_1 \cdot x^2 \quad (K_1 = \pm K \quad \text{constant})$$

Now we use the initial value $y(1) = 2$,

so $$\frac{2-1}{2+1} = K_1 \; 1^2 \implies K_1 = \frac{1}{3}$$

$$\frac{y-1}{y+1} = \frac{x^2}{3}$$

We can solve for y: $$1 - \frac{1}{y+1} = \frac{x^2}{3}$$

$$\frac{1}{y+1} = 1 - \frac{x^2}{3} = \frac{3 - x^2}{3}$$

$$y + 1 = \frac{3}{3 - x^2}$$

$$y = \frac{3}{3 - x^2} - 1 = \frac{x^2}{3 - x^2}$$

Equation without y

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An equation without y has the form

$$F(x, y') = 0$$

We may solve for x (in terms of y') $x = g(y')$

or solve for y' (in terms of x) $y' = h(x)$

If we have $y' = h(x)$: we have a separable equation

If we have $x = g(y')$:

we make a change of variable : put $t = y'$

and transform the original equation (not separable in x and y)

into an equation separable in t and y.

Examples of equations without y

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$$F(x, y') = 0$$

Example 1

$$y' = x \cdot e^x$$

this equation is separable

$$\frac{dy}{dx} = x e^x$$

$$\int dy = \int \underbrace{x \cdot e^x}_{\text{integrate by parts}} dx$$

$$y = xe^x - e^x + k.$$

Example 2 :

$$x = y' + y'^2$$

this equation is not separable

Change variable : put $t = y'(x) = \frac{dy}{dx}$

- $x = t + t^2$, $dx = (1 + 2t)\, dt$
- from $t = \frac{dy}{dx}$, we have

$$dy = t\, dx$$

$$dy = t(1 + 2t)\, dt \quad (\text{separable in } y \text{ and } t)$$

$$\int dy = \int t(1 + 2t)\, dt$$

$$y = \frac{t^2}{2} + \frac{2t^3}{3} + k \quad (k \text{ constant})$$

Therefore we have

$$\begin{cases} x = t + t^2 \\ y = \dfrac{t^2}{2} + \dfrac{2t^3}{3} + k \end{cases} \quad (k \text{ constant})$$

Is this a solution for $x = y' + y'^2$?

Check: $y' = y'(x) = \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$$= \frac{t + 2t^2}{1 + 2t} = t$$

So $y' + y'^2 = t + t^2 = x$ ✓

Yes, this is a solution in terms of a parameter t

"parametrized solution / parametric solution"

Equations without x

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$$F(y, y') = 0$$

We may solve for y in terms of y'. $y = g(y')$

or solve for y' in terms of y. $y' = h(y)$

If $y' = h(y)$, then $\frac{dy}{dx} = h(y)$

we have a separable equation.

If $y = g(y')$, we make a change of variables.

Put $t = y'$.

Then we transform the original equation without x

(not separable in x and y)

to an equation separable in x and t.

Examples of equations without x

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Example 1 $\quad y' = y^2 + 4$

$$\frac{dy}{dx} = y^2 + 4 \qquad \text{separable !}$$

$$\int \frac{dy}{y^2+4} = \int dx$$

$$\frac{1}{2}\arctan\frac{y}{2} = x + k \qquad (k \text{ constant})$$

$$y = 2\tan\left(2x + k_1\right) \qquad (k_1 = 2k \text{ constant})$$

Example 2 $\quad y = y' + y'^3$

This is not separable in x and y.

We change variables. Put $t = y'(x) = \frac{dy}{dx}$

- $y = t + t^3 \quad , \quad dy = (1 + 3t^2)\,dt$
- from $t = \frac{dy}{dx}$, we have

$$dy = t\,dx$$

$$(1+3t^2)\,dt = t\,dx \qquad (\text{separable in } x \text{ and } t)$$

$$\int\left(\frac{1}{t} + 3t\right)dt = \int dx$$

$$\ln|t| + \frac{3}{2}t^2 + k = x \qquad (k \text{ constant})$$

$$\ln|t| + \frac{3}{2}t^2 + k = x \qquad (k \text{ constant})$$

So
$$\begin{cases} x = \frac{3}{2}t^2 + \ln|t| + k \\ y = t + t^3 \end{cases} \qquad (k \text{ constant})$$

Is this a solution for $y = y' + y'^3$?

$y' = y'(x)$, not $y'(t)$!

$$y'(x) = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{1 + 3t^2}{3t + \frac{1}{t}} = t$$

therefore $y' + y'^3 = t + t^3 = y$. ✓

Thus we have a solution for $y = y' + y'^3$!

First-order Differential Equations

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$F(x, y, y') = 0$ first-order differential equation

- separable first-order differential equation

$$g(y)\,dy = h(x)\,dx$$

- DE without x. $F(y, y') = 0$

- DE without y. $F(x, y') = 0$

Homogenous first-order differential equations

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$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Put $u = \frac{y}{x}$ (change variable to make the equation separable)

$$y = ux$$

$$y' = u'x + u$$

$$u'x + u = f(u)$$

$$u'x = f(u) - u$$

$$\frac{du}{f(u) - u} = \frac{dx}{x} \quad : \text{separable}$$