Introduction

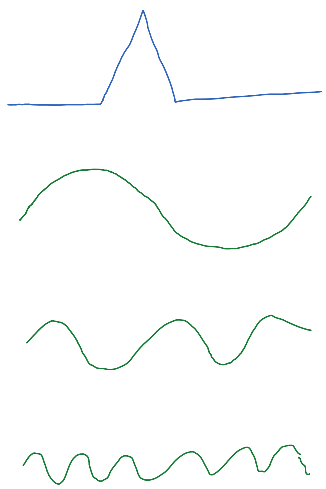
Tuesday, October 19, 2021 7:20 AM

Definition. A trigonometric series (chuỗi lượng giác) is a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

Jean Baptiste Joseph Fourier

Can we represent a periodic function as a sum of trigonometric functions?



$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) \quad (-\pi \le x \le \pi)$$

At first, this is a formal series, no convergence yet.

Suppose $S(x)$ is uniformly convergent, so $S(x)$ is a continuous function.

Question. How are the coefficients a_n, b_n expressed in terms of $S(x)$?

To answer this question, we look at

$$\int_{-\pi}^{\pi} S(x) \cos(mx)\, dx$$

$$\int_{-\pi}^{\pi} S(x) \sin(mx)\, dx$$

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$\int^{\pi} S(x) \cdot \cos mx\, dx = \frac{a_0}{2} \int^{\pi} \cos mx\, dx + \sum^{\infty} a_n \int^{\pi} \cos nx \cos mx\, dx$$

$$\int_{-\pi}^{\pi} S(x)\cdot \cos mx \, dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \cos mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx$$

Similarly for $\int_{-\pi}^{\pi} S(x)\cdot \sin mx \, dx$

Orthogonality of trigonometric functions.

$$\int_{-\pi}^{\pi} \cos nx \, dx = \begin{cases} 2\pi & (n=0) \\ 0 & (n \in \mathbb{Z}\setminus\{0\}) \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \, dx = 0$$

$(m, n \in \mathbb{Z})$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & (m \neq n) \\ 2\pi & (m = n = 0) \\ \pi & (m = n \neq 0) \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & (m \neq n) \\ 0 & (m = n = 0) \\ \pi & (m = n \neq 0) \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0$$

Suppose $S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is uniformly convergent in $[-\pi, \pi]$

Then

$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} S(x)\, dx$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} S(x) \cos nx \, dx$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} S(x) \sin nx \, dx$$

$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} S(x)\, dx = \right.$ average value of $S(x)$ in $[-\pi, \pi]$

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \text{average value of } f, \ldots \right.$$

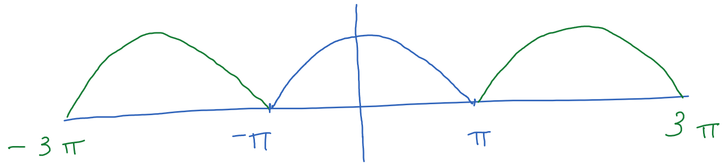
$$\left. \frac{2}{2\pi} \int_{-\pi}^{\pi} \cdots = \underline{\text{twice the average value}} \right) \text{(easier to remember)}$$

These are formulas for coefficients of trigonometric series.

Convergence of Fourier Series

Tuesday, October 19, 2021 8:08 AM

Suppose $f(x)$ is a periodic function with period 2π



"trigonometric Fourier series of $f(x)$"

Form the trigonometric series $S_f(x)$ "Fourier series of $f(x)$"

$$S_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Questions:

1. Does $S_f(x)$ converge ?

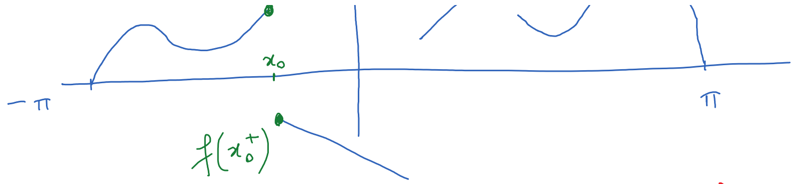
2. If yes, is it equal to $f(x)$?

Dirichlet's Theorem:

Dirichlet conditions: $f(x)$ is a periodic function with period 2π

- $f(x)$ is piecewise continuously differentiable

- $f(x)$ has at most finitely many discontinuities



Theorem: Suppose $f(x)$ satisfies Dirichlet conditions.

(i) At a point x_0 where $f(x)$ is continuous,

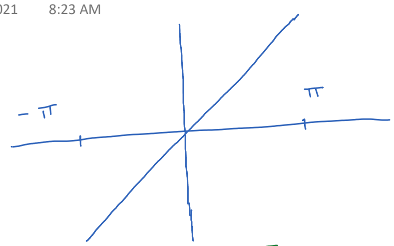
$S_f(x)$ converges to $f(x)$. $\left(S_f(x) = f(x)\right)$

(ii) At a discontinuity x_0,

$S_f(x)$ converges, $S_f(x_0) = \frac{1}{2}\left(f(x_0^-) + f(x_0^+)\right)$

Examples

Tuesday, October 19, 2021 8:23 AM

① 

$f(x) = x$ for $-\pi \le x < \pi$

Find the Fourier series $S_f(x)$.

$$a_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} x \, dx = 0 \qquad \text{“odd function”}$$

$$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} \underbrace{x}_{\text{odd}} \underbrace{\cos nx}_{\text{even}} \, dx = 0 \qquad \text{“odd function”}$$

$$b_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} \underbrace{x}_{\text{odd}} \underbrace{\sin nx}_{\text{odd}} \, dx = \frac{2}{\pi} \int_{0}^{\pi} x \sin nx \, dx$$

$$= \underbrace{\frac{2}{\pi} \left(x \cdot \frac{-\cos nx}{n} \right)_{x=0}^{x=\pi}} - \underbrace{\frac{2}{\pi} \int_0^{\pi} \frac{-\cos nx}{n} \, dx}_{0} \qquad (n \neq 0)$$

$$= (-1)^{n+1} \cdot \frac{2}{n}$$

Fourier series of $f(x) = x$ $(-\pi \le x < \pi)$ is

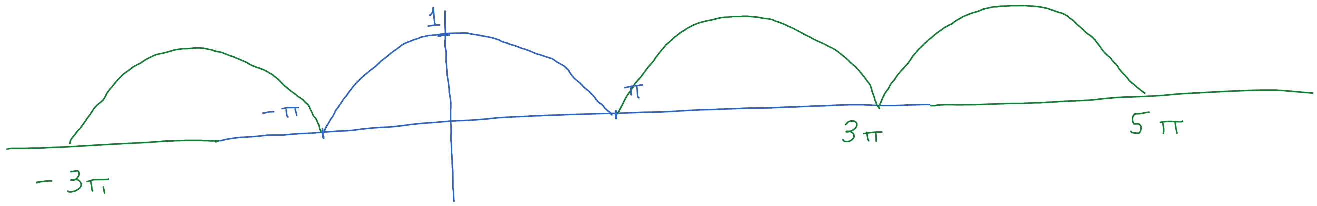
$$S_f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \sin nx$$

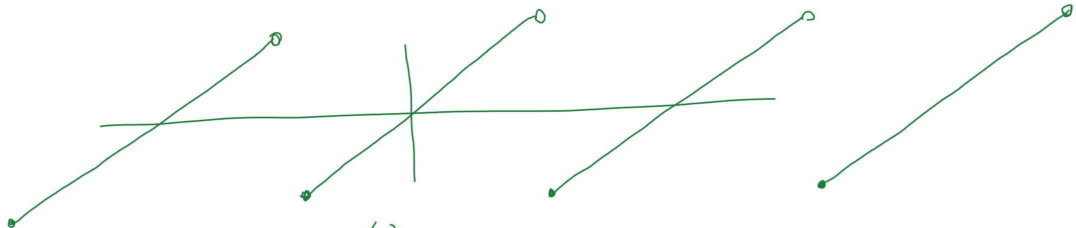
By Dirichlet's Theorem

$$f(x) = x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \qquad (-\pi < x < \pi)$$

---

② $f(x) = 1 - \dfrac{x^2}{\pi^2}$ $\quad (-\pi \le x \le \pi)$





$g(x) = x$

$$f(x) = 1 - \frac{x^2}{\pi^2} \qquad (-\pi \le x \le \pi)$$

$$a_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} \left(1 - \frac{x^2}{\pi^2}\right) dx = \frac{4}{3}$$

$$(n \neq 0) \quad a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} \left(1 - \frac{x^2}{\pi^2}\right) \cos(nx)\, dx = 4 \cdot \frac{(-1)^{n+1}}{n^2 \pi^2}$$

$$(n \neq 0) \quad b_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} \underbrace{\left(1 - \frac{x^2}{\pi^2}\right)}_{\text{even}} \underbrace{\sin(nx)}_{\text{odd}}\, dx = 0 \qquad \text{"odd function"}$$

The Fourier series of $f(x) = 1 - \frac{x^2}{\pi^2}$ $(-\pi \le x \le \pi)$ is

$$S_f(x) = \frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos nx}{n^2}$$

By Dirichlet's theorem, we deduce that

$$f(x) = \boxed{1 - \frac{x^2}{\pi^2} = \frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos nx}{n^2}} \quad (-\pi \le x \le \pi)$$

Take $x = \pi$

Take $x = \pi$

$$0 = \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{reciprocals of squares of natural numbers}} = \underbrace{\frac{\pi^2}{6}}_{\text{pi}} \qquad \text{"Euler's identity"}$$

Fourier Series of Odd and Even Functions

Tuesday, October 19, 2021 8:46 AM

- Suppose $f(x)$ is an odd function on $[-\pi, \pi]$. Then

$$S_f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{"sine series"}$$

where $$b_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

- Suppose $g(x)$ is an even function on $[-\pi, \pi]$. Then

$$S_g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{"cosine series"}$$

where $$a_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} g(x) \, dx$$

$$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} g(x) \cos nx \, dx$$

General Periodic Functions

Tuesday, October 19, 2021 8:50 AM

Suppose $f(x)$ is periodic with period $2L$.

Then we can consider its Fourier series

$$S_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

with Fourier coefficients

$$a_0 = \frac{2}{2L} \int_{-L}^{L} f(x) \, dx$$

$$a_n = \frac{2}{2L} \int_{-L}^{L} f(x) \cdot \cos \frac{n\pi x}{L} \, dx$$

$$b_n = \frac{2}{2L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{x} \, dx$$

Fourier Series in an Interval

Tuesday, October 19, 2021 8:53 AM

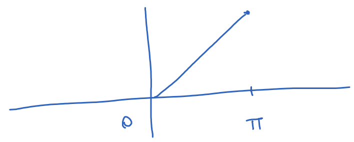
Suppose $f(x)$ is a function defined on $[r, s]$.

Extend $f(x)$ to a periodic function $\tilde{f}(x)$ with period $\geq s - r$

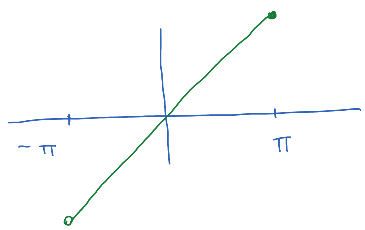
and find the Fourier series $S_{\tilde{f}}(x)$ of $\tilde{f}(x)$

For example

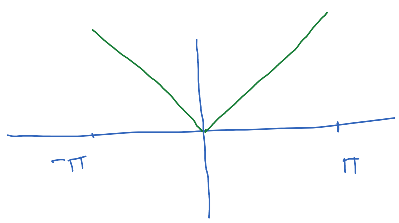
$f(x) = x \quad (0 \leq x \leq \pi)$



$f_1(x) = x \quad (-\pi < x \leq \pi)$



$f_2(x) = |x| \quad (-\pi \leq x \leq \pi)$



We may extend $f(x)$ to $f_1(x)$ or $f_2(x)$,

find $S_{f_1}(x)$ or $S_{f_2}(x)$,

and then look at the Fourier series $S_{f_1}(x)$ or $S_{f_2}(x)$ on $[0, \pi]$

and call this a Fourier series for $f(x)$ when $0 \leq x \leq \pi$