1 Path Integral

1.

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \exp\left(\sum_i -r_i x_i^2 + c_i x_i\right) \prod_{i=1}^{N} dx_i \\
= \; & \prod_i \int_{-\infty}^{\infty} \exp\left(-r_i x_i^2 + c_i x_i\right) dx_i \\
= \; & \int_{-\infty}^{\infty} \prod_i \left(\exp\left(-r_i (x_i - \frac{c_i}{2r_i})^2\right) \exp\left(\frac{c_i^2}{4r_i}\right) dx_i \right) \\
= \; & \prod_{i=1}^{N} \sqrt{\frac{\pi}{r_i}} \exp\left(\frac{1}{4} \sum_i \frac{c_i^2}{r_i}\right)
\end{aligned}
$$

2. The matrix form of

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \exp\left(\sum_i (-i a_i x_i^2 + i b_i x_i) \prod_i dx_i\right) \\
= \; & \prod_{i=1}^{N} \sqrt{\frac{\pi}{i a_i}} \exp\left(\frac{i}{4} \sum_i \frac{b_i^2}{a_i}\right)
\end{aligned}
$$

is

$$
\int_{-\infty}^{\infty} \exp\left(-iX^T A X + iBX\right) \prod_{i=1}^{N} dx_i \quad = \quad \sqrt{\frac{\pi^N}{\det(iA)}} \exp\left(\frac{i}{4} B^T A^{-1} B\right)
$$

We know that $A = O^T S O$,

$$
\int_{-\infty}^{\infty} \exp\left(-iX^T O^T S O X + iBO^T O X\right) \prod_{i=1}^{N} dx_i \quad = \quad \sqrt{\frac{\pi^N}{\det(S)}} \exp\left(\frac{i}{4} B^T O^T O A^{-1} O^T O B\right)
$$

Since $Y = OX$, $D = OB$,

$$
\int_{-\infty}^{\infty} \exp\left(-iY^T S Y + iD^T Y\right) \prod_{i=1}^{N} dy_i \quad = \quad \sqrt{\frac{\pi^N}{\det(S)}} \exp\left(\frac{i}{4} D^T S^{-1} D\right)
$$

3.

4.

5.

$$
\begin{aligned}
\langle q|e^{-itH}\rangle &= \int\int dp'dp''\langle q|p'\rangle\langle p'|e^{-ip^2/(2m\hbar)t}|p''\rangle\langle p''|0\rangle \\
&= \int dp'\frac{1}{(2\hbar\pi)^3}e^{-\frac{-it}{2m\hbar}p'^2}e^{iqp/\hbar} \\
&= \frac{1}{(2\pi\hbar)^3}\sqrt{\frac{\pi^3}{(it/(2m\hbar))}}e^{2mq^2/(2\hbar t)} \\
&= \left(\frac{m}{2\pi i\hbar t}\right)^{3/2}e^{imq^2/(2\hbar t)}
\end{aligned}
$$

6.

7.

$$
\begin{aligned}
S[q] &= \int_0^t\left(\frac{1}{2}m\dot{q}^2-\frac{1}{2}m\omega^2q^2\right)dt' \\
&= \int_0^t\frac{1}{2}m\left((-\omega q'\sin\omega t'+\dot{q}_0\cos\omega t')^2-\omega^2(q'\cos\omega t'+\frac{\dot{q}^2}{\omega}\sin\omega t')^2\right)dt' \\
&= \frac{m\omega}{2\sin(\omega t)}\left((q'^2+q''^2)\cos\omega t-2q'q''\right)
\end{aligned}
$$

8.

$$
\begin{aligned}
S[\delta q] &= \int_0^t dt'\left(\frac{1}{2}m\left(\sum_{n=1}a_n n\pi/t\cos\frac{n\pi t'}{t}\right)^2-\frac{1}{2}m\omega^2\left(\sum_{n=1}a_n\sin\frac{n\pi t'}{t}\right)^2\right) \\
&= \int_0^t dt'\left(\frac{1}{2}m\sum_{n=1}a_n^2\frac{n^2\pi^2}{t^2}\cos^2\frac{n\pi t'}{t}-\frac{1}{2}m\omega^2\sum_{n=0}a_n^2\sin^2\frac{n\pi t'}{t}\right) \\
&= \sum_{n=1}\frac{1}{2}ma_n^2\int_0^t dt'\left(\frac{n^2\pi^2}{t^2}\cos^2\frac{n\pi t'}{t}-\omega^2\sin^2\frac{n\pi t'}{t}\right) \\
&= \sum_{n=1}\frac{mt}{4}a_n^2\left(n^2\pi^2/t^2-\omega^2\right)
\end{aligned}
$$

9. When $q'=0$ and $q''=q$, it becomes

$$
\langle q|e^{-itH/\hbar}|0\rangle = \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}}\exp\left[i\frac{m\omega[q^2\cos\omega t]}{2\hbar\sin\omega t}\right]
$$

In the limit of $t\to 0$, the trigonometric functions used in our calculation becomes $\sin\omega t\to\omega t$ and $\cos\omega t\to 1$.

$$
\lim_{t\to 0}\langle q|e^{-itH/\hbar}|0\rangle = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left(\frac{imq^2}{2\hbar t}\right)
$$

10.

11.

$$\begin{aligned}
S_e[q] &= \int_0^\beta \left[\frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 q^2\right] dt \\
&= \int_0^\beta \frac{1}{2}m\left[(A\omega e^{\omega t} - B\omega e^{-\omega t})^2 + \omega(Ae^{\omega t} + Be^{-\omega t})^2\right] \\
&= \frac{1}{2}m\omega^2 \int_0^\beta 2(A^2 e^{2\omega t} + B^2 e^{-2\omega t})dt \\
&= m\omega^2\left[A^2(e^{2\omega t} - 1) - B^2(e^{-2\omega t} - 1)\right]
\end{aligned}$$

12.

$$\begin{aligned}
S_0[\phi] &= \int \frac{1}{2}[-\partial_a\phi(x)\partial^a\phi(x) - m^2\phi(x)]d^4x \\
&= \int \frac{1}{2}\left[-\int ip_a e^{ip'x}\tilde{\phi}(p')\frac{1}{(2\pi)^4}d^4p' \int \left(-ip_a e^{-ip''x}\tilde{\phi}(-p'')\right)\frac{d^4p''}{(2\pi)^4}\right. \\
&\quad \left. -m^2 \int\int \frac{dp'}{(2\pi)^4}\frac{dp''}{(2\pi)^4} e^{ip'-p''}x\tilde{\phi}(p')\tilde{\phi}(-p'')\right] d^4x \\
&= -\int d^4x e^{i(p'-p'')x} \int \frac{1}{2}(p^2 + m^2)\tilde{\phi}(p')\tilde{\phi}(-p'')\frac{d^4p'}{(2\pi)^4}\frac{d^4p''}{(2\pi)^4} \\
&= -\delta(p' - p'') \int \frac{1}{2}(p^2 + m^2)\tilde{\phi}(p')\tilde{\phi}(-p'')\frac{d^4p'}{(2\pi)^4}\frac{d^4p''}{(2\pi)^4} \\
&= -\frac{1}{2}\int |\tilde{\phi}(p)|^2(p^2 + m^2)\frac{d^4p}{(2\pi)^4}
\end{aligned}$$

13.

$$\begin{aligned}
&\lim_{\epsilon\to 0+} \epsilon \int_{-\infty}^{\infty} e^{-\epsilon|t|}dt \\
&= \lim_{\epsilon\to 0+} \left(\epsilon \int_{-\infty}^0 f(t)e^{\epsilon t}dt + \epsilon \int_0^\infty f(t)e^{-\epsilon t}dt\right) \\
&= \lim_{\epsilon\to 0+} \left(\int_{-\infty}^0 f(t)de^{\epsilon t} - \int_0^\infty f(t)de^{-\epsilon t}\right) \\
&= \lim_{\epsilon\to 0+} \left(f(t)e^{\epsilon t}|_{-\infty}^0 - \int_{-\infty}^0 e^{\epsilon t}df(t) - f(t)e^{-\epsilon t}|_0^\infty + \int_0^\infty e^{-\epsilon t}df(t)\right) \\
&= \lim_{\epsilon\to 0} \left(2f(0) + \int_0^\infty e^{-\epsilon t}df(t) - \int_{-\infty}^0 e^{\epsilon t}df(t)\right) \\
&= 2f(0) + f(\infty) - f(0) - f(0) + f(-\infty) \\
&= f(\infty) + f(-\infty)
\end{aligned}$$

14. Not Finished!!! Check out it after the other problems are solved.

Fourier transform of $\phi(\vec{x},t)$ and $\phi(p)$ are

$$\tilde{\phi}(\vec{p},t) = \int e^{-i\vec{p}\cdot\vec{x}}\phi(\vec{x},t)d^3x$$

$$\phi(\vec{x},t) = \int e^{i\vec{p}\cdot\vec{x}}e^{-ip_0t}\phi(p')\frac{d^4p'}{(2\pi)^4}$$

Then

$$\begin{aligned}\tilde{\phi}(\vec{p},t) &= \iint e^{-i\vec{p}\cdot\vec{x}}e^{-i\vec{p}'\cdot\vec{x}}e^{-ip_0t}\phi(p')\frac{d^4}{(2\pi)^4}d^3x \\ &= \iint e^{-i(\vec{p}-\vec{p}')\cdot\vec{x}}e^{-ip_0t}\phi(p')\frac{d^4p'}{(2\pi)^4}d^3x\end{aligned}$$

15.

$$\begin{aligned}S_0[\phi,\epsilon,j] &= -\frac{1}{2}\int\left[|\tilde{\phi}(p)|^2(p^2+m^2-i\epsilon)-\tilde{j}^*(p)\tilde{\phi}(p)-\tilde{\phi}^*(p)\tilde{j}(p)\right]\frac{d^4p}{(2\pi)^4} \\ &= -\frac{1}{2}\int\left[\left(\tilde{\psi}(p)+\frac{\tilde{j}(p)}{p^2+m^2-i\epsilon}\right)^*\left(\tilde{\psi}(p)+\frac{\tilde{j}(p)}{p^2+m^2-i\epsilon}\right)(p^2+m^2-i\epsilon)\right. \\ &\quad\left.-\tilde{j}^*(p)\left(\tilde{\psi}(p)+\frac{\tilde{j}(p)}{p^2+m^2-i\epsilon}\right)-\left(\tilde{\psi}^*(p)+\frac{\tilde{j}^*(p)}{p^2+m^2+i\epsilon}\right)\tilde{j}(p)\right]\frac{d^4}{(2\pi)^4} \\ &= -\frac{1}{2}\int\left[|\tilde{\psi}(p)|^2(p^2+m^2-i\epsilon)-\frac{\tilde{j}^*(p)\tilde{j}(p)}{p^2+m^2-i\epsilon}\right]\frac{d^4p}{(2\pi)^4} \\ &= S_0[\psi,\epsilon]+\frac{1}{2}\int\frac{\tilde{j}^*(p)\tilde{j}(p)}{p^2+m^2-i\epsilon}\frac{d^4p}{(2\pi)^4}\end{aligned}$$

16.

$$\begin{aligned}Z_0[j] &= \frac{\int\exp\left[i\int j(x)\phi(x)d^4x\right]e^{iS_0[\phi,\epsilon]}D\phi}{\int e^{iS_0[\phi,\epsilon]}D\phi} \\ &= \frac{\int e^{iS_0[\phi,\epsilon,j]}D\phi}{\int e^{iS_0[\phi,\epsilon]}D\phi} \\ &= \frac{\int e^{iS_0[\psi,\epsilon]}D\psi\cdot e^{\frac{i}{2}\int\frac{\tilde{j}^*(p)\tilde{j}(p)}{p^2+m^2-i\epsilon}\frac{d^4p}{(2\pi)^4}}}{\int e^{iS_0[\psi,\epsilon]}D\psi} \\ &= \exp\left[\frac{i}{2}\int\frac{|\tilde{j}(p)|^2}{p^2+m^2-i\epsilon}\frac{d^4p}{(2\pi)^4}\right]\end{aligned}$$

17. Applying

$$\begin{aligned}\tilde{j}(p) &= \int e^{-ipx} j(x) d^4x \\ \tilde{j}^*(p) &= \int e^{ipx'} j(x') d^4x'\end{aligned}$$

to $Z_0[j]$, we get

$$\begin{aligned}Z_0[j] &= \exp\left[\frac{i}{2}\int \frac{\iint d^4x d^4x' e^{ip(x-x')} j(x) j(x')}{p^2+m^2-i\epsilon}\frac{d^4p}{(2\pi)^4}\right] \\ &= \exp\left[\frac{i}{2}\int j(x)\Delta(x-x') j(x') d^4x d^4x'\right],\end{aligned}$$

in which $\Delta(x-x')$ is the Feynmann's propagrator.

18.

$$\begin{aligned}&\frac{1}{i^4}\frac{\delta^4 Z_0[j]}{\delta_j(x_1)\delta_j(x_2)\delta_j(x_3)\delta_j(x_4)}\bigg|_{j=0} \\ =\ &(Z_0[j])\end{aligned}$$