Note: In the previous lecture on counting the number of elementary operations, a lot of you are confused about whether or not the last comparison that breaks loop should be counted. Thus, from this lecture, if you have to choose a loop as the elementary operation then you can choose whatever is inside the loop instead.

Exercises on Big O:

Exercise 1:

1.1: Elementary operation: t := t + i + j (see the note above);

1.2: Running time: n^2

1.3: $O(n^2)$

1.4: The sum of all j's = $n.(1 + 2 + 3 + ... + n) = \frac{n.n.(n+1)}{2} = \frac{n^3+n^2}{2}$

The sum of all i's =

$n + 2n + 3n + ... + n.n = n.(1 + 2 + ... + n) = \frac{n^3+n^2}{2}$

Thus the sum of all i+j = $n^3 + n^2$

Exercise 2:

$f1 = 3nlog_2 n + 9n - 3log_2 n - 3$

Let $n_0 = 2$. Thus $nlog_2 n \geq n > log_2 n > 1 \quad (\forall n \geq 2)$

Now pick $c = 12$, we have

$f1 = 3nlog_2 n + 9n - 3log_2 n - 3 \leq 3nlog_2 n + 9nlog_2 n = 12nlog_2 n$

Thus $f1 \in O(nlog_2 n)$

Apply the same method for the rest of the exercise, we get:

$f2 = 2n^2 + nlog_3 n - 15 \in O(n^2)$

With $c = 3$ and $n_0 = 3$.

$f3 = 100n + \frac{(n+1)(n+5)}{2} + n^{\frac{3}{2}} \in O(n^2)$

With $c = 107$ and $n_0 = 1$.

$f4 = 1000n^2 + 2^n + 36nlogn + \frac{3}{2}^{n+1} \in O(2^n)$
With $c = 1040$ and $n_0 = 4$.

Exercise 3:

$n^2 \in O(n^3)$: True
You can try to find c and n_0

$2^n \in O(3^n)$: True:

$3^n \in O(2^n)$: False

$nlogn \in O(n^{\frac{3}{2}})$: True

$n^{\frac{3}{2}} \in O(nlogn)$: False

$2^{n+1} \in O(2^n)$: True

$O(2^{n+1}) = O(2^n)$: True

$O(2^n) = O(3^n)$: False

Exercise 4:

The running time = $(n - 1) + (n - 2) + ... + (n - k)$
= $nk - (1 + 2 + ... + k)$
= $nk - \frac{k(k+1)}{2}$
$\in O(nk)$

But the worst k can get is n. So that makes $O(n^2)$

Exercise 5:

The answer is in the lecture slide

Note: it's $n^{1+\epsilon}$ not $n^{1.\epsilon}$

Exercise 6:

The answer is in the lecture slide

Exercise 7:

Running time = $n(2n + n^2) \in O(n^3)$

Exercises on… the rest?

Exercise 1:

Let $n_0 = 1$ and $c = 22$. We have:

$f(n) = n^3 + 20n + 1 \leq n^3 + 20n^3 + n^3 = 22n^3 \quad (\forall n \geq 1)$

$\Rightarrow f(n) \in O(n^3)$

Exercise 2:

Assume that $f(n) = n^3 + 20n + 1 \leq cn^2 \quad (\forall n \geq n_0)$

Which means $f(n) = n^3 - cn^2 + 20n + 1 \leq 0 \quad (\forall n \geq n_0)$

We can see that, with a large enough n (namely n > c), $n^3 - cn^2 > 0$, $20n + 1 > 0$.

Thus we get $f(n) = n^3 - cn^2 + 20n + 1 > 0 \quad (\forall n > c)$, which violates the assumption. In conclusion, $f(n) = n^3 + 20n + 1 \notin O(n^2)$

Exercise 3:

Let $n_0 = 1$ and $c = 22$. We have:

$f(n) = n^3 + 20n + 1 \le n^4 + 20n^4 + n^4 = 22n^4 \quad (\forall n \ge 1)$

$\Rightarrow f(n) \in O(n^4)$

Exercise 4:

Let $n_0 = 1$ and $c = 1$. We have:

$f(n) = n^3 + 20n > n^3 \ge n^2 \quad (\forall n \ge 1)$

$\Rightarrow f(n) \in \Omega(n^3)$

Exercise 5:

Assume that $f(n) = \frac{1}{2}n^2 - 3n \ge cn^2 \quad (\forall n \ge n_0)$

$\Leftrightarrow (\frac{1}{2} - c)n^2 \ge 3n \quad (\forall n \ge n_0)$

Let assume n_0>0 so that n is positive.

$\Leftrightarrow \frac{1}{2} - c \ge \frac{3}{n} \quad (\forall n \ge n_0)$

We can see that when $n \to \infty$, $\frac{3}{n} \to 0$. Thus $\frac{1}{2} - c > 0$ is sufficient.

We can choose c = ¼ and a very large n_0 and get:

$f(n) = \frac{1}{2}n^2 - 3n \ge \frac{1}{4}n^2 \quad (\forall n \ge n_0)$

$\Rightarrow f(n) \in \Omega(n^2)$

Exercise 6:

Let $n_0 = \frac{7}{4}$, $c_1 = 1$, $c_2 = 5$, we have:

$f(n) = 5n^2 - 7n = n^2 + 4n^2 - 7n \ge n^2 \quad (\forall n \ge \frac{7}{4})$

And $f(n) = 5n^2 - 7n \le 5n^2 \quad (\forall n \ge \frac{7}{4})$

Thus $n^2 \le f(n) = 5n^2 - 7n = n^2 + 4n^2 - 7n \le 5n^2 \quad (\forall n \ge \frac{7}{4})$

$\Rightarrow f(n) \in \Theta(n^2)$

Exercise 7:

Let $n_0 = 1,\ c_1 = 13,\ c_2 = 36$, we have:

$f(n) = 23n^3 - 10n^2 logn + 7n + 6 = 13n^3 + 10n^2(n - logn) + 7n + 6 \geq 13n^3$ $(\forall n \geq 1)$

And

$f(n) = 23n^3 - 10n^2 logn + 7n + 6 \leq 23n^3 + 7n^3 + 6n^3 = 36n^3 \quad (\forall n \geq 1)$

Thus $13n^3 \leq f(n) = 23n^3 - 10n^2 logn + 7n + 6 \leq 36n^3 \quad (\forall n \geq 1)$

$\Rightarrow f(n) \in \Theta(n^3)$

Exercise 8:

Let $n_0 = 1$ and $c = 1$. We have:

$f(n) = 3n^3 - 2n^2 + 2 = n^3 + 2n^2(n - 1) + 2 \geq n^3 \quad (\forall n \geq 1)$

$\Rightarrow f(n) \in \Omega(n^3)$

Exercise 9:

9.1. $f(n) = n(2n + n^2) = n^3 + 2n^2$

9.2. Let $n_0 = 1,\ c_2 = 3$, we have:

$f(n) = n^3 + 2n^2 \leq n^3 + 2n^3 = 3n^3 \quad (\forall n \geq 1)$

$\Rightarrow f(n) \in O(n^3)$

9.3. Let $n_0 = 1,\ c_1 = 1$, we have:

$f(n) = n^3 + 2n^2 > n^3 \quad (\forall n \geq 1)$

$\Rightarrow f(n) \in \Omega(n^3)$

Furthermore, $f(n) \in \Theta(n^3)$, which is the time complexity of the algorithm.

Exercise 10:

$O(1) \subset O(loglogn) \subset O(logn) \subset O(\sqrt{n}) \subset O(n) \subset O(nlogn) \subset O(n^{\frac{3}{2}}) \subset O(n^2)$
$\subset O(n^3) \subset O(2^n) \subset O(3^n) \subset O(n!) \subset O(n^n)$

Exercise 11:

$O(n^{logn}) \subset O(n^{\sqrt{n}}) \subset O(2^{n}) = O(2^{n+1}) \subset O(2^{2n}) \subset O(n!) \subset O((n + 1)!) \subset O(n^{n})$