Brief review on Laplace transforms

Tuesday, December 21, 2021 7:31 AM

Laplace transform

$$f(t) \rightsquigarrow F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$F(s) = L\left[f(t)\right]$$

$$f(t) = L^{-1}\left[F(s)\right]$$

$f(t)$	$F(s)$
1	$\frac{1}{s}$ $(s>0)$
t^n $(n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$ $(s>0)$
e^{kt} $(k \in \mathbb{R})$	$\frac{1}{s-k}$ $(s>k)$
$\cos kt$	$\frac{s}{s^2+k^2}$ $(s>0)$
$\sin kt$	$\frac{k}{s^2+k^2}$ $(s>0)$
$\cosh kt$	$\frac{s}{s^2-k^2}$ $(s>\lvert k\rvert)$
$\sinh kt$	$\frac{k}{}$ $(s>\lvert k\rvert)$

$\sinh kt \qquad \dfrac{k}{s^2 - k^2} \quad (s > |k|)$

$u(t-k) \qquad \dfrac{e^{-ks}}{s} \quad (s > 0)$

Laplace transforms of derivatives

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<u>Theorem</u> : Suppose $f(t)$ is continuous and has exponential order.

Suppose $f'(t)$ is piecewise continuous on $[0, +\infty)$

Then there exists $L[f'(t)]$

and $L[f'(t)] = s \cdot L[f(t)] - f(0)$

<u>Proof</u> : Assume $[a, b] \subset [0, +\infty)$

$$\int_a^b f'(t) e^{-st} \, dt = \int_a^b e^{-st} \, d[f(t)]$$

$$= \Big[f(t) e^{-st}\Big]_a^b - \int_a^b f(t)(-s) e^{-st} \, dt$$

$$= f(b) e^{-sb} - f(a) e^{-sa} + s\int_a^b f(t) e^{-st} \, dt$$

Take $a = 0$ and let $b \to +\infty$

f is of exponential order, so if s is big enough then $f(b) e^{-sb} \xrightarrow{b \to +\infty} 0$

$$\Rightarrow \int_0^\infty f'(t) e^{-st} \, dt = -f(0) + s \int_0^{+\infty} f(t) e^{-st} \, dt$$

$$L[f'(t)] = -f(0) + s L[f(t)] \qquad \square$$

$*$ $L[f'] = -f(0) + s L[f]$.

Laplace transform of 1^{st} derivative

We iterate this formula

$$L[f''] = -f'(0) + s\,L[f']$$

$$= -f'(0) + s\left[-f(0) + s\,L[f]\right)$$

$$L[f''] = -\left(f'(0) + s f(0)\right) + s^2 L[f]$$

Laplace transform of 2^{nd} derivative

If we iterate $(n-1)$ times

$$L\left[f^{(n)}\right] = -\left(f^{(n-1)}(0) + s f^{(n-2)}(0) + \dots + s^{n-1} f(0)\right) + s^n L[f].$$

Linear ODEs

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Example: solve IVP:

$$\begin{cases} y' + 2y = 12e^{3t} \\ y(0) = 3 \end{cases}$$

Note. we can use Laplace transform.
The linearity is important because
now we only know $L[y^{(n)}]$. but not $L[(y^{(n)})^k]$ $(k \neq 1)$

$$L[y'] = s\,L[y] - y(0)$$

$$L[y'] = s\,L[y] - 3 = sY - 3$$

$$y' + 2y = 12e^{3t}$$

$$\Big\downarrow L$$

$$L[y'] + 2\,L[y] = 12\,L[e^{3t}].$$ Put $Y = L[y]$

$$(sY - 3) + 2Y = \frac{12}{s-3}$$

Note this is an algebraic equation of Y.

$$(s+2)\,Y = 3 + \frac{12}{s-3}$$

$$Y = \frac{3}{s+2} + \frac{12}{(s-3)(s+2)}$$

$$y = 3L^{-1}\left[\frac{1}{s+2}\right] + 12\,L^{-1}\Bigg[\underbrace{\frac{1}{(s-3)(s+2)}}_{= \frac{k}{s-3} + \frac{k'}{s+2}}\Bigg]$$

$$= \frac{1}{5(s-3)} - \frac{1}{5(s+2)}$$

$$y = 3L^{-1}\left[\frac{1}{s+2}\right] + \frac{12}{5}L^{-1}\left[\frac{1}{s-3}\right] - \frac{12}{5}L^{-1}\left[\frac{1}{s+2}\right]$$

$$y = 3\underbrace{L^{-1}\left[\frac{1}{s+2}\right]}_{e^{-2t}} + \frac{12}{5}\underbrace{L^{-1}\left[\frac{1}{s-3}\right]}_{e^{3t}} - \frac{12}{5}\underbrace{L^{-1}\left[\frac{1}{s+2}\right]}_{e^{-2t}}$$

$$y = \frac{12}{5}e^{3t} + \frac{3}{5}e^{-2t} .$$

Example : Solve the IVP

$$\begin{cases} y'' + 4y = t \\ y(0) = 0, \ y'(0) = 0 \end{cases}$$

Step 1 $L[y] = Y$

Step 2 . Solve for Y

Step 3 $L^{-1}[Y] = y$

$$L[y'] = sY - y(0) = sY$$

$$L[y''] = s^2 Y - \left(y'(0) + s\,y(0)\right) = s^2 Y$$

$$L[y''] + 4L[y] = L[t]$$

$$s^2 Y + 4Y = \frac{1}{s^2}$$

$$Y = \frac{1}{s^2(s^2+4)} = \frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)$$

$$y = \frac{1}{4}L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{4}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$y = \frac{t}{4} - \frac{\sin 2t}{8}$$

Example : Solve the IVP

$$\begin{cases} y'' + y' - 2y = 4 \end{cases}$$

Example: Solve the IVP

$$\begin{cases} y'' + y' - 2y = 4 \\ y(0) = 2 \ , \ y'(0) = 1 \end{cases}$$

Put $Y = L[y]$

$$L[y'] = sY - y(0) = sY - 2$$

$$L[y''] = s^2 Y - \Big(y'(0) + s\,y(0)\Big) = s^2 Y - (1 + 2s)$$

$$\Rightarrow \quad s^2 Y - (1 + 2s) + sY - 2 - 2Y = L[4]$$

$$Y(s^2 + s - 2) - (2s + 3) = \frac{4}{s}$$

$$\underbrace{Y(s^2 + s - 2)}_{Y(s-1)(s+2)} = \frac{4}{s} + (2s+3) = \frac{2s^2 + 3s + 4}{s}$$

$$\Rightarrow \quad Y = \frac{2s^2 + 3s + 4}{s(s-1)(s+2)} = \frac{k}{s} + \frac{k'}{s-1} + \frac{k''}{s+2}$$

$$Y = -\frac{2}{s} + \frac{3}{s-1} + \frac{1}{s+2}$$

$$y = -2\,L^{-1}\left[\frac{1}{s}\right] + 3\,L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{1}{s+2}\right]$$

$$y = -2 + 3e^{t} + e^{-2t}.$$

System of linear ODEs

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Example: Solve the system of (linear) ODEs

$$\begin{cases} 2x' + 3x + y = 0 \\ 2y' + x + 3y = 0 \\ x(0) = 2, \quad y(0) = 0 \end{cases}$$

Use Laplace transform!

(Previously, we had another method We eliminate 1 function to get a 2nd order ODE in 1 function

Now we have another method - thanks Laplace!)

Put $X = L[x]$, $Y = L[y]$

$$2x' + 3x + y = 0, \quad x(0) = 2$$

$$\downarrow$$

- $(2sX - 2) + 3X + Y = 0$

$$2y' + x + 3y = 0, \quad y(0) = 0$$

$$\downarrow$$

- $2sY + X + 3Y = 0$

Then we solve for X, Y

$$\begin{cases} X = \dfrac{2s+3}{(s+1)(s+2)} = \dfrac{1}{s+1} + \dfrac{1}{s+2} \\ Y = \dfrac{1}{(s+1)(s+2)} = \dfrac{1}{s+2} - \dfrac{1}{s+1} \end{cases}$$

$$\begin{cases} x = e^{-t} + e^{-2t} \\ \qquad\; e^{-2t} \quad -\, e^{-t} \end{cases}$$

$$\begin{cases} x = e \quad + e \\ y = e^{-2t} - e^{-t} \end{cases}$$

Example. Solve the system of ODEs

$$\begin{cases} x'' + y' - x' = -\frac{3}{4}x \\ y'' + x' - y' = -\frac{3}{4}y \\ x(0) = y(0) = 0 \\ x'(0) = 1, \quad y'(0) = -1 \end{cases}$$

Put $X = L[x]$, $Y = L[y]$

$$L[x'] = sX - x(0) = sX$$

$$L[x''] = s^2X - \left(x'(0) + s\,x(0)\right) = s^2X - 1$$

$$L[y'] = sY - y(0) = sY$$

$$L[y''] = s^2Y - \left(y'(0) + s\,y(0)\right) = s^2Y + 1$$

Plug the Laplace transforms in the ODEs

$$\begin{cases} (s^2X - 1) + s(Y - X) = -\frac{3X}{4} \\ (s^2Y + 1) + s(X - Y) = -\frac{3Y}{4} \end{cases}$$

Solve for X, Y.

$$\begin{cases} X = \frac{1}{s^2 - 2s + \frac{3}{4}} = \frac{1}{s - \frac{3}{2}} - \frac{1}{s - \frac{1}{2}} \\ Y = -X \end{cases}$$

$$Y = -X$$

$$\begin{cases} x = e^{\frac{3t}{2}} - e^{\frac{t}{2}} \\ y = -x \end{cases}$$

Summary

Tuesday, December 21, 2021 8:42 AM

Laplace transforms can be used to solve linear ODEs
or systems of linear ODEs

Laplace transforms turn linear ODEs into algebraic equations.
We solve algebraic equations,
and then take inverse Laplace transforms

2nd order linear ODEs

Tuesday, January 11, 2022 8:47 AM

We want to apply Laplace transform and convolution to solve

2nd-order linear ODEs

with constant coefficients

Example. Let $f(t)$ be a given continuous function

Solve $\begin{cases} 16y'' + y = f(t) \\ y(0) = -3, \quad y'(0) = 2 \end{cases}$

Cauchy problem

think of $f(t)$ as a parameter

[digression. $ax^2 + bx + c = 0 \rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

a, b, c : parameters]

[in the Cauchy problem, we, similarly, express y in terms of the 'parameter' f]

$16y'' + y = f \; ; \quad y(0) = -3, \; y'(0) = 2$

$\Big\downarrow$ Laplace transform $Y = L[y]$, $F = L[f]$

$$16(s^2 Y + 3s - 2) + Y = F$$

$$Y(16s^2 + 1) = -48s + 32 + F$$

$$Y = \frac{-48s}{16s^2 + 1} + \frac{32}{16s^2 + 1} + \frac{F}{16s^2 + 1}$$

$\Big\downarrow$ inverse transform L^{-1}

$$y = -3\, L^{-1}\left[\frac{s}{s^2 + \left(\frac{1}{4}\right)^2}\right] + 8\, L^{-1}\left[\frac{1/4}{s^2 + \left(\frac{1}{4}\right)^2}\right] + \frac{1}{4}\, L^{-1}\left[\frac{1/4}{s^2 + \left(\frac{1}{4}\right)^2} \cdot F\right]$$

$$y = -3\cos\frac{t}{4} + 8\sin\frac{t}{4} + \frac{1}{4}\left[\sin\frac{t}{4} * f(t)\right] \qquad \square$$

Next, we will consider 2^{nd}-order linear constant-coefficients ODEs with Heaviside functions.

(next week)

ODEs with Heaviside

Tuesday, January 11, 2022 10:09 AM

2nd-order linear constant-coefficient ODEs with Heaviside

<u>Example</u> solve $y'' + 4y = g(t) = \begin{cases} \sin t & (0 \le t < \frac{\pi}{2}) \\ 0 & (t \geq \frac{\pi}{2}) \end{cases}$

$$y(0) = y'(0) = 0$$

$$g(t) = \sin t \left(\underbrace{u(t-0)}_{1} - u\left(t - \frac{\pi}{2}\right) \right) \qquad (t \geq 0)$$

$$= \sin t - \sin t \ u\left(t - \frac{\pi}{2}\right) \qquad \text{(piecewise-defined function)}$$

$$y'' + 4y = \sin t - \sin t \cdot u\left(t - \frac{\pi}{2}\right), \quad y(0) = y'(0) = 0$$

$\Big\downarrow$ Laplace transform $Y = L[y]$

$$s^2 Y + 4Y = \frac{1}{s^2+1} - e^{-\frac{\pi s}{2}} \underbrace{L\left[\sin\left(t + \frac{\pi}{2}\right)\right]}_{\frac{s}{s^2+1}}$$

$$Y = \frac{1}{(s^2+1)(s^2+4)} - e^{-\frac{\pi s}{2}} \frac{s}{(s^2+1)(s^2+4)}$$

<u>Inverse transform</u>

$$\frac{1}{3}\left(\sin t - \frac{\sin 2t}{2}\right) \xleftarrow{L^{-1}} \frac{1}{(s^2+1)(s^2+4)} = \boxed{\frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)}$$

$$\frac{1}{3}(\cos t - \cos 2t) \xleftarrow{L^{-1}} \frac{s}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{s}{s^2+1} - \frac{s}{s^2+4}\right)$$

L^{-1} $-\frac{\pi}{2}$ s s

$$3 \left(\qquad s^2+4 \right)$$

$$\frac{u\left(t-\frac{\pi}{2}\right)}{3}\left[\cos\left(t-\frac{\pi}{2}\right) - \cos(2t-\pi)\right] \xleftarrow{L^{-1}} \boxed{e^{-\frac{\pi}{2}s} \cdot \frac{s}{(s^2+1)(s^2+4)}}$$

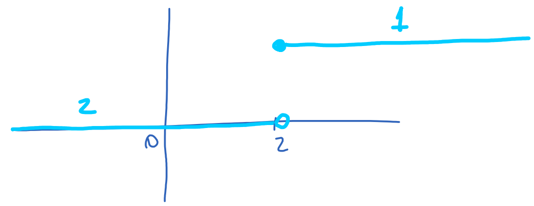
$$y = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \frac{u\left(t-\frac{\pi}{2}\right)}{3}\left[\cos\left(t-\frac{\pi}{2}\right) - \cos(2t-\pi)\right] \qquad \square$$

Cauchy's problem with Heaviside

Tuesday, January 18, 2022 7:42 AM

Example. Solve $y' - y = u(t-2)$
$y(0) = 0$

$$u(t-2) = \begin{cases} 0 & (t < 2) \\ 1 & (t \geq 2) \end{cases}$$



Write $u(t-2) = u_2(t) = u_2$

$$u(t-k) = u_k(t) = \begin{cases} 0 & (t < k) \\ 1 & (t \geq k) \end{cases}$$

$$y' - y = u_2(t), \quad y(0) = 0$$

$$\Big\downarrow \; Y = L[y]$$

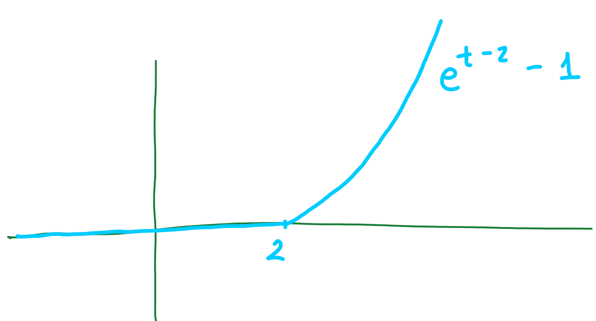
$$(s-1)Y = \frac{e^{-2s}}{s}$$

$$Y = e^{-2s} \cdot \frac{1}{s(s-1)}$$

$$e^t - 1 \longleftarrow X = \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s}$$

$$u_2 \cdot (e^{t-2} - 1) \longleftarrow Y = e^{-2s} \cdot X$$

So $$y(t) = u_2(t)\,(e^{t-2} - 1) = \begin{cases} 0 & (t < 2) \\ e^{t-2} - 1 & (t \geq 2) \end{cases}$$



$y(2^-) = 0 = y(2^+) = e^{2-2} - 1 = 0$ continuous at 2

$y'(2^-) = 0 \neq y'(2^+) = \left[e^{t-2}\right]_{t=2} = 1$ not differentiable at 2

This $y(t)$ is called a <u>generalized solution</u>

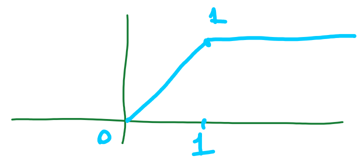
<u>With u_k</u>

$$f \longleftrightarrow F$$

$$u_k \, f(t-k) \longleftrightarrow e^{-ks} F(s) \quad (*)$$

$$u_k \cdot f(t) \longleftrightarrow e^{-ks} \, L\left[f(t+k) \right]$$

<u>Example</u> Let $f(t) = \begin{cases} t & (0 \le t < 1) \\ 1 & (t \geq 1) \end{cases}$



Solve $\quad y' + 3y = f(t)$
$\quad y(0) = 0$

Let $F = L[f]$

$$f = t(u_0 - u_1) + u_1 = t u_0 - (t-1) u_1$$

$$= t\, u(t) - (t-1)\, u(t-1)$$

$$F = L[f] = \frac{1}{s^2} - \frac{e^{-s}}{s^2} = (1 - e^{-s}) \cdot \frac{1}{s^2}$$

$$y' + 3y = f, \quad y(0) = 0$$

$\downarrow$ Let $Y = L[y]$

$$(s+3) Y = F = (1 - e^{-s}) \cdot \frac{1}{s^2}$$

$$Y = (1 - e^{-s}) \cdot \frac{1}{s^2 (s+3)}$$

$$x(t) = -\frac{1}{9} + \frac{t}{3} + \frac{e^{-3t}}{9} \longleftarrow\!\!| \quad X = \frac{1}{s^2(s+3)} = -\frac{1}{9s} + \frac{1}{3s^2} + \frac{1}{9(s+3)}$$

$$y(t) = x(t) - u_1 \cdot x(t-1) \longleftarrow\!\!| \quad Y = (1 - e^{-s}) X$$

$$y(t) = x(t)\cdot u_0 - u_1 \cdot x(t-1)$$

$$= \begin{cases} x(t) & (0 \le t < 1) \\ x(t) - x(t-1) & (t \geqslant 1) \end{cases}$$

$$= \begin{cases} -\frac{1}{9} + \frac{t}{3} + \frac{e^{-3t}}{9} & (0 \le t < 1) \\ \frac{1}{3} + \frac{1}{9}\left(e^{-3t} - e^{-3(t-1)}\right) & (t \geqslant 1) \end{cases}$$ generalized solution

Solve = find a solution
or a generalized solution