

# 1 Tensors and Local Symmetries

1. Transformation from euclidean (x,y) to polar (r,θ) is

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

Transformation from polar to euclidean coordinates is

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \cos \theta = x/r \\ \sin \theta = y/r \\ \theta = \arccos \frac{x}{\sqrt{x^2 + y^2}} & \text{for positive y} \\ \theta = \pi + \arccos \frac{x}{\sqrt{x^2 + y^2}} & \text{for negative y} \end{cases}$$

Partial derivatives

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{\partial \sqrt{x^2 + y^2}}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} \\ \frac{\partial x}{\partial r} &= \frac{\partial r \cos \theta}{\partial r} = \cos \theta = \frac{x}{r} \end{aligned}$$

$$\begin{aligned} \frac{\partial r}{\partial y} &= \frac{\partial \sqrt{x^2 + y^2}}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r} \\ \frac{\partial y}{\partial r} &= \frac{\partial r \sin \theta}{\partial r} = \sin \theta = \frac{y}{r} \end{aligned}$$

$$\begin{aligned} \frac{\partial \theta}{\partial x} &= \frac{\partial \arccos \frac{x}{\sqrt{x^2 + y^2}}}{\partial x} = \frac{-1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} \frac{y^2}{(x^2 + y^2)^{3/2}} = \frac{-y}{x^2 + y^2} = -\frac{y}{r^2} \\ \frac{\partial x}{\partial \theta} &= \frac{\partial r \cos \theta}{\partial \theta} = -r \sin \theta = -y \end{aligned}$$

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$$\begin{aligned} \frac{\partial \theta}{\partial y} &= \frac{\partial \arccos \frac{x}{\sqrt{x^2 + y^2}}}{\partial y} = \frac{-1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} \frac{-xy}{(x^2 + y^2)^{3/2}} = \frac{x}{x^2 + y^2} = \frac{x}{r^2} \\ \frac{\partial y}{\partial \theta} &= \frac{r \sin \theta}{\partial \theta} = r \cos \theta = x \end{aligned}$$

To summarize

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} \neq \frac{\partial x}{\partial \theta} = -y \quad (1)$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{r^2} \neq \frac{\partial y}{\partial \theta} = x \quad (2)$$

2.

3. Put

$$L = \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3)$$

into  $L^T \eta L$ .

$$L^T \eta L = \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4)$$

$$= \begin{pmatrix} -\gamma & \beta & 0 & 0 \\ -\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5)$$

$$= \begin{pmatrix} \beta^2 - \gamma^2 & 0 & 0 & 0 \\ 0 & -\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6)$$

Since  $\beta = \sqrt{\gamma^2 - 1}$ ,

$$L^T \eta L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta \quad (7)$$

4.

$$\eta = L \eta L^T = \Lambda^{-1} \eta (\Lambda^{-1})^T = \Lambda^{-1} \eta (\Lambda^T)^{-1} \quad (8)$$

Inverse both sides

$$\eta^{-1} = \Lambda^T \eta^{-1} \Lambda \quad (9)$$

We already have  $\eta^{-1} = \eta$ , then

$$\eta = \Lambda^T \eta \Lambda \quad (10)$$

5. To see the same physics means to have the same kinetic energy at center of mass system.

Coordinate transformation leaves scalar unchanged. In this problem,  $P_\mu P^\mu$  won't change when we change from one reference frame to another, i.e.,

$$-(E_1 + E_2)^2 + m^2 u_2^2 c^2 = -(E'_1 + E'_2)^2 + (m u'_1 + m u'_2)^2 c^2.$$

We also have

$$\begin{cases} E_2^2 = m^2 u_2^2 c^2 + m^2 c^4 \\ E_1 = m c^2 \\ E'_1 = E'_2 \\ u'_1 = -u'_2 \end{cases} \quad (11)$$

But the mass of proton is so small compared to the accelerator energy that we can drop  $m c^2$  term in our calculation.

Then the energy of the incoming proton in lab frame is

$$E_2 \approx \frac{2E_1^2}{m c^2} \approx 10^5 \text{TeV} \quad (12)$$

6. Take the divergence of both sides of Maxwell-Ampere law

$$\nabla \cdot \nabla \times \mathbf{B} = \mu_0 \nabla \cdot \mathbf{j} + \epsilon_0 \mu_0 \nabla \cdot \dot{\mathbf{E}} \quad (13)$$

$$0 = \nabla \cdot \mathbf{j} + \epsilon_0 \nabla \cdot \dot{\mathbf{E}} \quad (14)$$

Divergence of electric field is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (15)$$

Thus

$$0 = \nabla \cdot \mathbf{j} + \dot{\rho} \quad (16)$$

- 7.