1 Tensors and Local Symmetries

1. Transformation from euclidean (x,y) to polar (r,θ) is

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Transformation from polar to euclidean coordinates is

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \cos\theta = x/r \\ \sin\theta = y/r \\ \theta = \arccos\frac{x}{\sqrt{x^2+y^2}} \qquad \text{for positive y} \\ \theta = \pi + \arccos\frac{x}{\sqrt{x^2+y^2}} \qquad \text{for negative y} \end{cases}$$

Partial derivatives

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2}}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$
$$\frac{\partial x}{\partial r} = \frac{\partial r\cos\theta}{\partial r} = \cos\theta = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{\partial\sqrt{x^2+y^2}}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$
$$\frac{\partial y}{\partial r} = \frac{\partial r\sin\theta}{\partial r} = \sin\theta = \frac{y}{r}$$

$$\frac{\partial\theta}{\partial x} = \frac{\partial\arccos\frac{x}{\sqrt{x^2+y^2}}}{\partial x} = \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\frac{y^2}{(x^2+y^2)^{3/2}} = \frac{-y}{x^2+y^2} = -\frac{y}{r^2}$$
$$\frac{\partial x}{\partial\theta} = \frac{\partial r\cos\theta}{\partial\theta} = -r\sin\theta = -y$$

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$$\frac{\partial\theta}{\partial y} = \frac{\partial\arccos\frac{x}{\sqrt{x^2+y^2}}}{\partial y} = \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\frac{-xy}{(x^2+y^2)^{3/2}} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$
$$\frac{\partial y}{\partial\theta} = \frac{r\sin\theta}{\partial\theta} = r\cos\theta = x$$

To summarize

$$\frac{\partial\theta}{\partial x} = -\frac{y}{r^2} \quad \neq \quad \frac{\partial x}{\partial\theta} = -y \tag{1}$$
$$\frac{\partial\theta}{\partial y} = \frac{x}{r^2} \quad \neq \quad \frac{\partial y}{\partial\theta} = x \tag{2}$$

2.

3. Put

$$L = \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

into $L^T\eta L$.

$$L^T\eta L = \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$
$$= \begin{pmatrix} -\gamma & \beta & 0 & 0 \\ -\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$
$$= \begin{pmatrix} \beta^2 - \gamma^2 & 0 & 0 & 0 \\ 0 & -\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Since $\beta = \sqrt{\gamma^2 - 1}$,

$$L^T\eta L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta \tag{7}$$

4.