Series of Functions

Tuesday, October 5, 2021 7:22 AM

A series of numbers 'is' a number!

$$\sum_{n=1}^{\infty} x_n \longrightarrow \left(s_n = \sum_{j=1}^{n} x_j\right)_{n=1}^{\infty} \xrightarrow{\text{limit}} \lim_{n\to\infty} s_n$$

series of numbers $\longrightarrow$ sequence of partial sums

$$\sum_{n=1}^{\infty} f_n(x) \longrightarrow s_n(x) = \sum_{j=1}^{n} f_j(x) \xrightarrow{\text{limit}} \lim_{n\to\infty} s_n(x)$$

series of functions $\longrightarrow$ sequence of partial sums $\longrightarrow$ function

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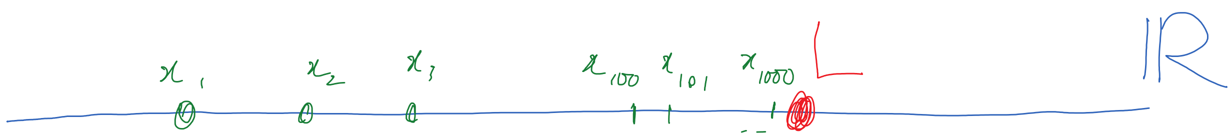
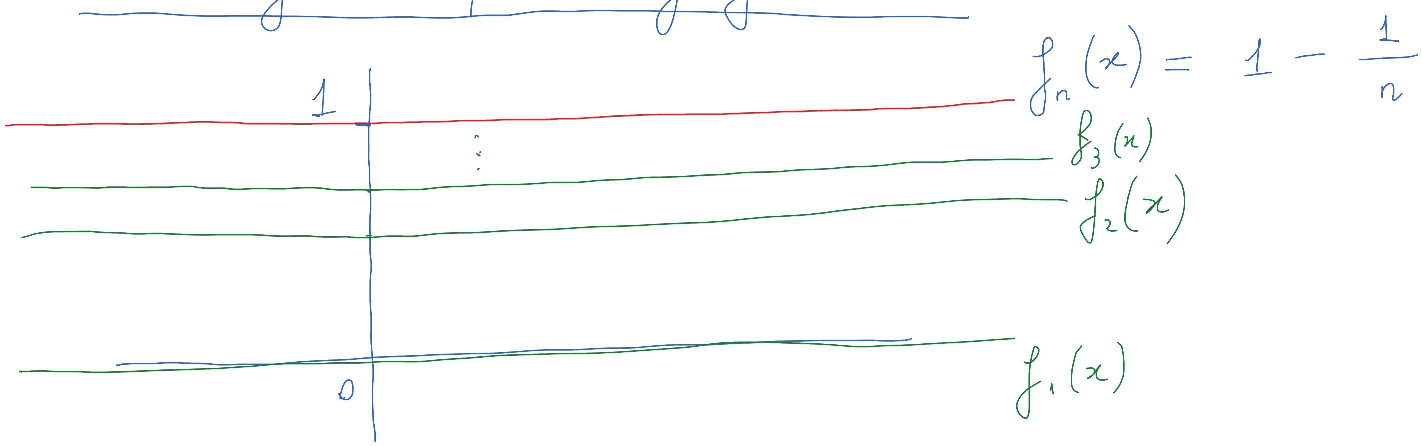
Pictures

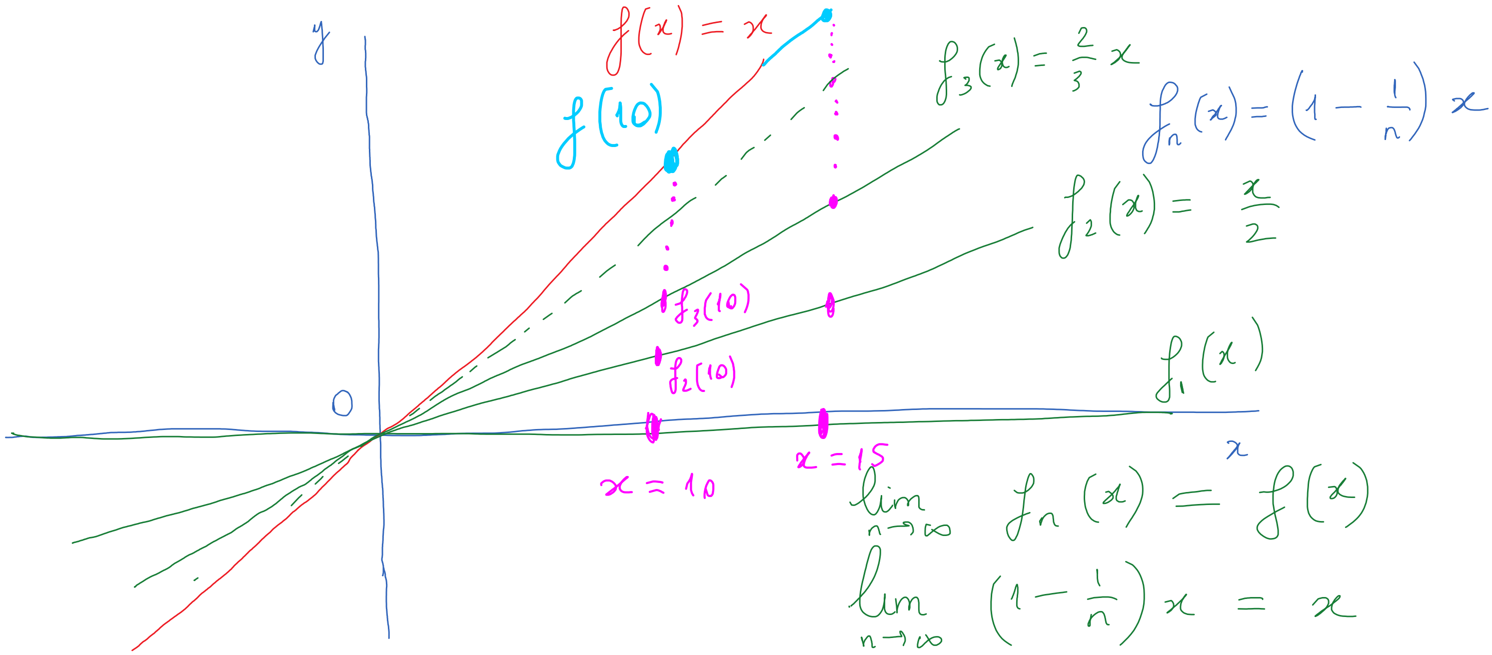
Limit of a sequence of numbers



$$\lim_{n\to\infty} x_n = L$$

Limit of a sequence of functions





$$\lim_{n \to \infty} f_n(x) = f(x)$$

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)x = x$$

① Think in pictures about

sequences of numbers,

sequences of functions!

② • A series of functions "is" a function!

• A function can often be expressed

as a series of functions!

(SERIES IS USEFUL !)

<u>Definition</u> Suppose $\left(f_n(x)\right)_{n=1}^{\infty}$ is a sequence of function

$$\sum_{n=1}^{\infty} f_n(x)$$ formal sum

"infinite addition of functions"

$n=1$ infinite addition of functions

$$\left(s_n(x) = \sum_{j=1}^{n} f_j(x) \right)_{n=1}^{\infty}$$ sequence of partial sums

sequence of functions

If $\lim_{n \to \infty} s_n(x) = f(x)$ exists,

the series $\sum_{n=1}^{\infty} f_n(x)$ is said to be <u>convergent</u>;

otherwise $\sum_{n=1}^{\infty} f_n(x)$ is said to be <u>divergent</u>.

<u>Domain of convergence</u>

The set of values of x for which $\sum_{n=1}^{\infty} f_n(x)$ converges

is called the <u>domain of convergence</u> (miền hội tụ)

<u>Examples</u>

① Find the domain of convergence of

$$\sum_{n=1}^{\infty} n^x = \sum_{n=1}^{\infty} f_n(x)$$

$$f_n(x) = n^x$$

We know $\sum_{n=1}^{\infty} n^x$ converges when $x < -1$

diverges when $x \geqslant -1$

Domain of convergence : $x < -1$

$(-\infty, -1)$

$\sum_{n=1}^{\infty} n^x$ has domain of convergence $(-\infty, -1)$, and is defined on $(-\infty, -1)$.

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② Find the domain of convergence of

$$\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} f_n(x) \qquad (x \text{ is a variable})$$

$$f_n(x) = x^n$$

$\sum_{n=1}^{\infty} x^n$ (geometric series) converges when $|x| < 1$

diverges when $|x| \geqslant -1$

So $\sum_{n=1}^{\infty} x^n$ has domain of convergence $(-1, 1)$

and is defined for $-1 < x < 1$

To find the domain of convergence of a series of functions, you use criteria of convergence of series of numbers.

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③ Power series

$$\sum_{n=1}^{\infty} a_n x^n \quad , \quad a_n \in \mathbb{R}$$

power series (chuỗi lũy thừa)

The domain of convergence of a power series is always an interval

Uniform Convergence

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The concept of Uniform Convergence is very useful, but it is hard to grasp at first!

Rate of convergence (tốc độ hội tụ)

Consider the following sequences (of numbers)

$$x_n = 1 - \frac{1}{n} \qquad \lim_{n\to\infty} x_n = 1$$

$$y_n = 1 - \frac{1}{n^2} \qquad \lim_{n\to\infty} y_n = 1$$

$\forall \varepsilon > 0$, there exists $N = N(\varepsilon)$ so that if $n > N$, then $|x_n - 1| < \varepsilon$

When $\underbrace{\varepsilon = 10^{-6}}_{\text{accuracy}}$, we want $|x_n - 1| < 10^{-6}$

$$\left|\left(1 - \frac{1}{n}\right) - 1\right| < 10^{-6}$$

$$\frac{1}{n} < 10^{-6}$$

$$n > 10^6$$

Choose $N(\varepsilon) = 10^6$

When $\varepsilon = 10^{-6}$, we want $|y_n - 1| < 10^{-6}$

$$\left|\left(1 - \frac{1}{n^2}\right) - 1\right| < 10^{-6}$$

$$\frac{1}{n^2} < 10^{-6}$$

$$n > 10^3 = 1000$$

Choose $N(\varepsilon) = 1000$

With $x_n = 1 - \frac{1}{n}$, to approach 1 within the accuracy 10^{-6},

we have to go to

$x_{10^6+1}, x_{10^6+2}, x_{10^6+3}, \dots$

With $y_n = 1 - \frac{1}{n^2}$, ______________________________

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$y_{1001}, y_{1002}, y_{1003}, \dots$

$\left(1 - \frac{1}{n^2}\right)_{n=1}^{\infty}$ approaches 1 faster than $\left(1 - \frac{1}{n}\right)_{n=1}^{\infty}$

rate of convergence (tốc độ hội tụ)

Uniform Convergence

Suppose $\sum_{n=1}^{\infty} f_n(x)$ is a series of functions.

Suppose $\sum_{n=1}^{\infty} f_n(x) = f(x)$ in a domain D.

This means, for every $\varepsilon > 0$,

there exists $N = N(\varepsilon, x)$ (N depends on ε and x)

so that, if $n > N$,

then $\left| \sum_{j=1}^{n} f_j(x) - f(x) \right| < \varepsilon$

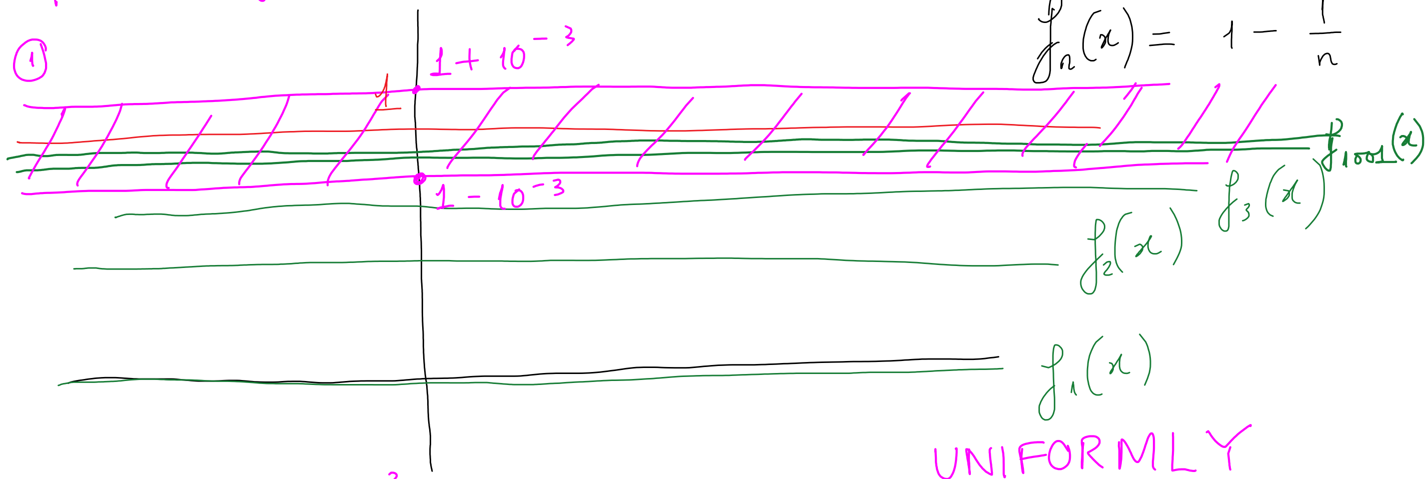
If $N = N(\varepsilon)$ depends only on ε but not x,

we say that $\sum_{n=1}^{\infty} f_n(x)$ <u>converges uniformly</u> to $f(x)$

<u>Examples</u> (uniform convergence of sequence of functions)

Examples (uniform convergence of sequence of functions)

① $f_n(x) = 1 - \frac{1}{n}$



UNIFORMLY CONVERGENT SEQUENCE

$$\varepsilon = 10^{-3}$$

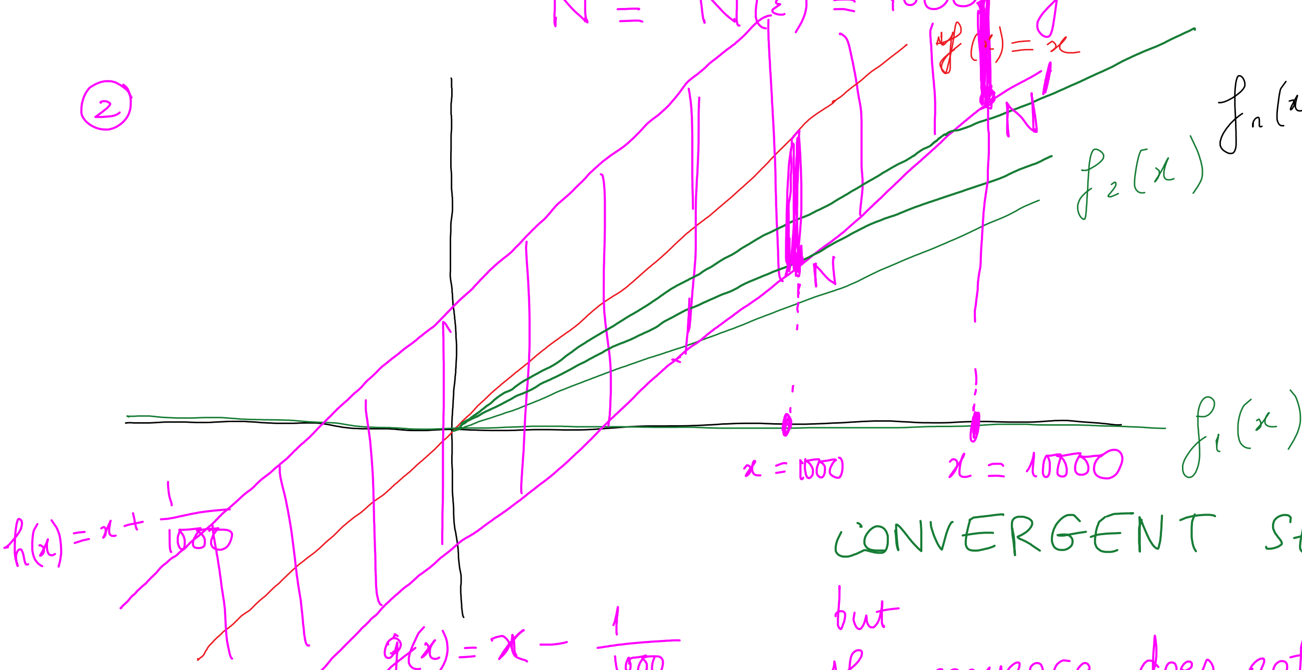
$$\left| f_n(x) - f(x) \right| < 10^{-3}$$

$$\left| \left(1 - \frac{1}{n}\right) - 1 \right| < 10^{-3}$$

$$n > 10^3 = 1000$$

$$N = N(\varepsilon) = 1000 \text{ for all } x$$

② $f_n(x) = \left(1 - \frac{1}{n}\right) x$



CONVERGENT SEQUENCE

but

this sequence does not converge uniformly

Weierstrass Criterion

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To say whether a series / sequence of function converges uniformly, we may use the definition (think in pictures and in terms of rate of convergence).

There is a useful criterion (a sufficient condition for uniform convergence)

Theorem (Weierstrass)

$$\sum_{n=1}^{\infty} f_n(x)$$

Suppose there is a sequence $(T_n)_{n=1}^{\infty}$ such that

(1) $|f_n(x)| \leq T_n$

(2) $\sum_{n=1}^{\infty} T_n$ converges

Then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly (hội tụ đều)

$\sum_{n=1}^{\infty} f_n(x)$ is 'compared with' $\sum_{n=1}^{\infty} T_n$

① If $|f_n(x)| < T_n$, then $\sum_{n=1}^{\infty} |f_n(x)| < \sum_{n=1}^{\infty} T_n$

② If $\sum^{\infty} T_n$ converges,

② If $\sum_{n=1}^{\infty} T_n$ converges,

then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly

Example Show that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2+x^2}$ converges uniformly

Use Weierstrass's criterion!

① $\left|\frac{\sin(nx)}{n^2+x^2}\right| \leq \frac{1}{n^2+x^2} \leq \frac{1}{n^2}$

depends on x — does not depend on x

(then compare $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2+x^2}$ with $\sum_{n=1}^{\infty} \frac{1}{n^2}$)

② We know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Therefore $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2+x^2}$ converges uniformly