1 Tensors and Local Symmetries

1. Transformation from euclidean (x,y) to polar (r,θ) is

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Transformation from polar to euclidean coordinates is

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \cos\theta = x/r \\ \sin\theta = y/r \\ \theta = \arccos\frac{x}{\sqrt{x^2+y^2}} & \text{for positive y} \\ \theta = \pi + \arccos\frac{x}{\sqrt{x^2+y^2}} & \text{for negative y} \end{cases}$$

Partial derivatives

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2}}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$
$$\frac{\partial x}{\partial r} = \frac{\partial r\cos\theta}{\partial r} = \cos\theta = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{\partial\sqrt{x^2+y^2}}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$
$$\frac{\partial y}{\partial r} = \frac{\partial r\sin\theta}{\partial r} = \sin\theta = \frac{y}{r}$$

$$\frac{\partial\theta}{\partial x} = \frac{\partial\arccos\frac{x}{\sqrt{x^2+y^2}}}{\partial x} = \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\frac{y^2}{(x^2+y^2)^{3/2}} = \frac{-y}{x^2+y^2} = -\frac{y}{r^2}$$
$$\frac{\partial x}{\partial\theta} = \frac{\partial r\cos\theta}{\partial\theta} = -r\sin\theta = -y$$

Page 432 equation 11.23?

$$\frac{\partial\theta}{\partial y} = \frac{\partial\arccos\frac{x}{\sqrt{x^2+y^2}}}{\partial y} = \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\frac{-xy}{(x^2+y^2)^{3/2}} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$
$$\frac{\partial y}{\partial\theta} = \frac{r\sin\theta}{\partial\theta} = r\cos\theta = x$$

To summarize

$$\begin{aligned}\frac{\partial\theta}{\partial x} = -\frac{y}{r^2} &\neq \frac{\partial x}{\partial\theta} = -y \\ \frac{\partial\theta}{\partial y} = \frac{x}{r^2} &\neq \frac{\partial y}{\partial\theta} = x\end{aligned}$$

2.

3. Put

$$L = \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

into $L^T\eta L$.

$$\begin{aligned} L^T\eta L &= \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\gamma & \beta & 0 & 0 \\ -\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \beta^2 - \gamma^2 & 0 & 0 & 0 \\ 0 & -\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Since $\beta = \sqrt{\gamma^2 - 1}$,

$$L^T\eta L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta \tag{2}$$

4.

$$\eta = L\eta L^T = \Lambda^{-1}\eta(\Lambda^{-1})^T = \Lambda^{-1}\eta(\Lambda^T)^{-1} \tag{3}$$

Inverse both sides

$$\eta^{-1} = \Lambda^T \eta^{-1} \Lambda \tag{4}$$

We already have $\eta^{-1} = \eta$, then

$$\eta = \Lambda^T \eta \Lambda \tag{5}$$

5. To see the same physics means to have the same kinetic energy at center of mass system.

   Coordinate transformation leaves scalar unchanged. In this problem, $P_\mu P^\mu$ won't change when we change from one reference frame to another, i.e.,

   $$-(E_1 + E_2)^2 + m^2 u_2^2 c^2 = -(E_1' + E_2')^2 + (mu_1' + mu_2')^2 c^2.$$

   We also have

   $$\begin{cases} E_2^2 = m^2 u_2^2 c^2 + m^2 c^4 \\ E_1 = mc^2 \\ E_1' = E_2' \\ u_1' = -u_2' \end{cases} \tag{6}$$

   But the mass of proton is so small compared to the accelerator energy that we can drop mc^2 term in our calculation.

   Then the energy of the incoming proton in lab frame is

   $$E_2 \approx \frac{2E_1^2}{mc^2} \approx 10^5 \mathrm{TeV} \tag{7}$$

6. Take the divergence of both sides of Maxwell-Ampere law

   $$\begin{aligned} \nabla \cdot \nabla \times \boldsymbol{B} &= \mu_0 \nabla \cdot \boldsymbol{j} + \epsilon_0 \mu_0 \nabla \cdot \dot{\boldsymbol{E}} \\ 0 &= \nabla \cdot \boldsymbol{j} + \epsilon_0 \nabla \cdot \dot{\boldsymbol{E}} \end{aligned}$$

   Divergence of electric field is

   $$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \tag{8}$$

   Thus

   $$0 = \nabla \cdot \boldsymbol{j} + \dot{\rho} \tag{9}$$

7.

$$M_{ik} = \frac{1}{2}(M_{ik} + M_{ki}) + \frac{1}{2}(M_{ik} - M_{ki}) \tag{10}$$

in which the first part is symmetric and the second is antisymmetric.

Exchange i and k,

$$\begin{aligned} & \frac{1}{2}(M_{ik} + M_{ki})\mathrm{d}x^i \wedge \mathrm{d}x^k = -\frac{1}{2}(M_{ki} + M_{ik})\mathrm{d}x^k \wedge \mathrm{d}x^i \\ \to \quad & (M_{ik} + M_{ki})\mathrm{d}x^i \wedge \mathrm{d}x^k = 0 \end{aligned}$$

This shows the symmetric part doesn't contribute to $M_{ik}\mathrm{d}x^i\mathrm{d}x^k$.

$$\frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i \wedge \mathrm{d}x^k = \frac{1}{2}(M_{ki} - M_{ik})\mathrm{d}x^k \wedge \mathrm{d}x^i = -\frac{1}{2}(M_{ki} - M_{ik})\mathrm{d}x^i \wedge \mathrm{d}x^k$$

Symmetric part contributes to $M_{ik}\mathrm{d}x^i\mathrm{d}x^k$.

For $M_{ik}\mathrm{d}x^i\mathrm{d}x^k = \frac{1}{2}(M_{ik} + M_{ki})\mathrm{d}x^i\mathrm{d}x^k + \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k$,

$$\begin{aligned} & \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k = \frac{1}{2}(M_{ki} - M_{ik})\mathrm{d}x^k\mathrm{d}x^i \\ \to \quad & \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k = -\frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^k\mathrm{d}x^i \\ \to \quad & \frac{1}{2}(M_{ik} - M_{ki})\mathrm{d}x^i\mathrm{d}x^k = 0 \end{aligned}$$

$$\frac{1}{2}(M_{ik} + M_{ki})\mathrm{d}x^i\mathrm{d}x^k = \frac{1}{2}(M_{ki} + M_{ik})\mathrm{d}x^k\mathrm{d}x^i$$

Antisymmetric part doesn't contribute to $M_{ik}\mathrm{d}x^i\mathrm{d}x^k$.

8. (On page 507, the other half of parentheses is lost.)

For any 3 dimensional vector,

$$
\begin{aligned}
(\nabla \times (\nabla \times \boldsymbol{E}))_i &= \sum_{j,k=1}^{3} \epsilon_{ijk}\partial_j(\nabla \times E)_k \\
&= \sum_{j,k,l,m=1}^{3} \epsilon_{ijk}\partial_j\epsilon_{klm}\partial_l E_m \\
&= \sum_{j,l,m=1}^{3} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l E_m \\
&= \sum_{j,l,m=1}^{3} \delta_{il}\delta_{jm}\partial_j\partial_l E_m - \sum_{j,l,m=1}^{3} \delta_{im}\delta_{jl}\partial_j\partial_l E_m \\
&= \sum_{j,l,m=1}^{3} \delta_{il}\delta_{jm}\partial_j\partial_l E_m - \sum_{j,l,m=1}^{3} \delta_{im}\delta_{jl}\partial_j\partial_l E_m \\
&= \partial_i(\nabla \times \boldsymbol{E}) - \partial_j\partial_j E_i \\
&= (\nabla(\nabla \times \boldsymbol{E}))_i - (\Delta \boldsymbol{E})_i
\end{aligned}
$$

9.

$$
\begin{aligned}
\partial_i F_{jk} &= \partial_i(\partial_j A_k - \partial_k A_j) \\
&= \partial_i\partial_j A_k - \partial_i\partial_k A_j
\end{aligned}
$$

$$
\begin{aligned}
\partial_k F_{ij} &= \partial_k(\partial_i A_j - \partial_j A_i) \\
&= \partial_k\partial_i A_j - \partial_k\partial_j A_i
\end{aligned}
$$

$$
\begin{aligned}
\partial_j F_{ki} &= \partial_j(\partial_k A_i - \partial_i A_k) \\
&= \partial_j\partial_k A_i - \partial_j\partial_i A_k
\end{aligned}
$$

Sum up

$$
\begin{aligned}
& \partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki} \\
= & \partial_i\partial_j A_k - \partial_i\partial_k A_j + \partial_k\partial_i A_j - \partial_k\partial_j A_i + \partial_j\partial_k A_i - \partial_j\partial_i A_k \\
= & 0
\end{aligned}
$$

10. Assume

$$
A = A_{i_1\cdots i_p} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \tag{11}
$$

$$B = B_{i_1\cdots i_q}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_q} \tag{12}$$

$$\begin{aligned}
\mathrm{d}(A\wedge B) &= \mathrm{d}((A_{i_1\cdots i_p}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p})\wedge(B_{i_1\cdots i_q}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_q}))\\
&= \partial_k A_{i_1\cdots i_p}\mathrm{d}x^k\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\wedge(B_{i_1\cdots i_q}\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_q})\\
&\quad +A_{i_1\cdots i_p}\partial_k B_{i_1\cdots i_q}\mathrm{d}x^k\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\wedge(\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_q})\\
&= \mathrm{d}A\wedge B\\
&\quad +A_{i_1\cdots i_p}(-1)^p\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\wedge(\partial_k B_{i_1\cdots i_q}\mathrm{d}x^k\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_q})\\
&= \mathrm{d}A\wedge B+(-1)^p A\wedge\mathrm{d}B
\end{aligned}$$

11.

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(\frac{1}{2}a_{ij})\mathrm{d}x^i\wedge\mathrm{d}x^j\\
&= \frac{1}{2}\partial_k a_{ij}\mathrm{d}x^k\wedge\mathrm{d}x^i\wedge\mathrm{d}x^j\\
&= \frac{1}{6}(\partial_k a_{ij}\mathrm{d}x^k\wedge\mathrm{d}x^i\wedge\mathrm{d}x^j+\partial_j a_{ki}\mathrm{d}x^j\wedge\mathrm{d}^k\wedge\mathrm{d}x^i+\partial_i a_{jk}\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k)\\
&= \frac{1}{3!}(\partial_k a_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k+\partial_j a_{ki}\mathrm{d}x^i\wedge\mathrm{d}^j\wedge\mathrm{d}x^k+\partial_i a_{jk}\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k)\\
&= \frac{1}{3!}(\partial_k a_{ij}+\partial_j a_{ki}+\partial_i a_{jk})\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k
\end{aligned}$$

12.

$$\delta^i{}_j = g'^{ik}\frac{\partial x^t}{\partial x'^k}g_{tu}\frac{\partial x^u}{\partial x'^j} \tag{13}$$

Multiply by g^{lt}

$$\begin{aligned}
g^{lt}\delta^i{}_j &= g'^{ik}\frac{\partial x^t}{\partial x'^k}g_{tu}\frac{\partial x^u}{\partial x'^j}g^{lt}\\
g^{lt}\delta^i{}_j &= g'^{ik}\frac{\partial x^t}{\partial x'^k}\delta^l{}_u\frac{\partial x^u}{\partial x'^j}\\
g^{lt}\delta^i{}_j &= g'^{ik}\frac{\partial x^t}{\partial x'^k}\frac{\partial x^l}{\partial x'^j}\\
g^{lt}\frac{\partial x'^j}{\partial x^l}\delta^i{}_j &= g'^{ik}\frac{\partial x^t}{\partial x'^k}\\
g^{lt}\frac{\partial x'^i}{\partial x^l} &= g'^{ik}\frac{\partial x^t}{\partial x'^k}\\
\frac{\partial x'^k}{\partial x^t}g^{lt}\frac{\partial x'^i}{\partial x^l} &= g'^{ik}
\end{aligned}$$

13.

$$\begin{aligned}
\mathrm{d}p &= \hat{\boldsymbol{x}}(\cos\phi\mathrm{d}\rho-\rho\sin\phi\phi)+\hat{\boldsymbol{y}}(\sin\phi\mathrm{d}\rho+\rho\cos\phi\mathrm{d}\phi)+\hat{\boldsymbol{z}}\mathrm{d}z\\
&= \hat{\boldsymbol{\rho}}\mathrm{d}\rho+\hat{\boldsymbol{\phi}}\rho\mathrm{d}\phi+\hat{\boldsymbol{z}}\mathrm{d}z
\end{aligned}$$

Collect the terms

$$\mathrm{d}\rho(\cos\phi\mathrm{d}\rho\hat{\boldsymbol{x}}+\sin\phi\mathrm{d}\rho\hat{\boldsymbol{y}})+\mathrm{d}\phi(-\rho\sin\phi\hat{\boldsymbol{x}}+\rho\cos\theta\hat{\boldsymbol{y}})+\hat{\boldsymbol{z}}\mathrm{d}z=\hat{\boldsymbol{\rho}}\mathrm{d}\rho+\hat{\boldsymbol{\phi}}\rho\mathrm{d}\phi+\hat{\boldsymbol{z}}\mathrm{d}z \quad (14)$$

The coefficients of each derivative should be the same

$$\begin{aligned}
\hat{\boldsymbol{\rho}} &= \cos\phi\hat{\boldsymbol{x}}+\sin\phi\hat{\boldsymbol{y}}\\
\hat{\boldsymbol{phi}} &= -\sin\phi\hat{\boldsymbol{x}}+\cos\phi\hat{\boldsymbol{y}}\\
\hat{\boldsymbol{z}} &= \hat{\boldsymbol{z}}
\end{aligned}$$

14.

$$\begin{aligned}
\mathrm{d}p &= \hat{\boldsymbol{x}}(-r\sin\theta\sin\phi\mathrm{d}\phi+r\cos\phi\cos\theta\mathrm{d}\theta+\sin\theta\cos\phi\mathrm{d}r)+\hat{\boldsymbol{y}}(r\sin\theta\cos\phi\mathrm{d}\phi\\
&\quad +r\sin\phi\cos\theta\mathrm{d}\theta+\sin\theta\sin\phi\mathrm{d}r)+\hat{\boldsymbol{z}}(-r\sin\theta\mathrm{d}\theta+\cos\theta\mathrm{d}r)\\
&= \hat{\boldsymbol{r}}\mathrm{d}r+\hat{\boldsymbol{\theta}}r\mathrm{d}\theta+\hat{\boldsymbol{\phi}}r\sin\theta\mathrm{d}\phi
\end{aligned}$$

Collect the terms

$$\begin{aligned}
&(-r\sin\theta\sin\phi\hat{\boldsymbol{x}}+r\sin\theta\cos\phi\hat{\boldsymbol{y}})\mathrm{d}\phi+(r\cos\phi\cos\theta\hat{\boldsymbol{x}}+r\sin\phi\cos\theta\hat{\boldsymbol{y}}-r\sin\theta\hat{\boldsymbol{z}})\mathrm{d}\theta\\
&+(\sin\theta\cos\phi\hat{\boldsymbol{x}}+\sin\theta\sin\phi\hat{\boldsymbol{y}}+\cos\theta\hat{\boldsymbol{z}})\mathrm{d}r\\
=&\hat{\boldsymbol{r}}\mathrm{d}r+\hat{\boldsymbol{\theta}}r\mathrm{d}\theta+\hat{\boldsymbol{\phi}}r\sin\theta\mathrm{d}\phi
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{\phi}} &= -\sin\phi\hat{\boldsymbol{x}}+\cos\phi\hat{\boldsymbol{y}}\\
\hat{\boldsymbol{\theta}} &= \cos\phi\cos\theta\hat{\boldsymbol{x}}+\sin\phi\cos\theta\hat{\boldsymbol{y}}-\sin\theta\hat{\boldsymbol{z}}\\
\hat{\boldsymbol{r}} &= \sin\theta\cos\phi\hat{\boldsymbol{x}}+\sin\theta\sin\phi\hat{\boldsymbol{y}}+\cos\theta\hat{\boldsymbol{z}}
\end{aligned}$$

15. A flat 3-space means the Riemann curvature is zero. But I have no idea about the metric. Any metric that is conformal with a flat metric indicates a flat space. But here I use $g_{ij}=\eta_{ij}$, which gives $\det g_{ij}=1$ and $s=1$. Am I wrong?

In this problem $g_{ij}=\eta_{ij}$, which gives $\det g_{ij}=1$ and $s=1$.

$$\begin{aligned}
**\mathrm{d}x^i &= *(\frac{1}{2}g^{il}\eta_{ljk}\mathrm{d}x^j\wedge\mathrm{d}x^k)\\
&= \frac{1}{2}\eta^{il}*(\mathrm{d}x^j\wedge\mathrm{d}x^k)\\
&= \frac{1}{2}\eta^{il}g^{jm}g^{kn}\eta_{mnt}\mathrm{d}x^t\\
&= \frac{1}{2}g^{il}\sqrt{g}\epsilon_{ljk}g^{jm}g^{kn}\sqrt{g}\epsilon_{mnt}\mathrm{d}x^t\\
&= \frac{1}{2}\epsilon^i{}_{jk}\epsilon^{jk}{}_t\mathrm{d}x^t\\
&= \mathrm{d}x^i
\end{aligned}$$

$$
\begin{aligned}
**\,(\mathrm{d}x^i \wedge \mathrm{d}x^k) &= *(g^{ik}g^{jl}\eta_{klm}\mathrm{d}x^m) \\
&= g^{ik}g^{jl}\eta_{klm} * \mathrm{d}x^m \\
&= g^{ik}g^{jl}\eta_{klm}\frac{1}{2}g^{mu}\eta_{uwt}\mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= g^{ik}g^{jl}\epsilon_{klm}\sqrt{g}\frac{1}{2}g^{mu}\epsilon_{uwt}\sqrt{g}\mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= \frac{1}{2}\epsilon^{ij}{}_m\epsilon^m{}_{wt}\mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= \mathrm{d}x^i \wedge \mathrm{d}x^j
\end{aligned}
$$

16.

$$
\begin{aligned}
**\,\mathrm{d}x^i &= *(\frac{1}{3!}g^{ik}\eta_{klmn}\mathrm{d}x^l \wedge \mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&= \frac{1}{3!}g^{ik}\eta_{klmn}g^{lr}g^{ms}g^{nt}\eta_{rstu}\mathrm{d}x^u \\
&= \frac{1}{3!}\eta^i{}_{lmn}\eta^{lmn}{}_u\mathrm{d}x^u \\
&= \mathrm{d}x^i
\end{aligned}
$$

$$
\begin{aligned}
**\,(\mathrm{d}x^i \wedge \mathrm{d}x^j) &= *(\frac{1}{2}g^{ik}g^{jl}\eta_{klmn}\mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&= \frac{1}{2}g^{ik}g^{jl}\eta_{klmn} * (\mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&= \frac{1}{2}g^{ik}g^{jl}\eta_{klmn}\frac{1}{2}g^{mr}g^{ns}\eta_{rswt}\mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= \frac{1}{4}\eta^{ij}{}_{mn}\eta^{mn}{}_{wt}\mathrm{d}x^w \wedge \mathrm{d}x^t \\
&= -\mathrm{d}x^i \wedge \mathrm{d}x^j
\end{aligned}
$$

$$
\begin{aligned}
**\,(\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k) &= *(g^{it}g^{ju}g^{kv}\eta_{tuvw}\mathrm{d}x^w) \\
&= g^{it}g^{ju}g^{kv}\eta_{tuvw}\frac{1}{3!}g^{wm}\eta_{mnrs}\mathrm{d}x^n \wedge \mathrm{d}x^r \wedge \mathrm{d}x^s \\
&= \frac{1}{3!}\eta^{ijk}{}_w\eta^w{}_{nrs}\mathrm{d}x^n \wedge \mathrm{d}x^r \wedge \mathrm{d}x^s \\
&= \mathrm{d}x^n \wedge \mathrm{d}x^r \wedge \mathrm{d}x^s
\end{aligned}
$$

$$
\begin{aligned}
**1 &= *(\frac{1}{4!}\eta_{klmn}\mathrm{d}x^k \wedge \mathrm{d}x^l \wedge \mathrm{d}x^m \wedge \mathrm{d}x^n) \\
&= \frac{1}{4!}\eta_{klmn} g^{ki} g^{lj} g^{mu} g^{nv} \eta_{ijuv} \\
&= \frac{1}{4!}\eta_{klmn}\eta^{klmn} \\
&= -1
\end{aligned}
$$

17. For simplicity, we'll use ij to denote ${\delta_i}^j$.

$$
\begin{aligned}
\epsilon_{klmn}\epsilon^{pwmn} &= \sum_{m,n=1}^{d} \begin{vmatrix} kp & kw & km & kn \\ lp & lw & lm & ln \\ mp & mw & mm & mn \\ np & nw & nm & nn \end{vmatrix} \\
&= d \cdot kw \cdot lp - d \cdot kp \cdot lw - kw \cdot lp + kp \cdot lw + kp \cdot lw - kw \cdot lp - d^2 \cdot kw \cdot lp \\
&\quad + d^2 \cdot kp \cdot lw + d \cdot kw \cdot lp - d \cdot lw \cdot kp - d \cdot kp \cdot lw + d \cdot lp \cdot kw + d \cdot kw \cdot lp - d \cdot k \\
&\quad - kw \cdot lp + lw \cdot kp + kp \cdot lw - lp \cdot kw - d \cdot kp \cdot lw + d \cdot kw \cdot lp + kp \cdot lw \\
&\quad - lp \cdot kw - kw \cdot lp + lw \cdot kp \\
&= (d^2 - 5d + 6)(kp \cdot lw - kw \cdot lp)
\end{aligned}
$$

Here are some useful equations

$$
\begin{aligned}
\sum_{m,n=1}^{d} {\delta_m}^m {\delta_n}^n &= d^2 \\
\sum_{m,n=1}^{d} {\delta_m}^n {\delta_n}^m &= d \\
\sum_{n=1}^{d} {\delta_m}^m &= d \\
\sum_{m=1}^{d} {\delta_l}^m {\delta_m}^p &= {\delta_l}^p
\end{aligned}
$$

For $d = 4$,

$$
\epsilon_{klmn}\epsilon^{pwmn} = 2!({\delta_k}^p {\delta_l}^w - {\delta_k}^w {\delta_l}^p) \tag{15}
$$

18. Should we make it clear the the dimension is 3 in this problem?

$$
\begin{aligned}
\epsilon_{lmn}\epsilon^{pmn} &= \sum_{m,n} \begin{vmatrix} lp & lm & ln \\ mp & mm & mn \\ np & nm & nn \end{vmatrix} \\
&= \sum_{m,n=0}^{x} (-lp \cdot mn \cdot nm + ln \cdot mp \cdot nm + lp \cdot mm \cdot nn \\
&\quad -lm \cdot mp \cdot nn - ln \cdot mm \cdot np + lm \cdot mn \cdot np) \\
&= \\
&= (d^2 - 3d + 2)lp
\end{aligned}
$$

For $d = 3$

$$
\epsilon_{lmn}\epsilon^{pmn} = 2\delta_l{}^p \tag{16}
$$

19.

$$
\begin{aligned}
\epsilon_{klmn}\epsilon^{plmn} &= \sum_{l,m,n=1}^{d} \begin{vmatrix} kp & kl & km & kn \\ lp & ll & lm & ln \\ mp & ml & mm & mn \\ np & nl & nm & nn \end{vmatrix} \\
&= kp(d^3 + 6d^2 + 11d - 6)
\end{aligned}
$$

For $d = 4$

$$
\epsilon_{klmn}\epsilon^{plmn} = 3!\delta_k{}^p \tag{17}
$$

20. To express the derivatives of $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$, we have to solve $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$

$$
\begin{aligned}
\hat{\boldsymbol{x}} &= \cos\phi(\sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}}) - \sin\phi\hat{\boldsymbol{\phi}} \\
\hat{\boldsymbol{y}} &= \sin\phi(\sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}}) + \cos\phi\hat{\boldsymbol{\phi}} \\
\hat{\boldsymbol{z}} &= \cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}
\end{aligned}
$$

Derivatives

$$
\begin{aligned}
\partial_\theta\hat{\boldsymbol{r}} &= \hat{\boldsymbol{x}}\cos\theta\cos\phi - \hat{\boldsymbol{z}}\sin\theta + \hat{\boldsymbol{y}}\sin\phi \\
&= \hat{\boldsymbol{\theta}} \\
\partial_\phi\hat{\boldsymbol{r}} &= \hat{\boldsymbol{y}}\cos\phi\sin\theta - \hat{\boldsymbol{x}}\sin\theta\sin\phi \\
&= \hat{\boldsymbol{\phi}}\sin\theta \\
\partial_r\hat{\boldsymbol{r}} &= 0
\end{aligned}
$$

$$\begin{aligned}
\partial_\theta \hat{\boldsymbol{\theta}} &= -\hat{\boldsymbol{z}}\cos\theta - \hat{\boldsymbol{x}}\cos\phi\sin\theta - \hat{\boldsymbol{y}}\sin\theta\sin\phi \\
&= -\hat{\boldsymbol{r}} \\
\partial_\phi \hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{y}}\cos\theta\cos\phi - \hat{\boldsymbol{x}}\cos\theta\sin\phi \\
&= \hat{\boldsymbol{\phi}}\cos\theta \\
\partial_r \hat{\boldsymbol{\theta}} &= 0
\end{aligned}$$

$$\begin{aligned}
\partial_\theta \hat{\boldsymbol{\phi}} &= 0 \\
\partial_\phi \hat{\boldsymbol{\phi}} &= -\hat{\boldsymbol{x}}\cos\phi - \hat{\boldsymbol{y}}\sin\phi \\
&= -\hat{\boldsymbol{\theta}}\cos\theta - \hat{\boldsymbol{r}}\sin\theta \\
\partial_r \hat{\boldsymbol{\phi}} &= 0
\end{aligned}$$

$$\boldsymbol{\nabla} f = \hat{\boldsymbol{r}}\partial_r f + \frac{\hat{\boldsymbol{\theta}}}{r}\partial_\theta f + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\partial_\phi f$$

Laplacian of f is

$$\begin{aligned}
\Delta f &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f \\
&= \hat{\boldsymbol{r}}\partial_r\left(\hat{\boldsymbol{r}}\partial_r f\right) + \hat{\boldsymbol{r}}\partial_r\left(\hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta f\right) + \hat{\boldsymbol{r}}\partial_r\left(\hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\partial_\phi f\right) \\
&\quad + \hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta\left(\hat{\boldsymbol{r}}\partial_r f\right) + \hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta\left(\hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta f\right) + \hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta\left(\hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\partial_\phi f\right) \\
&\quad + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\partial_\phi\left(\hat{\boldsymbol{r}}\partial_r f\right) + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\partial_\phi\left(\hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta f\right) + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\partial_\phi\left(\hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\partial_\phi f\right) \\
&= f_{,rr} + \frac{1}{r} + \frac{1}{r^2}f_{,\theta\theta} + \frac{1}{r\sin\theta}\sin\theta f_{,r} + \frac{1}{r\sin\theta}\cos\theta\frac{1}{r}f_{,\theta} + \frac{1}{r\sin\theta}\frac{1}{r\sin\theta}f_{,\phi\phi} \\
&= \frac{(r^2 f_{,r})_{,r}}{r^2} + \frac{(\sin\theta f_{,\theta})_{,\theta}}{r^2\sin\theta} + \frac{f_{,\phi\phi}}{r^2\sin^2\theta}
\end{aligned}$$

21. For any point

$$\boldsymbol{p} = \hat{\boldsymbol{x}}\cos\phi(R + r\sin\theta) + \hat{\boldsymbol{y}}\sin\phi(R + r\sin\theta) + \hat{\boldsymbol{z}}r\cos\theta$$

Differential of $\boldsymbol{p}$,

$$\begin{aligned}
\mathrm{d}\boldsymbol{p} &= \hat{\boldsymbol{x}}[-\sin\phi(R + r\sin\theta)\mathrm{d}\phi + \cos\phi r\cos\theta\mathrm{d}\theta] \\
&\quad + \hat{\boldsymbol{y}}[\cos\phi(R + r\sin\theta)\mathrm{d}\phi + \sin\phi r\cos\theta\mathrm{d}\theta] + \hat{\boldsymbol{z}}[-r\sin\theta\mathrm{d}\theta] \\
&= [-\sin\phi(R + r\sin\theta)\hat{\boldsymbol{x}} + \cos\phi(R + r\sin\theta)\hat{\boldsymbol{y}}]\mathrm{d}\phi \\
&\quad + [\cos\phi r\cos\theta\hat{\boldsymbol{x}} + \sin\phi r\cos\theta\hat{\boldsymbol{y}} - r\sin\theta\hat{\boldsymbol{z}}]\mathrm{d}\theta \\
&\equiv e_\theta\mathrm{d}\theta + e_\phi\mathrm{d}\phi
\end{aligned}$$

Orthonormal basis vectors are

$$\begin{aligned}\hat{\boldsymbol{\theta}} &= \cos\phi\cos\theta\hat{\boldsymbol{x}} + \sin\phi\cos\theta\hat{\boldsymbol{y}} - \sin\theta\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}\end{aligned}$$

We get

$$\begin{aligned}e_\theta &= r\hat{\boldsymbol{\theta}} \\ e_\phi &= -(R + r\sin\theta)\hat{\boldsymbol{\phi}}\end{aligned}$$

Metric of this torus is

$$\begin{aligned}g_{\theta\theta} &= r\hat{\boldsymbol{\theta}} \cdot r\hat{\boldsymbol{\theta}} \\ &= r^2 \\ g_{\theta\phi} &= r\hat{\boldsymbol{\theta}} \cdot (R + r\sin\theta)\hat{\boldsymbol{\phi}} \\ & \quad 0 \\ g_{\phi\phi} &= (R + r\sin\theta)\hat{\boldsymbol{\phi}} \cdot (R + r\sin\theta)\hat{\boldsymbol{\phi}} \\ &= (R + r\sin\theta)^2 \\ g_{\phi\theta} &= (R + r\sin\theta)\hat{\boldsymbol{\phi}} \cdot r\hat{\boldsymbol{\theta}} \\ &= 0\end{aligned}$$

Metric and its inverse are

$$\boldsymbol{g} = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r\sin\theta)^2 \end{pmatrix}$$

$$\boldsymbol{g}^{\mathbf{-1}} = \begin{pmatrix} 1/r^2 & 0 \\ 0 & 1/(R + r\sin\theta)^2 \end{pmatrix}$$

22. Contravariant basis vectors are

$$\begin{aligned}e^\theta &= \hat{\boldsymbol{\theta}}/r \\ e^\phi &= \hat{\boldsymbol{\phi}}/(R + r\sin\theta)\end{aligned}$$

Nozero connections are

$$\begin{aligned}\Gamma^\phi_{\theta\phi} &= \frac{r\cos\theta}{R + r\sin\theta} \\ \Gamma^\theta_{\phi\phi} &= -\frac{\cos\theta(R + r\sin\theta)}{r} \\ \Gamma^\phi_{\phi\theta} &= \frac{r\cos\theta}{R + r\sin\theta}\end{aligned}$$

23. Christoffel matrices are

$$\begin{aligned}
\Gamma_\theta &= \begin{pmatrix} 0 & 0 \\ 0 & r\cos\theta/(R+r\sin\theta) \end{pmatrix} \\
\Gamma_\phi &= \begin{pmatrix} r\cos\theta/(R+r\sin\theta) & -\cos\theta(R+r\sin\theta)/r \\ r\cos\theta/(R+r\sin\theta) & 0 \end{pmatrix}
\end{aligned}$$

Commutator is

$$[\Gamma_\theta, \Gamma_\phi] = \begin{pmatrix} 0 & \cos^2\theta \\ r^2\cos^\theta/(R+r\sin\theta)^2 & 0 \end{pmatrix}$$

Riemann's curvature tensor

$$\begin{aligned}
R^\theta_{\phi\theta\phi} &= \frac{\sin\theta(R+r\sin\theta)}{r} = -R^\theta_{\phi\phi\theta} \\
R^\phi_{\theta\theta\phi} &= -\frac{r\sin\theta}{R+r\sin\theta} = -R^\phi_{\theta\phi\theta}
\end{aligned}$$

All other components are zero.

24. Ricci scalar is

$$R = \frac{2\sin\theta}{r(R+r\sin\theta)}$$

25.

$$\begin{aligned}
\delta(g^{ik}g_{kl}) &= 0 \\
(\delta g^{ik})g_{kl} + g^{ik}(\delta g_{kl}) &= 0 \\
(\delta g^{ik})g_{kl}g^{lm} + g^{ik}(\delta g_{kl})g^{lm} &= 0 \\
(\delta g^{ik})\delta^m_k &= -g^{ik}(\delta g_{kl})g^{lm} \\
\delta g^{im} &= -g^{ik}g^{lm}\delta g_{kl}
\end{aligned}$$

Change the indices

$$\delta g^{ik} = -g^{is}g^{kt}\delta g_{st}$$

26. Schwarzchild radius of this sphere is

$$\begin{aligned}
r_S &= \frac{2MG}{c^2} \\
&= \frac{8}{3}\pi r_b^3\rho\frac{G}{c^2}
\end{aligned}$$

If the radius of the sphere is less than Schwarzchild radius, the sphere behaves like a black hole,

$$r_b < \frac{8}{3}\pi r_b^3\rho\frac{G}{c^2}$$

Simplify

$$r_b > \sqrt{\frac{3c^2}{8\pi\rho G}}$$

If the sphere is made of water,

$$r_b > 4 \times 10^1 1\text{m} \tag{18}$$

Dark energy mass density is about $7 \times 10^{-27}\text{kg/m}^3$,

$$r_b > 1.5 \times 10^{26}\text{m} \tag{19}$$

It's good to know that the observable universe is about 10^{26}m.

27. Derivative of the point

$$\begin{aligned}
\mathrm{d}\boldsymbol{p} &= \boldsymbol{\hat{t}}\mathrm{d}t + [\boldsymbol{\hat{x}}a\cos\chi\sin\theta\cos\phi + \boldsymbol{\hat{y}}a\cos\chi\sin\theta\sin\phi + \boldsymbol{\hat{z}}a\cos\chi\cos\theta - \boldsymbol{\hat{m}}a\sin\chi]\mathrm{d}\chi \\
&\quad +[\boldsymbol{\hat{x}}a\sin\chi\cos\theta\cos\phi + \boldsymbol{\hat{y}}\sin\chi\cos\theta\sin\phi - \boldsymbol{\hat{z}}a\sin\chi\sin\theta]\mathrm{d}\theta \\
&\quad +[-\boldsymbol{\hat{x}}a\sin\chi\sin\theta\sin\phi + \boldsymbol{\hat{y}}a\sin\chi\sin\theta\cos\phi]\mathrm{d}\phi
\end{aligned}$$

Define $\sin\chi = r$. Then $\mathrm{d}\chi = \mathrm{d}r/\sqrt{1-r^2}$. $(0 < \chi < \pi)$

$$\begin{aligned}
\mathrm{d}\boldsymbol{p} &\equiv e_t\mathrm{d}t + e_r\mathrm{d}r + e_\theta\mathrm{d}\theta + e_\phi\mathrm{d}\phi \\
&\equiv \boldsymbol{\hat{e}_t}\mathrm{d}t + \frac{a}{\sqrt{1-r^2}}\boldsymbol{\hat{e}_r}\mathrm{d}r + ar\boldsymbol{\hat{e}_\theta}\mathrm{d}\theta + ar\sin\theta\boldsymbol{\hat{e}_\phi}\mathrm{d}\phi
\end{aligned}$$

$$\begin{aligned}
g_{tt} &= e_t(-1)e_t = -1 \\
g_{rr} &= e_r e_r = \frac{a^2}{1-r^2} \\
g_{\theta\theta} &= e_\theta e_\theta = a^2r^2 \\
g_{\phi\phi} &= e_\phi e_\phi = a^2r^2\sin^2\theta
\end{aligned}$$

28.

$$\begin{aligned}
\mathrm{d}\boldsymbol{p} &= \boldsymbol{\hat{t}}\mathrm{d}t + \boldsymbol{\hat{x}}(a\sin\theta\cos\phi\mathrm{d}r + ar\cos\theta\cos\phi\mathrm{d}\theta - ar\sin\theta\sin\phi\mathrm{d}\phi) \\
&\quad +\boldsymbol{\hat{y}}(a\sin\theta\sin\phi\mathrm{d}r + ar\cos\theta\sin\phi\mathrm{d}\theta + ar\sin\theta\cos\phi\mathrm{d}\phi) \\
&\quad +\boldsymbol{\hat{z}}(a\cos\theta\mathrm{d}r - ar\sin\theta\mathrm{d}\theta) \\
&= \boldsymbol{\hat{t}}\mathrm{d}t + (\boldsymbol{\hat{x}}a\sin\theta\cos\phi + \boldsymbol{\hat{y}}a\sin\theta\sin\phi + \boldsymbol{\hat{z}}a\cos\theta)\mathrm{d}r \\
&\quad +(\boldsymbol{\hat{x}}ar\cos\theta\cos\phi + \boldsymbol{\hat{y}}ar\cos\theta\sin\phi - \boldsymbol{\hat{z}}ar\sin\theta)\mathrm{d}\theta \\
&\quad +(-\boldsymbol{\hat{x}}ar\sin\theta\sin\phi + \boldsymbol{\hat{y}}ar\sin\theta\cos\phi)\mathrm{d}\phi \\
&\equiv e_t\mathrm{d}t + e_r\mathrm{d}r + e_\theta\mathrm{d}\theta + e_\phi\mathrm{d}\phi \\
&\equiv \boldsymbol{\hat{e}_t}\mathrm{d}t + a\boldsymbol{\hat{e}_r}\mathrm{d}r + ar\boldsymbol{\hat{e}_\theta}\mathrm{d}\theta + ar\sin\theta\boldsymbol{\hat{e}_\phi}\mathrm{d}\phi
\end{aligned}$$

$$
\begin{aligned}
g_{tt} &= e_t(-1)e_t = -1 \\
g_{rr} &= e_r e_r = a^2 \\
g_{\theta\theta} &= e_\theta e_\theta = a^2 r^2 \\
g_{\phi\phi} &= e_\phi e_\phi = a^2 r^2 \sin^2\theta
\end{aligned}
$$

29.

$$
\begin{aligned}
\mathrm{d}\boldsymbol{p} &= \hat{\boldsymbol{t}}\mathrm{d}t + \hat{\boldsymbol{x}}(a\cosh\chi\sin\theta\cos\phi\mathrm{d}\chi + a\sinh\chi\cos\theta\cos\phi\mathrm{d}\theta - a\sinh\chi\sin\theta\sin\phi\mathrm{d}\phi) \\
&\quad +\hat{\boldsymbol{y}}(a\cosh\chi\sin\theta\sin\phi\mathrm{d}\chi + a\sinh\chi\cos\theta\sin\phi\mathrm{d}\theta + a\sinh\chi\sin\theta\cos\phi\mathrm{d}\phi) \\
&\quad +\hat{\boldsymbol{z}}(a\cosh\chi\cos\theta\mathrm{d}\chi - a\sinh\chi\sin\theta\mathrm{d}\theta) \\
&= \hat{\boldsymbol{t}}\mathrm{d}t + (\hat{\boldsymbol{x}}a\cosh\chi\sin\theta\cos\phi + \hat{\boldsymbol{y}}a\cosh\chi\sin\theta\sin\phi + \hat{\boldsymbol{z}}a\cosh\chi\cos\theta)\mathrm{d}\chi \\
&\quad +(\hat{\boldsymbol{x}}a\sinh\chi\cos\theta\cos\phi + \hat{\boldsymbol{y}}a\sinh\chi\cos\theta\sin\phi - \hat{\boldsymbol{z}}a\sinh\chi\sin\theta)\mathrm{d}\theta \\
&\quad +(-\hat{\boldsymbol{x}}a\sinh\chi\sin\theta\sin\phi + \hat{\boldsymbol{y}}a\sinh\chi\sin\theta\cos\phi)\mathrm{d}\phi
\end{aligned}
$$

Since we have $\sinh\chi = r$, the derivative $\mathrm{d}\chi = \mathrm{d}r/\cosh\chi = \mathrm{d}r/\sqrt{1+\sinh^2\chi}$.

$$
\mathrm{d}\boldsymbol{p} = \hat{\boldsymbol{e}}_{\boldsymbol{t}}\mathrm{d}t + \frac{a}{\sqrt{1+r^2}}\hat{\boldsymbol{e}}_{\boldsymbol{r}}\mathrm{d}r + ar\mathrm{d}\theta + ar\sin\theta\mathrm{d}\phi
$$

$$
\begin{aligned}
g_{tt} &= -1 \\
g_{rr} &= \frac{a^2}{1+r^2} \\
g_{\theta\theta} &= a^2 r^2 \\
g_{\phi\phi} &= a^2 r^2 \sin\theta
\end{aligned}
$$

30. Coordinates transform as

$$
\begin{aligned}
r &\to \sqrt{|k|}r, & \text{k} \neq 0 \\
a &\to \tfrac{a}{\sqrt{|k|}}, & \text{k} \neq 0 \\
k &\to \tfrac{k}{|k|} & \\
\theta &\to \theta & \\
\phi &\to \phi &
\end{aligned}
$$

31. Defination of $\Gamma^\sigma_{\mu\nu}$

$$
\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(g_{\rho\mu,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho})
$$

$$
\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2}g^{1\rho}(g_{\rho 2,2} + g_{2\rho,2} - g_{22,\rho}) \\
&= \frac{1}{2}\frac{1-kr^2}{a^2}(0 + 0 - 2a^2 r) \\
&= -r(1-kr^2) \\
\Gamma^1_{33} &= \frac{1}{2}g^{1\rho}(g_{\rho 3,3} + g_{3\rho,3} - g_{33,\rho}) \\
&= \frac{1}{2}\frac{1-kr^2}{a^2}(0 + 0 - 2a^2 r\sin^2\theta) \\
&= -r(1-kr^2)\sin^2\theta
\end{aligned}
$$

32.

$$
\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2}g^{2\rho}(g_{\rho 1,2} + g_{2\rho,1} - g_{12,\rho}) \\
&= \frac{1}{2}\frac{1}{a^2r^2}(0 + 2a^2 r - 0) \\
&= \frac{1}{r} \\
\Gamma^3_{13} &= \frac{1}{2}g^{3\rho}(g_{\rho 1,3} + g_{3\rho,1} - g_{13,\rho}) \\
&= \frac{1}{2}\frac{1}{a^2r^2\sin^2\theta}(0 + 2a^2 r\sin^2\theta - 0) \\
&= \frac{1}{r}
\end{aligned}
$$

The two lower indices are symmetric.

$$
\begin{aligned}
{\Gamma^2}_{21} = {\Gamma^2}_{12} &= \frac{1}{r} \\
{\Gamma^3}_{31} = {\Gamma^3}_{13} &= \frac{1}{r}
\end{aligned}
$$

33. Page491, equation 401, $\Gamma^3_{23} = \cos\theta = \Gamma^3_{32}$ should be $\Gamma^3_{23} = \cot\theta = \Gamma^3_{32}$

$$
\begin{aligned}
\Gamma^2_{33} &= \frac{1}{2}g^{2\rho}(g_{\rho 3,3} + g_{3\rho,3} - g_{33,\rho}) \\
&= \frac{1}{2}\frac{1}{a^2r^2}(0 + 0 - 2a^2r^2\sin\theta\cos\theta) \\
&= -\sin\theta\cos\theta \\
\Gamma^3_{23} &= \frac{1}{2}g^{3\rho}(g_{\rho 2,3} +_{3\rho,2} - g_{23,\rho}) \\
&= \frac{1}{2}\frac{1}{a^2r^2\sin^2\theta}(0 + 2a^2r^2\cos\theta\sin\theta - 0) \\
&= \cot\theta
\end{aligned}
$$

The two lower indices are symmetric.

$$\Gamma^3_{32} = \Gamma^2_{23} = \cot\theta$$

34. Ricci tensor is

$$R_{ij} = \begin{pmatrix} -3\ddot{a}/a & 0 & 0 & 0 \\ 0 & 2\dot{a}^2 + a\ddot{a} & 0 & 0 \\ 0 & 0 & r^2(2\dot{a}^2 + a\ddot{a}) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta(2\dot{a}^2 + a\ddot{a}) \end{pmatrix}$$

11 component of field equation

$$\begin{aligned} & R_{11} = -\frac{8\pi G}{c^4}(T_{11} - \frac{T}{2}g_{11}) \\ \Rightarrow \quad & R_{11} = -\frac{8\pi G}{c^4}(pg_{11} - \frac{1}{2}(-\rho + 3p)g_{11}) \\ \Rightarrow \quad & R_{11} = -\frac{8\pi G}{c^4}\frac{1}{2}(\rho - p)\frac{a^2r^2}{1-kr^2} \\ \Rightarrow \quad & \frac{A}{1-kr^2}\frac{1-kr^2}{a^2r^2} = \frac{4\pi G}{c^4}(\rho - p) \\ \Rightarrow \quad & \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) \end{aligned}$$

35. $|\Omega - 1|$ is proportional to $t^{2/3}$, then we have

$$\begin{aligned} \frac{|\Omega(t_0) - 1|}{|\Omega(t_i) - 1|} &= \frac{t_0^{2/3}}{t_i^{2/3}} \\ |\Omega(t_i) - 1| &= \frac{t_0^{2/3}}{t_i^{2/3}}|\Omega(t_i) - 1| \end{aligned}$$

In our calculation, $|\Omega(t_0) - 1| = 0.003 \pm 0.010$, $t_i = 1$ and $t_0 = 4.35 \times 10^{17}$.

$$\Omega(t_i) = (1.00 + 5.23 \times 10^{-15}) \pm 1.74 \times 10^{-14}$$

36. Assuming that w is constant, conservation of energy momentum leads to

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}a} &= -\frac{3}{a}(\rho + w\rho) \\ \mathrm{d}\rho &= -\frac{3}{a}(1+w)\rho\mathrm{d}a \\ \ln\rho &= \ln a^{-3(1+w)} + \text{Constant} \\ \rho &= e^{\text{Constant}}a^{-3(1+w)} \end{aligned}$$

The energy density of current era $a = \bar{a}$ is $\bar{\rho}$, that is

$$\bar{\rho} = e^{\text{Constant}} \bar{a}^{-3(1+w)}$$

Then we get

$$e^{\text{Constant}} = \frac{\bar{\rho}}{\bar{a}^{3(1+w)}}$$

Put this back into the conservation equation

$$\rho = \bar{\rho} \left(\frac{\bar{a}}{a} \right)^{3(1+w)}$$

37. Is it really (11.410 & 11.412)?

When $w = -1$, $g^2 = \frac{8\pi G\rho}{3}$ is constant.

$$\begin{aligned}
\left(\frac{\dot{a}}{a} \right)^2 &= g^2 - \frac{1}{a^2} \\
\frac{\mathrm{d}a}{\mathrm{d}t} &= \sqrt{g^2 a^2 - 1} \\
\frac{\mathrm{d}a}{\sqrt{g^2 a^2 - 1}} &= \mathrm{d}t \\
\frac{1}{g} \ln(ga + \sqrt{g^2 a^2 - 1}) &= t + \mathrm{C1} \\
ga + \sqrt{g^2 a^2 - 1} &= e^{gt + \mathrm{C1}} \\
ga - e^{gt + \mathrm{C1}} &= -\sqrt{g^2 a^2 - 1} \\
g^2 a^2 + e^{2gt + \mathrm{C2}} - 2gae^{gt + \mathrm{C2}} &= g^2 a^2 - 1 \\
a &= \frac{1}{g} \cosh(gt + \mathrm{C2})
\end{aligned}$$

$\cosh x$ is minimal at $x = 0$, thus C2 should be zero.

$$a(t) = \frac{\cosh(gt)}{g}$$

38. When $w = -1$, $g^2 = \frac{8\pi G\rho}{3}$ is constant.

$$\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= g^2 + \frac{1}{a^2} \\
\frac{\mathrm{d}a}{\mathrm{d}t} &= \sqrt{g^2a^2+1} \\
\frac{\mathrm{d}a}{\sqrt{g^2a^2+1}} &= \mathrm{d}t \\
\frac{1}{g}\ln(ga+\sqrt{g^2a^2+1}) &= t + \mathrm{C1} \\
ga+\sqrt{g^2a^2+1} &= e^{gt+\mathrm{C1}} \\
ga - e^{gt+\mathrm{C1}} &= -\sqrt{g^2a^2+1} \\
g^2a^2 + e^{2gt+\mathrm{C2}} - 2gae^{gt+\mathrm{C2}} &= g^2a^2+1 \\
a &= \frac{1}{g}\sinh(gt+\mathrm{C2})
\end{aligned}$$

Boundary condition is $\sinh 0 + \mathrm{C2} = 0$, then $\mathrm{C2} = 0$.

$$a(t) = \frac{\sinh(gt)}{g}$$

39. When $w = -1$, $g^2 = \frac{8\pi G\rho}{3}$ is constant.

$$\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= g^2 \\
\frac{\mathrm{d}a}{a\mathrm{d}t} &= \pm g \\
\ln(a) &= \pm gt + \mathrm{C1} \\
a &= e^{\pm gt+\mathrm{C1}}
\end{aligned}$$

Set $a(0) = e^{\mathrm{C1}} = 0$,

$$a(t) = a(0)\exp(\pm gt)$$

Assume $a = \mathrm{C1}\exp(gt) + \mathrm{C2}\exp(-gt)$.

$$\begin{aligned}
(\frac{\dot{a}}{a})^2 &= g^2 \\
\left(\frac{\mathrm{C1}g\exp(gt) - \mathrm{C2}g\exp(-gt)}{\mathrm{C1}\exp(gt)+\mathrm{C2}\exp(-gt)}\right)^2 &= g^2 \\
(\mathrm{C1}\exp(gt) - \mathrm{C2}\exp(-gt)) &= (\mathrm{C1}\exp(gt)+\mathrm{C2}\exp(-gt))^2 \\
\mathrm{C1C2} &= 0
\end{aligned}$$

Then we have $\mathrm{C1} = 0$ or $\mathrm{C2} = 0$.

40. 11.447? Is it 11.449?

For $w = 1/3$, $f^2 = \frac{8\pi G\rho a^4}{3}$ is constant.

Friedmann equation is

$$\begin{aligned} a^2\dot{a}^2 &= f^2 \\ a\dot{a} &= \pm f \\ a\mathrm{d}a &= \pm f\mathrm{d}t \\ \mathrm{d}(\frac{1}{2}a^2) &= \pm\mathrm{d}(ft) \\ a^2 &= \pm 2ft + \mathrm{C1} \\ a &= \sqrt{\pm 2ft + \mathrm{C1}} \end{aligned}$$

Boundary condition $a(0) = 0$ leads to $\sqrt{\mathrm{C1}} = 0$.

Finally,

$$a = \sqrt{2ft}$$

41. Defination of inverse of matrix is

$$UU^{-1} = I$$

Differentiate of this identity yields

$$\begin{aligned} (\partial_i U)U^{-1} + U\partial_i U^{-1} &= 0 \\ (\partial_i U)U^{-1} &= -U\partial_i U^{-1} \end{aligned}$$

42.

43. The defferential on both sides of

$$e^{a\dagger} \cdot e_c = \delta^a_c$$

shows

$$e^{a\dagger} \cdot e_{c,i} + e^{a\dagger}_{,i} \cdot e_c = 0.$$

That is

$$e^{a\dagger} \cdot e_{c,i} = -e^{a\dagger}_{,i} \cdot e_c$$

44. Defination of Faraday tensor is

$$F^a_{ijb} = [D_i, D_j]$$

The derivatives are

$$D_i = \partial_i + A_i$$

Thus

$$\begin{aligned}
F^a_{ijb} &= [\partial_i + A_i, \partial_j + A_j]^a{}_b \\
&= \partial_i A^a_{jb} - \partial_j A^a_{ib} + [A_i, A_j]^a{}_b \\
&= \partial_i(e^{a\dagger} \cdot e_{b,j}) - \partial_j(e^{a\dagger} \cdot e_{b,i}) + e^{a\dagger} \cdot e_{c,i} e^{c\dagger} \cdot e_{b,j} - e^{a\dagger} \cdot e_{c,j} e^{c\dagger} \cdot e_{b,i} \\
&= e^{a\dagger}_{,i} \cdot e_{b,j} - e^{a\dagger}_{,j} \cdot e_{b,i} - e^{a\dagger}_{,i} \cdot e_c e^{c\dagger} \cdot e_{b,j} + e^{a\dagger}_{,j} \cdot e_c e^{c\dagger} \cdot e_{b,i}
\end{aligned}$$