1 Tensors and Local Symmetries

1. Transformation from euclidean (x,y) to polar (r,θ) is

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Transformation from polar to euclidean coordinates is

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \cos\theta = x/r \\ \sin\theta = y/r \\ \theta = \arccos\frac{x}{\sqrt{x^2+y^2}} & \text{for positive y} \\ \theta = \pi + \arccos\frac{x}{\sqrt{x^2+y^2}} & \text{for negative y} \end{cases}$$

Partial derivatives

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2}}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$
$$\frac{\partial x}{\partial r} = \frac{\partial r\cos\theta}{\partial r} = \cos\theta = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{\partial\sqrt{x^2+y^2}}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$
$$\frac{\partial y}{\partial r} = \frac{\partial r\sin\theta}{\partial r} = \sin\theta = \frac{y}{r}$$

$$\frac{\partial\theta}{\partial x} = \frac{\partial\arccos\frac{x}{\sqrt{x^2+y^2}}}{\partial x} = \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\frac{y^2}{(x^2+y^2)^{3/2}} = \frac{-y}{x^2+y^2} = -\frac{y}{r^2}$$
$$\frac{\partial x}{\partial\theta} = \frac{\partial r\cos\theta}{\partial\theta} = -r\sin\theta = -y$$

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$$\frac{\partial\theta}{\partial y} = \frac{\partial\arccos\frac{x}{\sqrt{x^2+y^2}}}{\partial y} = \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\frac{-xy}{(x^2+y^2)^{3/2}} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$
$$\frac{\partial y}{\partial\theta} = \frac{r\sin\theta}{\partial\theta} = r\cos\theta = x$$

To summarize

$$\frac{\partial\theta}{\partial x} = -\frac{y}{r^2} \quad \neq \quad \frac{\partial x}{\partial\theta} = -y \tag{1}$$
$$\frac{\partial\theta}{\partial y} = \frac{x}{r^2} \quad \neq \quad \frac{\partial y}{\partial\theta} = x \tag{2}$$

2.

3. Put

$$L = \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

into $L^T\eta L$.

$$L^T\eta L = \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$
$$= \begin{pmatrix} -\gamma & \beta & 0 & 0 \\ -\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$
$$= \begin{pmatrix} \beta^2 - \gamma^2 & 0 & 0 & 0 \\ 0 & -\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Since $\beta = \sqrt{\gamma^2 - 1}$,

$$L^T\eta L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta \tag{7}$$

4.

$$\eta = L\eta L^T = \Lambda^{-1}\eta(\Lambda^{-1})^T = \Lambda^{-1}\eta(\Lambda^T)^{-1} \tag{8}$$

Inverse both sides

$$\eta^{-1} = \Lambda^T \eta^{-1} \Lambda \tag{9}$$

We already have $\eta^{-1} = \eta$, then

$$\eta = \Lambda^T \eta \Lambda \tag{10}$$

5. To see the same physics means to have the same kinetic energy at center of mass system.

   Coordinate transformation leaves scalar unchanged. In this problem, $P_\mu P^\mu$ won't change when we change from one reference frame to another, i.e.,

$$-(E_1 + E_2)^2 + m^2 u_2^2 c^2 = -(E_1' + E_2')^2 + (mu_1' + mu_2')^2 c^2.$$

   We also have

$$\begin{cases} E_2^2 = m^2 u_2^2 c^2 + m^2 c^4 \\ E_1 = mc^2 \\ E_1' = E_2' \\ u_1' = -u_2' \end{cases} \tag{11}$$

   But the mass of proton is so small compared to the accelerator energy that we can drop mc^2 term in our calculation.

   Then the energy of the incoming proton in lab frame is

$$E_2 \approx \frac{2E_1^2}{mc^2} \approx 10^5 \text{TeV} \tag{12}$$

6. Take the divergence of both sides of Maxwell-Ampere law

$$\begin{aligned} \nabla \cdot \nabla \times \mathbf{B} &= \mu_0 \nabla \cdot \mathbf{j} + \epsilon_0 \mu_0 \nabla \cdot \dot{\mathbf{E}} \qquad (13) \\ 0 &= \nabla \cdot \mathbf{j} + \epsilon_0 \nabla \cdot \dot{\mathbf{E}} \qquad (14) \end{aligned}$$

   Divergence of electric field is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{15}$$

   Thus

$$0 = \nabla \cdot \mathbf{j} + \dot{\rho} \tag{16}$$

7.

$$M_{ik} = \frac{1}{2}(M_{ik} + M_{ki}) + \frac{1}{2}(M_{ik} - M_{ki}) \tag{17}$$

in which the first part is symmetric and the second is antisymmetric.
Exchange i and k,

$$\frac{1}{2}(M_{ik} + M_{ki})\mathrm{d}x^i \wedge \mathrm{d}x^k = -\frac{1}{2}(M_{ki} + M_{ik})\mathrm{d}x^k \wedge \mathrm{d}x^i \tag{18}$$

$$\rightarrow \quad (M_{ik} + M_{ki})\mathrm{d}x^i \wedge \mathrm{d}x^k = 0 \tag{19}$$

This shows the symmetric part doesn't contribute to $M_{ik}\mathrm{d}x^i\mathrm{d}x^k$.

$$\frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i \wedge \mathrm{d}x^k = \frac{1}{2}(M_{ki} - M_{ik})\mathrm{d}x^k \wedge \mathrm{d}x^i = -\frac{1}{2}(M_{ki} - M_{ik})\mathrm{d}x^i \wedge \mathrm{d}x^k \tag{20}$$

Symmetric part contributes to $M_{ik}\mathrm{d}x^i\mathrm{d}x^k$.
For $M_{ik}\mathrm{d}x^i\mathrm{d}x^k = \frac{1}{2}(M_{ik} + M_{ki})\mathrm{d}x^i\mathrm{d}x^k + \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k$,

$$\frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k = \frac{1}{2}(M_{ki} - M_{ik})\mathrm{d}x^k\mathrm{d}x^i \tag{21}$$

$$\rightarrow \quad \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k = -\frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^k\mathrm{d}x^i \tag{22}$$

$$\rightarrow \quad \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k = 0 \tag{23}$$

$$\frac{1}{2}(M_{ik} + M_{ki})\mathrm{d}x^i\mathrm{d}x^k = \frac{1}{2}(M_{ki} + M_{ik})\mathrm{d}x^k\mathrm{d}x^i \tag{24}$$

Antisymmetric part doesn't contribute to $M_{ik}\mathrm{d}x^i\mathrm{d}x^k$.

8. (On page 507, the other half of parentheses is lost.)

For any 3 dimensional vector,

$$\begin{aligned}
(\nabla \times (\nabla \times \mathbf{E}))_i &= \sum_{j,k=1}^{3} \epsilon_{ijk} \partial_j (\nabla \times E)_k && (25)\\
&= \sum_{j,k,l,m=1}^{3} \epsilon_{ijk} \partial_j \epsilon_{klm} \partial_l E_m && (26)\\
&= \sum_{j,l,m=1}^{3} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_j \partial_l E_m && (27)\\
&= \sum_{j,l,m=1}^{3} \delta_{il}\delta_{jm} \partial_j \partial_l E_m - \sum_{j,l,m=1}^{3} \delta_{im}\delta_{jl} \partial_j \partial_l E_m && (28)\\
&= \sum_{j,l,m=1}^{3} \delta_{il}\delta_{jm} \partial_j \partial_l E_m - \sum_{j,l,m=1}^{3} \delta_{im}\delta_{jl} \partial_j \partial_l E_m && (29)\\
&= \partial_i (\nabla \times \mathbf{E}) - \partial_j \partial_j E_i && (30)\\
&= (\nabla(\nabla \times \mathbf{E}))_i - (\Delta \mathbf{E})_i && (31)
\end{aligned}$$

9.

$$\begin{aligned}
\partial_i F_{jk} &= \partial_i (\partial_j A_k - \partial_k A_j) && (32)\\
&= \partial_i \partial_j A_k - \partial_i \partial_k A_j && (33)
\end{aligned}$$

$$\begin{aligned}
\partial_k F_{ij} &= \partial_k (\partial_i A_j - \partial_j A_i) && (34)\\
&= \partial_k \partial_i A_j - \partial_k \partial_j A_i && (35)
\end{aligned}$$

$$\begin{aligned}
\partial_j F_{ki} &= \partial_j (\partial_k A_i - \partial_i A_k) && (36)\\
&= \partial_j \partial_k A_i - \partial_j \partial_i A_k && (37)
\end{aligned}$$

Sum up

$$\begin{aligned}
& \partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki} && (38)\\
= \; & \partial_i \partial_j A_k - \partial_i \partial_k A_j + \partial_k \partial_i A_j - \partial_k \partial_j A_i + \partial_j \partial_k A_i - \partial_j \partial_i A_k && (39)\\
= \; & 0 && (40)
\end{aligned}$$

10. Assume

$$A = A_{i_1 \cdots i_p} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \quad (41)$$

$$B = B_{i_1 \cdots i_q} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_q} \tag{42}$$

$$\begin{aligned}
\mathrm{d}(A \wedge B) &= \mathrm{d}((A_{i_1 \cdots i_p} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}) \wedge (B_{i_1 \cdots i_q} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_q})) \\
&= \partial_k A_{i_1 \cdots i_p} \mathrm{d}x^k \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \wedge (B_{i_1 \cdots i_q} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_q}) \\
&\quad + A_{i_1 \cdots i_p} \partial_k B_{i_1 \cdots i_q} \mathrm{d}x^k \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \wedge (\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_q}) \\
&= \mathrm{d}A \wedge B \\
&\quad + A_{i_1 \cdots i_p} (-1)^p \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \wedge (\partial_k B_{i_1 \cdots i_q} \mathrm{d}x^k \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_q}) \\
&= \mathrm{d}A \wedge B + (-1)^p A \wedge \mathrm{d}B
\end{aligned}$$

11.

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(\frac{1}{2} a_{ij}) \mathrm{d}x^i \wedge \mathrm{d}x^j \\
&= \frac{1}{2} \partial_k a_{ij} \mathrm{d}x^k \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j \\
&= \frac{1}{6} (\partial_k a_{ij} \mathrm{d}x^k \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j + \partial_j a_{ki} \mathrm{d}x^j \wedge \mathrm{d}^k \wedge \mathrm{d}x^i + \partial_i a_{jk} \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k) \\
&= \frac{1}{3!} (\partial_k a_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k + \partial_j a_{ki} \mathrm{d}x^i \wedge \mathrm{d}^j \wedge \mathrm{d}x^k + \partial_i a_{jk} \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k) \\
&= \frac{1}{3!} (\partial_k a_{ij} + \partial_j a_{ki} + \partial_i a_{jk}) \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k
\end{aligned}$$

12.

$$\delta^i{}_j = g'^{ik} \frac{\partial x^t}{\partial x'^k} g_{tu} \frac{\partial x^u}{\partial x'^j} \tag{43}$$

Multiply by g^{lt}

$$\begin{aligned}
g^{lt} \delta^i{}_j &= g'^{ik} \frac{\partial x^t}{\partial x'^k} g_{tu} \frac{\partial x^u}{\partial x'^j} g^{lt} \\
g^{lt} \delta^i{}_j &= g'^{ik} \frac{\partial x^t}{\partial x'^k} \delta^l{}_u \frac{\partial x^u}{\partial x'^j} \\
g^{lt} \delta^i{}_j &= g'^{ik} \frac{\partial x^t}{\partial x'^k} \frac{\partial x^l}{\partial x'^j} \\
g^{lt} \frac{\partial x'^j}{\partial x^l} \delta^i{}_j &= g'^{ik} \frac{\partial x^t}{\partial x'^k} \\
g^{lt} \frac{\partial x'^i}{\partial x^l} &= g'^{ik} \frac{\partial x^t}{\partial x'^k} \\
\frac{\partial x'^k}{\partial x^t} g^{lt} \frac{\partial x'^i}{\partial x^l} &= g'^{ik}
\end{aligned}$$

13.

$$\begin{aligned}
\mathrm{d}p &= \hat{\mathbf{x}}(\cos\phi \mathrm{d}\rho - \rho \sin\phi \phi) + \hat{\mathbf{y}}(\sin\phi \mathrm{d}\rho + \rho\cos\phi \mathrm{d}\phi) + \hat{\mathbf{z}} \mathrm{d}z \\
&= \hat{\rho} \mathrm{d}\rho + \hat{\phi} \rho \mathrm{d}\phi + \hat{\mathbf{z}} \mathrm{d}z
\end{aligned}$$

Collect the terms

$$\mathrm{d}\rho(\cos\phi\mathrm{d}\rho\hat{\mathbf{x}}+\sin\phi\mathrm{d}\rho\hat{\mathbf{y}})+\mathrm{d}\phi(-\rho\sin\phi\hat{\mathbf{x}}+\rho\cos\theta\hat{\mathbf{y}})+\hat{\mathbf{z}}\mathrm{d}z=\hat{\boldsymbol{\rho}}\mathrm{d}\rho+\hat{\boldsymbol{\phi}}\rho\mathrm{d}\phi+\hat{\mathbf{z}}\mathrm{d}z \quad (44)$$

The coefficients of each derivative equal