# On Exponential Tilting

In Monte Carlo estimation, exponential tilting is a distribution shifting technique used in rare-event simulation, rejection and importance sampling.

### 1 Exponential Tilting

Given a random variable X with probability distribution P, density f, and m.g.f.  $M_X(\theta) = E[e^{\theta X}] < \infty$ , the exponentially titled measure  $P_{\theta}$  is:

$$P_{\theta}(X \in dx) = \frac{E[e^{\theta X}1[X \in dx]]}{M_X(\theta)} = e^{\theta x - \kappa(\theta)}P(X \in dx) \tag{1}$$

where  $\kappa(\theta)$  is the c.g.f.  $\log E[e^{\theta X}]$ . Then  $P_{\theta}$  has density

$$f_{\theta}(x) = \frac{e^{\theta x} f(x)}{M_X(\theta)} = \exp\{\theta x - \kappa(\theta)\} f(x). \tag{2}$$

### 2 Esscher Transform

Let f(x) be a probability density. Then its Esscher transform is:

$$f_{\theta}(x) = \frac{e^{\theta x} f(x)}{\int_{-\infty}^{\infty} e^{\theta x} f(x) dx}$$
 (3)

The effect of the Esscher transform on the normal distribution is a mean shifting:

$$E_{\theta}(\mathcal{N}(\mu, \sigma^2)) = \mathcal{N}(\mu + \theta \sigma^2, \sigma^2). \tag{4}$$

#### 3 Gibbs and ReLU

Consider the distribution with parameters  $\theta$ :

$$p_{\theta}(X) = \frac{1}{Z(\theta)} \exp\left[f_{\theta}(X)\right] p_0(X) \tag{5}$$

where  $p_0$  is Gaussian noise as reference distribution and  $Z(\theta)$  denotes the partition function

$$Z(\theta) = \int \exp[f_{\theta}(X)]p_0(X)dX. \tag{6}$$

Then  $p_{\theta}(Y)$  is an exponential tilting of  $p_0$ . In the trivial case of  $f_{\theta}(X) = \theta X$ , equation (5) is a Gibbs distribution and the effect of exponential tilting on  $p_0(Y) \sim \mathcal{N}(\mu, \sigma^2)$  reduces to mean shifting  $p_{\theta}(X) \sim \mathcal{N}(\mu + \theta \sigma^2, \sigma^2)$ . If  $f_{\theta}(X)$  is a ConvNet with piecewise linear rectification, then  $f_{\theta}(X)$  is piecewise linear in X, and  $p_{\theta}$  is piecewise mean shifted Gaussian.

## References