Chapter 1 INFINITE SERIES

$$1 + 1 + 1 + 1 + \dots\dots = ? \quad \infty$$

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\dots = ? \quad 2$$

$$0.9999\dots\dots = 1 \; ? \quad \text{true}$$

To answer these questions rigorously, we need the language of sequences and series.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = ?$$

$$1 - 1 + 1 - 1 + 1 - 1 \dots = ?$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = ?$$

$$\boxed{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6} \quad \text{Euler}}$$

Let $(u_n)_{n=1}^{\infty}$ be a sequence of numbers.

$$s = u_1 + u_2 + u_3 + \dots = \sum_{n=1}^{\infty} u_n$$

This formal sum is an infinite series

Partial sum $s_n = u_1 + u_2 + \dots + u_n = \sum_{i=1}^{n} u_i$

If $\lim_{n \to \infty} s_n$ exists, the series s converges
(does not exist) (diverges)

- $1 + 1 + 1 + \dots = \infty$

$u_n = 1 \quad \forall n$

$$s_n = \underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = n$$

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} n = \infty \quad \text{diverges}$$

Example

$s = 1 + r + r^2 + r^3 + \dots \qquad (r \neq 1)$

$$u_n = r^{n-1}$$

$$s_n = \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} r^{i-1} = \frac{1 - r^n}{1 - r}$$

$$\lim_{n \to \infty} \frac{1 - r^n}{1 - r} \text{ exists} \iff |r| < 1$$

The series s converges $\iff |r| < 1$

In this case $s = \lim_{n \to \infty} \frac{1 - r^n}{1 - r} = \frac{1}{1 - r}$

$$s = a + ar + ar^2 + ar^3 + \dots \quad (a \neq 0, r \neq 1)$$

geometric series (chuỗi hình học)

s converges $\iff |r| < 1$

$$s = \frac{a}{1-r} \quad \text{when } |r| < 1.$$

Example p-series (Euler series)

$$\zeta(p) = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

zeta

$\zeta(p)$ converges $\iff p > 1$

When $p \leq 1$, $\zeta(p)$ diverges

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = 2$$

$$r = \frac{1}{2}, \quad a = 1$$

$$= \frac{1}{1 - \frac{1}{2}} = 2$$

Sequences / Series allow us to compute infinite additions.

Dictum $\boxed{\text{A series is a number!}}$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 2$$

$$1+1+1+1+\dots\dots = \infty$$

$$1-1+1-1+1-1\dots\dots \text{ `nonexistent'}$$

$s_1 = 1$, $s_2 = 0$, $s_3 = 1$, $s_4 = 0$, ...

$\lim_{n\to\infty} s_n$ does not exist

2 questions:

1. Does a series converge or diverge?

2. If a series converges, compute this series.

2 is often more difficult than 1.

Necessary condition for convergence

Tuesday, September 28, 2021 8:31 AM

Theorem (Necessary condition for convergence)

Let $s = \sum_{n=1}^{\infty} u_n$ be a series

If s converges, then $\lim_{n\to\infty} u_n = 0$.

$u_1 + u_2 + u_3 + u_4 + \cdots + u_{99} + u_{100} + \cdots$

$1 + 2 + 3 + 4 + \cdots$ diverge

$1 + 1 + 1 + \cdots$ diverges

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges

$u_n = \frac{1}{n} \longrightarrow 0$ (p-series, $p = 1$)

s converges $\Longrightarrow$ $u_n \longrightarrow 0$

s converges $\not\Longleftarrow$ $u_n \longrightarrow 0$

(necessary condition for convergence, not a sufficient condition)

(x_n), $\sum_{n=1}^{\infty} x_n$ converges

(y_n), $\sum_{n=1}^{\infty} y_n$ converges

$$\sum^{\infty} (x_n \pm y_n) = \sum_{n=1}^{\infty} x_n \pm \sum_{n=1}^{\infty} y_n$$

$$\sum_{n=1}^{\infty} (x_n \pm y_n) = \sum_{n=1} x_n \pm \sum_{n=1} y_n$$

$k \in \mathbb{R}$

$$\sum_{n=1}^{\infty} (k x_n) = k \sum_{n=1}^{\infty} x_n$$

Integral Test

Tuesday, September 28, 2021 8:42 AM

A series $\sum_{n=1}^{\infty} u_n$ is a non-negative series (positive series)

if $u_n \geq 0$.

$(u_n > 0)$

Integral Test

Theorem Suppose $f(x)$ is a continuous function on $[1, \infty)$; $f(x) \geq 0$; $f(x)$ is non-increasing

$u_n = f(n)$

(1) If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} u_n$ converges

(2) If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} u_n$ diverges

Example p-series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

$$u_n = \frac{1}{n^p}$$

Put $f(x) = \frac{1}{x^p}$, so $u_n = f(n)$

- $f(x)$ is continuous, $f(x) \geq 0$

- For $x \geq 1$, $f(x)$ is non-increasing

$s = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ converges / diverge ?

$$\int_1^\infty f(x)\,dx = \int_1^\infty x^{-p}\,dx$$

$$\int x^{-p}\,dx = \frac{x^{-p+1}}{-p+1} + c$$

- If $p = 1$, $\int_1^\infty x^{-1}\,dx = \left[\ln x\right]_1^\infty$ diverges
- If $p \neq 1$: $\int_1^\infty x^{-p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^\infty$

converge when $p > 1$

diverge when $p < 1$

$\int_1^\infty x^{-p}\,dx$ converge when $p > 1$

diverge when $p \leq 1$

$\Downarrow$ Integral Test

$$\boxed{\sum_{n=1}^\infty \frac{1}{n^p} \quad \text{converge when } p > 1, \quad \text{diverge when } p \leq 1}$$

Note: you need to verify the conditions / hypotheses before using the theorem !

$\underline{p=2}$

$$\boxed{1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots=\frac{\pi^2}{6}}$$

BEAUTIFUL!

There is a general formula for

$$1+\frac{1}{2^{2k}}+\frac{1}{3^{2k}}+\frac{1}{4^{2k}}+\dots=$$

zeta values

wikipedia

Comparison Tests

Tuesday, September 28, 2021 9:06 AM

Theorem (Convergence Test 1)

Let $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ be positive series.

Suppose $x_n \leq y_n$

(1) If $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ converges

(2) If $\sum_{n=1}^{\infty} x_n$ diverges, then $\sum_{n=1}^{\infty} y_n$ diverges.

Example $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $\qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$ converge

$$\left.\begin{array}{l} \frac{1}{n^2+1} < \frac{1}{n^2} \\ \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converge} \end{array}\right\} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ converge}$$

Theorem (Comparison Test 2)

Let $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ be 2 positive series

Suppose $\lim_{n\to\infty} \frac{x_n}{y_n} = L > 0$

Then $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$

have the same convergence / divergence

Example $\sum^{\infty} \frac{1}{}$

Example $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - 1}$

- Comparison Test 1 : $\dfrac{1}{n^2-1} > \dfrac{1}{n^2}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2-1} \text{ converges } \not\Longleftarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

- Comparison Test 2

compare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1 > 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges} \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2-1} \text{ converges}$$

Theorem of d'Alambert

Tuesday, September 28, 2021 9:23 AM

Theorem (d'Alambert)

Suppose $\sum_{n=1}^{\infty} x_n$ is a positive series

and $\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = L$ exists

(1) If $L < 1$: the series converges

(2) If $L > 1$: the series diverges

(3) If $L = 1$: inconclusive

Example $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$x_n = \frac{n}{2^n}$$

$$\frac{x_{n+1}}{x_n} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \underbrace{\frac{n+1}{n}}_{\to 1} \cdot \underbrace{\frac{2^n}{2^{n+1}}}_{\frac{1}{2}}$$

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \frac{1}{2} < 1$$

$$\xRightarrow{\text{d'Alambert}} \sum_{n=1}^{\infty} \frac{n}{2^n} \text{ converges}$$

Theorem of Cauchy

Tuesday, September 28, 2021 9:29 AM

Theorem (Cauchy)

Suppose $\sum_{n=1}^{\infty} x_n$ is a positive series

and $\lim_{n\to\infty} \sqrt[n]{x_n} = L$ exists

(1) If $L < 1$, the series converges

(2) If $L > 1$, the series diverges

(3) If $L = 1$, inconclusive.

Example $\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+1}\right)^n$

$$x_n = \left(\frac{2n+1}{3n+1}\right)^n > 0$$

$$\sqrt[n]{x_n} = \frac{2n+1}{3n+1} \xrightarrow{n\to\infty} \frac{2}{3} < 1$$

$\xRightarrow{\text{Cauchy}}$ the series converges

Note 1 Integral Test

2 Comparison Tests

d'Alambert's theorem

Cauchy's theorem

apply for positive series.

Cauchy's theorem

Alternating Series

Tuesday, September 28, 2021 9:36 AM

$$\left.\begin{array}{l} 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots \\ 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \end{array}\right\} \text{positive series}$$

$$\left(\begin{array}{l} 1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 \cdots \\ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots \end{array} \right)$$

An alternating series is a series of the form

$$\sum_{n=1}^{\infty} (-1)^n x_n$$

where $x_n > 0$

(chuỗi đan dấu)

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots :$$ alternating Series

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots :$$ NOT an alternating series

Absolute Convergence, Conditional Convergence

Tuesday, September 28, 2021 9:40 AM

A series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent (hội tụ tuyệt đối) if $\sum_{n=1}^{\infty} |x_n|$ converges

Theorem If $\sum_{n=1}^{\infty} |x_n|$ converges, then $\sum_{n=1}^{\infty} x_n$ converges

By the theorem, if a series is absolutely convergent, then $\sum_{n=1}^{\infty} |x_n|$ converges and $\sum_{n=1}^{\infty} x_n$ converges

Example $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ is absolutely convergent

because $\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$ converges

$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n^1}$ diverges

$\sum_{n=1} \left| (-1) \quad _n \right| = \sum_{n=1} \quad n$

harmonic series

(chuỗi điều hòa)

We say that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is conditionally convergent,

NOT absolutely convergent

alternating harmonic series

A series $\sum_{n=1}^{\infty} x_n$ is <u>conditionally convergent</u>

(bán hội tụ)

if $\sum_{n=1}^{\infty} |x_n|$ diverges

(absolute series)

but $\sum_{n=1}^{\infty} x_n$ converges

Theorem (Leibniz's criterion)

Suppose $\sum_{n=1}^{\infty} (-1)^n x_n$ is an alternating series ($x_n > 0$)

If $(x_n)_{n=1}^{\infty}$ is a decreasing sequence

and $\lim_{n \to \infty} x_n = 0$,

then the alternating series converges

Example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots -$

(alternating harmonic series)

(chuỗi điều hòa đan dấu)

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} x_n$$

($x_n = \frac{1}{n}$) is a decreasing sequence

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

Leibniz's criterion $\Longrightarrow$ the alternating harmonic series converges

Does it converge conditionally / ~~absolutely~~ ?

Generalised d'Alambert's criterion

Tuesday, September 28, 2021 9:57 AM

Generalised d'Alambert's criterion

Suppose $\sum_{n=1}^{\infty} x_n$ is a series such that

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = L \text{ exists}$$

(1) If $L < 1$, then the series converges (absolutely)

(2) If $L > 1$, then $\sum_{n=1}^{\infty} |x_n|$ diverges

(3) If $L = 1$, inconclusive

Generalised Cauchy's criterion

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Generalised Cauchy's criterion

Suppose $\sum_{n=1}^{\infty} x_n$ is a series such that

$$\lim_{n\to\infty} \sqrt[n]{|x_n|} = L \text{ exists}$$

(1) If $L < 1$, then the series converges (absolutely)

(2) If $L > 1$, then $\sum_{n=1}^{\infty} |x_n|$ diverges

(3) If $L = 1$, inconclusive !

Rearrangement of Series

Tuesday, September 28, 2021 10:05 AM

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots = \frac{\pi^2}{6}$$

$\frac{\pi^2}{6}$

$$1 + \frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{5^2} + \frac{1}{4^2} + \frac{1}{7^2} + \frac{1}{6^2} + \cdots = ?$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$$ converges by Leibniz's criterion

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{6} \cdots$$

rearrangement of the series

Theorem (Dirichlet)

Suppose $\sum_{n=1}^{\infty} x_n$ is an absolutely convergent series

Then any rearrangement of this series also converges absolutely to the same value as the original series

Applying Convergence Tests

Tuesday, September 28, 2021 10:11 AM

Until now, we have many convergence tests

For positive series, we have

- Integral Test
- 2 Comparison tests
- d'Alambert's criterion
- Cauchy's criterion

For alternating series, we have Leibniz's criterion

For arbitrary series, we have

- generalised d'Alambert's criterion
- generalised Cauchy's criterion

For an arbitrary series, you can use any of the above convergence tests/criteria.

Sometimes, you have to combine different convergence tests in order to check convergence/divergence.

Q & A

Tuesday, September 28, 2021 10:18 AM

What is the application of series?

- beautiful mathematics! (this is true, mathematicians study mathematics ~~because it is beautiful.~~)

- $\boxed{\text{A series is a number!}}$

$\cancel{0.9999\ldots = 1}$

$\boxed{\text{A Series of functions is a function!}}$

A FUNCTION CAN OFTEN BE REPRESENTED BY A SERIES OF FUNCTIONS!

These series representations allow us to compute functions, their derivatives, their integrals!

SPECIAL FUNCTIONS

Gamma, Bessel, zeta

$\longrightarrow$ use series to understand these functions

DICTUM! (1) A series is a number!

(2) A series of functions is a function!

(3) A function can often be represented by a series of functions!