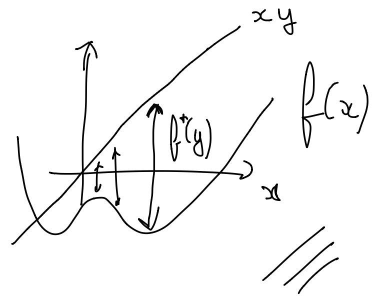
Lab n° 10 on duality:

$f^{:x} \to f^*(y) = \sup_x (\underbrace{x^T y - f(x)}_{y \mapsto x^T y - f_1(x)})$

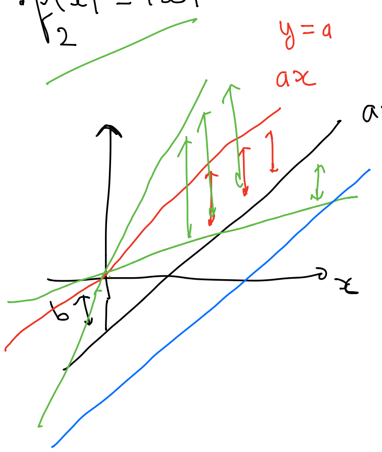


f^* is convex

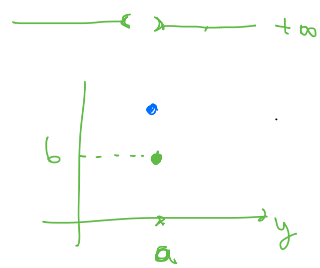
$f^* \in \mathbb{R} \cup \{+\infty\}$ $\qquad f^*(y) + f(x) \geq x^T y$

$\cdot f_1(x) = ax - b$ $\qquad f_1^*(y) =$ $|y - b|$ prop 1
$= \langle a, x \rangle - b$ $\qquad +b \ \forall y$ prop. 2

$\cdot f_2(x) = |x|$ $\qquad \begin{cases} 0 \text{ if } y = a \\ +\infty \text{ otherwise} \end{cases}$ prop 3



$\begin{cases} b \text{ if } y = a \\ +\infty \text{ otherwise} \end{cases}$



$f(\lambda x + (1-\lambda) y)$

convex ✓

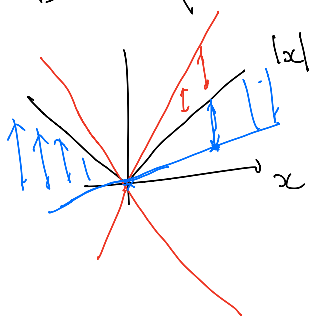
$f_2(x) = |x|$

$f_2^*(y) = \begin{cases} 0 & \text{if } |y| \leq 1 \\ +\infty & \text{otherwise} \end{cases}$ prop 1 ✓

prop 4 $\begin{cases} 0 & \text{if } y = \pm 1 \\ +\infty & \text{otherwise} \end{cases}$ $\qquad \begin{cases} 0 & \text{if } y \geq 0 \\ +\infty & \text{otherwise} \end{cases}$ prop 2

prop 5 $\{+\infty\}$ $\qquad \begin{cases} 0 & \text{if } 0 \leq y \leq 1 \\ +\infty & \text{otherwise} \end{cases}$ prop 3

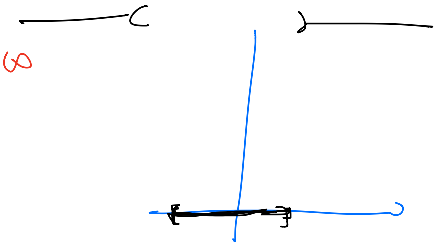
~~prop 6~~ ~~if $y \geq 0$~~



$y > 1$: then $+\infty$

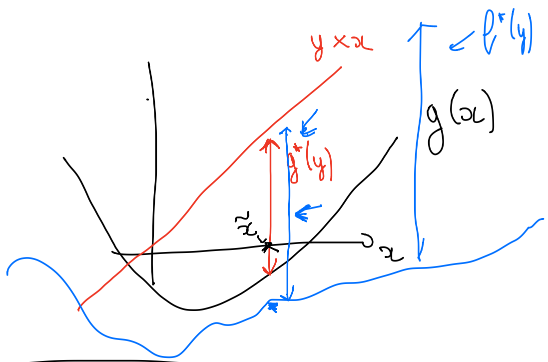
$y < -1$: then $+\infty$

$-1 \leq y \leq 1$: then 0



$yx - f(a)$

$$f \leq g \longrightarrow \begin{cases} f^* \leq g^* & (1) \\ g^* \leq f^* & (2) \; ✓ \end{cases}$$



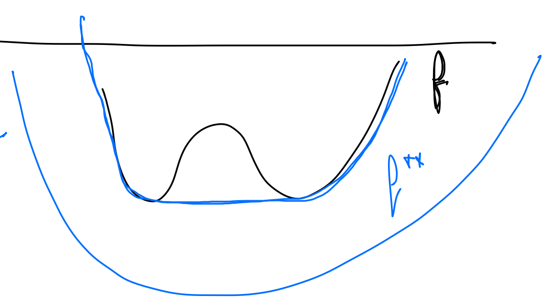
$f^*(y) \geq g^*(y)$

$\forall y$

$$f \leq g \Rightarrow g^* \leq f^* \Rightarrow f^{**} \leq g^{**}$$

$$f^{**} \overset{(1)}{\leq} f$$
$$\underset{(2)}{\geq}$$

$$f^{**} \leq f$$



$f(x)$ $\quad f^{**}(x)$

$f^*(y)$

$$f^*(y) = \sup_x x^T y - f(x)$$

$$\forall y, x \quad f^*(y) + f(x) \geq x^T y$$

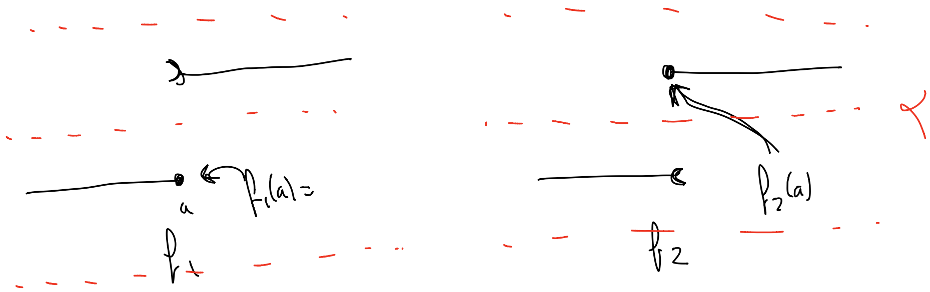
$$\forall x, \forall y : \underbrace{f(x) \geq x^T y - f^*(y)}$$

$$f(x) \geq \sup_y x^T y - f^*(y) = f^{**}(x)$$

$f^{**} = f$

f is convex, closed $\Longrightarrow f^{**} = f$

$\forall \alpha \quad \{x \text{ s.t. } f(x) \leq \alpha\}$ is closed

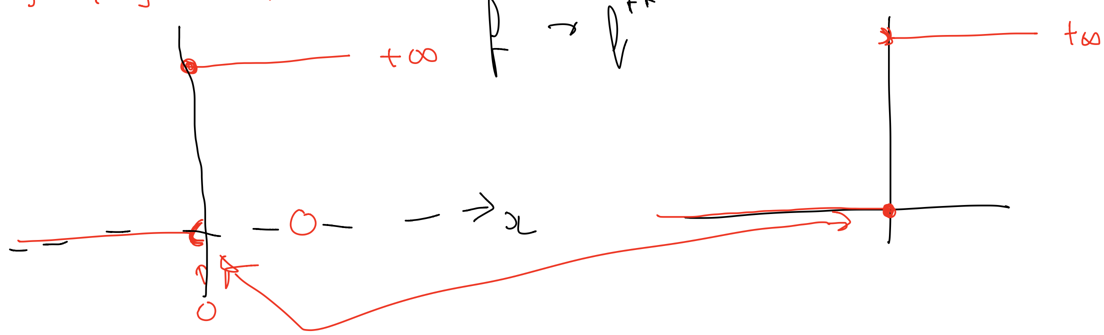


closed

$]-\infty, a[$ ← not closed

$]-\infty, a]$ $\quad \mathbb{R}$

$f \to f^{**}$



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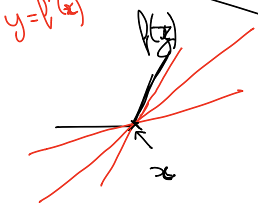
f is closed and convex ($f^{**} = f$)

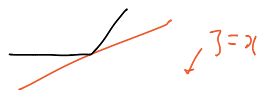
$$y \in \partial f(x) \iff x \in \partial f^*(y) \iff f(x) + f^*(y) \overset{\geq}{=} x^T y$$

$y \in \partial f(x)$, hence: $f(y) \geq$ ~~$f(x) + \langle$~~ $>$

$y = f'(x)$



$$\forall z: \; f(z) \geq f(x) + \langle y, z - x \rangle$$

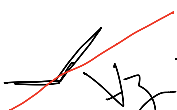


$$\forall z \quad y^T x - f(x) \geq y^T z - f(z)$$

$$y^T x - f(x) \geq \sup_z \left[y^T z - f(z) \right] = f^*(y)$$

$$y^T x - f(x) = f^*(y)$$

$$\boxed{f(x) + f^*(y) = y^T x}$$



$$\forall z \quad f^*(z) = \sup_w \left(w^T z - f(w) \right) \geq \begin{cases} x^T z - f(x) & ① \\ \cancel{y^T z - f(y)} & ② \end{cases}$$

$$\geq x^T z - f(x)$$

$$= x^T z - y^T x + f^*(y) = f^*(y) + \langle x, z - y \rangle$$

$$\forall z \quad f^*(z) \geq f^*(y) + \langle x, z - y \rangle \xleftarrow{\text{def}} x \in \partial f^*(y)$$

gt

$$y \in \partial f(x) \iff f(x) + f^*(y) = x^T y \iff x \in \partial f^*(y)$$

$f^{**} = f$

$\Longrightarrow y \in \partial f^{**}(x)$ $\Big\|$ closed convex

$\Longrightarrow y \in \partial f(x)$

assume x, y are s.t

$f(x) + f^*(y) = x^T y \iff y \in \partial f(x)$

$f^*(y) = \sup_{z} (z^T y - f(z)) = x^T y - f(x) \geq z^T y - f(z)$

$\Longrightarrow f(z) \geq f(x) + \langle y, z - x \rangle \quad \forall z$

$\Longrightarrow y \in \partial f(x)$

$n \in \mathbb{N}$ $\qquad$ $n \in \mathbb{R}$

$x \in \mathbb{R}$