MATH 521 (Analysis I) Course Notes

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This document was made from notes taken in Professor Hung Tran's MATH 521 lectures during the Spring 2020 semester at UW-Madison. The course covered the majority of Chapters 1 - 5 and a bit of Chapter 7 from Walter Rudin's *Principles of Mathematical Analysis* (link). We did not discuss complex numbers in this course.

This document is organized in terms of content discussed in each lecture, not by chapters in the textbook. This course was taught without direct reference to the textbook, so the notes are designed to not require a supporting text.

I welcome feedback on this document. Please feel free to reach out to me by email at eegalles@gmail.com to share your thoughts or concerns. This document was generated on September 13, 2024; for the most recent edition, please visit GitHub (link).

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Table of Contents

Foreword

As I took this course, I was very lucky to have friends that were always open to discussing difficult proofs or talking about homework problems together. I accredit the majority of my learning to the conversations we had, so to all of those who I crossed paths with, especially Dan and John, thank you for being patient with me throughout the semester.

To Professor Tran, thank you for teaching this course with such a great passion for the subject. I would've struggled much more if not for your enthusiasm and flexibility with how you presented each topic and answered our questions. Your teaching made me very excited for what is to come with other math courses; thank you for helping me open the door.

- *Emmett Galles*

Lecture 1: 01/21/20

This lecture discusses proof by induction and introduces sets and ordered sets. Reviewing inequalities would not be a bad idea before starting this lecture.

1.1 Induction

We typically look to use induction when we have to prove that a claim P is true for all $n \in \mathbb{N}$, or for all $n \in \mathbb{N}$ with $n \geq n_0$. Denoting that the claim P is true for a specific n can be denoted by $P(n)$.

Definition 1.1.1: Induction

To construct a valid proof by induction, we must perform the following:

i. Show that $P(1)$, or $P(n_0)$, is true (verifying the base case).
ii. Assume that $P(k)$ is true for $k \in \mathbb{N}$ and show that $P(k+1)$ is true. In other words, show that $P(k)$ true $\implies P(k+1)$ true (verifying the induction hypothesis).

Why does induction work?

We can think of induction as an iterative process. The base case establishes that we can start from the bottom level, and the induction hypothesis allows us to build upon this bottom level. For each $n \in \mathbb{N}$, we can eventually build up to it from the base case and our induction hypothesis.

Example 1.1.2: Induction

Prove that $3^n \geq n^3$ for all $n \in \mathbb{N}$, $n \geq 3$.

Proof: Since we are looking at proving elements in $\mathbb{N}$, we think to use induction. Therefore, we must prove that the base case and the induction hypothesis hold.

i. Base case: When $n = 3$, we have

$$3^n = 3^3 = n^3$$

so the base case is verified.
ii. We assume that $3^k \geq k^3$ for some $k \in \mathbb{N}$, $k \geq 3$. We must show that $3^{k+1} \geq$

$(k+1)^3$. We tackle this below:

$$(k+1)^3 = k^3 \cdot \left(\frac{k+1}{k}\right)^3 = k^3 \cdot \left(1+\frac{1}{k}\right)^3 \leq k^3 \cdot \left(1+\frac{1}{3}\right)^3 = k^3 \cdot \frac{64}{27}$$
$$< k^3 \cdot 3 \leq 3^k \cdot 3 = 3^{k+1}$$

Thus the inductive hypothesis is verified.

We have proven both the base case and the inductive hypothesis, so we are done. □

Note that when we try to prove the $k+1$ case, we often call upon the assume we make in the induction hypothesis. This was seen in Example 1.1.2 when we declared $k^3 \cdot 3 \leq 3^k \cdot 3$.

1.2 Ordered sets

We view sets as a collection of objects, but in most cases, we would like to establish an ordering (denoted by $\prec$) of the elements in the set.

Definition 1.2.1: Ordered set

Assume that $S \neq \varnothing$. To create an ordered set $(S, \prec)$, the following must be true:

- For any $x, y \in S$, only one of the following three points hold: $x = y$, $x \prec y$, $y \prec x$.
- For $x, y, z \in S$, if $x \prec y$ and $y \prec z$, then $x \prec z$ (transitivity).

When we say that $x = y$ this means that x, y are the same element from the set S.

At its core, an ordered set basically means that we can compare any two elements in a set and declare that they are either the same element or that they are different with one being preferred to the other.

Please note that sets can have many different orderings, so just because a certain ordering fails on a set doesn't mean the set is incapable of becoming ordered. We see this in action in the following examples.

Example 1.2.2: Ordered sets

$(\mathbb{Z}, \prec)$ with $\prec$ declaring, for $m, n \in \mathbb{Z}$, $m \prec n$ if $n - m \in \mathbb{N}$. Is this an ordered set?

This is an ordered set.

Example 1.2.3: Ordered sets

$(\mathbb{Z}, \prec)$ with $\prec$ declaring, for $x, y \in \mathbb{Z}$, $x \prec y$ if $|x| < |y|$. Is this an ordered set?

This is NOT an ordered set as we cannot compare -1 and 1.

Example 1.2.4: Ordered sets

$(\mathbb{Z}, \prec)$ with $\prec$ declaring $0 \prec -1 \prec 1 \prec -2 \prec 2 \prec \dots$. Is this an ordered set?

This is an ordered set.

Example 1.2.5: Ordered sets

$(\mathbb{Z}^2, \prec)$ with $\prec$ declaring, for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{Z}^2$, $x \prec y$ if $x_1 < y_1$. Is this an ordered set?

This is NOT an ordered set as we can't compare $(1, -1)$, $(1, 0)$, $(1, 1)$, ...

Traditional (and typical) ordering

We typically consider sets with 1-dimensional elements. We use the notation $(S, <)$ to indicate that for $a, b \in S$, $a \prec b \iff a < b$. If an ordering is not defined on a set, we assume it is of this type.

Recognize how we can't use such a simple ordering with higher dimensional elements, since each element would consist of more than one real number to compare with other elements.

1.3 Bounded sets

Determining whether or not a set is bounded or not is very important, as it can allow us to say whether or not a set breaches a certain threshold or not.

Definition 1.3.1: Bounded set

Let $(S, \prec)$ be an ordered set and have $A \subset S$. We can declare the boundedness of A in the following ways:

i. A is bounded from above if we can find a $d \in S$ such that $a \preceq d$ for all $a \in A$.
ii. A is bounded from below if we can find a $c \in S$ such that $c \preceq a$ for all $a \in A$.
iii. A is bounded if it's bounded from both below and above.

Note that $\preceq$ means $\prec$ or $=$.

Please note that there could exist many upper and lower bounds of a set, as many different elements in our set S could satisfy the requirements of Definition 1.3.1.

Lecture 2: 01/23/20

This lecture covers more on ordered and bounded sets from Lecture 1 and introduces a way to optimize upper and lower bounds. Please make sure to understand Definition 1.3.1 and Definition 1.2.1 before continuing with this lecture.

2.1 More on bounded sets

Recall our discussion on bounded sets, particularly when we defined it with Definition 1.3.1. We will now look at some examples.

Example 2.1.1: Bounded sets

Consider the parent set S to be $(\mathbb{Z}, <)$ and let $A = 3\mathbb{Z} = \{3k : k \in \mathbb{Z}\}$. We claim that A is not bounded from above or below.

Proof: Let's assume that A is bounded from above by $d \in \mathbb{Z}$. However, we see that $3 \cdot |d| \in A$ and $3 \cdot |d| > d$, which means that d cannot be an upper bound of A.

Similarly, if we declare that $c \in \mathbb{Z}$ bounds A from below, we know that $3 \cdot -|c| \in A$ and $3 \cdot -|c| < c$, so c cannot be a lower bound of A.

We have shown that for any arbitrary point $z \in \mathbb{Z}$, z cannot be an upper or lower bound of A, meaning that A is not bounded from above or below. $\square$

Example 2.1.2: Bounded sets

Consider the parent set S to be $(\mathbb{Z}, <)$ and let $B = \mathbb{N}$. We claim that B is not bounded from above but is bounded from below.

Proof: From Example 2.1.1 we can deduce a similar process for showing how B is not bounded from above.

For a lower bound, consider $c = -1 \in \mathbb{Z}$. Since B does not contain any negative numbers, we can say that for any $b \in B$, $c < b$, which means that c is a lower bound of B, so B is bounded from below. $\square$

Notice how there are many lower bounds to choose from to prove that B is bounded from below.

Example 2.1.3: Bounded sets

Consider the parent set S to be $(\mathbb{Q}, <)$ and let $C = \{\frac{1}{k} : k \in \mathbb{N}\}$. We claim that we cannot find a positive lower bound of C.

Proof: Assume that we can find a $q \in \mathbb{Q}$ with $q > 0$ such that q is a lower bound of of C. We are free to say $q = \frac{m}{n}$ where $m, n \in \mathbb{N}$. Notice that $\frac{1}{n+1} \in C$ and $\frac{1}{1+n} < \frac{m}{n}$. This contradicts the assumption that q is a lower bound of C. □

Example 2.1.4: Bounded sets

Consider the parent set S to be $(\mathbb{Q}, <)$ and let $D = \{q^2 < 2 : q \in \mathbb{Q}\}$. We claim that D is bounded from above by 2 and from below by -2.

Proof: Observe that

$$q^2 < 2 \iff -\sqrt{2} < q < \sqrt{2}$$

From here it is clear to see that 2 bounds D from above and -2 bounds D from below. □

2.2 Supremum and infimum

We've learned about upper and lower bounds and gathered that there could be many different values that bound a set. Supremum and infimum act more or less like an optimization of upper and lower bounds.

Definition 2.2.1: Supremum

Let $A \subset S$ with $A \neq \varnothing$ be bounded from above. We say that $\beta \in S$ is a least upper bound of A if:

- β is an upper bound of A.
- For any other upper bound $d \in S$ of A, we have $\beta \preceq d$.

We declare $\beta = \sup A$.

Definition 2.2.2: Infimum

Let $A \subset S$ with $A \neq \varnothing$ be bounded from below. We say that $\alpha \in S$ is a greatest lower bound of A if:

- α is an lower bound of A.
- For any other lower bound $c \in S$ of A, we have $c \preceq \alpha$.

We declare $\alpha = \inf A$.

With these two definitions, we retrieve the following lemmas.

Lemma 2.2.3: Uniqueness of supremum

If $\sup A$ exists, then it is unique.

Proof: Assume A has at least two different least upper bounds $\beta, \gamma \in S$ where $\beta \neq \gamma$. By definition of least upper bound, $\beta \preceq \gamma$ and $\gamma \preceq \beta$, which forces $\beta = \gamma$, contradicting the assumption that $\beta \neq \gamma$. $\square$

Lemma 2.2.4: Uniqueness of infimum

If $\inf A$ exists, then it is unique.

Proof: Assume A has at least two different greatest lower bounds $\alpha, \lambda \in S$ where $\alpha \neq \lambda$. By definition of greatest lower bound, $\alpha \preceq \lambda$ and $\lambda \preceq \alpha$, which forces $\alpha = \lambda$, contradicting the assumption that $\alpha \neq \lambda$. $\square$

Note that we used the fact that the supremum of a set has to be less than all other upper bounds and the infimum of a set has to be greater than all other lower bounds. For proofs incorporating supremum and infimum this is a common technique.

Example 2.2.5: Supremum and infimum

Let E, F be nonempty bounded sets in $\mathbb{R}$ such that $E \subset F$. Show that

$$\inf F \leq \inf E \leq \sup E \leq \sup F$$

Proof: We can use Definition 1.3.1 to see that $\inf F \leq \sup F$ and $\inf E \leq \sup E$. Since $E \subset F$ and $\sup F$ is an upper bound of F, then $\sup F$ is an upper bound of E. Since $\sup E$ is the least upper bound of E, $\sup E \leq \sup F$. By a similar process, we declare that $\inf F \leq \inf E$, leaving us with $\inf F \leq \inf E \leq \sup E \leq \sup F$. $\square$

Example 2.2.6: Supremum and infimum

Let E, F be nonempty bounded sets in $\mathbb{R}$. Find $\sup(E \cup F)$ and $\inf(E \cup F)$ in terms of $\sup E$, $\sup F$, $\inf E$, and $\inf F$.

Proof: Let's say $\alpha = \max\{\sup E, \sup F\}$. We will prove that $\alpha = \sup(E \cup F)$.

Since $E \subset (E \cup F)$ and $F \subset (E \cup F)$, by Example 2.2.5 we know that $\sup(E \cup F) \geq$

$\sup E$ and $\sup(E \cup F) \geq \sup F$, so we can say $\alpha \leq \sup(E \cup F)$.

We know that for all $x \in (E \cup F)$, $x \in E$ or $x \in F$. Therefore, it's certainly true that $x \leq \max\{\sup E, \sup F\} = \alpha$, so α is an upper bound of $(E \cup F)$, and by Definition 2.2.1 we know $\alpha \geq \sup(E \cup F)$.

We have shown that $\alpha \geq \sup(E \cup F)$ and $\alpha \leq \sup(E \cup F)$, which means that $\alpha = \max\{\sup E, \sup F\} = \sup(E \cup F)$. $\square$

We can use a similar process to show that $\beta = \min\{\inf E, \inf F\} = \inf(E \cup F)$.

By thinking about the supremum and infimum in a geometric sense, the following remark comes naturally.

Remark 2.2.7: Supremum and infimum interval

Given that $\sup A$, $\inf A$ exist, we have $A \subset [\inf A, \sup A]$.

Theorem 2.2.8: Supremum and infimum of finite sets

Let $A \subset S$ be a finite set. Then, $\min A = \inf A$ and $\max A = \sup A$.

Proof: Let's first consider $\min A = \inf A$. Declare $\alpha = \min A$. By definition of minimum, we have that there does not exist an $s \in S$ such that $s < \alpha$, so α is a lower bound of A. Surely $\alpha \in A$, so for any $p \in S$ with $\alpha < p$, we know $p \neq \inf A$ since p is not a lower bound of A. When $p < \alpha$, it is true that p is a lower bound of A, but since α is a lower bound of A and $p < \alpha$, then certainly p is not the greatest lower bound.

We do a similar process for $\max A = \sup A$. Declare $\beta = \max A$. By definition of maximum, we have that there does not exist an $t \in S$ such that $\beta < t$, so β is an upper bound of A. Surely $\beta \in A$, so for any $q \in S$ with $q < \beta$, we know $q \neq \sup A$ since q is not an upper bound of A. When $\beta < q$, it is true that q is an upper bound of A, but since β is an upper bound of A and $\beta < q$, then certainly q is not the least upper bound. $\square$

Please note that in general, a set does not have a maximum or a minimum. Furthermore, Theorem 2.2.8 extends to any set that contains maximum or minimum values.

Example 2.2.9: Supremum and infimum

Let the parent set $S = (\mathbb{Q}, <)$, and let $B = \{\frac{1}{k} : k \in \mathbb{N}\}$. We claim that $\inf B = 0$.

Proof: First, notice how 0 is a lower bound of B as $0 < \frac{1}{k}$ for all $k \in \mathbb{N}$. We will show that it's the greatest lower bound.

Assume that we can find a $q > 0$ where $q \in \mathbb{Q}$ such that q is a lower bound of B. However, if we let $k = \lceil \frac{1}{q} \rceil + 1$, we see that $\frac{1}{k} < q$, meaning that q is not a lower bound of B. Therefore, $\inf B = 0$. $\square$

Example 2.2.10: Supremum and infimum

Let $A \subset \{x \in \mathbb{R} : x > 0\}$ be a bounded set and let $B = \{x^2 : x \in A\}$. Show that $\sup B = (\sup A)^2$.

Proof: Note that for all $x \in A$, we have that $0 < x \leq \sup A$, meaning that $0 < x^2 \leq (\sup A)^2$, or that $(\sup A)^2$ is an upper bound of B.

Let z be an upper bound of B. This means that $x^2 \leq z$ for all $x \in A$. Since $x > 0$, we know that $0 < x \leq \sqrt{z}$ for all $x \in A$. This means that $\sqrt{z}$ is an upper bound of A, so

$$\sqrt{z} \geq \sup A > 0 \implies z \geq (\sup A)^2$$

This means that $(\sup A)^2$ is less than or equal to all other upper bounds of B, so $\sup B = (\sup A)^2$. $\square$

2.3 Least upper bound property (LUB)

Although it may not be explicitly mentioned later in the course, the least upper bound property allows us to build many powerful theorems and gives us great insight on the real numbers.

Definition 2.3.1: Least upper bound property (LUB)

Assume that $S \neq \varnothing$. We say that $(S, \prec)$ has the least upper bound property if for any set $A \neq \varnothing$ where $A \subset S$ and A is bounded from above, we have that $\sup A$ exists.

Remark 2.3.2: Integers and LUB

$(\mathbb{Z}, <)$ has the least upper bound property.

We will discuss two lemmas that give us some more information on the rational numbers and allow us to construct a powerful theorem.

Lemma 2.3.3: $\sqrt{2}$ is irrational

There does not exist a $t \in \mathbb{Q}$ such that $t^2 = 2$.

Proof: Let's assume that t exists. Since $t \in \mathbb{Q}$, we can write it in the form $t = \frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ and $\frac{p}{q}$ is irreducible.

Since $t^2 = 2$, we can say

$$\frac{p^2}{q^2} = 2 \implies p^2 = 2 \cdot q^2 \implies p \text{ is even}$$

Therefore, we can say $p = 2 \cdot k$ for some $k \in \mathbb{Z}$. We now have

$$\frac{(2 \cdot k)^2}{q^2} = \frac{4 \cdot k^2}{q^2} = 2 \implies 2 \cdot k^2 = q^2 \implies q \text{ is even}$$

However, the conclusion that p and q are both divisible by 2 is a contradiction of $\frac{p}{q}$ being irreducible. $\square$

Note that in Lemma 2.3.3 our contradiction did not directly oppose the definition of a rational number. However, an implication of rational numbers is that we can put them into an irreducible form. Our contradiction we came to was of this implication, which means that $\sqrt{2}$ is not a rational number.

Lemma 2.3.4: Upper bounds of subsets of $\mathbb{Q}$

Let $D = \{q \in \mathbb{Q} : q^2 < 2\}$. Assume that $d \in \mathbb{Q}$ where $d \leq 2$ is an upper bound of D. Then, there exists an $r \in \mathbb{Q}$ where $r < d$ such that r is an upper bound of D.

Proof: Notice that in order for d to be an upper bound of D, it must be true that $d^2 \geq 2$. By Lemma 2.3.3 we see that $d^2 \neq 2$, so $d^2 > 2$. Furthermore, we can find a $k \in \mathbb{N}$ such that $d^2 > 2 + \frac{1}{k}$.

Let $r = d - \frac{1}{4 \cdot k}$. We shall verify that $r^2 > 2$ as well.

$$r^2 = \left(d - \frac{1}{4 \cdot k}\right)^2 = d^2 - \frac{d}{2 \cdot k} + \frac{1}{16 \cdot k^2} \geq d^2 - \frac{d}{2 \cdot k} \geq d^2 - \frac{1}{k} > 2$$

Therefore, $r \in \mathbb{Q}$ and $r^2 > 2$, which means that r is another upper bound of D and $r < d$. $\square$

A trick we used in the proof for Lemma 2.3.4 was finding a number in between a strict inequality (strict meaning, for example, we have $<$ instead of $\leq$). Note that we can't do this in all sets. For example, if we have two consecutive integers i and j where $i < j$, we can't find an integer between i and j.

Theorem 2.3.5: $(\mathbb{Q}, <)$ does not have LUB

$(\mathbb{Q}, <)$ does not have the least upper bound property.

Proof: Recall from Definition 2.3.1 that in order for a set to have the least upper bound property, all nonempty and bounded from above subsets have a supremum. However, by Lemma 2.3.4, we found a set where for each upper bound, there exists a smaller upper bound, implying that a supremum does not exist. Therefore, $(\mathbb{Q}, <)$ does not have the least upper bound property. $\square$

Lecture 3: 01/28/20 (*substitute*)

This is an extremely odd lecture. A substitute professor gave this lecture and it was hard to find connections from the content covered here to other parts of the course. We prove an important theorem regarding the least upper bound property defined in Definition 2.3.1 at the beginning of the lecture, but then start discussing fields which is the part of the lecture that wasn't referred to during the rest of the course.

3.1 Greatest lower bound property (GLB)

We did not mention this in the past lecture, but just as we have a least upper bound property, we also have a greatest lower bound property.

Definition 3.1.1: Greatest lower bound property (GLB)

Assume that a set $S \neq \varnothing$. We say that $(S, \prec)$ has the greatest lower bound property if for any set $B \neq \varnothing$, $B \subset S$ being bounded from below, we have that $\inf B$ exists.

We will now use our knowledge of the least upper bound and greatest lower bound properties and the following theorem to show how they are connected.

Theorem 3.1.2: LUB $\Longrightarrow$ GLB

Let $\boldsymbol{S}$ be a set with the least upper bound property. Take a set $\boldsymbol{B \neq \varnothing}$, $\boldsymbol{B \subset S}$ being bounded from below. Set $\boldsymbol{L = \{x \in S : x \text{ is a lower bound of } B\}}$. We declare that $\boldsymbol{\sup L}$ and $\boldsymbol{\inf B}$ exist and $\boldsymbol{\sup L = \inf B}$.

Proof: We have to unpack a couple things of things to prove this theorem. Let's take a look at each important part.

i. $\sup L = \alpha$ exists. We know this to be true since $L \subset S$, $L \neq \varnothing$, and L is bounded from above (any element $b \in B$ is an upper bound of L).
ii. α is a lower bound of B. If this were not true, there would exist an $x \in B$ such that $\alpha > x$. Since x is an upper bound of L, this contradicts the fact that α is the least upper bound of L.
iii. α is the greatest lower bound of B. Surely for any lower bound y of B, $y \in L$. Since $\alpha = \sup L$, we see that $\alpha \geq y \in L$.

This confirms that $\alpha = \sup L = \inf B$, so we are done. $\square$

Note that we can do a similar proof to show that GUB $\implies$ LUB, meaning that LUB $\iff$ GUB.

3.2 Fields

Definition 3.2.1: Field

A field F is a set equipped with the operations addition $(+)$ and multiplication $(\cdot)$ that satisfy the following criteria:
Addition:

- For all $x, y \in F$, $x + y \in F$.
- $x + y = y + x$ (commutative).
- $(x + y) + z = x + (y + z)$ (associative).
- There exists an element $0 \in F$ such that for all $x \in F$, $0 + x = x$.
- For all $x \in F$, there exists a $-x \in F$ such that $x + (-x) = 0$ (additive inverse).

Multiplication:

- For all $x, y \in F$, $x \cdot y \in F$.
- $x \cdot y = y \cdot x$ (commutative).
- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (associative).
- There exists an element $1 \in (F \setminus \{0\}$ such that for all $x \in F$, $1 \cdot x = x$.
- For all $x \in F \setminus \{0\}$, there exists an $x^{-1} = \frac{1}{x} \in F$ such that $x \cdot x^{-1} = 1$ (multiplicative inverse).
- $(x + y) \cdot z = x \cdot z + y \cdot z$.

So the following remarks are all properties of fields. They are grouped into one big remark because the proofs are extremely basic (all one line or less) and are not imperative to cover individually.

Remark 3.2.2: Properties of fields

The following are all properties of fields:

- $x + y = x + z \implies y = z$. In particular, $x + y = x \implies y = 0$.
- If $x + y =$, then $y = -x$.
- $-(-x) = x$.
- If $x \neq 0$ and $x \cdot y = x \cdot z$, then $y = z$.

- If $x \cdot y = 1$, then $y = x^{-1}$.
- $0 \cdot x = 0$.
- If $x \neq 0$, $\left(x^{-1}\right)^{-1} = x$.
- If $x, y \neq 0$, then $x \cdot y \neq 0$.
- $(-x) \cdot y = x \cdot (-y) = -(x \cdot y)$.
- $(-x) \cdot (-y) = x \cdot y$.

In addition to a field, we can go one step further and define an ordered field that satisfy more properties.

Definition 3.2.3: Ordered field

An ordered field G is a field with an order that satisfies the following:

- For all $x, y, z \in G$, $y \prec z \implies x + y \prec x + z$.
- For $x, y \in G$, if $x, y \succ 0$, then $x \cdot y \succ 0$.

It's important to note that for an $x \in G$, if $x \succ 0$ we say x is positive and if $x \prec 0$ we say x is negative.

We will once again lump all field information into one big remark. These proofs are a touch longer (maybe two lines instead of one) but we won't go through them.

Remark 3.2.4: Properties of ordered fields

The following are all properties of ordered fields (assume that $x, y, z \in G$):

- For $x \in G$, if $x \succ 0$ then $-x \prec 0$.
- If $x \succ 0$, then $x \succ z \implies x \cdot y \prec x \cdot z$.
- If $x \prec 0$, then $y \prec z \implies x \cdot y \succ x \cdot z$.
- If $x \neq 0$, then $x \cdot x = x^2 > 0$. In particular, $1 = 1 \cdot 1 > 0$.
- If $0 \prec x \prec y$, then $0 \prec y^{-1} \prec x^{-1}$.

Some examples of ordered fields are the real numbers $\mathbb{R}$ and the rational numbers $\mathbb{Q}$, both of which are heavily used in this course. A non-example of a field is the set of integers $\mathbb{Z}$ since we cannot find a multiplicative inverse of all elements (for example, $2 \in \mathbb{Z}$ but there is not a $z \in \mathbb{Z}$ such that $2 \cdot z = 1$).

Lecture 4: 01/30/20

In the next two lectures, we will take a closer look at a method of constructing the real number line, something that we often think just exists on its own. This is more or less a very interesting tangent in the course but it's worthwhile to cover, if not for application then at least for an example of reconsidering what we deem to be an inherent truth.

4.1 Dedekind cuts

We've discussed sets in past lectures, but we will now give a name to a set with particular properties.

Definition 4.1.1: Dedekind cuts

Let $\alpha \subset \mathbb{Q}$. We say that α is a Dedekind cut (or simply a cut) if:

i. $\alpha \neq \varnothing$ and $\alpha \neq \mathbb{Q}$.
ii. If $p \in \alpha$, then for all $q \in \mathbb{Q}$ with $q < p$ we have $q \in \alpha$.
iii. If $p \in \alpha$, then we can find an $r \in \mathbb{Q}$ such that $r > p$ and $r \in \alpha$.

Example 4.1.2: Dedekind cuts

Let $\boldsymbol{A = \{q \in \mathbb{Q} : -1 \leq q \leq 1\}}$. Is $\boldsymbol{A}$ a cut?
A is not a cut since $-2 < 1$ but $-2 \notin A$.

By examining Example 4.1.2 and imagining similar situations, we retrieve the following remark.

Remark 4.1.3: Dedekind cuts

A cut must contain all rational numbers to the left of a specific marker.

Example 4.1.4: Dedekind cuts

Let $\boldsymbol{B = \{q \in \mathbb{Q} : q \leq 0\}}$. Is $\boldsymbol{B}$ a cut?
B is not a cut as $0 \in B$ but we cannot find any $r > 0$ such that $r \in B$.

By examining Example 4.1.4 and imagining similar situations, we retrieve the following remark.

Remark 4.1.5: Dedekind cuts

A cut must not contain its right endpoint (its marker).

Example 4.1.6: Dedekind cuts

Let $C = \{q \in \mathbb{Q} : q < 0\}$. Is C a cut?

Proof: To show that C is a cut, we need to verify the three properties outlined in Definition 4.1.1.

i. $C \neq \varnothing$ as $-1 \in C$, and $C \neq \mathbb{Q}$ as $1 \notin C$.
ii. Let $p \in C$ be given, which means that $p \in \mathbb{Q}$ and $p < 0$. For any $q \in \mathbb{Q}$ with $q < p$, we certainly have that $q < 0$, which means that $q \in C$ since $q < p < 0$.
iii. Pick $p \in C$, which means that $p \in \mathbb{Q}$ and $p < 0$. Let $r = \frac{p}{2}$. We see that $r \in \mathbb{Q}$, and since $p < r < 0$, we have that $r \in C$.

Therefore, C is a Dedekind cut. $\square$

A typical trick to pick an element c between two points a and b is to let $c = \frac{a+b}{2}$. In Example 4.1.6, we had $a = p$ and $b = 0$.

By looking at Example 4.1.6, we are able to make a useful remark about Dedekind cuts.

Remark 4.1.7: Dedekind cuts

For any $a \in \mathbb{Q}$, we can define $a^* = \{q \in \mathbb{Q} : q < a\}$ where a^* is a Dedekind cut.

Please note that Dedekind cuts made by employing the technique outlined in Remark 4.1.7 are not the only cuts that can exist. In the following lemma, we construct a cut from an irrational number.

Lemma 4.1.8: $\sqrt{2}$ as a cut

Let $\alpha = \{q \in \mathbb{Q} :$ there is an $r \in \mathbb{Q}$ with $r > 0$ and $r^2 < 2$ such that $q < r\}$. We claim that α is a cut.

Proof: To show that α is a cut, we need to verify the three properties outlined in Definition 4.1.1.

i. $\alpha \neq \varnothing$ as $0 \in \alpha$, and $\alpha \neq \mathbb{Q}$ since $1.5 \notin \alpha$.
ii. Let $p \in \alpha$ be given. We know that $p < r$, so for any $q \in \mathbb{Q}$ with $q < p$, we have $q < r$, so $q \in \alpha$ as well.
iii. Let $q \in \alpha$. We have two cases to consider:

    a. If $q \leq 0$, pick $r = 1$. Then, $q < r$ and $r \in \alpha$.

b. If $q > 0$, then $q = \frac{m}{n}$ with $m, n \in \mathbb{N}$ and $q^2 < 2$. Let $r = q + \frac{1}{k}$ with $k \in N$ to be chosen later. We then have

$$\left(q + \frac{1}{k}\right)^2 = q^2 + 2 \cdot q \cdot \frac{1}{k} + \frac{1}{k^2}$$

Furthermore, we see that

$$r^2 < q^2 + 2 \cdot 1.5 \cdot \frac{1}{k} + \frac{1}{k^2} \leq q^2 + \frac{3}{k} + \frac{1}{k} = q^2 + \frac{4}{k}$$

Since $2 - q^2 > 0$, we can pick a $k \in \mathbb{N}$ large enough so that

$$\frac{4}{k} < 2 - q^2 \implies q^2 + \frac{4}{k} < 2 \implies r^2 < 2$$

Therefore, α is a Dedekind cut. □

Once again, in Lemma 4.1.8 the right end point of our cut is NOT a rational number.

Constructing $\mathbb{R}$ from Dedekind cuts

We began this lecture discussing how Dedekind cuts can be used to craft the real numbers. With work done in Lemma 4.1.8 and the conclusion from Remark 4.1.7, we can declare the real numbers $\mathbb{R}$ to be the set that contains all possible Dedekind cuts. In other words, the real numbers is a set where each one of its elements is a Dedekind cut. As Dedekind cuts are sets of rational numbers, we declare the real numbers $\mathbb{R}$ to be a set of sets.

By Remark 4.1.5 we know that a Dedekind cut does not contain its right endpoint. We use this right endpoint to identify each particular Dedekind cut in $\mathbb{R}$. For example, the Dedekind cut made in Example 4.1.6 would be labeled as 0 and the Dedekind cut made in Lemma 4.1.8 would be labeled as $\sqrt{2}$.

How do we define an ordering of the cuts that make up $\mathbb{R}$? We examine that in the following definition, and then go on to show that $\mathbb{R}$ has the least upper bound property.

Definition 4.1.9: Ordering of cuts

Let α, β be two distinct cuts. We say that $\alpha < \beta$ if $\alpha \subset \beta$.

Theorem 4.1.10: $\mathbb{R}$ has LUB

$\mathbb{R}$ has the least upper bound property.

Proof: Let $A \subset \mathbb{R}$ be a nonempty set that is bounded from above. This means A is a set that contains some cuts, and there is a cut β such that for all $\alpha \in A$, we have $\alpha \leq \beta$. By using Definition 4.1.9, this means that $\alpha \subset \beta$ for all $\alpha \in A$.

Define $\gamma = \bigcup_{\alpha \in A} \alpha$. Notice that γ is a cut. We claim that $\gamma = \sup A$.

Since $\gamma = \bigcup_{\alpha \in A} \alpha$, we see that $\alpha \subset \gamma$ for all $\alpha \in A \implies \gamma$ is an upper bound. Pick any upper bound η of A. As η is an upper bound of A, we have $\alpha \leq \eta$ for all $\alpha \in A$. By definition, this means that $\bigcup_{\alpha \in A} \alpha \subset \eta \iff \gamma \subset \eta$. This verifies that γ is the supremum of A.

Since our set A was arbitrary and we found a supremum for it, we are done. $\square$

Lecture 5: 02/04/20

We begin this lecture with a recap of Dedekind cuts (defined in Definition 4.1.1). We then introduce some inequalities that we call upon frequently in this course.

5.1 More on Dedekind cuts

Recall that cuts are subsets of $\mathbb{Q}$ but the right end point of the cut (which is not contained in the cut itself) need not be in $\mathbb{Q}$. Also, we were able to construct the real number line $\mathbb{R}$ by considering it to be a set of all possible Dedekind cuts. It's important to note that this means that each element of $\mathbb{R}$ is a subset of $\mathbb{Q}$, but $\mathbb{R}$ itself is NOT a subset of $\mathbb{Q}$.

Theorem 5.1.1: Archimedean properties of $\mathbb{R}$

We make two claims:

- **i. If $x, y \in \mathbb{R}$ and $x > 0^*$, then there is an $n \in \mathbb{N}$ such that $y < n \cdot x$.**
- **ii. If $x, y \in \mathbb{R}$ and $x < y$, then there is an $a \in \mathbb{Q}$ such that $x < a^* < y$. This implies that $\mathbb{Q}$ is dense in $\mathbb{R}$.**

Proof: We will break the proof up into two parts.

- i. As $x > 0^*$, that means $0^* \subset x$ and $0^* \neq x$. In other words, there is a $p \in \mathbb{Q}$ with $p > 0$ such that $p \in x$. Furthermore, since y is a cut, this implies that $y \neq \mathbb{Q}$, which means that we can find a $q \in \mathbb{Q}$ such that $q \notin y$. We then can take $n \in \mathbb{N}$ large enough so that $q < n \cdot p \implies q \in n \cdot x \implies y < n \cdot x$.
- ii. $x < y$ means that $x \subset y$ and $x \neq y$. This means there exists a $p \in \mathbb{Q}$ such that $p \in y$ and $p \notin x$. Therefore, $x \leq p^* < y$. As $p \in y$, then by the third property of cuts (see Definition 4.1.1), we can find an $a \in \mathbb{Q}$ with $a > p$ such that $a \in y$. Then, $x \leq p^* < a^* < y$.

This concludes the proof. $\square$

Please take note of the inequality $x \leq p^* < y$ seen in (ii.). Although $p \notin x$, we might have $x = p^*$ since p could be the right endpoint of x.

Note that we mentioned dense sets in Theorem 5.1.1. We explicitly define dense sets later in the course, specifically with Definition 8.1.2.

One of the big takeaways from Theorem 5.1.1 is that between any two rational numbers, we can find another rational number. We use this fact to construct the following theorem.

Theorem 5.1.2: Rationals and irrationals in any interval of $\mathbb{R}$

Let $a, b \in \mathbb{R}$ such that $a < b$. There are both rational and irrational numbers in the interval (a, b).

Proof: By (ii.) of Theorem 5.1.1, we can find a rational number number a_* between a and b. We can use this process again to find a rational number b_* between a_* and b. We now have an interval $(a_*, b_*) \subset (a, b)$. We will find an irrational number between a_* and b_*.

Recognize that $\frac{1}{\sqrt{2}}$ is an irrational number and that $0 < \frac{1}{\sqrt{2}} < 1$. Furthermore, note that the product of a rational number and an irrational number is an irrational number, and the difference between two rational numbers is a rational number. We then see that $a_* < a_* + \frac{1}{\sqrt{2}} \cdot (b_* - a_*) < b_*$ and $\frac{1}{\sqrt{2}} \cdot (b_* - a_*) = c_*$ is an irrational number.

We have found both a rational (a_*) and an irrational number (c_*) in an arbitrary interval (a, b), so we are done. □

Note that instead of thinking about the real numbers as a set of cuts, we could just take the endpoints of the cuts and consider those endpoints to make up the real numbers; this is typically what we imagine $\mathbb{R}$ to be. Combining this thinking with cuts allows us to make a remark that we usually take for granted.

Remark 5.1.3: $\mathbb{Q}$ and $\mathbb{R}$

The rational numbers is a subset of the real numbers.

Let $a \in \mathbb{Q}$. The right endpoint of a^* is just a, so $a \in \mathbb{R} \implies \mathbb{Q} \subset \mathbb{R}$.

It goes without saying that Dedekind cuts are a bit unruly to deal with. **Therefore, for the remainder of the course, assume that the real numbers contain strictly the right endpoints of all possible Dedekind cuts (each element of $\mathbb{R}$ is just a single number, not a cut) unless otherwise stated.**

5.2 Decimal representations of the real numbers

In general, for $x \in \mathbb{R}$ and $x > 0$, x has a decimal representation. This decimal representation can be seen as $x = a_0.a_1a_2 \ldots a_n \ldots$ where $a_0 \in (\mathbb{N} \cup \{0\})$ and $a_k \in \{0, 1, \ldots 9\}$ for $k \in \mathbb{N}$. Since $x > 0$, there must exist a nonzero element in the decimal expansion for x. Let $x_n = a_0.a_1a_2 \ldots a_n \in \mathbb{Q}$ for each $n \in \mathbb{N}$. This is fair game because every finite decimal representation can be put into the $\frac{p}{q}$ form we use for rational numbers.

In terms of cuts, we see that $x = \bigcup_{n\in\mathbb{N}} x_n^* = \sup\{x_n^* : n \in \mathbb{N}\}$. This leads us to make the following remark.

Remark 5.2.1

For any $x \in \mathbb{R}$ with $x > 0$, we have that $x = \sup\{x_n : n \in \mathbb{N}\}$.

Recall that $x_n = a_0.a_1a_2\ldots a_n$ where $a_k \in \{0, 1, \ldots, 9\}$ for $k \geq 1$.

A common question that is asked is why $0.99999\ldots = 1$. However, when we look at this question in terms of cuts, we see that both of these numbers represent the same cut. In general, a real number might have different decimal representations.

From previous coursework, we know that $e = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots$. We can let $x_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$ for $n \in \mathbb{N}$. Then, we can simply define $e = \sup\{x_n : n \in \mathbb{N}\}$.

5.3 Important inequalities of the real numbers

We will use these quite frequently throughout the course.

Theorem 5.3.1: Cauchy inequality

For $a, b \in \mathbb{R}$, we have that $a^2 + b^2 \geq 2 \cdot a \cdot b$.

Proof: We see that

$$a^2 + b^2 \geq 2 \cdot a \cdot b \iff a^2 + b^2 - 2 \cdot a \cdot b \geq 0 \iff (a-b)^2 \geq 0$$

Note that the latter is always true, so we are done. $\square$

We obtain an equality if and only if $a = b$.

Theorem 5.3.2: Cauchy-Schwarz inequality

For $n \in \mathbb{N}$ with $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{R}$ we have that

$$(a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) \geq (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2$$

Proof: Observe that the right hand side (RHS) of the inequality is

$$\text{RHS} = \left(\sum_{i=1}^{n} a_i \cdot b_i\right) \cdot \left(\sum_{j=1}^{n} a_j \cdot b_j\right) = \sum_{i,j=1}^{n} a_i \cdot b_i \cdot a_j \cdot b_j$$

Similarly, the left hand side (LHS) of the inequality is

$$\text{LHS} = \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{j=1}^{n} b_j\right) = \sum_{i,j=1}^{n} a_i^2 \cdot b_j^2 = \frac{1}{2} \cdot \sum_{i,j=1}^{n} a_i^2 \cdot b_j^2 + \frac{1}{2} \cdot \sum_{i,j=1}^{n} a_j^2 \cdot b_i^2$$

By subtraction, we see that

$$\text{LHS} - \text{RHS} = \frac{1}{2} \cdot \sum_{i,j=1}^{n} (a_i^2 \cdot b_j^2 + a_j^2 \cdot b_i^2 - 2 \cdot a_i \cdot b_i \cdot a_j \cdot b_j) = \frac{1}{2} \cdot \sum_{i,j=1}^{n} (a_i \cdot b_j - a_j \cdot b_i)^2 \geq 0$$

This implies that LHS $\geq$ RHS, thus concluding the proof. $\square$

Although we are most familiar with the one-dimensional real number line, we can also look at the real numbers in higher dimensions. We define this below.

Definition 5.3.3: Elements of $\mathbb{R}^n$

Let $x \in \mathbb{R}^n$. This means that $x = (x_1, x_2, \ldots, x_n)$ and $|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$.

Theorem 5.3.4: Triangle inequality

For $\boldsymbol{x} \in \mathbb{R}^n$, we have $|\boldsymbol{x} + \boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$.

Proof: By Definition 5.3.3, we see that $x + y = (x_1 + y_1, \ldots, x_n + y_n)$. We see that

$$\sqrt{(x_1 + y_1)^2 + \ldots + (x_n + y_n)^2} \leq \sqrt{x_1^2 + \ldots + x_n^2} + \sqrt{y_1^2 + \ldots + y_n^2}$$

$$\iff$$

$$(x_1+y_1)^2+\ldots+(x_n+y_n)^2 \leq x_1^2+\ldots+x_n^2+y_1^2+\ldots+y_n^2+2\cdot\sqrt{(x_1^2 + \ldots + x_n^2) \cdot (y_1^2 + \ldots + y_n^2)}$$

$$\iff$$

$$2 \cdot (x_1 \cdot y_1 + \ldots + x_n \cdot y_n) \leq 2 \cdot \sqrt{(x_1^2 + \ldots + x_n^2) \cdot (y_1^2 + \ldots + y_n^2)}$$

$$\iff$$

$$(x_1 \cdot y_1 + ... + x_n \cdot y_n) \leq \sqrt{(x_1^2 + ... + x_n^2) \cdot (y_1^2 + ... + y_n^2)}$$

Observe that this is always true by Theorem 5.3.2, so we are done. $\square$

Lecture 6: 02/06/20

In this lecture, we define a measurement of sorts used to evaluate sets and categorize sets based on this measurement. We call this measurement cardinality. Please keep in mind that unless otherwise mentioned, we are considering the real numbers $\mathbb{R}$ to be made up of single numbers, not Dedekind cuts.

6.1 Cardinality

Definition 6.1.1: Cardinality

Let A and B be two different sets. We say that A and B have the same cardinality if there exists a bijective function (one-to-one correspondence) from A to B.

We denote this by $A \sim B$, where $\sim$ symbolizes an equivalent relation. Furthermore,

- $A \sim A$.
- If $A \sim B$, then $B \sim A$.
- If $A \sim B$ and $B \sim C$, then $A \sim C$ (transitivity).

Discussions on cardinality

Below are some facts on cardinality and terminology we will use. Assume that the A we use below is a given set.

- If A has no elements, then $A = \varnothing$.
- If $A \sim J_n = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$, we say that A has n elements and write $|A| = n$.
- If $A \sim \mathbb{N}$, we say that A is countable.
- If A is finite (meaning it has either 0 elements or n elements) or A is countable, we say that A is at most countable.
- If A is not at most countable and A is not countable, we say that A is uncountable.

Example 6.1.2: Cardinality

$\mathbb{Z} \sim \mathbb{N}$. In other words, $\mathbb{Z}$ is countable.

Proof: In order to prove that a set is countable, we have to design a bijection from $\mathbb{N}$ to our set of interest. In our case, the set of interest is $\mathbb{Z}$.

Let the function $f : \mathbb{N} \to \mathbb{Z}$ be as follows

$$f(x) = \begin{cases} f(1) = 0 & \\ f(2k) = k & \text{for } k \in \mathbb{N} \\ f(2k+1) = -k & \text{for } k \in \mathbb{N} \end{cases}$$

From here, it is clear that f is onto and one-to-one; each element of $\mathbb{Z}$ is covered, and each element of $\mathbb{Z}$ has a unique input from $\mathbb{N}$. □

Looking at the claim for Example 6.1.2 and its proof is sometimes hard to get over. After all, we know that $\mathbb{N} \subset \mathbb{Z}$ and $\mathbb{Z} \not\subset \mathbb{N}$, but we claim they have the same cardinality. This leads us to make the following remark.

Remark 6.1.3: Cardinality and subsets

Cardinality is not equivalent to comparing how many possible subsets exist on each set.

Lemma 6.1.4: $\mathbb{N}^2 \sim \mathbb{N}$

We claim that $\mathbb{N}^2 \sim \mathbb{N}$.

Proof: Like in Example 6.1.2, we must design a bijective function $f : \mathbb{N}^2 \to \mathbb{N}$.

For any $(i, j) \in \mathbb{N}^2$, we can define $k = i + j \geq 2$. Therefore, we can say that $f(i, j) = f(i, k - i)$. For all $k, i \in \mathbb{N}$, we can define $f(i, k - i)$ to be

$$f(i, k-i) = 1 + 2 + ... + (k-2) + (k-1) + 1 - i = \frac{k \cdot (k-1)}{2} + 1 - i$$

By examining f, we can see that it is bijective. We are done. □

We can use induction to show that $\mathbb{N}^k \sim \mathbb{N}$ for any $k \in \mathbb{N}$.

Note that in Lemma 6.1.4, we are basically putting $\mathbb{N}^2$ onto a grid and our mapping makes diagonal lines to uniquely cover all elements.

Lemma 6.1.5: Surjectivity and countability

Let A be a set such that there is an onto (surjective) map $f : \mathbb{N} \to A$. A is at most countable.

Proof: Note that we don't have that f is one-to-one, so it could be the case that f maps a lot of elements in $\mathbb{N}$ to a single element in A. Starting from $n \in \mathbb{N}$, look at $f(n)$ and compare it to $f(1), f(2), \dots, f(n-1)$. If $f(n)$ is the same as any one of these, delete n from the set.

By continuing this process, we end up with a set $K \subset \mathbb{N}$ such that $f : K \to A$ is bijective. Since K is at most countable and $A \sim K$, A is at most countable. $\square$

The following remark can be seen as the counterpart of Lemma 6.1.5.

Remark 6.1.6: Injectivity and countability

Let B be a set such that there is a 1-to-1 (injective) map $g : B \to \mathbb{N}$. B is at most countable.

Theorem 6.1.7: Union of countable sets

Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of countable sets. We have that $C = \bigcup_{k=1}^{\infty} A_k$ is also countable.

Proof: Note that since A_1 is infinite and $A_1 \subset C$, this implies that C is infinite as well.

As A_k is countable, we can write $A_k = \{a_{k_1}, a_{k_2}, \dots, a_{k_n}, \dots\}$ for each $k \in \mathbb{N}$. Recall that in Lemma 6.1.4 we were able to arrange the elements of $\mathbb{N}^2$ on a grid. By imagining a similar grid, let A_1 go in the first row, A_2 go in the second row, ..., A_n go in in the nth row, ...

By a similar process used in Lemma 6.1.4, we can find an onto map $h : \mathbb{N} \to C$. Note that h might not by one-to-one as an element in A_i might be the same as another element in A_j. This implies that C is at most countable. However, we previously deduced that C is infinite, so we conclude that C must be countable. $\square$

Lecture 7: 02/11/20

We now have all the pieces that are required to begin talking about sets in more depth. The discussions we have and the definitions we present in this lecture permeate through much of this course. We discuss metric distances, metric spaces, and define open and closed sets.

7.1 Metric distances

In this section, we seek to establish a way to measure the distance between elements of a set. In many ways, the distance measurement between elements of a set will largely determine how a particular set can be manipulated and evaluated. However, no matter what the measurement is, it must obey the following definition.

Definition 7.1.1: Metric distance

Let $X \neq \varnothing$ be a given set. We say that $d : X \times X \to \mathbb{R}$ is a metric distance if

i. $d(x, y) \geq 0$ for all $x, y \in X$, where $d(x, y) = 0 \iff x = y$.
ii. $d(x, y) = d(y, x)$ for all $x, y \in X$.
iii. For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

Note that we call Theorem 5.3.4 and (iii.) fromDefinition 7.1.1 by the same name. Therefore, when we say "triangle inequality" we have to look at the context of the situation to know what we are referencing.

Once we have an established metric distance, the definition of a metric space follows naturally.

Definition 7.1.2: Metric space

A metric space (X, d) is established if d is a valid metric distance on X.

We can look to prove some common metric distances as well as some examples of invalid distance metrics.

Lemma 7.1.3: Euclidean distance metric in $\mathbb{R}$

Let $X = \mathbb{R}$. Set $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. We claim that (X, d) is a metric space.

Proof: In order to verify that $(X, d) = (\mathbb{R}, d)$ is a metric space, we must show that the three conditions from Definition 7.1.1 hold.

i. It is certainly true that $d(x,y) = |x-y| \geq 0$. Furthermore, $d(x,y) = |x-y| = 0 \iff x-y=0 \iff x=y$.

ii. $d(x,y) = |x-y| = |y-x| = d(y,x)$.

iii. $d(x,y) = |x-y| = |(x-z)+(z-y)| \leq |x-z| + |z-y|$ by Theorem 5.3.4.

This concludes the proof. □

Note that $\widetilde{d}(x,y) = c \cdot |x-y|$ is a distance metric as well, where $c \in \mathbb{R}$ and $c > 0$.

Example 7.1.4: Metric distances

Let $X = \mathbb{R}$. Set $\rho(x,y) = \sqrt{|x-y|}$ for $x, y \in \mathbb{R}$. Is ρ a metric distance?

Proof: By work done in Lemma 7.1.3 we can see that the first two conditions of a metric distance hold for ρ. We will now verify the triangle inequality.

$$\rho(x,y) \leq \rho(x,z) + \rho(z,y) \iff \sqrt{|x-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|}$$

$$\iff$$

$$|x-y| \leq |x-z| + |z-y| + 2 \cdot \sqrt{|x-z| \cdot |z-y|}$$

This is certainly true since $|x-y| \leq |x-z| + |z-y|$, and we are simply adding another non-negative value to the right hand side. This concludes the proof. □

Example 7.1.5: Metric distances

Let $X = \mathbb{R}$. Set $\gamma(x,y) = |x-y|^2$. Is γ a metric distance?

Proof: Let's take a look at the triangle inequality. In order for the triangle inequality to hold for γ, we would need $|x-y|^2 \leq |x-z|^2 + |z-y|^2$. Let $x = 0$, $z = \frac{1}{2}$, $y = 1$. Certainly $1 \not\leq \frac{1}{2}$, so γ is not a metric distance. □

Example 7.1.6: Metric distances

For $X = \mathbb{R}^2$ with $x = (x_1, x_2)$ and $y = (y_1, y_2)$, show that $\lambda(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ is a distance metric.

Proof: We must verify the three conditions outlined in Definition 7.1.1

i. We know that the absolute value function is always non-negative. Furthermore,

$$\lambda(x,y) = 0 \iff x_1 = y_1 \text{ and } x_2 = y_2 \iff x = y$$

ii. For any two real numbers a and b, we know that $|a-b| = |b-a|$, so we declare $\lambda(x,y) = \lambda(y,x)$.

iii. Let $|x_1 - z_1| = a$, $|x_2 - z_2| = b$, $|z_1 - y_1| = c$, and $|z_2 - y_2| = d$. By Theorem 5.3.4 we know that

$$|x_1 - y_1| = |x_1 - z_1 + z_1 - y_1| \leq |x_1 - z_1| + |z_1 - y_1| = a + c$$

and by a similar process, we know $|x_2 - y_2| \leq b + d$. We see that

$$\lambda(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \leq \max\{a+c, b+d\} \leq \max\{a,b\} + \max\{c,d\}$$

since $a,b,c,d \geq 0$. Note that $\max\{a,b\} = \lambda(x,z)$ and $\max\{c,d\} = \lambda(z,y)$, so $\lambda(x,y) \leq \lambda(x,z) + \lambda(z,y)$.

This concludes the proof. □

Lemma 7.1.7: Euclidean distance metric in $\mathbb{R}^n$

Let $X = \mathbb{R}^n$. Set $d(x,y) = |x-y| = \sqrt{(x_1-y_1)^2 + \ldots + (x_n - y_n)^2}$ for $x_n, y_n \in \mathbb{R}^n$. We claim that (X,d) is a metric space.

Proof: We must show that the three conditions from Definition 7.1.1 hold. We won't go through the first two conditions (they are straightforward to prove) so we will jump straight to the triangle inequality.

Let $a = x - z$ and $b = z - y$ (this means $a, b \in \mathbb{R}^n$). Our problem then simplifies to showing that $|a+b| \leq |a| + |b|$. By Theorem 5.3.4 we know this to be a true statement, so we are done. □

It's important to note that if we're told that a set is a subset of $\mathbb{R}$ or $\mathbb{R}^n$ and a distance metric is not specified, we assume that the distance metric used is the Euclidean distance metric.

In general, we will typically assume that our metric space has an abstract framework (X,d). In other words, X is an arbitrary set and d is an arbitrary distance metric on that set.

7.2 Open and closed sets

Similar to how we used cardinality to categorize sets, we can also categorize sets by using distance metrics. We discuss two of the most common categorization of sets (open and closed) in this section.

Definition 7.2.1: Open ball

Let a metric space (X, d) be given and fix $x \in X$ and $r > 0$. We define an open ball of center x and radius r to be $\{y \in X : d(y, x) < r\}$.

Denote an open ball of center x and radius r by $B(x, r)$.

Definition 7.2.2: Closed ball

Let a metric space (X, d) be given and fix $x \in X$ and $r > 0$. We define a closed ball of center x and radius r to be $\{y \in X : d(y, x) \leq r\}$.

Denote a closed ball of center x and radius r by $\overline{B}(x, r)$.

Observe that both an open and closed ball contain the point x so neither are empty. Furthermore, other than making $r > 0$, we don't set a restriction on what r has to be, meaning that it can be any positive real number.

We now start looking at classifying specific points within a set. This will set up the framework for defining an open and closed set.

Definition 7.2.3: Limit point

Let a metric space (X, d) be given and have $E \subset X$ and $p \in X$. We say that p is a limit point of E if for all $r > 0$, we have

$$(E \setminus \{p\}) \cap B(p, r) = E \cap (B(p, r) \setminus \{p\}) \neq \varnothing$$

Denote the limit points of a set E by the set E'.

Example 7.2.4: Limit points

Consider $X = \mathbb{R}$, $d =$ Euclidean, and let $E = (0, 1)$. Is $p = 0$ a limit point of E?

Proof: We claim that p is a limit point of E. Observe that $(E \setminus \{0\}) \cap B(0, r) = (0, 1) \cap (-r, r) \neq \varnothing$ regardless of what we choose our $r > 0$ to be. ☐

Example 7.2.5: Limit points

Consider $X = \mathbb{R}$, $d =$ Euclidean, and let $F = (0,1) \cup \{2\}$. Is $q = 2$ a limit point of F?

Proof: We claim that q is NOT a limit point of F. Set $r = \frac{1}{2}$. We then see that $(F \setminus \{2\}) \cap B(2, \frac{1}{2}) = (0,1) \cap (1.5, 2.5) = \varnothing$. $\square$

We are now ready to use our knowledge of limit points to define a closed set.

Definition 7.2.6: Closed set

$E \subset X$ is closed if E contains all of its limit points. If E does not have any limit points, then E is automatically closed.

Remark 7.2.7: Vacuous truth on closed sets

The empty set $\varnothing$ is a closed set.

The governing set X is a closed set.

We now turn to interior points which will help us to construct the definition of an open set.

Definition 7.2.8: Interior point

Let $E \subset X$ and $p \in E$. We say that p is an interior point of E if we can find a radius $r > 0$ such that $B(p, r) \subset E$.

Denote the interior points of a set E by the set E^o.

Please note that in Definition 7.2.8, we are free to change r depending on what point p we are examining. In other words, if p is an interior point, that means we have found an $r_p > 0$ such that $B(p, r_p) \subset E$.

Definition 7.2.9: Open set

$E \subset X$ is open if every point in E is an interior point of E.

Remark 7.2.10: Vacuous truth on open sets

The empty set $\varnothing$ is an open set.

The governing set X is an open set.

A common mistake is to think that if a set E contains all of its interior points, then E is open. This is incorrect, since Definition 7.2.9 states that **every point in $\boldsymbol{E}$** must be an interior point.

Example 7.2.11: Open and closed sets

Let $\boldsymbol{X} = \mathbb{R}$ and have $\boldsymbol{E} = (\mathbf{0}, \mathbf{1}]$. Is $\boldsymbol{E}$ open in $\mathbb{R}$?
We claim that E is NOT open in $\mathbb{R}$. This means we need to find a point $p \in E$ such that p is not an interior point. If we look at $p = 1$, we see that for any $r > 0$, we have $1 + \frac{r}{2} \notin E$, so p is not an interior point, hence E is not open.

In fact, E is not closed as well since 0 is a limit point of E but $0 \notin E$.

We see from Example 7.2.11 that we can have sets that are neither open nor closed, and from Remark 7.2.7 and Remark 7.2.10 we can have sets that are both open and closed. This leads us to an important remark.

Remark 7.2.12: Open and closed relationship

In general, for a set $\boldsymbol{A} \subset \boldsymbol{X}$, $\boldsymbol{A}$ open $\not\Leftrightarrow$ $\boldsymbol{A}$ not closed.

In general, for a set $\boldsymbol{A} \subset \boldsymbol{X}$, $\boldsymbol{A}$ closed $\not\Leftrightarrow$ $\boldsymbol{A}$ not open.

Lecture 8: 02/13/20

In Lecture 7, we started to talk about metric spaces and defined opened and closed sets. We continue the discussion in this lecture and define some other categorizations of sets.

8.1 Metric spaces

We would like to categorize a couple more sets that we refer to later on in the course.

Definition 8.1.1: Bounded set

Let a set $C \subset X$ be given. C is bounded if there exists an $x \in X$ and $R > 0$ such that $C \subset B(x, R)$.

Please note how Definition 8.1.1 is just an extension of Definition 1.3.1.

Definition 8.1.2: Dense set

Let a set $D \subset X$ be given. D is dense in X if $X = D \cup D'$.

Recall that in Theorem 5.1.1, we declared that $\mathbb{Q}$ is dense in $\mathbb{R}$. If it is not clear why, please review Theorem 5.1.1 and try to see why this statement is true.

Definition 8.1.3: Perfect set

Let a set $E \subset X$ be given. E is perfect if E is closed and $E = E'$.

We now start to dive into open and closed sets some more. The following theorems will help us understand more about what an open or closed set must possess, and will give us tricks to declare whether a set is open or closed.

Theorem 8.1.4: Open ball is open

$B(x, r)$ is an open set.

Proof: Pick a point $z \in B(x, r)$. This means that $d(x, z) < r$. Let $s = r - d(x, z)$. We claim that $B(z, s) \subset B(x, r)$, which would make z an interior point of $B(x, r)$.

Take $p \in B(z, s)$. We need to show that $p \in B(x, r)$. Since $p \in B(z, s)$, we know that $d(z, p) < s = r - d(x, z)$, which means that $d(x, z) + d(z, p) < r$. By the triangle inequality, we know that $d(x, p) \leq d(x, z) + d(z, p) < r$. Therefore, $p \in B(x, r)$.

We have found a radius s for each point $z \in B(x, r)$ such that $B(z, s) \subset B(x, r)$, which means that each point in $B(x,r)$ is an interior point, so $B(x,r)$ is an open set. $\square$

Theorem 8.1.5: Closed ball is closed

$\overline{B}(x,r)$ is a closed set.

Proof: Take any point $z \notin \overline{B}(x,r)$. We claim that z is not a limit point of $\overline{B}(x,r)$.

Since $z \notin \overline{B}(x,r)$, we know that $d(x,z) > r$. Let's set $s = d(x,z) - r$. We make the claim that $B(z,s) \cap \overline{B}(x,r) = \varnothing$.

Let's take $y \in \overline{B}(x,r)$. This means that $d(x,y) \leq r$. By the triangle inequality, we see that

$$d(x,y) + d(y,z) \geq d(x,z) \implies d(y,z) \geq d(x,z) - d(x,y) \geq d(x,z) - r = s$$

$$\implies y \notin B(z,s)$$

Therefore, it is true that $B(z,s) \cap \overline{B}(x,r) = \varnothing$.

We have proven that all points outside of $\overline{B}(x,r)$ are not limit points, meaning that $\overline{B}(x,r)$ either contains all of its limit points or that is doesn't have any limit points. Either situation implies that $\overline{B}(x,r)$ is closed, so we are done. $\square$

Before we move any further, it's important to understand some notation. Please note that if $A \subset X$, then $X \setminus A$ is equivalent to A^c. Both denote the complement of A.

Theorem 8.1.6: Open set implies closed complement

Let $A \subset X$ be given. A is open $\iff$ $X \setminus A$ is closed.

Proof: Since we are looking to prove an if-and-only-if statement, we must show that each side implies the other.

i. Assume that A is open. Since A is open, we see that for each $x \in A$, there is an $r > 0$ such that $B(x,r) \subset A$. This means that $B(x,r) \cap A^c = \varnothing$, meaning that x is not a limit point of A^c. Since $x \notin A^c$, this means that A^c is closed.

ii. Assume that A^c is closed. This means that for $x \notin A^c$, we have that x is not a limit point of A^c. This implies that there exists an $r > 0$ such that

$$B(x,r) \cap (A^c \setminus \{x\}) = \varnothing \implies B(x,r) \cap A^c = \varnothing \implies B(x,r) \subset A$$

Note that since $x \notin A^c$, this means that $x \in A$. Therefore, x is an interior point of A, so A is open.

This concludes the proof. $\square$

Theorem 8.1.7: Limit points and infinite sets

Let $\boldsymbol{A \subset X}$ be given, and let $\boldsymbol{p \in X}$ be a limit point of $\boldsymbol{A}$. For each $\boldsymbol{r > 0}$, we have $\boldsymbol{B(p, r) \cap A}$ to be an infinite set (contains infinitely many elements).

Proof: Assume that there is an $R > 0$ such that $B(p, R) \cap A$ is a finite set. This means that $B(p, R) \cap A = \{a_1, a_2, \ldots, a_k\}$. For all values $i \in \{1, 2, \ldots, k\}$ where $a_i \neq p$, let $r = \min\{d(p, a_i)\}$ and then let $s = \frac{r}{2}$. This results in $B(p, s) \cap (A \setminus \{p\}) = \varnothing$, which contradicts the definition of p being a limit point (see Definition 7.2.3).

This concludes the proof. $\square$

The above proof implies that if A has a limit point p, then A contains infinitely many elements. Furthermore, for $k \in \mathbb{N}$, pick $a_k \in B(p, \frac{1}{k}) \cap (A \setminus \{p\})$. We can find a sequence $\{a_k\} \subset (A \setminus \{p\})$ such that $\{a_k\} \to p$. This is proven directly in Lemma 13.1.2.

The following remark is a direct consequence of Theorem 8.1.7.

Remark 8.1.8: Finite sets are closed

If a set $\boldsymbol{A \subset X}$ is finite, then $\boldsymbol{A}$ has no limit points, so $\boldsymbol{A}$ is closed.

We now will introduce some more notation. Let the set I be an abstract set and let $i \in I$ be a running variable. Essentially, we are not restricting I to be anything, so it can morph into whatever it likes to fit the scenario. We use this notation in the following remark regarding complements with open and closed sets.

Remark 8.1.9: De Morgan's laws

$$\boldsymbol{X \setminus \left(\bigcup_{i \in I} \Theta_i\right) = \bigcap_{i \in I} \left(X \setminus \Theta_i\right)}$$

$$\boldsymbol{X \setminus \left(\bigcap_{i \in I} \Theta_i\right) = \bigcup_{i \in I} \left(X \setminus \Theta_i\right)}$$

We will now examine properties of unions of open and closed sets.

Theorem 8.1.10: Union of open sets is open

If $\{U_i\}_{i\in I}$ is a collection of open sets, then $\bigcup_{i\in I} U_i$ is open.

Proof: We would like to prove that any point in the union $\bigcup_{i\in I} U_i$ is an interior point. Pick any point $x \in \bigcup_{i\in I} U_i$. This means that there exists a $k \in I$ such that $x \in U_k$. Since U_k is open, this means that there exists an $r > 0$ such that $B(x, r) \subset U_k$. Since $U_k \subset \bigcup_{i\in I} U_i$, this means that $B(x, r) \subset U_k \subset \bigcup_{i\in I} U_i$, so x is an interior point. $\square$

Theorem 8.1.11: Intersection of closed sets is closed

If $\{F_i\}_{i\in I}$ is a collection of closed sets, then $\bigcap_{i\in I} F_i$ is closed.

Proof: Note from Theorem 8.1.6 that F_i being closed means F_i^c is open. From Theorem 8.1.10, we know that $\bigcup_{i\in I} F_i^c$ is open. By Remark 8.1.9, we see that $\bigcup_{i\in I} F_i^c = X \setminus (\bigcap_{i\in I} F_i)$, which is the complement of $\bigcap_{i\in I} F_i$. By referencing Theorem 8.1.6 again, we see that $\bigcap_{i\in I} F_i$ is closed since $X \setminus (\bigcap_{i\in I} F_i)$ is open. $\square$

Theorem 8.1.12: Finite intersection of open sets is open

If $U_1, U_2, \ldots, U_k$ are open, then $\bigcap_{i=1}^k U_i$ is open.

Proof: To make things nicer, define $K = \{1, 2, \ldots, k\}$. Let's pick $x \in \bigcap_{i=1}^k U_i$. This means that $x \in U_i$ for all $i \in K$. Since U_i is open, there exists an $r_i > 0$ such that $B(x, r_i) \subset U_i$ for $i \in K$. Let $r = \min_{i\in K} r_i$. We then see that $B(x, r) \subset B(x, r_i) \subset U_i$ for any $i \in K$. This implies that $B(x, r) \subset \bigcap_{i=1}^k U_i$, so x is an interior point of $\bigcap_{i=1}^k U_i$, meaning $\bigcap_{i=1}^k U_i$ is an open set. $\square$

A common question is to ask about the outcome of an infinite intersection of open sets. Let's look at the collection of sets $\{U_k\}$ where $U_k \subset \mathbb{R}$ and $U_k = B(0, \frac{1}{k})$. When we take the intersection $\bigcap_{k\in\mathbb{N}} U_k$, we get the set $\{0\}$. There cannot exist another point p in the set besides 0 because there will always be a $k \in \mathbb{N}$ where the ball shrinks enough so that $p \notin \bigcap_{k\in\mathbb{N}} U_k$. Notice further that $\{0\}$ is closed but not open (0 is not an interior point of $\{0\}$), so the conclusion made in Theorem 8.1.12 requires a FINITE intersection.

Theorem 8.1.13: Finite union of closed sets is closed

If $F_1, F_2, \ldots, F_k$ are closed, then $\bigcup_{i=1}^k F_i$ is closed.

Proof: From Remark 8.1.9, we know that $X \setminus \bigcup_{i=1}^k F_i = \bigcap_{i=1}^k (X \setminus F_i)$. By Theorem 8.1.6, we know that $X \setminus F_i$ is an open set (since F_i is closed), so by Theorem 8.1.12

we know that $\bigcap_{i=1}^{k}(X \setminus F_i)$ is open, which means that $\bigcup_{i=1}^{k} F_i$ is closed (using Theorem 8.1.6 again). $\square$

Example 8.1.14: Metric spaces

Let (X, d) be a metric space and assume $x, y \in X$ satisfies $d(x, y) = R > 0$. Prove that

$$\big(B(x, 2R) \cup B(y, 3R)\big) \subset B(x, 4R)$$

Proof: Let $p \in \big(B(x, 2R) \cup B(y, 3R)\big)$ be given. This means that $p \in B(x, 2R)$ or $p \in B(y, 3R)$. We will examine each case.

i. Suppose that $p \in B(x, 2R)$. This means

$$p \in \{a : d(x, a) < 2R\} \implies d(x, p) < 2R < 4R$$

so p is contained in $B(x, 4R)$.

ii. Suppose $p \in B(y, 3R)$. This means that $p \in \{b : d(b, y) < 3R\}$. By the triangle inequality, we know that

$$d(x, p) \leq d(x, y) + d(y, p) \implies d(x, p) < R + 3R = 4R$$

so p is contained in $B(x, 4R)$.

Therefore, $\big(B(x, 2R) \cup B(y, 3R)\big) \subset B(x, 4R)$. $\square$

Example 8.1.15: Metric spaces

Let (X, d) be a metric space and let $E \subset X$. Show that E^o, or the set of all interior points of E, is an open set.

Proof: Let $p \in E^o$, meaning that p is an interior point of E. This means that there exists some r' such that $B(p, r') \subset E$. Let $q \in B(p, r')$. We can say that $B\big(q, r' - d(p, q)\big) \subset B(p, r') \subset E$, which means that $q \in E^o$. In summary, for every point $p \in E^o$, we know $B(p, r') \subset E^o$, so E^o is an open set. $\square$

A point of confusion may be why $B\big(q, r' - d(p, q)\big) \subset B(p, r')$. Let $f \in B\big(q, r' - d(p, q)\big)$. This means that $d(f, q) < r' - d(p, q)$. By the triangle inequality,

$$d(p, f) \leq d(p, q) + d(q, f) < d(p, q) + r' - d(p, q) = r'$$

Therefore, any point $f \in B\big(q, r' - d(p,q)\big)$ is less than a distance r' away from p, meaning that $B\big(q, r' - d(p,q)\big) \subset B(p, r')$.

Lecture 9: 02/18/20

In this lecture, we continue talking about metric spaces and introduce metric subspaces and compactness. Please make sure to review Lecture 7 and Lecture 8 before continuing since a lot of this stuff, especially compactness, refers to what we discussed and defined in past lectures.

9.1 More on metric spaces

Definition 9.1.1: Closure of a set

Let $E \subset X$. We define the closure of E to be the union of E with its limit points E'.

Denote the closure of a set E by the $\overline{E}$. In set notation, $\overline{E} = E \cup E'$.

Note that we also used the notation $E \cup E'$ when we defined dense sets in Definition 8.1.2. This means that a set is dense if and only if its closure is the governing set.

Theorem 9.1.2: Properties of the closure

The following hold:

i. $\overline{E}$ is closed.

ii. E is closed $\iff$ $E = \overline{E}$.

iii. If $E \subset F$ and F is closed, then $\overline{E} \subset F$.

Proof: We will break this proof up into three parts corresponding to the three different statements made.

i. Surely $\overline{E}$ contains all limit points of E, but it's not clear whether $\overline{E}$ contains all limit points of $\overline{E}$. Let $p \notin \overline{E}$. Since $p \notin E$ and $p \notin E'$, we can find an $r > 0$ such that $B(p, r) \cap E = \varnothing$. We still need to make sure that $B(p, r)$ does not contain any elements of E'. For every $z \in B(p, r)$, let $s = r - d(p, z)$. It's then true that $B(z, s) \subset B(p, r)$. This means that $B(z, s) \cap E = \varnothing$, so $z \notin E$ and $z \notin E'$, meaning that $B(p, r) \cap \overline{E} = \varnothing$. Therefore, p is not a limit point of $\overline{E}$, so $\overline{E}$ is closed.

ii. We're proving an if-and-only-if statement so we must show that each side implies the other. If E is closed, then surely $E' \subset E$ (since E contains all of its limit points), so $\overline{E} = E \cup E' = E$. If $E = \overline{E}$, then by (i.) we have $\overline{E}$ to be closed, so certainly $E = \overline{E}$ is closed as well.

iii. We have $E \subset F$ and F being closed. This means that $F' \subset F$, and by (ii.) we have that $\overline{F} = F$. Please note that since $E \subset F$, then $E' \subset F'$, since if all balls around a point p has a nonempty intersection with E, then surely it has a nonempty intersection with F. Therefore, $E \cup E' \subset F \cup F'$, and since $F' \subset F$ due to F being closed, this means that $\overline{E} \subset F$.

This concludes the proof. $\square$

Theorem 9.1.3: Supremum contained in the closure

Let a set $E \subset \mathbb{R}$ be bounded. We have that $\sup E \in \overline{E}$.

Proof: It is straightforward that if $\sup E \in E$, then $\sup E \in \overline{E}$. Let's examine the other case.

If $\sup E \notin E$, we need $\sup E \in E'$ for the theorem to hold. Take any $r > 0$ and let $c = \sup E - \frac{r}{2} < \sup E$. By Definition 2.2.1 we know c is not an upper bound of E, so we can find an $x \in E$ such that $c \leq x < \sup E$. Therefore, we know that $E \cap (B(\sup E, r) \setminus \sup E) \neq \varnothing$, and since r is arbitrary, we have that $\sup E$ is a limit point, so $\sup E \in \overline{E}$. $\square$

Although we do not prove it, we could follow a similar process to show that $\inf E \in \overline{E}$ as well.

A common hurdle to get over with the proof for Theorem 9.1.3 is that we can find an $x \in E$ such that $c \leq x < \sup E$. Consider if this were not the case, meaning that we couldn't find such an x. This would essentially mean that $\sup E$ is on an island by itself, meaning that $\sup E = \max E$, and thus would fall under the case where $\sup E \in E$.

There's a common misconception regarding the interior points and their closure. The following remark discusses this topic.

Remark 9.1.4: Closure of interior points

Please note that $\overline{E^o} \neq \overline{E}$.

To emphasize this, let our metric space be $\mathbb{R}$ with the Euclidean distance metric and let $E = \mathbb{N}$. Surely $E^o = \varnothing \implies \overline{E^o} = \varnothing \neq \overline{E} = \mathbb{N}$.

9.2 Metric subspaces

Metric subspaces is basically restricting the amount of points that we can select from our governing set X. This stuff gets kind of dicey, so please take note of the notation with the following definitions.

Definition 9.2.1: Metric subspace

Let (X, d) be a metric space and a set $Y \neq \varnothing$ such that $Y \subset X$. A metric subspace (Y, d_Y) can be defined where $d_Y(p, q) = d(p, q)$ for $p, q \in Y$.

Definition 9.2.2: Open ball in a metric subspace

Let $x \in Y$ and $r > 0$. We declare $B_Y(x, r) = \{y \in Y : d_Y(x, y) = d(x, y) < r\} = Y \cap B(x, r)$.

Take note that even though we're dealing with a metric subspace, we don't restrict the values that r can take. It's also important to see that a ball can have a very odd shape with metric subspaces depending on what Y is and where we choose our $x \in Y$.

Example 9.2.3: Metric subspaces

Let $\boldsymbol{X = \mathbb{R}}$ with the Euclidean distance metric and have $\boldsymbol{Y = [1, 9)}$. Find $\boldsymbol{B_Y(2, 3)}$.

From Definition 9.2.2, we see that $B_Y(2, 3) = Y \cap B(2, 3)$. We know $B(2, 3) = (-1, 5)$ and $Y = [1, 9)$, so $B_Y(2, 3) = [1, 9) \cap (-1, 5) = [1, 5)$.

9.3 Compactness

Compactness is often cited as an extremely profound and important concept in mathematics. One way to grasp the mystique that surrounds this concept is that no matter how we openly surround (or cover) a set, compactness allows us to reduce our analysis to a finite number of sets. We state this explicitly in the following definition.

Definition 9.3.1: Compactness

Let (X, d) be a metric space and let $K \subset X$. We say that K is compact if for any given collection of open sets $\{U_\alpha\}_{\alpha \in A}$ such that $K \subset \bigcup_{\alpha \in A} U_\alpha$, we can extract out a finite subcover. In other words, there exist an $\alpha_1, \alpha_2, \ldots, \alpha_m \in A$ such that $K \subset \bigcup_{i=1}^{m} U_{\alpha_i}$.

It should be clear by Definition 9.3.1 that if we can find an open cover of $K \subset X$ such that a finite subcover cannot be extracted, then K is not compact.

Example 9.3.2: Compactness

$\boldsymbol{K = \{0\}}$ **is compact.**

Proof: Let $\{U_\alpha\}_{\alpha \in A}$ be a family of open sets in $\mathbb{R}$ such that $\{0\} \subset \bigcup_{\alpha \in A} U_\alpha$. There is a $\beta \in A$ such that $0 \in U_\beta \implies \{0\} \subset U_\beta$. Therefore, we can find a finite subcover of just one element, specifically U_β, that covers $\{0\}$. $\square$

We can use the thinking we employed in Example 9.3.2 to craft the proof for the following theorem.

Theorem 9.3.3: Any finite set is compact

Let $\boldsymbol{(X, d)}$ be a metric space. Any finite set $\boldsymbol{A \subset X}$ is compact.

Proof: Since A is finite, we can say that $A = \{a_1, a_2, \dots, a_n\}$ where $n \in \mathbb{N}$. Let U be an arbitrary open cover of A such that $U = \bigcup_{i \in I} \Theta_i$ where Θ_i is an open set, I is an abstract set, and i is a running variable.

Since U is an open cover, we know that for all $k \in \mathbb{N}$ with $k \leq n$, there exists an $i \in I$ such that $a_k \in \Theta_i$. Let $a_k \in \Theta_{i_j}$ where $j \in \{1, 2, \dots, k\} = K$ and $i_j \in I$. Therefore, $A \subset \bigcup_{j \in K} \Theta_{i_j}$, which is precisely a finite subcover of A. $\square$

Example 9.3.4: Compactness

$\boldsymbol{Y = \mathbb{R}}$ **is not compact.**

Note that $\mathbb{R} = \bigcup_{k \in \mathbb{Z}} (k-1, k+1) = \bigcup_{k \in \mathbb{Z}} B(k, 1)$. We can't extract a finite subcover with this particular open covering, and this is enough to show that Y is not compact.

Lecture 10: 02/25/20

We continue our discussion on compactness and spend time debunking a common assumption about compact sets. Please review Section 9.3 before starting this lecture.

10.1 More on compactness

A set being compact can help us out in many ways. We will go over some of the implications from a set being compact.

Lemma 10.1.1: Compactness implies boundedness

If a set K is compact, then K is bounded.

Proof: Let's look at our metric space X. If we take $z \in X$, we see that $X = \bigcup_{n\in\mathbb{N}} B(z,n)$. In particular, we see that $K \subset \bigcup_{n\in\mathbb{N}} B(z,n)$. Note that $\bigcup_{n\in\mathbb{N}} B(z,n)$ is an open cover of K.

Since K is compact, we can find $n_1, n_2, \dots, n_m \in \mathbb{N}$ such that $K \subset \bigcup_{i=1}^{m} B(z,n_i)$. Define $M = \{1, 2, \dots, m\}$, and let $N = \max_{i\in M} n_i$. We then see that $K \subset B(z,N)$, so K is bounded. □

Lemma 10.1.2: Compactness implies closedness

If a set K is compact, then K is closed.

Proof: Assume that K is not closed. This means there exists a $p \notin K$ where p is a limit point of K. For each $x \in K$, we know that $d(x,p) > 0$, so let $r_x = \frac{1}{4} \cdot d(x,p) > 0$. Of course $x \in B(x,r_x)$, so $K \subset \bigcup_{x\in K} B(x,r_x)$.

By compactness, we can find $x_1, x_2, \dots, x_m \in K$ such that $K \subset \bigcup_{i=1}^{m} B(x_i, r_{x_i})$. Define $M = \{1, 2, \dots, m\}$ and let $r = \min_{i\in M} r_{x_i} > 0$. Then, $B(x_i, r_{x_i}) \cap B(p,r) = \varnothing$ for each $i \in M$, meaning that

$$\left(\bigcup_{i=1}^{m} B(x_i, r_{x_i})\right) \cap B(p,r) = \varnothing \implies K \cap B(p,r) = \varnothing$$

This contradicts the statement that p is a limit point of K, so we conclude that K is closed. □

Theorem 10.1.3: Compactness implies closedness and boundedness

If a set K is compact, then K is closed and bounded.

Proof: This is a direct consequence of Lemma 10.1.2 and Lemma 10.1.1, so the proof is already done for us. $\square$

Note that in a general metric space, a set K being closed and bounded does NOT imply that K is compact.

The following example builds upon the comment we made at the end of Theorem 10.1.3.

Example 10.1.4: Closed and bounded $\not\Longrightarrow$ compact

Find a closed and bounded set that is not compact.

Consider the metric space where $X = \mathbb{R}$ and the distance metric is as follows:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

We claim that $\mathbb{R}$ is closed and bounded but not compact.

i. By Remark 7.2.7, we know that $X = \mathbb{R}$ is a closed set. Furthermore, by our distance metric, we see that $\mathbb{R} \subset B(0, 2)$, so $\mathbb{R}$ is bounded.

ii. We know that for $x \in \mathbb{R}$, $B(x, \frac{1}{2}) = \{x\}$, so $\mathbb{R} = \bigcup_{x \in \mathbb{R}} B(x, \frac{1}{2})$, which is an open cover of $\mathbb{R}$. If we extract a finite number of balls from our open cover, we will not be able to cover $\mathbb{R}$ since each ball contains only its center. Therefore, $\mathbb{R}$ is not compact.

Lemma 10.1.5: Closed subsets of compact sets are compact

Let $C \subset K \subset X$. Assume that K is compact and C is closed. This implies that C is compact.

Proof: Take an arbitrary open cover $\{\Theta_i\}_{i \in I}$ of C. In other words, $C \subset \bigcup_{i \in I} \Theta_i$. As C is closed, we know that $X \setminus C$ is open. In addition, we have that $X = (\bigcup_{i \in I} \Theta_i) \cup (X \setminus C)$. In particular, we have that $K \subset (\bigcup_{i \in I} \Theta_i) \cup (X \setminus C)$, which is an open cover of K.

Since K is compact, we can extract a finite subcover of K. We now take this finite subcover and remove $X \setminus C$ from it if it's in there. From our arbitrary open cover $\{\Theta_i\}_{i \in I}$ of C we have found a finite subcover, so the proof is done. $\square$

Lemma 10.1.6: Nonempty intersection of compact sets

Let $\{K_\alpha\}_{\alpha\in A}$ be a family of compact sets such that the intersection of every finite subcollection of $\{K_\alpha\}_{\alpha\in A}$ is nonempty. We then have that $\bigcap_{\alpha\in A} K_\alpha \neq \varnothing$.

Proof: Let's employ contradiction and assume otherwise that $\bigcap_{\alpha\in A} K_\alpha = \varnothing$.

Fix $\alpha_1 \in A$ and let $B = A \setminus \{\alpha_1\}$. We see that

$$K_{\alpha_1} \cap \bigcap_{\alpha\in B} K_\alpha = \varnothing \implies K_{\alpha_1} \subset X \setminus \bigcap_{\alpha\in B} K_\alpha = \bigcup_{\alpha\in B} (X \setminus K_\alpha)$$

By Theorem 10.1.3 we know that K_α is closed, so $X \setminus K_\alpha$ is open, meaning that we have found an open cover of K_{α_1}.

By compactness, we can find $\alpha_2, \alpha_3, \dots, \alpha_m \in A$ such that

$$K_{\alpha_1} \subset \bigcup_{i=2}^{m} (X \setminus K_{\alpha_i}) = X \setminus \left(\bigcap_{i=2}^{m} K_{\alpha_i}\right) \implies K_{\alpha_1} \cap \bigcap_{i=2}^{m} K_{\alpha_i} = \bigcap_{i=1}^{m} K_{\alpha_i} = \varnothing$$

However, this violates the declaration that every finite subcollection of $\{K_\alpha\}_{\alpha\in A}$ is nonempty, so the proof is concluded. $\square$

Furthermore, if we let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty compact sets such that $K_1 \supset K_2 \supset \dots \supset K_m \supset \dots$, then $\bigcap_{n\in\mathbb{N}} \neq \varnothing$.

Lemma 10.1.6 gets pretty tricky with the implications. Please realize that we assumed that $\bigcap_{\alpha\in A} K_\alpha = \varnothing$, which means that for K_{α_a}, it must be a subset of $X \setminus \bigcap_{\alpha\in B} K_\alpha$. We then find our finite subcover and basically work backwards to see that we arrive at a contradiction.

Below is a lemma we did not prove in lecture but was something we used in the course.

Lemma 10.1.7: Limit points and infinite subsets

A set K is compact if and only if every infinite set $H \subset K$ has a limit point in K.

Proof: Since we are proving an if-and-only-if statement, we have to show that each side implies the other.

i. If K is compact, let our infinite subset H be given. Assume that H has no limit points in K. Therefore, for each point $k \in K$, we could find a radius r_k such that $B(k, r_k)$ contains at most one point $h \in H$. Note if $B(k, r_k)$ contains h, then $h = k$. We see that $\bigcup_{k \in K} B(k, r_k)$ is an open cover of K, so we can extract a finite number of balls that cover K. However, this would mean that H is finite, which is a contradiction.

ii. Proving the other implication is quite difficult. It was an exercise in the textbook so there are solutions online, but it was not clear enough to include the proof in this text.

This concludes the proof. $\square$

Lecture 11: 02/27/20

In this lecture, we continue our discussion on compactness and spend a fair amount of time examining compactness in $\mathbb{R}^n$.

11.1 Even more on compactness

This last section of general compactness looks at unions and intersections of compact sets.

Theorem 11.1.1: Finite union of compact sets is compact

Let (X, d) be a metric space, and let $\{K_n\}_{n\in\mathbb{N}}$ be a family of compact sets in X. We have that $K_1 \cup K_2 \cup \ldots \cup K_m$ is compact for any $m \in \mathbb{N}$.

Proof: Let $C = \bigcup_{i\in I} \Theta_i$ be an open covering of $K_1 \cup K_2 \cup \ldots \cup K_m$. This implies that C is an open cover of $K_1, K_2, \ldots, K_m$. Since we have that K_i is compact for all $i \in \mathbb{N}$, we can find a finite subcover from C of K_j where $j \in \mathbb{N}$ and $j \leq m$. Let's label these finite subcovers as G_j.

Let $W = G_1 \cup G_2 \cup \ldots \cup G_m$. W covers $K_1 \cup K_2 \cup \ldots \cup K_m$ and W contains finitely many sets from C (since W it is finite union of finitely many sets from C). Therefore, from an arbitrary open cover of $K_1 \cup K_2 \cup \ldots \cup K_m$, we have found a finite subcover, meaning that $K_1 \cup K_2 \cup \ldots \cup K_m$ is compact. □

Note that this union of compact sets MUST be finite. To emphasize this, consider $X = \mathbb{R}$ and the union $\bigcup_{n\in\mathbb{N}}[0, n]$ with the open covering of $\bigcup_{n\in\mathbb{N}}(-1, n+1)$. A finite subcover does not exist with this open covering of an infinite union of compact sets.

Theorem 11.1.2: Intersection of compact sets is compact

Let $\{F_{i\in I}\}$ be a family of compact sets where I is an abstract set with i being a running variable. We have that $\bigcap_{i\in I} F_i$ is compact.

Proof: By nature of the intersection, for all $i \in I$ we have that $F_i \supset \bigcap_{i\in I} F_i$. Furthermore, by Theorem 10.1.3, F_i being compact means that F_i is closed, and by Theorem 8.1.11 we know $\bigcap_{i\in I} F_i$ is closed since any intersection of closed sets is closed.

By Lemma 10.1.5, if $C \subset K \subset X$ with C closed and K compact, then C is compact. Let $K = F_1$ and $C = \bigcap_{i\in I} F_i$. Since $\bigcap_{i\in I} F_i$ is closed and F_1 is compact, we have $\bigcap_{i\in I} F_i$ to be compact. □

11.2 Compactness of the reals

We often find ourselves working in the real number line $\mathbb{R}$ or its counterpart $\mathbb{R}^n$. In this section, we examine how compactness operates while in these two parent sets.

Theorem 11.2.1: Closed cell in $\mathbb{R}^n$ is compact

Consider $X = \mathbb{R}^n$ with the Euclidean distance metric. A cell $C = [a_1, b_1] \times ... \times [a_n, b_n]$ is compact.

Proof: We will only consider $n = 1$ for our proof, but it could naturally be extended for other values of n. Therefore, our setting is that $X = \mathbb{R}$ with the Euclidean distance metric and $C_0 = [a, b] \subset \mathbb{R}$. We will assume otherwise that C_0 is not compact.

We do the following process iteratively:

i. Divide C_0 into two equal subintervals of length $(\frac{b-a}{2})$.
ii. At least one of the two intervals does not have a finite subcover from an open cover $\{\Theta_i\}_{i \in I}$ (since we assumed C_0 is not compact). Name this subinterval C_1.

We then get a sequence of intervals $\{C_k\}$ such that

- $C_k = [a_k, b_k]$ with length $\frac{b-a}{2^k}$.
- $C_0 \supset C_1 \supset ... \supset C_k \supset ...$.
- C_k does not have a finite subcover from $\{\Theta_i\}_{i \in I}$.

Observe that $[a_0, b_0] \supset [a_1, b_1] \supset ...$, which implies that $a_0 \leq a_1 \leq ... \leq a_n \leq ... \leq b_k$ for each $k \in \mathbb{N}$. Let $A = \{a_n : n \geq 0\}$. We see that A is bounded (above by b_k and below by a_0), and since $A \subset \mathbb{R}$, we know that $\sup A$ exists (by Theorem 4.1.10). Let $x = \sup A$. Certainly $x = \sup A \leq b_k$ since b_k is an upper bound of A, meaning that

$$a_k \leq x \leq b_k \implies x \in C_k = [a_k, b_k] \implies x \in \bigcap_{k \in \mathbb{N}} C_k$$

Can $\bigcap_{k \in \mathbb{N}} C_k$ have more than one point? Assume that there are $x \neq y$ such that $x, y \in \bigcap_{k \in \mathbb{N}} C_k$. This implies that $|x - y| \leq$ length of $C_k = \frac{b-a}{2^k}$. This is absurd though, since for n large enough we could certainly have $\frac{b-a}{2^n} < |x - y|$. We then conclude that $\bigcap_{k \in \mathbb{N}} C_k = \{x\}$.

As $x \in [a, b] \subset \bigcup_{i \in I} \Theta_i$, we can find a $j \in I$ such that $x \in \Theta_j$. Since Θ_j is open, x is an interior point of Θ_j, implying the existence of an $r > 0$ such that $B(x, r) \subset \Theta_j$. For $k \in \mathbb{N}$ large enough, we have that the length of $C_k = \frac{b-a}{2^k} < r$. Therefore,

$x \in C_k \subset B(x, r) \subset \Theta_j$, which is a contradiction of C_k not having a finite subcover. Since this contradicted an implication of the cell C not being compact, we conclude that C is compact. □

A common question is if the proof for Theorem 11.2.1 works for an open cell like $U = (a, b)$. We can't repeat the proof since if we have $U_k = (a_k, b_k)$, it could be that $\bigcap_{k \in \mathbb{N}} U_k = \varnothing$. For example, we can have $U_k = (0, \frac{1}{2^k})$, but $\bigcap_{k \in \mathbb{N}} (0, \frac{1}{2^k}) = \varnothing$.

We now discuss a theorem that is often attempted to be applied to compactness in general. Please note that we have a specific setting to the theorems in this section and the following is no exception.

Theorem 11.2.2: Heine-Borel Theorem

Let $X = \mathbb{R}^n$ with the Euclidean metric and let $E \subset \mathbb{R}^n$. The following are equivalent:

i. E is compact.
ii. E is closed and bounded.
iii. Every infinite subset of E has a limit point in E.

Proof: We are going to set up a chain of implications to prove this theorem. Note that we did (ii.) $\implies$ (i.) in Theorem 11.2.1 and (i.) $\implies$ (iii.) was done in Lemma 10.1.7, so all we have to prove is (iii.) $\implies$ (ii.) to complete the proof.

If E is not bounded, then for each $n \in \mathbb{N}$ we can find an $x_n \in E$ such that $|x_n| > n$. Then $I = \{x_n : n \in \mathbb{N}\}$ does not have a limit point in E, which contradicts our assumption that every infinite subset of E has a limit point in E, so E must be bounded.

To show that E is closed, pick any point p that is a limit point of E. Then, for each $n \in \mathbb{N}$, we can find a $y_n \in (B(p, \frac{1}{n}) \setminus \{p\}) \cap E$. Let $J = \{y_n : n \in \mathbb{N}\}$. J has only one limit point, which is p, so by (iii.) we know $p \in E \implies E$ is closed.

We have shown that (iii.) $\implies$ (ii.) $\implies$ (i.) $\implies$ (iii.), so the proof is complete. □

Lecture 12: 03/03/20

We look more into perfect sets. Perfect sets were not tested on during the course, so there may not be much connection with that part of lecture throughout the rest of the notes. At the end of the lecture we introduce sequences.

12.1 Perfect sets

Recall from Definition 8.1.3 that a perfect set P is a closed set where every point $p \in P$ is a limit point of P. In other words, $P' = P = \overline{P}$. If we let $X = \mathbb{R}$ with the Euclidean metric, we see that $A = \mathbb{R}$ and $B = [0, 5]$ are perfect sets whereas $C = [0, 6)$ and $D = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ are not perfect sets.

Theorem 12.1.1: Perfect sets and uncountability in $\mathbb{R}^n$

Let $X = \mathbb{R}^n$ with the Euclidean distance metric and let $A \subset \mathbb{R}^n$ be a perfect set. This implies that A is uncountable.

Proof: We will assume that A is at most countable and arrive at a contradiction. Surely A cannot be finite by the comment made at the end of Theorem 8.1.7, so A must be countable.

Since A is countable, we can write $A = \{x_n : n \in \mathbb{N}\}$. Based on the setting of the problem, A is closed and x_n is a limit point of A for every $n \in \mathbb{N}$. We can do the following process iteratively:

i. Let $y_1 = x_1$ and $r_1 = 1$. By Theorem 8.1.7, we know that $(B(y_1, r_1) \setminus \{y_1\}) \cap A$ is nonempty and infinite.
ii. In the intersection $(B(y_1, r_1) \setminus \{y_1\}) \cap A$, pick x_k such that k is the smallest possible index. Let $y_2 = x_k$ and have $r_2 = \frac{1}{2} \cdot \min\{|y_2 - y_1|, 1 - |y_2 - y_1|\}$.
iii. We have $B(y_2, r_2) \subset B(y_1, r_1)$ and $y_1 \notin \overline{B}(y_2, r_2)$. Select y_3 in a similar manner as how we chose y_2 and let $r_3 = \frac{1}{2} \cdot \min\{|y_3 - y_2|, r_2 - |y_3 - y_2|\}$.

Through this process, we've constructed a collection of balls $\{B(y_k, r_k)\}_{k \in \mathbb{N}}$ such that

i. $B(y_1, r_1) \supset B(y_2, r_2) \supset ... \supset B(y_k, r_k) \supset ...$.
ii. $\overline{B}(y_1, r_1) \supset \overline{B}(y_2, r_2) \supset ... \supset \overline{B}(y_k, r_k) \supset ...$.
iii. $\overline{B}(y_k, r_k)$ does not contain $y_1, y_2, \ldots, y_{k-1}$.

Since $\overline{B}(y_k, r_k)$ is closed and bounded, we have that it is compact by Theorem 11.2.2. Define $F_k = \overline{B}(y_k, r_k) \cap A$, which is compact (since it's closed and bounded) and nonempty. We have a collection of compact sets $F_1 \supset F_2 \supset \ldots \supset F_k \supset \ldots$. By using thinking from Theorem 11.2.1, we see that $\bigcap_{k \in \mathbb{N}} F_k$ is nonempty (in fact, it contains only one point).

Recall that F_k does not contain $y_1, y_2, \ldots, y_{k-1}$ for $k \geq 2$, which means that F_k does not contain $x_1, x_2, \ldots, x_{k-1}$ for $k \geq 2$. We then see that $\bigcap_{k \in \mathbb{N}} F_k$ does not contain $x_1, x_2, \ldots, x_{k-1}$ for $k \geq 2$, meaning that $\bigcap_{n \in \mathbb{N}} F_k$ does not contain any element in A. This is our contradiction, so the proof is concluded. □

12.2 Cantor sets (not discussed)

We had a brief discussion in lecture about Cantor sets. There was not a lot of time devoted to it and its main points or topics were not clear enough to include in this text, so they will not be covered.

12.3 Introducing sequences

Sequences are basically an infinite list of terms that need not be unique. We can define sequences in a more precise way below.

Definition 12.3.1: Sequence

Let (X, d) be a metric space. We can craft a sequence $\{x_n\} \subset X$ if for every $n \in \mathbb{N}$, we have $x_n \in X$. In other words, each element of the natural numbers corresponds to an element in X.

There can be multiple elements of $\mathbb{N}$ that map to the same element of X.

An important outcome of Definition 12.3.1 is that there is no such thing as a finite sequence; a sequence cannot be terminated after $k \in \mathbb{N}$ terms. Although there can be a finite number of distinct terms of X that are used in a sequence, we still must assign each element of $\mathbb{N}$ to those terms.

Furthermore, we introduce a concept of a list. Recall that with sets, we merely had a grouping of elements that we could imagine to be jumbled together. With sequences, there is such a thing called the first term, the second term, ... , the nth term, ... This becomes important in the following definition.

Definition 12.3.2: Convergence of a sequence

Let a sequence $\{x_n\} \subset X$ be given. We say that $\{x_n\}$ converges to $x \in X$ if for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n > N$, we have $d(x_n, x) < \varepsilon$. Denote by $\{x_n\} \to x$, or simply $x_n \to x$.

Note that we can use $> N$ or $\geq N$, and similarly, we can use $d(x_n, x) < \varepsilon$ or $d(x_n, x) \leq \varepsilon$.

We will be doing a fair bit with distance metrics, so it is important to review this topic; we introduced this topic in Section 7.1. One thing to note is that for all sequences, we are looking at the distance between two points and wanting that distance to be arbitrarily small. This allows us to always introduce the real numbers $\mathbb{R}$ to convergence, which could be more manageable than considering points in some crazy metric space we might encounter.

The following lemma will highlight an important fact on converging sequences.

Lemma 12.3.3: Converging to a unique point

If $\{x_n\}$ converges to x and y, then $x = y$.

Proof: Since $\{x_n\}$ converges to x and y, by Definition 12.3.2 we have that $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y) = 0$, which implies that $\lim_{n\to\infty}[d(x_n, x) + d(x_n, y)] = 0$. By the triangle inequality, we know $d(x_n, x) + d(x_n, y) \geq d(x, y)$, meaning

$$\lim_{n\to\infty} [d(x_n, x) + d(x_n, y)] = 0 \geq \lim_{n\to\infty} d(x, y) \implies d(x, y) = 0$$

This concludes the proof. $\square$

Lecture 13: 03/05/20

In this lecture, we continue our discussion on sequences and introduce subsequences. Please review the definitions covered in Section 12.3 before continuing with this lecture.

13.1 Converging sequences

We use sequences a lot in this course. We will look at some facts we can use to either find a converging sequence or to extract information from a converging sequence.

Lemma 13.1.1: Convergence implies boundedness

If $\{x_n\} \to x$ in X, then $\{x_n\}$ is bounded.

Proof: By Definition 12.3.2 we can find an $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon = 1$ for all $N \geq N$. This implies that $x_n \in B(x, 1)$ for all $n \geq N$. Note that we don't know the locations of the first $N-1$ points in our sequence, but we know that there are only $N-1$ of them. Define $K = \{k \in \mathbb{N} : 1 \leq k \leq N-1\}$. Let $R = [\max_{k \in K} d(x_k, x)] + 1$. We then have that $x_n \in B(x, R)$ for all $n \in \mathbb{N}$, meaning that $\{x_n\}$ is bounded. $\square$

Lemma 13.1.2: Limit points and convergence

If $E \subset X$ and $p \in X$ is a limit point of E, we can find a sequence $\{x_n\} \subset E$ such that $\{x_n\} \to p$.

Proof: As p is a limit point of E, for each $r > 0$, we have $(B(p, r) \setminus \{p\}) \cap E \neq \varnothing$. For any $n \in \mathbb{N}$, let $r = \frac{1}{n}$. We can then find an $x_n \in (B(p, \frac{1}{n}) \setminus \{p\}) \cap E$, implying that $x_n \in E$ and $d(x_n, p) < \frac{1}{n}$ for $n \in \mathbb{N}$. We then see that $\lim_{n \to \infty} d(x_n, p) = 0$, which means that $\{x_n\} \to p$. $\square$

Theorem 13.1.3: Convergence in $\mathbb{R}^d$

Let $X = \mathbb{R}^d$ with the Euclidean distance metric, meaning that for $x \in \mathbb{R}^d$, $x = (x_1, x_2, \ldots, x_d)$. For $\{x_n\} \subset \mathbb{R}^d$, $\{x_n\} \to x \in \mathbb{R}^d \iff \{|x_{in} - x_i|\} \to 0$ for all $1 \leq i \leq d$.

Proof: Since we are looking to prove an if-and-only-if statement, we need to prove that each side implies the other.

i. Assume that $\{x_n\} \to x \in \mathbb{R}^d$. We know by Definition 12.3.2 that $\{x_n\} \to x \in \mathbb{R}^d \iff \{|x_n - x|\} \to 0$ as $n \to \infty$. In other words, $\{|x_n - x|\} =$

$\left\{\sqrt{(x_{1n}-x_1)^2+...+(x_{dn}-x_d)^2}\right\} \to 0$. For each $1 \leq i \leq d$, it's certainly true that

$$|x_n - x| \geq \sqrt{(x_{in}-x_i)^2} = |x_{in} - x_i| \geq 0$$

Therefore, if $\{|x_n - x|\} \to 0$, then by the above inequality we have $\{|x_{in} - x_i|\} \to 0$ as $n \to \infty$ for each $1 \leq i \leq d$.

ii. Assume that $\{|x_{in} - x_i|\} \to 0$ for all $1 \leq i \leq d$. Notice that

$$|x_n - x| = \sqrt{(x_{1n}-x_1)^2+...+(x_{dn}-x_d)^2} \leq |x_{1n} - x_1| + ... + |x_{dn} - x_d|$$

We have that $\lim_{n\to\infty}[|x_{1n} - x_1| + ... + |x_{dn} - x_d|] = 0$, so by the above inequality we have $\lim_{n\to\infty} |x_n - x| \leq 0$, resulting in $\lim_{n\to\infty} |x_n - x| = 0$.

This concludes the proof. □

Please note that x_{in} is notating the ith slot (or dimension) of x_n where $x_n \in \{x_n\}$.

Example 13.1.4: Converging sequences

Let $X = \mathbb{R}^2$. Does the sequence $\{y_n\} = (-n, \frac{1}{n})$ converge?

This directly uses Theorem 13.1.3. Since $\{-n\}$ does not converge in $\mathbb{R}$, we certainly have that $\{y_n\}$ does not converge.

13.2 Subsequences

From Definition 12.3.1, we see that our indices take the form of each element of $\mathbb{N}$. We can craft another sequence from an existing sequence, something we take a look at in this section.

Definition 13.2.1: Subsequence

Let $\{x_n\} \subset X$. We can pick out the indices of $\mathbb{N}$ in a monotone order such that $n_1 < n_2 < ... < n_k < ...$. We then declare that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Note that n_i is an element of $\mathbb{N}$, meaning that x_{n_i} corresponds to a specific element of $\{x_n\}$.

The important thing about subsequences is that the indices must be strictly increasing; we can't backtrack to a previous index.

Example 13.2.2: Subsequences

Let a sequence $\{x_n\} \subset X$ be given, and let

$$n_k = \begin{cases} 2 \cdot k & \text{for } k \text{ even} \\ 3 \cdot k & \text{for } k \text{ odd} \end{cases}$$

Is $\{x_{n_k}\}$ a subsequence?

This is NOT an example of a subsequence since $n_3 > n_4$.

Just like we had nice facts about converging sequences, we look to prove useful characteristics about subsequences in the lemmas and theorems that follow.

Lemma 13.2.3: Subsequence indices

If $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, then $n_k \geq k$ for all $k \in \mathbb{N}$.

Proof: We can use induction.

i. Base case: $n_1 \in \mathbb{N}$ so of course $n_1 \geq 1$.
ii. Induction hypothesis: We assume that $n_k \geq k$ for some $k \in \mathbb{N}$. We need to show that $n_{k+1} \geq k+1$. Note that $n_{k+1} > n_k \implies n_{k+1} \geq n_k + 1$ since the natural numbers increase in a step-wise fashion, and then it's clear by our assumption for the induction hypothesis that $n_{k+1} \geq k+1$.

This concludes the proof. $\square$

Lemma 13.2.4: Subsequences of a converging sequence

If $\{x_n\} \to x$ and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, then $\{x_{n_k}\} \to x$ as well.

Proof: Pick any $\varepsilon > 0$. We need to show that there is a $K \in \mathbb{N}$ such that $d(x_{n_k}, x) \leq \varepsilon$ for all $k \geq K$.

We have that $\{x_n\} \to x$, so for our chosen ε, there is an $N \in \mathbb{N}$ such that $d(x_n, x) \leq \varepsilon$ for $n \geq N$. Choose $K = N$. Then, for $k \geq K = N$, we know by Lemma 13.2.3 that $n_k \geq k \geq K = N$. Therefore, as $n_k \geq N$, we have that $d(x_{n_k}, x) \leq \varepsilon$. $\square$

Theorem 13.2.5: Convergence of subsequences

$\{x_n\}$ is convergent $\iff$ all subsequences of $\{x_n\}$ converge to a single point.

Proof: Since we are looking to prove an if-and-only-if statement, we need to prove that each side implies the other.

i. When we assume that $\{x_n\}$ is convergent, we proved our desired result in Lemma 13.2.4.

ii. When we assume that all subsequences of $\{x_n\}$ converge to a single point, note that $\{x_n\}$ is a subsequence of $\{x_n\}$, so certainly $\{x_n\}$ converges to the same single point.

We have proved that each side implies the other of the if-and-only-if statement, so we are done. □

Example 13.2.6: Subsequences

Let $\{x_n\}$ be a given sequence in $\mathbb{R}$. Assume that the subsequences $\{x_{2n}\}$, $\{x_{2n-1}\}$, and $\{x_{3n}\}$ converge. Show that $\{x_n\}$ is convergent as well.

Proof: Let's say that $\{x_{2n}\} \to a$, $\{x_{2n-1}\} \to b$, and $\{x_{3n}\} \to c$. Please note that $\{x_{6n}\}$ is a subsequence of $\{x_{3n}\}$ and $\{x_{2n}\}$ while $\{x_{6n-3}\}$ is a subsequence of $\{x_{3n}\}$ and $\{x_{2n-1}\}$. Since $\{x_{2n}\} \to a$, this implies that $\{x_{6n}\} \to a$ by Lemma 13.2.4. Furthermore, $\{x_{3n}\} \to c \implies \{x_{6n}\} \to c$, so by Lemma 12.3.3 we know that $a = c$. We can use a similar process with $\{x_{2n-1}\}$ and $\{x_{3n}\}$ to show that $b = c$, which means $a = b = c$, or that the three subsequences given in the problem statement converge to the same point.

Note that if $\{x_n\}$ is convergent to a, this means that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$, we have $d(x_n, a) < \varepsilon$. Please note that the subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ cover $\{x_n\}$, meaning that for some $x_i \in \{x_n\}$, $x_i \in \{x_{2n}\}$ if i is even or $x_i \in \{x_{2n-1}\}$ if i is odd.

Let $\varepsilon > 0$ be given. Since $\{x_{2n}\} \to a$, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $d(x_{2n}, a) < \varepsilon$. We can find a similar marker $N_2 \in \mathbb{N}$ for $\{x_{2n-1}\}$. Let $N = \max\{N_1, N_2\}$. Since $N \geq N_1$ and $N \geq N_2$, we know that for all $n \geq N$, we have $d(x_{2n}, a) < \varepsilon$ and $d(x_{2n-1}, a) < \varepsilon$, meaning that $d(x_n, a) < \varepsilon$. This is precisely what we need for proving $\{x_n\} \to a$, so we are done. □

Compactness is something we often refer to throughout this class as its properties can help us to deduce useful conclusions. The following theorem links compactness and converging subsequences.

Theorem 13.2.7: Convergent subsequence in a compact metric space

Let (X, d) be a compact metric space. Then for every sequence $\{x_n\} \subset X$, there exists a convergent subsequence $\{x_{n_k}\}$.

Proof: Let a set V be given such that $V = \{x_n \in X : n \in \mathbb{N}\} \subset X$. We will examine the two different cases where V is either finite or infinite.

i. Suppose that V is finite. This means that there exists an $a \in V$ such that $x_n = a$ for infinitely many indices. Let the index set $I = \{n \in \mathbb{N} : x_n = a\}$. We then have that $I \subset \mathbb{N}$ and I is infinite. Pick $n_1 = \min I$, $n_2 = \min(I \setminus \{n_1\})$, $n_3 = \min(I \setminus \{n_1, n_2\}), \ldots$. We have crafted a subsequence $\{x_{n_k}\}$ where $\{x_{n_k}\} \to a$.

ii. Suppose that V is infinite. Note that since $V \subset X$ and X is compact, by Lemma 10.1.7 we know that V has a limit point $p \in X$. By Lemma 13.1.2, we can find a sequence in V that converges to p. By utilizing the peeling strategy from (i.) where we choose the smallest index that satisfies a distance condition, we can find a subsequence $\{x_{n_k}\} \to p$.

This concludes the proof. □

Note that we were able to find the minimum index of an infinite set that contained only natural numbers. This is fair to do because the natural numbers themselves have a minimum value (which is 1), so we can think about incrementing our way up from 1 until we reach the minimum value of our set.

We now look at a more specific case of sequences, specifically bounded sequences in $\mathbb{R}^n$.

Theorem 13.2.8: Bolzano-Weirstrass Theorem

Let $\{x_n\} \subset \mathbb{R}^n$ be a bounded sequence. $\{x_n\}$ has a convergent subsequence.

Proof: Since $\{x_n\}$ is a bounded sequence, there exists an $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. This means that $\{x_n\} \subset [-M, M]$. We now perform the following steps:

i. Divide the interval $[-M, M]$ into two closed intervals of equal length. We know that one of the intervals must have infinitely many terms, but note that these terms need not be distinct. Call this interval J_1 and let $x_{n_1} \in J_1$.

ii. Divide the interval J_1 into two closed intervals of equal length. One of these closed intervals must have infinitely many terms of $\{x_n\}$ in it. Call that interval J_2 and let $x_{n_2} \in J_2$ such that $n_2 > n_1$.

We continue this process iteratively (similar to what we did in Theorem 11.2.1) and realize that $\bigcap_{k\in\mathbb{N}} J_k = \{x\}$ where $x \in \{x_n\}$. We know the length of the interval J_k is $M \cdot (\frac{1}{2})^{k-1}$. If we made a sequence $\left\{M \cdot (\frac{1}{2})^{k-1}\right\} \subset \mathbb{R}$, we see that it converges to 0. This means for any $\varepsilon > 0$, there exists some $K \in \mathbb{N}$ such that $|M \cdot (\frac{1}{2})^{k-1} - 0| < \varepsilon$ for all $k \geq K$.

Please note that $x_{n_k} \in J_k$ and $x \in J_k$, so therefore $|x_{n_k} - x| < \varepsilon$ for all $k \geq K$, thus implying $\{x_{n_k}\} \to x$. $\square$

Lecture 14: 03/10/20

We discussed converging sequences in detail in Lecture 13. In this lecture, we look at another way to characterize a sequence by defining a Cauchy sequence. We also discuss complete metric spaces, monotone sequences, and begin talking about limsup and liminf.

14.1 Cauchy sequences

As mentioned in the introduction to this lecture, Cauchy sequences is another classification of sequences.

Definition 14.1.1: Cauchy sequence

Let a sequence $\{x_n\} \subset X$ be given. $\{x_n\}$ is called a Cauchy sequence if for each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for $m, n > N$.

Please realize how this is different from convergence. With convergent sequences, we fix a point and look at the distances between that points and elements of our sequences beyond a certain marker. With Cauchy sequences, we still have the idea of establishing a marker, but now we look at the distance **between elements** of our sequence beyond that marker.

Theorem 14.1.2: Convergent implies Cauchy

If $\{x_n\} \to x \in X$, then $\{x_n\}$ is Cauchy.

Proof: Let's fix $\varepsilon > 0$ and then find $\frac{\varepsilon}{4} > 0$. Since $\{x_n\} \to x$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < \frac{\varepsilon}{4}$ for all $n > N$. This means that $d(x_n, x) < \frac{\varepsilon}{4}$ and $d(x_m, x) < \frac{\varepsilon}{4}$ for $m, n > N$. Therefore, for $m, n > N$, the triangle inequality tells us

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon$$

We have shown that for any arbitrary $\varepsilon > 0$, we can find a marker $N \in \mathbb{N}$ such that for all $m, n > N$, we have $d(x_m, x_n) < \varepsilon$, so we are done. $\square$

Please note that convergence implies Cauchy, but in a general metric space, Cauchy does NOT imply convergence.

For emphasis, **a sequence being Cauchy does NOT imply that the sequence converges.** Example 14.1.4 is a little technical by how we declare the sequence to not converge, but is worth covering to emphasize this statement.

We again see how compactness permeates through the entire course with the following theorem.

Theorem 14.1.3: Cauchy sequence in a compact metric space

If $\{x_n\}$ is Cauchy and X is compact, then $\{x_n\}$ converges.

Proof: By Theorem 13.2.7, there is a convergent subsequence $\{x_{n_k}\} \to x \in X$. We claim that $\{x_n\} \to x \in X$.

Fix $\varepsilon > 0$. Two facts that we will exploit are

i. $\{x_{n_k}\} \to x$. As $\frac{\varepsilon}{5} > 0$, we can find a $K \in \mathbb{N}$ such that $d(x_{n_k}, x) < \frac{\varepsilon}{5}$ for $k \geq K$.
ii. $\{x_n\}$ is Cauchy. We can find an $M \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\varepsilon}{5}$ for $m, n \geq M$.

Choose $N = \max\{M, n_K\}$. For $n \geq N$, we have

i. $d(x_{n_N}, x) < \frac{\varepsilon}{5}$ since $N \geq n_K \geq K$.
ii. $d(x_n, x_{n_N}) < \frac{\varepsilon}{5}$ since $n, n_N \geq N \geq M$.

By the triangle inequality, we retrieve

$$d(x_n, x) \leq d(x_{n_N}, x) + d(x_n, x_{n_N}) < \frac{\varepsilon}{5} + \frac{\varepsilon}{5} < \varepsilon$$

This leads us to conclude that $\{x_n\} \to x$, thus $\{x_n\}$ is convergent. $\square$

Example 14.1.4: Cauchy sequences

Find two sequences that are Cauchy but do not converge.

i. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ with the Euclidean distance metric. Consider the sequence $\{x_n\} \subset X$ where $x_n = \frac{1}{n}$ for $n \in \mathbb{N}$. Note that $\{x_n\}$ is Cauchy since $|x_n - x_m| = |\frac{1}{n} - \frac{1}{m}| < \frac{1}{N} < \varepsilon$ for $m, n > N > \frac{1}{\varepsilon}$. However, $\{x_n\}$ is NOT convergent in X since $0 \notin X$.
ii. Let $X = \mathbb{Q}$ with the Euclidean distance metric. Pick $x_n \in \mathbb{Q}$ such that $\sqrt{2} - \frac{1}{n} < x_n < \sqrt{2}$ for $n \in \mathbb{N}$. We see that $\{x_n\}$ is Cauchy but NOT convergent in $\mathbb{Q}$ since $\sqrt{2} \notin \mathbb{Q}$.

In addition to showing how a Cauchy sequence could not be a convergent sequence, Example 14.1.4 highlights how a sequence must converge to a point in its metric space for it to be considered a convergent sequence. In other words, a sequence could get arbitrarily close to a point, but if that point is not in the metric space of the sequence, then the sequence is not convergent.

14.2 Complete metric spaces

Complete metric spaces are extremely nice since they provide a direct link between convergent and Cauchy sequences.

Definition 14.2.1: Complete metric space

Let a metric space (X, d) be given. We say that (X, d) is complete if every Cauchy sequence in X is convergent.

In complete metric spaces, a given sequence $\{x_n\}$ converges $\iff$ $\{x_n\}$ is Cauchy.

Remark 14.2.2: Compactness and completeness

Compactness of a metric space X implies that X is complete, but the completeness of X does not imply compactness.

Another way to express this is X compact $\implies$ X complete, but X complete $\not\Longrightarrow$ X compact.

Theorem 14.2.3: Completeness of the reals

$X = \mathbb{R}^d$ with the Euclidean distance is complete.

Proof: Let a Cauchy sequence $\{x_n\} \subset \mathbb{R}^d$ be given and pick $\varepsilon = 1$. Since $\{x_n\}$ is Cauchy, there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) = |x_n - x_m| < 1$ for $n, m \geq N$. Please note that this implies $|x_n - x_N| < 1$ for $n \geq N$. By the triangle inequality, we can say $|x_n| \leq |x_n - x_N| + |x_N| < |x_N| + 1$ for all $n \geq N$.

Let $P = \{n \in \mathbb{N} : n \leq N\}$ and have $C = \max_{p \in P}(1 + |x_p|)$. We have that $|x_n| \leq C$ for all $n \in \mathbb{N}$, and this implies $\{x_n\} \subset [-C, C]^d$. We know that $[-C, C]^d$ is compact by Theorem 11.2.1, so we can use Theorem 14.1.3 to conclude that $\{x_n\}$ converges. $\square$

Example 14.2.4: Complete metric spaces

Assume that $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in a metric space (X, d). Show that $d(x_n, y_n)$ converges.

Proof: Please note how the sequence $\{d(x_n, y_n)\}$ is a subset of $\mathbb{R}$. Since a distance metric is not given, we assume the Euclidean distance metric is used. By Theorem 14.2.3, we know $\{d(x_n, y_n)\}$ converges $\iff$ $\{d(x_n, y_n)\}$ is a Cauchy sequence.

Let $\varepsilon > 0$ be given. Find the Cauchy sequence markers N and M for $\{x_n\}$ and $\{y_n\}$, respectively, for $\frac{\varepsilon}{4}$ and let $A = \max\{N, M\}$. Therefore, for all $n, m > A$, we have $d(x_n, x_m) < \frac{\varepsilon}{4}$ and $d(y_n, y_m) < \frac{\varepsilon}{4}$.

When $n, m > A$, the triangle inequality tells us

$$d(x_m, y_m) < d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m) < \frac{\varepsilon}{2} + d(x_n, y_n)$$

$$\Longrightarrow$$

$$d(x_m, y_m) - d(x_n, y_n) < \frac{\varepsilon}{2}$$

We can do a similar process to show that $d(x_n, y_n) - d(x_m, y_m) < \frac{\varepsilon}{2}$. This means that $|d(x_n, y_n) - d(x_m, y_m)| < \frac{\varepsilon}{2} < \varepsilon$, so $\{d(x_n, y_n)\}$ is Cauchy and therefore convergent. $\square$

14.3 Monotone sequences of real numbers

A sequence being monotone means that it is nonincreasing or nondecreasing. The properties of something being nonincreasing and nondecreasing frequently arise in everyday life, thus a basic understanding of these types of sequences is of interest.

Definition 14.3.1: Nondecreasing sequence

Let $\{x_n\} \subset \mathbb{R}$. We say that $\{x_n\}$ is nondecreasing if $x_1 \leq x_2 \leq ... \leq x_n \leq ...$.

Definition 14.3.2: Nonincreasing sequence

Let $\{x_n\} \subset \mathbb{R}$. We say that $\{x_n\}$ is nonincreasing if $x_1 \geq x_2 \geq ... \geq x_n \geq ...$.

Theorem 14.3.3: Bounded monotone sequences

Let $\{x_n\} \subset \mathbb{R}$ be bounded. If $\{x_n\}$ is nonincreasing or nondecreasing then $\{x_n\}$ is convergent.

Proof: Let $\{x_n\}$ be nondecreasing and have $S = \{x_n \in \mathbb{R} : n \in \mathbb{N}\}$. S is bounded since $\{x_n\}$ is bounded. We claim that $\{x_n\} \to a = \sup S$.

For each $\varepsilon > 0$, $a - \varepsilon$ is NOT an upper bound of S. Thus, we can find an $N \in \mathbb{N}$ such that $a - \varepsilon < x_N \leq a$. Since $\{x_n\}$ is nondecreasing, we know that for $n \geq N$ we have $a - \varepsilon < x_N \leq x_n \leq a$. This implies that $|x_n - a| \leq \varepsilon$ for $n \geq N$, so $\{x_n\}$ converges to $a = \sup S$. $\square$

Please note that although it is not shown, the process for a nonincreasing sequence is very similar except that it converges to $b = \inf S$.

14.4 Limsup and liminf

Using limsup and liminf is essentially optimizing a sequence regardless of whether it is convergent or not. We will only be looking at sequences of real numbers when we discuss limsup and liminf.

Definition 14.4.1: Limsup

Let $\{x_n\} \subset \mathbb{R}$ be a bounded sequence. Let the set E be the set of all possible subsequential limit points of $\{x_n\}$. We define $\operatorname{limsup}_{n\to\infty} x_n = \sup E$.

Definition 14.4.2: Liminf

Let $\{x_n\} \subset \mathbb{R}$ be a bounded sequence. Let the set E be the set of all possible subsequential limit points of $\{x_n\}$. We define $\operatorname{liminf}_{n\to\infty} x_n = \inf E$.

Lecture 15: 03/12/20

This lecture wraps up the introduction to limsup and liminf with an important theorem, and we begin our discussion on series in $\mathbb{R}$.

15.1 More on limsup and liminf

Recall that limsup and liminf allow us to optimize all possible subsequences of a bounded sequence in $\mathbb{R}$. This allows us to make powerful conclusions on all types of sequences regardless of whether they converge or not.

Theorem 15.1.1: Finding limsup and liminf

Let a sequence $\{x_n\} \subset \mathbb{R}$ be bounded. We declare $\limsup_{n\to\infty} x_n = \lim_{n\to\infty}[\sup\{x_m : m \geq n\}]$ and $\liminf_{n\to\infty} x_n = \lim_{n\to\infty}[\inf\{x_m : m \geq n\}]$.

Proof: Let's define $y_n = \sup\{x_m : m \geq n\}$. We see that

- $\{y_n\}$ is bounded since $\{x_n\}$ is bounded.
- $y_1 \geq y_2 \geq ... \geq y_n \geq ...$, so $\{y_n\}$ is nonincreasing.

By Theorem 14.3.3, we declare that $\{y_n\}$ converges. Furthermore, by how we structured our notation, we have $\lim_{n\to\infty} y_n = \lim_{n\to\infty}[\sup\{x_m : m \geq n\}]$.

Take an arbitrary subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \to \alpha$ (we know at least one exists by Theorem 13.2.8). Observe that $x_{n_k} \leq y_{n_k} = \sup\{x_{n_k}, x_{n_{k+1}}, ...\}$. Therefore, $\alpha = \lim_{k\to\infty} x_{n_k} \leq \lim_{k\to\infty} y_{n_k} = \lim_{n\to\infty} y_n$ (look over Lemma 13.2.4 if the last equality is not clear). This means that $\lim_{n\to\infty} y_n$ is an upper bound of E, where E is defined in Definition 14.4.1.

We will construct a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = \lim_{n\to\infty} y_n$ by induction.

i. Note that $y_1 = \sup\{x_1, x_2, ...\}$. We can find $n_1 \in \mathbb{N}$ such that $x_{n_1} \leq y_1 \leq x_{n_1} + 1$.
ii. Note that $y_{(n_1+1)} = \sup\{x_{(n_1+1)}, x_{(n_1+2)}, ...\}$. We can find an $n_2 > n_1$ such that $x_{n_2} \leq y_{(n_1+1)} \leq x_{n_2} + \frac{1}{2}$.

Therefore, induction tells us that we can find an $n_j > n_{j-1}$ such that $x_{n_j} \leq y_{(n_{j-1}+1)} \leq x_{n_j} + \frac{1}{j}$ for $j \in \mathbb{N}$. This leads us to conclude $\lim_{j\to\infty} x_{n_j} = \lim_{j\to\infty} y_{(n_{j-1}+1)} = \lim_{n\to\infty} y_n$. We have found a subsequence that converges to an upper bound of E

(the upper bound being $\lim_{n\to\infty} y_n$), meaning that

$$\sup E = \lim_{n\to\infty} y_n = \lim_{n\to\infty} [\sup\{x_m : m \geq n\}] = \text{limsup}_{n\to\infty} x_n$$

This concludes the proof. $\square$

The proof for $\text{liminf}_{n\to\infty} x_n = \lim_{n\to\infty}[\inf\{x_m : m \geq n\}]$ is very similar.

We can see that limsup and liminf are a bit more involved than simple limits. Furthermore, we must often consider limsup and liminf with certain types of proofs. However, the following remark simplifies situations where we have converging sequences.

Remark 15.1.2: Limsup and liminf of a converging sequence

$\{x_n\} \to \alpha \iff E = \{\alpha\} \iff \text{limsup}_{n\to\infty} x_n = \text{liminf}_{n\to\infty} x_n = \alpha$

Something important to note is that although a bounded sequence in $\mathbb{R}$ might not converge (meaning its limit does not exist), a bounded sequence in $\mathbb{R}$ always has a limsup and a liminf.

15.2 Series of real numbers

We have looked at sequences of numbers, but now we will look at the summation of every point in a sequence. Although no human or computer can do infinitely many additions, we can look at the partial sum of n many terms of a sequence and take the limit as $n \to \infty$.

Definition 15.2.1: Convergence of a series

Let a sequence $\{a_i\} \subset \mathbb{R}$ be given. Let a partial sum s_n be as follows:

$$s_n = a_1 + a_2 + ... + a_n = \sum_{i=1}^{n} a_i$$

If a sequence of partial sums $\{s_n\} \to s \in \mathbb{R}$, we say that $\sum_{i=1}^{\infty} a_i$ converges.

Denote the convergence of a series to a point s by $\sum_{i=1}^{\infty} a_i = s = \lim_{n\to\infty} s_n$.

Theorem 15.2.2: Cauchy criterion of converging series

A series $\sum_{i=1}^{\infty} a_i$ converges $\iff$ for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\left|\sum_{i=n+1}^{m} a_i\right| < \varepsilon$ for $m > n > N$.

Proof: Note that $\sum_{i=1}^{\infty}$ converges if and only if $\{s_n\}$ converges in $\mathbb{R}$. By Theorem 14.2.3 we know that $\{s_n\}$ converges in $\mathbb{R}$ if and only if $\{s_n\}$ is Cauchy, meaning that for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|s_m - s_n| < \varepsilon$ for $m > n > N$. $\square$

Please note that $\left|\sum_{i=n+1}^{m} a_i\right| = |s_m - s_n|$.

Theorem 15.2.2 basically says that a series converges if and only if the tail of the series is small. In other words, we need the sum of the terms after a certain marker to be arbitrarily small. We don't really care about the terms before the marker because there is only a finite number of them, so surely the sum of them is finite as well.

Note that instead of writing $\sum_{i=1}^{\infty} a_i$ to indicate the summation of all terms of $\{a_i\}$, we may just write $\sum a_i$. These two notations are equivalent.

We will now look at some properties of converging series as well as some tests to help us identify a convergent series.

Theorem 15.2.3: Convergence of series and sequences

If $\sum a_i$ converges, then $\lim_{i\to\infty} a_i = 0$.

Proof: If $\sum a_i$ converges, then $\lim_{n\to\infty} s_n = s = \lim_{n\to\infty} s_{n+1}$. This means

$$\lim_{n\to\infty}(s_{n+1} - s_n) = 0 \implies \lim_{n\to\infty} a_{n+1} = 0 \implies \lim_{n\to\infty} a_n = 0$$

This concludes the proof. $\square$

In general, $\lim_{i\to\infty} a_i = 0 \;\not\Longrightarrow\; \sum a_i$ converges.

Lemma 15.2.4: Nonzero series and bounded partial series

Assume that $a_i \geq 0$ for all $i \in \mathbb{N}$. Then $\sum a_i$ converges $\iff$ $\{s_n\}$ is bounded.

Proof: As $a_i \geq 0$ for all $i \in \mathbb{N}$, we have that $s_n = a_1 + a_2 + ... + a_n \geq 0$ and also $s_1 \leq s_2 \leq ... \leq s_n \leq ...$, which means $\{s_n\}$ is nondecreasing. By Theorem 14.3.3 we conclude that $\{s_n\}$ is convergent $\iff$ $\{s_n\}$ is bounded. This concludes the proof. $\square$

Theorem 15.2.5: Geometric series convergence

For $x \in \mathbb{R}$, consider $\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + \dots$. Then, $\sum_{i=0}^{\infty} x^i$ converges $\iff |x| < 1$. Furthermore, if $|x| < 1$, then $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$.

Proof: We will look at two cases:

i. Suppose $x = 1$. It is clear that this series diverges ($1 + 1 + 1 + \dots = \infty$).
ii. Suppose $x \neq 1$. We know the partial sum $s_n = 1 + x + x^2 + \dots + x^n$ and $x \cdot s_n = x + x^2 + x^3 + \dots + x^n + x^{n+1}$. This means $(1-x) \cdot s_n = 1 - x^{n+1} \implies s_n = \frac{1-x^{n+1}}{1-x}$. This means that s_n converges $\iff$ x^{n+1} converges as $n \to \infty \iff |x| < 1$. When this occurs, we see $\lim_{n\to\infty} s_n = \frac{1}{1-x}$.

This concludes the proof. □

Theorem 15.2.6: Comparison test

Given two series $\sum a_i$ and $\sum c_i$:

i. If $|a_i| \leq c_i$ for all $i \geq M$ for some given $M \in \mathbb{N}$ and $\sum c_i$ converges, then $\sum a_i$ converges.
ii. If $a_i \geq c_i > 0$ for all $i \geq M$ for some $M \in \mathbb{N}$ and $\sum c_i$ diverges, then $\sum a_i$ diverges.

Proof: We will prove each part of the comparison test:

i. By Theorem 15.2.2, we know that for a given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ where $N \geq M$ such that for $m > n > N$, we have

$$\left| \sum_{i=n+1}^{m} a_i \right| \leq \sum_{i=n+1}^{m} |a_i| \leq \sum_{i=n+1}^{m} c_i \leq \varepsilon$$

which means that $\sum a_i$ converges by Theorem 15.2.2.
ii. Note that by work done for the proof of (i.) we see that if $\sum a_i$ converges then $\sum c_i$ must converge. However, we know $\sum c_i$ diverges, so $\sum a_i$ cannot converge, hence $\sum a_i$ diverges.

This concludes the proof. □

Note that for (ii.) we are dealing with nonnegative terms.

Lecture 16: 03/24/20 (online)

The main thing we look to address with series is if they converge or not. In this lecture, we discuss more tests to see if a particular series converges or not. Please note that in this course we don't spend a lot of time figuring out where a series converges to, just whether or not it will converge.

16.1 More on series

Theorem 16.1.1: Root test

Let $\sum a_n$ be a given series. Set $\alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. We have three situations to consider:

i. If $0 \leq \alpha < 1$, the series converges.
ii. If $\alpha > 1$, the series diverges.
iii. If $\alpha = 1$, no conclusions can be made.

Proof: We will prove the claims of convergence and divergence.

i. If $0 \leq \alpha < 1$, we can pick a β such that $\alpha < \beta < 1$. For example, let $\beta = \frac{1+\alpha}{2}$. Then, by virtue of limsup, there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $|a_n|^{\frac{1}{n}} < \beta \implies |a_n| < \beta^n$. We know $\sum \beta^n$ converges since $\beta < 1$ (see Theorem 15.2.5), so by the comparison test, we conclude $\sum a_n$ converges.

ii. If $\alpha > 1$, we can find a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\lim_{k\to\infty} |a_{n_k}|^{\frac{1}{n_k}} = \alpha > 1 \implies \lim_{k\to\infty} |a_{n_k}| = \infty$. This means we don't have $\lim_{n\to\infty} a_n = 0$, so by Theorem 15.2.3 we know the series diverges.

This concludes the proof. $\square$

We tend to use this test when we see powers in the terms we are summing together. Furthermore, we can always use the fact that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Note that if we say $\alpha = |a_n|^{\frac{1}{n}}$, this implies $\alpha^n = |a_n|$. We realize that α^n is in the geometric series form, so if $0 \leq \alpha < 1$, surely $\sum \alpha^n$ converges. This is basically imagining the worst case scenario for convergence of our series, so if it converges, then certainly the original series converges as well.

Example 16.1.2: Root test

Let $\{a_n\} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, ...\}$. Does $\sum a_n$ converge?

Proof: Note that we can cover all terms of $\{a_n\}$ with the two subsequences $\{a_{2n-1}\} = \frac{1}{2^n}$ and $\{a_{2n}\} = \frac{1}{3^n}$. We will utilize Remark 15.1.2 and only calculate limits to prove that $\sum a_n$ converges.

We find $\lim_{n\to\infty} |a_{2n-1}|^{\frac{1}{2n-1}} = \frac{1}{\sqrt{2}}$ and $\lim_{n\to\infty} |a_{2n}|^{\frac{1}{2n}} = \frac{1}{\sqrt{3}}$. Since these two subsequences cover all terms of $\{a_n\}$, our findings imply $\text{limsup}_{m\to\infty} |a_m|^{\frac{1}{m}} = \frac{1}{\sqrt{2}} < 1$, so $\sum a_n$ converges. □

Theorem 16.1.3: Ratio test

Let $\sum a_n$ be a given series. We have two situations to consider:

i. If $\text{limsup}_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, the series converges.

ii. If $\frac{|a_{n+1}|}{|a_n|} \geq 1$ for $n \geq M$ for some $M \in \mathbb{N}$, the series diverges.

Proof: We will prove both situations.

i. If $\text{limsup}_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \alpha < 1$, we can pick a β such that $\alpha < \beta < 1$. For example, let $\beta = \frac{1+\alpha}{2}$. Therefore, there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $\frac{|a_{n+1}|}{|a_n|} < \beta \implies |a_{N+k}| < |a_N| \cdot \beta^k$ for $k \in \mathbb{N}$. We don't care about the first N terms when considering convergence since the sum of those terms is finite, so we just look at $\sum_{n=N+1}^{\infty} a_n$. Since $\sum_{k=1}^{\infty} |a_N| \cdot \beta^k$ converges since $|a_N|$ is a constant and $\beta < 1$, then $\sum_{n=N+1}^{\infty} a_n$ converges by the comparison test. Therefore, $\sum a_n$ converges.

ii. Assume that $\frac{|a_{n+1}|}{|a_n|} \geq 1$ for $n \geq M$ for some $M \in \mathbb{N}$. Therefore, $0 < |a_M| \leq |a_{M+1}| \leq ... \leq |a_n| \leq ...$, so $\lim_{n\to\infty} a_n \neq 0$, so by Theorem 15.2.3 we know the series diverges.

This concludes the proof. □

Note that if (ii.) is satisfied, this means $\frac{|a_{n+1}|}{|a_n|}$ is well defined for $n \geq M$, so $a_n \neq 0$ for $n \geq M$.

Something that might be troubling in the proof for (i.) of Theorem 16.1.3 is the inequality $|a_{N+k}| < |a_N| \cdot \beta^k$. A way to think about this is that the jump between $|a_{N+1}|$ and $|a_N|$ certainly has to be less than β, the jump between $|a_{N+2}|$ and $|a_{N+1}|$ has to be less than β as well, and so on. This leaves us with multiplying $|a_N|$ by β^k.

Theorem 16.1.4: Integral test

Let $f : [1, \infty) \to [0, \infty)$ be a continuous, decreasing function. Let $\sum a_n$ be a given series such that $a_n = f(n)$ for $n \in \mathbb{N}$. $\sum a_n$ converges $\iff$ $\int_1^\infty f(x)\,\mathrm{d}x$ converges.

Proof: Note that $a_1 = f(1) \geq \int_1^2 f(x)\,\mathrm{d}x \geq f(2) = a_2$, or that $f(1) \geq f(x) \geq f(2)$ where $1 \leq x \leq 2$ since $f(x)$ is decreasing. The equality holds if and only if $f(x) = f(1)$ when $1 \leq x \leq 2$.

Therefore, for $m \in \mathbb{N}$, $\sum_{n=1}^m a_n \geq \int_1^m f(x)\,\mathrm{d}x \geq \sum_{n=2}^m a_n$. Therefore, if $\sum a_n$ converges, then $\int_1^\infty f(x)\,\mathrm{d}x$ converges since $\int_1^\infty f(x)\,\mathrm{d}x \leq \sum a_n$. Similarly, if $\int_1^\infty f(x)\,\mathrm{d}x$ converges, then $\sum_{n=2}^\infty a_n$ converges, which means $\sum a_n$ converges because we are just adding one more finite term. $\square$

Please keep in mind that we do not cover integration in this course so use your knowledge from previous calculus courses. We introduce continuity in Lecture 17.

Theorem 16.1.5: p-test

For $p > 0$, consider $\sum \frac{1}{n^p}$. This series converges if and only if $p > 1$.

This is a direct consequence of the integral test. Although we will not go through the proof, we could prove this theorem by defining $f(x) = \frac{1}{x^p}$ for $x \geq 1$ and using the integral test.

Using this with the comparison test is very powerful and is frequently utilized.

An important thing to note is that sometimes when we do a test, the result will be inconclusive, meaning we can't confirm if a series converges or diverges. When this occurs, we must look to use another test. In the cases where a test confirms convergence or divergence, there is no need to conduct another test; there won't be a situation where one test confirms convergence while another confirms divergence.

Lecture 17: 03/26/20 (online)

We continue to talk about convergence of series and introduce the concept of continuity towards the end of the lecture.

17.1 Even more on series

We will bring forth even more tests to confirm convergence and divergence of a series.

Theorem 17.1.1: Alternating series test

Let a series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ be given.

i. If $a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots \geq 0$ and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Proof: We will prove the convergence of the partial sum sequence $\{s_n\}$ by showing that $\{s_{2n}\}$ and $\{s_{2n-1}\}$ converge to the same limit.

Observe that $0 \leq s_1 = a_1$ and $0 \leq s_3 = a_1 - a_2 + a_3 = a_1 - (a_2 - a_3) \leq a_1 = s_1$. Inductively, we see that $s_1 \geq s_3 \geq \ldots \geq s_{2n-1} \geq \ldots \geq 0$. Therefore, $\{s_{2n-1}\}$ is nonincreasing and bounded from below by 0, so $\{s_{2n-1}\}$ is convergent by Theorem 14.3.3.

Similarly, observe that $s_4 = a_1 - a_2 + a_3 - a_4 = a_1 - a_2 + (a_3 - a_4) = s_2 + (a_3 - a_4) \geq s_2 \geq 0$. Inductively, this leads us to declare $s_1 \geq \ldots \geq s_{2n} \geq \ldots \geq s_6 \geq s_4 \geq s_2 \geq 0$. Therefore, $\{s_{2n}\}$ is nondecreasing and bounded from above by s_1, so $\{s_{2n}\}$ is convergent.

We now see that

$$\lim_{n\to\infty}[s_{2n} - s_{2n-1}] = \lim_{n\to\infty}[-a_{2n}] = 0 \implies \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n-1}$$

Since $\{s_{2n}\}$ and $\{s_{2n-1}\}$ cover all elements of $\{s_n\}$ and converge to the same limit, $\{s_n\}$ also converges. $\square$

We can also declare $\sum_{n=1}^{\infty}(-1)^n a_n$ to be convergent by a similar proof. In addition, this theorem implies $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p}$ for any fixed $p > 0$ is convergent.

Before we go through another test, we will define a certain type of series and a parameter frequently associated with the series.

Definition 17.1.2: Power series

Let $\{c_n\} \subset \mathbb{R}$ and $x \in \mathbb{R}$. The corresponding power series is $\sum_{n=0}^{\infty} c_n x^n$. In other words, for each fixed $x \in \mathbb{R}$, we have $\sum_{n=0}^{\infty} a_n$ for $a_n = c_n x^n$.

Definition 17.1.3: Radius of convergence

Let $\beta = \operatorname{limsup}_{n\to\infty} |c_n|^{\frac{1}{n}}$. We define the radius of convergence R to be $R = \frac{1}{\beta}$.

If $\beta = 0$ then $R = \infty$, and if $\beta = \infty$ then $R = 0$.

Theorem 17.1.4: Power series test

The power series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ and diverges for $|x| > R$.

Proof: Note that $\operatorname{limsup}_{n\to\infty} |a_n|^{\frac{1}{n}} = |x| \cdot \operatorname{limsup}_{n\to\infty} |c_n|^{\frac{1}{n}} = |x| \cdot \beta = \frac{|x|}{R}$. By Theorem 16.1.1 we see that when $\frac{|x|}{R} < 1$ we have convergence and when $\frac{|x|}{R} > 1$ we have divergence. $\square$

Note that we use strict inequalities for comparing $|x|$ and R. When $x = \pm R$, we need to check for convergence directly. The following example shows what that looks like.

Example 17.1.5: Power series test

Find the interval of convergence for the power series of $c_n = \frac{1}{n^2}$, which is $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

To find β, we do the following:

$$\beta = \operatorname{limsup}_{n\to\infty} |c_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{2}{n}}} = \lim_{n\to\infty} \frac{1}{(n^{\frac{1}{n}})^2} = \frac{1}{1^2} = 1$$

This means that $R = 1$, so a subset of our interval of convergence is $(-1, 1)$. Let's now examine the endpoints of -1 and 1.

By the p-test from Theorem 16.1.5, we know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, and by the comparison test from Theorem 15.2.6, we can declare that $\sum_{n=1}^{\infty} \frac{-1}{n^2}$ is convergent as well. Therefore, our interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is $[-1, 1]$.

Note that we can have a power series that starts at values other than $n = 0$. In this case, we started at $n = 1$ since $n = 0$ does not make any sense.

17.2 Limits of functions

Before we start discussing continuous functions, we must first learn about the limits of functions. The following definition gives two options for how to characterize limits of functions, either can be used depending on convenience or familiarity.

Definition 17.2.1: Limits of functions

Let $E \subset X$ be given and assume $p \in X$ is a limit point of E. Consider a function $f : E \to Y$. We say $f(x) \to q$ as $x \to p$ (equivalently written as $\lim_{x \to p} f(x) = q$) if for each $\varepsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), q) < \varepsilon$ for all $x \in E$ with $0 < d_x(x, p) < \delta$.

Let $E \subset X$ be given and assume $p \in X$ is a limit point of E. Consider a function $f : E \to Y$. We say $f(x) \to q$ as $x \to p$ (equivalently written as $\lim_{x \to p} f(x) = q$) if for each $\varepsilon > 0$, there is a $\delta > 0$ such that $f(x) \in B_Y(q, \varepsilon)$ for all $x \in E \cap (B_X(p, \delta) \setminus \{p\})$.

We introduce some notation in Definition 17.2.1 that may not be obvious. Let (X, d_X) and (Y, d_Y) be two metric spaces. For balls in X we write B_X and for balls in Y we write B_Y. We will typically look at functions of the form $f : X \to Y$.

Theorem 17.2.2: Limit of functions and sequences

We have $\lim_{x \to p} f(x) = q \iff$ for every sequence $\{p_n\} \subset (E \setminus \{p\})$ with $\{p_n\} \to p$, we have $\lim_{n \to \infty} f(p_n) = q$.

Proof: Since we are proving an if-and-only-if statement, we need to show that each side implies the other.

i. Assume that $\lim_{x \to p} f(x) = q$. Fix $\varepsilon > 0$. By Definition 17.2.1, there exists a $\delta > 0$ such that $d_Y(f(x), q) < \varepsilon$ for all $x \in E$ with $0 < d_X(x, p) < \delta$. Now, as $\delta > 0$ and $\{p_n\} \to p$, we can find an $N \in \mathbb{N}$ such that $d_X(p_n, p) < \delta$ for all $n > N$. This implies $d_Y(f(p_n), q) < \varepsilon$ for all $n > N$, which means that $\{f(p_n)\} \to q$.

ii. Assume that for every sequence $\{p_n\} \subset (E \setminus \{p\})$ with $\{p_n\} \to p$, we have $\lim_{n \to \infty} f(p_n) = q$. If we don't have $\lim_{x \to p} f(x) = q$, then there exists an $\varepsilon_0 > 0$ such that for all $\delta > 0$, we can find an $x \in E$ with $d_X(x, p) < \delta$ but $d_Y(f(x), q) > \varepsilon_0$. Now, for all $n \in \mathbb{N}$, consider $\delta = \frac{1}{n}$. Then, we can find a $p_n \in (E \setminus \{p\})$ with $d_X(p_n, p) < \frac{1}{n}$ but $d_Y(f(p_n), q) > \varepsilon_0$. Therefore, we have $\{p_n\} \to p$ but $\{f(p_n)\}$ does not converge to q, which is a contradiction of our assumption.

This concludes the proof. □

The following remark allows us to take, for lack of a better phrase, a shortcut when finding limits. The proofs are not given but are relatively straightforward.

Remark 17.2.3: Operators of limits

Take $E \subset X$ and assume $p \in X$ is a limit point of E. Let $f, g : E \to \mathbb{R}$ such that $\lim_{x\to p} f(x) = \alpha$ and $\lim_{x\to p} g(x) = \beta$. The following are true:

i. $\lim_{x\to p}(f+g)(x) = \alpha + \beta$
ii. $\lim_{x\to p}(f \cdot g)(x) = \alpha \cdot \beta$
iii. $\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{\alpha}{\beta}$

Note that with limits of functions, we require that $x \neq p$.

17.3 Continuous functions

The rules that govern continuous functions are a simple extension of the limits of functions except now we examine when $x = p$.

Definition 17.3.1: Continuous functions

Let a function $f : X \to Y$ be given and fix $p \in X$ (p need not be a limit point). We say that f is continuous at p if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y\big(f(x), f(p)\big) < \varepsilon$ for all $x \in X$ with $d_X(x,p) < \delta$. If f is continuous at all points $p \in X$, we say that f is continuous on X.

This can be written as $\lim_{x\to p} f(x) = f(p)$ since the above statement is automatically true when $x = p$.

An important realization is that we could have the limit of a function exist at a certain point, but the function need not be continuous at that point. A common example of this is a hole discontinuity.

Theorem 17.3.2: Continuity and sequences

f is continuous at $p \iff$ for all sequences $\{x_n\} \subset X$ with $\{x_n\} \to p$, we have $\lim_{n\to\infty} f(x_n) = f(p)$.

We will not prove this theorem, but it is very similar to the proof for Theorem 17.2.2.

Example 17.3.3: Continuous functions

Let $f, g : X \to Y$ be continuous functions and let E be a dense subset of X. Assume that $f(x) = g(x)$ for all $x \in E$. Show that $f(x) = g(x)$ for all $x \in X$.

Proof: Please note by Definition 8.1.2 we have that $X = \overline{E} = E \cup E'$. Since we already know that $f(x) = g(x)$ for all $x \in E$, we just need to prove that $f(p) = g(p)$ for all $p \in E'$.

Let a point $p \in E'$ be given. Since p is a limit point of E, by Lemma 13.1.2 we can find a sequence $\{x_k\} \subset E$ such that $\{x_k\} \to p$. By Theorem 17.3.2 we know that $\lim_{k\to\infty} f(x_k) = f(p)$ and $\lim_{k\to\infty} g(x_k) = g(p)$. Since $f(x) = g(x)$ for all $x \in E$, we see that

$$f(x_k) = g(x_k) \implies \lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} g(x_k) \implies f(p) = g(p)$$

This concludes the proof. □

Example 17.3.4: Continuous functions

Let $f : X \to \mathbb{R}$ be a continuous function. Show that for every set $E \subset X$, $f(\overline{E}) \subset \overline{f(E)}$.

Proof: By Definition 9.1.1 we know that $f(E) \subset \overline{f(E)}$, so we just need to show that $f(E') \subset \overline{f(E)}$.

Let $x \in E'$. By Lemma 13.1.2, there exists a sequence $\{x_n\} \subset E$ such that $\{x_n\} \to x$. By Definition 17.3.1, we see that $f(\{x_n\}) \to f(x) \implies f(x)$ is a limit point of $f(E)$ since we have infinitely many points tending to $f(x)$. Therefore, $f(x) \in \overline{f(E)}$. □

Lecture 18: 03/31/20 (online)

We continue our discussion on continuity and introduce how continuity operates in compact metric spaces towards the end of the lecture.

18.1 More on continuous functions

We often enjoy dealing with continuous functions because of the nice properties they have. Below is a theorem that helps us identify continuous functions.

Theorem 18.1.1: Composition of continuous functions

Let (X, d_X), (Y, d_Y), and (Z, d_Z) be given metric spaces. Assume that $f : X \to Y$ is continuous at $p \in X$ and $g : Y \to Z$ is continuous at $f(p) \in Y$. We declare $h = g \circ f : X \to Z$ to be continuous at p.

Proof: Take an arbitrary sequence $\{x_n\} \subset X$ with $\{x_n\} \to p$. By Theorem 17.3.2 we know that $\{f(x_n\} \to f(p)$ since f is continuous at p. Moreover, $\{g\big(f(x_n)\big)\} \to g\big(f(p)\big)$ as g is continuous at $f(p)$. Note that $\{f(x_n)\}$ is the sequence we use as an input for g.

We wanted to show that $\{h(x_n)\} \to h(p)$ since that would imply continuity. Notice that $g\big(f(x_n)\big) \equiv h(x_n)$ and $g\big(f(p)\big) \equiv h(p)$, so we are done. $\square$

In particular, if f is continuous on X and g is continuous on Y, then h is continuous on X.

Although the typical ε - δ definition of continuity works, we can look to define continuity in a more topological sense. We do just this in the following theorem.

Theorem 18.1.2: Continuity and preimages

Let (X, d_X) and (Y, d_Y) be given metric spaces. A function $f : X \to Y$ is continuous on $X \iff f^{-1}(V)$ is open in X for every V open in Y.

Proof: Since we are proving an if-and-only-if statement, we need to show that each side implies the other.

i. Assume that $f : X \to Y$ is continuous on X. Let V be an open set in Y and let $p \in X$ such that $f(p) \in V$, that is, $p \in f^{-1}(V)$. As V is open, $f(p)$ is an interior point of V, so there exists an $\varepsilon > 0$ such that $B_Y(f(p), \varepsilon) \subset V$. By definition of continuity, there exists a $\delta > 0$ such that $d_Y\big(f(x), f(p)\big) < \varepsilon$ for all $x \in X$ such that $d_X(x,p) < \delta$, or that $f(x) \in B_Y(f(p), \varepsilon) \subset V$ for all $x \in X$ such that $d_X(x,p) < \delta$. This means that $B_X(p, \delta) \subset f^{-1}(V)$, implying that p is an interior point of $f^{-1}(V)$, so $f^{-1}(V)$ is open.

ii. Assume that $f^{-1}(V)$ is open in X for every V open in Y. Fix $p \in X$ and $\varepsilon > 0$, and let $V = B_Y(f(p), \varepsilon)$, which is an open set in Y by Theorem 8.1.4. Now, $f^{-1}(V) = f^{-1}\big(B_Y(f(p), \varepsilon)\big)$ is open in X based on our assumption, and $p \in f^{-1}(V)$ since $f(p) \in B_Y(f(p), \varepsilon)$. Since $f^{-1}\big(B_Y(f(p), \varepsilon)\big)$ is open and $p \in f^{-1}\big(B_Y(f(p), \varepsilon)\big) = f^{-1}(V)$, p is an interior point of $f^{-1}(V)$. Therefore, there exists a $\delta > 0$ such that $B_X(p, \delta) \subset f^{-1}(V)$. It follows that for $x \in X$ such that $d_X(x,p) < \delta$, we have $d_Y\big(f(x), f(p)\big) < \varepsilon$ because $x \in B_X(p, \delta)$ which is a subset of the preimage of a ball centered around $f(p)$ with radius ε. This is precisely the ε - δ condition from Definition 17.3.1, so f is continuous.

This concludes the proof. $\square$

Note that we can also say that a function $f : X \to Y$ is continuous on $X \iff f^{-1}(C)$ is closed in X for every C closed in Y.

This is natural as C closed $\iff$ $(Y \setminus C)$ is open. Moreover, $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$ is an open set since $(Y \setminus C)$ is open, meaning it's preimage must be open because f is continuous. Therefore, $f^{-1}(C)$ is closed.

Let's now look at functions that go from a general metric space X to $\mathbb{R}^n$. In other words, $f : X \to \mathbb{R}^n$. For these types of functions, we write $f(x) = \big(f_1(x), f_2(x), \dots, f_n(x)\big)$ where $f_i : X \to \mathbb{R}$ for $1 \leq i \leq n$.

Remark 18.1.3: Continuity in $\mathbb{R}^n$

For $f : X \to \mathbb{R}^n$, f is continuous on $X \iff f_1, f_2, \dots, f_n$ are continuous on X.

Let $f, g : X \to \mathbb{R}^n$ be continuous functions. Then, $f + g$ and $f \cdot g$ are continuous as well.

Note that $f(x) \cdot g(x) = f_1(x) \cdot g_1(x) + \dots + f_n(x) \cdot g_n(x)$.

18.2 Continuity and compactness

This section combines two properties that we like to encounter, continuity and compactness. Luckily for us, there are many relationships between the two.

Theorem 18.2.1: Compact image

Let $f : X \to Y$ be a continuous function and assume X is compact. This implies that $f(X)$ is compact.

Proof: Pick a sequence of distinct points $\{y_n\} \subset f(X)$. Then $y_n = f(x_n)$ for some $x_n \in X$. By definition of a function, $\{x_n\}$ is a sequence of distinct points in X.

Since X is compact, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \to \overline{x}$ for some $\overline{x} \in X$ (see Theorem 13.2.7). As f is continuous, $\{y_{n_k}\} = \{f(x_{n_k})\} \to f(\overline{x})$, which means that our general infinite subset $\{y_n\}$ of $f(X)$ has a limit point in $f(X)$. Therefore, by Lemma 10.1.7, $f(X)$ is compact. □

Theorem 18.2.2: Closed and bounded image in $\mathbb{R}^n$

Let $f : X \to \mathbb{R}^n$ be a continuous function and assume X is compact. This implies that $f(X)$ is closed and bounded in $\mathbb{R}^n$.

Note that this is a direct consequence of Theorem 18.2.1 since when we are in $\mathbb{R}^n$ with the Euclidean distance metric, a set E is compact $\iff$ E is closed and bounded.

Theorem 18.2.3: Extrema of images in $\mathbb{R}$

Let $f : X \to \mathbb{R}$ be a continuous function and assume X is compact. This implies the existence of $p, q \in X$ such that $f(p) = \max_{x\in X} f(x)$ and $f(q) = \min_{x\in X} f(x)$.

Proof: We know that $f(X)$ is closed and bounded in $\mathbb{R}$ from Theorem 18.2.2. Therefore, by Theorem 9.1.3, we know $\sup f(X)$ and $\inf f(X)$ are in $f(X)$. In particular, we can find $p, q \in X$ such that $f(p) = \sup f(X) = \max_{x\in X} f(x)$ and $f(q) = \inf f(X) = \min_{x\in X} f(x)$. □

Theorem 18.2.4: Continuous bijections

Let $f : X \to Y$ be a continuous function such that f is 1-to-1 and onto (f is bijective). Assume that X is compact. This implies that $g = f^{-1} : Y \to X$ is continuous.

Proof: We only need to show that for C closed in X, $g^{-1}(C)$ is closed in Y (see Theorem 18.1.2). Note that C being closed and a subset of a compact metric space implies that C is compact (see Lemma 10.1.5). Therefore, by Theorem 18.2.1, $g^{-1}(C) = f(C)$ is compact, which means that $g^{-1}(C)$ is closed (see Lemma 10.1.2), so g is continuous. $\square$

Lecture 19: 04/07/20 (online)

We will discuss uniform continuity, the intermediate value theorem, and monotone functions in this lecture.

19.1 Uniform continuity

Uniform continuity is more or less an elitist version of continuity we defined in Definition 17.3.1. We still have an ε - δ condition but now it's a little more stringent. Below are two equivalent definitions of uniform continuity.

Definition 19.1.1: Uniform continuity

A function $f : X \to Y$ is uniformly continuous if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y\big(f(x), f(y)\big) < \varepsilon$ for all $x, y \in X$ with $d_X(x, y) < \delta$.

A function $f : X \to Y$ is uniformly continuous if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(y) \in B_Y(f(x), \varepsilon)$ for all $x, y \in X$ with $y \in B_X(x, \delta)$.

Notice the subtlety in differences from a function being continuous and uniformly continuous. With uniform continuity, δ is independent of the position of x and y, meaning it only depends on ε.

Example 19.1.2: Uniform continuity

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x$. Is f uniformly continuous?
f is uniformly continuous since for each $\varepsilon > 0$, we can let $\delta = \frac{\varepsilon}{2}$, meaning that $|f(x) - f(y)| = |x - y| < \frac{\varepsilon}{2} < \varepsilon$.

Example 19.1.3: Uniform continuity

Let $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = x^2$. Is g uniformly continuous?
g is NOT uniformly continuous. Let's assume that it is and fix $\varepsilon = 1$. This means that there exists a $\delta > 0$ such that $|x^2 - y^2| < \varepsilon = 1$ for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Pick $y = n$ and $x = n + \frac{\delta}{2}$ for some $n \in \mathbb{N}$ to be chosen. We then see that

$$|x^2 - y^2| = \left(n + \frac{\delta}{2}\right)^2 - n^2 = n \cdot \delta + \frac{\delta^2}{4} > n \cdot \delta > 1$$

provided that $n > \frac{1}{\delta}$.

Example 19.1.4: Uniform continuity

Let $f : (0,1) \to \mathbb{R}$ be uniformly continuous. Show that $f\big((0,1)\big)$ is a bounded set in $\mathbb{R}$.

Proof: Let $\varepsilon > 0$ be given and find a corresponding δ. Next, collect a set $V = \{\delta, 2\delta, \ldots, N\delta\}$ such that $(N+1)\delta > 1$. Since V is a finite set, let $C = \max\{|f(\delta)|, |f(2\delta)|, \ldots, |f(N\delta)|\}$. Please note that each point $a \in (0,1)$ is within δ from a point in $v \in V$. By uniform continuity, we then know that

$$|f(v) - f(a)| < \varepsilon \implies |f(a)| < C + \varepsilon$$

Since a is arbitrary, we know that $f\big((0,1)\big)$ is bounded by $C + \varepsilon$. □

We now establish a link between compactness and uniform continuity.

Theorem 19.1.5: Uniform continuity and compactness

Let $f : X \to Y$ be a continuous function and assume that X is compact. This implies that f is uniformly continuous.

Proof: Fix $\varepsilon > 0$ and find $\frac{\varepsilon}{2}$. For each $p \in X$, as f is continuous at p, we can find a $\delta_p > 0$ such that $d_Y\big(f(x), f(p)\big) < \frac{\varepsilon}{2}$ for all $x \in B_X(p, \delta_p)$. Please note that $X = \bigcup_{p \in X} B_X(p, \frac{\delta_p}{2})$. As X is compact, we can find a finite number of points $p_1, p_2, \ldots, p_k \in X$ such that $X = \bigcup_{i=1}^{k} B_X(p_i, \frac{\delta_{p_i}}{2})$.

Let $K = \{1, 2, \ldots, k\}$ and have $\delta = \min_{i \in K} \frac{\delta_{p_i}}{2}$. We now show that, for any $x, y \in X$ with $d_X(x,y) < \delta$, we have $d_Y\big(f(x), f(y)\big) < \varepsilon$. Certainly $x \in B_X(p_i, \frac{\delta_{p_i}}{2})$ for some $i \in K$. As $d_X(x,y) < \delta \leq \frac{\delta_{p_i}}{2}$, we see that $x, y \in B_X(p_i, \delta_{p_i})$. Therefore,

$$d_Y\big(f(x), f(y)\big) \leq d_Y\big(f(x), f(p_i)\big) + d_Y\big(f(p_i), f(y)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This concludes the proof. □

19.2 Intermediate value theorem

The intermediate value theorem is so important that it deserves a section by itself. This is probably one of the most cited theorems in everyday life and a very well-known consequence of continuous functions.

Theorem 19.2.1: Intermediate value theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, for any value y between $f(a)$ and $f(b)$, there exists a $c \in [a, b]$ such that $f(c) = y$.

Proof: Without loss of generality, we assume $f(a) \leq f(b)$. Then, $f(a) \leq y \leq f(b)$. Cases of equality are straightforward, so we will only consider $f(a) < y < f(b)$.

Let $S = \{x \in [a, b] : f([a, x]) \subset (-\infty, y]\}$. Certainly $a \in S \subset [a, b]$. Let $c = \sup S$. By continuity of f we know that $f(c) \leq y$ and therefore $c \in S$. We also know that $c < b$ as $f(c) \leq y < f(b)$.

As $c = \sup S$, we have that $c + \varepsilon \notin S$ for any $\varepsilon > 0$. This means that there exists a sequence $\{x_n\} \subset [c, b]$ such that $c < x_n < c + \frac{1}{n}$ and $f(x_n) > y$. Therefore, by continuity, $f(c) = \lim_{n\to\infty} f(x_n) \geq y$.

We have arrived at the claims $y \leq f(c)$ and $y \geq f(c)$, which means that $y = f(c)$. $\square$

Note that we can set $[a, b] = \mathbb{R}$ and this theorem would still apply.

What the intermediate value theorem tells us is that if we choose a point in between two outputs of a real-valued and continuous function, we can find an input such that our function will send that input to our chosen point in the output space.

19.3 Monotone functions of real numbers

Although these are more-or-less colloquial terms for limits of a function, left-hand and right-hand limits serve an important purpose when discussing monotone functions. We define these below.

Definition 19.3.1: Left-hand limit

Let $f : (a, b) \to \mathbb{R}$ be a given function and let $c \in (a, b)$. We say that $\lim_{x\to c^-} f(x) = \alpha$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - \alpha| < \varepsilon$ for all $x \in (a, b) \cap (c - \delta, c)$.

Definition 19.3.2: Right-hand limit

Let $f : (a, b) \to \mathbb{R}$ be a given function and let $c \in (a, b)$. We say that $\lim_{x\to c^+} f(x) = \beta$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - \beta| < \varepsilon$ for all $x \in (a, b) \cap (c, c + \delta)$.

Remark 19.3.3: Left-hand and right-hand agreement

If $\lim_{x\to c^+} \neq \lim_{x\to c^-}$, then $\lim_{x\to c}$ does not exist.

This means that $\lim_{x\to c} f(x) = A \iff \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = A$.

Monotone functions of real numbers are extremely similar to monotone sequences of real numbers (discussed in Section 14.3). Like monotone sequences, we see these a lot outside of the classroom, so the following definitions and their consequences are of interest to us.

Definition 19.3.4: Nondecreasing function

Let $f : (a, b) \to \mathbb{R}$ be a given function. f is nondecreasing if $f(x) \leq f(y)$ for all $x \leq y$ where $x, y \in (a, b)$.

Definition 19.3.5: Nonincreasing function

Let $f : (a, b) \to \mathbb{R}$ be a given function. f is nonincreasing if $f(x) \geq f(y)$ for all $x \geq y$ where $x, y \in (a, b)$.

Note that with both of these definitions, f need not be continuous.

Theorem 19.3.6: Left-hand and right-hand limits of monotonic functions

Let $f : (a, b) \to \mathbb{R}$ be a nondecreasing function. Then, for each $c \in (a, b)$, we have $\lim_{x\to c^-} f(x) = \sup_{a<x<c} f(x) \leq f(c)$ and $\lim_{x\to c^+} f(x) = \inf_{c<x<b} f(x) \geq f(c)$.
A proof is not given for this theorem.

The next theorem takes us way back to countability. Please see Section 6.1 for a refresher on this concept.

Theorem 19.3.7: Discontinuities of monotonic functions

Let $f : (a, b) \to \mathbb{R}$ be a nondecreasing function. Then, the set of discontinuities of f is at most countable.

Proof: Let E be the set of discontinuities of f. For each $c \in E$, it must be the case that $\lim_{x\to c^-} f(x) = \sup_{a<x<c} f(x) < \lim_{x\to c^+} f(x) = \inf_{c<x<b} f(x)$. Take $h(c) \in \mathbb{Q}$ so that $\lim_{x\to c^-} f(x) < h(c) < \lim_{x\to c^+} f(x)$. In other words, $h(c)$ is a rational number that stays in the gap between the left-hand and right-hand limits.

For $c \neq d$ with $c, d \in E$, we have that $h(c) \neq h(d)$. We have designed a map $h : E \to \mathbb{Q}$

where $c \mapsto h(c) \in \mathbb{Q}$. This map is 1-to-1 and since $\mathbb{Q}$ is countable, by Remark 6.1.6 we see that E is at most countable. $\square$

Lecture 20: 04/09/20 (online)

We introduce differentiation in this lecture. In fact, this whole lecture is devoted to learning more about differentiation.

20.1 Differentiation

It's important to recognize that our main object is a function $f : (a, b) \to \mathbb{R}$. Note that we can also let $(a, b) = \mathbb{R}$ and what we prove will still hold.

Definition 20.1.1: Differentiability

Let a function $f : (a, b) \to \mathbb{R}$ be given. For $x \in (a, b)$, we say that f is differentiable at x if the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$ exists.

We can also write $f'(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x}$.

Theorem 20.1.2: Differentiability implies continuity

Let $f : (a, b) \to \mathbb{R}$ be a given function and assume that f is differentiable at a given $x \in (a, b)$. This implies that f is continuous at x.

Proof: We have

$$\lim_{h\to 0}[f(x+h) - f(x)] = \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h} \cdot h\right] = f'(x) \cdot 0 = 0$$

This implies that $\lim_{h\to 0} f(x + h) = \lim_{h\to 0} f(x) = f(x)$. Although not explicitly in ε - δ form, this is an equivalent statement, so continuity is verified. □

Observe that the conclusion made in Theorem 20.1.2 means that claiming a function is differentiable is a stronger claim than claiming a function is continuous. These two properties should not be used interchangeably.

Definition 20.1.3: Remainder term definition of differentiability

Recall Definition 20.1.1 where we had $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. If we drop the limit, we can say that $\frac{f(x+h)-f(x)}{h} = f'(x) + \omega(h)$ where $\omega(h)$ is a remainder term and $\lim_{h\to 0} \omega(h) = 0$.

A useful arrangement of the above expression can be $f(x+h) = f(x)+f'(x)\cdot h+\omega(h)\cdot h$. This allows us to see that to get to $f(x+h)$, we can start at $f(x)$, move along a line tangent to $f(x)$ by a distance $f'(x)\cdot h$, then make up the difference by adding $\omega(h)\cdot h$.

The following theorem seems unnecessary, but it allows us to link the comments we made in Definition 20.1.3 to something more concrete.

Theorem 20.1.4: Different form of differentiability

$\boldsymbol{f : (a,b) \to \mathbb{R}}$ **is differentiable at** $\boldsymbol{x \iff f}$ **can be written as** $\boldsymbol{f(x+h) = f(x) + c\cdot h + \omega(h)\cdot h}$ **for** $\boldsymbol{|h| < \min\{b-x, x-a\}}$ **and** $\boldsymbol{c \in \mathbb{R}}$**.**

Proof: Note that the theorem assumes $c \in \mathbb{R}$ is a given number and $\omega : \mathbb{R} \to \mathbb{R}$ such that $\lim_{h\to 0}\omega(h) = 0$. Furthermore, if we look at the form given in this theorem and Definition 20.1.3, we see that $c = f'(x)$.

Note that our statement can be rearranged as $\frac{f(x+h)-f(x)}{h} = c+\omega(h) \to c$ as $h \to 0$. Since $c = f'(x)$, we are done. $\square$

Theorem 20.1.5: Operators of differentiable functions

Let $\boldsymbol{f, g : (a,b) \to \mathbb{R}}$ **be differentiable at** $\boldsymbol{x \in (a,b)}$**. The following hold:**

- **i.** $\boldsymbol{(f+g)'(x) = f'(x) + g'(x)}$
- **ii.** $\boldsymbol{(f\cdot g)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)}$
- **iii.** $\boldsymbol{(\frac{f}{g})'(x) = \frac{\left(f'(x)\cdot g(x) - f(x)\cdot g'(x)\right)}{g(x)^2}}$ **given** $\boldsymbol{g(x) \neq 0}$

Proof: We will only prove the second part, known as the product rule.

ii. We will solve this by introducing something called a connection term. Observe that

$$(f\cdot g)'(x) = \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

is equivalent to

$$\lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x+h)\cdot g(x) + f(x+h)\cdot g(x) - f(x)\cdot g(x)}{h}$$

which in turn is equivalent to

$$\lim_{h\to 0}\left[f(x+h)\cdot\frac{g(x+h)-g(x)}{h} + g(x)\cdot\frac{f(x+h)-f(x)}{h}\right]$$

Note that $\lim_{h\to 0} f(x+h) = f(x)$ and that f and g are differentiable. This leaves us with

$$f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

This concludes the proof we will do for this theorem. $\square$

We will now introduce the chain rule, which something we often see in common applications of mathematics (physics, engineering, economics, etc.). There are often incorrect proofs done for the chain rule that appear a lot shorter than what we are about to embark on, but what follows is, if not clear, at least accurate.

Theorem 20.1.6: Chain rule

Let $f : (a, b) \to (c, d)$ and $g : (c, d) \to \mathbb{R}$ be two given functions. Assume that f is differentiable at x and g is differentiable at $y = f(x)$. This implies that $\phi = g \circ f$ is differentiable at x and $\phi'(x) = g'\big(f(x)\big) \cdot f'(x)$.

Proof: Since f is differentiable at x, we can write

$$f(x+h) = f(x) + f'(x) \cdot h + \omega_1(h) \cdot h$$

where $\lim_{h\to 0} \omega_1(h) = 0$. Similarly, since g is differentiable at $y = f(x)$, we can write

$$g(y+s) = g(y) + g'(y) \cdot s + \omega_2(s) \cdot s$$

where $\lim_{s\to 0} \omega_2(s) = 0$.

Let $s = f(x+h) - f(x)$. Observe that $y + s = f(x+h)$ since $y = f(x)$. We see that

$$\phi(x+h) = g\big(f(x+h)\big) = g(y+s) = g(y) + g'(y) \cdot s + \omega_2(s) \cdot s$$

$$=$$

$$g(y) + g'(y) \cdot [f(x+h) - f(x)] + \omega_2(s) \cdot [f(x+h) - f(x)]$$

$$=$$

$$g(y) + g'(y) \cdot [f'(x) \cdot h + \omega_1(h) \cdot h] + \omega_2(s) \cdot [f'(x) \cdot h + \omega_1(h) \cdot h]$$

$$=$$

$$\phi(x) + [g'\big(f(x)\big) \cdot f'(x)] \cdot h + \big[[f'(x) + \omega_1(h)] \cdot \omega_2(s) + g'(y) \cdot \omega_1(h)\big] \cdot h$$

If we let

$$\omega(h) = [f'(x) + \omega_1(h)] \cdot \omega_2(s) + g'(y) \cdot \omega_1(h)$$

we can see that $\lim_{h \to 0} \omega(h) = 0$ and

$$\phi(x+h) = \phi(x) + [g'\big(f(x)\big) \cdot f'(x)] \cdot h + \omega(h) \cdot h$$

This concludes the proof. $\square$

Lecture 21: 04/14/20 (online)

We continue our talks on differentiation and how to use it to find meaningful results. In this lecture, we discuss local minimums and maximums and the mean value theorem.

21.1 Local minimum and maximum

Although we have an intuitive grasp on what a maximum and a minimum are, the following definitions explicitly detail what we look for.

Definition 21.1.1: Local maximum

Let $f : (a, b) \to \mathbb{R}$ be a given function and $c \in (a, b)$. We say that c is a local min of f if there exists an $r > 0$ such that $f(c) \leq f(x)$ for all $x \in (c - r, c + r) \cap (a, b)$.

Definition 21.1.2: Local minimum

Let $f : (a, b) \to \mathbb{R}$ be a given function and $c \in (a, b)$. We say that c is a local max of f if there exists an $r > 0$ such that $f(c) \geq f(x)$ for all $x \in (c - r, c + r) \cap (a, b)$.

Please note that local maximum $\not\Longrightarrow$ global maximum and similarly for local and global minimum. Also, we could let $(a, b) = \mathbb{R}$ and not have any issues.

Local maximums and minimums are precisely a form of optimization. The next theorem gives us a trick to find these optimal points.

Theorem 21.1.3: Local extrema and derivatives

Let $f : (a, b) \to \mathbb{R}$ be a given function and have $c \in (a, b)$ be a local max of f. If f is differentiable at c, then $f'(c) = 0$.

Proof: We have

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

since $f(c + h) - f(c) \leq 0$ and $h > 0$ when $h \to 0^+$. Furthermore,

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

since $f(c + h) - f(c) \leq 0$ and $h < 0$ when $h \to 0^-$.

We have $f'(c) \leq 0 \leq f'(c)$, which means $f'(c) = 0$. □

A similar method can be used to show that $f'(c) = 0$ when $c \in (a, b)$ is a local min of f.

Theorem 21.1.4: Rolle's Theorem

Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function such that f is differentiable in (a, b). Assume that $f(a) = f(b)$. Then, there exists a $c \in (a, b)$ such that $f'(c) = 0$.

Proof: Since $[a, b]$ is compact, we know f attains its maximum and minimum value on $[a, b]$ by Theorem 18.2.3. We will look at two cases:

i. Suppose $\max_{[a,b]} f = \min_{[a,b]} f$. This means that $f(x) = f(a) = f(b)$ for all $x \in (a, b)$. Therefore, $f'(x) = 0$ for all $x \in (a, b)$.
ii. Suppose $\max_{[a,b]} f > \min_{[a,b]} f$. Then, either the maximum value of f or the minimum value of f is not $f(a)$ (meaning the endpoint a cannot map to both the maximum and minimum value). Without loss of generality, assume $\max_{[a,b]} f \neq f(a) = f(b)$. We can then find a $c \in (a, b)$ such that $f(c) = \max_{[a,b]} f$. In particular, c is a local max of f, so therefore, $f'(c) = 0$.

These two cases cover all possibilities, so the proof is concluded □

Theorem 21.1.4 tells us that if we have a differentiable function with two distinct points in our input space that map to the same point in our output space, the function goes flat (meaning $f' = 0$) in at least one point between our two inputs.

21.2 Generalized mean value theorem

Similar to the intermediate value theorem from Theorem 19.2.1, the mean value theorem is more or less a staple in teachings of differentiable functions. In the following theorem we provide a more general mean value theorem than what is usually encountered in calculus classes.

Theorem 21.2.1: Generalized mean value theorem

Let $a < b$ and $f, g : [a, b] \to \mathbb{R}$ be given continuous functions such that f, g are differentiable on (a, b). There exists a $c \in (a, b)$ such that $[f(b) - f(a)] \cdot g'(c) = [g(b) - g(a)] \cdot f'(c)$.

Proof: Observe that since a, b are fixed, the only variable terms that are free to change are $f'(c)$ and $g'(c)$. Also, please take note that we can simply rearrange our desired conclusion to be $h'(c) = [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0$ for some $c \in (a, b)$, where $h(x) = [f(b) - f(a)] \cdot g(x) - [g(b) - g(a)] \cdot f(x)$.

Since h is simply a composition of differentiable functions, we know that h is both differentiable and continuous in (a, b). If we stare at h long enough and cancel out terms, we see that $h(a) = h(b)$. Therefore, we can apply Theorem 21.1.4 to find that there exists a $c \in (a, b)$ such that $h'(c) = 0$. This concludes the proof. □

Please note that in the case where $g(x) = x$ for $x \in [a, b]$, then we deduce the usual mean value theorem of $f'(c) = \frac{f(b)-f(a)}{b-a}$, which denotes that f' attains the average slope at some point $c \in (a, b)$.

We can employ the mean value theorem right away with monotonic functions. This might seem rather straightforward and obvious but it is worth covering due to the frequent encounter of monotonic functions.

Theorem 21.2.2: Monotonic functions and derivatives

Let $a < b$, $f : [a, b] \to \mathbb{R}$ be given continuous functions such that f is differentiable in (a, b). Then,

i. If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is non-decreasing.
ii. If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is non-increasing.
iii. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

Proof: We will only prove the first claim.

i. For any $x, y \in [a, b]$ with $x < y$, there exists a $z \in (x, y)$ such that $f(y) - f(x) = (y - x)f'(z)$ and since $(y - x)f'(z) \geq 0$, this implies $f(y) \geq f(x)$, so f is non-decreasing.

The proofs for (ii.) and (iii.) follow naturally from the proof for (i.), so this concludes the proof. □

Example 21.2.3: Generalized mean value theorem

Suppose f is defined and differentiable for every $x > 0$ and $f'(x) \to 0$ as $x \to \infty$. Let $g(x) = f(x + 1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$.

Proof: First, let's examine the nature of $f'(x)$. Since $\lim_{x\to\infty} f'(x) = 0$, for each $\varepsilon > 0$, we can find a marker $R \in \mathbb{R}$ such that for all $x > R$, we have

$$|f'(x) - 0| < \varepsilon \implies |f'(x)| < \varepsilon$$

Fix $\varepsilon > 0$, find a corresponding R, and let $x > R$. By Theorem 21.2.1 we can find a $c \in (x, x+1)$ such that

$$f'(c) = \frac{f(x+1) - f(x)}{x + 1 - x} = f(x+1) - f(x)$$

Observe that this is precisely the form of $g(x)$, so $f'(c) = g(x)$ for our chosen $x > R$. Furthermore, since $c \in (x, x+1)$ and $x > R$, this implies that $c > R$ so

$$f'(c) = g(x) < \varepsilon \implies |g(x) - 0| < \varepsilon$$

Therefore, we have shown that $g(x) \to 0$ as $x \to \infty$, thus concluding the proof. □

The following is an interesting example of the mean value theorem in action. What follows is not a formal proof, but there is enough information where the proof should be able to be written.

Example 21.2.4: Generalized mean value theorem

Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree $k \in \mathbb{N}$. That is, for $x \in \mathbb{R}$, we can write

$$\boldsymbol{P(x) = a_k \cdot x^k + a_{k-1} \cdot x^{k-1} + \ldots + a_1 \cdot x + a_0}$$

Here, $a_i \in \mathbb{R}$ for all $0 \leq i \leq k$ and $a_k \neq 0$. Can P have more than k distinct real roots?

We claim that P cannot have more than k distinct real roots, a real root being some $c \in \mathbb{R}$ such that $P(c) = 0$. First, note that if $P(y) = P(z) = 0$ for some $y < z$, then $P'(v) = 0$ for some $v \in (y, z)$ by Theorem 21.2.1. Therefore, if P had more than k distinct real roots, then P' has more than $k - 1$ distinct real roots, P'' has more than $k - 2$ distinct real roots, ... , $P^{(k)}$ has more than 0 distinct real roots. However, please note that $P^{(k)} = a_k \cdot k!$ which is a constant and does not equal zero (since $a_k \neq 0$). This is a contradiction, so therefore, P cannot have more than k distinct roots.

Lecture 22: 04/16/20 (online)

In this lecture we talk about L'Hôpital's rule, which is a method for evaluating limits of specific forms, and higher order derivatives.

22.1 L'Hôpital's rule

We sometimes encounter situations where we evaluate limits of the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$. In situations like these, we can employ the following theorem.

Theorem 22.1.1: L'Hôpital's rule

Let two numbers $-\infty \leq a < b \leq \infty$ such that $f, g : (a, b) \to \mathbb{R}$ are given differentiable functions with $g' \neq 0$ in (a, b). Suppose that $\lim_{x\to a} \frac{f'(x)}{g'(x)} = A \in \mathbb{R} \cup \{-\infty, \infty\}$. Assume further that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$. Then, $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f(x)}{g(x)} = A$

Proof: We look at the two different cases of $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Although the theorem works for values of $-\infty$ and ∞, we will only prove the situations where $a, b, A \in \mathbb{R}$.

i. Consider when $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. First, let's fix $\varepsilon > 0$. Since $\frac{f'(p)}{g'(p)}$ converges to A as $p \to a$, we can find a $\delta > 0$ such that $|\frac{f'(p)}{g'(p)} - A| < \frac{\varepsilon}{2}$ for all $p \in (a, a + \delta)$ where $a + \delta \in (a, b)$. For $a < x < y < a + \delta$, the mean value theorem (Theorem 21.2.1) tells us that there exists a $z \in (x, y)$ such that

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(z)}{g'(z)} \in \left(A - \frac{\varepsilon}{2}, A + \frac{\varepsilon}{2}\right)$$

Now let $x \to a$. As $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, we get, for $a < z < y < a+\delta$,

$$\frac{f(y)}{g(y)} = \frac{f'(z)}{g'(z)} \in \left(A - \frac{\varepsilon}{2}, A + \frac{\varepsilon}{2}\right) \implies \left|\frac{f(y)}{g(y)} - A\right| < \frac{\varepsilon}{2}$$

This is precisely an ε - δ condition so we are done with this case.

ii. Consider when $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$. Once again, fix $\varepsilon > 0$. Since $\frac{f'(p)}{g'(p)}$ converges to A as $p \to a$, we can find a $\delta > 0$ such that $|\frac{f'(p)}{g'(p)} - A| < \frac{\varepsilon}{2}$ for all $p \in (a, a + \delta)$ where $a + \delta \in (a, b)$. For $a < x < y < a + \delta$, there exists a $z \in (x, y)$ such that

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(z)}{g'(z)} \in \left(A - \frac{\varepsilon}{2}, A + \frac{\varepsilon}{2}\right)$$

Now fix $y \in (a, a+\delta)$. This means that $f(y), g(y)$ are just fixed numbers. After a lot of rearranging, we can now say

$$\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f(y)-f(x)}{-g(x)} \cdot \frac{-g(x)}{g(y)-g(x)} = \left(\frac{f(y)}{-g(x)} + \frac{f(x)}{g(x)}\right) \cdot \frac{-g(x)}{g(y)-g(x)}$$

As $\lim_{x\to a} g(x) = \infty$, we see that $\lim_{x\to a} \frac{f(y)}{-g(x)} = 0$ and $\lim_{x\to a} \frac{-g(x)}{g(y)-g(x)} = 1$. Next, we can choose a $\overline{\delta} \in (0, \delta)$ and let $a < x < a + \overline{\delta}$. Please note that we are NOT saying $x \to a$, but rather that x is at most a distance $\overline{\delta}$ away from a. Therefore, for $a < x < a + \overline{\delta}$, $\lim_{x\to a} \frac{f(y)}{-g(x)} \approx 0$ and $\lim_{x\to a} \frac{-g(x)}{g(y)-g(x)} \approx 1$. Let's now look at our original expression. We can say that for $a < x < z < y < a + \overline{\delta}$,

$$\frac{f(x)}{g(x)} \approx \frac{f'(z)}{g'(z)} \in (A-\varepsilon, A+\varepsilon)$$

Therefore, we would like to make sure that our quantity $\frac{f(x)}{g(x)}$ is within a prescribed distance from A. Therefore, we choose our $\overline{\delta}$ so that $\frac{f(x)}{g(x)} \in (A-\varepsilon, A+\varepsilon)$. In other words, we added some insurance in the form of ε to A due to the fact that we didn't let $x \to a$. This, once again, is precisely an ε - δ condition, so we are done with this case.

This concludes the proof. □

This rule is also valid for $\lim x \to c$ where $c \in (a, b)$. However, note that this eliminates the possibility of $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ since our function needs to be well defined in (a, b).

Please note that we have some odd notation in Theorem 22.1.1, mainly $A \in \mathbb{R} \cup (-\infty, \infty)$ and $\infty \leq a < b \leq \infty$. What this means is that A can be either $-\infty$ or ∞ as well as any real number, and a or b could be an infinite value.

Let's now look at an example that uses L'Hôpital's Rule.

Example 22.1.2: L'Hôpital's rule

Suppose that $f : (-1, 1) \to \mathbb{R}$ such that $f''(x)$ exists for all $x \in (-1, 1)$. Show that

$$\lim_{h\to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} = f''(0)$$

Proof: Please note that when we look at $\lim_{h\to 0} \frac{f(h)+f(-h)-2f(0)}{h^2}$, we see that it approaches $\frac{0}{0}$. Therefore, we will look to use Theorem 22.1.1. We will define $g(h) = f(h) + f(-h) - 2f(0)$ and $j(h) = h^2$. To use L'Hôpital's rule, we must verify that $\lim_{h\to 0} \frac{g'(h)}{j'(h)}$ is well defined.

Let's first define our derivatives. We see that $g'(h) = f'(h) - f'(-h)$ and $j'(h) = 2h$, so

$$\lim_{h\to 0} \frac{g'(h)}{j'(h)} = \lim_{h\to 0} \frac{f'(h) - f'(-h)}{2h} = \lim_{h\to 0} \frac{f'(h) - f'(0) - [f'(-h) - f'(0)]}{2h}$$
$$=$$
$$\lim_{h\to 0} \frac{f'(h) - f'(0)}{2h} + \lim_{h\to 0} \frac{f'(-h) - f'(0)}{2(-h)} = \frac{f''(0)}{2} + \frac{f''(0)}{2} = f''(0)$$

Therefore,

$$\lim_{h\to 0} \frac{g(h)}{j(h)} = \lim_{h\to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2} = f''(0)$$

by L'Hôpital's rule. □

22.2 Higher order derivatives

It's common after taking the derivative of a function to wonder if we can do the same differentiation process over and over again. We look into exactly that in this section.

Definition 22.2.1: Higher order derivatives

Let $f : (a.b) \to \mathbb{R}$ be a given function. We can define higher order derivatives iteratively:

$$f'' = (f')', f''' = (f'')', \dots, f^{(n)} = (f^{(n-1)})'$$

We've learned about continuous functions in Section 17.3 and differentiable functions in Section 20.1. We now look at another categorization of functions that we label as smooth.

Definition 22.2.2: Smooth function

Let $f : \mathbb{R} \to \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$. We say that f is in C^k if $f^{(k)}$ exists and is continuous in $\mathbb{R}$. We declare $C^k(\mathbb{R})$ to be the set of all such functions in $\mathbb{R}$. We say that f is smooth (or C^∞) if $f^{(n)}$ exists for all $n \in \mathbb{N}$. We declare $C^\infty(\mathbb{R})$ to be the set of all smooth functions in $\mathbb{R}$.

In the case where $k = 0$, $C^0(\mathbb{R}) = C(\mathbb{R})$ is the set of all continuous functions in $\mathbb{R}$.

We now look to prove a rather well known theorem that is commonly used but frequently misunderstood.

Theorem 22.2.3: Taylor's theorem

Let a function $f : (a, b) \to \mathbb{R}$ such that $f^{(n)}$ exists for some given $n \in \mathbb{N}$. Take two numbers $a < c < d < b$. Then, there exists an $x \in (c, d)$ such that

$$f(d) = \sum_{i=0}^{n-1} \frac{f^{(i)}(c)}{i!}(d-c)^i + \frac{f^{(n)}(x)}{n!}(d-c)^n$$

Proof: Let's define a function $g(t)$ such that

$$g(t) = f(t) - \sum_{i=0}^{n-1} \frac{f^{(i)}(c)}{i!}(t-c)^i - M(t-c)^n$$

where $M \in \mathbb{R}$ is a constant being chosen so that

$$f(d) = \sum_{i=0}^{n-1} \frac{f^{(i)}(c)}{i!}(d-c)^i + M(d-c)^n$$

In other words, we want to choose M to make sure that $g(d) = 0$. Please note that $g(c) = g(d) = 0$ and $g'(c) = g''(c) = ... = g^{(n-1)}(c) = 0$. Finally, $g^{(n)}(t) = f^{(n)}(t) - Mn!$. These facts can be confirmed by straightforward computation. This is the end of the complicated set up.

Now we can use Theorem 21.1.4 a number of times:

$$g(c) = g(d) = 0 \implies g'(c_1) = 0$$

for some $c_1 \in (c, d)$,

$$g'(c) = g'(c_1) = 0 \implies g''(c_2) = 0$$

for some $c_2 \in (c, c_1)$, and this continues until

$$g^{(n-1)}(c) = g^{(n-1)}(c_{n-1}) = 0 \implies g^{(n)}(c_n) = 0$$

for some $c_n \in (c, c_{n-1})$. Let $x = c_n$. Then,

$$g^{(n)}(x) = f^{(n)}(x) - Mn! = 0 \implies M = \frac{f^{(n)}(x)}{n!}$$

We have now defined what M is and it is precisely the form we outlined in the theorem statement, so we are done. $\square$

Taylor's theorem is often used to make polynomial approximations around a certain point in a function. Below is an application of this theorem.

Theorem 22.2.4: Local extrema and second derivatives

Let $f \in C^2(\mathbb{R})$ be a given function. Assume that there exists $c \in \mathbb{R}$ such that $f'(c) = 0$. The following are true:

i. If $f''(c) > 0$, then c is a local minimum of f.
ii. If $f''(c) < 0$, then c is a local maximum of f.
iii. If $f''(c) = 0$, a conclusion can't be made.

Proof: We will only consider the case $f'(c) = 0$ and $f''(c) > 0$. All other situations should follow naturally.

i. Since f'' is continuous and $f''(c) > 0$, we can find an $r > 0$ such that $f''(y) > 0$ for all $y \in (c-r, c+r)$. For each each $x \in (c-r, c+r)$, we can use Theorem 22.2.3 to declare that there exists a y between c and x such that

$$f(x) = f(c) + f'(c) \cdot (x - c) + \frac{f''(y)}{2} \cdot (x - c)^2$$

$$=$$

$$f(c) + \frac{f''(y)}{2} \cdot (x - c)^2 \geq f(c)$$

since $f'(c) = 0$ and $f''(y) \geq 0$. This means that for all $x \in (c - r, c + r)$, we have $f(x) \geq f(c)$, so $f(c)$ is a local minimum.

This concludes the work we will do for this theorem. $\square$

Lecture 23: 04/21/20 (online)

In this lecture we define vector-valued functions and some of their properties. This follows naturally from the work we did on differentiation in Section 20.1 so please review before continuing.

23.1 Differentiation of vector-valued functions

Before we dive into differentiation of vector-valued functions, let's make sure that a vector-valued function in $\mathbb{R}^n$ is well defined.

Definition 23.1.1: Vector-valued function

Let a function $f : (a, b) \to \mathbb{R}^n$ be given. We declare f to be a vector-valued function with n components where $f(t) = \big(f_1(t), f_2(t), \dots, f_n(t)\big)$ for $t \in (a, b)$.

Definition 23.1.2: Differentiable vector-valued function

Let a function $f : (a, b) \to \mathbb{R}^n$ be given. We say that f is differentiable at $x \in (a, b)$ if $f'(x) = \lim_{t\to 0} \frac{f(x+t)-f(x)}{t}$ exists.

Note that if $f'(x)$ exists, then it is also a vector in $\mathbb{R}^n$.

Note how similar Definition 23.1.2 is to Definition 20.1.1. Further analysis leads us to see that

$$\lim_{t\to 0} \frac{f(x-t) - f(x)}{t} = \lim_{t\to 0} \left(\frac{f_1(x+t) - f_1(x)}{t}, \dots, \frac{f_n(x+t) - f_n(x)}{t}\right)$$

Therefore, in order for this limit to exist, we must have that $\lim_{t\to 0} \frac{f_i(x+t)-f_i(x)}{t}$ exists for all $1 \le i \le n$.

Remark 23.1.3: Existence of vector-valued derivative

For a vector-valued function $f : (a, b) \to \mathbb{R}^n$, $f'(x)$ exists $\iff$ $f_i'(x)$ exists for all $1 \le i \le n$.

Furthermore,

$$f'(x) = \big(f_1'(x), f_2'(x), \dots, f_n'(x)\big)$$

Similar to how we've defined operators such as addition and multiplication in one-dimensional functions, we define a new operator for vector-valued functions below.

Definition 23.1.4: Dot product of vector-valued functions

Let $f, g : (a, b) \to \mathbb{R}^n$ be two vector-valued functions. We define the dot product between f and g to be as follows:

$$f(x) \bullet g(x) = f_1(x) \cdot g_1(x) + \ldots + f_n(x) \cdot g_n(x)$$

It is very subtle, but please recognize the difference between the notation of a dot product symbol $\bullet$ and the multiplication symbol $\cdot$. Typically there is no distinction between the two and it will be up to the reader to determine the interpretation based on the context of the problem.

The following remark can be seen as an extension of Theorem 20.1.5 to vector-valued functions.

Remark 23.1.5: Vector-valued differentiation rules

For a function $f : (a, b) \to \mathbb{R}^n$, the following are true:

i. $(f + g)'(x) = f'(x) + g'(x) = \big(f_1'(x) + g_1'(x), \ldots, f_n'(x) + g_n'(x)\big)$

ii. $(c \cdot f)'(x) = c \cdot f'(x) = c \cdot \big(f_1'(x), \ldots, f_n'(x)\big)$

iii. $\big(f(x) \bullet g(x)\big)' = \big(f_1(x) \cdot g_1(x) + \ldots + f_n(x) \cdot g_n(x)\big)' = f_1'(x) \cdot g_1(x) + f_1(x) \cdot g_1'(x) + \ldots + f_n'(x) \cdot g_n(x) + f_n(x) \cdot g_n'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

An important realization of vector-valued functions is that the mean value theorem (Theorem 21.2.1) FAILS if we do not adapt it to handle vector-valued functions. For example, let $f(t) = (\cos t, \sin t)$ for $t \in \mathbb{R}$. We can show that $|f(t)| = |f'(t)| = 1$ for all $t \in \mathbb{R}$ (by using the identity $\cos^2 t + \sin^2 t = 1$). Observe that

$$f(0) = f(2\pi) \implies f(2\pi) - f(0) = \vec{0}$$

where $\vec{0}$ is a vector of all zeros (called the zero vector). However, we see that $\vec{0} = \frac{f(2\pi) - f(0)}{2\pi} \neq f'(t)$ for any $t \in (0, 2\pi)$, so we have a contradiction.

We will discuss further why the mean value theorem fails for vector-valued functions and provide a correction for it in the following lecture.

Lecture 24: 04/23/20 (online)

As promised, we will talk more about the mean value theorem for vector-valued functions. In addition, we talk about how L'Hôpital's rule (Theorem 22.1.1) fails for vector-valued functions.

24.1 Mean value theorem for vector-valued functions

We showed in the last lecture how the mean value theorem we learned in Theorem 21.2.1 fails for vector-valued functions. Let's take a closer look as to why that is.

Assume that $f : \mathbb{R} \to \mathbb{R}^n$ is differentiable, and let us consider $f(1) - f(0)$, which can be written as

$$f(1) - f(0) = \big(f_1(1) - f_1(0), \dots, f_n(1) - f_n(0)\big)$$

We can certainly apply the mean value theorem for each individual slot $f_i(1) - f_i(0)$ (since they are not vector-valued) and see that there exists an $x_i \in (0,1)$ such that $\frac{f_i(1)-f_i(0)}{1} = f_i'(x_i)$. However, $x_1, x_2, \dots, x_n$ need not be the same. This means that we can only expect to have

$$\frac{f(1) - f(0)}{1} = \big(f_1'(x_1), \dots, f_n'(x_n)\big)$$

Example 24.1.1: Failure of the MVT for vector-valued functions

Consider $n \geq 3$ and $f(x) = (x, x^2, \dots, x^n)$ for $x \in \mathbb{R}$. Show why Theorem 21.2.1 fails.

Proof: Surely the derivative of $f(x)$ is

$$f'(x) = (1, 2 \cdot x, \dots, n \cdot x^{n-1})$$

We see that $f(1) - f(0) = (1, 1, \dots, 1)$. This leads us declare that for all $x \in (0,1)$:

$$f(1) - f(0) = (1, 1, \dots, 1) \neq f'(x) = (1, 2 \cdot x, \dots, n \cdot x^{n-1})$$

Why is this true? Suppose that such an x existed. In the second slot of $f'(x)$, we declare that $x = \frac{1}{2}$ since $2 \cdot x = 1$. However, in the third slot, $3 \cdot (\frac{1}{2})^2 = \frac{3}{4} \neq 1$, which is a contradiction. $\square$

Remark 24.1.2

L'Hôpital's rule (Theorem 22.1.1) fails for vector-valued functions.

The counterexample for this remark involves complex numbers, which we do not cover in this course. However, for those curious, the two functions we would use are as follows:

Consider $f : (0, \infty) \to \mathbb{R}$ and $g : (0, \infty) \to \mathbb{C}$. Let

$$f(x) = x$$

and

$$g(x) = \big(x + x^2 \cdot \cos(\frac{1}{x^2})\big) + i\big(x^2 \cdot \sin(\frac{1}{x^2})\big)$$

The following theorem is an adaptation of the mean value theorem for vector-valued functions.

Theorem 24.1.3: Mean value theorem for vector-valued functions

Let $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^n$ be a differentiable vector-valued function and fix $\boldsymbol{v} \in \mathbb{R}^n$. For each $\boldsymbol{a} < \boldsymbol{b}$, there exists a $\boldsymbol{c} \in (\boldsymbol{a}, \boldsymbol{b})$ such that

$$\big(\boldsymbol{f}(\boldsymbol{b}) - \boldsymbol{f}(\boldsymbol{a})\big) \bullet \boldsymbol{v} = (\boldsymbol{b} - \boldsymbol{a}) \cdot [\boldsymbol{f}'(\boldsymbol{c}) \bullet \boldsymbol{v}]$$

Proof: Let us define $\phi(x) = f(x) \bullet v$. Note that

$$\phi(x) = v_1 \cdot f_1(x) + ... + v_n \cdot f_n(x)$$

We see that $\phi : \mathbb{R} \to \mathbb{R}$, meaning that we can apply Theorem 21.2.1 to ϕ, which is precisely what the theorem states. $\square$

We can now use Theorem 24.1.3 to prove the following theorem.

Theorem 24.1.4: Application of the vector-valued MVT

Let $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^n$ be a differentiable vector-valued function. Fix $\boldsymbol{a} < \boldsymbol{b}$ and assume that $\sup_{\boldsymbol{x} \in (\boldsymbol{a}, \boldsymbol{b})} |\boldsymbol{f}'(\boldsymbol{x})| = \boldsymbol{M} < \infty$. This implies that $|\boldsymbol{f}(\boldsymbol{b}) - \boldsymbol{f}(\boldsymbol{a})| \leq \boldsymbol{M} \cdot (\boldsymbol{b} - \boldsymbol{a})$.

Proof: Note that if $f(b) - f(a) = \vec{0}$ there is nothing to prove, so we will consider the case where $v = f(b) - f(a) \neq \vec{0}$.

Let $\phi(x) = f(x) \bullet v$. By Theorem 24.1.3, we know that there exists a $c \in (a, b)$ such that

$$\big(f(b) - f(a)\big) \bullet v = (b - a) \cdot [f'(c) \bullet v]$$

Since $v = f(b) - f(a)$, we can use Theorem 5.3.2 to deduce that

$$|f(b) - f(a)|^2 = (b - a) \cdot [f'(c) \bullet v] \leq (b - a) \cdot |f'(c)| \cdot |v|$$

Therefore,

$$|f(b) - f(a)| \leq (b - a) \cdot |f'(c)| \leq (b - a) \cdot M$$

and we are done. $\square$

Lecture 25 : 04/28/20 (online)

We now turn to the last main topics of this course which are spaces of functions and sequences of functions. We set up the framework for these ideas in this lecture.

25.1 Spaces of functions

Something important to note is that our main function space will be functions that are continuous and bounded. This condition is described in the bottom half of the following definition.

Definition 25.1.1: Function space

Let (G, d_G) and (H, d_H) be two given metric spaces. A function space S is a set that contains all functions $f : G \to H$.

MAIN FUNCTION SPACE IN THIS COURSE:

Let (E, d) be a metric space and consider functions $f : E \to \mathbb{R}$. We collect functions that fit the following condition:

$$Y = C(E, \mathbb{R}) = \{f : E \to \mathbb{R} : f \text{ is continuous and bounded}\}$$

In other words, the set Y contains all functions that are continuous and bounded from $E \to \mathbb{R}$.

What's important to note is that every element of a function space is a function itself. In this course, the functions in our main function space Y must be continuous and bounded.

We can define metric distances on Y just like we did with metric spaces in Section 7.1. Even though the makeup of Y is different since each element is a function, a metric distance on Y must have the same properties that we outlined in Definition 7.1.1.

Definition 25.1.2: Metric distance d_∞

Let a general metric space (E, d) be given and let $f, g : E \to \mathbb{R}$ be two functions. We define the distance metric d_∞ to be as follows:

$$d_\infty(f, g) = \sup_{e \in E} |f(e) - g(e)|$$

We can verify that d_∞ is a metric distance:

i. It is clear that $d_\infty(f,g) = \sup_{e\in E}|f(e)-g(e)| \geq 0$. Furthermore, equality happens only when $f(e)-g(e)=0$ for all $e \in E$ if and only if $f=g$.
ii. We have that $d_\infty(f,g) = d_\infty(g,f)$ since $|f(e)-g(e)| = |g(e)-f(e)|$ for all $e \in E$.
iii. Let $f,g,h : E \to \mathbb{R}$ be given and pick $z \in E$ such that $d_\infty(f,g) = \sup_{e\in E}|f(e)-g(e)| = |f(z)-g(z)|$. We then have

$$d_\infty(f,g) = |f(z)-g(z)| \leq |f(z)-h(z)| + |h(z)-g(z)| \leq d_\infty(f,h) + d_\infty(h,g)$$

We now look for a way to characterize sequences in a function space. We do this in the following definition of uniform convergence.

Definition 25.1.3: Uniform convergence of $\{f_n\}$

Let a sequence of functions $\{f_n\}$ be given such that each function in $\{f_n\}$ is of the form $f_i : G \to \mathbb{R}$ where (G, d_G) is a metric space. We say that $\{f_n\}$ uniformly converges to a function $f : G \to \mathbb{R}$ if for each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$, $d_\infty(f_n, f) < \varepsilon$, which is equivalent to saying that for $n > N$, $|f_n(g) - f(g)| < \varepsilon$ for all $g \in G$.

In other words, $\{f_n\} \to f$ if $\lim_{n\to\infty} d_\infty(f_n, f) = 0$.

Something to note is that we look at ALL points $g \in G$. This means that N does not signify a marker for inputs to the functions in $\{f_n\}$, but rather a marker for entire functions in $\{f_n\}$. One way to think about this is that we set a thickness around a function $f : G \to \mathbb{R}$. For all functions past a certain point in the sequence $\{f_n\}$, they fall within this thickness.

Theorem 25.1.4: Uniform convergence of continuous and bounded functions

Let $\{f_n\} \subset Y$ be given such that $\lim_{n\to\infty} d_\infty(f_n, f) = 0$ where $f : E \to \mathbb{R}$ is some function. This implies that $f \in Y$.

Proof: We need to show that f is bounded and continuous.

i. Fix $\varepsilon > 0$ and then find $\frac{\varepsilon}{3} > 0$. Since $\lim_{n\to\infty} d_\infty(f_n, f) = 0$, we can find an $N \in \mathbb{N}$ such that for $n \geq N$,

$$d_\infty(f_n, f) = \sup_{e\in E}|f_n(e) - f(e)| < \frac{\varepsilon}{3}$$

Therefore, we have that

$$|f(e)| \leq |f_N(x)| + \frac{\varepsilon}{3} \text{ for all } x \in E$$

which means that f is bounded.

ii. Recall that

$$d_\infty(f_N, f) = \sup_{e \in E} |f_n(e) - f(e)| < \frac{\varepsilon}{3}$$

Since f_N is continuous at each point $p \in E$, we can find a $\delta > 0$ such that

$$|f_N(e) - f_N(p)| < \frac{\varepsilon}{3} \text{ for all } x \in B(p, \delta)$$

By the triangle inequality, for all $x \in B(p, \delta)$ we have

$$|f(x) - f(p)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(p)| + |f_N(p) - f(p)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Therefore, f is both bounded and continuous, so $f \in Y$. $\square$

Lecture 26: 04/30/20 (online)

For the final lecture of the course, we will continue our discussion on function spaces and prove a theorem based on the particular function space we outlined in Definition 25.1.1.

26.1 More on function spaces

We were able to construct a form of convergence with Definition 25.1.3. We will now look to understand Cauchy sequences in a function space.

Definition 26.1.1: Cauchy sequence in a function space

Let a sequence of functions $\{f_n\} \subset S$ be given where S is a function space. We say that $\{f_n\} \subset S$ is a Cauchy sequence if for each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$d_\infty(f_n, f_m) < \varepsilon \quad \text{for all } n, m > N$$

Recall from Definition 14.2.1 that a metric space is complete if every Cauchy sequence in the metric space is also a convergent sequence. This leads us to the final theorem we will prove in this course.

Theorem 26.1.2: (Y, d_∞) is a complete metric space

The metric space $(\boldsymbol{Y}, \boldsymbol{d_\infty})$ (where $\boldsymbol{Y}$ is defined in the bottom half of Definition 25.1.1) is a complete metric space.

Proof: Let $\{f_n\} \subset Y$ be a Cauchy sequence. We need to show that it converges to some function $f \in Y$. This proof uses lots of notation defined in Definition 25.1.1 so please refer to it if confusion arises.

Fix $z \in E$. We claim that $\{f_n(z)\} \subset \mathbb{R}$ is a Cauchy sequence. This is true since

$$|f_n(z) - f_m(z)| \leq \sup_{e \in E} |f_n(e) - f_m(e)| = d_\infty(f_n, f_m)$$

Therefore, by Definition 26.1.1, we know that $\{f_n(z)\} \subset \mathbb{R}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, we know that $\{f_n(z)\}$ is convergent and will denote its limit by $f(z)$. In other words, we know $f_n(z) \to f(z)$ as $n \to \infty$. We now need to show that $d_\infty(f_n, f) \to 0$ as $n \to \infty$.

As $\{f_n\} \subset Y$ is a Cauchy sequence, we that that for each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$

such that

$$d_\infty(f_n, f_m) < \varepsilon \quad \text{for all } n, m > N$$

In particular, for every $z \in E$ and $m, n > N$, we have that $|f_n(z) - f_m(z)| < \varepsilon$. Therefore,

$$|f_n(z) - f(z)| = \lim_{m\to\infty} |f_n(z) - f_m(z)| \leq \varepsilon$$

Hence, for $n > N$, we know that $d_\infty(f_n, f) \leq \varepsilon$. By Theorem 25.1.4, we know that $f \in Y$. This concludes the proof. $\square$