Hanoi University of Science and Technology **Dr. Bui Xuan Dieu**
School of Applied Mathematics and Informatics **ICT**

Math1 Exercises

1 Symbolic Logic

Exercise 1.1. *Show that the following propositions are tautology*

a) $[(A \to B) \wedge (B \to C)] \to (A \to C)$.

Exercise 1.2. *Which of the following propositions are tautology, contradiction*

a) $(p \vee q) \to (p \wedge q)$,

b) $(p \wedge q) \vee (p \to q)$,

c) $p \to (q \to p)$,

d) $q \to (q \to p)$,

e) $(p \to q) \to q$,

f) $(p \wedge q) \leftrightarrow (q \updownarrow p)$.

Exercise 1.3. *Prove that*

a) $A \leftrightarrow B$ *and* $(A \wedge B) \vee (\overline{A} \wedge \overline{B})$ *are logically equivalent.*

b) $(A \to B) \to C$ *and* $A \to (B \to C)$ *are not logically equivalent.*

Exercise 1.4. *Find the negation* p *if*

a) $p =$ "$\forall \epsilon > 0, \exists \delta > 0 : \forall x, |x - x_0| < \delta, |f(x) - f(x_0)| < \epsilon$."

b) $p = \lim\limits_{n \to +\infty} x_n = \infty \Leftrightarrow \forall M > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n| > M$.

c) $p = \lim\limits_{n \to +\infty} x_n = L \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - L| < \epsilon$.

2 Sets

Exercise 2.1. *Let*

$$A = \{x \in \mathbb{R} | x^2 - 4x + 3 \leq 0\}, B = \{x \in \mathbb{R} || x - 1| \leq 1\},$$

and

$$C = \{x \in \mathbb{R} | x^2 - 5x + 6 \leq 0\}.$$

Compute $(A \cup B) \cap C$ *and* $(A \cap B) \cup C$.

Exercise 2.2. *Let* A, B, C *be arbitrary sets. Prove that*

a) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

b) $A \cup (B \setminus A) = A \cup B$.

c) If $(A \cap C) \subset (A \cap B)$ *and* $(A \cup C) \subset (A \cup B)$*, then* C.

d) $A \setminus (A \setminus B) = A \cap B$.

e) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

f) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

g) $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

h) *Is it true that* $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$. *If not, give a counterexample.*

Exercise 2.3. *Let A be a set with n elements. Determine the total number of subsets of A.*

Exercise 2.4. *How many numbers are not divisible by $3, 4, 5$ between 1 and 1500?*

3 Maps

Exercise 3.1. *Let $f : X \to Y$. Prove that*

a $f(A \cup B) = f(A) \cup f(B),\ A, B \subset X$

b $f(A \cap B) \subset f(A) \cap f(B),\ A, B \subset X$. *Give an example to show that the converse is not true.*

c *f is injective if and only if for any $A, B \subset X, f(A \cap B) = f(A) \cap f(B)$.*

d $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B),\ A, B \subset Y$

e $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B),\ A, B \subset Y$

f $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B),\ A, B \subset Y$

g $A \subset f^{-1}(f(A)), A \subset X$,

h $B \supset f(f^{-1}(B)), B \subset Y$.

Exercise 3.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (2x, 2y)$ and $A = \{(x, y) \in \mathbb{R}^2 | (x-4)^2 + y^2 = 4\}$. Find $f(A), f^{-1}(A)$.*

Exercise 3.3. *Which of the following maps are injective, surjective, bijective?*

a) $f : \mathbb{R} \to \mathbb{R}, f(x) = 3 - 2x$,

b) $f : (-\infty, 0] \to [4, +\infty), f(x) = x^2 + 4$,

c) $f : (1, +\infty) \to (-1, +\infty), f(x) = x^2 - 2x$,

d) $f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{3\}, f(x) = \frac{3x+1}{x-1}$,

e) $f : [4, 9] \to [21, 96], f(x) = x^2 + 2x - 3$,

f) $f : \mathbb{R} \to \mathbb{R} f(x) = 3x - 2|x|$,

g) $f : (-1, 1) \to \mathbb{R}, f(x) = \ln \frac{1+x}{1-x}$, *h)* $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = \frac{1}{x}$,

i) $f : \mathbb{R} \to \mathbb{R}, g(x) = \frac{2x}{1+x^2}$.

Exercise 3.4. *Let* $f(x) = -x^2 - 2x + 3$.

a) *Find* a *such that* $f : \mathbb{R} \to (-\infty, a]$ *is surjective.*

b) *Find* b *such that* $f : [b, +\infty) \to (-\infty, 3]$ *is injective.*

Exercise 3.5. *Let* X, Y, Z *be sets and* $f : X \to Y, \quad g : Y \to Z$. *Prove that*

a) *if* f *is surjective and* $g \circ f$ *is injective, then* g *is injective,*

b) *give an example to show that* $g \circ f$ *is injective, but* g *is not,*

c) *if* g *is is injective and* $g \circ f$ *is surjective, then* f *is surjective,*

d) *give an example to show that* $g \circ f$ *is surjective but* f *is not.*

Exercise 3.6. *Let* $f : X \to Y$ *be a map. Prove that*

a) f *is surjective iff there exists* $g : Y \to X$ *such that* $f \circ g = \mathrm{Id}_Y$,

b) f *is injective iff there exists* $g : Y \to X$ *such that* $f \circ g = \mathrm{Id}_X$.

Exercise 3.7. *Let* X *be a set such that* $\operatorname{card} X = n$. *Find the total number of bijective from* X *to itself.*

Exercise 3.8. *Let* X, Y *be sets such that* $\operatorname{card} X = m, \operatorname{card} Y = n$. *Find the total number of maps from* X *to* Y.

Exercise 3.9. *Let* X, Y *be sets such that* $\operatorname{card} X = m, \operatorname{card} Y = n$ *and* $m < n$. *Find the total number of injective from* X *to* Y.

Exercise 3.10. *Let*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 2 & 1 & 5 & 7 & 6 & 9 & 10 & 8 \end{pmatrix},$$

and

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 1 & 2 & 5 & 7 & 6 & 9 & 8 & 10 \end{pmatrix}$$

i) *Compute* σ^{-1} *and* $\tau \circ \sigma$.

ii) *Write* σ, τ *as a product of disjoint cycles.*

iii) *Compute* $\operatorname{sign}(\sigma), \operatorname{sign}(\tau)$.

Exercise 3.11. *Let* $|X| = n$ *and* f *be a bijection from* X *to* X. *Prove that there exists* $k \in \mathbb{N}$ *such that* $f^k = \mathrm{Id}_X$, *where* $f^k = f \circ f \cdots \circ f$ *(k-times).*

4 Binary relations

Exercise 4.1. *Let X be a set and $P(X)$ be the collection of all subsets of X. We define a relation $\leq$ on $P(X)$ as follow: $A \leq B \Leftrightarrow A \subset B$.*

a) Prove that this is an order relation on $P(X)$.

b) Is it a total order relation?

c) Find the maximal and minimal element of $P(X)$.

Exercise 4.2. *Let $\leq$ be an order relation on X. Prove the following statements.*

a) The greatest element, if exists, is unique.

b) The least element, if exists, is unique.

c) Find an example of $(X, \leq)$ such that the greatest (least) element does not exist.

d) If x is the greatest (least) element, then x is also a maximal (minimal) element.

e) In totally ordered set, the terms maximal element and greatest element coincide.

Exercise 4.3. *Let A, B be sets and $\leq$ be a total order relation on B. Assume that $f : A \to B$ is a map. We define a relation on A as follow:*

$$a_1 \preceq a_2 \Leftrightarrow f(a_1) \leq f(a_2).$$

a) Prove that if f is injective, then $\preceq$ is an order relation on A.

b) Give an example to show that $\preceq$ is not an order relation on A.

Exercise 4.4. *Let S be an order relation on $X \times X$. The inverse relation of S, denoted by S^{-1}, defined by $xS^{-1}y \Leftrightarrow ySx$. Prove that S^{-1} is an order relation.*

Exercise 4.5. *Let $X = \mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Consider a relation $\sim$ on $\mathbb{X}$ as follow $(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$. Prove that $\sim$ is an equivalent relation.*

Exercise 4.6. *Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ and $X = \mathbb{Z} \times Z^*$. Consider a relation on X as follow $(a, b) \sim (c, d) \Leftrightarrow ad = bc$. Prove that this is an equivalent relation.*

Exercise 4.7. *Let S be an order relation on $X \times X$. The inverse relation of S, denoted by S^{-1}, defined by $xS^{-1}y \Leftrightarrow ySx$. Prove that S^{-1} is an order relation.*

Exercise 4.8. *Let R_1, R_2 be relations on $\mathbb{Z}$ defined as follow*

$$xR_1y \text{ if } x + y \text{ is an odd number}, \quad xR_2y \text{ if } x + y \text{ is an even number}$$

Determine whether R_1, R_2 are order or equivalence relations?

Exercise 4.9. *Consider the relations R_1, R_2 on $\mathbb{R}^2$ as follow*

$$(x_1, x_2)R_1(y_1, y_2) \Leftrightarrow x_1^2 + x_2^2 = y_1^2 + y_2^2,$$

$$(x_1, x_2)R_2(y_1, y_2) \Leftrightarrow x_1^1 + x_2^2 \leq y_1^2 + y_2^2.$$

Determine whether R_1, R_2 are order or equivalence relations?

Exercise 4.10. *Consider the commutativity, associativity of the following binary operator $*$ on $\mathbb{R}$ and $\circ$ on $\mathbb{R}^2$ and find the identity element, the inverse element.*

a) $x * y := xy + 1,$

b) $x * y := \frac{1}{2}xy,$

c) $x * y := |x|^y.$

d) $(x_1, x_2) \circ (y_1, y_2) := \left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}\right).$

Exercise 4.11. *Let X, Y be sets, $* : Y \times Y \to Y$ is a commutative, associative binary operator with identity element e and $f : X \to Y$ be an bijection. Consider the binary operator on X as follow: $x_1 \circ x_2 = f^{-1}(f(x_1) * f(x_2))$. Prove that $\circ$ is a commutative, associative binary operator with identity element.*

Exercise 4.12. *Which of the following are groups?*

a) $(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{N}, +)$, $(\mathbb{Z}/n, +)$..

b) $(\mathbb{Z}^* = \{\pm 1\}, \times), (\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}, \times), (\mathbb{R}^*, \times).$

c) $(S_n, \circ)$*, where S_n is the set of all permutations on n elements.*

d) $(m\mathbb{Z}, +)$*, where $m\mathbb{Z} = \{n \in \mathbb{Z} | n$ is divisible by $m\}$.*

e) $(2^{\mathbb{Z}}, \times)$*, where $2^{\mathbb{Z}} = \{2^n, n \in \mathbb{Z}\}$.*

f) $(P_n(X), +)$*, where $P_n(X)$ is the all real polynomials of degree not exceeding n.*

Exercise 4.13. *Let X be arbitrary set and consider the binary operator $x * y = x, \forall x, y \in X$. Prove that $(X, *)$ is a semigroup.*

Exercise 4.14. *Lett X be a semigroup with the multiplication.*

a) Prove that if $ab = ba \forall a, b \in X$, then $(ab)^n = a^n b^n, n > 1$.

b) Let $a, b \in X$ such that $(ab)^2 = a^2 b^2$. Can we conclude that $ab = ba$?

Exercise 4.15. *Prove that*

a) $(\mathbb{Z}, +, \times), (\mathbb{Q}, +, \times)$ *are commutative rings with identity.*

b) $(\mathbb{N}, +, \times)$ *is not a ring.*

c) $(\mathbb{Z}/n, +, \times)$ *is a commutative ring with identity.* [1]

Exercise 4.16. *Let* $X = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$ *and* $Y = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$. *Are* X, Y *rings with addition and multiplication?*

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2},$$

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2},$$

Exercise 4.17. *Prove that*

a) $(\mathbb{Q}, +, \times)$ *is a field.*

b) The ring $(\mathbb{Z}, +, \times)$ *is not a field.*

Exercise 4.18. *Let* $X = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$ *and* $Y = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$. *Are* X, Y *fields with addition and multiplication?*

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2},$$

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2},$$

Exercise 4.19. *Find* $\mathrm{GCD}(3195, 630), \mathrm{GCD}(1243, 3124), \mathrm{GCD}(123456789, 987654321)$

Exercise 4.20. *Find integers* a, b *such that* $1243a + 3124b = 11$.

Exercise 4.21. *Presentation the following numbers by the base* 6*:*

a) 2011,

b) 3125.

Exercise 4.22. *Perform the following operations*

a) $3145_{(7)} + 5436_{(7)}$,

b) $6145_{(7)} - 5451_{(7)}$,

c) $3142_{(7)} : 6_{(7)}$,

d) $3142_{(7)} \times 54_{(7)}$.

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	11	13	15
3	3	6	12	15	21	24
4	4	11	15	22	26	33
5	5	13	21	26	34	42
6	6	15	24	33	42	51

The multiplication table with base 7

Exercise 4.23. *Write the following complex numbers in the canonical form.*

[1] it is called the multiplicative group of integers modulo n.

a) $(1+i\sqrt{3})^9$,

b) $\sqrt[8]{1-i\sqrt{3}}$,

c) $\frac{(1+i)^{21}}{(1-i)^{13}}$,

d) $(2+i\sqrt{12})^5(\sqrt{3}-i)^{11}$.

Exercise 4.24. *Solve the following equations*

a) $z^2+z+1=0$,

b) $z^2+2iz-5=0$,

c) $z^4-3iz^2+4=0$,

d) $z^6-7z^3-8=0$,

e) $\frac{(z+i)^4}{(z-i)^4}=1$,

f) $z^8(\sqrt{3}+i)=1-i$.

Exercise 4.25. *Prove that if* $z+\frac{1}{z}=2\cos\varphi$, *then* $z^n+\frac{1}{z^n}=2\cos n\varphi, \forall n\in\mathbb{N}$.

Exercise 4.26. *a) Find the sum of* n*-roots of the complex number* 1.

b) Find the sum of n*-roots of an arbitrary complex number* $z\neq 0$.

c) Let $\epsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}, k=0,1,\ldots,n-1$. *Compute* $S=\sum_{k=0}^{n-1}\epsilon_k^m,\ (m\in\mathbb{N})$.

Exercise 4.27. *Consider the equation* $\frac{(z+1)^9-1}{z}=0$.

a) Solve the above equation.

b) Compute the moduli of the solutions.

c) Compute the product of its solutions and $\prod_{k=1}^{8}\sin\frac{k\pi}{9}$.

Exercise 4.28. *Solve the following equation*

a) $\overline{z^7}=\frac{1}{z^3}$,

b) $z^4=z+\overline{z}$.

Exercise 4.29. *Let* x,y,z *be complex numbers that satisfy* $|x|=|y|=|z|=1$. *Compare the modulus of* $x+y+z$ *and* $xy+yz+zx$.