Probability and Statistics – Homework 3

Instructions: For each problem, remember you must briefly explain/justify how you obtained your answer, as correct answers without an explanation will receive **no credit**. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide. It is fine for your answers to include summations, products, factorials, exponentials, or combinations; you don't need to calculate those all out to get a single numeric answer.

Submission: You must upload your working in PDF format to Canvas MATH2010

Question 1 (4 points):
Let $X_1, X_2, \dots, X_n$ be a random sample from distributions with the given probability density functions. In each case, find the maximum likelihood estimator $\hat{\theta}_{MLE}$.

a) $f(x;\ \theta) = \left(\frac{1}{\theta^2}\right) x\, e^{-x/\theta}, 0 < x < \infty, 0 < \theta < \infty.$

b) $f(x;\ \theta) = \left(\frac{1}{2}\right) e^{-|x-\theta|}, -\infty < x < \infty, -\infty < \theta < \infty.$

Question 2 (4 points):
Let the independent normal random variables $Y_1, Y_2, \dots, Y_n$ have the respective distributions $N(\mu, \gamma^2 x_i^2),\ i = 1, 2, \dots, n$, where $x_1, x_2, \dots, x_n$ are known but not all the same and no one of which is equal to zero.

a) Compute the likelihood function of μ and γ^2.

b) Find the maximum likelihood estimators for μ and γ^2.

Question 3 (4 points):
Let the pdf of the random variable X be defined by:

$$f_X(x) = \begin{cases} \left(\frac{4}{\theta}\right)^2 x & 0 < x < \frac{\theta}{2} \\ -\left(\frac{4}{\theta^2}\right) x + \frac{4}{\theta} & \frac{\theta}{2} \le x < \theta \\ 0 & otherwise \end{cases}$$

where the range of θ is $0 < \theta \le 2$.

a) Computing $E[X]$.

b) Let $x_1, x_2, \dots, x_n$ be the iid samples of X, find an estimator for θ by the method of moments.

Question 4 (4 points):

a) Let X be a continuous random variable with the following PDF:

$$f_X(x) = \begin{cases} 6x(1-x) & if\ 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

Suppose that $Y|X = x \sim Geometric(x)$. Find the posterior density of X given $Y = 2$.

b) Let X be a continuous random variable with the following PDF:

$$f_X(x) = \begin{cases} 3x^2 & if\ 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

Suppose that $Y|X = x \sim Geometric(x)$. Find the MAP estimate of X given $Y = 5$.

Question 5 (4 points):

Remember that the posterior distribution, $\pi_\Theta(\theta|x)$ contains all the knowledge that we have about the unknown quantiy Θ. Therefore, to find an estimate of Θ, we can just choose a summary statistic of the posterior distribution $\pi_\Theta(\theta|x)$ such its mean, median, or mode. If we choose the mode (the value of θ that maximizes $\pi_\Theta(\theta|x)$), we obtain the MAP estimate of Θ.

Another option would be to choose the posterior mean, i.e.,

$$\hat{\theta} = E[\Theta|x]$$

It is shown that $E[\Theta|x]$ will give the best estimate of Θ in terms of the mean squared error. For this reason, the conditional expectation is called the minimum mean squared error (MMSE) estimate of Θ.

Let $X \sim Normal(0, 1)$ and

$$Y = 2X + W,$$

where $W \sim Normal(0, 1)$ is independent of X.

a) Find the MMSE estimator $\hat{X}_{MMSE}$ of X given $Y = y$.
b) Find the mean squared error (MSE) of this estimator, in which MSE is defined as following:

$$MSE = E\left[\left(X - \hat{X}_{MMSE}\right)^2\right]$$