Exercises

11.1 Compute the derivatives (11.22 & 11.23).

11.2 Show that the transformation $x \to x'$ defined by (11.16) is a rotation and a reflection.

11.3 Show that the matrix (11.40) satisfies the Lorentz condition (11.39).

11.4 If $\eta = L\,\eta\,L^{\mathsf{T}}$, show that $\Lambda = L^{-1}$ satisfies the definition (11.39) of a Lorentz transformation $\eta = \Lambda^{\mathsf{T}}\eta\,\Lambda$.

11.5 The LHC is designed to collide 7 TeV protons against 7 TeV protons for a total collision energy of 14 TeV. Suppose one used a linear accelerator to fire a beam of protons at a target of protons at rest at one end of the accelerator. What energy would you need to see the same physics as at the LHC?

11.6 Use Gauss's law and the Maxwell-Ampère law (11.87) to show that the microscopic (total) current 4-vector $j = (c\rho, \boldsymbol{j})$ obeys the continuity equation $\dot{\rho} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$.

11.7 Show that if M_{ik} is a covariant second-rank tensor with no particular symmetry, then only its antisymmetric part contributes to the 2-form $M_{ik}\, dx^i \wedge dx^k$ and only its symmetric part contributes to the quantity $M_{ik}\, dx^i dx^k$.

11.8 In rectangular coordinates, use the Levi-Civita identity (1.449) to derive the curl-curl equations (11.90.

11.9 Derive the Bianchi identity (11.92) from the definition (11.79) of the Faraday field-strength tensor, and show that it implies the two homogeneous Maxwell equations (11.82).

11.10 Show that if A is a p-form, then $d(AB) = dA \wedge B + (-1)^p A \wedge dB$.

11.11 Show that if $\omega = a_{ij} dx^i \wedge dx^j/2$ with $a_{ij} = -\,a_{ji}$, then

$$d\omega = \frac{1}{3!}\left(\partial_k a_{ij} + \partial_i a_{jk} + \partial_j a_{ki}\right)\, dx^i \wedge dx^j \wedge dx^k. \tag{11.504}$$

11.12 Using tensor notation throughout, derive (11.147) from (11.145 & 11.146).

11.13 Use the flat-space formula (11.168) to compute the change $\boldsymbol{dp}$ due to $d\rho$, $d\phi$, and dz, and so derive the expressions (11.169) for the orthonormal basis vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{z}}$.

11.14 Similarly, derive (11.175) from (11.174).

11.15 Use the definition (11.191) to show that in flat 3-space, the dual of the Hodge dual is the identity: $**\,dx^i = dx^i$ and $**\,(dx^i \wedge dx^k) = dx^i \wedge dx^k$.

11.16 Use the definition of the Hodge star (11.202) to derive (a) two of the four identities (11.203) and (b) the other two.

11.17 Show that Levi-Civita's 4-symbol obeys the identity (11.207).

11.18 Show that $\epsilon_{\ell mn}\, \epsilon^{pmn} = 2\, \delta^p_\ell$.

11.19 Show that $\epsilon_{k\ell mn}\, \epsilon^{p\ell mn} = 3!\, \delta^p_k$.

11.20 Using the formulas (11.175) for the basis vectors of spherical coordinates in terms of those of rectangular coordinates, compute the derivatives of the unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ with respect to the variables r, θ, and ϕ. Your formulas should express these derivatives in terms of the basis vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$. (b) Using the formulas of (a), derive the formula (11.297) for the laplacian $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$.

11.21 Consider the torus with coordinates θ, ϕ labeling the arbitrary point

$$\boldsymbol{p} = (\cos\phi(R + r\sin\theta), \sin\phi(R + r\sin\theta), r\cos\theta) \tag{11.505}$$

in which $R > r$. Both θ and ϕ run from 0 to 2π. (a) Find the basis vectors e_θ and e_ϕ. (b) Find the metric tensor and its inverse.

11.22 For the same torus, (a) find the dual vectors e^θ and e^ϕ and (b) find the nonzero connections Γ^i_{jk} where i, j, & k take the values θ & ϕ.

11.23 For the same torus, (a) find the two Christoffel matrices Γ_θ and Γ_ϕ, (b) find their commutator $[\Gamma_\theta, \Gamma_\phi]$, and (c) find the elements $R^\theta_{\theta\theta\theta}$, $R^\phi_{\theta\phi\theta}$, $R^\theta_{\phi\theta\phi}$, and $R^\phi_{\phi\phi\phi}$ of the curvature tensor.

11.24 Find the curvature scalar R of the torus with points (11.505). **Hint:** In these four problems, you may imitate the corresponding calculation for the sphere in Sec. 11.42.

11.25 By differentiating the identity $g^{ik}\, g_{k\ell} = \delta^i_\ell$, show that $\delta g^{ik} = -g^{is} g^{kt} \delta g_{st}$.

11.26 Just to get an idea of the sizes involved in black holes, imagine an isolated sphere of matter of uniform density ρ that as an initial condition is all at rest within a radius r_b. Its radius will be less than its Schwarzschild radius if

$$r_b < \frac{2MG}{c^2} = 2\left(\frac{4}{3}\pi r_b^3 \rho\right)\frac{G}{c^2}. \tag{11.506}$$

If the density ρ is that of water under standard conditions (1 gram per cc), for what range of radii r_b might the sphere be or become a black hole? Same question if ρ is the density of dark energy.

11.27 For the points (11.392), derive the metric (11.395) with $k = 1$. Don't forget to relate $d\chi$ to dr.

11.28 For the points (11.393), derive the metric (11.395) with $k = 0$.

11.29 For the points (11.394), derive the metric (11.395) with $k = -1$. Don't forget to relate $d\chi$ to dr.

11.30 Suppose the constant k in the Roberson-Walker metric (11.391 or 11.395) is some number other than 0 or ± 1. Find a coordinate transformation such that in the new coordinates, the Roberson-Walker metric has $k = k/|k| = \pm 1$.

11.31 Derive the affine connections in Eq.(11.399).

11.32 Derive the affine connections in Eq.(11.400).

11.33 Derive the affine connections in Eq.(11.401).

11.34 Derive the spatial Einstein equation (11.411) from (11.375, 11.395, 11.406, 11.408, & 11.409).

11.35 Assume there had been no inflation and that there were no dark energy. In this case, the magnitude of the difference $|\Omega - 1|$ would have increased as $t^{2/3}$ over the past 13.8 billion years. Show explicitly how close to unity Ω would have had to have been at $t = 1\,\mathrm{s}$ so as to satisfy the observational constraint $\Omega_0 = 1.003 \pm 0.010$ on the present value of Ω.

11.36 Derive the relation (11.431) between the energy density ρ and the Robertson-Walker scale factor $a(t)$ from the conservation law (11.427) and the equation of state $p = w\rho$.

11.37 Use the Friedmann equations (11.410 & 11.412) with $\rho = -p$ and $k = 1$ to derive (11.438) subject to the boundary condition that $a(t)$ has its minimum at $t = 0$.

11.38 Use the Friedmann equations (11.410 & 11.412) with $w = -1$ and $k = -1$ to derive (11.439) subject to the boundary condition that $a(0) = 0$.

11.39 Use the Friedmann equations (11.410 & 11.412) with $w = -1$ and $k = 0$ to derive (11.440). Show why a linear combination of the two solutions (11.440) does not work.

11.40 Use the Friedmann equations (11.410 & 11.412) with $w = 1/3$ and $k = 0$ to derive (11.447) subject to the boundary condition that $a(0) = 0$.

11.41 Show that if the matrix $U(x)$ is nonsingular, then

$$(\partial_i U)\, U^{-1} = -\, U\, \partial_i U^{-1}. \tag{11.507}$$

11.42 The gauge-field matrix is a linear combination $A_k = -ig\, t^b\, A_k^b$ of the generators t^b of a representation of the gauge group. The generators obey the commutation relations

$$[t^a, t^b] = i f_{abc} t^c \tag{11.508}$$

in which the f_{abc} are the structure constants of the gauge group. Show that under a gauge transformation (11.474)

$$A_i' = U A_i U^{-1} - (\partial_i U)\, U^{-1}. \tag{11.509}$$

by the unitary matrix $U = \exp(-ig\lambda^a t^a)$ in which λ^a is infinitesimal, the gauge-field matrix A_i transforms as

$$-igA_i'^a t^a = -igA_i^a t^a - ig^2 f_{abc}\lambda^a A_i^b t^c + ig\partial_i\lambda^a t^a. \tag{11.510}$$

Show further that the gauge field transforms as

$$A_i'^a = A_i^a - \partial_i\lambda^a - g f_{abc} A_i^b \lambda^c. \tag{11.511}$$

11.43 Show that if the vectors $e_a(x)$ are orthonormal, then $e^{a\dagger}\cdot e_{c,i} = -e^{a\dagger}_{,i}\cdot e_c$.

11.44 Use the identity of exercise 11.43 to derive the formula (11.489) for the nonabelian Faraday tensor.