1 Path Integral

1.

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \exp\left(\sum_i -r_i x_i^2 + c_i x_i\right) \prod_{i=1}^{N} \mathrm{d}x_i \\
= & \prod_i \int_{-\infty}^{\infty} \exp\left(-r_i x_i^2 + c_i x_i\right) \mathrm{d}x_i \\
= & \int_{-\infty}^{\infty} \prod_i \left(\exp\left(-r_i (x_i - \frac{c_i}{2r_i})^2\right) \exp\left(\frac{c_i^2}{4r_i}\right) \mathrm{d}x_i\right) \\
= & \prod_{i=1}^{N} \sqrt{\frac{\pi}{r_i}} \exp\left(\frac{1}{4} \sum_i \frac{c_i^2}{r_i}\right)
\end{aligned}
$$

2. The matrix form of

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \exp\left(\sum_i (-i a_i x_i^2 + i b_i x_i) \prod_i \mathrm{d}x_i\right) \\
= & \prod_{i=1}^{N} \sqrt{\frac{\pi}{i a_i}} \exp\left(\frac{i}{4} \sum_i \frac{b_i^2}{a_i}\right)
\end{aligned}
$$

is

$$
\int_{-\infty}^{\infty} \exp\left(-iX^T A X + iBX\right) \prod_{i=1}^{N} \mathrm{d}x_i = \sqrt{\frac{\pi^N}{\det(iA)}} \exp\left(\frac{i}{4} B^T A^{-1} B\right)
$$

We know that $A = O^T S O$,

$$
\int_{-\infty}^{\infty} \exp\left(-iX^T O^T S O X + iBO^T O X\right) \prod_{i=1}^{N} \mathrm{d}x_i = \sqrt{\frac{\pi^N}{\det(S)}} \exp\left(\frac{i}{4} B^T O^T O A^{-1} O^T O B\right)
$$

Since $Y = OX$, $D = OB$,

$$
\int_{-\infty}^{\infty} \exp\left(-iY^T S Y + iD^T Y\right) \prod_{i=1}^{N} \mathrm{d}y_i = \sqrt{\frac{\pi^N}{\det(S)}} \exp\left(\frac{i}{4} D^T S^{-1} D\right)
$$

3.

4.

5.

$$
\begin{aligned}
\langle q|e^{-itH}\rangle &= \int\int \mathrm{d}p'\mathrm{d}p''\langle q|p'\rangle\langle p'|e^{-ip^2/(2m\hbar)t}|p''\rangle\langle p''|0\rangle \\
&= \int \mathrm{d}p'\frac{1}{(2\hbar\pi)^3}e^{-\frac{-it}{2m\hbar}p'^2}e^{iqp/\hbar} \\
&= \frac{1}{(2\pi\hbar)^3}\sqrt{\frac{\pi^3}{(it/(2m\hbar))}}e^{2mq^2/(2\hbar t)} \\
&= \left(\frac{m}{2\pi i\hbar t}\right)^{3/2}e^{imq^2/(2\hbar t)}
\end{aligned}
$$

6.

7.

$$
\begin{aligned}
S[q] &= \int_0^t \left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right)\mathrm{d}t' \\
&= \int_0^t \frac{1}{2}m\left((-\omega q'\sin\omega t' + \dot{q}_0\cos\omega t')^2 - \omega^2(q'\cos\omega t' + \frac{\dot{q}^2}{\omega}\sin\omega t')^2\right)\mathrm{d}t' \\
&= \frac{m\omega}{2\sin(\omega t)}\left((q'^2 + q''^2)\cos\omega t - 2q'q''\right)
\end{aligned}
$$

8.

$$
\begin{aligned}
S[\delta q] &= \int_0^t \mathrm{d}t'\left(\frac{1}{2}m\left(\sum_{n=1} a_n n\pi/t\cos\frac{n\pi t'}{t}\right)^2 - \frac{1}{2}m\omega^2\left(\sum_{n=1} a_n\sin\frac{n\pi t'}{t}\right)^2\right) \\
&= \int_0^t \mathrm{d}t'\left(\frac{1}{2}m\sum_{n=1} a_n^2\frac{n^2\pi^2}{t^2}\cos^2\frac{n\pi t'}{t} - \frac{1}{2}m\omega^2\sum_{n=0} a_n^2\sin^2\frac{n\pi t'}{t}\right) \\
&= \sum_{n=1}\frac{1}{2}ma_n^2\int_0^t \mathrm{d}t'\left(\frac{n^2\pi^2}{t^2}\cos^2\frac{n\pi t'}{t} - \omega^2\sin^2\frac{n\pi t'}{t}\right) \\
&= \sum_{n=1}\frac{mt}{4}a_n^2\left(n^2\pi^2/t^2 - \omega^2\right)
\end{aligned}
$$

9.