1 Tensors and Local Symmetries

1. Transformation from euclidean (x,y) to polar (r,θ) is

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Transformation from polar to euclidean coordinates is

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \cos\theta = x/r \\ \sin\theta = y/r \\ \theta = \arccos\frac{x}{\sqrt{x^2+y^2}} & \text{for positive y} \\ \theta = \pi + \arccos\frac{x}{\sqrt{x^2+y^2}} & \text{for negative y} \end{cases}$$

Partial derivatives

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2}}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$
$$\frac{\partial x}{\partial r} = \frac{\partial r\cos\theta}{\partial r} = \cos\theta = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{\partial\sqrt{x^2+y^2}}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$
$$\frac{\partial y}{\partial r} = \frac{\partial r\sin\theta}{\partial r} = \sin\theta = \frac{y}{r}$$

$$\frac{\partial\theta}{\partial x} = \frac{\partial\arccos\frac{x}{\sqrt{x^2+y^2}}}{\partial x} = \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\frac{y^2}{(x^2+y^2)^{3/2}} = \frac{-y}{x^2+y^2} = -\frac{y}{r^2}$$
$$\frac{\partial x}{\partial\theta} = \frac{\partial r\cos\theta}{\partial\theta} = -r\sin\theta = -y$$

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$$\frac{\partial\theta}{\partial y} = \frac{\partial\arccos\frac{x}{\sqrt{x^2+y^2}}}{\partial y} = \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\frac{-xy}{(x^2+y^2)^{3/2}} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$
$$\frac{\partial y}{\partial\theta} = \frac{r\sin\theta}{\partial\theta} = r\cos\theta = x$$

To summarize

$$\frac{\partial\theta}{\partial x} = -\frac{y}{r^2} \neq \frac{\partial x}{\partial\theta} = -y \tag{1}$$

$$\frac{\partial\theta}{\partial y} = \frac{x}{r^2} \neq \frac{\partial y}{\partial\theta} = x \tag{2}$$

2.

3. Put

$$L = \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

into $L^T \eta L$.

$$\begin{aligned} L^T \eta L &= \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && (4) \\ &= \begin{pmatrix} -\gamma & \beta & 0 & 0 \\ -\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && (5) \\ &= \begin{pmatrix} \beta^2 - \gamma^2 & 0 & 0 & 0 \\ 0 & -\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && (6) \end{aligned}$$

Since $\beta = \sqrt{\gamma^2 - 1}$,

$$L^T \eta L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta \tag{7}$$

4.

$$\eta = L\eta L^T = \Lambda^{-1}\eta(\Lambda^{-1})^T = \Lambda^{-1}\eta(\Lambda^T)^{-1} \tag{8}$$

Inverse both sides

$$\eta^{-1} = \Lambda^T \eta^{-1} \Lambda \tag{9}$$

We already have $\eta^{-1} = \eta$, then

$$\eta = \Lambda^T \eta \Lambda \tag{10}$$

5. To see the same physics means to have the same kinetic energy at center of mass system.

   Coordinate transformation leaves scalar unchanged. In this problem, $P_\mu P^\mu$ won't change when we change from one reference frame to another, i.e.,

   $$-(E_1 + E_2)^2 + m^2 u_2^2 c^2 = -(E_1' + E_2')^2 + (mu_1' + mu_2')^2 c^2.$$

   We also have

   $$\begin{cases} E_2^2 = m^2 u_2^2 c^2 + m^2 c^4 \\ E_1 = mc^2 \\ E_1' = E_2' \\ u_1' = -u_2' \end{cases} \tag{11}$$

   But the mass of proton is so small compared to the accelerator energy that we can drop mc^2 term in our calculation.

   Then the energy of the incoming proton in lab frame is

   $$E_2 \approx \frac{2E_1^2}{mc^2} \approx 10^5 \text{TeV} \tag{12}$$

6. Take the divergence of both sides of Maxwell-Ampere law

   $$\begin{aligned} \nabla \cdot \nabla \times \mathbf{B} &= \mu_0 \nabla \cdot \mathbf{j} + \epsilon_0 \mu_0 \nabla \cdot \dot{\mathbf{E}} & (13) \\ 0 &= \nabla \cdot \mathbf{j} + \epsilon_0 \nabla \cdot \dot{\mathbf{E}} & (14) \end{aligned}$$

   Divergence of electric field is

   $$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{15}$$

   Thus

   $$0 = \nabla \cdot \mathbf{j} + \dot{\rho} \tag{16}$$

7.

$$M_{ik} = \frac{1}{2}(M_{ik} + M_{ki}) + \frac{1}{2}(M_{ik} - M_{ki}) \tag{17}$$

in which the first part is symmetric and the second is antisymmetric.

Exchange i and k,

$$\frac{1}{2}(M_{ik} + M_{ki})\mathrm{d}x^i \wedge \mathrm{d}x^k = -\frac{1}{2}(M_{ki} + M_{ik})\mathrm{d}x^k \wedge \mathrm{d}x^i \tag{18}$$

$$\rightarrow \quad (M_{ik} + M_{ki})\mathrm{d}x^i \wedge \mathrm{d}x^k = 0 \tag{19}$$

This shows the symmetric part doesn't contribute to $M_{ik}\mathrm{d}x^i\mathrm{d}x^k$.

$$\frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i \wedge \mathrm{d}x^k = \frac{1}{2}(M_{ki} - M_{ik})\mathrm{d}x^k \wedge \mathrm{d}x^i = -\frac{1}{2}(M_{ki} - M_{ik})\mathrm{d}x^i \wedge \mathrm{d}x^k \tag{20}$$

Symmetric part contributes to $M_{ik}\mathrm{d}x^i\mathrm{d}x^k$.

$$\frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k = \frac{1}{2}(M_{ki} - M_{ik})\mathrm{d}x^k\mathrm{d}x^i \tag{21}$$

$$\rightarrow \quad \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k = -\frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^k\mathrm{d}x^i \tag{22}$$

$$\rightarrow \quad \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k = 0 \tag{23}$$

$$\frac{1}{2}(M_{ik} + M_{ki})\mathrm{d}x^i\mathrm{d}x^k = \frac{1}{2}(M_{ki} + M_{ik})\mathrm{d}x^k\mathrm{d}x^i \tag{24}$$

Antisymmetric part doesn't contribute to $M_{ik}\mathrm{d}x^i\mathrm{d}x^k$.