1 Tensors and Local Symmetries

1. Defination of $\Gamma^{\sigma}_{\mu\nu}$

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(g_{\rho\mu,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho})$$

$$\begin{aligned}
\Gamma^{1}_{22} &= \frac{1}{2} g^{1\rho}(g_{\rho 2,2} + g_{2\rho,2} - g_{22,\rho}) \\
&= \frac{1}{2}\frac{1-kr^2}{a^2}(0+0-2a^2 r) \\
&= -r(1-kr^2) \\
\Gamma^{1}_{33} &= \frac{1}{2} g^{1\rho}(g_{\rho 3,3} + g_{3\rho,3} - g_{33,\rho}) \\
&= \frac{1}{2}\frac{1-kr^2}{a^2}(0+0-2a^2 r\sin^2\theta) \\
&= -r(1-kr^2)\sin^2\theta
\end{aligned}$$

2.

$$\begin{aligned}
\Gamma^{2}_{12} &= \frac{1}{2} g^{2\rho}(g_{\rho 1,2} + g_{2\rho,1} - g_{12,\rho}) \\
&= \frac{1}{2}\frac{1}{a^2 r^2}(0+2a^2 r-0) \\
&= \frac{1}{r} \\
\Gamma^{3}_{13} &= \frac{1}{2} g^{3\rho}(g_{\rho 1,3} + g_{3\rho,1} - g_{13,\rho}) \\
&= \frac{1}{2}\frac{1}{a^2 r^2\sin^2\theta}(0+2a^2 r\sin^2\theta-0) \\
&= \frac{1}{r}
\end{aligned}$$

The two lower indices are symmetric.

$$\begin{aligned}
{\Gamma^2}_{21} = {\Gamma^2}_{12} &= \frac{1}{r} \\
{\Gamma^3}_{31} = {\Gamma^3}_{13} &= \frac{1}{r}
\end{aligned}$$

3. Page491, equation 401, $\Gamma^{3}_{23} = \cos\theta = \Gamma^{3}_{32}$ should be $\Gamma^{3}_{23} = \cot\theta = \Gamma^{3}_{32}$

$$
\begin{aligned}
\Gamma^2_{33} &= \frac{1}{2} g^{2\rho}(g_{\rho 3,3} + g_{3\rho,3} - g_{33,\rho}) \\
&= \frac{1}{2}\frac{1}{a^2r^2}(0 + 0 - 2a^2r^2 \sin\theta\cos\theta) \\
&= -\sin\theta\cos\theta \\
\Gamma^3_{23} &= \frac{1}{2} g^{3\rho}(g_{\rho 2,3} +_{3\rho,2} - g_{23,\rho}) \\
&= \frac{1}{2}\frac{1}{a^2r^2\sin^2\theta}(0 + 2a^2r^2\cos\theta\sin\theta - 0) \\
&= \cot\theta
\end{aligned}
$$

The two lower indices are symmetric.

$$\Gamma^3_{32} = \Gamma^2_{23} = \cot\theta$$

4. Ricci tensor is

$$
R_{ij} = \begin{pmatrix}
-3\ddot{a}/a & 0 & 0 & 0 \\
0 & 2\dot{a}^2 + a\ddot{a} & 0 & 0 \\
0 & 0 & r^2(2\dot{a}^2 + a\ddot{a}) & 0 \\
0 & 0 & 0 & r^2\sin^2\theta(2\dot{a}^2 + a\ddot{a})
\end{pmatrix}
$$

11 component of field equation

$$
\begin{aligned}
& R_{11} = -\frac{8\pi G}{c^4}(T_{11} - \frac{T}{2} g_{11}) \\
\Rightarrow \quad & R_{11} = -\frac{8\pi G}{c^4}(p g_{11} - \frac{1}{2}(-\rho + 3p) g_{11}) \\
\Rightarrow \quad & R_{11} = -\frac{8\pi G}{c^4}\frac{1}{2}(\rho - p)\frac{a^2r^2}{1 - kr^2} \\
\Rightarrow \quad & \frac{A}{1 - kr^2}\frac{1 - kr^2}{a^2r^2} = \frac{4\pi G}{c^4}(\rho - p) \\
\Rightarrow \quad & \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p)
\end{aligned}
$$