Probability and Statistics – Problem Set 7

Question 0:

I have a bag that contains 3 balls. Each ball is either red or blue, but I have no information in addition to this. Thus, the number of blue balls, call it θ, might be 0, 1, 2, or 3. I am allowed to choose 4 balls at random from the bag with replacement. We define the random variables X_1, X_2, X_3, and X_4 as follows:

$$X_i = \begin{cases} 1 & \text{if the i} - \text{th chosen ball is blue} \\ 0 & \text{if the i} - \text{th chosen ball is red} \end{cases}$$

Note that $X_i s$ are i.i.d. and $X_i \sim Bernoulli\left(\frac{\theta}{3}\right)$. After doing my experiment, I observe the following values for $X_i's$.

$$x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 1.$$

Thus, I observe 3 blue balls and 1 red balls.

1. For each possible value of θ, find the probability of the observed sample, $(x_1, x_2, x_3, x_4) = (1,0,1,1)$.
2. For which value of θ is the probability of the observed sample is the largest?

Question 1.

Suppose that X is a discrete random variable with the following probability mass function: where $0 \leq \theta \leq 1$ is a parameter.

X	0	1	2	3
$P(X)$	$\frac{2\theta}{3}$	$\frac{\theta}{3}$	$\frac{2(1-\theta)}{3}$	$\frac{1-\theta}{3}$

The following 10 independent observations were taken from such a distribution: $(3, \ 0, 2, 1, 3, 2, 1, 0, 2, 1)$. What is the maximum likelihood estimate of θ.

Question 2:

Suppose $x_1, x_2, \dots, x_n$ are i.i.d. (independently identically distributed) samples of a random variable X with density function:

$$f_X(x|\sigma) = \frac{1}{2\sigma} e^{\left(-\frac{|x|}{\sigma}\right)}$$

Find the maximum likelihood estimate of σ.

Question 3:

Use the maximum likelihood estimation method to estimate the parameters μ and σ of the normal density:

$$f_X(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}$$

based on the i.i.d. sample $x_1, x_2, \dots, x_n$.

Question 4:

The Pareto distribution has been used in economics as a model for a density function with a slowly decaying tail:

$$f_X(x|x_0,\theta) = \theta\, x_0^{\theta} x^{-\theta-1}, \forall\, x \geq\, x_0, \theta > 1$$

Assume that $x_0 > 0$ is given and that $x_1, x_2, \dots, x_n$ is an i.i.d. sample. Find the MLE of θ .