

1 Tensors and Local Symmetries

1. Transformation from euclidean (x,y) to polar (r,θ) is

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

Transformation from polar to euclidean coordinates is

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \cos \theta = x/r \\ \sin \theta = y/r \\ \theta = \arccos \frac{x}{\sqrt{x^2 + y^2}} & \text{for positive y} \\ \theta = \pi + \arccos \frac{x}{\sqrt{x^2 + y^2}} & \text{for negative y} \end{cases}$$

Partial derivatives

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{\partial \sqrt{x^2 + y^2}}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} \\ \frac{\partial x}{\partial r} &= \frac{\partial r \cos \theta}{\partial r} = \cos \theta = \frac{x}{r} \end{aligned}$$

$$\begin{aligned} \frac{\partial r}{\partial y} &= \frac{\partial \sqrt{x^2 + y^2}}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r} \\ \frac{\partial y}{\partial r} &= \frac{\partial r \sin \theta}{\partial r} = \sin \theta = \frac{y}{r} \end{aligned}$$

$$\begin{aligned} \frac{\partial \theta}{\partial x} &= \frac{\partial \arccos \frac{x}{\sqrt{x^2 + y^2}}}{\partial x} = \frac{-1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} \frac{y^2}{(x^2 + y^2)^{3/2}} = \frac{-y}{x^2 + y^2} = -\frac{y}{r^2} \\ \frac{\partial x}{\partial \theta} &= \frac{\partial r \cos \theta}{\partial \theta} = -r \sin \theta = -y \end{aligned}$$

Page 432 equation 11.23?

$$\begin{aligned} \frac{\partial \theta}{\partial y} &= \frac{\partial \arccos \frac{x}{\sqrt{x^2 + y^2}}}{\partial y} = \frac{-1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} \frac{-xy}{(x^2 + y^2)^{3/2}} = \frac{x}{x^2 + y^2} = \frac{x}{r^2} \\ \frac{\partial y}{\partial \theta} &= \frac{r \sin \theta}{\partial \theta} = r \cos \theta = x \end{aligned}$$

To summarize

$$\begin{aligned}\frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} \neq \frac{\partial x}{\partial \theta} = -y \\ \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} \neq \frac{\partial y}{\partial \theta} = x\end{aligned}$$

2.

3. Put

$$L = \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

into $L^T \eta L$.

$$\begin{aligned}L^T \eta L &= \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\gamma & \beta & 0 & 0 \\ -\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \beta^2 - \gamma^2 & 0 & 0 & 0 \\ 0 & -\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

Since $\beta = \sqrt{\gamma^2 - 1}$,

$$L^T \eta L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta \tag{2}$$

4.

$$\eta = L \eta L^T = \Lambda^{-1} \eta (\Lambda^{-1})^T = \Lambda^{-1} \eta (\Lambda^T)^{-1} \tag{3}$$

Inverse both sides

$$\eta^{-1} = \Lambda^T \eta^{-1} \Lambda \quad (4)$$

We already have $\eta^{-1} = \eta$, then

$$\eta = \Lambda^T \eta \Lambda \quad (5)$$

5. To see the same physics means to have the same kinetic energy at center of mass system.

Coordinate transformation leaves scalar unchanged. In this problem, $P_\mu P^\mu$ won't change when we change from one reference frame to another, i.e.,

$$-(E_1 + E_2)^2 + m^2 u_2^2 c^2 = -(E'_1 + E'_2)^2 + (m u'_1 + m u'_2)^2 c^2.$$

We also have

$$\begin{cases} E_2^2 = m^2 u_2^2 c^2 + m^2 c^4 \\ E_1 = m c^2 \\ E'_1 = E'_2 \\ u'_1 = -u'_2 \end{cases} \quad (6)$$

But the mass of proton is so small compared to the accelerator energy that we can drop $m c^2$ term in our calculation.

Then the energy of the incoming proton in lab frame is

$$E_2 \approx \frac{2E_1^2}{m c^2} \approx 10^5 \text{ TeV} \quad (7)$$

6. Take the divergence of both sides of Maxwell-Ampere law

$$\begin{aligned} \nabla \cdot \nabla \times \mathbf{B} &= \mu_0 \nabla \cdot \mathbf{j} + \epsilon_0 \mu_0 \nabla \cdot \dot{\mathbf{E}} \\ 0 &= \nabla \cdot \mathbf{j} + \epsilon_0 \nabla \cdot \dot{\mathbf{E}} \end{aligned}$$

Divergence of electric field is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (8)$$

Thus

$$0 = \nabla \cdot \mathbf{j} + \dot{\rho} \quad (9)$$

7.

$$M_{ik} = \frac{1}{2}(M_{ik} + M_{ki}) + \frac{1}{2}(M_{ik} - M_{ki}) \quad (10)$$

in which the first part is symmetric and the second is antisymmetric.

Exchange i and k ,

$$\begin{aligned} \frac{1}{2}(M_{ik} + M_{ki})dx^i \wedge dx^k &= -\frac{1}{2}(M_{ki} + M_{ik})dx^k \wedge dx^i \\ \rightarrow (M_{ik} + M_{ki})dx^i \wedge dx^k &= 0 \end{aligned}$$

This shows the symmetric part doesn't contribute to $M_{ik}dx^i dx^k$.

$$\frac{1}{2}(M_{ik} - M_{ki})dx^i \wedge dx^k = \frac{1}{2}(M_{ki} - M_{ik})dx^k \wedge dx^i = -\frac{1}{2}(M_{ki} - M_{ik})dx^i \wedge dx^k$$

Symmetric part contributes to $M_{ik}dx^i dx^k$.

For $M_{ik}dx^i dx^k = \frac{1}{2}(M_{ik} + M_{ki})dx^i dx^k + \frac{1}{2}(M_{ik} - M_{ki})dx^i dx^k$,

$$\begin{aligned} \frac{1}{2}(M_{ik} - M_{ki})dx^i dx^k &= \frac{1}{2}(M_{ki} - M_{ik})dx^k dx^i \\ \rightarrow \frac{1}{2}(M_{ik} - M_{ki})dx^i dx^k &= -\frac{1}{2}(M_{ik} - M_{ki})dx^k dx^i \\ \rightarrow \frac{1}{2}(M_{ik} - M_{ki})dx^i dx^k &= 0 \end{aligned}$$

$$\frac{1}{2}(M_{ik} + M_{ki})dx^i dx^k = \frac{1}{2}(M_{ki} + M_{ik})dx^k dx^i$$

Antisymmetric part doesn't contribute to $M_{ik}dx^i dx^k$.

8. (On page 507, the other half of parentheses is lost.)

For any 3 dimensional vector,

$$\begin{aligned}
(\nabla \times (\nabla \times \mathbf{E}))_i &= \sum_{j,k=1}^3 \epsilon_{ijk} \partial_j (\nabla \times \mathbf{E})_k \\
&= \sum_{j,k,l,m=1}^3 \epsilon_{ijk} \partial_j \epsilon_{klm} \partial_l E_m \\
&= \sum_{j,l,m=1}^3 (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j \partial_l E_m \\
&= \sum_{j,l,m=1}^3 \delta_{il} \delta_{jm} \partial_j \partial_l E_m - \sum_{j,l,m=1}^3 \delta_{im} \delta_{jl} \partial_j \partial_l E_m \\
&= \sum_{j,l,m=1}^3 \delta_{il} \delta_{jm} \partial_j \partial_l E_m - \sum_{j,l,m=1}^3 \delta_{im} \delta_{jl} \partial_j \partial_l E_m \\
&= \partial_i (\nabla \times \mathbf{E}) - \partial_j \partial_j E_i \\
&= (\nabla (\nabla \times \mathbf{E}))_i - (\Delta \mathbf{E})_i
\end{aligned}$$

9.

$$\begin{aligned}
\partial_i F_{jk} &= \partial_i (\partial_j A_k - \partial_k A_j) \\
&= \partial_i \partial_j A_k - \partial_i \partial_k A_j
\end{aligned}$$

$$\begin{aligned}
\partial_k F_{ij} &= \partial_k (\partial_i A_j - \partial_j A_i) \\
&= \partial_k \partial_i A_j - \partial_k \partial_j A_i
\end{aligned}$$

$$\begin{aligned}
\partial_j F_{ki} &= \partial_j (\partial_k A_i - \partial_i A_k) \\
&= \partial_j \partial_k A_i - \partial_j \partial_i A_k
\end{aligned}$$

Sum up

$$\begin{aligned}
&\partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki} \\
&= \partial_i \partial_j A_k - \partial_i \partial_k A_j + \partial_k \partial_i A_j - \partial_k \partial_j A_i + \partial_j \partial_k A_i - \partial_j \partial_i A_k \\
&= 0
\end{aligned}$$

10. Assume

$$A = A_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \quad (11)$$

$$B = B_{i_1 \dots i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q} \quad (12)$$

$$\begin{aligned} d(A \wedge B) &= d((A_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}) \wedge (B_{i_1 \dots i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q})) \\ &= \partial_k A_{i_1 \dots i_p} dx^k dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge (B_{i_1 \dots i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q}) \\ &\quad + A_{i_1 \dots i_p} \partial_k B_{i_1 \dots i_q} dx^k dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge (dx^{i_1} \wedge \dots \wedge dx^{i_q}) \\ &= dA \wedge B \\ &\quad + A_{i_1 \dots i_p} (-1)^p dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge (\partial_k B_{i_1 \dots i_q} dx^k dx^{i_1} \wedge \dots \wedge dx^{i_q}) \\ &= dA \wedge B + (-1)^p A \wedge dB \end{aligned}$$

11.

$$\begin{aligned} d\omega &= d\left(\frac{1}{2}a_{ij}\right)dx^i \wedge dx^j \\ &= \frac{1}{2}\partial_k a_{ij} dx^k \wedge dx^i \wedge dx^j \\ &= \frac{1}{6}(\partial_k a_{ij} dx^k \wedge dx^i \wedge dx^j + \partial_j a_{ki} dx^j \wedge dx^k \wedge dx^i + \partial_i a_{jk} dx^i \wedge dx^j \wedge dx^k) \\ &= \frac{1}{3!}(\partial_k a_{ij} dx^i \wedge dx^j \wedge dx^k + \partial_j a_{ki} dx^i \wedge dx^j \wedge dx^k + \partial_i a_{jk} dx^i \wedge dx^j \wedge dx^k) \\ &= \frac{1}{3!}(\partial_k a_{ij} + \partial_j a_{ki} + \partial_i a_{jk})dx^i \wedge dx^j \wedge dx^k \end{aligned}$$

12.

$$\delta^i_j = g^{ik} \frac{\partial x^t}{\partial x'^k} g_{tu} \frac{\partial x^u}{\partial x'^j} \quad (13)$$

Multiply by g^{lt}

$$\begin{aligned} g^{lt} \delta^i_j &= g^{ik} \frac{\partial x^t}{\partial x'^k} g_{tu} \frac{\partial x^u}{\partial x'^j} g^{lt} \\ g^{lt} \delta^i_j &= g^{ik} \frac{\partial x^t}{\partial x'^k} \delta^l_u \frac{\partial x^u}{\partial x'^j} \\ g^{lt} \delta^i_j &= g^{ik} \frac{\partial x^t}{\partial x'^k} \frac{\partial x^l}{\partial x'^j} \\ g^{lt} \frac{\partial x'^j}{\partial x^l} \delta^i_j &= g^{ik} \frac{\partial x^t}{\partial x'^k} \\ g^{lt} \frac{\partial x'^i}{\partial x^l} &= g^{ik} \frac{\partial x^t}{\partial x'^k} \\ \frac{\partial x'^k}{\partial x^t} g^{lt} \frac{\partial x'^i}{\partial x^l} &= g^{ik} \end{aligned}$$

13.

$$\begin{aligned} dp &= \hat{\mathbf{x}}(\cos \phi d\rho - \rho \sin \phi d\phi) + \hat{\mathbf{y}}(\sin \phi d\rho + \rho \cos \phi d\phi) + \hat{\mathbf{z}}dz \\ &= \hat{\boldsymbol{\rho}}d\rho + \hat{\boldsymbol{\phi}}\rho d\phi + \hat{\mathbf{z}}dz \end{aligned}$$

Collect the terms

$$d\rho(\cos\phi d\rho\hat{\mathbf{x}} + \sin\phi d\rho\hat{\mathbf{y}}) + d\phi(-\rho\sin\phi\hat{\mathbf{x}} + \rho\cos\phi\hat{\mathbf{y}}) + \hat{\mathbf{z}}dz = \hat{\rho}d\rho + \hat{\phi}d\phi + \hat{\mathbf{z}}dz \quad (14)$$

The coefficients of each derivative should be the same

$$\begin{aligned}\hat{\rho} &= \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}} \\ \hat{\phi} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}}\end{aligned}$$

14.

$$\begin{aligned}dp &= \hat{\mathbf{x}}(-r\sin\theta\sin\phi d\phi + r\cos\phi\cos\theta d\theta + \sin\theta\cos\phi dr) + \hat{\mathbf{y}}(r\sin\theta\cos\phi d\phi \\ &\quad + r\sin\phi\cos\theta d\theta + \sin\theta\sin\phi dr) + \hat{\mathbf{z}}(-r\sin\theta d\theta + \cos\theta dr) \\ &= \hat{\mathbf{r}}dr + \hat{\theta}r d\theta + \hat{\phi}r\sin\theta d\phi\end{aligned}$$

Collect the terms

$$\begin{aligned}&(-r\sin\theta\sin\phi\hat{\mathbf{x}} + r\sin\theta\cos\phi\hat{\mathbf{y}})d\phi + (r\cos\phi\cos\theta\hat{\mathbf{x}} + r\sin\phi\cos\theta\hat{\mathbf{y}} - r\sin\theta\hat{\mathbf{z}})d\theta \\ &\quad + (\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})dr \\ &= \hat{\mathbf{r}}dr + \hat{\theta}r d\theta + \hat{\phi}r\sin\theta d\phi\end{aligned}$$

$$\begin{aligned}\hat{\phi} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} \\ \hat{\theta} &= \cos\phi\cos\theta\hat{\mathbf{x}} + \sin\phi\cos\theta\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \\ \hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\end{aligned}$$

15. A flat 3-space means the Riemann curvature is zero. But I have no idea about the metric. Any metric that is conformal with a flat metric indicates a flat space. But here I use $g_{ij} = \eta_{ij}$, which gives $\det g_{ij} = 1$ and $s = 1$. Am I wrong?

In this problem $g_{ij} = \eta_{ij}$, which gives $\det g_{ij} = 1$ and $s = 1$.

$$\begin{aligned}**dx^i &= *(\frac{1}{2}g^{il}\eta_{ljk}dx^j \wedge dx^k) \\ &= \frac{1}{2}\eta^{il}*(dx^j \wedge dx^k) \\ &= \frac{1}{2}\eta^{il}g^{jm}g^{kn}\eta_{mnt}dx^t \\ &= \frac{1}{2}g^{il}\sqrt{g}\epsilon_{ljk}g^{jm}g^{kn}\sqrt{g}\epsilon_{mnt}dx^t \\ &= \frac{1}{2}\epsilon^i_{jk}\epsilon^{jk}_t dx^t \\ &= dx^i\end{aligned}$$

$$\begin{aligned}
**(\mathrm{d}x^i \wedge \mathrm{d}x^k) &= *(g^{ik} g^{jl} \eta_{klm} \mathrm{d}x^m) \\
&= g^{ik} g^{jl} \eta_{klm} * \mathrm{d}x^m \\
&= g^{ik} g^{jl} \eta_{klm} \frac{1}{2} g^{mu} \eta_{uwt} \mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= g^{ik} g^{jl} \epsilon_{klm} \sqrt{g} \frac{1}{2} g^{mu} \epsilon_{uwt} \sqrt{g} \mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= \frac{1}{2} \epsilon^{ij}_{m} \epsilon^m_{wt} \mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= \mathrm{d}x^i \wedge \mathrm{d}x^j
\end{aligned}$$

16.

$$\begin{aligned}
**\mathrm{d}x^i &= *(\frac{1}{3!} g^{ik} \eta_{klmn} \mathrm{d}x^l \wedge \mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&= \frac{1}{3!} g^{ik} \eta_{klmn} g^{lr} g^{ms} g^{nt} \eta_{rstu} \mathrm{d}x^u \\
&= \frac{1}{3!} \eta^i_{lmn} \eta^{lmn}_{u} \mathrm{d}x^u \\
&= \mathrm{d}x^i
\end{aligned}$$

$$\begin{aligned}
**(\mathrm{d}x^i \wedge \mathrm{d}x^j) &= *(\frac{1}{2} g^{ik} g^{jl} \eta_{klmn} \mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&= \frac{1}{2} g^{ik} g^{jl} \eta_{klmn} *(\mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&= \frac{1}{2} g^{ik} g^{jl} \eta_{klmn} \frac{1}{2} g^{mr} g^{ns} \eta_{rswt} \mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= \frac{1}{4} \eta^{ij}_{mn} \eta^{mn}_{wt} \mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= -\mathrm{d}x^i \wedge \mathrm{d}x^j
\end{aligned}$$

$$\begin{aligned}
**(\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k) &= *(g^{it} g^{ju} g^{kv} \eta_{tuvw} \mathrm{d}x^w) \\
&= g^{it} g^{ju} g^{kv} \eta_{tuvw} \frac{1}{3!} g^{wm} \eta_{mnrs} \mathrm{d}x^n \wedge \mathrm{d}x^r \wedge \mathrm{d}x^s \\
&= \frac{1}{3!} \eta^{ijk}_{w} \eta^w_{nrs} \mathrm{d}x^n \wedge \mathrm{d}x^r \wedge \mathrm{d}x^s \\
&= \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k
\end{aligned}$$

$$\begin{aligned}
**1 &= *(\frac{1}{4!}\eta_{klmn}dx^k \wedge dx^l \wedge dx^m \wedge dx^n) \\
&= \frac{1}{4!}\eta_{klmn}g^{ki}g^{lj}g^{mu}g^{nv}\eta_{ijuv} \\
&= \frac{1}{4!}\eta_{klmn}\eta^{klmn} \\
&= -1
\end{aligned}$$

17. For simplicity, we'll use ij to denote δ_i^j .

$$\begin{aligned}
\epsilon_{klmn}\epsilon^{pwmn} &= \sum_{m,n=1}^d \begin{vmatrix} kp & kw & km & kn \\ lp & lw & lm & ln \\ mp & mw & mm & mn \\ np & nw & nm & nn \end{vmatrix} \\
&= d \cdot kw \cdot lp - d \cdot kp \cdot lw - kw \cdot lp + kp \cdot lw + kp \cdot lw - kw \cdot lp - d^2 \cdot kw \cdot lp \\
&\quad + d^2 \cdot kp \cdot lw + d \cdot kw \cdot lp - d \cdot lw \cdot kp - d \cdot kp \cdot lw + d \cdot lp \cdot kw + d \cdot kw \cdot lp - d \cdot k \\
&\quad - kw \cdot lp + lw \cdot kp + kp \cdot lw - lp \cdot kw - d \cdot kp \cdot lw + d \cdot kw \cdot lp + kp \cdot lw \\
&\quad - lp \cdot kw - kw \cdot lp + lw \cdot kp \\
&= (d^2 - 5d + 6)(kp \cdot lw - kw \cdot lp)
\end{aligned}$$

Here are some useful equations

$$\begin{aligned}
\sum_{m,n=1}^d \delta_m^m \delta_n^n &= d^2 \\
\sum_{m,n=1}^d \delta_m^n \delta_n^m &= d \\
\sum_{n=1}^d \delta_m^m &= d \\
\sum_{m=1}^d \delta_l^m \delta_m^p &= \delta_l^p
\end{aligned}$$

For $d = 4$,

$$\epsilon_{klmn}\epsilon^{pwmn} = 2!(\delta_k^p \delta_l^w - \delta_k^w \delta_l^p) \quad (15)$$

18. Should we make it clear the the dimension is 3 in this problem?

$$\begin{aligned}
\epsilon_{lmn}\epsilon^{pmn} &= \sum_{m,n} \begin{vmatrix} lp & lm & ln \\ mp & mm & mn \\ np & nm & nn \end{vmatrix} \\
&= \sum_{m,n=0}^x (-lp \cdot mn \cdot nm + ln \cdot mp \cdot nm + lp \cdot mm \cdot nn \\
&\quad -lm \cdot mp \cdot nn - ln \cdot mm \cdot np + lm \cdot mn \cdot np) \\
&= \\
&= (d^2 - 3d + 2)lp
\end{aligned}$$

For $d = 3$

$$\epsilon_{lmn}\epsilon^{pmn} = 2\delta_l^p \quad (16)$$

19.

$$\begin{aligned}
\epsilon_{klmn}\epsilon^{plmn} &= \sum_{l,m,n=1}^d \begin{vmatrix} kp & kl & km & kn \\ lp & ll & lm & ln \\ mp & ml & mm & mn \\ np & nl & nm & nn \end{vmatrix} \\
&= kp(d^3 + 6d^2 + 11d - 6)
\end{aligned}$$

For $d = 4$

$$\epsilon_{klmn}\epsilon^{plmn} = 3!\delta_k^p \quad (17)$$

20. To express the derivatives of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$, we have to solve $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$

$$\begin{aligned}
\hat{\mathbf{x}} &= \cos \phi (\sin \theta \hat{\mathbf{r}} + \cos \theta \hat{\boldsymbol{\theta}}) - \sin \phi \hat{\boldsymbol{\phi}} \\
\hat{\mathbf{y}} &= \sin \phi (\sin \theta \hat{\mathbf{r}} + \cos \theta \hat{\boldsymbol{\theta}}) + \cos \phi \hat{\boldsymbol{\phi}} \\
\hat{\mathbf{z}} &= \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}
\end{aligned}$$

Derivatives

$$\begin{aligned}
\partial_\theta \hat{\mathbf{r}} &= \hat{\mathbf{x}} \cos \theta \cos \phi - \hat{\mathbf{z}} \sin \theta + \hat{\mathbf{y}} \sin \phi \\
&= \hat{\boldsymbol{\theta}} \\
\partial_\phi \hat{\mathbf{r}} &= \hat{\mathbf{y}} \cos \phi \sin \theta - \hat{\mathbf{x}} \sin \phi \sin \theta \\
&= \hat{\boldsymbol{\phi}} \sin \theta \\
\partial_r \hat{\mathbf{r}} &= 0
\end{aligned}$$

$$\begin{aligned}
\partial_\theta \hat{\boldsymbol{\theta}} &= -\hat{\mathbf{z}} \cos \theta - \hat{\mathbf{x}} \cos \phi \sin \theta - \hat{\mathbf{y}} \sin \theta \sin \phi \\
&= -\hat{\mathbf{r}} \\
\partial_\phi \hat{\boldsymbol{\theta}} &= \hat{\mathbf{y}} \cos \theta \cos \phi - \hat{\mathbf{x}} \cos \theta \sin \phi \\
&= \hat{\boldsymbol{\phi}} \cos \theta \\
\partial_r \hat{\boldsymbol{\theta}} &= 0
\end{aligned}$$

$$\begin{aligned}
\partial_\theta \hat{\boldsymbol{\phi}} &= 0 \\
\partial_\phi \hat{\boldsymbol{\phi}} &= -\hat{\mathbf{x}} \cos \phi - \hat{\mathbf{y}} \sin \phi \\
&= -\hat{\boldsymbol{\theta}} \cos \theta - \hat{\mathbf{r}} \sin \theta \\
\partial_r \hat{\boldsymbol{\phi}} &= 0
\end{aligned}$$

$$\nabla f = \hat{\mathbf{r}} \partial_r f + \frac{\hat{\boldsymbol{\theta}}}{r} \partial_\theta f + \frac{\hat{\boldsymbol{\phi}}}{r \sin \theta} \partial_\phi f$$

Laplacian of f is

$$\begin{aligned}
\Delta f &= \nabla \cdot \nabla f \\
&= \hat{\mathbf{r}} \partial_r (\hat{\mathbf{r}} \partial_r f) + \hat{\mathbf{r}} \partial_r \left(\hat{\boldsymbol{\theta}} \frac{1}{r} \partial_\theta f \right) + \hat{\mathbf{r}} \partial_r \left(\hat{\boldsymbol{\phi}} \frac{1}{r \sin \theta} \partial_\phi f \right) \\
&\quad + \hat{\boldsymbol{\theta}} \frac{1}{r} \partial_\theta (\hat{\mathbf{r}} \partial_r f) + \hat{\boldsymbol{\theta}} \frac{1}{r} \partial_\theta \left(\hat{\boldsymbol{\theta}} \frac{1}{r} \partial_\theta f \right) + \hat{\boldsymbol{\theta}} \frac{1}{r} \partial_\theta \left(\hat{\boldsymbol{\phi}} \frac{1}{r \sin \theta} \partial_\phi f \right) \\
&\quad + \hat{\boldsymbol{\phi}} \frac{1}{r \sin \theta} \partial_\phi (\hat{\mathbf{r}} \partial_r f) + \hat{\boldsymbol{\phi}} \frac{1}{r \sin \theta} \partial_\phi \left(\hat{\boldsymbol{\theta}} \frac{1}{r} \partial_\theta f \right) + \hat{\boldsymbol{\phi}} \frac{1}{r \sin \theta} \partial_\phi \left(\hat{\boldsymbol{\phi}} \frac{1}{r \sin \theta} \partial_\phi f \right) \\
&= f_{,rr} + \frac{1}{r} + \frac{1}{r^2} f_{,\theta\theta} + \frac{1}{r \sin \theta} \sin \theta f_{,r} + \frac{1}{r \sin \theta} \cos \theta \frac{1}{r} f_{,\theta} + \frac{1}{r \sin \theta} \frac{1}{r \sin \theta} f_{,\phi\phi} \\
&= \frac{(r^2 f_{,r})_{,r}}{r^2} + \frac{(\sin \theta f_{,\theta})_{,\theta}}{r^2 \sin \theta} + \frac{f_{,\phi\phi}}{r^2 \sin^2 \theta}
\end{aligned}$$

21. For any point

$$\mathbf{p} = \hat{\mathbf{x}} \cos \phi (R + r \sin \theta) + \hat{\mathbf{y}} \sin \phi (R + r \sin \theta) + \hat{\mathbf{z}} r \cos \theta$$

Differential of \mathbf{p} ,

$$\begin{aligned}
d\mathbf{p} &= \hat{\mathbf{x}} [-\sin \phi (R + r \sin \theta) d\phi + \cos \phi r \cos \theta d\theta] \\
&\quad + \hat{\mathbf{y}} [\cos \phi (R + r \sin \theta) d\phi + \sin \phi r \cos \theta d\theta] + \hat{\mathbf{z}} [-r \sin \theta d\theta] \\
&= [-\sin \phi (R + r \sin \theta) \hat{\mathbf{x}} + \cos \phi (R + r \sin \theta) \hat{\mathbf{y}}] d\phi \\
&\quad + [\cos \phi r \cos \theta \hat{\mathbf{x}} + \sin \phi r \cos \theta \hat{\mathbf{y}} - r \sin \theta \hat{\mathbf{z}}] d\theta \\
&\equiv e_\theta d\theta + e_\phi d\phi
\end{aligned}$$

Orthonormal basis vectors are

$$\begin{aligned}\hat{\boldsymbol{\theta}} &= \cos \phi \cos \theta \hat{\mathbf{x}} + \sin \phi \cos \theta \hat{\mathbf{y}} - \sin \theta \hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}\end{aligned}$$

We get

$$\begin{aligned}e_\theta &= r \hat{\boldsymbol{\theta}} \\ e_\phi &= -(R + r \sin \theta) \hat{\boldsymbol{\phi}}\end{aligned}$$

Metric of this torus is

$$\begin{aligned}g_{\theta\theta} &= r \hat{\boldsymbol{\theta}} \cdot r \hat{\boldsymbol{\theta}} \\ &= r^2 \\ g_{\theta\phi} &= r \hat{\boldsymbol{\theta}} \cdot (R + r \sin \theta) \hat{\boldsymbol{\phi}} \\ &= 0 \\ g_{\phi\phi} &= (R + r \sin \theta) \hat{\boldsymbol{\phi}} \cdot (R + r \sin \theta) \hat{\boldsymbol{\phi}} \\ &= (R + r \sin \theta)^2 \\ g_{\phi\theta} &= (R + r \sin \theta) \hat{\boldsymbol{\phi}} \cdot r \hat{\boldsymbol{\theta}} \\ &= 0\end{aligned}$$

Metric and its inverse are

$$\mathbf{g} = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r \sin \theta)^2 \end{pmatrix}$$

$$\mathbf{g}^{-1} = \begin{pmatrix} 1/r^2 & 0 \\ 0 & 1/(R + r \sin \theta)^2 \end{pmatrix}$$

22. Contravariant basis vectors are

$$\begin{aligned}e^\theta &= \hat{\boldsymbol{\theta}}/r \\ e^\phi &= \hat{\boldsymbol{\phi}}/(R + r \sin \theta)\end{aligned}$$

Nozero connections are

$$\begin{aligned}\Gamma_{\theta\phi}^\phi &= \frac{r \cos \theta}{R + r \sin \theta} \\ \Gamma_{\phi\phi}^\theta &= -\frac{\cos \theta (R + r \sin \theta)}{r} \\ \Gamma_{\phi\theta}^\phi &= \frac{r \cos \theta}{R + r \sin \theta}\end{aligned}$$

23. Christoffel matrices are

$$\begin{aligned}\Gamma_\theta &= \begin{pmatrix} 0 & 0 \\ 0 & r \cos \theta / (R + r \sin \theta) \end{pmatrix} \\ \Gamma_\phi &= \begin{pmatrix} r \cos \theta / (R + r \sin \theta) & -\cos \theta (R + r \sin \theta) / r \\ r \cos \theta / (R + r \sin \theta) & 0 \end{pmatrix}\end{aligned}$$

Commutator is

$$[\Gamma_\theta, \Gamma_\phi] = \begin{pmatrix} 0 & \cos^2 \theta \\ r^2 \cos \theta / (R + r \sin \theta)^2 & 0 \end{pmatrix}$$

Riemann's curvature tensor

$$\begin{aligned}R_{\phi\theta\phi}^\theta &= \frac{\sin \theta (R + r \sin \theta)}{r} = -R_{\phi\phi\theta}^\theta \\ R_{\theta\theta\phi}^\phi &= -\frac{r \sin \theta}{R + r \sin \theta} = -R_{\theta\phi\theta}^\phi\end{aligned}$$

All other components are zero.

24. Ricci scalar is

$$R = \frac{2 \sin \theta}{r(R + r \sin \theta)}$$

25.

$$\begin{aligned}\delta(g^{ik}g_{kl}) &= 0 \\ (\delta g^{ik})g_{kl} + g^{ik}(\delta g_{kl}) &= 0 \\ (\delta g^{ik})g_{kl}g^{lm} + g^{ik}(\delta g_{kl})g^{lm} &= 0 \\ (\delta g^{ik})\delta_k^m &= -g^{ik}(\delta g_{kl})g^{lm} \\ \delta g^{im} &= -g^{ik}g^{lm}\delta g_{kl}\end{aligned}$$

Change the indices

$$\delta g^{ik} - g^{is}g^{kt}\delta g_{st}$$