

# 1 Tensors and Local Symmetries

1. Transformation from euclidean (x,y) to polar (r,θ) is

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

Transformation from polar to euclidean coordinates is

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \cos \theta = x/r \\ \sin \theta = y/r \\ \theta = \arccos \frac{x}{\sqrt{x^2 + y^2}} & \text{for positive y} \\ \theta = \pi + \arccos \frac{x}{\sqrt{x^2 + y^2}} & \text{for negative y} \end{cases}$$

Partial derivatives

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{\partial \sqrt{x^2 + y^2}}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} \\ \frac{\partial x}{\partial r} &= \frac{\partial r \cos \theta}{\partial r} = \cos \theta = \frac{x}{r} \end{aligned}$$

$$\begin{aligned} \frac{\partial r}{\partial y} &= \frac{\partial \sqrt{x^2 + y^2}}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r} \\ \frac{\partial y}{\partial r} &= \frac{\partial r \sin \theta}{\partial r} = \sin \theta = \frac{y}{r} \end{aligned}$$

$$\begin{aligned} \frac{\partial \theta}{\partial x} &= \frac{\partial \arccos \frac{x}{\sqrt{x^2 + y^2}}}{\partial x} = \frac{-1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} \frac{y^2}{(x^2 + y^2)^{3/2}} = \frac{-y}{x^2 + y^2} = -\frac{y}{r^2} \\ \frac{\partial x}{\partial \theta} &= \frac{\partial r \cos \theta}{\partial \theta} = -r \sin \theta = -y \end{aligned}$$

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$$\begin{aligned} \frac{\partial \theta}{\partial y} &= \frac{\partial \arccos \frac{x}{\sqrt{x^2 + y^2}}}{\partial y} = \frac{-1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} \frac{-xy}{(x^2 + y^2)^{3/2}} = \frac{x}{x^2 + y^2} = \frac{x}{r^2} \\ \frac{\partial y}{\partial \theta} &= \frac{r \sin \theta}{\partial \theta} = r \cos \theta = x \end{aligned}$$

To summarize

$$\begin{aligned}\frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} \neq \frac{\partial x}{\partial \theta} = -y \\ \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} \neq \frac{\partial y}{\partial \theta} = x\end{aligned}$$

2.

3. Put

$$L = \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

into  $L^T \eta L$ .

$$\begin{aligned}L^T \eta L &= \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\gamma & \beta & 0 & 0 \\ -\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \beta^2 - \gamma^2 & 0 & 0 & 0 \\ 0 & -\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

Since  $\beta = \sqrt{\gamma^2 - 1}$ ,

$$L^T \eta L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta \tag{2}$$

4.

$$\eta = L \eta L^T = \Lambda^{-1} \eta (\Lambda^{-1})^T = \Lambda^{-1} \eta (\Lambda^T)^{-1} \tag{3}$$

Inverse both sides

$$\eta^{-1} = \Lambda^T \eta^{-1} \Lambda \quad (4)$$

We already have  $\eta^{-1} = \eta$ , then

$$\eta = \Lambda^T \eta \Lambda \quad (5)$$

5. To see the same physics means to have the same kinetic energy at center of mass system.

Coordinate transformation leaves scalar unchanged. In this problem,  $P_\mu P^\mu$  won't change when we change from one reference frame to another, i.e.,

$$-(E_1 + E_2)^2 + m^2 u_2^2 c^2 = -(E'_1 + E'_2)^2 + (m u'_1 + m u'_2)^2 c^2.$$

We also have

$$\begin{cases} E_2^2 = m^2 u_2^2 c^2 + m^2 c^4 \\ E_1 = m c^2 \\ E'_1 = E'_2 \\ u'_1 = -u'_2 \end{cases} \quad (6)$$

But the mass of proton is so small compared to the accelerator energy that we can drop  $m c^2$  term in our calculation.

Then the energy of the incoming proton in lab frame is

$$E_2 \approx \frac{2E_1^2}{m c^2} \approx 10^5 \text{ TeV} \quad (7)$$

6. Take the divergence of both sides of Maxwell-Ampere law

$$\begin{aligned} \nabla \cdot \nabla \times \mathbf{B} &= \mu_0 \nabla \cdot \mathbf{j} + \epsilon_0 \mu_0 \nabla \cdot \dot{\mathbf{E}} \\ 0 &= \nabla \cdot \mathbf{j} + \epsilon_0 \nabla \cdot \dot{\mathbf{E}} \end{aligned}$$

Divergence of electric field is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (8)$$

Thus

$$0 = \nabla \cdot \mathbf{j} + \dot{\rho} \quad (9)$$

7.

$$M_{ik} = \frac{1}{2}(M_{ik} + M_{ki}) + \frac{1}{2}(M_{ik} - M_{ki}) \quad (10)$$

in which the first part is symmetric and the second is antisymmetric.

Exchange  $i$  and  $k$ ,

$$\begin{aligned} \frac{1}{2}(M_{ik} + M_{ki})dx^i \wedge dx^k &= -\frac{1}{2}(M_{ki} + M_{ik})dx^k \wedge dx^i \\ \rightarrow (M_{ik} + M_{ki})dx^i \wedge dx^k &= 0 \end{aligned}$$

This shows the symmetric part doesn't contribute to  $M_{ik}dx^i dx^k$ .

$$\frac{1}{2}(M_{ik} - M_{ki})dx^i \wedge dx^k = \frac{1}{2}(M_{ki} - M_{ik})dx^k \wedge dx^i = -\frac{1}{2}(M_{ki} - M_{ik})dx^i \wedge dx^k$$

Symmetric part contributes to  $M_{ik}dx^i dx^k$ .

For  $M_{ik}dx^i dx^k = \frac{1}{2}(M_{ik} + M_{ki})dx^i dx^k + \frac{1}{2}(M_{ik} - M_{ki})dx^i dx^k$ ,

$$\begin{aligned} \frac{1}{2}(M_{ik} - M_{ki})dx^i dx^k &= \frac{1}{2}(M_{ki} - M_{ik})dx^k dx^i \\ \rightarrow \frac{1}{2}(M_{ik} - M_{ki})dx^i dx^k &= -\frac{1}{2}(M_{ik} - M_{ki})dx^k dx^i \\ \rightarrow \frac{1}{2}(M_{ik} - M_{ki})dx^i dx^k &= 0 \end{aligned}$$

$$\frac{1}{2}(M_{ik} + M_{ki})dx^i dx^k = \frac{1}{2}(M_{ki} + M_{ik})dx^k dx^i$$

Antisymmetric part doesn't contribute to  $M_{ik}dx^i dx^k$ .

8. (On page 507, the other half of parentheses is lost.)

For any 3 dimensional vector,

$$\begin{aligned}
(\nabla \times (\nabla \times \mathbf{E}))_i &= \sum_{j,k=1}^3 \epsilon_{ijk} \partial_j (\nabla \times \mathbf{E})_k \\
&= \sum_{j,k,l,m=1}^3 \epsilon_{ijk} \partial_j \epsilon_{klm} \partial_l E_m \\
&= \sum_{j,l,m=1}^3 (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j \partial_l E_m \\
&= \sum_{j,l,m=1}^3 \delta_{il} \delta_{jm} \partial_j \partial_l E_m - \sum_{j,l,m=1}^3 \delta_{im} \delta_{jl} \partial_j \partial_l E_m \\
&= \sum_{j,l,m=1}^3 \delta_{il} \delta_{jm} \partial_j \partial_l E_m - \sum_{j,l,m=1}^3 \delta_{im} \delta_{jl} \partial_j \partial_l E_m \\
&= \partial_i (\nabla \times \mathbf{E}) - \partial_j \partial_j E_i \\
&= (\nabla (\nabla \times \mathbf{E}))_i - (\Delta \mathbf{E})_i
\end{aligned}$$

9.

$$\begin{aligned}
\partial_i F_{jk} &= \partial_i (\partial_j A_k - \partial_k A_j) \\
&= \partial_i \partial_j A_k - \partial_i \partial_k A_j
\end{aligned}$$

$$\begin{aligned}
\partial_k F_{ij} &= \partial_k (\partial_i A_j - \partial_j A_i) \\
&= \partial_k \partial_i A_j - \partial_k \partial_j A_i
\end{aligned}$$

$$\begin{aligned}
\partial_j F_{ki} &= \partial_j (\partial_k A_i - \partial_i A_k) \\
&= \partial_j \partial_k A_i - \partial_j \partial_i A_k
\end{aligned}$$

Sum up

$$\begin{aligned}
&\partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki} \\
&= \partial_i \partial_j A_k - \partial_i \partial_k A_j + \partial_k \partial_i A_j - \partial_k \partial_j A_i + \partial_j \partial_k A_i - \partial_j \partial_i A_k \\
&= 0
\end{aligned}$$

10. Assume

$$A = A_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \quad (11)$$

$$B = B_{i_1 \dots i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q} \quad (12)$$

$$\begin{aligned} d(A \wedge B) &= d((A_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}) \wedge (B_{i_1 \dots i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q})) \\ &= \partial_k A_{i_1 \dots i_p} dx^k dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge (B_{i_1 \dots i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q}) \\ &\quad + A_{i_1 \dots i_p} \partial_k B_{i_1 \dots i_q} dx^k dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge (dx^{i_1} \wedge \dots \wedge dx^{i_q}) \\ &= dA \wedge B \\ &\quad + A_{i_1 \dots i_p} (-1)^p dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge (\partial_k B_{i_1 \dots i_q} dx^k dx^{i_1} \wedge \dots \wedge dx^{i_q}) \\ &= dA \wedge B + (-1)^p A \wedge dB \end{aligned}$$

11.

$$\begin{aligned} d\omega &= d\left(\frac{1}{2}a_{ij}\right)dx^i \wedge dx^j \\ &= \frac{1}{2}\partial_k a_{ij} dx^k \wedge dx^i \wedge dx^j \\ &= \frac{1}{6}(\partial_k a_{ij} dx^k \wedge dx^i \wedge dx^j + \partial_j a_{ki} dx^j \wedge dx^k \wedge dx^i + \partial_i a_{jk} dx^i \wedge dx^j \wedge dx^k) \\ &= \frac{1}{3!}(\partial_k a_{ij} dx^i \wedge dx^j \wedge dx^k + \partial_j a_{ki} dx^i \wedge dx^j \wedge dx^k + \partial_i a_{jk} dx^i \wedge dx^j \wedge dx^k) \\ &= \frac{1}{3!}(\partial_k a_{ij} + \partial_j a_{ki} + \partial_i a_{jk})dx^i \wedge dx^j \wedge dx^k \end{aligned}$$

12.

$$\delta^i_j = g^{ik} \frac{\partial x^t}{\partial x'^k} g_{tu} \frac{\partial x^u}{\partial x'^j} \quad (13)$$

Multiply by  $g^{lt}$

$$\begin{aligned} g^{lt} \delta^i_j &= g^{ik} \frac{\partial x^t}{\partial x'^k} g_{tu} \frac{\partial x^u}{\partial x'^j} g^{lt} \\ g^{lt} \delta^i_j &= g^{ik} \frac{\partial x^t}{\partial x'^k} \delta^l_u \frac{\partial x^u}{\partial x'^j} \\ g^{lt} \delta^i_j &= g^{ik} \frac{\partial x^t}{\partial x'^k} \frac{\partial x^l}{\partial x'^j} \\ g^{lt} \frac{\partial x'^j}{\partial x^l} \delta^i_j &= g^{ik} \frac{\partial x^t}{\partial x'^k} \\ g^{lt} \frac{\partial x'^i}{\partial x^l} &= g^{ik} \frac{\partial x^t}{\partial x'^k} \\ \frac{\partial x'^k}{\partial x^t} g^{lt} \frac{\partial x'^i}{\partial x^l} &= g^{ik} \end{aligned}$$

13.

$$\begin{aligned} dp &= \hat{\mathbf{x}}(\cos \phi d\rho - \rho \sin \phi d\phi) + \hat{\mathbf{y}}(\sin \phi d\rho + \rho \cos \phi d\phi) + \hat{\mathbf{z}}dz \\ &= \hat{\boldsymbol{\rho}}d\rho + \hat{\boldsymbol{\phi}}\rho d\phi + \hat{\mathbf{z}}dz \end{aligned}$$

Collect the terms

$$d\rho(\cos\phi d\rho\hat{\mathbf{x}} + \sin\phi d\rho\hat{\mathbf{y}}) + d\phi(-\rho\sin\phi\hat{\mathbf{x}} + \rho\cos\phi\hat{\mathbf{y}}) + \hat{\mathbf{z}}dz = \hat{\rho}d\rho + \hat{\phi}d\phi + \hat{\mathbf{z}}dz \quad (14)$$

The coefficients of each derivative should be the same

$$\begin{aligned}\hat{\rho} &= \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}} \\ \hat{\phi} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}\end{aligned}$$

14.

$$\begin{aligned}dp &= \hat{\mathbf{x}}(-r\sin\theta\sin\phi d\phi + r\cos\phi\cos\theta d\theta + \sin\theta\cos\phi dr) + \hat{\mathbf{y}}(r\sin\theta\cos\phi d\phi \\ &\quad + r\sin\phi\cos\theta d\theta + \sin\theta\sin\phi dr) + \hat{\mathbf{z}}(-r\sin\theta d\theta + \cos\theta dr) \\ &= \hat{\mathbf{r}}dr + \hat{\theta}r d\theta + \hat{\phi}r\sin\theta d\phi\end{aligned}$$

Collect the terms

$$\begin{aligned}&(-r\sin\theta\sin\phi\hat{\mathbf{x}} + r\sin\theta\cos\phi\hat{\mathbf{y}})d\phi + (r\cos\phi\cos\theta\hat{\mathbf{x}} + r\sin\phi\cos\theta\hat{\mathbf{y}} - r\sin\theta\hat{\mathbf{z}})d\theta \\ &+ (\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})dr \\ &= \hat{\mathbf{r}}dr + \hat{\theta}r d\theta + \hat{\phi}r\sin\theta d\phi\end{aligned}$$

$$\begin{aligned}\hat{\phi} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} \\ \hat{\theta} &= \cos\phi\cos\theta\hat{\mathbf{x}} + \sin\phi\cos\theta\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \\ \hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\end{aligned}$$

15. A flat 3-space means the Riemann curvature is zero. But I have no idea about the metric. Any metric that is conformal with a flat metric indicates a flat space. But here I use  $g_{ij} = \eta_{ij}$ , which gives  $\det g_{ij} = 1$  and  $s = 1$ . Am I wrong?

In this problem  $g_{ij} = \eta_{ij}$ , which gives  $\det g_{ij} = 1$  and  $s = 1$ .

$$\begin{aligned}**dx^i &= *(\frac{1}{2}g^{il}\eta_{ljk}dx^j \wedge dx^k) \\ &= \frac{1}{2}\eta^{il}*(dx^j \wedge dx^k) \\ &= \frac{1}{2}\eta^{il}g^{jm}g^{kn}\eta_{mnt}dx^t \\ &= \frac{1}{2}g^{il}\sqrt{g}\epsilon_{ljk}g^{jm}g^{kn}\sqrt{g}\epsilon_{mnt}dx^t \\ &= \frac{1}{2}\epsilon^i_{jk}\epsilon^{jk}_t dx^t \\ &= dx^i\end{aligned}$$

$$\begin{aligned}
**(\mathrm{d}x^i \wedge \mathrm{d}x^k) &= *(g^{ik}g^{jl}\eta_{klm}\mathrm{d}x^m) \\
&= g^{ik}g^{jl}\eta_{klm}*\mathrm{d}x^m \\
&= g^{ik}g^{jl}\eta_{klm}\frac{1}{2}g^{mu}\eta_{uwt}\mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= g^{ik}g^{jl}\epsilon_{klm}\sqrt{g}\frac{1}{2}g^{mu}\epsilon_{uwt}\sqrt{g}\mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= \frac{1}{2}\epsilon^{ij}{}_m\epsilon^m{}_{wt}\mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= \mathrm{d}x^i \wedge \mathrm{d}x^j
\end{aligned}$$

16.

$$\begin{aligned}
**\mathrm{d}x^i &= *(\frac{1}{3!}g^{ik}\eta_{klmn}\mathrm{d}x^l \wedge \mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&= \frac{1}{3!}g^{ik}\eta_{klmn}g^{lr}g^{ms}g^{nt}\eta_{rstu}\mathrm{d}x^u \\
&= \frac{1}{3!}\eta^i{}_{lmn}\eta^{lmn}{}_u\mathrm{d}x^u \\
&= \mathrm{d}x^i
\end{aligned}$$

$$\begin{aligned}
**(\mathrm{d}x^i \wedge \mathrm{d}x^j) &= *(\frac{1}{2}g^{ik}g^{jl}\eta_{klmn}\mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&=
\end{aligned}$$