1 Tensors and Local Symmetries

1. Derivative of the point

$$\begin{aligned}
\mathrm{d}\boldsymbol{p} &= \hat{\boldsymbol{t}}\mathrm{d}t + [\hat{\boldsymbol{x}}a\cos\chi\sin\theta\cos\phi + \hat{\boldsymbol{y}}a\cos\chi\sin\theta\sin\phi + \hat{\boldsymbol{z}}a\cos\chi\cos\theta - \hat{\boldsymbol{m}}a\sin\chi]\mathrm{d}\chi \\
&\quad +[\hat{\boldsymbol{x}}a\sin\chi\cos\theta\cos\phi + \hat{\boldsymbol{y}}\sin\chi\cos\theta\sin\phi - \hat{\boldsymbol{z}}a\sin\chi\sin\theta]\mathrm{d}\theta \\
&\quad +[-\hat{\boldsymbol{x}}a\sin\chi\sin\theta\sin\phi + \hat{\boldsymbol{y}}a\sin\chi\sin\theta\cos\phi]\mathrm{d}\phi
\end{aligned}$$

Define $\sin\chi = r$. Then $\mathrm{d}\chi = \mathrm{d}r/\sqrt{1-r^2}$. $(0 < \chi < \pi)$

$$\begin{aligned}
\mathrm{d}\boldsymbol{p} &\equiv e_t\mathrm{d}t + e_r\mathrm{d}r + e_\theta\mathrm{d}\theta + e_\phi\mathrm{d}\phi \\
&\equiv \hat{\boldsymbol{e}}_{\boldsymbol{t}}\mathrm{d}t + \frac{a}{\sqrt{1-r^2}}\hat{\boldsymbol{e}}_{\boldsymbol{r}}\mathrm{d}r + ar\hat{\boldsymbol{e}}_{\boldsymbol{\theta}}\mathrm{d}\theta + ar\sin\theta\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}\mathrm{d}\phi
\end{aligned}$$

$$\begin{aligned}
g_{tt} &= e_t(-1)e_t = -1 \\
g_{rr} &= e_re_r = \frac{a^2}{1-r^2} \\
g_{\theta\theta} &= e_\theta e_\theta = a^2r^2 \\
g_{\phi\phi} &= e_\phi e_\phi = a^2r^2\sin^2\theta
\end{aligned}$$

2.

$$\begin{aligned}
\mathrm{d}\boldsymbol{p} &= \hat{\boldsymbol{t}}\mathrm{d}t + \hat{\boldsymbol{x}}(a\sin\theta\cos\phi\mathrm{d}r + ar\cos\theta\cos\phi\mathrm{d}\theta - ar\sin\theta\sin\phi\mathrm{d}\phi) \\
&\quad +\hat{\boldsymbol{y}}(a\sin\theta\sin\phi\mathrm{d}r + ar\cos\theta\sin\phi\mathrm{d}\theta + ar\sin\theta\cos\phi\mathrm{d}\phi) \\
&\quad +\hat{\boldsymbol{z}}(a\cos\theta\mathrm{d}r - ar\sin\theta\mathrm{d}\theta) \\
&= \hat{\boldsymbol{t}}\mathrm{d}t + (\hat{\boldsymbol{x}}a\sin\theta\cos\phi + \hat{\boldsymbol{y}}a\sin\theta\sin\phi + \hat{\boldsymbol{z}}a\cos\theta)\mathrm{d}r \\
&\quad +(\hat{\boldsymbol{x}}ar\cos\theta\cos\phi + \hat{\boldsymbol{y}}ar\cos\theta\sin\phi - \hat{\boldsymbol{z}}ar\sin\theta)\mathrm{d}\theta \\
&\quad +(-\hat{\boldsymbol{x}}ar\sin\theta\sin\phi + \hat{\boldsymbol{y}}ar\sin\theta\cos\phi)\mathrm{d}\phi \\
&\equiv e_t\mathrm{d}t + e_r\mathrm{d}r + e_\theta\mathrm{d}\theta + e_\phi\mathrm{d}\phi \\
&\equiv \hat{\boldsymbol{e}}_{\boldsymbol{t}}\mathrm{d}t + a\hat{\boldsymbol{e}}_{\boldsymbol{r}}\mathrm{d}r + ar\hat{\boldsymbol{e}}_{\boldsymbol{\theta}}\mathrm{d}\theta + ar\sin\theta\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}\mathrm{d}\phi
\end{aligned}$$

$$\begin{aligned}
g_{tt} &= e_t(-1)e_t = -1 \\
g_{rr} &= e_re_r = a^2 \\
g_{\theta\theta} &= e_\theta e_\theta = a^2r^2 \\
g_{\phi\phi} &= e_\phi e_\phi = a^2r^2\sin^2\theta
\end{aligned}$$

3.

$$\begin{aligned}
\mathrm{d}\boldsymbol{p} &= \hat{\boldsymbol{t}}\mathrm{d}t + \hat{\boldsymbol{x}}(a\cosh\chi\sin\theta\cos\phi\mathrm{d}\chi + a\sinh\chi\cos\theta\cos\phi\mathrm{d}\theta - a\sinh\chi\sin\theta\sin\phi\mathrm{d}\phi) \\
&\quad +\hat{\boldsymbol{y}}(a\cosh\chi\sin\theta\sin\phi\mathrm{d}\chi + a\sinh\chi\cos\theta\sin\phi\mathrm{d}\theta + a\sinh\chi\sin\theta\cos\phi\mathrm{d}\phi) \\
&\quad +\hat{\boldsymbol{z}}(a\cosh\chi\cos\theta\mathrm{d}\chi - a\sinh\chi\sin\theta\mathrm{d}\theta) \\
&= \hat{\boldsymbol{t}}\mathrm{d}t + (\hat{\boldsymbol{x}}a\cosh\chi\sin\theta\cos\phi + \hat{\boldsymbol{y}}a\cosh\chi\sin\theta\sin\phi + \hat{\boldsymbol{z}}a\cosh\chi\cos\theta)\mathrm{d}\chi \\
&\quad +(\hat{\boldsymbol{x}}a\sinh\chi\cos\theta\cos\phi + \hat{\boldsymbol{y}}a\sinh\chi\cos\theta\sin\phi - \hat{\boldsymbol{z}}a\sinh\chi\sin\theta)\mathrm{d}\theta \\
&\quad +(-\hat{\boldsymbol{x}}a\sinh\chi\sin\theta\sin\phi + \hat{\boldsymbol{y}}a\sinh\chi\sin\theta\cos\phi)\mathrm{d}\phi
\end{aligned}$$

Since we have $\sinh\chi = r$, the derivative $\mathrm{d}\chi = \mathrm{d}r/\cosh\chi = \mathrm{d}r/\sqrt{1+\sinh^2\chi}$.

$$\mathrm{d}\boldsymbol{p} = \hat{\boldsymbol{e}}_{\boldsymbol{t}}\mathrm{d}t + \frac{a}{\sqrt{1+r^2}}\hat{\boldsymbol{e}}_{\boldsymbol{r}}\mathrm{d}r + ar\mathrm{d}\theta + ar\sin\theta\mathrm{d}\phi$$

$$\begin{aligned}
g_{tt} &= -1 \\
g_{rr} &= \frac{a^2}{1+r^2} \\
g_{\theta\theta} &= a^2r^2 \\
g_{\phi\phi} &= a^2r^2\sin\theta
\end{aligned}$$

4. Coordinates transform as

$$\begin{aligned}
r &\to kr \\
a &\to \frac{a}{k}, \qquad \mathrm{k} \neq 0 \\
\theta &\to \theta \\
\phi &\to \phi
\end{aligned}$$

5.