

# On Exponential Tilting

In Monte Carlo estimation, exponential tilting is a distribution shifting technique used in rare-event simulation, rejection and importance sampling.

## 1 Exponential Tilting

Given a random variable  $X$  with probability distribution  $P$ , density  $f$ , and m.g.f.  $M_X(\theta) = E[e^{\theta X}] < \infty$ , the exponentially tilted measure  $P_\theta$  is:

$$P_\theta(X \in dx) = \frac{E[e^{\theta X} 1[X \in dx]]}{M_X(\theta)} = e^{\theta x - \kappa(\theta)} P(X \in dx) \quad (1)$$

where  $\kappa(\theta)$  is the c.g.f.  $\log E[e^{\theta X}]$ . Then  $P_\theta$  has density

$$f_\theta(x) = \frac{e^{\theta x} f(x)}{M_X(\theta)} = \exp\{\theta x - \kappa(\theta)\} f(x). \quad (2)$$

## 2 Esscher Transform

Let  $f(x)$  be a probability density. Then its Esscher transform is:

$$f_h(x) = \frac{e^{hx} f(x)}{\int_{-\infty}^{\infty} e^{hx} f(x) dx} \quad (3)$$

The effect of the Esscher transform on the normal distribution is a mean shifting:

$$E_h(\mathcal{N}(\mu, \sigma^2)) = \mathcal{N}(\mu + h\sigma^2, \sigma^2). \quad (4)$$

## 3 ReLU

Consider the distribution

$$p_\theta(X) = \frac{1}{Z(\theta)} \exp[f_\theta(X)] p_0(X) \quad (5)$$

where  $p_0$  is Gaussian noise as reference distribution and  $Z(\theta)$  denotes the partition function

$$Z(\theta) = \int \exp[f_\theta(X)] p_0(X) dX. \quad (6)$$

Then  $p_\theta(Y)$  is an exponential tilting of  $p_0$ . In the trivial case of  $f_\theta(X) = \theta X$ , equation (5) is a Gibbs distribution and the effect of exponential tilting on  $p_0(Y) \sim \mathcal{N}(\mu, \sigma^2)$  reduces to mean shifting  $p_\theta(X) \sim \mathcal{N}(\mu + \theta\sigma^2, \sigma^2)$ . If  $f_\theta(X)$  is a ConvNet with piecewise linear rectification, then  $f_\theta(X)$  is piecewise linear in  $X$ , and  $p_\theta$  is piecewise mean shifted Gaussian.

## References