System of Ordinary Differential Equations of Order 1

Tuesday, December 7, 2021 7:36 AM

A system of ordinary differential equations of order 1 in canonical form has the form

$$\begin{cases} y_1' = f_1(x, y_1, y_2, \dots, y_n) \\ y_2' = f_2(x, y_1, y_2, \dots, y_n) \\ \vdots \\ y_n' = f_n(x, y_1, y_2, \dots, y_n) \end{cases} \qquad (*)$$

Cauchy's problem: find $y_1, \dots, y_n$

satisfying $(*)$

$$y_1(x = x_0) = a_1$$
$$y_2(x = x_0) = a_2$$
$$\vdots$$
$$y_n(x = x_0) = a_n$$

Theorem Suppose f_i, $\frac{\partial f_i}{\partial y_j}$ are continuous in an open box D in $\mathbb{R}^{n+1}$.

Suppose $(x_0, a_1, \dots, a_n) \in D$

Then there is an open neighborhood of x_0 such that Cauchy's problem has a unique solution.

We will learn how to solve a system of ODEs of order 1.

System of ODE 's of order 1 and high-order ODE

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Suppose we have an ODE of order n

$$z^{(n)} = f\left(x, z, z', z'', \dots, z^{(n-1)}\right) \quad (1)$$

We can transform the ODE (1) to a system of ODEs of order 1:

Put
$$\begin{aligned} y_1 &= z \\ y_2 &= z' \\ &\vdots \\ y_n &= z^{(n-1)} \end{aligned} \implies \begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \vdots \\ y_{n-1}' = y_n \\ y_n' = f(x, y_1, y_2, \dots, y_n) \end{cases}$$

Reversely, from a system of ODEs of order 1, say of $x, y_1, \dots, y_n$,
we can eliminate $(n-1)$ functions y_i
to get an ODE of order n

Examples

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Goal - solve a system of ODEs of order 1

Example Solve

$$\begin{cases} y' = 5y + 4z & (1) \\ z' = 4y + 5z & (2) \end{cases}$$

We will eliminate 1 function y or z to get an ODE of order 2

Differentiate (1), eliminate z

$$z = \frac{1}{4}(y' - 5y)$$

$$y'' = 5y' + 4z'$$

$$= 5(5y + 4z) + 4(4y + 5z)$$

$$\Rightarrow y'' - 10y' + 9y = 0$$

linear ODE of order 2, homogeneous

$$r^2 - 10r + 9 = 0 = (r-1)(r-9)$$

$$r = 1, r = 9$$

$$y = k_1 e^x + k_2 e^{9x} \quad (k_1, k_2 \text{ constants})$$

but $z = \frac{1}{4}(y' - 5y)$

$$\Rightarrow z = \frac{1}{4}\left(k_1 e^x + 9k_2 e^{9x} - 5k_1 e^x - 5k_2 e^{9x}\right)$$

$$\begin{cases} z = -k_1 e^x + k_2 e^{9x} \\ y = k_1 e^x + k_2 e^{9x} \end{cases}$$

Example. solve

$$\begin{cases} y' = y + z & (1) \\ z' = y + z + x & (2) \end{cases}$$

Differentiate (1), eliminate z

$$y'' = y' + z' = y' + y + z + x$$

but $z = y' - y$ by (1)

$$= y' + \cancel{y} + y' - \cancel{y} + x = 2y' + x$$

$y'' - 2y' = x$ (I) linear ODE of order 2, inhomogeneous

$y'' - 2y' = 0$ (H) $\quad r^2 - 2r = 0$, $r = 0, 2$

(H) has $y_c = y_h = k_1 + k_2 e^{2x}$ (k_1, k_2 constants)

- undetermined coefficients (*)
- variation of parameters

$$y'' - 2y = x \cdot e^{0x} \quad (I)$$

(H) has $y_h = k_1 + k_2 e^{2x}$

(I) has $y_p = x(Ax + B)e^{0x}$

plug in and solve for A, B

$$A = -\frac{1}{4}, \quad B = -\frac{1}{4}$$

(I) has general solution. $y = k_1 + k_2 e^{2x} - \frac{1}{4}x(x+1)$

but $z = y' - y \Rightarrow$ solve for z

$$\cdot\ z = -k_1 + k_2 e^{2x} + \frac{1}{4}(x^2 - x - 1)$$

We also have (2) $z' = y + z + x$

$$2\cancel{k_2}e^{2x} + \frac{1}{4}(\cancel{2x} - 1) = \cancel{k_1} + \cancel{k_2}e^{2x} - \frac{1}{4}\cancel{x}(\cancel{x} + 1)$$

$$-\cancel{k_1} + \cancel{k_2}e^{2x} + \frac{1}{4}(\cancel{x^2} \cancel{-} x - 1)$$

$$-\cancel{k}_1 + \cancel{k_2}\cancel{e}^{2x} + \frac{1}{4}\left(\cancel{x^2}\cancel{-} x - 1\right)$$
$$+ \cancel{x}$$

So we have solution

$$y = k_1 + k_2 e^{2x} - \frac{1}{4} x (x+1)$$

$$z = -k_1 + k_2 e^{2x} + \frac{1}{4}\left(x^2 - x - 1\right)$$

System of Linear ODEs of order 1

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A system of linear ODEs of order 1 in canonical form is

$$\begin{cases} y_1' = a_{11} y_1 + a_{12} y_2 + \dots + a_{1n} y_n + g_1(x) \\ y_2' = a_{21} y_1 + a_{22} y_2 + \dots + a_{2n} y_n + g_2(x) \\ \vdots \\ y_n' = a_{n1} y_1 + a_{n2} y_2 + \dots + a_{nn} y_n + g_n(x) \end{cases}$$

It's best to represent a linear system using matrices

$$\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \vec{y}' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & \cdots & \vdots \\ \vdots & & \vdots \\ \cdot & \cdots & a_{nn} \end{pmatrix}, \quad \vec{g} = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$$

$$\boxed{\vec{y}' = A\vec{y} + \vec{g}}$$

$\underline{a_{ij} \text{ coefficients}}$, they may be functions

If $\vec{g} = \vec{0}$: homogeneous system

$\Rightarrow$ we have superposition of solutions

$\vec{Y}_1, \vec{Y}_2$ are solutions $\Rightarrow k_1\vec{Y}_1 + k_2\vec{Y}_2$ is sol.

We will study system of linear ODEs of order 1

not for general coefficients

but for constant coefficients.

Eigenvalue and eigenvector

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$$A = \begin{pmatrix} a_{11} & \cdots & \cdot \\ \vdots & & \vdots \\ \cdot & \cdots & a_{nn} \end{pmatrix} \qquad a_{ij} \in \mathbb{R}$$

$$A\vec{v} = \lambda \vec{v} \qquad (\vec{v} \neq \vec{0})$$

λ eigenvalue, $\vec{v}$ eigenvector for λ

<u>Linear Algebra</u> $\rightsquigarrow$ find eigenvalue, eigenvector

System of Linear ODEs of Order 1, constant coefficients and homogeneous

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$A = (a_{ij}) \qquad a_{ij} \in \mathbb{R}$

$\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$A\vec{y} = \vec{y}'$

Suppose A has <u>distinct eigenvalues</u> $\lambda_1, \ldots, \lambda_n$

and λ_k has eigenvector $\vec{p_k} = (p_{1k}, \ldots, p_{nk})$

Then $A\vec{y} = \vec{y}'$ has n linearly independent solutions $\vec{Y_1}, \vec{Y_2}, \ldots, \vec{Y_n}$

where $\vec{Y_1} = \begin{pmatrix} p_{11} e^{\lambda_1 x} \\ p_{21} e^{\lambda_1 x} \\ \vdots \\ p_{n1} e^{\lambda_1 x} \end{pmatrix} \quad \ldots \quad \vec{Y_n} = \begin{pmatrix} p_{1n} e^{\lambda_n x} \\ p_{2n} e^{\lambda_n x} \\ \vdots \\ p_{nn} e^{\lambda_n x} \end{pmatrix}$

general solution is $k_1 \vec{Y_1} + k_2 \vec{Y_2} + \ldots + k_n \vec{Y_n}$.

Example

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Solve $\begin{cases} y' = y + 2z \\ z' = 4y + 3z \end{cases}$

$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}$

eigenvalues. $\det(A - \lambda I) = 0$

$= (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5$

$= (\lambda - 5)(\lambda + 1)$

$\underbrace{\lambda = 5}$, $\underbrace{\lambda = -1}$

$\downarrow \qquad \downarrow$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vec{Y}_1 = \begin{pmatrix} 1 \; e^{5x} \\ 2 \; e^{5x} \end{pmatrix}$ $\qquad \vec{Y}_2 = \begin{pmatrix} 1 \; e^{-x} \\ -1 \; e^{-x} \end{pmatrix}$

general solution $\vec{Y} = k_1 \vec{Y}_1 + k_2 \vec{Y}_2$

$= \begin{pmatrix} k_1 e^{5x} + k_2 e^{-x} \\ 2k_1 e^{5x} - k_2 e^{-x} \end{pmatrix}$

The reason is not difficult,

but it takes a smart idea to find out the formula.

* explain this next week

We also have the case where the eigenvalues are not distinct.

We will study this case next week.