1 Tensors and Local Symmetries

1. Transformation from euclidean (x,y) to polar (r,θ) is

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Transformation from polar to euclidean coordinates is

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \cos\theta = x/r \\ \sin\theta = y/r \\ \theta = \arccos\frac{x}{\sqrt{x^2+y^2}} & \text{for positive y} \\ \theta = \pi + \arccos\frac{x}{\sqrt{x^2+y^2}} & \text{for negative y} \end{cases}$$

Partial derivatives

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2}}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$
$$\frac{\partial x}{\partial r} = \frac{\partial r\cos\theta}{\partial r} = \cos\theta = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{\partial\sqrt{x^2+y^2}}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$
$$\frac{\partial y}{\partial r} = \frac{\partial r\sin\theta}{\partial r} = \sin\theta = \frac{y}{r}$$

$$\frac{\partial\theta}{\partial x} = \frac{\partial\arccos\frac{x}{\sqrt{x^2+y^2}}}{\partial x} = \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\frac{y^2}{(x^2+y^2)^{3/2}} = \frac{-y}{x^2+y^2} = -\frac{y}{r^2}$$
$$\frac{\partial x}{\partial\theta} = \frac{\partial r\cos\theta}{\partial\theta} = -r\sin\theta = -y$$

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$$\frac{\partial\theta}{\partial y} = \frac{\partial\arccos\frac{x}{\sqrt{x^2+y^2}}}{\partial y} = \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\frac{-xy}{(x^2+y^2)^{3/2}} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$
$$\frac{\partial y}{\partial\theta} = \frac{r\sin\theta}{\partial\theta} = r\cos\theta = x$$

To summarize

$$\begin{aligned} \frac{\partial \theta}{\partial x} = -\frac{y}{r^2} &\neq \frac{\partial x}{\partial \theta} = -y \\ \frac{\partial \theta}{\partial y} = \frac{x}{r^2} &\neq \frac{\partial y}{\partial \theta} = x \end{aligned}$$

2.

3. Put

$$L = \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

into $L^T \eta L$.

$$\begin{aligned} L^T \eta L &= \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\gamma & \beta & 0 & 0 \\ -\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \beta^2 - \gamma^2 & 0 & 0 & 0 \\ 0 & -\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Since $\beta = \sqrt{\gamma^2 - 1}$,

$$L^T \eta L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta \tag{2}$$

4.

$$\eta = L \eta L^T = \Lambda^{-1} \eta (\Lambda^{-1})^T = \Lambda^{-1} \eta (\Lambda^T)^{-1} \tag{3}$$

Inverse both sides

$$\eta^{-1} = \Lambda^T \eta^{-1} \Lambda \tag{4}$$

We already have $\eta^{-1} = \eta$, then

$$\eta = \Lambda^T \eta \Lambda \tag{5}$$

5. To see the same physics means to have the same kinetic energy at center of mass system.

   Coordinate transformation leaves scalar unchanged. In this problem, $P_\mu P^\mu$ won't change when we change from one reference frame to another, i.e.,

$$-(E_1 + E_2)^2 + m^2 u_2^2 c^2 = -(E_1' + E_2')^2 + (mu_1' + mu_2')^2 c^2.$$

   We also have

$$\begin{cases} E_2^2 = m^2 u_2^2 c^2 + m^2 c^4 \\ E_1 = mc^2 \\ E_1' = E_2' \\ u_1' = -u_2' \end{cases} \tag{6}$$

   But the mass of proton is so small compared to the accelerator energy that we can drop mc^2 term in our calculation.

   Then the energy of the incoming proton in lab frame is

$$E_2 \approx \frac{2E_1^2}{mc^2} \approx 10^5 \text{TeV} \tag{7}$$

6. Take the divergence of both sides of Maxwell-Ampere law

$$\begin{aligned} \nabla \cdot \nabla \times \boldsymbol{B} &= \mu_0 \nabla \cdot \boldsymbol{j} + \epsilon_0 \mu_0 \nabla \cdot \dot{\boldsymbol{E}} \\ 0 &= \nabla \cdot \boldsymbol{j} + \epsilon_0 \nabla \cdot \dot{\boldsymbol{E}} \end{aligned}$$

   Divergence of electric field is

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \tag{8}$$

   Thus

$$0 = \nabla \cdot \boldsymbol{j} + \dot{\rho} \tag{9}$$

7.

$$M_{ik} = \frac{1}{2}(M_{ik} + M_{ki}) + \frac{1}{2}(M_{ik} - M_{ki}) \tag{10}$$

in which the first part is symmetric and the second is antisymmetric.
Exchange i and k,

$$\begin{aligned} & \frac{1}{2}(M_{ik} + M_{ki})\mathrm{d}x^i \wedge \mathrm{d}x^k = -\frac{1}{2}(M_{ki} + M_{ik})\mathrm{d}x^k \wedge \mathrm{d}x^i \\ \rightarrow \quad & (M_{ik} + M_{ki})\mathrm{d}x^i \wedge \mathrm{d}x^k = 0 \end{aligned}$$

This shows the symmetric part doesn't contribute to $M_{ik}\mathrm{d}x^i\mathrm{d}x^k$.

$$\frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i \wedge \mathrm{d}x^k = \frac{1}{2}(M_{ki} - M_{ik})\mathrm{d}x^k \wedge \mathrm{d}x^i = -\frac{1}{2}(M_{ki} - M_{ik})\mathrm{d}x^i \wedge \mathrm{d}x^k$$

Symmetric part contributes to $M_{ik}\mathrm{d}x^i\mathrm{d}x^k$.
For $M_{ik}\mathrm{d}x^i\mathrm{d}x^k = \frac{1}{2}(M_{ik} + M_{ki})\mathrm{d}x^i\mathrm{d}x^k + \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k$,

$$\begin{aligned} & \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k = \frac{1}{2}(M_{ki} - M_{ik})\mathrm{d}x^k\mathrm{d}x^i \\ \rightarrow \quad & \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k = -\frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^k\mathrm{d}x^i \\ \rightarrow \quad & \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k = 0 \end{aligned}$$

$$\frac{1}{2}(M_{ik} + M_{ki})\mathrm{d}x^i\mathrm{d}x^k = \frac{1}{2}(M_{ki} + M_{ik})\mathrm{d}x^k\mathrm{d}x^i$$

Antisymmetric part doesn't contribute to $M_{ik}\mathrm{d}x^i\mathrm{d}x^k$.

8. (On page 507, the other half of parentheses is lost.)

For any 3 dimensional vector,

$$
\begin{aligned}
(\nabla \times (\nabla \times \boldsymbol{E}))_i &= \sum_{j,k=1}^{3} \epsilon_{ijk}\partial_j(\nabla \times E)_k \\
&= \sum_{j,k,l,m=1}^{3} \epsilon_{ijk}\partial_j\epsilon_{klm}\partial_l E_m \\
&= \sum_{j,l,m=1}^{3} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l E_m \\
&= \sum_{j,l,m=1}^{3} \delta_{il}\delta_{jm}\partial_j\partial_l E_m - \sum_{j,l,m=1}^{3} \delta_{im}\delta_{jl}\partial_j\partial_l E_m \\
&= \sum_{j,l,m=1}^{3} \delta_{il}\delta_{jm}\partial_j\partial_l E_m - \sum_{j,l,m=1}^{3} \delta_{im}\delta_{jl}\partial_j\partial_l E_m \\
&= \partial_i(\nabla \times \boldsymbol{E}) - \partial_j\partial_j E_i \\
&= (\nabla(\nabla \times \boldsymbol{E}))_i - (\Delta\boldsymbol{E})_i
\end{aligned}
$$

9.

$$
\begin{aligned}
\partial_i F_{jk} &= \partial_i(\partial_j A_k - \partial_k A_j) \\
&= \partial_i\partial_j A_k - \partial_i\partial_k A_j
\end{aligned}
$$

$$
\begin{aligned}
\partial_k F_{ij} &= \partial_k(\partial_i A_j - \partial_j A_i) \\
&= \partial_k\partial_i A_j - \partial_k\partial_j A_i
\end{aligned}
$$

$$
\begin{aligned}
\partial_j F_{ki} &= \partial_j(\partial_k A_i - \partial_i A_k) \\
&= \partial_j\partial_k A_i - \partial_j\partial_i A_k
\end{aligned}
$$

Sum up

$$
\begin{aligned}
& \partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki} \\
= & \partial_i\partial_j A_k - \partial_i\partial_k A_j + \partial_k\partial_i A_j - \partial_k\partial_j A_i + \partial_j\partial_k A_i - \partial_j\partial_i A_k \\
= & 0
\end{aligned}
$$

10. Assume

$$
A = A_{i_1\cdots i_p}\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \tag{11}
$$

$$B = B_{i_1\cdots i_q}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_q} \tag{12}$$

$$\begin{aligned}
\mathrm{d}(A\wedge B) &= \mathrm{d}((A_{i_1\cdots i_p}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p})\wedge(B_{i_1\cdots i_q}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_q}))\\
&= \partial_k A_{i_1\cdots i_p}\mathrm{d}x^k\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\wedge(B_{i_1\cdots i_q}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_q})\\
&\quad +A_{i_1\cdots i_p}\partial_k B_{i_1\cdots i_q}\mathrm{d}x^k\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\wedge(\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_q})\\
&= \mathrm{d}A\wedge B\\
&\quad +A_{i_1\cdots i_p}(-1)^p\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\wedge(\partial_k B_{i_1\cdots i_q}\mathrm{d}x^k\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_q})\\
&= \mathrm{d}A\wedge B+(-1)^pA\wedge\mathrm{d}B
\end{aligned}$$

11.

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(\frac{1}{2}a_{ij})\mathrm{d}x^i\wedge\mathrm{d}x^j\\
&= \frac{1}{2}\partial_k a_{ij}\mathrm{d}x^k\wedge\mathrm{d}x^i\wedge\mathrm{d}x^j\\
&= \frac{1}{6}(\partial_k a_{ij}\mathrm{d}x^k\wedge\mathrm{d}x^i\wedge\mathrm{d}x^j+\partial_j a_{ki}\mathrm{d}x^j\wedge\mathrm{d}^k\wedge\mathrm{d}x^i+\partial_i a_{jk}\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k)\\
&= \frac{1}{3!}(\partial_k a_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k+\partial_j a_{ki}\mathrm{d}x^i\wedge\mathrm{d}^j\wedge\mathrm{d}x^k+\partial_i a_{jk}\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k)\\
&= \frac{1}{3!}(\partial_k a_{ij}+\partial_j a_{ki}+\partial_i a_{jk})\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k
\end{aligned}$$

12.

$$\delta^i{}_j = g'^{ik}\frac{\partial x^t}{\partial x'^k}g_{tu}\frac{\partial x^u}{\partial x'^j} \tag{13}$$

Multiply by g^{lt}

$$\begin{aligned}
g^{lt}\delta^i{}_j &= g'^{ik}\frac{\partial x^t}{\partial x'^k}g_{tu}\frac{\partial x^u}{\partial x'^j}g^{lt}\\
g^{lt}\delta^i{}_j &= g'^{ik}\frac{\partial x^t}{\partial x'^k}\delta^l{}_u\frac{\partial x^u}{\partial x'^j}\\
g^{lt}\delta^i{}_j &= g'^{ik}\frac{\partial x^t}{\partial x'^k}\frac{\partial x^l}{\partial x'^j}\\
g^{lt}\frac{\partial x'^j}{\partial x^l}\delta^i{}_j &= g'^{ik}\frac{\partial x^t}{\partial x'^k}\\
g^{lt}\frac{\partial x'^i}{\partial x^l} &= g'^{ik}\frac{\partial x^t}{\partial x'^k}\\
\frac{\partial x'^k}{\partial x^t}g^{lt}\frac{\partial x'^i}{\partial x^l} &= g'^{ik}
\end{aligned}$$

13.

$$\begin{aligned}
\mathrm{d}p &= \hat{\boldsymbol{x}}(\cos\phi\mathrm{d}\rho-\rho\sin\phi\phi)+\hat{\boldsymbol{y}}(\sin\phi\mathrm{d}\rho+\rho\cos\phi\mathrm{d}\phi)+\hat{\boldsymbol{z}}\mathrm{d}z\\
&= \hat{\boldsymbol{\rho}}\mathrm{d}\rho+\hat{\boldsymbol{\phi}}\rho\mathrm{d}\phi+\hat{\boldsymbol{z}}\mathrm{d}z
\end{aligned}$$

Collect the terms

$$\mathrm{d}\rho(\cos\phi\mathrm{d}\rho\hat{\boldsymbol{x}}+\sin\phi\mathrm{d}\rho\hat{\boldsymbol{y}})+\mathrm{d}\phi(-\rho\sin\phi\hat{\boldsymbol{x}}+\rho\cos\theta\hat{\boldsymbol{y}})+\hat{\boldsymbol{z}}\mathrm{d}z=\hat{\boldsymbol{\rho}}\mathrm{d}\rho+\hat{\boldsymbol{\phi}}\rho\mathrm{d}\phi+\hat{\boldsymbol{z}}\mathrm{d}z \quad (14)$$

The coefficients of each derivative should be the same

$$\begin{aligned}
\hat{\boldsymbol{\rho}} &= \cos\phi\hat{\boldsymbol{x}}+\sin\phi\hat{\boldsymbol{y}}\\
\hat{\boldsymbol{phi}} &= -\sin\phi\hat{\boldsymbol{x}}+\cos\phi\hat{\boldsymbol{y}}\\
\hat{\boldsymbol{z}} &= \hat{\boldsymbol{z}}
\end{aligned}$$

14.

$$\begin{aligned}
\mathrm{d}p &= \hat{\boldsymbol{x}}(-r\sin\theta\sin\phi\mathrm{d}\phi+r\cos\phi\cos\theta\mathrm{d}\theta+\sin\theta\cos\phi\mathrm{d}r)+\hat{\boldsymbol{y}}(r\sin\theta\cos\phi\mathrm{d}\phi\\
&\quad +r\sin\phi\cos\theta\mathrm{d}\theta+\sin\theta\sin\phi\mathrm{d}r)+\hat{\boldsymbol{z}}(-r\sin\theta\mathrm{d}\theta+\cos\theta\mathrm{d}r)\\
&= \hat{\boldsymbol{r}}\mathrm{d}r+\hat{\boldsymbol{\theta}}r\mathrm{d}\theta+\hat{\boldsymbol{\phi}}r\sin\theta\mathrm{d}\phi
\end{aligned}$$

Collect the terms

$$\begin{aligned}
&(-r\sin\theta\sin\phi\hat{\boldsymbol{x}}+r\sin\theta\cos\phi\hat{\boldsymbol{y}})\mathrm{d}\phi+(r\cos\phi\cos\theta\hat{\boldsymbol{x}}+r\sin\phi\cos\theta\hat{\boldsymbol{y}}-r\sin\theta\hat{\boldsymbol{z}})\mathrm{d}\theta\\
&+(\sin\theta\cos\phi\hat{\boldsymbol{x}}+\sin\theta\sin\phi\hat{\boldsymbol{y}}+\cos\theta\hat{\boldsymbol{z}})\mathrm{d}r\\
=&\hat{\boldsymbol{r}}\mathrm{d}r+\hat{\boldsymbol{\theta}}r\mathrm{d}\theta+\hat{\boldsymbol{\phi}}r\sin\theta\mathrm{d}\phi
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{\phi}} &= -\sin\phi\hat{\boldsymbol{x}}+\cos\phi\hat{\boldsymbol{y}}\\
\hat{\boldsymbol{\theta}} &= \cos\phi\cos\theta\hat{\boldsymbol{x}}+\sin\phi\cos\theta\hat{\boldsymbol{y}}-\sin\theta\hat{\boldsymbol{z}}\\
\hat{\boldsymbol{r}} &= \sin\theta\cos\phi\hat{\boldsymbol{x}}+\sin\theta\sin\phi\hat{\boldsymbol{y}}+\cos\theta\hat{\boldsymbol{z}}
\end{aligned}$$

15. A flat 3-space means the Riemann curvature is zero. But I have no idea about the metric. Any metric that is conformal with a flat metric indicates a flat space. But here I use $g_{ij}=\eta_{ij}$, which gives $\det g_{ij}=1$ and $s=1$. Am I wrong?

In this problem $g_{ij}=\eta_{ij}$, which gives $\det g_{ij}=1$ and $s=1$.

$$\begin{aligned}
**\mathrm{d}x^i &= *(\frac{1}{2}g^{il}\eta_{ljk}\mathrm{d}x^j\wedge\mathrm{d}x^k)\\
&= \frac{1}{2}\eta^{il}*(\mathrm{d}x^j\wedge\mathrm{d}x^k)\\
&= \frac{1}{2}\eta^{il}g^{jm}g^{kn}\eta_{mnt}\mathrm{d}x^t\\
&= \frac{1}{2}g^{il}\sqrt{g}\epsilon_{ljk}g^{jm}g^{kn}\sqrt{g}\epsilon_{mnt}\mathrm{d}x^t\\
&= \frac{1}{2}\epsilon^i{}_{jk}\epsilon^{jk}{}_t\mathrm{d}x^t\\
&= \mathrm{d}x^i
\end{aligned}$$

$$
\begin{aligned}
**\,(\mathrm{d}x^i \wedge \mathrm{d}x^k) &= *(g^{ik} g^{jl} \eta_{klm} \mathrm{d}x^m) \\
&= g^{ik} g^{jl} \eta_{klm} * \mathrm{d}x^m \\
&= g^{ik} g^{jl} \eta_{klm} \frac{1}{2} g^{mu} \eta_{uwt} \mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= g^{ik} g^{jl} \epsilon_{klm} \sqrt{g} \frac{1}{2} g^{mu} \epsilon_{uwt} \sqrt{g} \mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= \frac{1}{2} \epsilon^{ij}{}_{m} \epsilon^{m}{}_{wt} \mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= \mathrm{d}x^i \wedge \mathrm{d}x^j
\end{aligned}
$$

16.

$$
\begin{aligned}
**\,\mathrm{d}x^i &= *(\frac{1}{3!} g^{ik} \eta_{klmn} \mathrm{d}x^l \wedge \mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&= \frac{1}{3!} g^{ik} \eta_{klmn} g^{lr} g^{ms} g^{nt} \eta_{rstu} \mathrm{d}x^u \\
&= \frac{1}{3!} \eta^{i}{}_{lmn} \eta^{lmn}{}_{u} \mathrm{d}x^u \\
&= \mathrm{d}x^i
\end{aligned}
$$

$$
\begin{aligned}
**\,(\mathrm{d}x^i \wedge \mathrm{d}x^j) &= *(\frac{1}{2} g^{ik} g^{jl} \eta_{klmn} \mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&= \frac{1}{2} g^{ik} g^{jl} \eta_{klmn} * (\mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&= \frac{1}{2} g^{ik} g^{jl} \eta_{klmn} \frac{1}{2} g^{mr} g^{ns} \eta_{rswt} \mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= \frac{1}{4} \eta^{ij}{}_{mn} \eta^{mn}{}_{wt} \mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= -\mathrm{d}x^i \wedge \mathrm{d}x^j
\end{aligned}
$$

$$
\begin{aligned}
**\,(\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k) &= *(g^{it} g^{ju} g^{kv} \eta_{tuvw} \mathrm{d}x^w) \\
&= g^{it} g^{ju} g^{kv} \eta_{tuvw} \frac{1}{3!} g^{wm} \eta_{mnrs} \mathrm{d}x^n \wedge \mathrm{d}x^r \wedge \mathrm{d}x^s \\
&= \frac{1}{3!} \eta^{ijk}{}_{w} \eta^{w}{}_{nrs} \mathrm{d}x^n \wedge \mathrm{d}x^r \wedge \mathrm{d}x^s \\
&= \mathrm{d}x^n \wedge \mathrm{d}x^r \wedge \mathrm{d}x^s
\end{aligned}
$$

$$
\begin{aligned}
**1 &= *(\frac{1}{4!}\eta_{klmn}\mathrm{d}x^k \wedge \mathrm{d}x^l \wedge \mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&= \frac{1}{4!}\eta_{klmn} g^{ki} g^{lj} g^{mu} g^{nv} \eta_{ijuv} \\
&= \frac{1}{4!}\eta_{klmn}\eta^{klmn} \\
&= -1
\end{aligned}
$$