

RB-SFA: High Harmonic Generation in the Strong Field Approximation via *Mathematica*

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Usage and Examples

Loading the package

You can use this software

- within the RB-SFA notebook itself by simply running the initialization cells of that notebook, or
- from an external notebook by loading it as a package.

In the latter case, place a copy of the package file RB-SFA.m on the same directory as your notebook and run the loading command

```
Needs["RBSFA`", FileNameJoin[{NotebookDirectory[], "RB-SFA.m"}]]
```

You can also call the package from another directory by suitably modifying the directory call. If you plan on using this package in the long term you can use the File > Install prompt, in which case the package is simply loaded as `Needs["RBSFA"]`, though this is not particularly recommended. (A better choice is to include a soft link called `RBSFA.m` in your `$UserBaseDirectory/Applications/` directory to the file `RB-SFA.m`. This works just fine and is easy to undo if required.)

To print the version of the package in use, use the command

```
$RBSFAversion
RB-SFA v2.1.0, Wed 11 May 2016 14:37:55
```

There are also commands to get the `$RBSFAtimestamp` directly, as well as the git `$RBSFACommit` hash and message.

Simple usage

For basic usage, simply call the main numerical integrator, `makeDipoleList`, with the vector potential you want to use, and provide any parameters you wish to specify using the `FieldParameters` option.

```
AbsoluteTiming[
  simpleDipole = makeDipoleList[
    VectorPotential → Function[t, {F ω Sin[ω t], 0, 0}], FieldParameters → {F → 0.05, ω → 0.057}];
]
{2.57624, Null}
```

Calling the function with insufficient parameters will produce error messages:

```

makeDipoleList[VectorPotential → Function[t, {F Sin[ω t], 0, 0}]]
makeDipoleList::pot:
The vector potential A provided as VectorPotential→Function[t, {F Sin[ω t], 0, 0}] is incorrect or is missing FieldParameters.
Its usage as A[4.989144044218128] returns {F Sin[4.98914 ω], 0, 0} and should return a list of numbers.
$Aborted

```

The symbol ω is taken to be the carrier frequency, and is set by default to $\omega = 0.057$ atomic units, corresponding to a wavelength of 800 nm. If the carrier frequency is changed, this must be specified on **both** the field parameters and the explicit option for the integrator, as

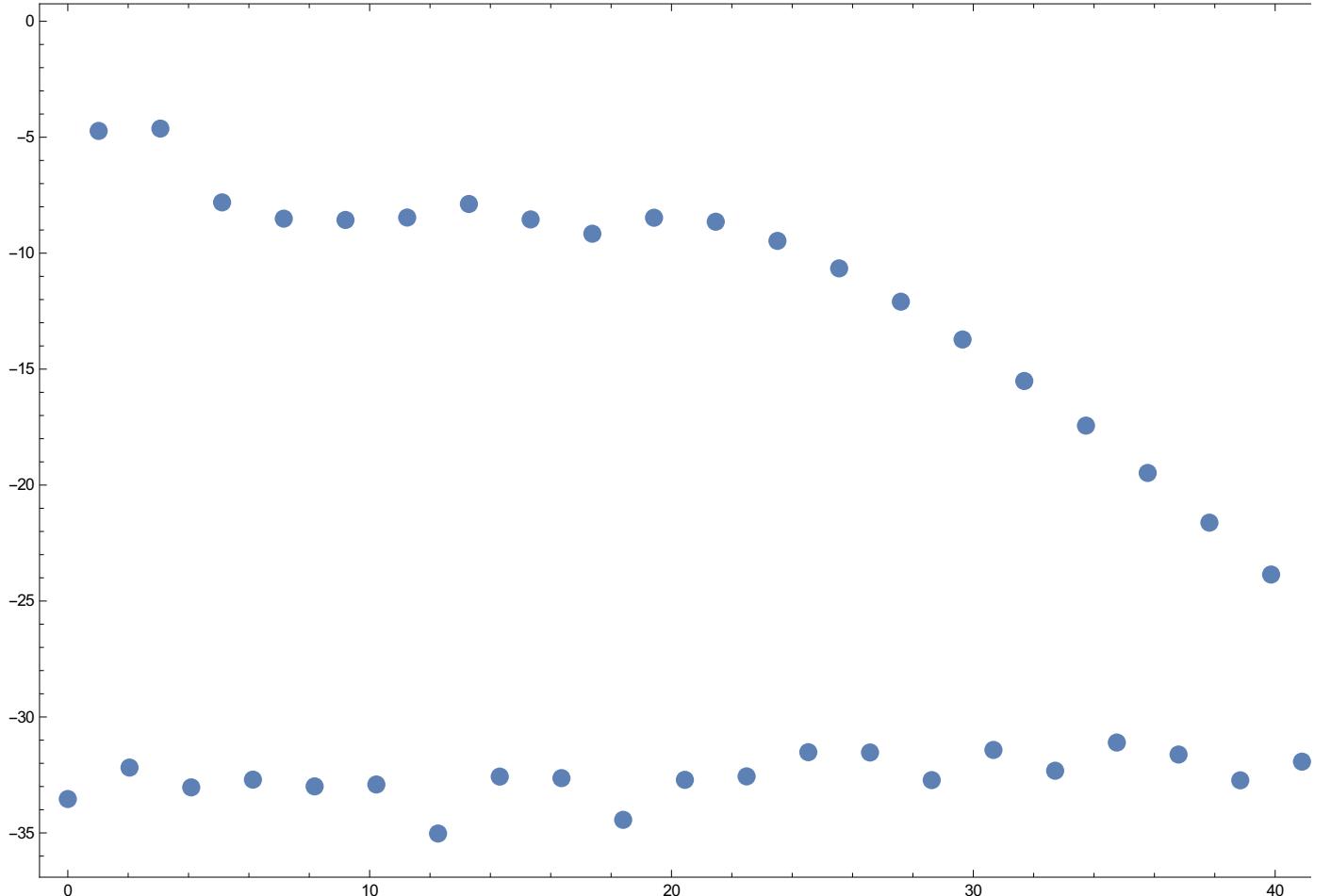
```

makeDipoleList[VectorPotential → Function[t, {F Sin[ω t], 0, 0}],
FieldParameters → {F → 0.05, ω → 0.0456}, CarrierFrequency → 0.0456]

```

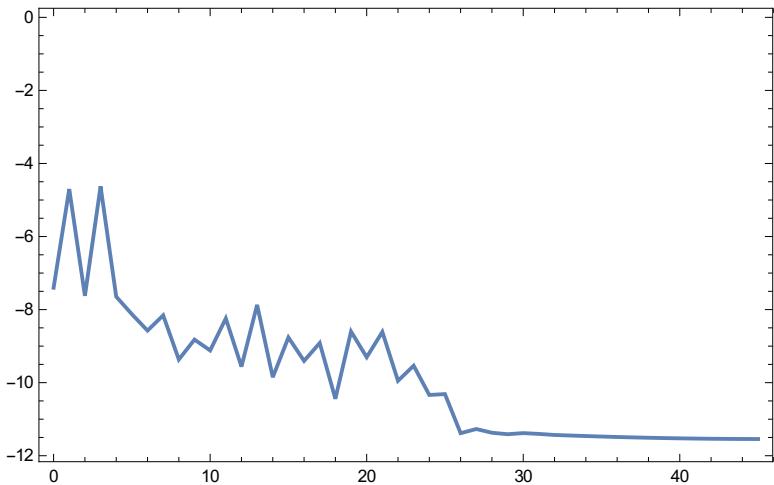
To see the spectrum, use the `getSpectrum` and the `spectrumPlotter` commands, such as

```
spectrumPlotter[getSpectrum[Most[simpleDipole]], Joined → False]
```



Note here the use of `Most` on the dipole when a monochromatic field is indicated. This ensures that the signal is actually periodic (i.e. it eliminates repetition between the initial and final points, which are separated by exactly one period). If this is not done, the spectrum is much noisier:

```
spectrumPlotter[getSpectrum[simpleDipole], ImageSize -> 400]
```

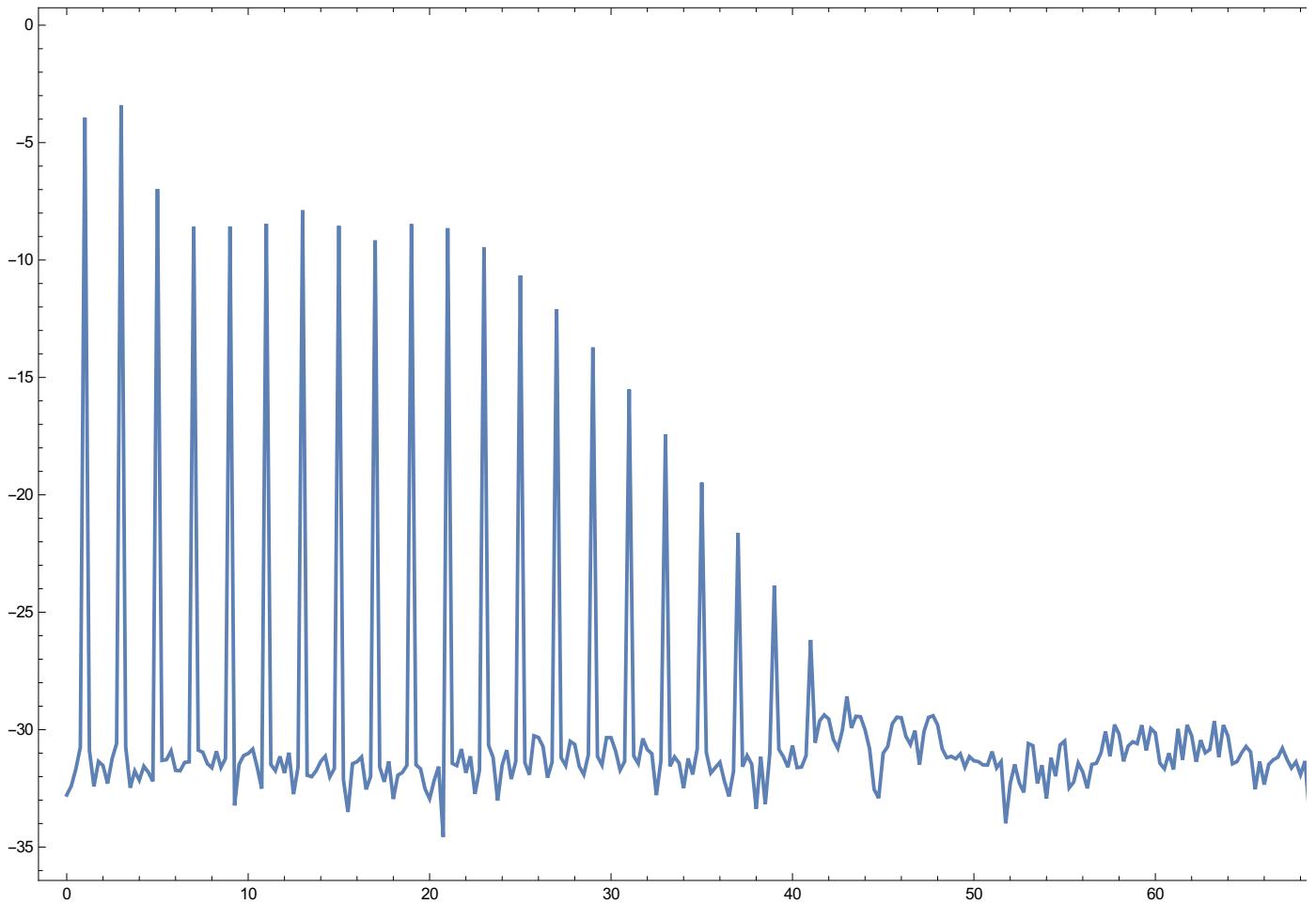


The default options are built for a periodic pulse for which simple functions of the vector potential can be integrated analytically, and for which only a single period of integration is necessary. More periods can be specified using the `TotalCycles` option. Similarly, the `PointsPerCycle` option controls the number of points per period.

```
AbsoluteTiming[
  biggerDipole = makeDipoleList[VectorPotential -> Function[t, {F ω Sin[ω t], 0, 0}],
    FieldParameters -> {F -> 0.05, ω -> 0.057}, TotalCycles -> 4, PointsPerCycle -> 150];
]
{28.0018, Null}
```

To get a correct spectrum plot, give these settings to the spectrum plotter.

```
spectrumPlotter[getSpectrum[Most[biggerDipole]], TotalCycles → 4, PointsPerCycle → 150]
```



You can specify a Target chemical species using the option

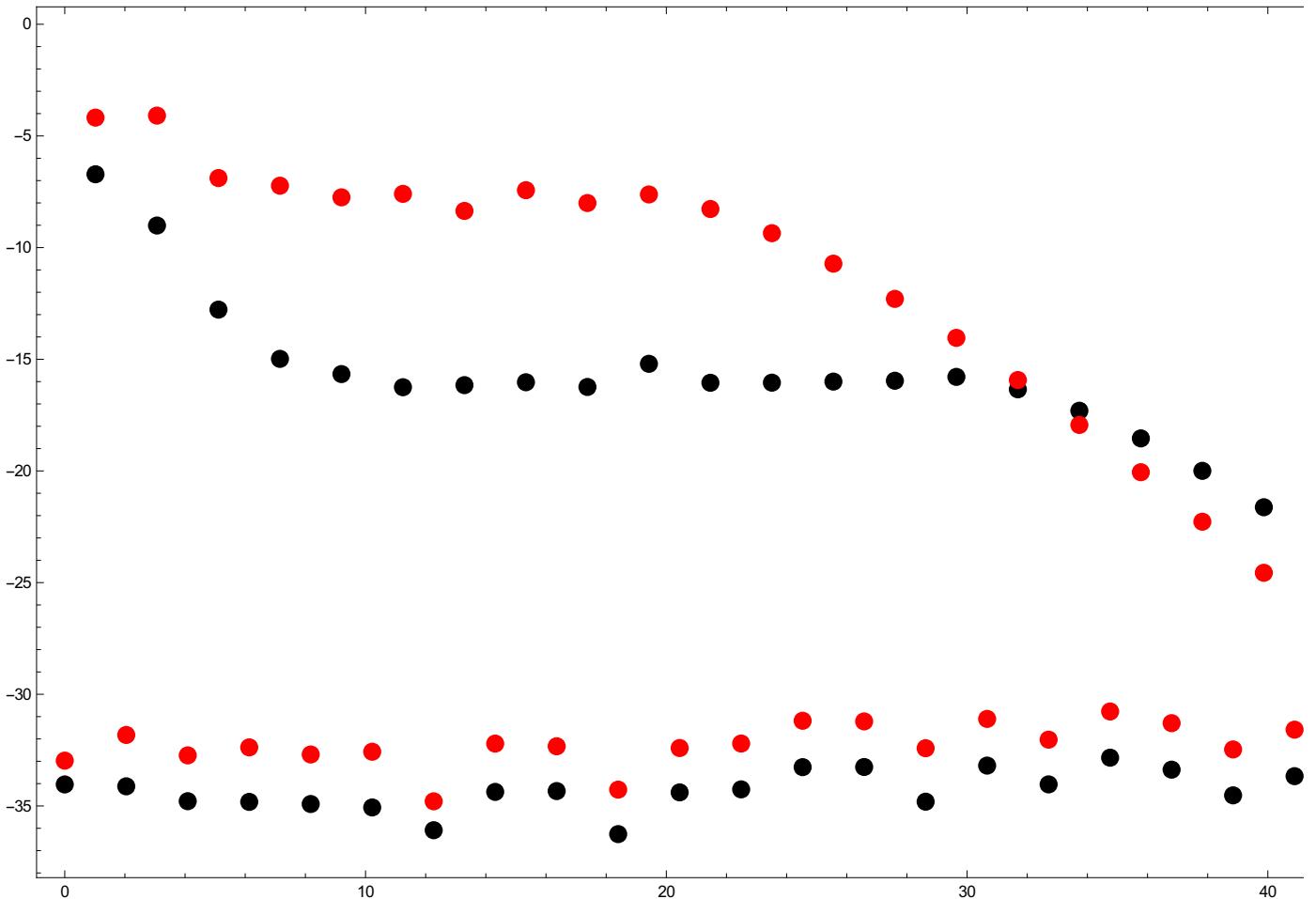
? Target

Target is an option for makeDipoleList which specifies chemical species producing the HHG emission, pulling the ionization potential from the Wolfram ElementData curated data set.

i.e. using the syntax

```
AbsoluteTiming[
  heliumDipole = makeDipoleList[VectorPotential → Function[t, {F Sin[ω t], 0, 0}], 
    FieldParameters → {F → 0.05, ω → 0.057}, Target → "Helium"];
  xenonDipole = makeDipoleList[VectorPotential → Function[t, {F Sin[ω t], 0, 0}], 
    FieldParameters → {F → 0.05, ω → 0.057}, Target → "Xenon"]];
{9.49704, Null}
```

```
Show[{  
    spectrumPlotter[getSpectrum[Most[heliumDipole]], Joined → False, PlotStyle → Black],  
    spectrumPlotter[getSpectrum[Most[xenonDipole]], Joined → False, PlotStyle → Red]  
}]
```



For convenience, the function `getIonizationPotential` gives a public-facing access to this functionality, via

```
? getIonizationPotential
```

`getIonizationPotential[Target]` returns the ionization potential of an atomic target, e.g. "Hydrogen", in atomic units.

`getIonizationPotential[Target,q]` returns the ionization potential of the q-th ion of the specified Target, in atomic units.

so that e.g.

```
{"H", #, UnitConvert[Quantity[#, "Hartrees"], "Electronvolts"]} &[  
  getIonizationPotential["Hydrogen"]]  
{"He+", #, UnitConvert[Quantity[#, "Hartrees"], "Electronvolts"]} &[  
  getIonizationPotential["Helium", 1]]  
{H, 0.49971, 13.598 eV}  
{He+, 1.9998, 54.418 eV}
```

An ionization potential can also be specified directly:

? IonizationPotential

IonizationPotential is an option for `makeDipoleList` which specifies the ionization potential I_p of the target.

To see the available options for this function (and others), use

```
Options[makeDipoleList]
{PointsPerCycle → 90, TotalCycles → 1, CarrierFrequency → 0.057,
 VectorPotential → Automatic, FieldParameters → {}, VectorPotentialGradient → None,
 Preintegrals → Analytic, ReportingFunction → Identity, Gate → SineSquaredGate[ $\frac{1}{2}$ ],
 nGate →  $\frac{3}{2}$ , εCorrection → 0.1, IonizationPotential → 0.5,
 Target → Automatic, DipoleTransitionMatrixElement → hydrogenicDTME,
 PointNumberCorrection → 0, Verbose → 0, IntegrationPointsPerCycle → Automatic}
```

All options have suitable information messages.

? VectorPotential

VectorPotential is an option for `makeDipole` list which specifies the field's vector potential. Usage should be `VectorPotential→A`, where `A[t]//.pars` must yield a list of numbers for numeric `t` and parameters indicated by `FieldParameters→pars`.

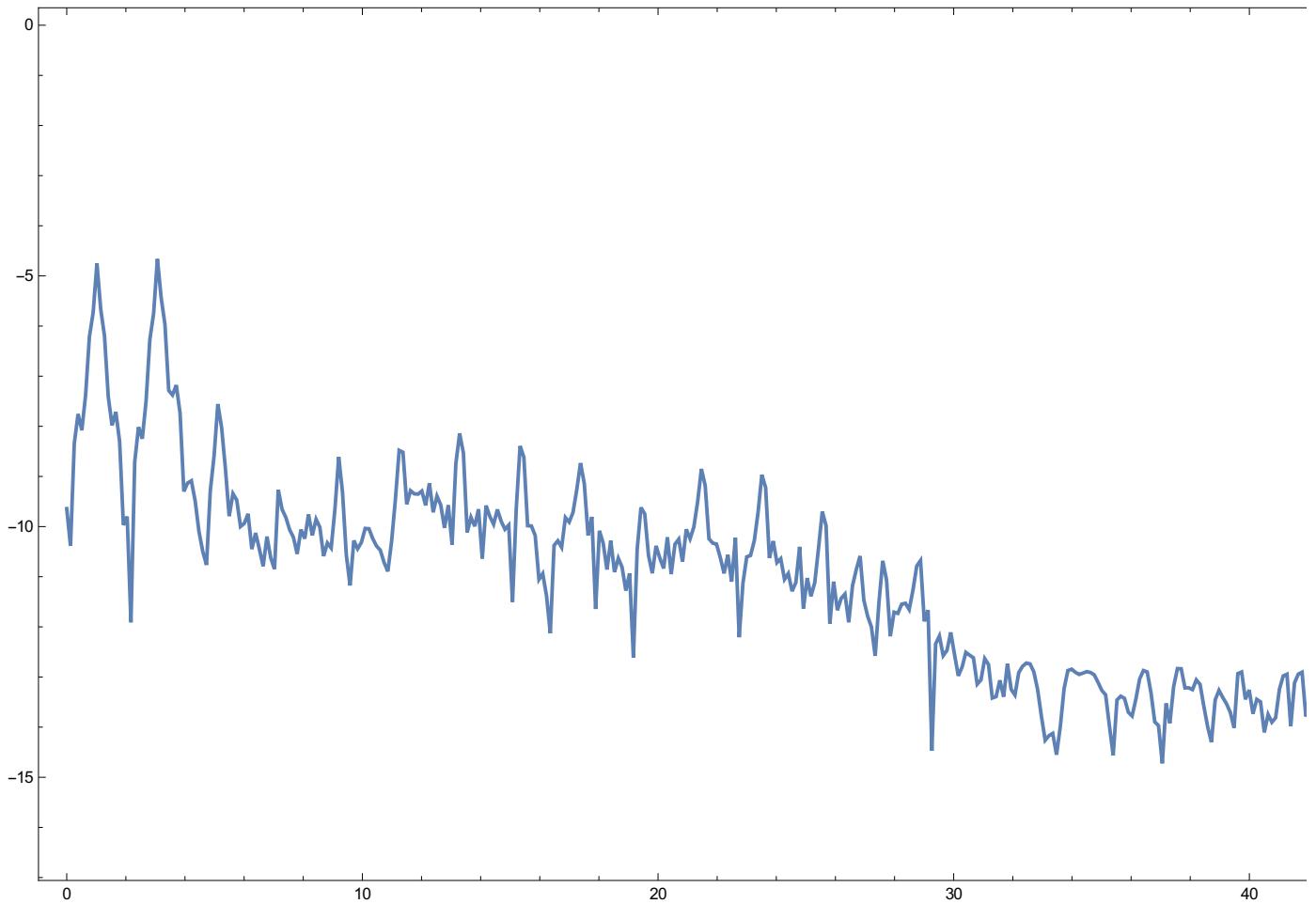
Using numerical integration for the preintegrals

Dipole case

To simulate a pulse with an envelope, it can be convenient to perform the preintegrals numerically, using the option `Preintegrals→"Numeric"`. These cases are generally slower but mainly because they require many more periods of integration.

```
AbsoluteTiming[
numericallyIntegratedDipole =
  makeDipoleList[VectorPotential → Function[t, { $\frac{F}{\omega}$  envelope[t] Sin[ω t], 0, 0}],
    FieldParameters → {ω → 0.057, F → 0.055, envelope → cosPowerFlatTop[0.057, 8, 16]},
    TotalCycles → 8,
    Preintegrals → "Numeric"];
]
{30.2574, Null}
```

```
spectrumPlotter[getSpectrum[numericallyIntegratedDipole]]
```



When using flat top pulses, and other waveforms that depend on Piecewise functions, it is possible that the function will return errors caused by an Indeterminate derivative being evaluated at the corners of the envelope.

```
AbsoluteTiming[
flatTopPulseDipole =
  makeDipoleList[VectorPotential → Function[t, {F/ω envelope[t] Sin[ω t], 0, 0}],
    FieldParameters → {ω → 0.057, F → 0.055, envelope → flatTopEnvelope[0.057, 8, 2]},
    TotalCycles → 8, Preintegrals → "Numeric"];
]
{32.3575, Null}
```

In these cases, use a numeric test to diagnose what's happened

```
Tally[flatTopPulseDipole /. _?NumberQ → √]
{{{√, √, √}, 721}}
```

and if the function is returning non-numeric values, it can help to fiddle with the PointNumberCorrection option.

```
? PointNumberCorrection
```

PointNumberCorrection is an option for makeDipoleList and timeAxis which specifies an extra number of points to be integrated over, which is useful to prevent Indeterminate errors when a Piecewise envelope is being differentiated at the boundaries.

Nondipole case

The numerical Preintegrals can be used in the nondipole case but they're obviously much slower. The number of preintegrals to find numerically increases from two in the dipole case ($\int \mathbf{A}(\tau) d\tau$ and $\int \mathbf{A}(\tau)^2 d\tau$) to eight with the nondipole contributions, three of them parametrized by t' . The main load, however, is not in numerically calculating these integrals via NDSolve constructs, but rather in the added strain of accessing the preintegrals as InterpolatingFunction objects once they've been calculated, from the main integration loop.

```
DateString[]
AbsoluteTiming[
  numericallyIntegratedNonDipoleCaseDipole = makeDipoleList[
    VectorPotential → Function[t, {F ω envelope[t] Sin[ω t], 0, 0}],
    VectorPotentialGradient →
      Function[t, {{0, 0, 0}, {0, 0, 0}, {-k F ω envelope[t] Sin[ω t], 0, 0}}],
    FieldParameters → {ω → 0.057, F → 0.055, envelope → cosPowerFlatTop[0.057, 8, 16],
      k → ω ω, α → 1/20},
    , TotalCycles → 8, Preintegrals → "Numeric"
  ];
]
DateString[]
Beep[]

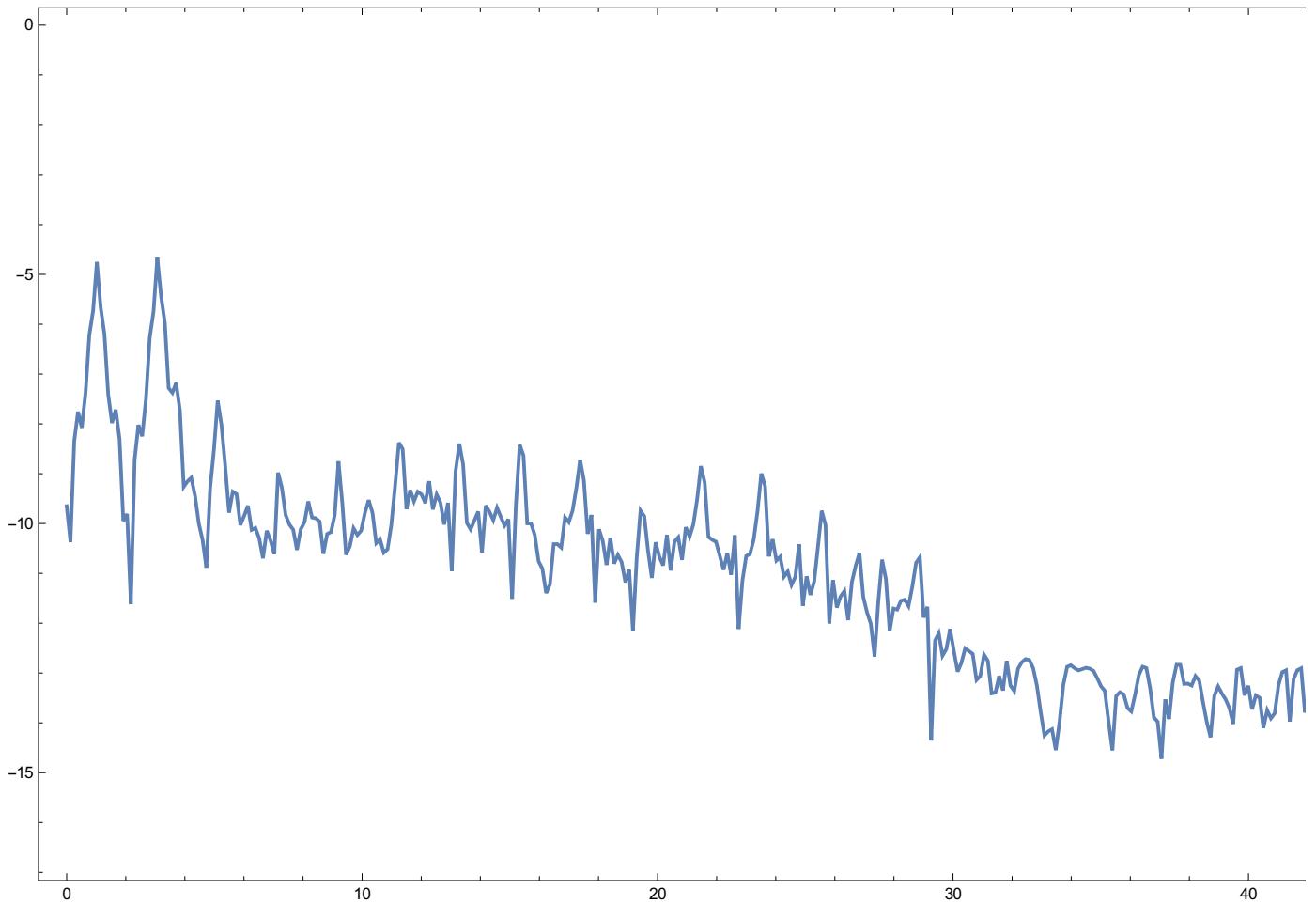
Mon 2 May 2016 15:17:00

$Aborted

Mon 2 May 2016 15:25:37
```

This requires some attention - currently taking too long to calculate.

```
spectrumPlotter[getSpectrum[numericallyIntegratedNonDipoleCaseDipole]]
```



Parallelization

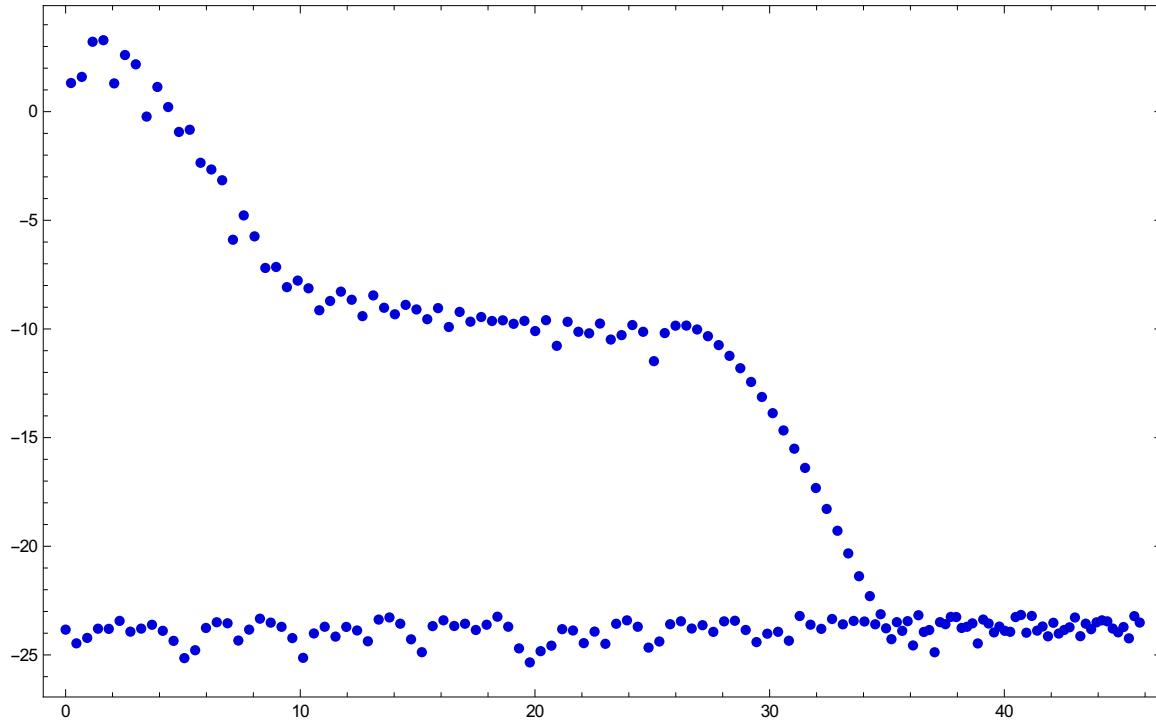
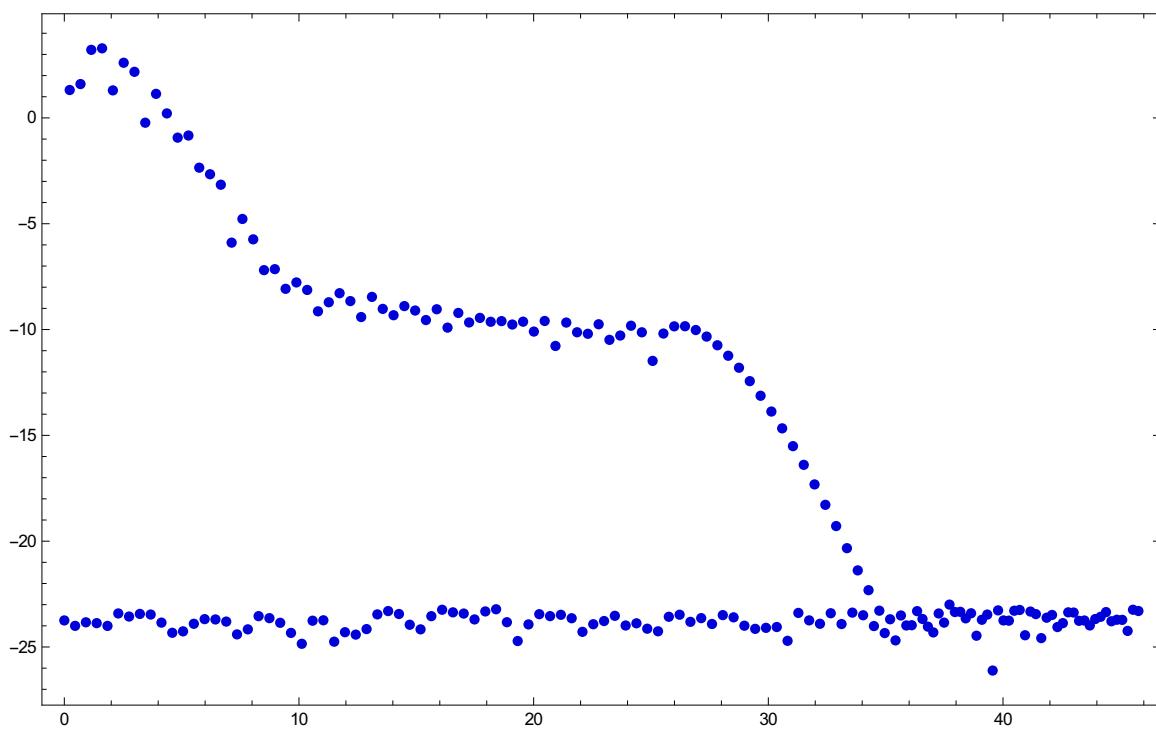
Parallelizing single instances

For faster evaluation of a single instance, it is possible to parallelize the evaluation, by adding the option `RunInParallel → True`.

```
AbsoluteTiming[directDipole = makeDipoleList[VectorPotential → Function[t, {F ω Sin[ω t], 0, 0}],  
FieldParameters → {F → Sqrt[10] 0.05, ω → 0.057}, PointsPerCycle → 400, RunInParallel → False];]  
AbsoluteTiming[parallelizedDipole = makeDipoleList[  
VectorPotential → Function[t, {F ω Sin[ω t], 0, 0}],  
FieldParameters → {F → Sqrt[10] 0.05, ω → 0.057}, PointsPerCycle → 400, RunInParallel → True];]  
{50.7121, Null}  
{52.8681, Null}
```

NOTE that this requires further attention - it appears that the parallelization is not working at the moment.

```
Row[{
  spectrumPlotter[getSpectrum[directDipole[[1 ;; -2]]],
  Joined → False, PlotStyle → Darker[Blue, 0.15], ImageSize → 600],
  spectrumPlotter[getSpectrum[parallelizedDipole[[1 ;; -2]]],
  Joined → False, PlotStyle → Darker[Blue, 0.15], ImageSize → 600]
}]
```



Unfortunately, the in-package single-instance parallelization can be unstable on occasion; this is probably due to a bug in `ParallelTable` (which can, under enough load, return different results to `Table`, in which case results typically differ run-to-run) that has proven so far very difficult to diagnose.

In such cases, the `RunInParallel` option takes a third possibility - an explicit set of commands, `{TableCommand, SumCommand}`, to use in the iteration.

? `RunInParallel`

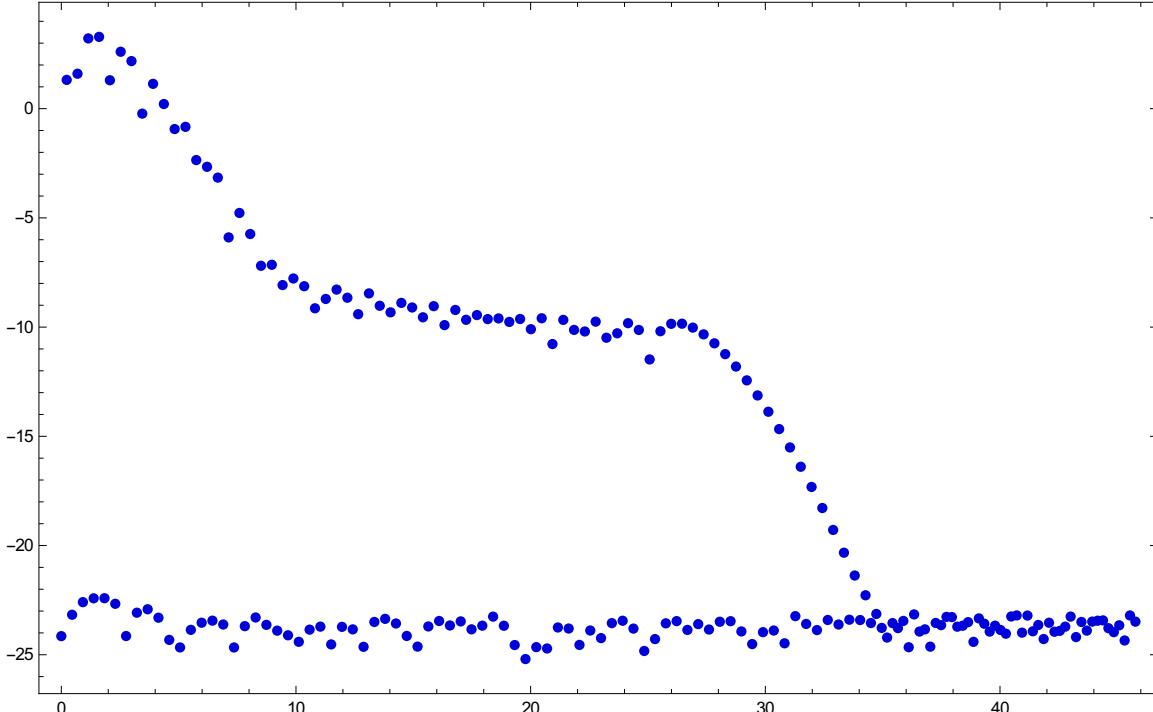
`RunInParallel` is an option for `makeDipoleList` which controls whether each RB-SFA instance is parallelized. It accepts `False` as the (Automatic) option, `True`, to parallelize each instance, or a pair of functions `{TableCommand, SumCommand}` to use for the iteration and summing, which could be e.g. `{Inactive[ParallelTable], Inactive[Sum]}`.

This is meant to be used by changing those commands to dud versions which can be sprung up later. The ideal use case (in v10 and up) is via `Inactive` commands, which return as

```
makeDipoleList[VectorPotential → Function[t, {F Sin[ω t], 0, 0}],  
FieldParameters → {F → Sqrt[10] 0.05, ω → 0.057}, Gate → (1 &), PointsPerCycle → 400  
, RunInParallel → {Inactive[ParallelTable], Inactive[Sum]}  
] /. {RBSFA`Private`t → t, RBSFA`Private`τ → τ}  
  
ParallelTable[0.275578 Sum[ $\left\{ \left( (56.7185 + 0. i) e^{-i \left( -\frac{(-48.6654 \cos[0.057 t] + 48.6654 \cos[0.057 (t-\tau)])^2}{(0.-0.1 i)+\tau} + \frac{1}{2} \tau \left( 1 + \frac{(-48.6654 \cos[0.057 t] + 48.6654 \cos[0.057 (t-\tau)])^2}{((0.-0.1 i)+\tau)^2} \right) + \frac{1}{2} (7.69468 (0.5 t - 4.38596 \sin[0.114 t] - 0.057 \tau) + 2.77393 \sin[0.057 t]) \right) / ((0.-0.1 i)+\tau) + 2.77393 \sin[0.057 (t-\tau)] + (0.-((0.+0.56941 i) \cos[0.057 (t-\tau)] - ((-48.6654 \cos[0.057 t] + 48.6654 \cos[0.057 (t-\tau)]) / ((0.-0.1 i)+\tau)) + 2.77393 \sin[0.057 (t-\tau)]) / ((0.-0.1 i)+\tau) + 2.77393 \sin[0.057 (t-\tau)]) \right) / (1. + (-((-48.6654 \cos[0.057 t] + 48.6654 \cos[0.057 (t-\tau)]) / ((0.-0.1 i)+\tau)) + 2.77393 \sin[0.057 (t-\tau)])^2)^{3/2} \right)$   
/ ((1. + (-((-48.6654 \cos[0.057 t] + 48.6654 \cos[0.057 (t-\tau)]) / ((0.-0.1 i)+\tau)) + 2.77393 \sin[0.057 (t-\tau)])^2)^{3/2} / ((0.+0.i) \left( \frac{1}{0.1+i \tau} \right)^{3/2} (0.-((0.+0.56941 i) \cos[0.057 (t-\tau)] - ((-48.6654 \cos[0.057 t] + 48.6654 \cos[0.057 (t-\tau)]) / ((0.-0.1 i)+\tau)) + 2.77393 \sin[0.057 (t-\tau)]) / ((0.-0.1 i)+\tau) + 2.77393 \sin[0.057 (t-\tau)]) / ((1. + (-((-48.6654 \cos[0.057 t] + 48.6654 \cos[0.057 (t-\tau)]) / ((0.-0.1 i)+\tau)) + 2.77393 \sin[0.057 (t-\tau)])^2)^{3/2} / ((0.+0.i) \left( \frac{1}{0.1+i \tau} \right)^{3/2} (0.-((0.+0.56941 i) \cos[0.057 (t-\tau)] - ((-48.6654 \cos[0.057 t] + 48.6654 \cos[0.057 (t-\tau)]) / ((0.-0.1 i)+\tau)) + 2.77393 \sin[0.057 (t-\tau)]) / ((0.-0.1 i)+\tau) + 2.77393 \sin[0.057 (t-\tau)]) / ((1. + (-((-48.6654 \cos[0.057 t] + 48.6654 \cos[0.057 (t-\tau)]) / ((0.-0.1 i)+\tau)) + 2.77393 \sin[0.057 (t-\tau)])^2)^{3/2}) \right) / ((0., 0, 165.347, 0.275578)], {t, 0, 110.231, 0.275578}]
```

and which can then be sprung into action using `Activate`:

```
AbsoluteTiming[postActivatedDipole = Activate[
  makeDipoleList[VectorPotential → Function[t, {F ω Sin[ω t], 0, 0}], 
  FieldParameters → {F → Sqrt[10] 0.05, ω → 0.057}, PointsPerCycle → 400,
  RunInParallel → {Inactive[ParallelTable], Inactive[Sum]}]
];
spectrumPlotter[getSpectrum[postActivatedDipole[[1 ;; -2]]],
 Joined → False, PlotStyle → Darker[Blue, 0.15], ImageSize → 600]
{6.14601, Null}]
```



This looks like it shouldn't change anything, but it can help fix a noisy `ParallelTable` output. It can also, inexplicably, be rather faster than the in-package parallelization. (For a cleaner example of the latter, see this [mathematica.stackexchange](#) question.)

Multiple instances in parallel

Alternatively, one can also parallelize over each run by using `ParallelTable` and similar commands. In general, this requires some careful handling of contexts; in this package this has been resolved by calling

```
DistributeDefinitions["RBSFA`"];
```

at the end of the package, which should pre-load all the RB-SFA definitions into each subkernel as it is launched, even if a kernel dies or is re-spawned. If parallelization seems not to be working (calculations taking too long and using a single core), there is a chance that the parallel kernels are returning the `RBSFA`` functions unevaluated to the main kernel, which is then doing the work. This can be diagnosed by inserting an appropriate print statement of the `$KernelId` to force the calculating kernel to identify itself (0 for the main kernel, >0 for subkernels), and it can be fixed by suitable use of the `DistributeDefinitions` function and the `$DistributedContexts` variable.

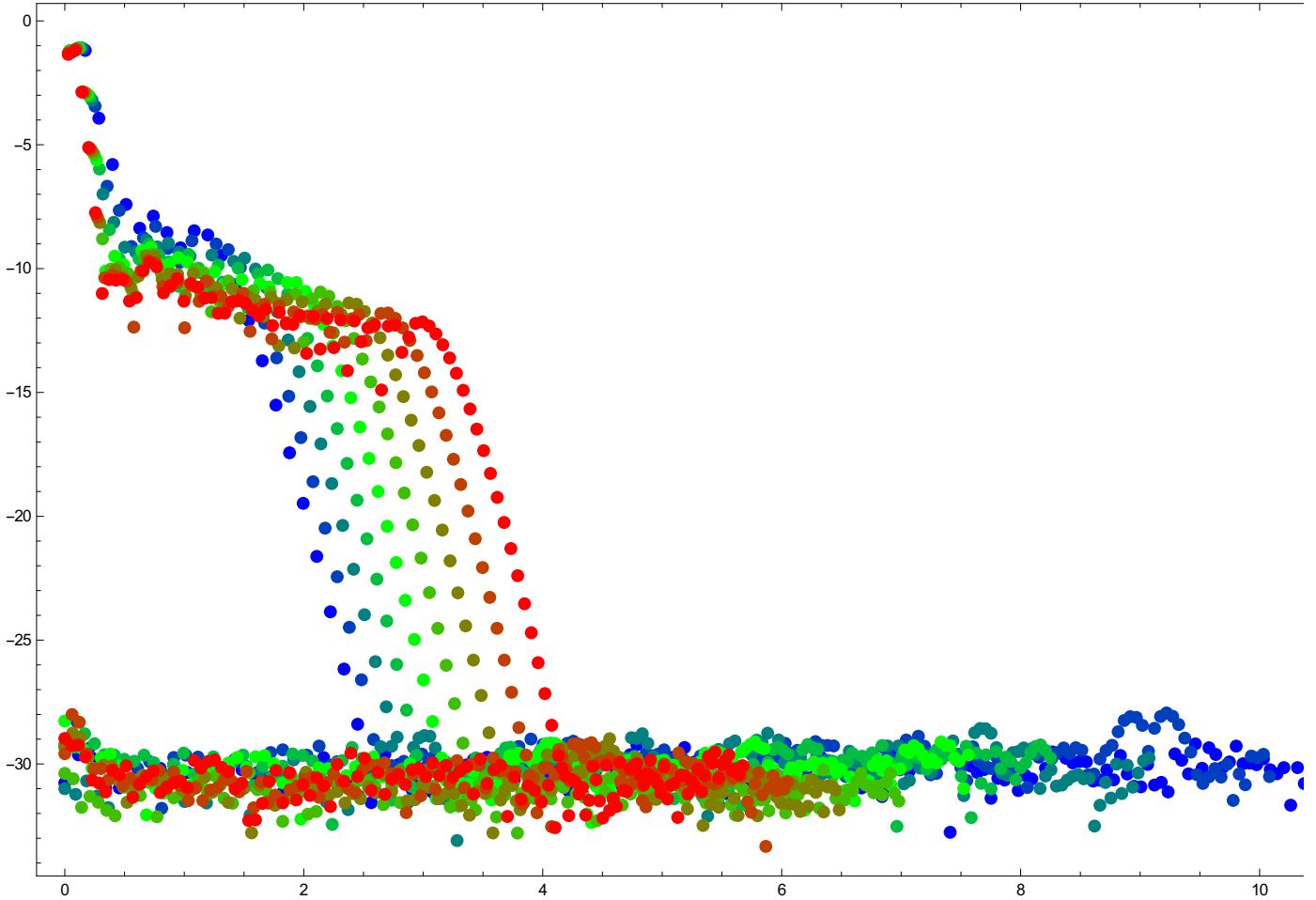
In addition to this, if a variable or function is used to store the results, this must be synchronized using `SetShared``.

Function or SetSharedVariable, as usual.

```
SetSharedFunction[wavelengthScanDipole];

ParallelTable[
 Print[AbsoluteTiming[
 wavelengthScanDipole[λ] =
 makeDipoleList[VectorPotential → Function[t, {F ω Sin[ω t], 0, 0}], FieldParameters →
 {F → 0.05, ω → 45.6 / λ}, CarrierFrequency → 45.6 / λ, PointsPerCycle → 400];
 ]]
 , {λ, 800, 1600, 100}]
{58.5366, Null}
{59.3303, Null}
{60.1489, Null}
{61.4793, Null}
{60.5053, Null}
{61.4832, Null}
{61.1334, Null}
{61.0625, Null}
{52.589, Null}
{Null, Null, Null, Null, Null, Null, Null, Null}
```

```
Show[Table[
  spectrumPlotter[getSpectrum[Most[wavelengthScanDipole[λ]]],
  PlotStyle → Blend[{Blue, Green, Red}, λ/800 - 1], CarrierFrequency → 45.6/λ,
  Joined → False, FrequencyAxis → "Frequency", PointsPerCycle → 400]
, {λ, 800, 1600, 100}]]
```



Writing output to file

For very large calculations (many integration points per cycle, in particular), the limiting factor is available memory. In these situations, it can help to write the data directly to a file on disk. This is slower (by a factor of about 2) but it has a roughly constant RAM footprint, so it enables calculations of a bigger size than would be possible otherwise. (Of course, this can also be done from non-parallelized calls!) This is done via the `ReportingFunction` option:

`? ReportingFunction`

ReportingFunction is an option for `makeDipole` list which specifies a function used to report the results, either internally (by the default, `Identity`) or to an external file.

In essence, the integration loop consists of a `Table` construct, which goes over the time t at which the integral is performed, and an inner integration construct. Setting an option `ReportingFunction → f` interposes the function `f` between these two steps, as

```
Table[ f[ integrator[t] ] , {t, tInitial, tFinal}]
```

The default is `f=Identity`, which returns its input untouched, but it can also be replaced by a `Write` construct that can

shunt its input to the hard disk without telling the kernel what it is, so it is not kept in memory.

`Quit`

```
directory = NotebookDirectory[];
filename[F_] := FileNameJoin[{directory, "Field scan data at F=" <> ToString[F] <> ".txt"}];

ParallelTable[
Print[AbsoluteTiming[
makeDipoleList[VectorPotential -> Function[t, {F Sin[w t], 0, 0}], 
FieldParameters -> {w -> 0.057}, CarrierFrequency -> 0.057, PointsPerCycle -> 400,
ReportingFunction -> Function[Write[filename[F], #]]]
];
],
{F, 0.05, 0.2, 0.025}]

{61.0728, Null}
{61.6888, Null}
{61.9032, Null}
{63.9092, Null}
{54.0083, Null}
{55.1511, Null}
{55.1227, Null}
{Null, Null, Null, Null, Null, Null}
```

The data in the files can then be pulled in quite simply using e.g.

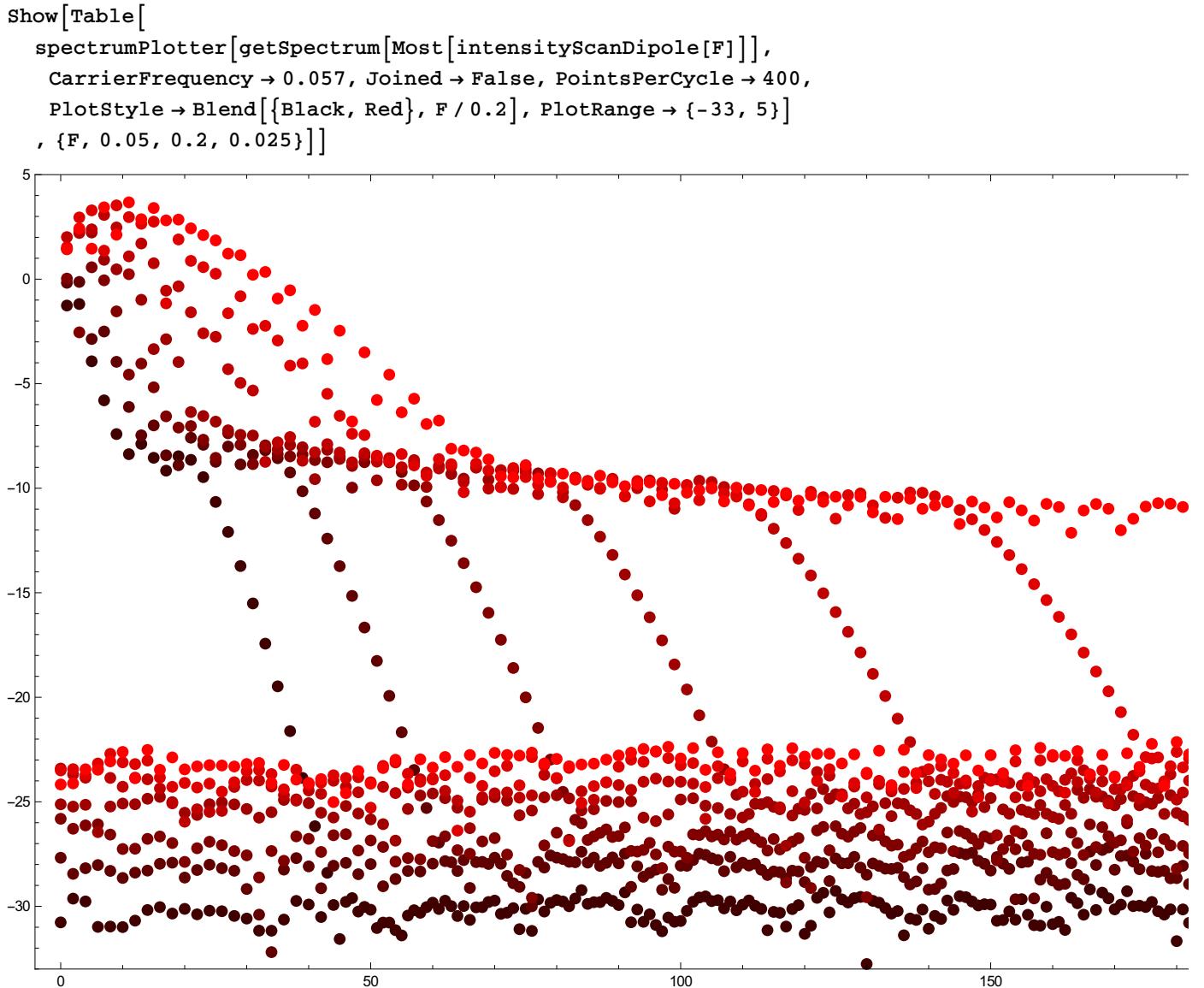
```
Do[ intensityScanDipole[F] = ReadList[filename[F]], {F, 0.05, 0.2, 0.025}]
```

This tends to litter the directories by creating lots of files for different parameters, so it is usually cleaner to `Save` them into a single file, e.g. using

```
Save[FileNameJoin[{NotebookDirectory[], "Field scan collected data.txt"}],
intensityScanDipole]
```

which in turn can then be pulled in using

```
<< (FileNameJoin[{NotebookDirectory[], "Field scan collected data.txt"}]);
```



As written, though, this has the disadvantage that each subkernel must access the hard drive for every timestep of the computation, which obviously responsible for (at least most of) the slowdown. A middle ground is also possible by choosing an appropriate `ReportingFunction`: a function which will cache a specific number k of results on RAM, and then write them to file all in one go. This is on the development to do (wish) list, and will hopefully be implemented soon - if time allows.

Time and memory use

Benchmark evaluation

The benchmarks below were taken on a desktop machine with 8-thread, 4-core Intel i7-3770 CPU at 3.40GHz, 16GB RAM, running *Mathematica* 10.0.1 over Ubuntu 14.04. The time taken per computation depends most strongly on the `PointsPerCycle` used to sample and integrate, and the dependence is therefore quadratic.

```

timingsList = Table[
  {n, AbsoluteTiming[
    MaxMemoryUsed[makeDipoleList[VectorPotential → Function[t, {F/ω Sin[ω t], 0, 0}], 
      FieldParameters → {F → Sqrt[n/100] 0.053, ω → 0.057}, PointsPerCycle → n]]]
  },
  {n,
   100,
   1000,
   1000}
]

{{100, {2.36167, 4905296}}, {200, {9.40602, 19018888}}, 
 {300, {20.6878, 43497000}}, {400, {37.2461, 78976528}}, {500, {57.5847, 118770848}}, 
 {600, {82.7101, 173233496}}, {700, {112.082, 232621352}}, {800, {150.096, 301125912}}, 
 {900, {187.298, 387140808}}, {1000, {233.122, 473885368}}}

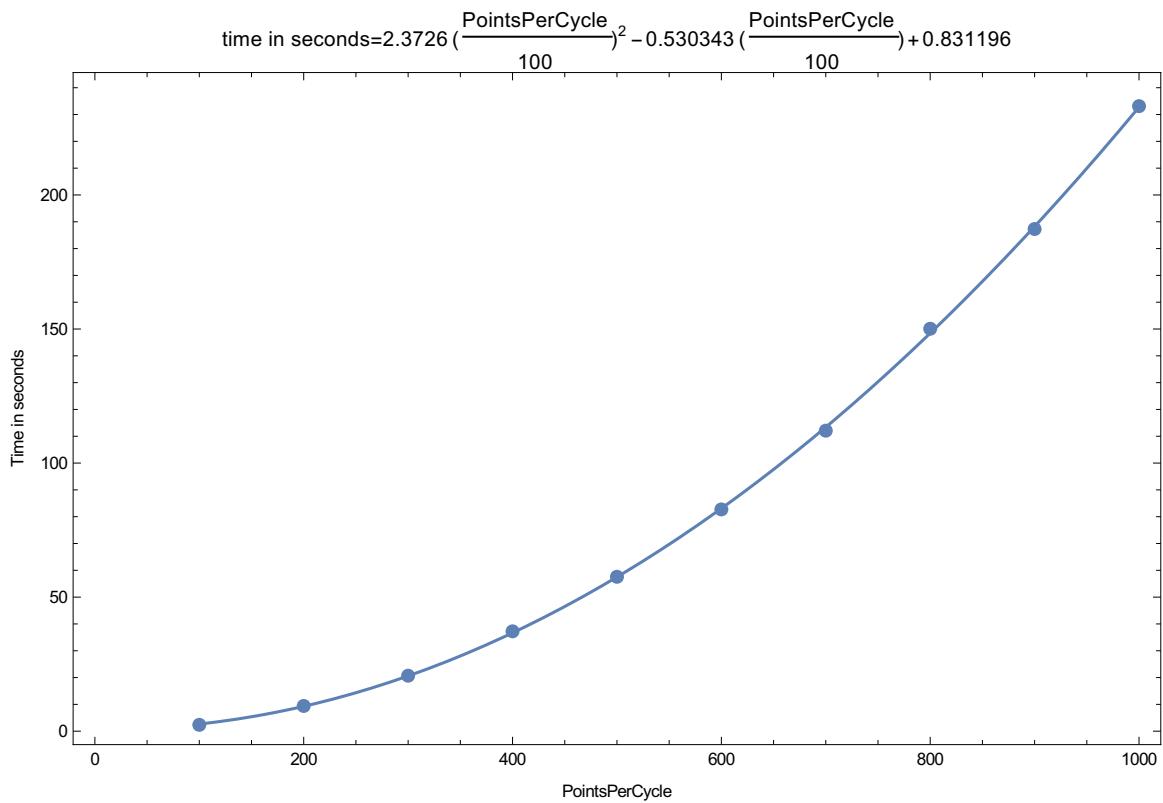
```

Timings

```

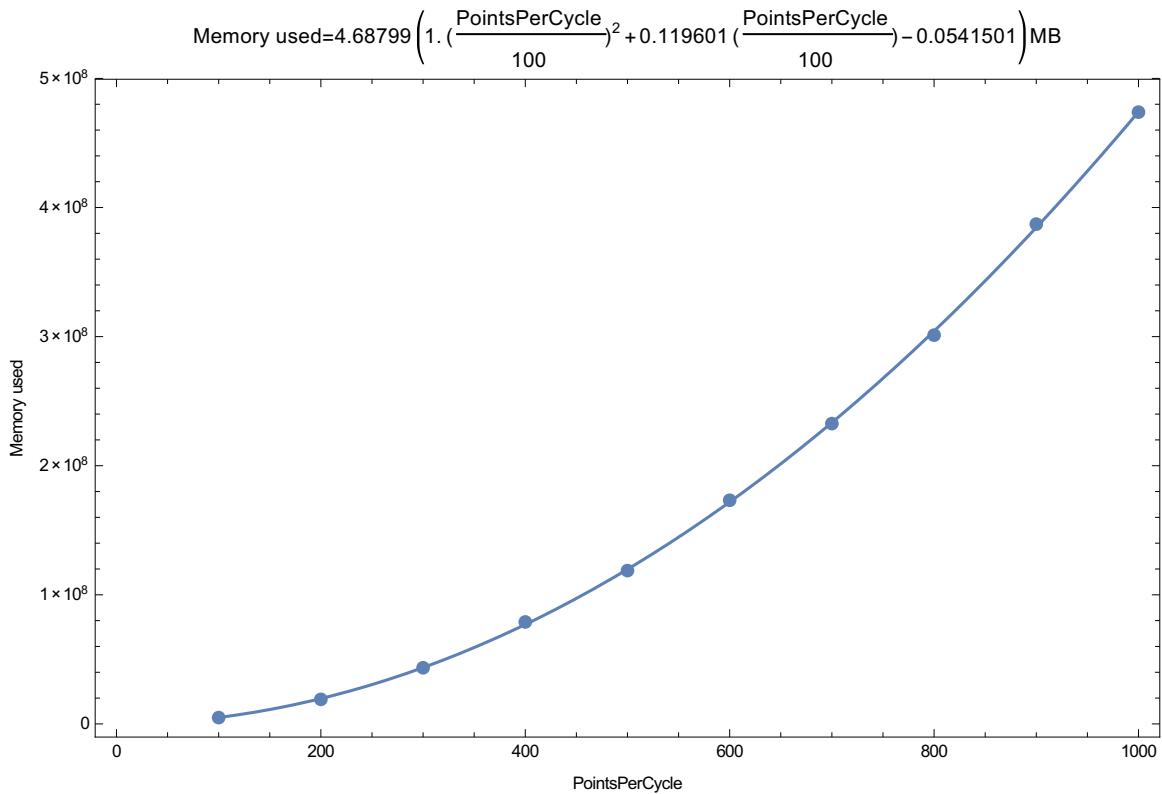
timingsModel = LinearModelFit[(Flatten/@timingsList) [[All, {1, 2}]], {1, n, n^2}, {n}];
Show[{
  ListPlot[
    (Flatten/@timingsList) [[All, {1, 2}]]
  ],
  Plot[timingsModel[n], {n, timingsList[[1, 1]], timingsList[[-1, 1]]}]
}
, Frame → True, PlotLabel → Row[{"time in seconds=", timingsModel[100] (PointsPerCycle/100)}],
FrameLabel → {"PointsPerCycle", "Time in seconds"}, ImageSize → 600
]

```



Maximum memory used

```
memoryModel = LinearModelFit[(Flatten/@timingsList) [[All, {1, 3}], {1, n, n^2}, {n}]];
Show[{
  ListPlot[
    (Flatten/@timingsList) [[All, {1, 3}]]
  ],
  Plot[memoryModel[n], {n, timingsList[[1, 1]], timingsList[[-1, 1]]}]
}
, Frame → True,
PlotLabel → Row[{"Memory used=", Simplify[memoryModel[100. " (PointsPerCycle)"] 10^-6 "MB"]}],
FrameLabel → {"PointsPerCycle", "Memory used"}, ImageSize → 600
]
```



In parallel

Inside parallel environments the timings are somewhat slower, by a factor of about 1.8. The timings below were taken with 7 *Mathematica* kernels running in parallel.

```
DistributeDefinitions["RBSFA`"];
```

```

parallelTimingsList = ParallelTable[
  {n, AbsoluteTiming[MaxMemoryUsed[
    makeDipoleList[VectorPotential -> Function[t, {F Sin[w t], 0, 0}], FieldParameters ->
      {F -> Sqrt[n/100] 0.053, w -> 45.6 / λ}, CarrierFrequency -> 45.6 / λ, PointsPerCycle -> n]]]}],
  , {λ, 770, 830, 10}, {n, 100, 1000, 100}]

{{{100, {4.71833, 7949312}}, {200, {18.2567, 19028920}}, {300, {39.6818, 43507248}}, {400, {68.0825, 75529984}}, {500, {108.27, 118772272}}, {600, {154.871, 173234856}}, {700, {211.182, 232622488}}, {800, {275.072, 301126208}}, {900, {350.061, 387140984}}, {1000, {424.31, 473885664}}}, {{100, {3.7169, 7949264}}, {200, {17.4824, 19028912}}, {300, {40.1556, 43507248}}, {400, {68.1348, 75529984}}, {500, {109.399, 118772272}}, {600, {153.631, 173234976}}, {700, {211.956, 232622248}}, {800, {279.29, 301126208}}, {900, {353.1, 387141104}}, {1000, {429.039, 473885664}}}, {{100, {4.70831, 7949264}}, {200, {17.3451, 19028912}}, {300, {40.1826, 43507248}}, {400, {68.9951, 75529984}}, {500, {109.87, 118772272}}, {600, {156.143, 173234976}}, {700, {209.361, 232622488}}, {800, {279.29, 301126208}}, {900, {350.784, 387141104}}, {1000, {418.914, 473885664}}}, {{100, {4.27991, 7949264}}, {200, {18.6273, 19028912}}, {300, {40.4553, 43507248}}, {400, {67.5833, 75529984}}, {500, {107.493, 118772272}}, {600, {151.738, 173234976}}, {700, {211.565, 232622368}}, {800, {276.237, 301126208}}, {900, {352.376, 387141104}}, {1000, {429.842, 473885664}}}, {{100, {4.80072, 7949264}}, {200, {17.9797, 19028912}}, {300, {38.2944, 43507248}}, {400, {68.7844, 75529984}}, {500, {105.406, 118772272}}, {600, {151.986, 173234976}}, {700, {216.018, 232622488}}, {800, {279.116, 301126208}}, {900, {361.033, 387141104}}, {1000, {424.589, 473885664}}}, {{100, {4.24956, 7949264}}, {200, {17.1782, 19028912}}, {300, {40.1991, 43507248}}, {400, {67.2693, 75529864}}, {500, {108.852, 118772032}}, {600, {156.147, 173234736}}, {700, {210.96, 232622008}}, {800, {279.971, 301126208}}, {900, {352.399, 387141104}}, {1000, {423.345, 473885664}}}, {{100, {4.4809, 7949264}}, {200, {17.6272, 19028912}}, {300, {39.652, 43507248}}, {400, {67.7538, 75529984}}, {500, {107.413, 118772272}}, {600, {156.833, 173234976}}, {700, {213.902, 232622488}}, {800, {276.711, 301126208}}, {900, {352.447, 387140984}}, {1000, {423.858, 473885544}}}}}

parallelTimingsListAveraged =
  Table[{parallelTimingsList[[k, 1, 1]], Mean[parallelTimingsList[[k, All, 2]]]}, {k, Length[parallelTimingsList]}]

{{100, {4.42209, 55644896/7}}, {200, {17.7852, 133202392/7}}, {300, {39.803, 43507248}}, {400, {68.0862, 528709768/7}}, {500, {108.101, 831405664/7}}, {600, {154.479, 1212644472/7}}, {700, {212.135, 232622368}}, {800, {277.955, 301126208}}, {900, {353.171, 2709987488/7}}, {1000, {424.842, 3317199528/7}}}

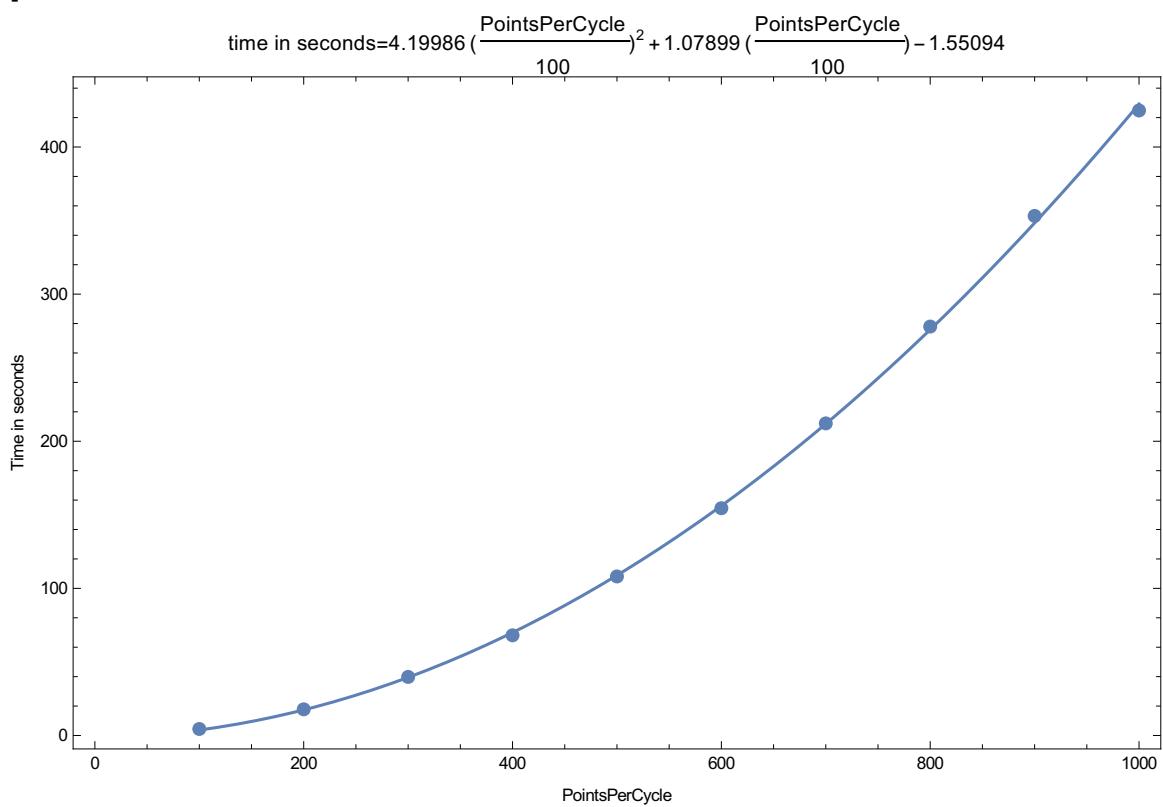
```

Timings

```

parallelTimingsModel =
  LinearModelFit[(Flatten/@parallelTimingsListAveraged)[[All, {1, 2}]], {1, n, n^2}, {n}];
Show[
  ListPlot[
    (Flatten/@parallelTimingsListAveraged)[[All, {1, 2}]]
  ],
  Plot[parallelTimingsModel[n],
    {n, parallelTimingsListAveraged[[1, 1]], parallelTimingsListAveraged[[-1, 1]]}]
]
, Frame → True,
PlotLabel → Row[{"time in seconds=", parallelTimingsModel[100 "(\frac{PointsPerCycle}{100})"]}],
FrameLabel → {"PointsPerCycle", "Time in seconds"}, ImageSize → 600
]

```

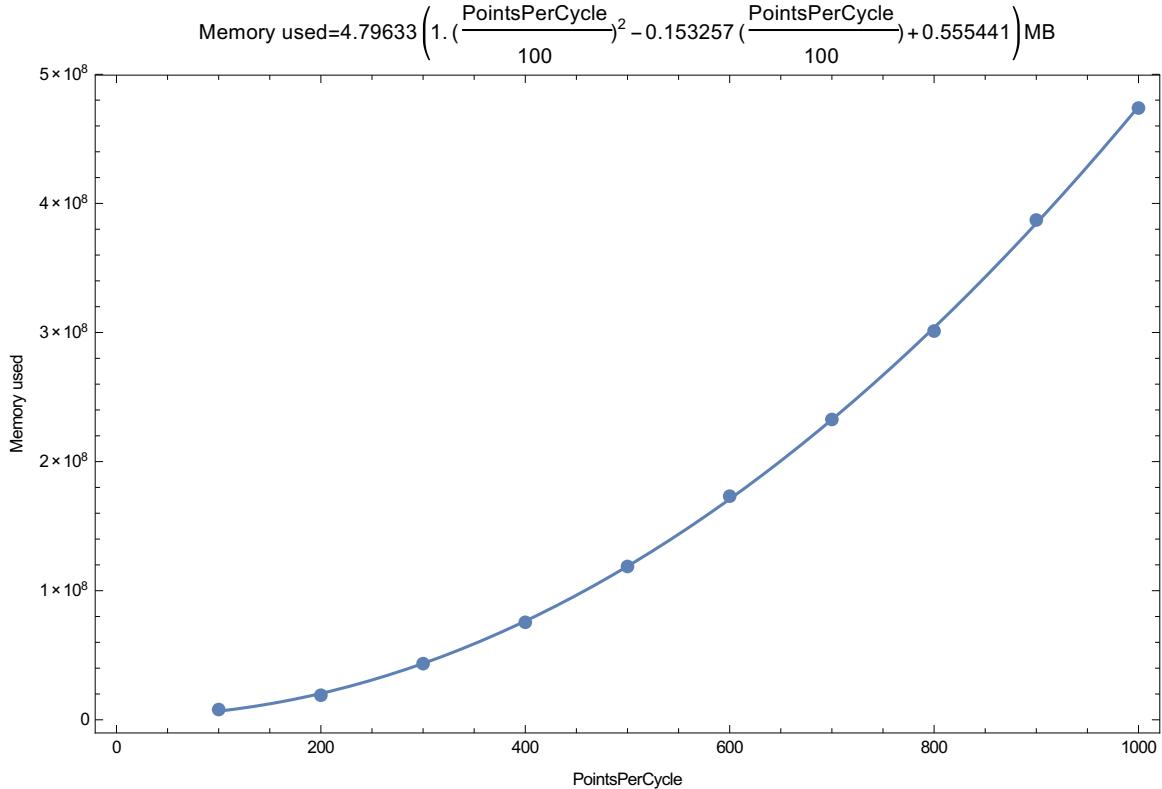


Memory

```

parallelMemoryModel =
  LinearModelFit[(Flatten/@parallelTimingsListAveraged)[[All, {1, 3}]], {1, n, n2}, {n}];
Show[{
  ListPlot[
    (Flatten/@parallelTimingsListAveraged)[[All, {1, 3}]]],
  Plot[parallelMemoryModel[n],
    {n, parallelTimingsListAveraged[[1, 1]], parallelTimingsListAveraged[[-1, 1]]}]
}
, Frame → True, PlotLabel →
  Row[{"Memory used=", Simplify[parallelMemoryModel[100. (PointsPerCycle/100)]] 10-6 "MB"]}],
FrameLabel → {"PointsPerCycle", "Memory used"}, ImageSize → 600
]

```



Decoupling integration and sampling

If the quadratic scaling with respect to `PointsPerCycle` becomes too onerous, the sampling rate for the evaluation and for the numerical integration can be decoupled by providing an explicit option for `IntegrationPointsPerCycle`, which will set how many points are used per cycle in the numerical integration.

```
? IntegrationPointsPerCycle
```

IntegrationPointsPerCycle is an option for `makeDipoleList` which controls the number of points per cycle to use for the integration. Set to Automatic, to follow `PointsPerCycle`, or to an integer.

Obviously, if this option is taken, then the outcome should be checked for numerical convergence with respect to `IntegrationPointsPerCycle`.

```

DateString[]
decoupledTimingsList = Table[
  {nSampling, nIntegration, AbsoluteTiming[
    MaxMemoryUsed[makeDipoleList[VectorPotential -> Function[t, {F
      \frac{F}{\omega} Sin[\omega t], 0, 0}]]]
    , FieldParameters -> {F -> Sqrt[nSampling/100] 0.053, \omega -> 0.057}]
    , PointsPerCycle -> nSampling, IntegrationPointsPerCycle -> nIntegration
  ]]}
, {nSampling, 100, 500, 100}, {nIntegration, 100, 500, 100}]
DateString[]
Fri 5 Feb 2016 17:08:36
{{{100, 100, {2.39139, 4 789 112}}, {100, 200, {4.76862, 9 473 168}}, {100, 300, {7.07336, 14 241 584}}, {100, 400, {9.39992, 19 138 656}}, {100, 500, {11.7964, 24 035 936}}}, {{200, 100, {4.72937, 9 447 040}}, {200, 200, {9.3758, 19 017 072}}, {200, 300, {14.1647, 28 478 336}}, {200, 400, {18.8825, 38 195 832}}, {200, 500, {23.4484, 47 916 856}}}, {{300, 100, {7.02226, 14 177 120}}, {300, 200, {14.067, 28 434 560}}, {300, 300, {21.5271, 43 233 320}}, {300, 400, {28.4153, 56 993 480}}, {300, 500, {35.2399, 70 747 672}}}, {{400, 100, {9.41508, 19 027 632}}, {400, 200, {18.6957, 38 108 104}}, {400, 300, {28.159, 56 949 288}}, {400, 400, {37.5268, 75 259 432}}, {400, 500, {46.83, 95 676 600}}}, {{500, 100, {11.6672, 23 882 240}}, {500, 200, {23.5225, 47 786 456}}, {500, 300, {35.0966, 70 660 928}}, {500, 400, {46.7308, 95 634 048}}, {500, 500, {58.5078, 118 501 720}}}}

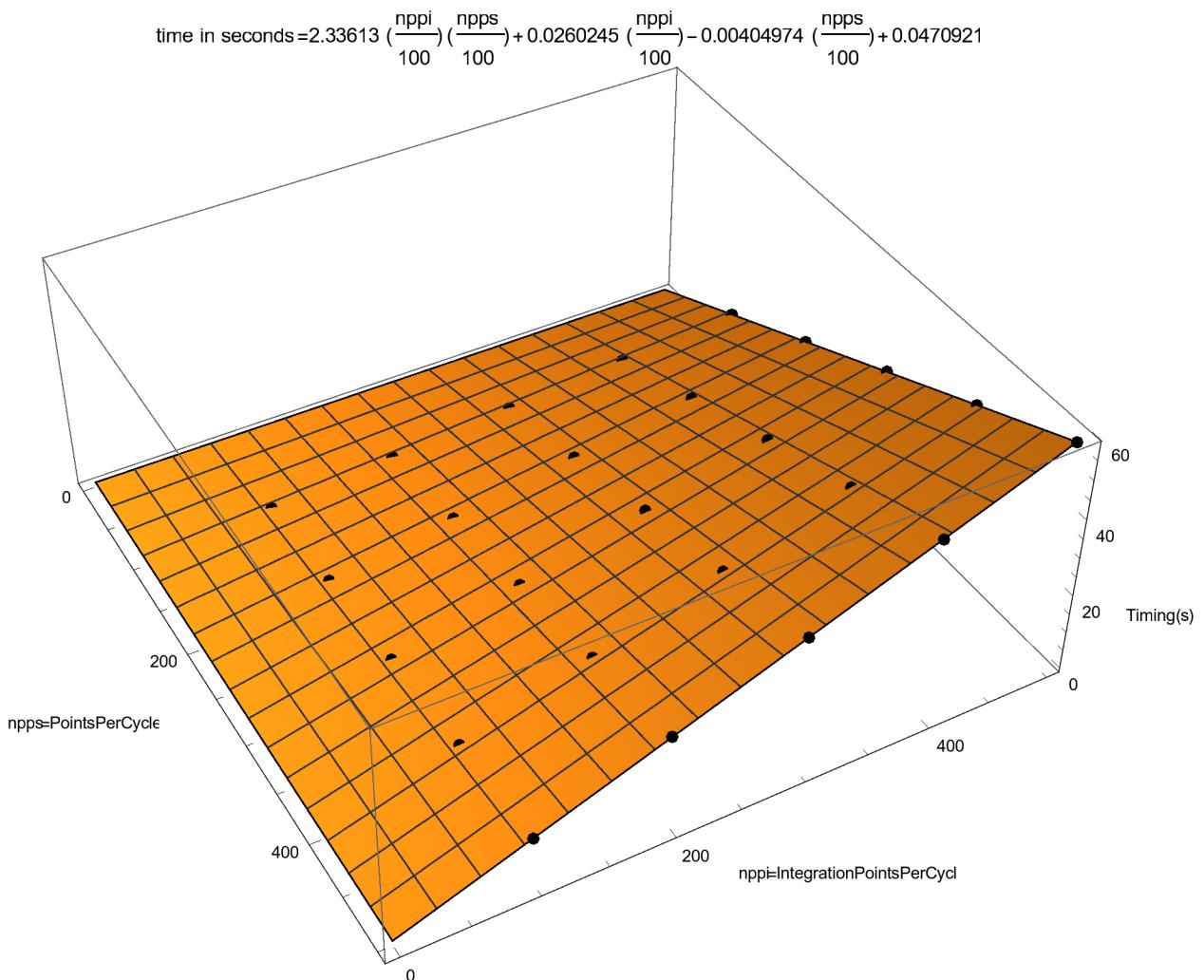
```

Fri 5 Feb 2016 17:17:24

```

decoupledTimingsModel = LinearModelFit[
  Flatten @ Flatten[decoupledTimingsList, {{1, 2}}][[All, {1, 2, 3}]],
  {1, nSampling, nIntegration, nSampling × nIntegration}, {nSampling, nIntegration}];
Show[{
  Plot3D[
    decoupledTimingsModel[nSampling, nIntegration]
    , {nSampling, 0, 500}, {nIntegration, 0, 500}
    , AxesLabel → {"npps=PointsPerCycle", "nppi=IntegrationPointsPerCycle", "Timing (s)"}
    ,
    PlotLabel → Row[{"time in seconds=", decoupledTimingsModel[100 "(\frac{npps}{100})", 100 "(\frac{nppi}{100})"]}]
    , ImageSize → 650
  ],
  ListPointPlot3D[
    (Flatten @ Flatten[decoupledTimingsList, {{1, 2}}])[[All, {1, 2, 3}]]
    , PlotStyle → {{Black, PointSize[Large]}}
  ]
}]

```

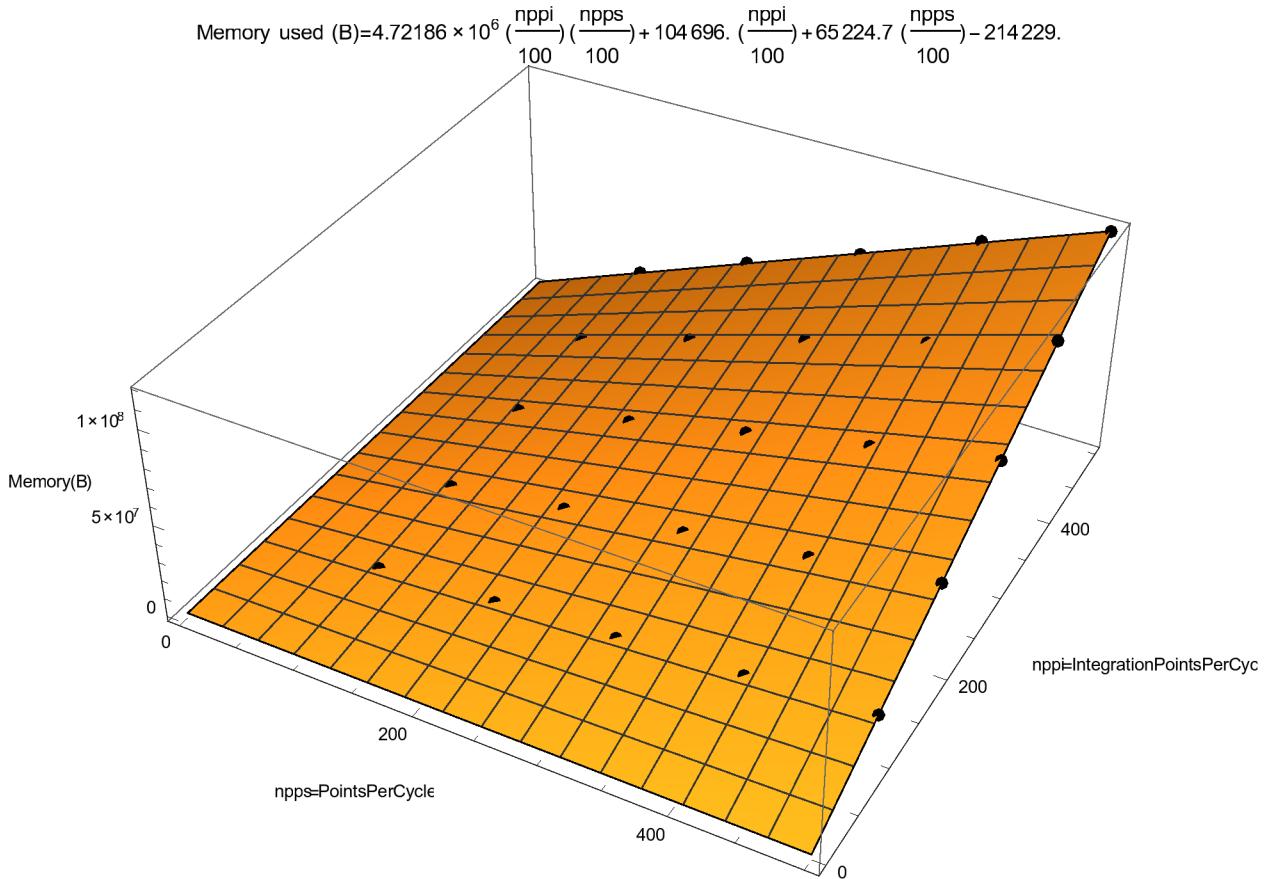


```

decoupledMemoryModel = LinearModelFit[
  (Flatten /@ Flatten[decoupledTimingsList, {{1, 2}}])[[All, {1, 2, 4}]]  

  , {1, nSampling, nIntegration, nSampling × nIntegration}, {nSampling, nIntegration}];
Show[{
  Plot3D[
    decoupledMemoryModel[nSampling, nIntegration]
    , {nSampling, 0, 500}, {nIntegration, 0, 500}
    , AxesLabel → {"npps=PointsPerCycle", "nppi=IntegrationPointsPerCycle", "Memory (B)"}
    , PlotLabel → Row[{"Memory used (B)=", decoupledMemoryModel[100 "(\frac{npps}{100})", 100 "(\frac{nppi}{100})"]}]
    , ImageSize → 650
  ],
  ListPointPlot3D[
    (Flatten /@ Flatten[decoupledTimingsList, {{1, 2}}])[[All, {1, 2, 4}]]
    , PlotStyle → {{Black, PointSize[Large]}}]
}
]
}

```



Cutting off the long trajectories

This section shows, as an example of the use of the package, the use of the integration gate to eliminate the contribution from long trajectories. This can be tested by the reduced presence of quantum path interference patterns in the spectrum, and more practically by examining the dependence of the quantum phase on the field intensity.

The gating cutoff time

Given the classical trajectory,

```
trajectory[wt_, wt0_] := (x[wt] /. First@DSolve[
  {x''[wt] == Cos[wt], x'[wt0] == 0, x[wt0] == 0},
  x, wt
])
```

the recollision kinetic energy and excursion time can be found as

```
recollisionKE[wt0_?NumericQ] := (D[trajectory[wtt, wt0], wtt]^2 /. wtt -> wt) /.
  First[Quiet[NSolve[{trajectory[wt, wt0] == 0, π/2 <= wt < 2 π}, wt]]]
recollisionExcursionTime[wt0_?NumericQ] :=
  (wt - wt0) /. First[Quiet[NSolve[{trajectory[wt, wt0] == 0, π/2 <= wt < 2 π}, wt]]]
```

and the excursion time at the cutoff can be found by maximizing the kinetic energy.

```
FindMaximum[recollisionKE[wt0], {wt0, 0.3}]
recollisionExcursionTime[wt0] /. Last[%]
 2 π
{1.58657, {wt0 -> 0.313408}}
0.650239
```

In other words, the cutoff trajectories occur at excursion times of $\omega \tau = 0.65 \times 2\pi$, i.e. at a gate number of 0.65 cycles.

Calculation

This calculation runs a standard linearly-polarized field with intensity between 0.8 and $2 \times 10^{14} \text{ W/cm}^2$, with a fine intensity resolution. We compare the standard, non-gated calculation against a calculation with nGate set to 0.65, as per the above, and a sharp \sin^2 cutoff of 0.05 cycles.

```
intRange = Range[0.8, 2., 0.002];
nppsl = 150;
SetSharedFunction[quantumPhaseScan, fourierDipole];
LaunchKernels[];
nFlat = 0.65;
nGateRamp = 0.05;
LaunchKernels::nodef : Some subkernels are already running. Not launching default kernels again. >>
```

The actual calculation,

```

DateString[]
AbsoluteTiming[
ParallelTable[
  quantumPhaseScan[trajectories, int] = makeDipoleList[
    VectorPotential → Function[t, {F ω Sin[ω t], 0, 0}],
    FieldParameters → {F → Sqrt[int] 0.053, ω → 0.057},
    PointsPerCycle → nppsl,
    If[trajectories === "Short", Sequence @@ {
      Gate → SineSquaredGate[nGateRamp], nGate → (nFlat + nGateRamp)}], ## &[], ## &[]]
  ];
  , {int, intRange}, {trajectories, {"Short", "Long"}}];
]
]
DateString[]

```

Wed 2 Dec 2015 19:28:50

{1216.8, Null}

Wed 2 Dec 2015 19:49:07

and the energy-domain dipole.

```

AbsoluteTiming[
Table[
  fourierDipole[trajectories, int] = Fourier[
    Re[quantumPhaseScan[trajectories, int][[1 ;; -2, 1]]]
    , FourierParameters → {-1, 1}];
  , {int, intRange}, {trajectories, {"Short", "Long"}}];
]
{0.093762, Null}

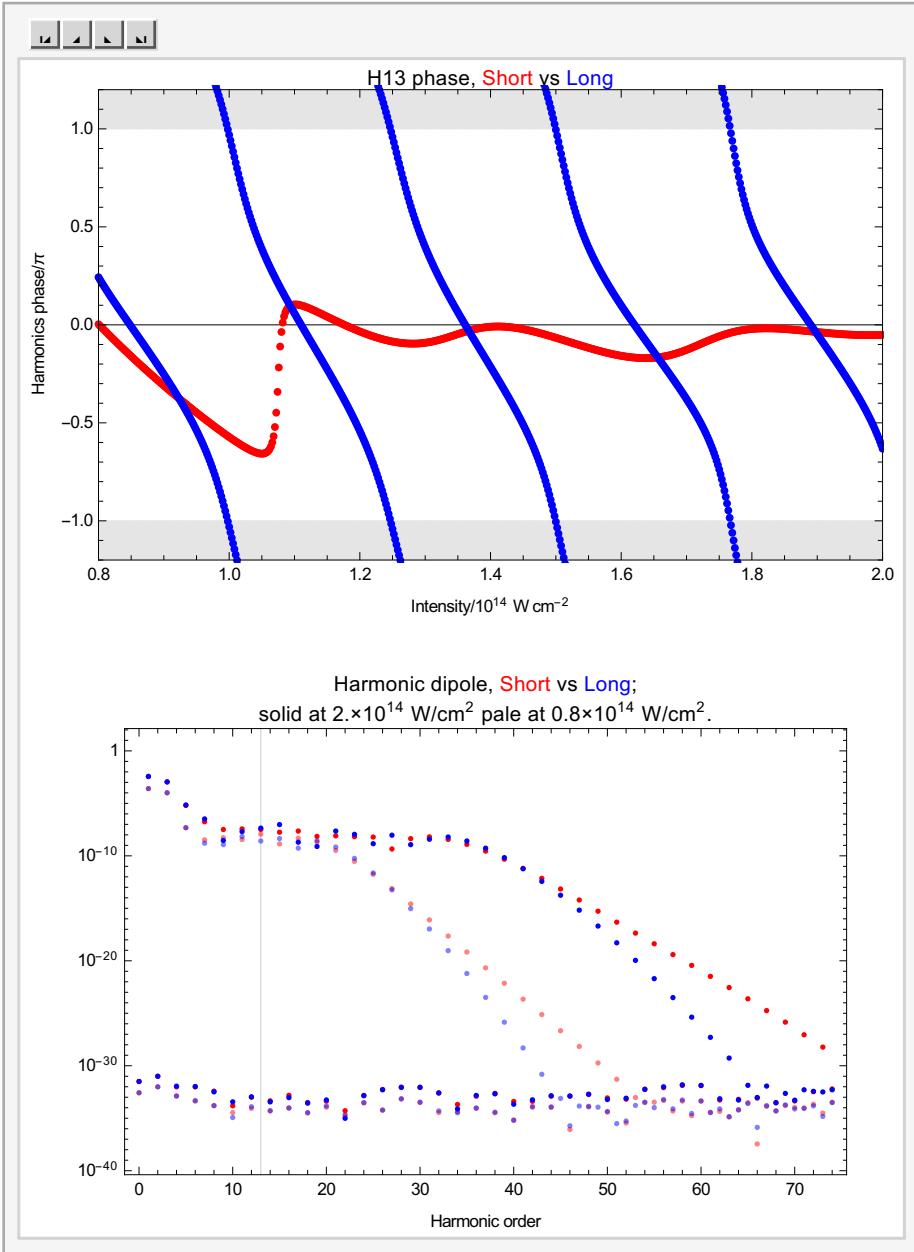
```

Analysis

```

Block[{background},
background = ListLogPlot[
  Flatten[Table[
    {Range[0, nppsl/2 - 1], Abs[fourierDipole[trajectories, m[intRange]]][1 ;; nppsl/2]}^2}^t
    , {m, {Min, Max}}, {trajectories, {"Short", "Long"}}, 1]
  , ImageSize → 420
  , PlotStyle → {{Red, Opacity[0.5]}, {Blue, Opacity[0.5]}, {Red}, {Blue}}
  , PlotLabel → "Harmonic dipole, Short vs Long;\\nsolid at " <> ToString[Max[intRange]] <>
    "×1014 W/cm2 pale at " <> ToString[Min[intRange]] <> "×1014 W/cm2."
  , FrameLabel → {"Harmonic order", ""}
  , Frame → True
];
SlideView[Table[
  Row[{{
    Show[{{
      RegionPlot[Abs[ϕ] > 1, {int, Min[intRange], Max[intRange]}, {ϕ, -1.2, 1.2},
        PlotStyle → GrayLevel[0.9], Method → {"AxesInFront" → False}, BoundaryStyle → None],
      ListPlot[
        Table[
          Flatten[Table[
            {#, {#[[1]], #[[2]] + 2}, {#[[1]], #[[2]] - 2}} &@
              {int,  $\frac{1}{\pi}$  Arg[fourierDipole[trajectories, int][HO + 1]]}
            , {int, intRange[[1 ;; -1]]}, 1]
            , {trajectories, {"Short", "Long"}}, ]
        , PlotStyle → {{PointSize[0.01], Red}, {PointSize[0.01], Blue}}
        , Joined → False
      ]
    ]}
  }}, PlotRange → 1.2 {-1, 1}, AspectRatio → 0.6
  , PlotRangePadding → {None, Automatic}
  , Axes → True
  , ImageSize → 450
  , PlotLabel → "H" <> ToString[HO] <> " phase, Short vs Long"
  , FrameLabel → {"Intensity/1014 W cm-2", "Harmonics phase/π"}
  ],
  Show[{background}, GridLines → {{HO}, None}]
}]
, {HO, 1, nppsl/2, 2}], 7]
]

```



The short- and long-trajectory calculations are in red and blue respectively. It is clear that, in the plateau regions, the gated calculation has a much smoother dependence of the harmonic phase on the field intensity. On the other hand, the cutoff is perfectly preserved. These are the hallmarks that the contributions from long trajectories have been mostly eliminated.

Nondipole contributions

Nondipole contributions can be specified by setting a nonzero vector potential gradient:

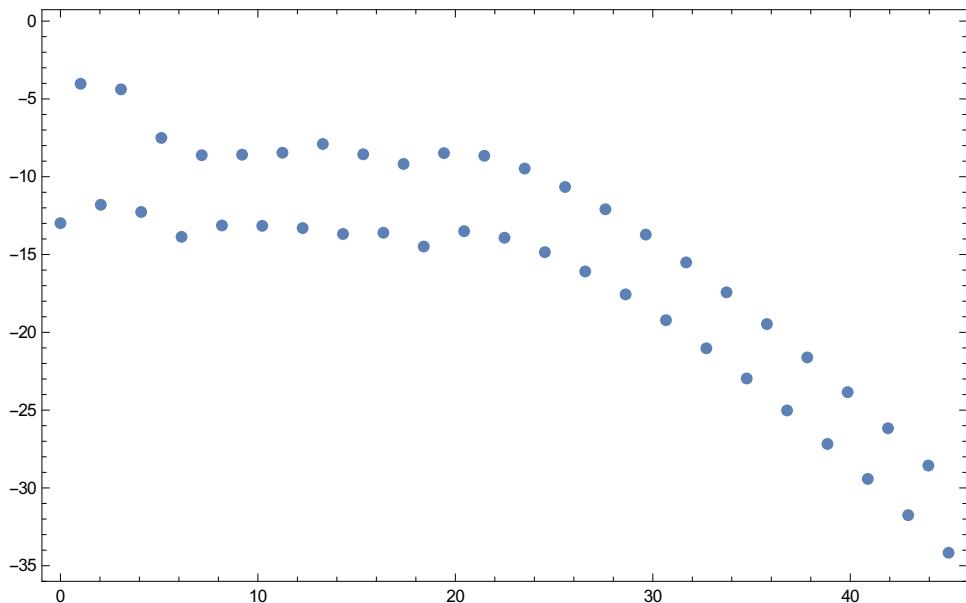
? VectorPotentialGradient

"VectorPotentialGradient is an option for makeDipole list which specifies the gradient of the field's vector potential. Usage should be VectorPotentialGradient→GA, where GA[t]//.pars must yield a square matrix of the same dimension as the vector potential for numeric t and parameters indicated by FieldParameters→pars. The indices must be such that GA[t][[i,j]] returns $\partial_i A_j[t]$."

If, for example, the travelling-wave form of the vector potential is of the form $\mathbf{A}(\mathbf{r}, t) = \frac{E}{\omega} \hat{\mathbf{x}} \cos(kz - \omega t)$, then at the

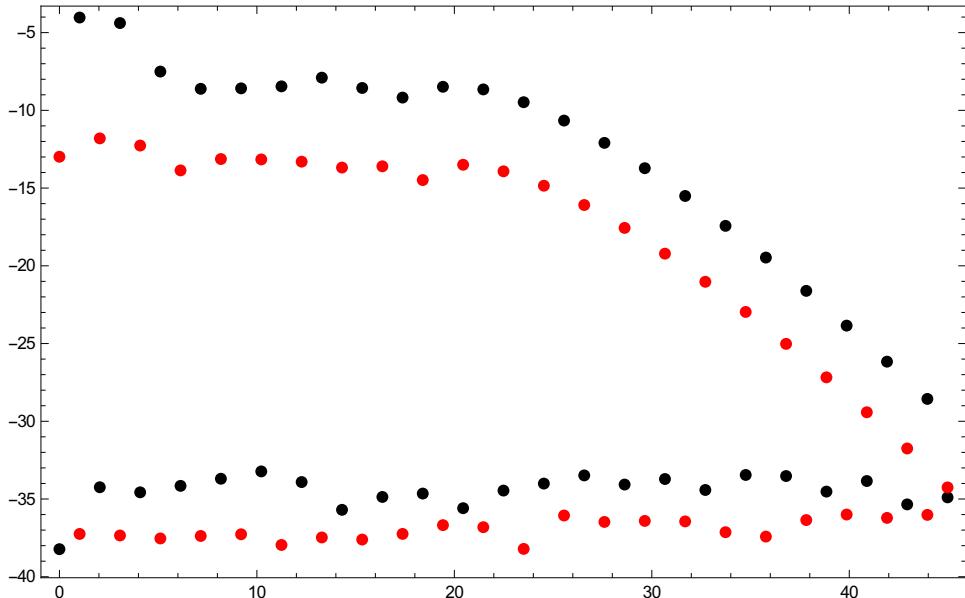
origin the vector potential is $\mathbf{A}(\mathbf{0}, t) = \frac{F}{\omega} \hat{\mathbf{x}} \cos(\omega t)$ and it has a single nonzero entry in its gradient matrix $\nabla \mathbf{A}$, i.e. $\partial_z A_x = -\frac{kF}{\omega} \sin(\omega t)$. This is entered into the VectorPotentialGradient option as

```
nonDipoleContributions = makeDipoleList[
  VectorPotential → Function[t, {F ω Cos[ω t], 0, 0}],
  VectorPotentialGradient → Function[t, {{0, 0, 0}, {0, 0, 0}, {-k F ω Sin[ω t], 0, 0}}],
  FieldParameters → {F → 0.05, ω → 0.057, k → ω / c, c → 137}
];
spectrumPlotter[getSpectrum[Most[nonDipoleContributions]], Joined → False, ImageSize → 500]
```



At low wavelengths, the first obvious effect is the appearance of even harmonics. This is the expected behaviour, with the harmonics along the laser propagation direction. (Informally, the magnetic pushing on the wavepacket acts on the propagation direction on both halves of each laser period. This off-axis recollision causes the dipole to oscillate in the propagation direction with an even symmetry. More formally, the dynamical symmetries of the problem permit even (but not odd) harmonics along this direction.) This is indeed what is observed:

```
Show[{
  spectrumPlotter[getSpectrum[nonDipoleContributions[[1 ;; -2, {1, 2}]], 
    Joined → False, PlotStyle → Black],
  spectrumPlotter[getSpectrum[nonDipoleContributions[[1 ;; -2, {3}]], 
    Joined → False, PlotStyle → Red]
}, PlotRange → All, ImageSize → 500]
```



Benchmarking the nondipole contributions

Nondipole contributions in a crossed-beam setup

This section explores the harmonic emission by a crossed-beam setup, with nondipole contributions, as a benchmarking step for the latter. The crossed-beam setup was proposed by X.-M. Tong and S.-I. Chu in *Phys. Rev. A* **58** no .4, R2656 (1998), and it was explored in a nondipole setting by V. Averbukh et al. in *Phys Rev. A* **65**, 063402 (2002). The results below reproduce those of Averbukh et al.

In short, we consider the harmonic emission by a circularly polarized pulse propagating along the z direction, at frequency ω , and a linearly polarized pulse of frequency $r \omega$ propagating along the x direction and polarized along the z direction.

```

crossedBeamsA[x_, z_] = Function[t,
  {
     $\frac{F1}{\omega} \cos[kz - \omega t], \frac{F1}{\omega} \sin[kz - \omega t], \frac{F2}{r\omega} \sin[rkx - r\omega t + \theta_0]$ 
  }
] [t] // MatrixForm
crossedBeamsGA[x_] = Function[t, Evaluate[{
  D[crossedBeamsA[x, z][t], x] /. {z → 0},
  D[crossedBeamsA[x, z][t], y] /. {z → 0},
  D[crossedBeamsA[x, z][t], z] /. {z → 0}
}]]
) [t] // MatrixForm

$$\begin{pmatrix} \frac{F1 \cos[kz - \omega t]}{\omega} \\ \frac{F1 \sin[kz - \omega t]}{\omega} \\ \frac{F2 \sin[krx + \theta_0 - rt\omega]}{r\omega} \end{pmatrix}$$


$$\begin{pmatrix} 0 & 0 & \frac{F2 k \cos[krx + \theta_0 - rt\omega]}{\omega} \\ 0 & 0 & 0 \\ \frac{F1 k \sin[t\omega]}{\omega} & \frac{F1 k \cos[t\omega]}{\omega} & 0 \end{pmatrix}$$


```

The dipole selection rules allow harmonic orders of the form $2r/\pm 1$, with $r=0, 1, 2, 3, \dots$, with polarization in the x, y plane, and harmonics of order $r(2r+1)$, with $r=0, 1, 2, 3, \dots$, polarized along the z direction.

```

allowedHarmonics[r_, {1, 2}] := Select[Union[2r Range[0, 100] + 1, 2r Range[0, 100] - 1], # > 0 &]
allowedHarmonics[r_, {3}] := r (2 Range[0, 100] + 1)

```

For the calculation, then, some preliminaries,

```

αRange = {0, 1/137};
nppcb = 240;
crossedBeamsParameters[rr_] := {F1 → 0.1, F2 → 0.2, ω → 0.057, θ0 → 0, r → rr, k → αω};
DistributeDefinitions["RBSFA`"];
SetSharedFunction[crossedBeamsResults];

```

and the calculation itself for $r=2$ and $r=5$.

```

Print[DateString[]]
ParallelTable[AbsoluteTiming[
  crossedBeamsResults[r, α] = makeDipoleList[
    VectorPotential → crossedBeamsA[0, 0], VectorPotentialGradient → crossedBeamsGA[0],
    FieldParameters → crossedBeamsParameters[r],
    DipoleTransitionMatrixElement → Function[{p, κ}, gaussianDTME[p, 1/1.3]],
    CarrierFrequency → 0.057, PointsPerCycle → nppcb
  ];
], {r, {2, 5}}, {α, αRange}]
Print[DateString[]]

```

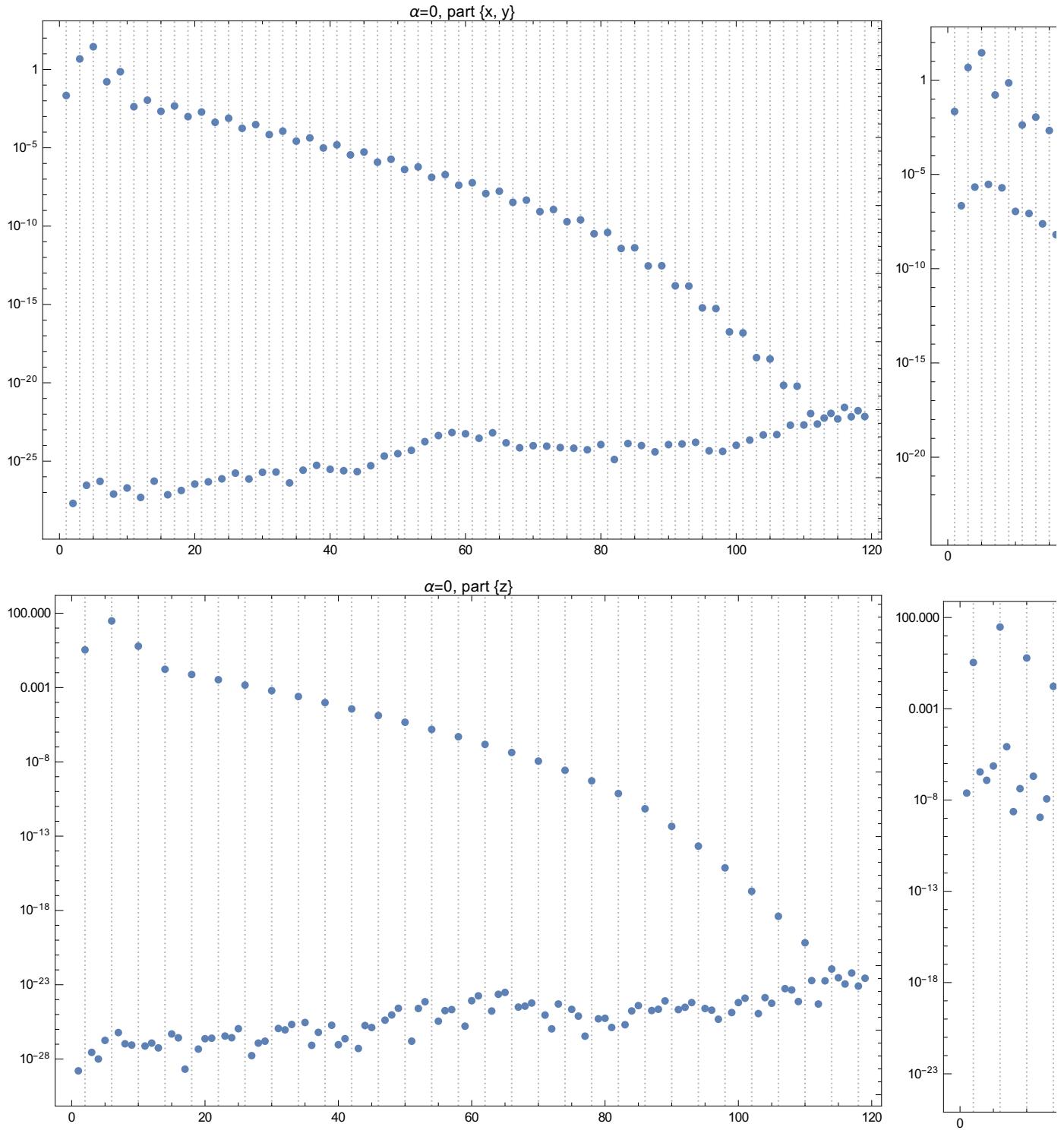
Mon 2 May 2016 15:26:02

$\{\{20.7392, \text{Null}\}, \{196.329, \text{Null}\}\}, \{\{20.962, \text{Null}\}, \{199.187, \text{Null}\}\}\}$

Mon 2 May 2016 15:29:44

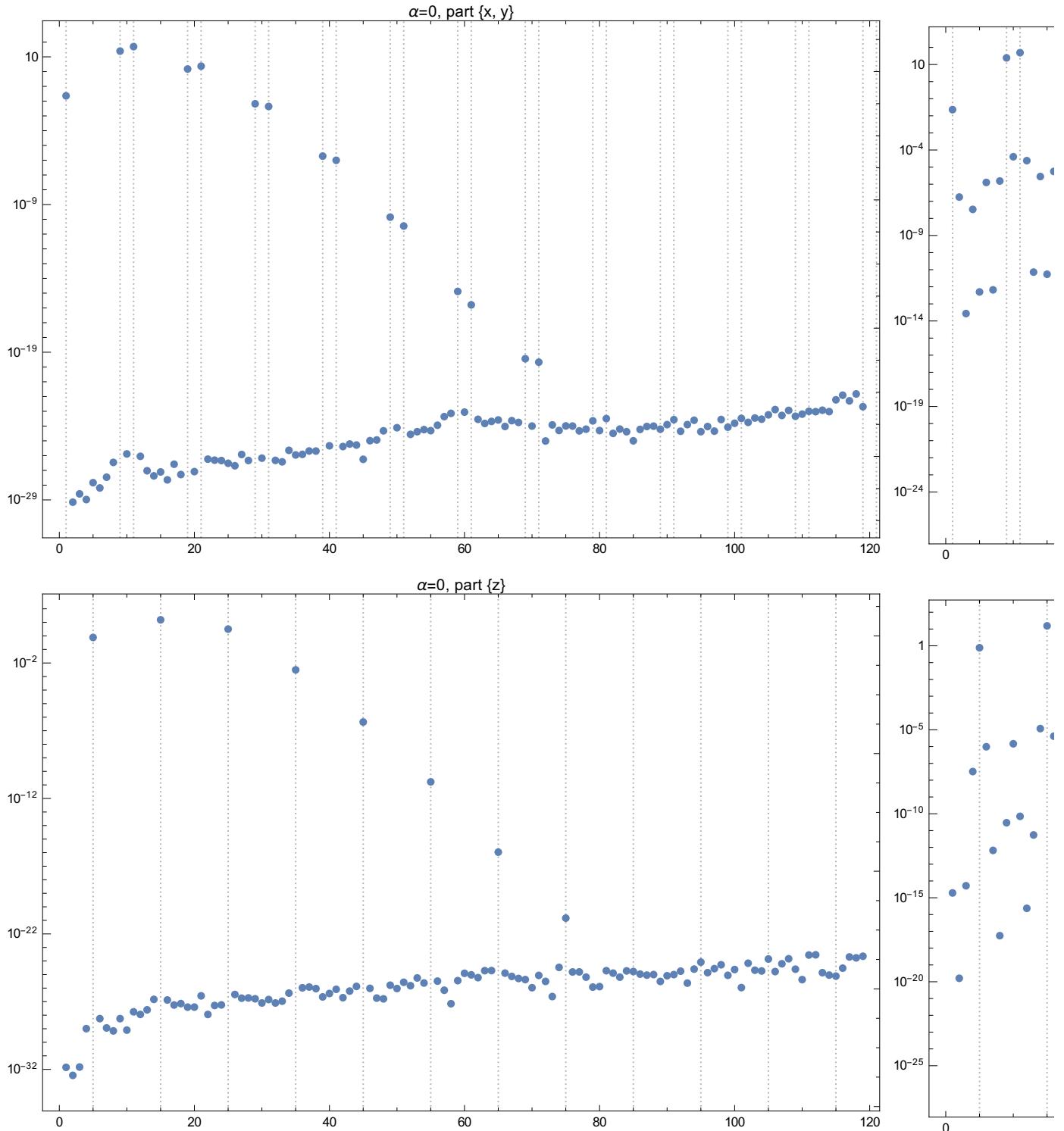
Results for $r=2$, comparable to Fig. 1 in Averbukh et al. Dashed lines mark the dipole-allowed harmonics. The left-hand column has nondipole contributions turned off ($\alpha=0$), and the right-hand column includes the nondipole contributions and observes a massive increase in the amplitude of the dipole-forbidden harmonics.

```
Grid[Table[
  ListLogPlot[
    getSpectrum[crossedBeamsResults[2,  $\alpha$ ][1 ;; -2, part], wPower  $\rightarrow$  2][[2 ;;]]
    , Joined  $\rightarrow$  False, ImageSize  $\rightarrow$  600, PlotTheme  $\rightarrow$  "Detailed", PlotRange  $\rightarrow$  Full
    , GridLines  $\rightarrow$  {allowedHarmonics[2, part], None}
    , PlotLabel  $\rightarrow$  Row[{ $\alpha$  = " $\alpha$ ",  $\alpha$ , ", part ", {"x", "y", "z"}[[part]]}]
  ], {part, {{1, 2}, {3}}}, { $\alpha$ ,  $\alpha$ Range}]]
```



Results for $r = 5$, comparable to Fig. 2 in Averbukh et al.

```
Grid[Table[
  ListLogPlot[
    getSpectrum[crossedBeamsResults[5,  $\alpha$ ][1 ;; -2, part], wPower  $\rightarrow$  2][[2 ;;]]
    , Joined  $\rightarrow$  False, ImageSize  $\rightarrow$  600, PlotTheme  $\rightarrow$  "Detailed", PlotRange  $\rightarrow$  Full
    , GridLines  $\rightarrow$  {allowedHarmonics[5, part], None}
    , PlotLabel  $\rightarrow$  Row[{ $\alpha$  = " $\alpha$ ",  $\alpha$ , ", part ", {"x", "y", "z"}[[part]]}]
  ], {part, {{1, 2}, {3}}}, { $\alpha$ ,  $\alpha$ Range}]]
```



Multiple plateaus in HHG in ions

This section benchmarks this code against the results of N.J. Kylstra et al. reported in *J. Phys B: At. Mol. Opt. Phys.* **34** no. 3, L55 (2001);. In particular, we study HHG in the He⁺ ion at high intensity ($I = 5.6 \times 10^{15} \text{ W cm}^{-2}$) and reasonable (800nm) wavelength.

```

kylstraA[z_] = Function[t, {F Sin[(\omega t - k z)/4]^2 Sin[\omega t - k z], 0, 0}][t] // MatrixForm
(kylstraGA = Function[t, Evaluate[{
    D[kylstraA[z][t], x] /. {z → 0},
    D[kylstraA[z][t], y] /. {z → 0},
    D[kylstraA[z][t], z] /. {z → 0}
}]])
) [t] // MatrixForm

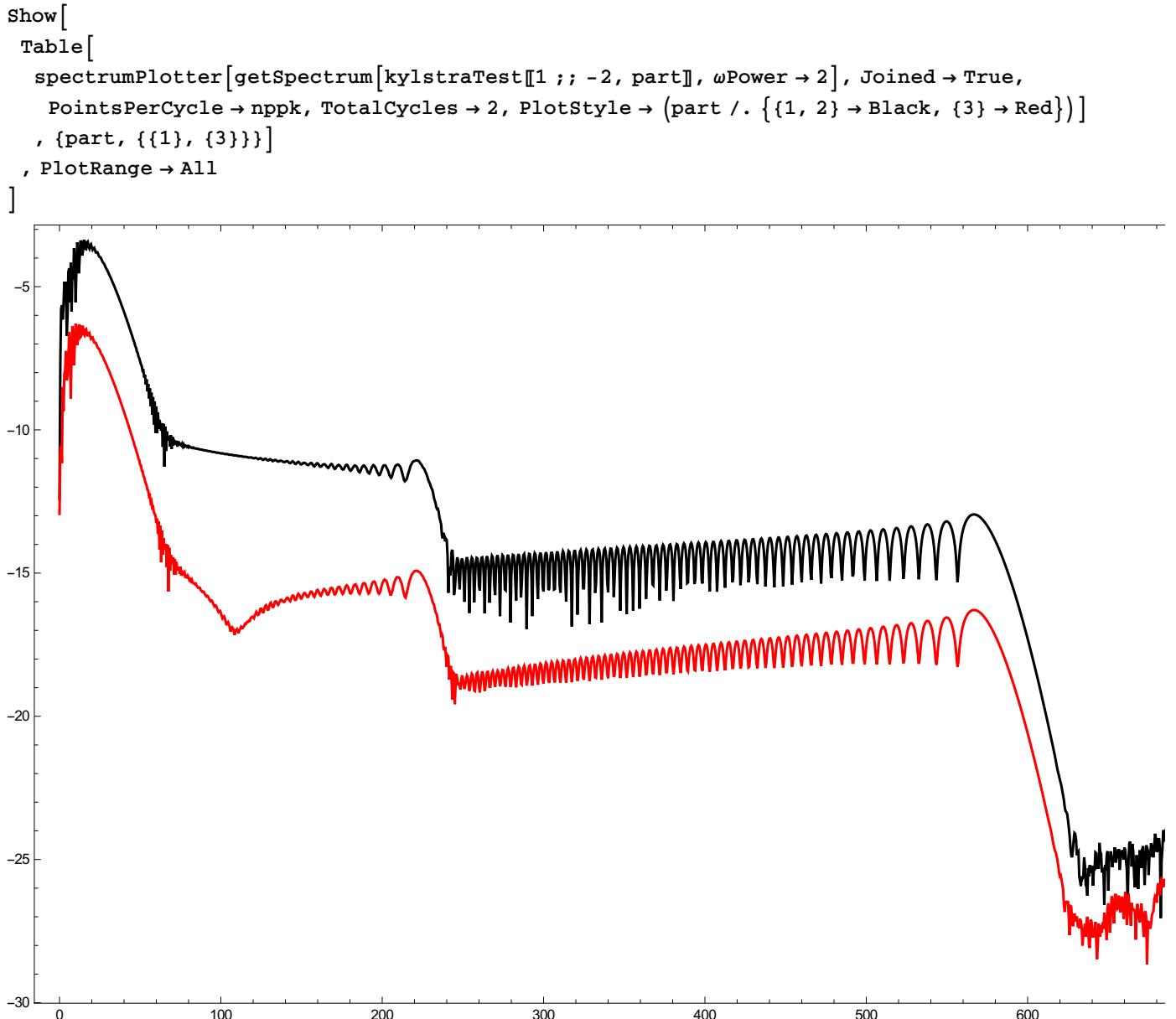
$$\left( \begin{array}{ccc} -\frac{F \sin[\frac{1}{4}(-k z+t \omega)]^2}{\omega} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)$$


$$\left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{F k \cos[t \omega] \sin[\frac{t \omega}{4}]^2}{\omega} - \frac{F k \cos[\frac{t \omega}{4}] \sin[\frac{t \omega}{4}] \sin[t \omega]}{2 \omega} & 0 & 0 \end{array} \right)$$

nppk = 1500;
DateString[]
AbsoluteTiming[
  kylstraTest = makeDipoleList[
    VectorPotential → kylstraA[0], VectorPotentialGradient → kylstraGA,
    FieldParameters → {F → Sqrt[5.6*10^15/10^14] 0.053, \omega → 0.057, k → \alpha \omega, \alpha → 1/137},
    IonizationPotential → 2,
    PointsPerCycle → nppk, TotalCycles → 2
  ];
]
DateString[]
Thu 1 Oct 2015 18:25:47
{1619.16, Null}
Thu 1 Oct 2015 18:52:46

```

Plotting the results. The x component (along the laser polarization) is in black, the z component (along the laser propagation) is in red.



The results are a good qualitative match to the dipoles reported by Kylstra et al., with the notable exception of the low-order harmonics below $n \leq 50$.

On the other hand, taken naively this code cannot be applied to the harder targets described in that paper (Li^{2+} and Be^{3+} , at intensities between 0.9 and $3.6 \times 10^{17} \text{ W cm}^{-2}$), which have cutoffs of order as high as 35 000, which requires several days to several months of calculation using the naive scaling. (That said, using a smarter choice of `ReportingFunction`, judicious use of parallelization and lots of waiting, those targets are probably within reach of this code.)

Periodicity of nondipole contributions

`Quit`

```

crossedBeamsA[t_] = First /@ Sum[
  
$$\frac{F}{\omega} \begin{pmatrix} \cos[\theta] \\ 0 \\ -s \sin[\theta] \end{pmatrix} \cos[k \{s \sin[\theta], 0, \cos[\theta]\}. \{x, y, z\} - \omega t - s \phi]$$

, {s, {-1, 1}}]
] // MatrixForm;

(crossedBeamsGA[t_] = Evaluate[{
  D[crossedBeamsA[t], x],
  D[crossedBeamsA[t], y],
  D[crossedBeamsA[t], z]
}]) // MatrixForm;

crossedBeamsParameters = {x → 0, y → 0, z → 0, F → Sqrt[0.5] 0.053, ω → 0.057, θ → 10. °, k → α ω};

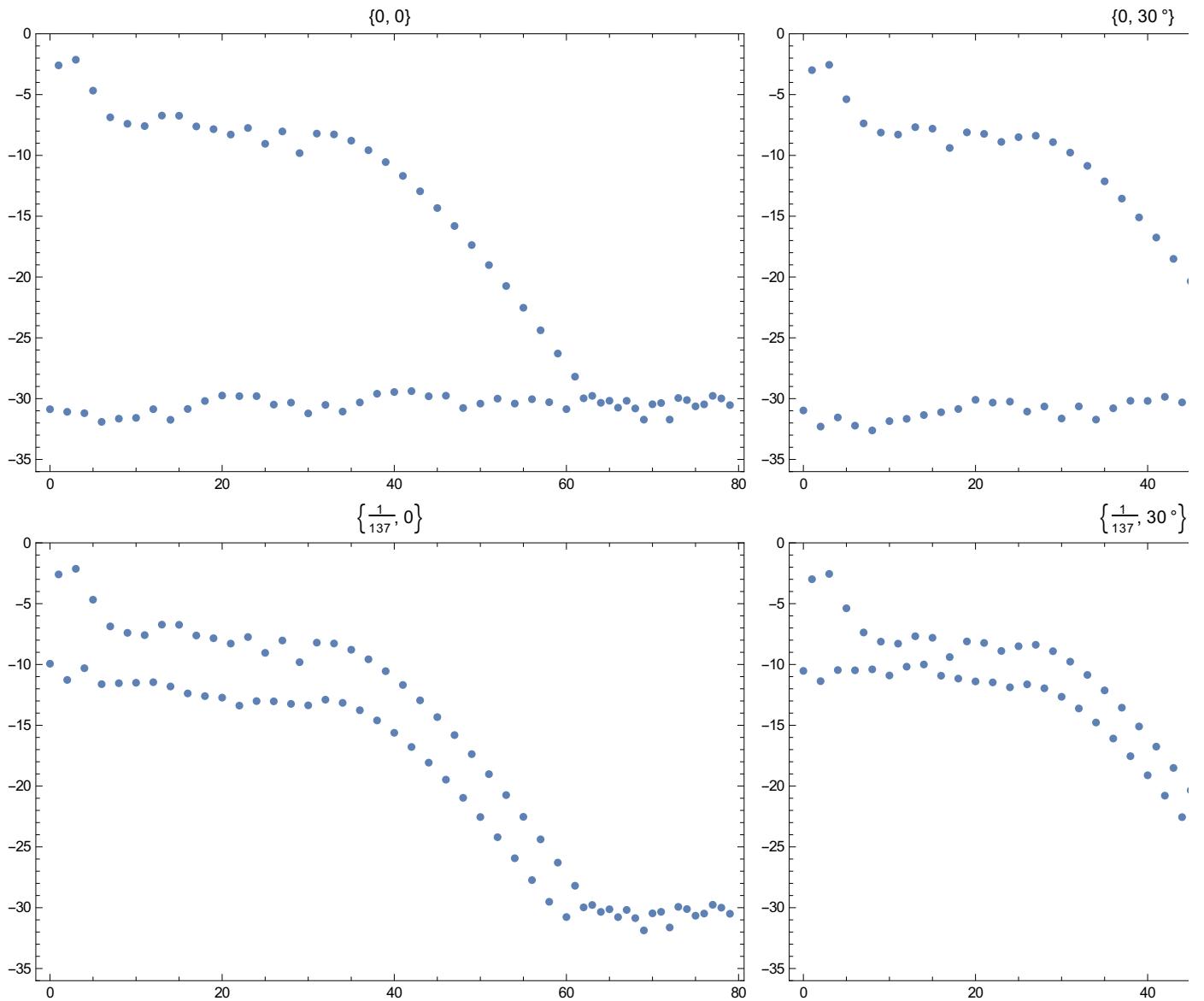
DateString[]
Table[
  First[AbsoluteTiming[
    symmetryTestDipole[α, φ] = makeDipoleList[
      VectorPotential → crossedBeamsA,
      VectorPotentialGradient → crossedBeamsGA, FieldParameters → crossedBeamsParameters,
      PointsPerCycle → 160
    ];
  ]]
, {α, {0, 1/137}}, {φ, {0, 30 °}}]
DateString[]

Mon 2 May 2016 16:23:09
{{8.75144, 9.80629}, {11.9277, 16.5271}]

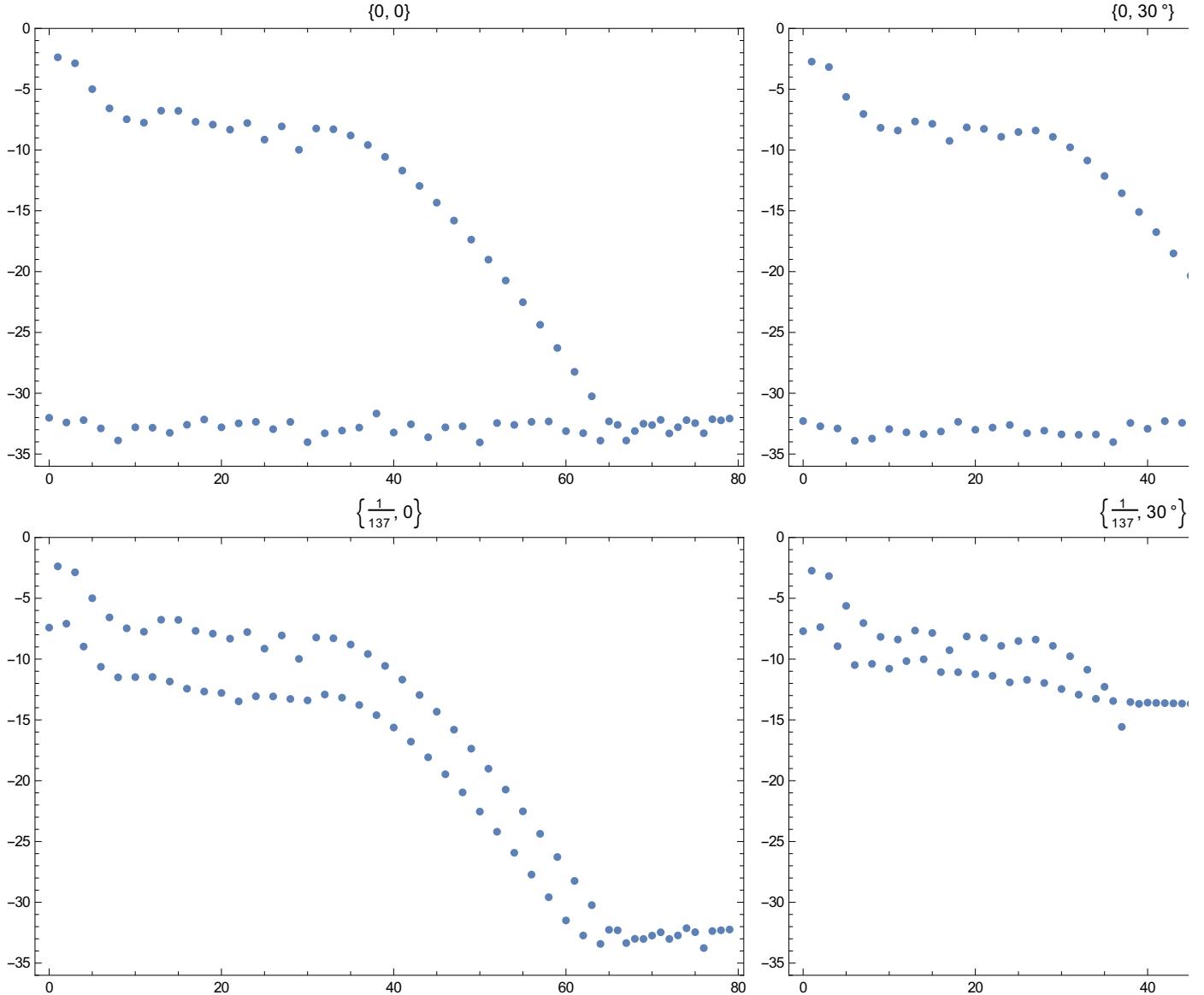
Mon 2 May 2016 16:23:56

```

```
Grid[Table[
  spectrumPlotter[
    getSpectrum[symmetryTestDipole[\alpha, \phi][1 ;; -2, {1, 2, 3}]],
    Joined → False, PointsPerCycle → 160,
    ImageSize → 450, PlotLabel → {\alpha, \phi}, PlotRange → {-36, 0}
  ]
, {\alpha, {0, 1/137}}, {\phi, {0, 30 °}}]]
```



Compare this with the unacceptably high noise floor from the previous version for the case $\alpha = 1/137$, $\phi = 30^\circ$.



Debugging and benchmarking tools

If something goes funny with your calls, then before you start taking `makeDipoleList` apart you can try using its `Verbose` option to diagnose the internal functions it is using. In particular:

- Setting `Verbose → 1` makes `makeDipoleList` print the information of the key internal functions it is using, before it goes on to the integration loop.

```
makeDipoleList[VectorPotential → Function[t, {F Sin[ω t], 0, 0}],  
FieldParameters → {F → 0.05, ω → 0.057}, Verbose → 1][1 ;; 10]
```

```
RBSFA`Private`A
```

```
RBSFA`Private`A[RBSFA`Private`t$_] = {0.877193 Sin[0.057 RBSFA`Private`t$], 0, 0}
```

```
RBSFA`Private`GA
```

```
RBSFA`Private`GA[RBSFA`Private`t$_] = {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
RBSFA`Private`ps
```

```
RBSFA`Private`ps[RBSFA`Private`t, RBSFA`Private`tt] =
{ -15.3894 Cos[0.057 RBSFA`Private`t]+15.3894 Cos[0.057 RBSFA`Private`tt],
  (0.-0.1 i)+RBSFA`Private`t-RBSFA`Private`tt, 0, 0}

RBSFA`Private`ps[RBSFA`Private`t_, RBSFA`Private`tt_] :=
RBSFA`Private`ps[RBSFA`Private`t, RBSFA`Private`tt] = -1
Inverse[IdentityMatrix[Length[RBSFA`Private`A[RBSFA`Private`tInit]]]] -
(RBSFA`Private`GAIntInt[RBSFA`Private`t, RBSFA`Private`tt] +
Transpose[RBSFA`Private`GAIntInt[RBSFA`Private`t, RBSFA`Private`tt]]) /
(RBSFA`Private`t - RBSFA`Private`tt - i RBSFA`Private`e).
(RBSFA`Private`AInt[RBSFA`Private`t, RBSFA`Private`tt] -
RBSFA`Private`bigPScorrectionInt[RBSFA`Private`t, RBSFA`Private`tt])
```

```
RBSFA`Private`pi
```

```
RBSFA`Private`pi[RBSFA`Private`p_, RBSFA`Private`t_, RBSFA`Private`tt_] :=
RBSFA`Private`p + RBSFA`Private`A[RBSFA`Private`t] -
RBSFA`Private`GAInt[RBSFA`Private`t, RBSFA`Private`tt].RBSFA`Private`p -
RBSFA`Private`GAdotAInt[RBSFA`Private`t, RBSFA`Private`tt]
```

```
RBSFA`Private`S
```

```
RBSFA`Private`S[RBSFA`Private`t_, RBSFA`Private`tt_] =
(-15.3894 Cos[0.057 RBSFA`Private`t]+15.3894 Cos[0.057 RBSFA`Private`tt])^2
((0.-0.1 i)+RBSFA`Private`t-RBSFA`Private`tt) + 1/2 (RBSFA`Private`t - RBSFA`Private`tt)
(1. + (-15.3894 Cos[0.057 RBSFA`Private`t] + 15.3894 Cos[0.057 RBSFA`Private`tt]))^2 /
((0.-0.1 i)+RBSFA`Private`t - RBSFA`Private`tt)^2) +
1/2 (0.769468 (0.5 RBSFA`Private`t - 4.38596 Sin[0.114 RBSFA`Private`t]) -
0.769468 (0.5 RBSFA`Private`tt - 4.38596 Sin[0.114 RBSFA`Private`tt]))
```

(abridged.)

```
{{-0.0181258 - 0.0000997423 i, 0. + 0. i, 0. + 0. i}, {-0.018315 - 1.44441 i, 0. + 0. i, 0. + 0. i},
{-0.0180896 - 5.34162 i, 0. + 0. i, 0. + 0. i}, {-0.0171312 - 10.575 i, 0. + 0. i, 0. + 0. i},
{-0.0153823 - 15.8078 i, 0. + 0. i, 0. + 0. i}, {-0.013213 - 19.9497 i, 0. + 0. i, 0. + 0. i},
{-0.0111113 - 22.4178 i, 0. + 0. i, 0. + 0. i}, {-0.0091716 - 23.1444 i, 0. + 0. i, 0. + 0. i},
{-0.00705106 - 22.4188 i, 0. + 0. i, 0. + 0. i}, {-0.00454356 - 20.6872 i, 0. + 0. i, 0. + 0. i}}
```

- Setting Verbose→2 makes makeDipoleList output its key internal functions and shut down before the integration takes place. Its results can be caught as follows:

```

{A[t_], GA[t_], ps[t_, tt_], pi[p_, t_], S[t_, tt_], AInt[t_], AInt[t_, tt_], A2Int[t_],
 A2Int[t_, tt_], GAIInt[t_], GAIInt[t_, tt_], GAdotAInt[t_], GAdotAInt[t_, tt_], AdotGAInt[t_],
 AdotGAInt[t_, tt_], GAIIntInt[t_], GAIIntInt[t_, tt_], bigPScorrectionInt[t_],
 bigPScorrectionInt[t_, tt_], AdotGAdotAInt[t_], AdotGAdotAInt[t_, tt_], integrand[t_, τ_]
} = makeDipoleList[VectorPotential → Function[t, {F Sin[ω t], 0, 0}],
 FieldParameters → {F → 0.05, ω → 0.057}, Verbose → 2];

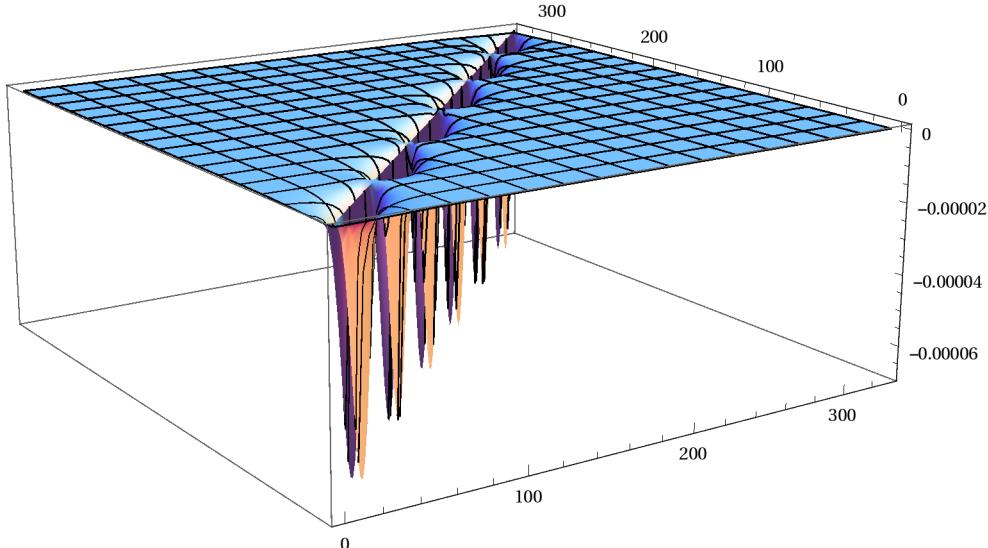
```

This then enables examination of e.g. the action:

```

Block[{ω = 0.057},
 Plot3D[
 Im[S[t, tt]]
 , {t, 0, 3 2 π / ω}, {tt, 0, 3 2 π / ω}
 , PlotRange → Full, ImageSize → 600, PlotTheme → "Classic", PlotPoints → 100
 ]
]

```



See the implementation notes in the code for `makeDipoleList` for further definitions of what each term entails.

- Setting `Verbose→3` makes `makeDipoleList` return as pure functions the prefactor and action of the SFA integrand, as functions of the recombination time t and the ionization time $t' = tt$, which can then be used for e.g. quantum orbit calculations.

Note that the precise name of this setting is subject to change in future versions.

```

Block[{prefactor, S},
{prefactor, S} = makeDipoleList[VectorPotential → Function[t, {F/ω Sin[ω t], 0, 0}],
FieldParameters → {F → 0.05, ω → 0.057}, Verbose → 3];
Simplify[prefactor[t, tt] Exp[-i S[t, tt]]]
]
{- \left( \left( (0. + 10.2129 i) e^{(0.-0.192367 i) t + (0.+0.192367 i) tt - \frac{1}{2} i (t-tt)} \left( 1. + \frac{(15.3894 \cos[0.057 t] - 15.3894 \cos[0.057 tt])^2}{((0.-0.1 i)+t-1. tt)^2} \right) + \frac{i (15.3894 \cos[0.057 t] - 15.3894 \cos[0.057 tt])^2}{(0.-0.1 i)+t-1. tt} + (0. - 0.192367 i) t + (0.+0.192367 i) tt - \frac{1}{2} i (t-tt) \right) \left( 1. + \frac{(15.3894 \cos[0.057 t] - 15.3894 \cos[0.057 tt])^2}{((0.-0.1 i)+t-1. tt)^2} \right) + 0.877193 \sin[0.057 t] \right) \\
\left( \frac{15.3894 \cos[0.057 t] - 15.3894 \cos[0.057 tt]}{(0.-0.1 i)+t-1. tt} + 0.877193 \sin[0.057 tt] \right) \\
\left( \frac{15.3894 \cos[0.057 t] - 15.3894 \cos[0.057 tt]}{(0.-0.1 i)+t-1. tt} + 0.877193 \sin[0.057 tt] \right) \right) / \\
\left( \left( 1. + ((15.3894 \cos[0.057 t] - 15.3894 \cos[0.057 tt]) / ((0.-0.1 i)+t-1. tt) + 0.877193 \sin[0.057 t])^2 \right)^3 \right. \\
\left. \left( 1. + ((15.3894 \cos[0.057 t] - 15.3894 \cos[0.057 tt]) / ((0.-0.1 i)+t-1. tt) + 0.877193 \sin[0.057 tt])^2 \right)^3 \right), 0, 0 \}

```

In this case it is not necessary for the vector potential to return numeric values, so a purely symbolic vector potential can be used.

```

Block[{prefactor, S},
{prefactor, S} =
makeDipoleList[VectorPotential → Function[t, {F/ω Sin[ω t], 0, 0}], Verbose → 3];
Simplify[prefactor[t, tt] Exp[-i S[t, tt]]]
]
{- \left( \left( (0. + 204.258 i) e^{\frac{1}{8} i (-4 (t-tt))} \left( 1. + \frac{F^2 (\cos[t \omega] - 1. \cos[tt \omega])^2}{((0.-0.1 i)+t-1. tt)^2 \omega^4} + \frac{8 F^2 (\cos[t \omega] - 1. \cos[tt \omega])^2}{((0.-0.1 i)+t-1. tt) \omega^4} + \frac{F^2 (-2 t \omega + \sin[2 t \omega])}{\omega^3} + \frac{F^2 (2 tt \omega - \sin[2 tt \omega])}{\omega^3} \right) F^3 \right. \\
\left( \frac{1}{0.1 + i (t - tt)} \right)^{3/2} \cos[tt \omega] (\cos[t \omega] - 1. \cos[tt \omega] + ((0.-0.1 i)+t-1. tt) \omega \sin[t \omega]) \\
(\cos[t \omega] - 1. \cos[tt \omega] + ((0.-0.1 i)+t-1. tt) \omega \sin[tt \omega]) \right) / ((0.-0.1 i)+t-1. tt)^2 \\
\omega^4 \left( 1. + \left( F^2 (\cos[t \omega] - 1. \cos[tt \omega] + ((0.-0.1 i)+t-1. tt) \omega \sin[t \omega])^2 \right) / \left( ((0.-0.1 i)+t-1. tt)^2 \omega^4 \right)^3 \right. \\
\left. \left( 1. + \left( F^2 (\cos[t \omega] - 1. \cos[tt \omega] + ((0.-0.1 i)+t-1. tt) \omega \sin[tt \omega])^2 \right) / \left( ((0.-0.1 i)+t-1. tt)^2 \omega^4 \right)^3 \right) \right), 0, 0 \}

```

It is also important to note, particularly if these functions are used for quantum orbits calculations, that the action does *not* include the Fourier-transform factor of $e^{-i\Omega t}$, which affects both the phase and the amplitude of the harmonic dipole when evaluated at a complex saddle-point time $t = t_s$.

Bicircular fields

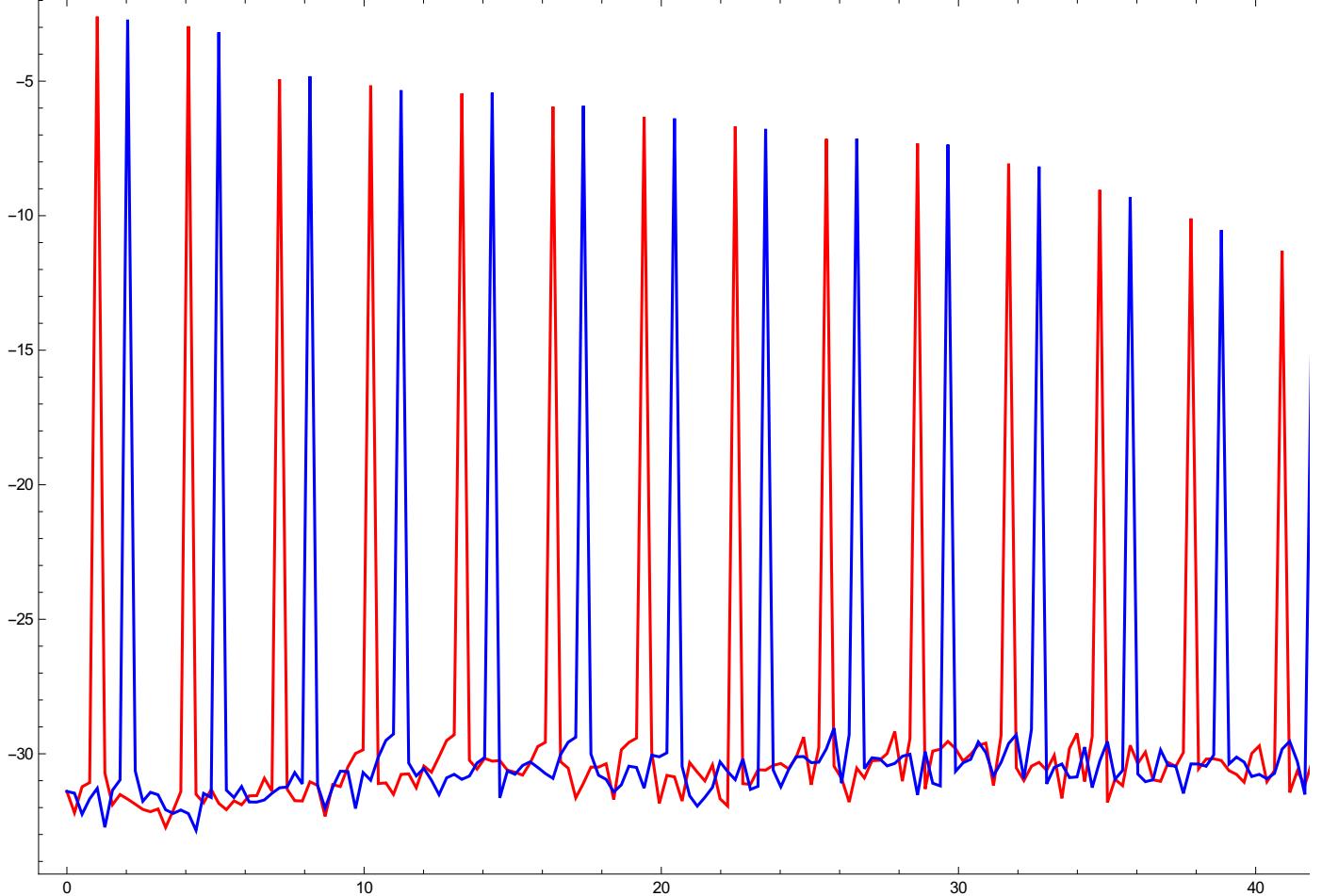
As a slightly less trivial example, consider a bicircular field: two counter-rotating, circularly polarized fields of different frequencies. The ‘standard’ case - as first demonstrated experimentally - has one field as the second harmonic of the fundamental, with both at equal intensities. The resultant harmonics appear at all integer orders except those divisible by three, with the $3n + 1$ harmonics polarized as the fundamental, and the $3n - 1$ harmonics polarized as the second-harmonic driver.

Quit

```
bicircularA[t_] :=  
  
$$\left( \frac{F1}{\omega_1} \{ \cos[t \omega_1] \sin[\alpha], -\cos[\alpha] \sin[t \omega_1] \} + \frac{F2}{\omega_2} \{ \cos[\beta] \cos[\omega_2 t], \sin[\beta] \sin[\omega_2 t] \} \right)$$
  
bicircularParameters = {F1 → 0.075, F2 → 0.075, α → 45°, β → 45°, ω1 → 45.6 / 800, ω2 → 45.6 / 400};  
AbsoluteTiming[bicircularTest = makeDipoleList[VectorPotential → bicircularA,  
  FieldParameters → bicircularParameters, TotalCycles → 4];]  
{10.7277, Null}
```

The function `biColorSpectrum` takes the spectrum and plots it, separating the two circular polarizations into different colours.

```
biColorSpectrum[Most[bicircularTest]]
```



Bicircular fields with a sine-squared envelope

To benchmark the original calculations, we compared them with the output of full MCTDH calculations. Here we used a \sin^2 envelope as the TDSE numerics require a finite pulse; the calculations take correspondingly longer but they are still very manageable (two/three minutes per calculation for a fifteen-cycle pulse, resolving up to ~70 har-

monics). One distinctive feature is that the harmonics near the cutoff are broader, because less cycles contribute to those energies.

```
bicircularEnvelopeA[t_] := cosPowerFlatTop[\omega1, TotalCycles, 2][t]

$$\left( \frac{F1}{\omega1} \{ \cos[t \omega1] \sin[\alpha], -\cos[\alpha] \sin[t \omega1] \} + \frac{F2}{\omega2} \{ \cos[\beta] \cos[\omega2 t], \sin[\beta] \sin[\omega2 t] \} \right);$$

bicircularParameters = {F1 → 0.075, F2 → 0.075, α → 45 °, β → 45 °, ω1 → 45.6 / 800, ω2 → 45.6 / 400};
```

If (as in this case) the field depends on a number-of-cycles parameter, care must be taken that it matches the `num` option of the main call.

```
AbsoluteTiming[
  bicircularEnvelopeDipole =
    makeDipoleList[VectorPotential → bicircularEnvelopeA, FieldParameters →
      Join[bicircularParameters, {TotalCycles → 15}], PointsPerCycle → 150, TotalCycles → 15];
]
{173.565, Null}
```

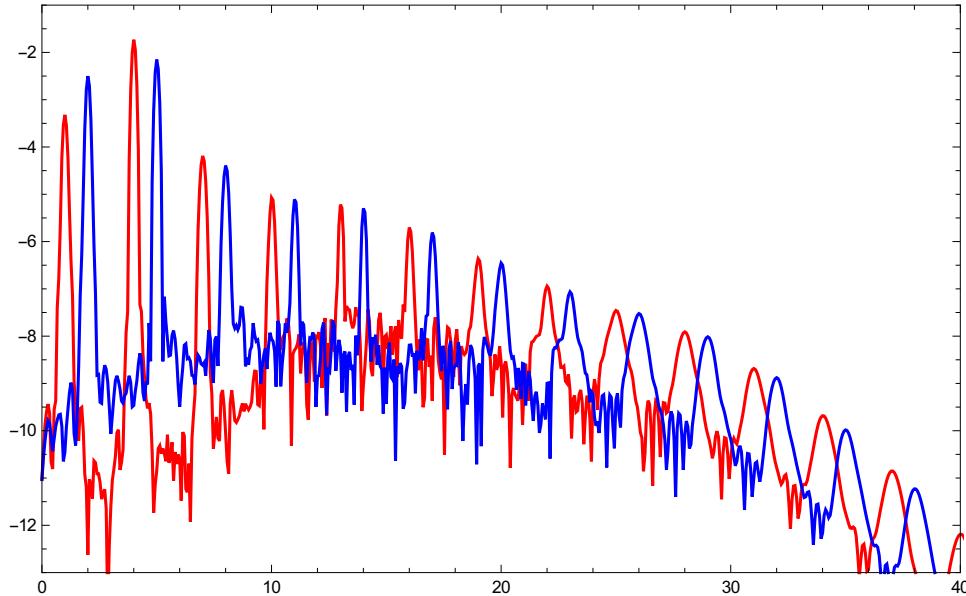
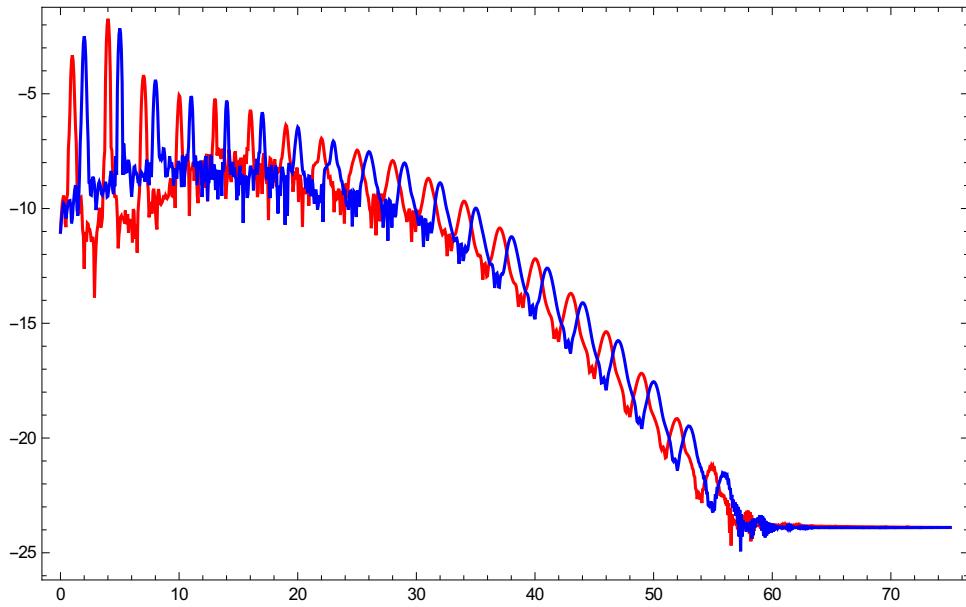
```
{141.302, Null}
```

Plotting the spectrum, and a zoom at the plateau:

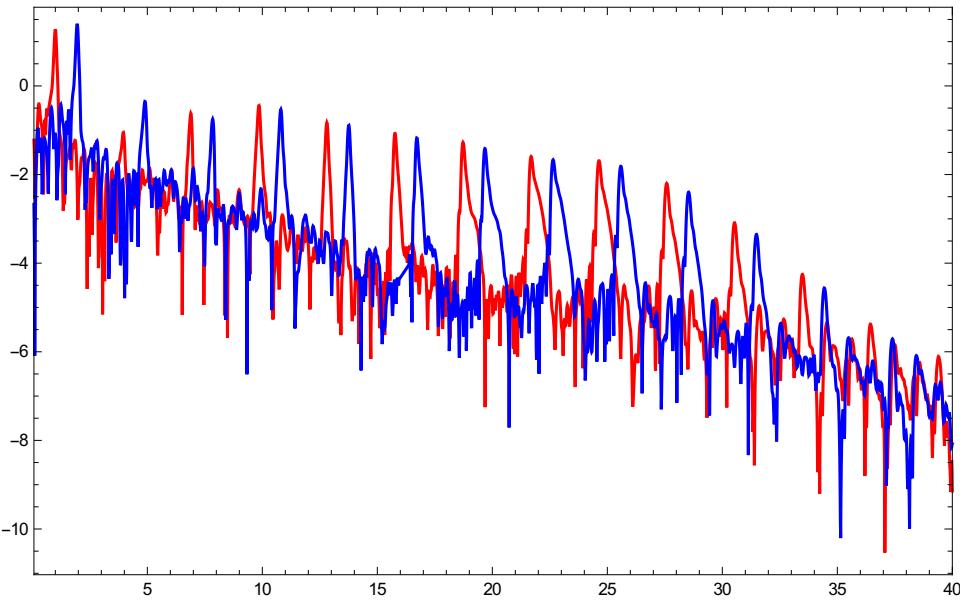
```

biColorSpectrum[bicircularEnvelopeDipole,
  PointsPerCycle → 150, TotalCycles → 15, ImageSize → 500]
biColorSpectrum[bicircularEnvelopeDipole, PointsPerCycle → 150,
  TotalCycles → 15, ImageSize → 500, PlotRange → {{0, 40}, {-13, -1}}]

```



The comparable MCTDH spectrum, for identical conditions, looks like this:



Original RB-SFA: 'rotating' bicircular fields

Calculation

Here the fundamental laser driver has been set at an elliptical polarization (as in the original experiment, A. Fleischner et al., *Nature Photon.* **8**, 543 (2014)), which helps investigate the spin-angular-momentum conservation properties of HHG. In the model proposed in the original paper (*Phys. Rev. A* **90**, 043829 (2014)), the photon model is validated by splitting the elliptical field itself into two circular components, which can then be tuned independently:

$$\text{rotatingBicircularA}[t_] := \text{envelope}[t] \left(\frac{F2}{\omega2} \{ \cos[\beta] \cos[\omega2 t - \phi1], \sin[\beta] \sin[\omega2 t - \phi1] \} + \frac{F1}{\sqrt{2}} \left(\frac{1}{\omega1} \cos\left[\alpha - \frac{\pi}{4}\right] \{ \cos[\omega1 t + \phi1], -\sin[\omega1 t + \phi1] \} + \frac{1}{(1+\delta) \omega1} \sin\left[\alpha - \frac{\pi}{4}\right] \{ \cos[(1+\delta) \omega1 t - \phi1 + \phi2], +\sin[(1+\delta) \omega1 t - \phi1 + \phi2] \} \right) \right),$$

```
DistributeDefinitions["RBSFA`"];
directory = FileNameJoin[{NotebookDirectory[], "Temp Data"}];
filename[δ_] :=
  FileNameJoin[{directory, "data 25.09 detuning scan at δ=" <> ToString[δ] <> ".txt"}];
Length[δRange = Range[0, 0.25, 0.001]]
```

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To test the validity of the photon model, we ran a scan over the detuning δ , using the calculation below.

```

DateString[]
Print["Total = ", Length[δRange], " points at ~230s/point will be done at approximately ",
DateString[AbsoluteTime[] + Length[δRange] * 230. / 7], "."]
ParallelTable[
Print[AbsoluteTiming[
makeDipoleList[
VectorPotential → rotatingBicircularA,
FieldParameters → {α → 35 °, β → 45 °, F1 → 0.075, F2 → 0.075, ω1 → 0.057,
ω2 → 1.95 × 0.057, φ1 → 0, φ2 → 0, envelope → flatTopEnvelope[ω1, 26, 3]}, CarrierFrequency → 0.057, TotalCycles → 26, PointsPerCycle → 115,
nGate → 1.8, PointNumberCorrection → 1, Preintegrals → "Numeric",
ReportingFunction → Function[Write[filename[δ], #]]]
];]];
Print[DateString[]];
, {δ, δRange}];
DateString[]
NotebookSave[]

```

Total time 2h 32min. (Desktop machine with 8-thread, 4-core Intel i7-3770 CPU at 3.40GHz, 16GB RAB, running 7 *Mathematica* kernels in parallel.)
 Expand this cell to see the calculation log.

The results can be pulled in from the files using this:

```
Do[detunedDipole[δ] = ReadList[filename[δ]], {δ, δRange}]
```

Or saved into a single location using this:

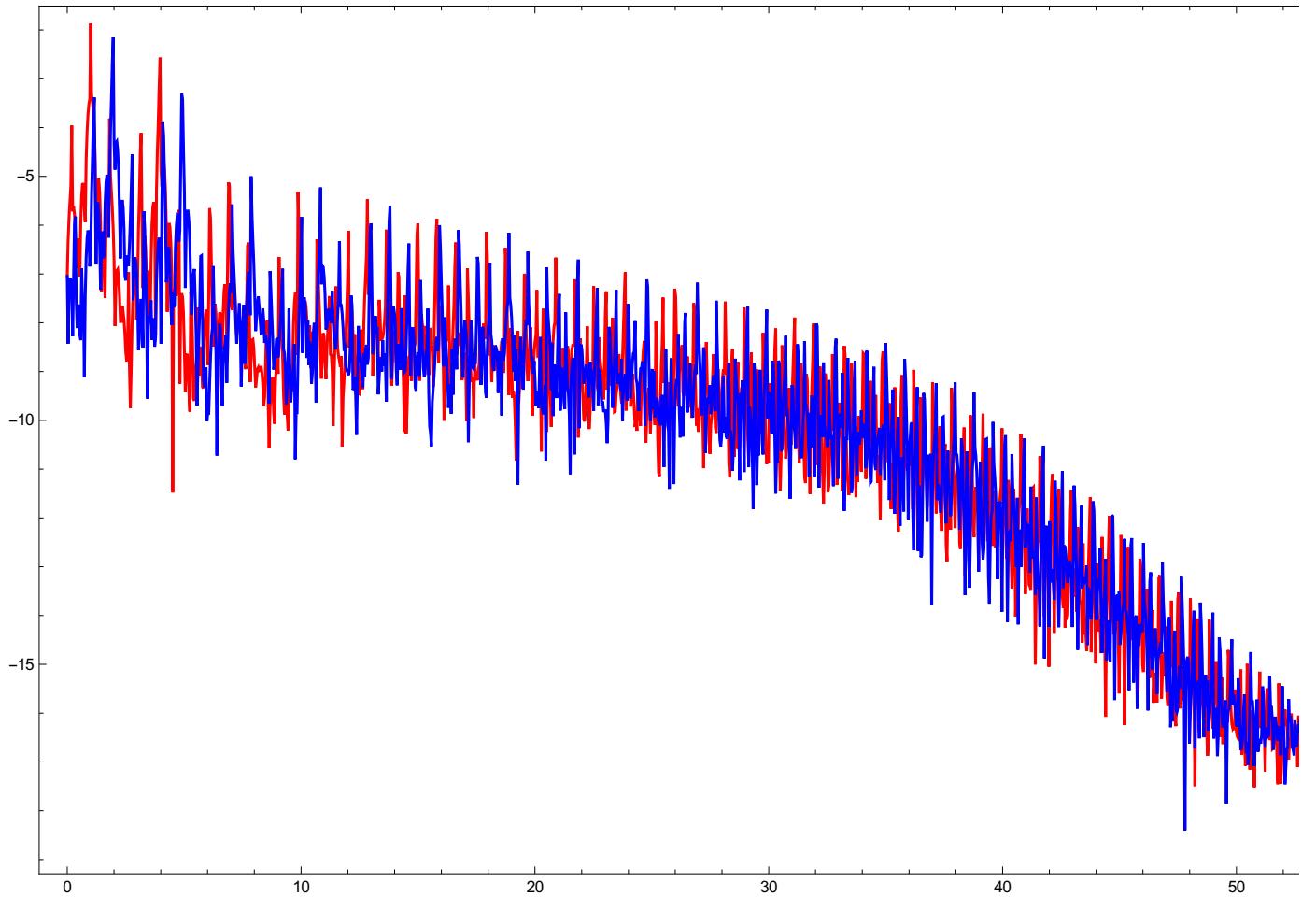
```
Save[FileNameJoin[{NotebookDirectory[], "Detuning scan collected data.txt"}], detunedDipole]
DumpSave[
FileNameJoin[{NotebookDirectory[], "Detuning scan collected data.mx"}], detunedDipole];
```

and pulled in from the single location using this:

```
<< (FileNameJoin[{NotebookDirectory[], "Detuning scan collected data.txt"}]);
```

A sample spectrum looks like this:

```
biColorSpectrum[detunedDipole[0.001 RandomInteger[{0, 1000}]],
CarrierFrequency → 45.6 / 800, TotalCycles → 3, PointsPerCycle → 115]
```



Plots from the original paper

The plots from the original paper were produced using the code below. For simplicity we pre-define an interpolation function.

```
conditions := Sequence[CarrierFrequency → 45.6 / 800, TotalCycles → 26, PointsPerCycle → 115]
```

```

Remove[detuningInterpolation]
With[{length = Length[getSpectrum[detunedDipole[0.], Polarization -> {1, i}]]},
  AbsoluteTiming[
    Table[
      detuningInterpolation[ $\epsilon$ ] = Interpolation[
        Flatten[Table[
          {{harmonicOrderAxis[TargetLength -> length, conditions],
            Table[ $\delta$ , {length}]}
          }T,
          Log[10, getSpectrum[detunedDipole[ $\delta$ ], Polarization -> {1,  $\epsilon$  i}]]
          }T,
          , { $\delta$ ,  $\delta$ Range}], 1]
        , { $\epsilon$ , {1, -1}}];
    ]
  ]
{2.99829, Null}

```

Some plotting admin:

```

CMRmap = Function[x, Blend[{█, █, █, █, █, █, █, █}, x]];
CMRwithMin[minIn_, minOut_: 1./9] :=
  Function[x, CMRmap[If[x < minIn,  $\frac{\text{minOut}}{\text{minIn}}$  x, minOut + (1 - minOut)  $\frac{x - \text{minIn}}{1 - \text{minIn}}$ ]]];
min =  $6 \times 10^{-9}$ ;
max =  $5 \times 10^{-7}$ ;
colorfunction = CMRwithMin[min/max];
HOTicks[ $\epsilon$ _] :=
  ({#, If[ $\epsilon$  == 1, Style[#, Black], ""], {0.02, 0}, {Thickness[0.005], Gray}} & /@ Range[12, 18, 1]) ~
  Join ~ ({#, "", {0.01, 0}, {Thickness[0.004], Gray}} & /@ Range[ $11 + \frac{1}{2}$ ,  $18 + \frac{1}{4}$ , 1/4])
downTicks = {{0, Style[0, Black], 0}, {0.25, Style[0.25, Black], 0}} ~Join ~
  ({#, Style[#, Black], {0.015, 0}, {Thickness[0.005], Gray}} & /@ Range[0.05, 0.20, 0.05]) ~
  Join ~ ({#, "", {0.01, 0}, {Thickness[0.004], Gray}} & /@ Range[0.01, 0.24, 0.01]);
upTicks = ({#, "", {0.015, 0}, {Thickness[0.005], Gray}} & /@ Range[0.05, 0.20, 0.05]) ~
  Join ~ ({#, "", {0.01, 0}, {Thickness[0.004], Gray}} & /@ Range[0.01, 0.24, 0.01]);

```

The plot itself:

```

Row[Table[
  splittingsScan[ $\epsilon$ ] = RegionPlot[
    True
    , { $\delta$ , 0, 0.25}, {HO, 11.25, 18.5}
    , AspectRatio -> 1.2
    , PlotRangePadding -> None
    , ImagePadding -> 1 {{35 + 15  $\epsilon$ , 20}, {70, 6}}
    , ImageSize -> {Automatic, 550}
    , PlotPoints -> 600
    , FrameStyle -> Automatic
    ,
    FrameLabel -> {Style[" $\frac{\omega'}{\omega} - 1$ ", Black, 12], If[ $\epsilon == 1$ , Style["Harmonic Order", Black, 16], ""]}
    , ColorFunctionScaling -> False
    , FrameTicks -> {{HOTicks[1], HOTicks[-1]}, {downTicks, upTicks}}
    , ColorFunction -> Function[{ $\delta$ , HO}, colorfunction[ $\frac{10^{\detuningInterpolation[\epsilon][HO, \delta]}}{\max}$ ]]
    , PlotLabel ->
      Style[StringJoin[ $\epsilon /. \{1 \rightarrow \text{Right}, -1 \rightarrow \text{Left}\}$ , "-circular harmonics"], Black, 16]
  ]
  , { $\epsilon$ , {1, -1}}
]
]

```

(Removed to keep file size low.)

Quantum-orbit functionality (experimental)

The following functions are experimental and should not be considered fixed at this stage. The function usage messages are reasonably well-documented, and the examples below show a reasonably complete use case from getting the action through to calculating a spectrum, but they are not at present fully documented.

Getting the action and prefactor

```

Quit

parameters =
{F -> Sqrt[int 0.053], w -> 45.6 /  $\lambda$ , int -> 1,  $\lambda$  -> 800, Ip -> getIonizationPotential["Helium", 0]};

```

```

AbsoluteTiming[
{prefactor, s} = makeDipoleList[
  VectorPotential → Function[t, {0, 0, F ω Cos[ω t]}], FieldParameters → parameters
  , CarrierFrequency → (ω /. parameters), IonizationPotential → (Ip /. parameters)
  , DipoleTransitionMatrixElement → {hydrogenicDTMERegularized, hydrogenicDTME}
  , Verbose → 3
  , Simplifier → Simplify
]
]

{0.094035, {Function[{t, tt}, {(0. + 0. i) (1/(0.1 + i (t - tt)))^(3/2)
  (0.929825 Cos[0.057 tt] - (16.3127 Sin[0.057 t] - 16.3127 Sin[0.057 tt])/(0. - 0.1 i) + t - tt
  Sin[0.057 tt], (0. + 0. i) (1/(0.1 + i (t - tt)))^(3/2)
  (0.929825 Cos[0.057 tt] - (16.3127 Sin[0.057 t] - 16.3127 Sin[0.057 tt])/(0. - 0.1 i) + t - tt
  Sin[0.057 tt], ((0. + 47.5253 i) (1/(0.1 + i (t - tt)))^(3/2)
  (0.929825 Cos[0.057 t] - (16.3127 Sin[0.057 t] - 16.3127 Sin[0.057 tt])/(0. - 0.1 i) + t - tt
  Sin[0.057 tt] - (16.3127 Sin[0.057 t] - 16.3127 Sin[0.057 tt])/(0. - 0.1 i) + t - tt
  Sin[0.057 tt])]/((1.8071 + (0.929825 Cos[0.057 t] - (16.3127 Sin[0.057 t] - 16.3127 Sin[0.057 tt]))/
  ((0. - 0.1 i) + t - tt))^2)]},
Function[{t, tt}, (1/2 (0.432287 t - 0.432287 tt + 3.79199 Sin[0.114 t] +
  (t - tt) (1.8071 + (16.3127 Sin[0.057 t] - 16.3127 Sin[0.057 tt]))^2/((0. - 0.1 i) + t - 1. tt)^2) -
  532.209 (1. Sin[0.057 t] - 1. Sin[0.057 tt])^2/(0. - 0.1 i) + t - 1. tt)]]}]
]

$HistoryLength = 10;
LaunchKernels[16 - Length[Kernels[]]];

```

Getting saddle points

```

DateString[]
Block[{ω, Ip, κ, U, γ},
{ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω}/. parameters;
ΩRange = Range[Ceiling[Ip, ω], Ceiling[1.32 Ip + 3.17 U] + 5 ω, 0.1 ω];

saddlePoints = GetSaddlePoints[ΩRange, S, {
  {{(0 ° - 1.5 i γ)/ω, (450 ° + 1.5 i γ)/ω}, {(-90 ° + 0.6 i γ)/ω, (0 ° + 1.2 i γ)/ω}},
  {{(180 ° + 0 ° - 1.5 i γ)/ω, (180 ° + 450 ° + 1.5 i γ)/ω}, {(180 ° + (-90 ° + 0.6 i γ)/ω, (180 ° + 0 ° + 1.2 i γ)/ω)}},
  , IndependentVariables → {"t", "tt"}
  , Tolerance → 10^-5/ω, Seeds → 75
  , Jacobian → FiniteDifference
  }
]
[[1 ;; 3]] // AbsoluteTiming
DateString[]

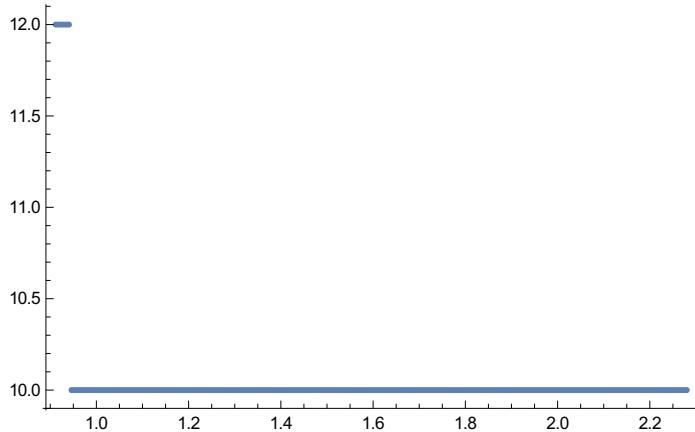
Wed 11 May 2016 17:17:56

{13.0059,
<| 0.912 → {{63.7226 - 14.708 i, 27.8501 - 40.9741 i}, {8.60697 - 14.708 i, 27.8501 - 40.9741 i},
{12.2456 - 11.286 i, 31.4941 - 37.5465 i}, {67.3613 - 11.286 i, 31.4941 - 37.5465 i},
{137.474 + 0.11658 i, 108.706 - 20.4102 i}, {82.3584 + 0.11658 i, 108.706 - 20.4102 i},
{87.3152 + 0.120837 i, 113.662 - 20.406 i}, {142.431 + 0.120837 i, 113.662 - 20.406 i},
{129.795 - 0.232336 i, 154.386 - 20.9851 i}, {184.91 - 0.232336 i, 154.386 - 20.9851 i},
{134.957 - 0.211504 i, 159.548 - 20.9644 i}, {190.073 - 0.211504 i, 159.548 - 20.9644 i}},
0.9177 → {{8.15841 - 15.2318 i, 27.3882 - 41.4809 i}, {63.2741 - 15.2318 i, 27.3882 - 41.4809 i},
{12.8665 - 10.8162 i, 32.1077 - 37.053 i}, {67.9821 - 10.8162 i, 32.1077 - 37.053 i},
{81.6545 + 0.117763 i, 107.992 - 20.4101 i}, {136.77 + 0.117763 i, 107.992 - 20.4101 i},
{88.0875 + 0.123561 i, 114.424 - 20.4045 i}, {143.203 + 0.123561 i, 114.424 - 20.4045 i},
{128.998 - 0.237015 i, 153.598 - 20.9882 i}, {184.114 - 0.237015 i, 153.598 - 20.9882 i},
{135.699 - 0.209637 i, 160.297 - 20.9612 i}, {190.815 - 0.209637 i, 160.297 - 20.9612 i}},
0.9234 → {{7.81051 - 15.6581 i, 27.0265 - 41.8909 i}, {62.9262 - 15.6581 i, 27.0265 - 41.8909 i},
{68.5012 - 10.4436 i, 32.6205 - 36.6562 i}, {13.3856 - 10.4436 i, 32.6205 - 36.6562 i},
{136.201 + 0.119053 i, 107.413 - 20.41 i}, {81.0858 + 0.119053 i, 107.413 - 20.41 i},
{143.842 + 0.126278 i, 115.052 - 20.403 i}, {88.7262 + 0.126278 i, 115.052 - 20.403 i},
{128.341 - 0.241246 i, 152.949 - 20.9908 i}, {183.457 - 0.241246 i, 152.949 - 20.9908 i},
{191.416 - 0.208306 i, 160.906 - 20.9585 i}, {136.301 - 0.208306 i, 160.906 - 20.9585 i}}]>}

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```

```
ListPlot[Length /@ saddlePoints[[1 ;; -1]]]
```



```
With[{data = Compress[saddlePoints]},
  Button["Restore example saddle points", Set[saddlePoints, Uncompress[saddlePoints]];
    saddlePoints;]
]
```

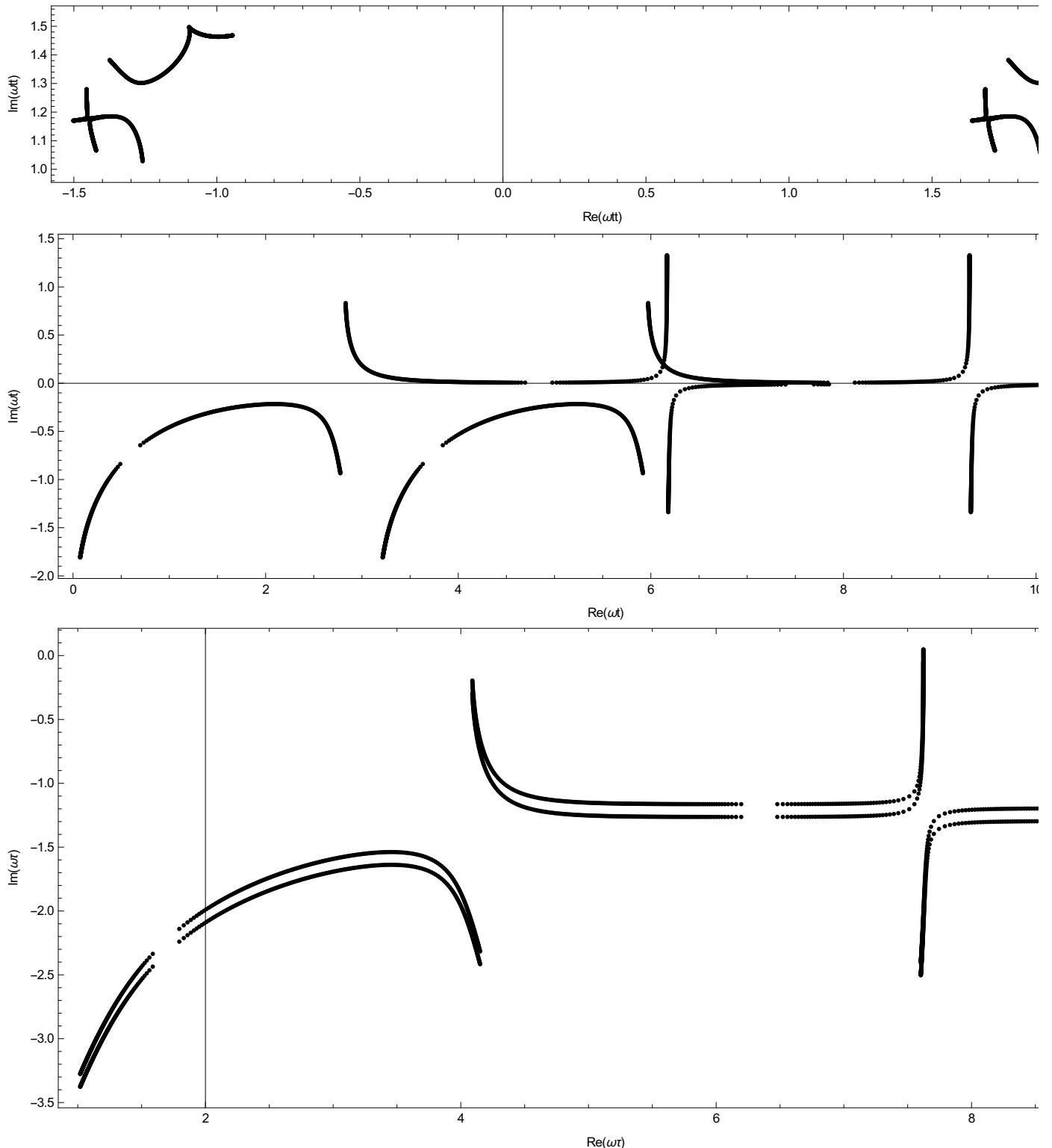
Restore example saddle points

Global saddle-points map

```
Block[{ω, Ip, κ, U, γ, saddles},
  {ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω/F} //.
    parameters;
  saddles = saddlePoints;
  Column[{Show[
    Graphics[{Table[
      Map[
        Apply[Function[{t, τ},
          Tooltip[Point[ReIm[ω (t - τ)]], {Ω/ω, ω {t, τ}, Floor[ω Re[t - τ], π]}]
        ]], saddles[Ω][[All]]]
       , {Ω, Keys[saddles]}]
      }]
     , Frame → True, Axes → True
     , ImageSize → 800
     , FrameLabel → {"Re(ωtt)", "Im(ωtt)"}]
    ]
   , Show[
    Graphics[{Table[
      Map[
        
```

```

    Apply[Function[{t, \tau},
      Tooltip[Point[ReIm[\omega t]], {\Omega/\omega, \omega {t, \tau}, \frac{Floor[\omega Re[t - \tau], \pi]}{\pi}}]
    ], saddles[\Omega][All]]
  , {\Omega, Keys[saddles]}}
]
, Frame → True, Axes → True
, ImageSize → 800
, FrameLabel → {"Re(\omega t)", "Im(\omega t)"}
]
, Show[
Graphics[{
  Table[
    Map[
      Apply[Function[{t, \tau},
        Tooltip[
          Point[ReIm[\omega \tau + 0.1 i \frac{Floor[\omega Re[t - \tau], \pi]}{\pi}]], {\Omega/\omega, \omega {t, \tau}, \frac{Floor[\omega Re[t - \tau], \pi]}{\pi}}]
        ]
      ], saddles[\Omega][All]]
    , {\Omega, Keys[saddles]}}
  ]
, Frame → True, Axes → True
, ImageSize → 800
, FrameLabel → {"Re(\omega \tau)", "Im(\omega \tau)"}
]
}
]
]
]
```



Syntax options for GetSaddlePoints

Single energy, single range:

```

Block[{ω, Ip, κ, U, γ},
{ω, Ip, κ, U, γ} = {ω, Ip, √2 Ip, F^2/(4 ω^2), κ ω}/. parameters;
ΩRange = Range[Ceiling[Ip, ω] + ω, Ceiling[1.32 Ip + 3.17 U] + ω, 2 ω];
GetSaddlePoints[ΩRange[[1]], s, {0 - 1.5 i γ, 3 π + 1.5 i γ}, {0 - 2.5 i γ, 2 π + 2.5 i γ}
, Tolerance → 10^-5/ω, Seeds → 30
, SelectionFunction → Function[{t, τ, SS, Ω}, Im[t - τ] > 0]
, IndependentVariables → {"tt", "τ"}
, IndependentVariables → {"RecombinationTime", "ExcursionTime"}]
]
]
{{61.3283 - 17.8854 i, 25.3117 - 44.0085 i},
{171.56 - 17.8854 i, 25.3117 - 44.0085 i}, {71.4328 - 8.63528 i, 35.5268 - 34.6444 i},
{181.664 - 8.63528 i, 35.5268 - 34.6444 i}, {126.549 - 8.63528 i, 35.5268 - 34.6444 i},
{133.234 + 0.130964 i, 104.361 - 20.4077 i}, {243.466 + 0.130964 i, 104.361 - 20.4077 i}}

```

Single energy, multiple ranges:

```

Block[{ω, Ip, κ, U, γ},
{ω, Ip, κ, U, γ} = {ω, Ip, √2 Ip, F^2/(4 ω^2), κ ω}/. parameters;
ΩRange = Range[Ceiling[Ip, ω] + ω, Ceiling[1.32 Ip + 3.17 U] + ω, 2 ω];
GetSaddlePoints[ΩRange[[1]], s, {{0 - 1.5 i γ, 3 π + 1.5 i γ}, {0 - 2.5 i γ, 2 π + 2.5 i γ}},
{{0 - 1.5 i γ, 3 π + 1.5 i γ}, {2 π - 2.5 i γ, 4 π + 2.5 i γ}}}]
, Tolerance → 10^-5/ω, Seeds → 30
, SelectionFunction → Function[{t, τ, SS, Ω}, Im[t - τ] > 0]
, IndependentVariables → {"tt", "τ"}
, IndependentVariables → {"RecombinationTime", "ExcursionTime"}]
]
{{116.444 - 17.8854 i, 25.3117 - 44.0085 i},
{126.549 - 8.63528 i, 35.5268 - 34.6444 i}, {71.4328 - 8.63528 i, 35.5268 - 34.6444 i},
{181.664 - 8.63528 i, 35.5268 - 34.6444 i}, {188.35 + 0.130964 i, 104.361 - 20.4077 i},
{243.466 + 0.130964 i, 104.361 - 20.4077 i}, {202.548 + 0.150768 i, 118.55 - 20.3896 i},
{147.432 + 0.150768 i, 118.55 - 20.3896 i}, {290.012 - 0.272736 i, 149.346 - 21.0081 i},
{194.593 - 0.203933 i, 164.14 - 20.9434 i}, {249.708 - 0.203933 i, 164.14 - 20.9434 i},
{304.824 - 0.203933 i, 164.14 - 20.9434 i}, {242.257 + 0.0334796 i, 214.037 - 20.4209 i}}

```

List of energies, single range:

```

Block[{ω, Ip, κ, U, γ},
{ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω/F} //.
parameters;
ΩRange = Range[Ceiling[Ip, ω] + ω, Ceiling[1.32 Ip + 3.17 U] + ω, 2 ω];
GetSaddlePoints[ΩRange[[1 ;; 2]], s, {0 - 1.5 I γ/ω, 3 π + 1.5 I γ/ω}, {0 - 2.5 I γ/ω, 2 π + 2.5 I γ/ω}
, Tolerance → 10^-5/ω, Seeds → 30
, SelectionFunction → Function[{t, τ, SS, Ω}, Im[t - τ] > 0]
, IndependentVariables → {"tt", "τ"}
, IndependentVariables → {"RecombinationTime", "ExcursionTime"}]
]
]
]

<| 0.969 → {{61.3283 - 17.8854 I, 25.3117 - 44.0085 I},
{171.56 - 17.8854 I, 25.3117 - 44.0085 I}, {71.4328 - 8.63528 I, 35.5268 - 34.6444 I},
{181.664 - 8.63528 I, 35.5268 - 34.6444 I}, {126.549 - 8.63528 I, 35.5268 - 34.6444 I},
{133.234 + 0.130964 I, 104.361 - 20.4077 I}, {243.466 + 0.130964 I, 104.361 - 20.4077 I}},
1.083 → {{59.5902 - 21.0124 I, 23.275 - 46.9513 I}, {169.822 - 21.0124 I, 23.275 - 46.9513 I},
{131.403 - 6.48193 I, 40.4479 - 31.9638 I}, {186.519 - 6.48193 I, 40.4479 - 31.9638 I},
{239.05 + 0.168505 I, 99.7229 - 20.3967 I}, {128.819 + 0.168505 I, 99.7229 - 20.3967 I}}|>

```

List of energies, multiple ranges:

```

Block[{ω, Ip, κ, U, γ},
{ω, Ip, κ, U, γ} = {ω, Ip, √2 Ip, F^2/(4 ω^2), κ ω}/. parameters;
ΩRange = Range[Ceiling[Ip, ω] + ω, Ceiling[1.32 Ip + 3.17 U] + ω, 2 ω];

GetSaddlePoints[ΩRange[[1 ;; 2]], s, {{{{0 - 1.5 i γ, 3 π + 1.5 i γ}, {0 - 2.5 i γ, 2 π + 2.5 i γ}}, {{0 - 1.5 i γ, 3 π + 1.5 i γ}, {2 π - 2.5 i γ, 4 π + 2.5 i γ}}}}, , Tolerance → 10^-5/ω, Seeds → 30
, SelectionFunction → Function[{t, τ, SS, Ω}, Im[t - τ] > 0]
, IndependentVariables → {"t", "τ"}
, IndependentVariables → {"RecombinationTime", "ExcursionTime"}]
]
]

<| 0.969 → {{116.444 - 17.8854 i, 25.3117 - 44.0085 i}, {126.549 - 8.63528 i, 35.5268 - 34.6444 i}, {71.4328 - 8.63528 i, 35.5268 - 34.6444 i}, {181.664 - 8.63528 i, 35.5268 - 34.6444 i}, {188.35 + 0.130964 i, 104.361 - 20.4077 i}, {243.466 + 0.130964 i, 104.361 - 20.4077 i}, {133.234 + 0.130964 i, 104.361 - 20.4077 i}, {202.548 + 0.150768 i, 118.55 - 20.3896 i}, {147.432 + 0.150768 i, 118.55 - 20.3896 i}, {290.012 - 0.272736 i, 149.346 - 21.0081 i}, {194.593 - 0.203933 i, 164.14 - 20.9434 i}, {249.708 - 0.203933 i, 164.14 - 20.9434 i}, {304.824 - 0.203933 i, 164.14 - 20.9434 i}, {242.257 + 0.0334796 i, 214.037 - 20.4209 i}}, 1.083 → {{114.706 - 21.0124 i, 23.275 - 46.9513 i}, {186.519 - 6.48193 i, 40.4479 - 31.9638 i}, {131.403 - 6.48193 i, 40.4479 - 31.9638 i}, {76.2875 - 6.48193 i, 40.4479 - 31.9638 i}, {183.935 + 0.168505 i, 99.7229 - 20.3967 i}, {128.819 + 0.168505 i, 99.7229 - 20.3967 i}, {239.05 + 0.168505 i, 99.7229 - 20.3967 i}, {154.268 + 0.278908 i, 125.093 - 20.3022 i}, {209.383 + 0.278908 i, 125.093 - 20.3022 i}, {283.079 - 0.408886 i, 142.658 - 21.0949 i}, {309.722 - 0.206451 i, 169.188 - 20.9196 i}, {254.606 - 0.206451 i, 169.188 - 20.9196 i}, {199.49 - 0.206451 i, 169.188 - 20.9196 i}, {237.464 + 0.0449743 i, 209.127 - 20.4165 i}}|>

```

Reperioding of saddle points

Getting saddle points over a t, τ box including multiple ionization bursts

```

DateString[]
Block[{ω, Ip, κ, U, γ},
{ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω/F} //.
parameters;
ΩRange = Range[Ceiling[Ip, ω], Ceiling[1.32 Ip + 3.17 U] + 5 ω, 0.1 ω];
saddlePoints =
GetSaddlePoints[ΩRange, s, {0° - 1.5 i γ, 2 π + 1.5 i γ}, {0 - 2.5 i γ, 3 π + 0.1 i γ}
, Tolerance → 10^-5 /ω, Seeds → 300
, SelectionFunction → Function[{t, τ, SS, Ω}, Im[t - τ] > 0]
]
] [[1 ;; 3]] // AbsoluteTiming
DateString[]
Wed 11 May 2016 13:43:26

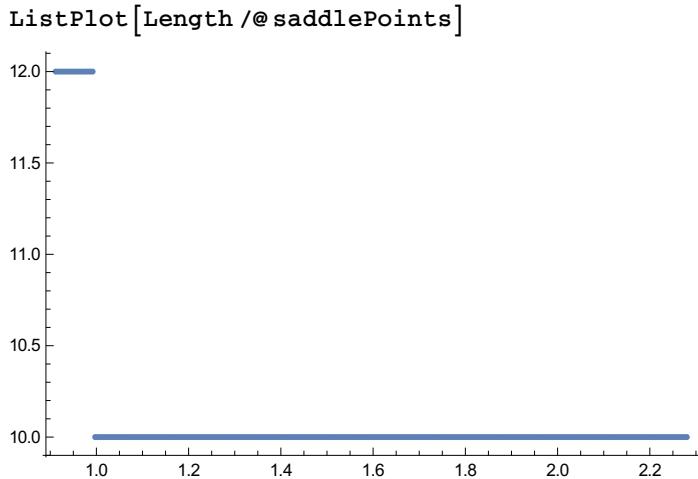
FindRoot::cvmit : Failed to converge to the requested accuracy or precision within 100 iterations.

GetSaddlePoints::error : Errors encountered for frequency Ω=1.0716`

{26.1461,
<|0.912 → {{8.60697 - 14.708 i, 27.8501 - 40.9741 i}, {63.7226 - 14.708 i, 27.8501 - 40.9741 i},
{67.3613 - 11.286 i, 31.4941 - 37.5465 i}, {12.2456 - 11.286 i, 31.4941 - 37.5465 i},
{27.2427 + 0.11658 i, 108.706 - 20.4102 i}, {82.3584 + 0.11658 i, 108.706 - 20.4102 i},
{32.1996 + 0.120837 i, 113.662 - 20.406 i}, {87.3152 + 0.120837 i, 113.662 - 20.406 i},
{19.5634 - 0.232336 i, 154.386 - 20.9851 i}, {74.6791 - 0.232336 i, 154.386 - 20.9851 i},
{79.8412 - 0.211504 i, 159.548 - 20.9644 i}, {24.7256 - 0.211504 i, 159.548 - 20.9644 i},
0.9177 → {{63.2741 - 15.2318 i, 27.3882 - 41.4809 i}, {8.15841 - 15.2318 i, 27.3882 - 41.4809 i},
{12.8665 - 10.8162 i, 32.1077 - 37.053 i}, {67.9821 - 10.8162 i, 32.1077 - 37.053 i},
{26.5389 + 0.117763 i, 107.992 - 20.4101 i}, {81.6545 + 0.117763 i, 107.992 - 20.4101 i},
{32.9718 + 0.123561 i, 114.424 - 20.4045 i}, {88.0875 + 0.123561 i, 114.424 - 20.4045 i},
{18.7672 - 0.237015 i, 153.598 - 20.9882 i}, {73.8828 - 0.237015 i, 153.598 - 20.9882 i},
{25.4677 - 0.209637 i, 160.297 - 20.9612 i}, {80.5833 - 0.209637 i, 160.297 - 20.9612 i},
0.9234 → {{7.81051 - 15.6581 i, 27.0265 - 41.8909 i}, {62.9262 - 15.6581 i, 27.0265 - 41.8909 i},
{68.5012 - 10.4436 i, 32.6205 - 36.6562 i}, {13.3856 - 10.4436 i, 32.6205 - 36.6562 i},
{81.0858 + 0.119053 i, 107.413 - 20.41 i}, {25.9702 + 0.119053 i, 107.413 - 20.41 i},
{88.7262 + 0.126278 i, 115.052 - 20.403 i}, {33.6106 + 0.126278 i, 115.052 - 20.403 i},
{73.2256 - 0.241246 i, 152.949 - 20.9908 i}, {18.1099 - 0.241246 i, 152.949 - 20.9908 i},
{81.1851 - 0.208306 i, 160.906 - 20.9585 i}, {26.0694 - 0.208306 i, 160.906 - 20.9585 i}}]>
}

Wed 11 May 2016 13:43:52

```



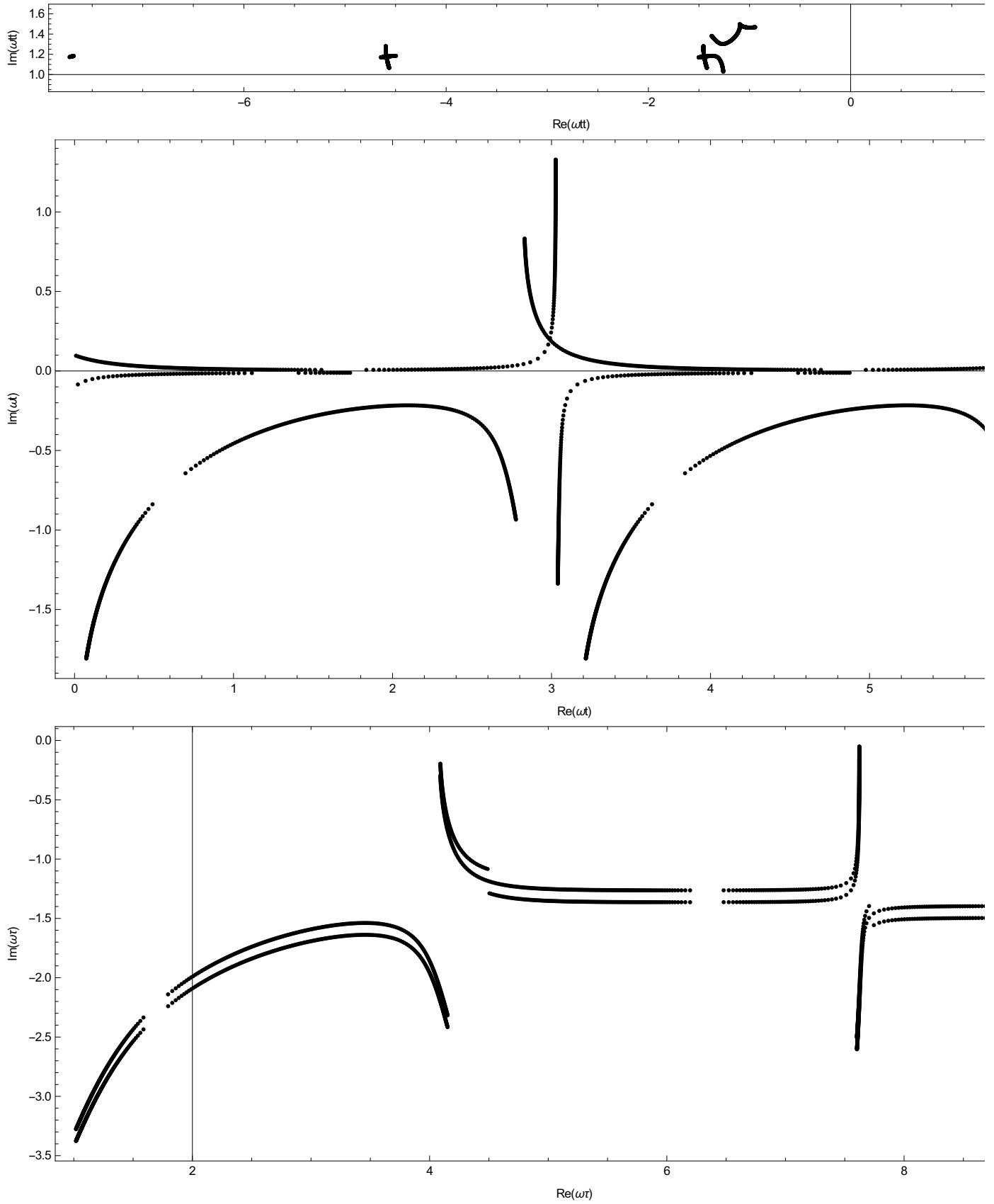
```
With[{data = Compress[saddlePoints]},
  Button["Restore saddlePoints", Set[saddlePoints, Uncompress[data]]]]
]
saddlePoints;
```

Global saddle-points map

```
Block[{\omega, Ip, \kappa, U, \gamma, saddles},
  {\omega, Ip, \kappa, U, \gamma} = {\omega, Ip, \sqrt{2 Ip}, F^2/(4 \omega^2), \kappa \omega}/. parameters;
  saddles = saddlePoints;
  Column[{
    Show[
      Graphics[{
        Point[
          Flatten[Table[
            Map[
              Apply[Function[{t, \tau},
                ReIm[\omega (t - \tau)]]
            ],
            saddles[\Omega][[All]]]
          , {\Omega, Keys[saddles]}], 1]
        ]
      }]
    ,
    Frame -> True, Axes -> True
    , ImageSize -> 800
    , FrameLabel -> {"Re(\omega_{tt})", "Im(\omega_{tt})"}
  ]
  ,
  Show[
    Graphics[{
      Point[
        Flatten[Table[
          Map[
            Apply[Function[{t, \tau},
              ReIm[\omega t]
            ]
          ]
        ]
      ]
    ]]
```

```

        , saddles[ $\Omega$ ][[All]]
        , { $\Omega$ , Keys[saddles]}], 1]
    }
}
, Frame → True, Axes → True
, ImageSize → 800
, FrameLabel → {"Re( $\omega t$ )", "Im( $\omega t$ )"}
]
, Show[
Graphics[{(
Point[
Flatten[Table[
Map[
Apply[Function[{t,  $\tau$ },
ReIm[ $\omega \tau + 0.1 i \frac{\text{Floor}[\omega \text{Re}[t - \tau], \pi]}{\pi}$ ]
]]]
, saddles[ $\Omega$ ][[All]]
, { $\Omega$ , Keys[saddles]}], 1]
]
}
]
, Frame → True, Axes → True
, ImageSize → 800
, FrameLabel → {"Re( $\omega \tau$ )", "Im( $\omega \tau$ )"}
]
}
]
]
```



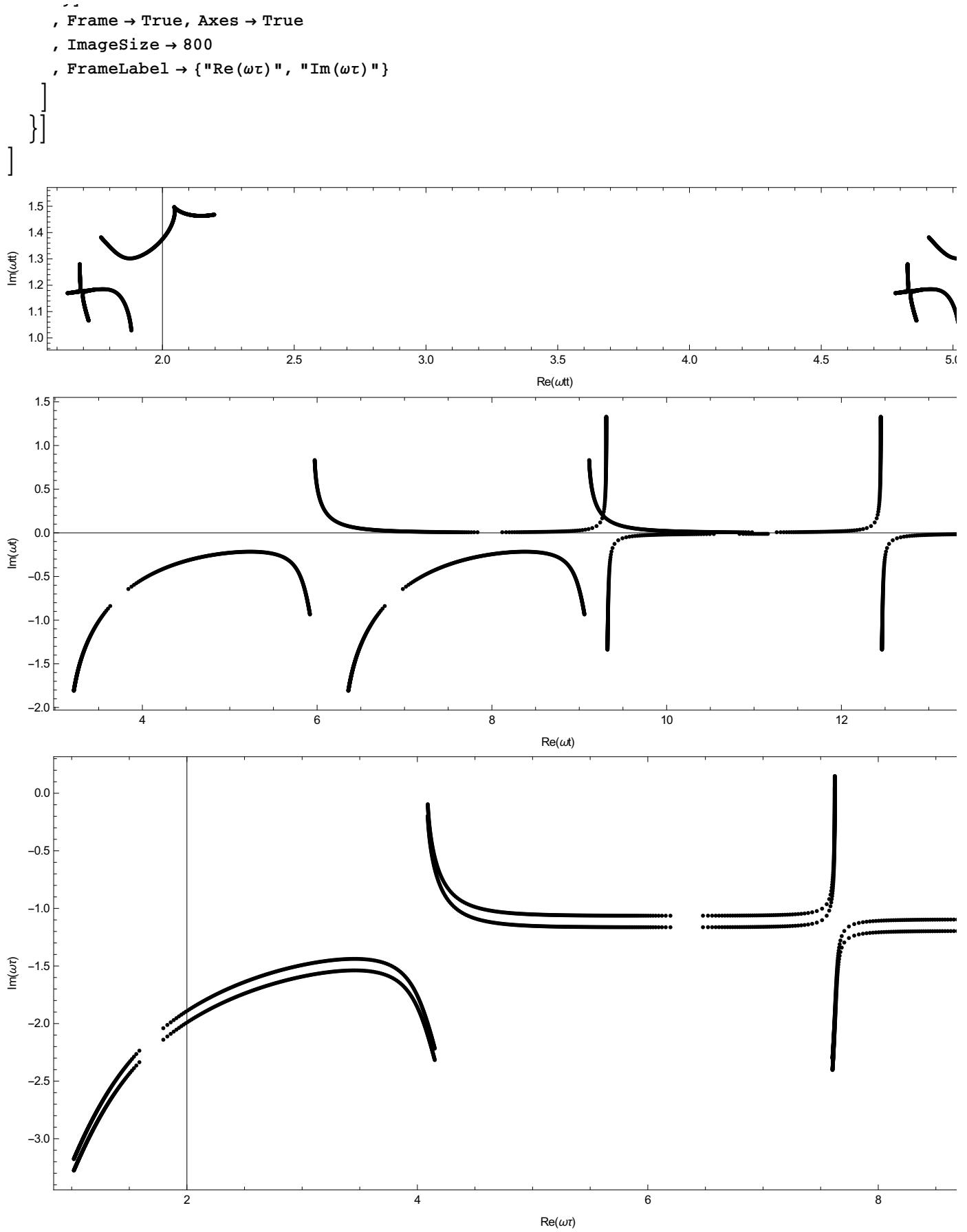
Reperioding a saddle-point set

```
Block[{\omega, Ip, κ, U, γ, saddles},  

 {ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω/F} // . parameters;
```

```

saddles = ReperiodSaddles[saddlePoints, Function[{t, \tau}, {t - Floor[Re[t - \tau], \frac{2\pi}{\omega}], \tau}]];
Column[{
  Show[
    Graphics[{
      Point[
        Flatten[Table[
          Map[
            Apply[Function[{t, \tau},
              ReIm[\omega (t - \tau)]]
            ],
            saddles[\Omega][[All]]]
          , {\Omega, Keys[saddles]}], 1]
        ]
      ]
    },
    Frame \rightarrow True, Axes \rightarrow True
    , ImageSize \rightarrow 800
    , FrameLabel \rightarrow {"Re(\omega_{tt})", "Im(\omega_{tt})"}
  ]
  ,
  Show[
    Graphics[{
      Point[
        Flatten[Table[
          Map[
            Apply[Function[{t, \tau},
              ReIm[\omega t]
            ],
            saddles[\Omega][[All]]]
          , {\Omega, Keys[saddles]}], 1]
        ]
      ]
    },
    Frame \rightarrow True, Axes \rightarrow True
    , ImageSize \rightarrow 800
    , FrameLabel \rightarrow {"Re(\omega_t)", "Im(\omega_t)"}
  ]
  ,
  Show[
    Graphics[{
      Point[
        Flatten[Table[
          Map[
            Apply[Function[{t, \tau},
              ReIm[\omega \tau + 0.1 i \frac{Floor[\omega Re[t - \tau], \pi]}{\pi}]]
            ],
            saddles[\Omega][[All]]]
          , {\Omega, Keys[saddles]}], 1]
        ]
      ]
    }
  ]
}
]
```



Saddle-point classification

Classified saddles using points-based map

```

Block[{ω, Ip, κ, U, γ, selection, classifierFunction, sortingFunction, keyColour},
{ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω}/. parameters;
classifierFunction = Function[{t, τ, Ω}, Which[
And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == -1], "A",
And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == 0], "B",
And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == 0], "C",
And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == -1], "D",
True, "Discard"
]];
sortingFunction =
Function[list, SortBy[list, Function[Re[ω #[[1]] - Floor[ω Re[#[[1]] - #[[2]]], 2 π]]]]];
keyColour = <|"A" → <|1 → Black, 2 → Blue|>, "B" → <|1 → Red, 2 → Magenta|>,
"C" → <|1 → Darker[Green], 2 → Orange|>, "D" → <|1 → Darker[Cyan], 2 → Purple|>|>;
keyColour["D", _] = Cyan;

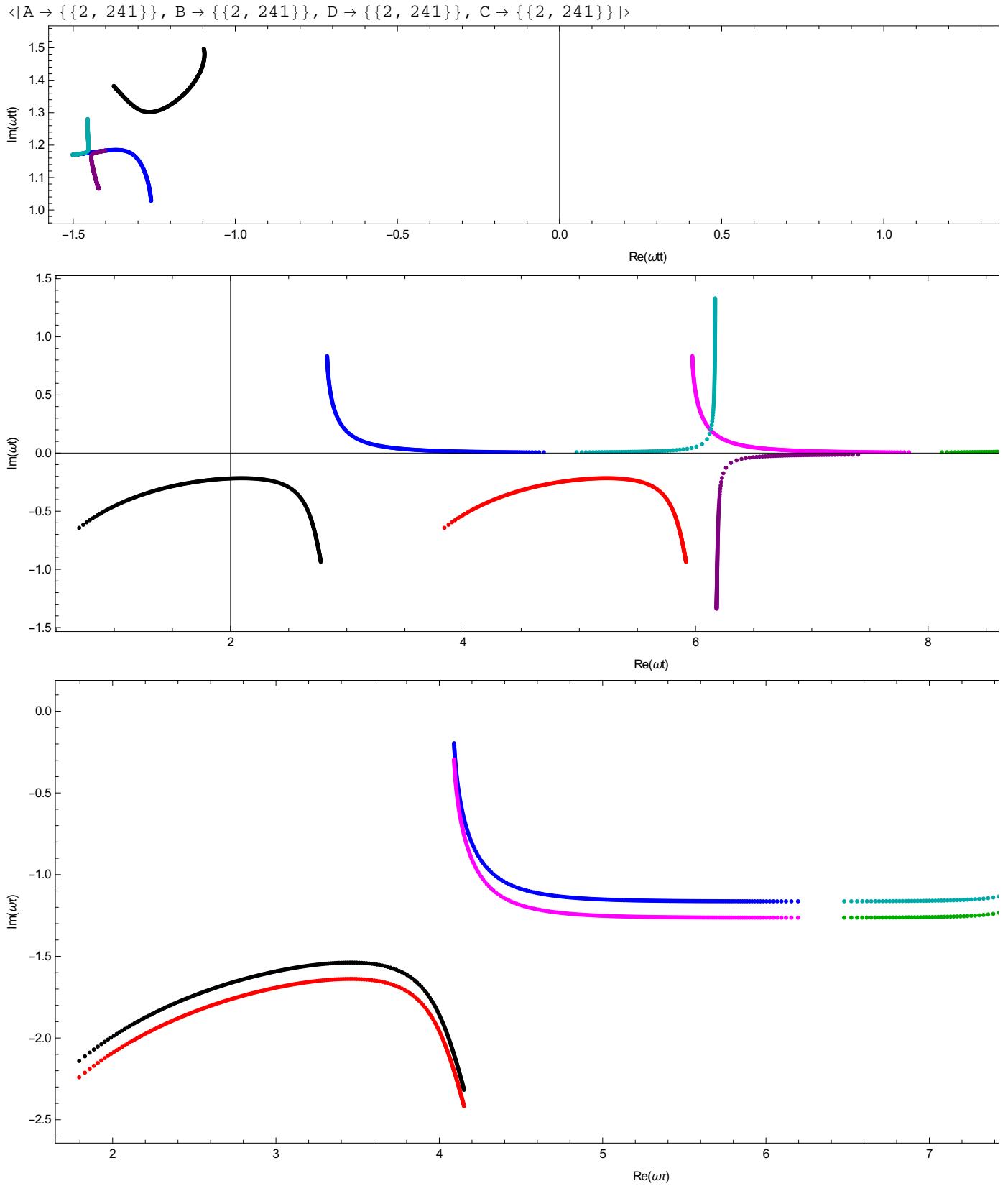
selection = ClassifyQuantumOrbits[saddlePoints,
classifierFunction, sortingFunction, DiscardedLabels → {"Discard"}];

Column[{
Tally /@ Map[Length, selection, {2}],
Show[
Graphics[
Table[Table[{{
KeyValueMap[
Function[{n, t, τ},
{keyColour[index, n] /. Missing[_] → Gray,
Tooltip[Point[ReIm[ω (t - τ) (*+Floor[ω Re[τ - t], 2 π]*)], {
Ω/ω, index, n, ω {t, τ}, 1/π Floor[ω Re[t - τ], π]}]
] ] @* Apply[Sequence] @* Flatten @* List
, selection[index, Ω]
}], {Ω, Keys[selection[index]]}], {index, Keys[selection]}]
}, {Ω, Keys[selection[index]]}], {index, Keys[selection]}]
],
Frame → True, Axes → True
, ImageSize → 900
]
]

```

```

, FrameLabel -> {"Re(ωtt)", "Im(ωtt)"}
]
, Show[
Graphics[
Table[Table[{{
KeyValueMap[
Function[{n, t, τ},
{keyColour[index, n] /. Missing[_] -> Gray,
Tooltip[Point[ReIm[ωt]], {Ω/ω, index, n, ω{t, τ}, 1/π Floor[ω Re[t - τ], π]}]]}]*Apply[Sequence]*Flatten@*List
, selection[index, Ω]]}
}, {Ω, Keys[selection[index]]}], {index, Keys[selection]}]
]
, Frame -> True, Axes -> True
, ImageSize -> 900
, FrameLabel -> {"Re(ωt)", "Im(ωt)"}
]
, Show[
Graphics[
Table[Table[{{
KeyValueMap[
Function[{n, t, τ},
{keyColour[index, n] /. Missing[_] -> Gray,
Tooltip[Point[ReIm[ωτ + 0.1 i 1/π Floor[ω Re[τ - t], π]], {Ω/ω, index, n, ω{t, τ}, 1/π Floor[ω Re[t - τ], π]}]]}]*Apply[Sequence]*Flatten@*List
, selection[index, Ω]]}
}, {Ω, Keys[selection[index]]}], {index, Keys[selection]}]
]
, Frame -> True, Axes -> True
, ImageSize -> 900
, FrameLabel -> {"Re(ωτ)", "Im(ωτ)"}
]
]
}
]
```



Classified saddles using lines-based map

```
Block[{\omega, Ip, κ, U, γ, selection, classifierFunction, sortingFunction, keyColour, d2S},
{ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω/F} //.
parameters;
```

```

d2S[t_, tt_] = Derivative[0, 2][S][t, tt];

classifierFunction = Function[{t, τ, Ω}, Which[
  And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == -1], "A",
  And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == 0], "B",
  And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == 0], "C",
  And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == -1], "D",
  True, "Discard"
]];
];
sortingFunction =
  Function[list, SortBy[list, Function[Re[ω #[[1]] - Floor[ω Re[#[[1]] - #[[2]]], 2 π]]]]];
keyColour = <|"A" → <|1 → Black, 2 → Blue|>, "B" → <|1 → Red, 2 → Magenta|>,
  "C" → <|1 → Darker[Green], 2 → Orange|>, "D" → <|1 → Darker[Cyan], 2 → Purple|>
, "Bad" → <|1 → Brown, 2 → Brown, 3 → Brown|>
|>;
selection = selectionCache = ClassifyQuantumOrbits[saddlePoints,
  classifierFunction, sortingFunction(*, DiscardedLabels → {"Discard"}*)];

Column[{
  Show[
    Table[
      Values[
        MapIndexed[
          Function[{assoc, key},
            Graphics[{keyColour[index, key[[1, 1]]] /. Missing[_] → Gray, Thickness[0.003],
              Tooltip[Line[Values[assoc]], {index, key[[1, 1]]}]]]
          ]]
        ],
        DeleteMissing[Query[Transpose][
          Apply[
            Function[{t, τ},
              ReIm[ω (t - τ)]
            ]
          , selection[[index]], {2}]
        ], 2]
      ]
    ],
    {index, Keys[selection]}]
  , Frame → True, Axes → True
  , PlotRangePadding → None, PlotRangeClipping → True
  , ImageSize → {{1200}, {700}}
  , FrameLabel → {"Re(ωtt)", "Im(ωtt)"}
]

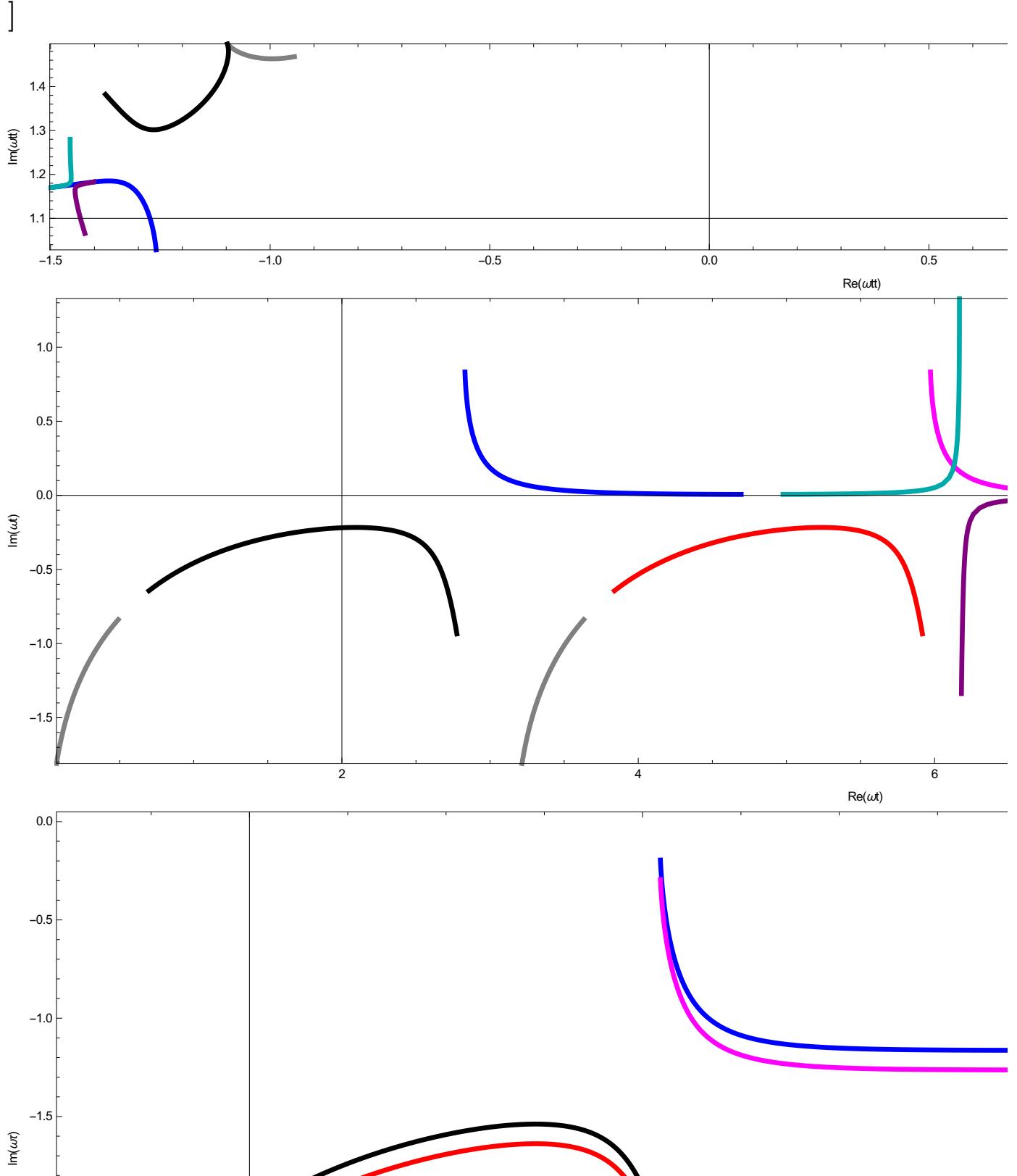
```

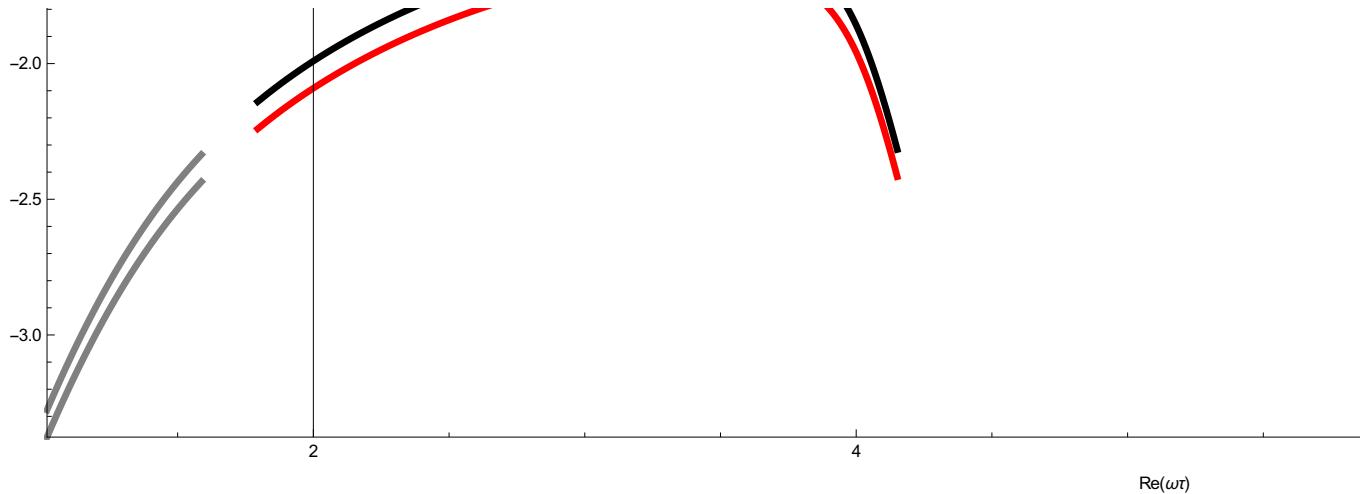
```

, Show[
Table[
Values[
MapIndexed[
Function[{assoc, key}],
Graphics[{{
keyColour[index, key[[1, 1]]] /. Missing[_] → Gray, Thickness[0.003],
Tooltip[Line[Values[assoc]], {index, key[[1, 1]]}]
}}]
],
DeleteMissing[Query[Transpose][
Apply[
Function[{t, τ},
ReIm[ω t]
]
, selection[index], {2}]
], 2]
]
]
, {index, Keys[selection]}]
, Frame → True, Axes → True
, PlotRangePadding → None, PlotRangeClipping → True
, ImageSize → {{1200}, {700}}
, FrameLabel → {"Re(ωt)", "Im(ωt)"}
]
]
, Show[
Table[
Values[
MapIndexed[
Function[{assoc, key}],
Graphics[{{
keyColour[index, key[[1, 1]]] /. Missing[_] → Gray, Thickness[0.003],
Tooltip[Line[Values[assoc]], {index, key[[1, 1]]}]
}}]
],
DeleteMissing[Query[Transpose][
Apply[
Function[{t, τ},
ReIm[ω τ + 0.1 i Floor[ω Re[τ - t], π]/π]
]
, selection[index], {2}]
], 2]
]
]
, {index, Keys[selection]}]
, Frame → True, Axes → True
]
```

```

, PlotRangePadding -> None, PlotRangeClipping -> True
, ImageSize -> {{1200}, {700}}
, FrameLabel -> {"Re(\omega\tau)", "Im(\omega\tau)"}
]
}
]
```





Watching the Hessian for branch cuts

```

Block[{ω, Ip, κ, U, γ, selection, classifierFunction, sortingFunction, keyColour},
{ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω/F} //.
parameters;

classifierFunction = Function[{t, τ, Ω}, Which[
And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == -1], "A",
And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == 0], "B",
And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == 0], "C",
And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == -1], "D",
True, "Discard"
]];
sortingFunction =
Function[list, SortBy[list, Function[Re[ω #[[1]] - Floor[ω Re[#[[1]] - #[[2]]], 2 π]]]]];
keyColour = <|"A" → <|1 → Black, 2 → Blue|>, "B" → <|1 → Red, 2 → Magenta|>,
"C" → <|1 → Darker[Green], 2 → Orange|>, "D" → <|1 → Darker[Cyan], 2 → Purple|>|>;
keyColour["D", _] = Cyan;

selection = ClassifyQuantumOrbits[saddlePoints,
classifierFunction, sortingFunction(*, DiscardedLabels → {"Discard"}*)];

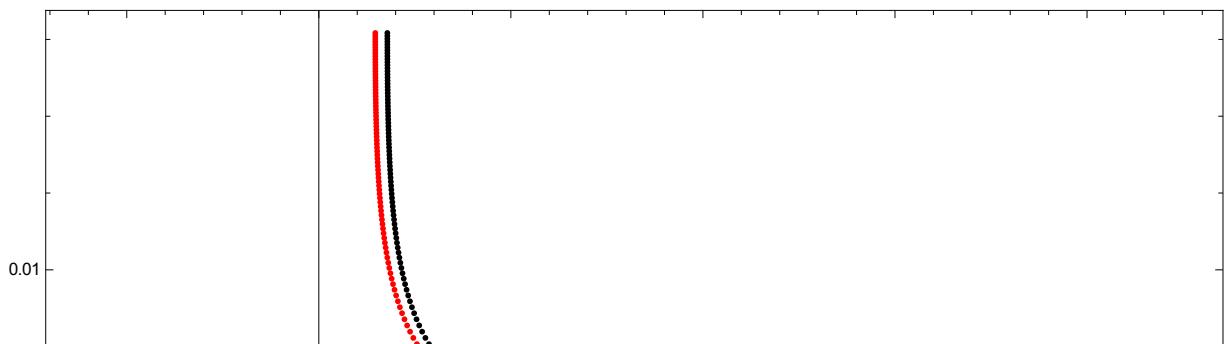
Column[{
Show[
Graphics[
Table[Table[{{
KeyValueMap[
Function[{n, t, τ},
{keyColour[index, n] /. Missing[_] → Gray,

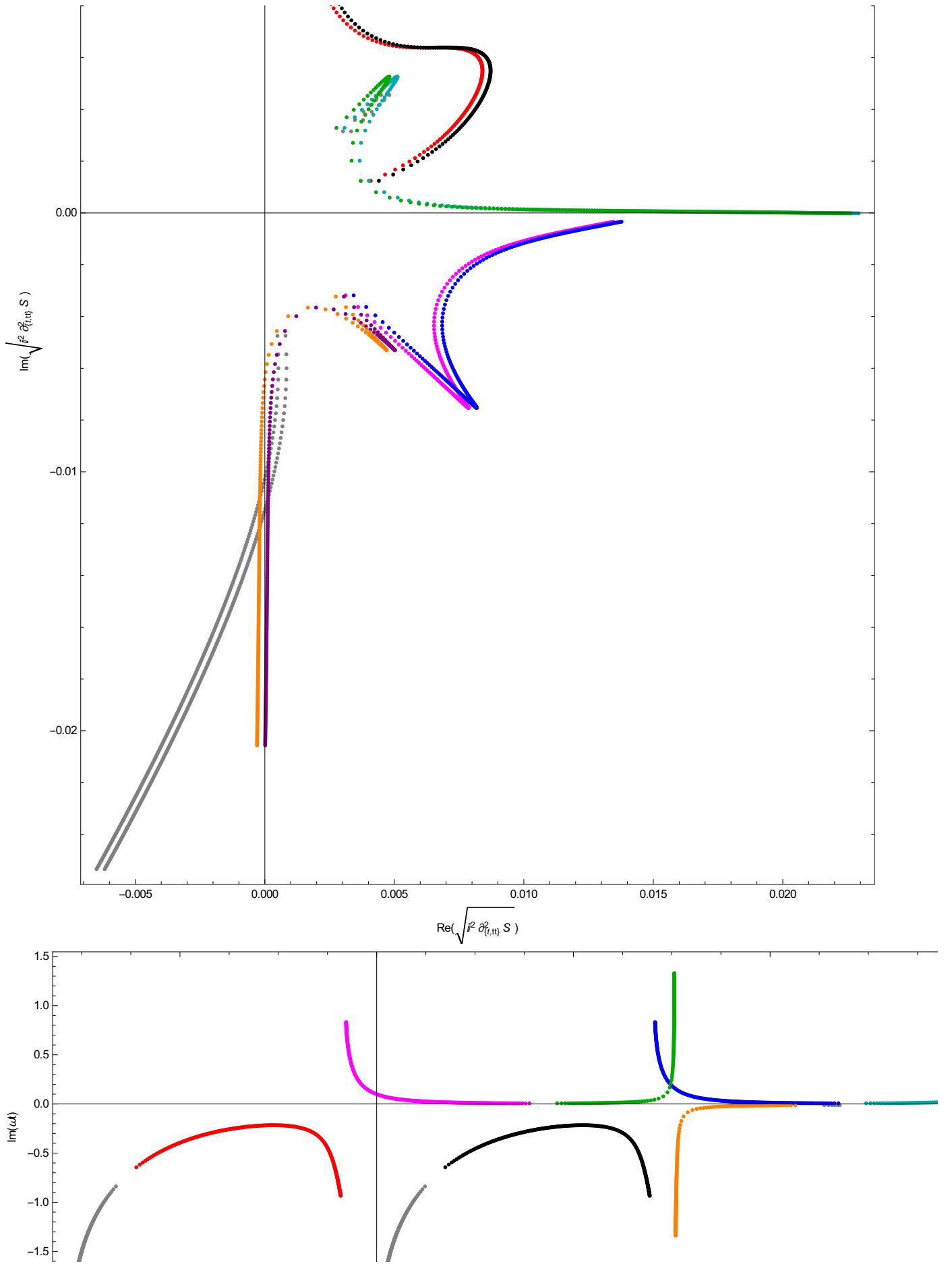
```

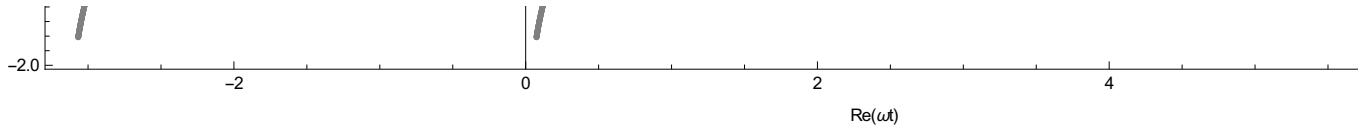
```

`Tooltip[Point[
  ReIm[
    1
    HessianRoot[S, t, τ]
  ]
], {Ω/ω, index, n, ω{t, τ}, 1/π Floor[ω Re[t - τ], π]}]
] @*Apply[Sequence] @*Flatten @*List
, selection[index, Ω]
}, {Ω, Keys[selection[index]]}], {index, Keys[selection]}]
]
, Frame → True, Axes → True
(*, AxesOrigin → {0, 0}*)
, ImageSize → {{900}, {900}}
, FrameLabel → {"Re(√i² ∂²_{t,tt} S)", "Im(√i² ∂²_{t,tt} S)"}
]
, Show[
Graphics[
Table[Table[{{
  KeyValueMap[
    Function[{n, t, τ},
      keyColour[index, n] /. Missing[_] → Gray,
      Tooltip[Point[ReIm[ω t + Floor[ω Re[τ - t], 2 π]]],
        {Ω/ω, index, n, ω{t, τ}, 1/π Floor[ω Re[t - τ], π]}]
    ] @*Apply[Sequence] @*Flatten @*List
    , selection[index, Ω]
  }, {Ω, Keys[selection[index]]}], {index, Keys[selection]}]
]
, Frame → True, Axes → True
, ImageSize → 900
, FrameLabel → {"Re(wt)", "Im(wt)"}
]
]
]
]
]

```







Demonstrating the Stokes transitions

```

Block[{ω, Ip, κ, U, γ, selection, classifierFunction, sortingFunction,
  keyColour, transitions, secondClassifierFunction, zoomPlot},
  {ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω/F} //.
    parameters;
  classifierFunction = Function[{t, τ, Ω}, Which[
    And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == -1], "A",
    And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == 0], "B",
    And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == 0], "C",
    And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == -1], "D",
    True, "Discard"
  ]];
  sortingFunction =
    Function[list, SortBy[list, Function[Re[ω #[[1]] - Floor[ω Re[#[[1]] - #[[2]]], 2 π]]]]];
  (*keyColour=⟨|"A"→⟨1→Black,2→Blue|⟩,"B"→⟨1→Red,2→Magenta|⟩,
   "C"→⟨1→Darker[Green],2→Orange|⟩,"D"→⟨1→Darker[Cyan],2→Purple|⟩|>;*)
  selection = ClassifyQuantumOrbits[saddlePoints,
    classifierFunction, sortingFunction, DiscardedLabels → {"Discard"}];
  transitions = FindStokesTransitions[S, selection];

  secondClassifierFunction = Function[{t, τ, Ω}, Which[
    And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == -1, Ω < transitions[["A", 1]]], "A1",
    And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == -1,
      transitions[["A", 1]] ≤ Ω < transitions[["A", 2]]], "A2",
    And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == -1, transitions[["A", 2]] ≤ Ω], "A3",
    And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == 0, Ω < transitions[["B", 1]]], "B1",
    And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == 0,
      transitions[["B", 1]] ≤ Ω < transitions[["B", 2]]], "B2",
    And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == 0, transitions[["B", 2]] ≤ Ω], "B3"]
  ];

```

```

And[6.3 < ω Re[τ] < 8.9,  $\frac{\text{Floor}[\omega \text{Re}[t - \tau], \pi]}{\pi} = 0, \Omega < \text{transitions}["C", 1]$ ], "C1",
And[6.3 < ω Re[τ] < 8.9,  $\frac{\text{Floor}[\omega \text{Re}[t - \tau], \pi]}{\pi} = 0,$ 
     $\text{transitions}["C", 1] \leq \Omega < \text{transitions}["C", 2]$ ], "C2",
And[6.3 < ω Re[τ] < 8.9,  $\frac{\text{Floor}[\omega \text{Re}[t - \tau], \pi]}{\pi} = 0, \text{transitions}["C", 2] \leq \Omega$ ], "C3",
And[6.3 < ω Re[τ] < 8.9,  $\frac{\text{Floor}[\omega \text{Re}[t - \tau], \pi]}{\pi} = -1, \Omega < \text{transitions}["D", 1]$ ], "D1",
And[6.3 < ω Re[τ] < 8.9,  $\frac{\text{Floor}[\omega \text{Re}[t - \tau], \pi]}{\pi} = -1,$ 
     $\text{transitions}["D", 1] \leq \Omega < \text{transitions}["D", 2]$ ], "D2",
And[6.3 < ω Re[τ] < 8.9,  $\frac{\text{Floor}[\omega \text{Re}[t - \tau], \pi]}{\pi} = -1, \text{transitions}["D", 2] \leq \Omega$ ], "D3",
True, "Discard"
];
keyColour = <|
"A1" → <|1 → Black, 2 → Blue|>, "A2" → <|1 → Blue, 2 → Black|>, "A3" → <|1 → Black, 2 → Blue|>,
"B1" → <|1 → Red, 2 → Magenta|>,
"B2" → <|1 → Magenta, 2 → Red|>, "B3" → <|1 → Red, 2 → Magenta|>,
"C1" → <|1 → Darker[Green], 2 → Orange|>, "C2" → <|1 → Orange, 2 → Darker[Green]|>,
"C3" → <|1 → Darker[Green], 2 → Orange|>,
"D1" → <|1 → Darker[Cyan], 2 → Purple|>, "D2" → <|1 → Purple, 2 → Darker[Cyan]|>,
"D3" → <|1 → Darker[Cyan], 2 → Purple|>
|>;
selection = ClassifyQuantumOrbits[saddlePoints,
secondClassifierFunction, sortingFunction, DiscardedLabels → {"Discard"}];
Column[{  

  transitions,  

  Show[  

    Graphics[  

      Table[Table[{  

          KeyValueMap[  

            Function[{n, t, τ},  

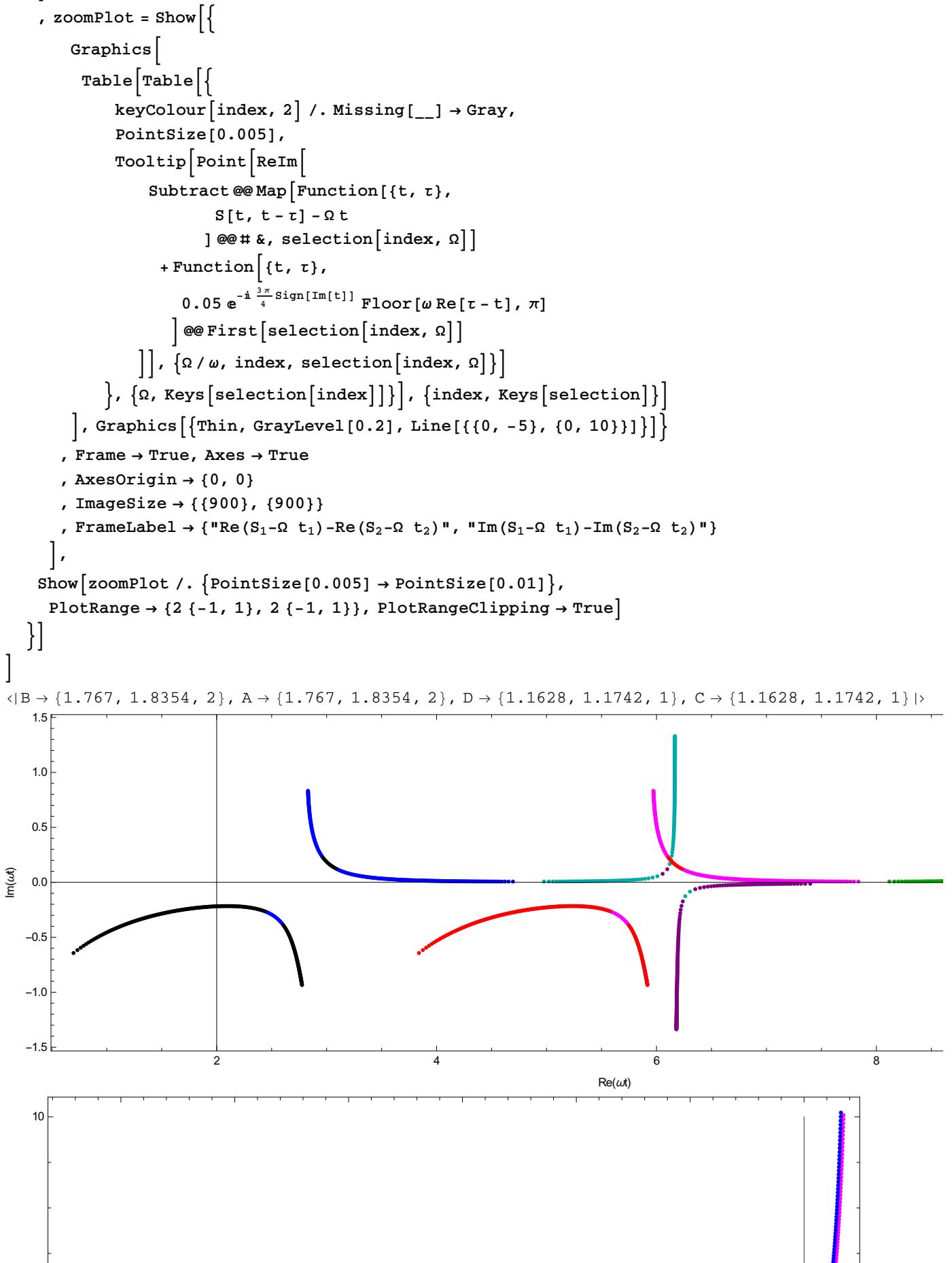
              {keyColour[index, n] /. Missing[_] → Gray,  

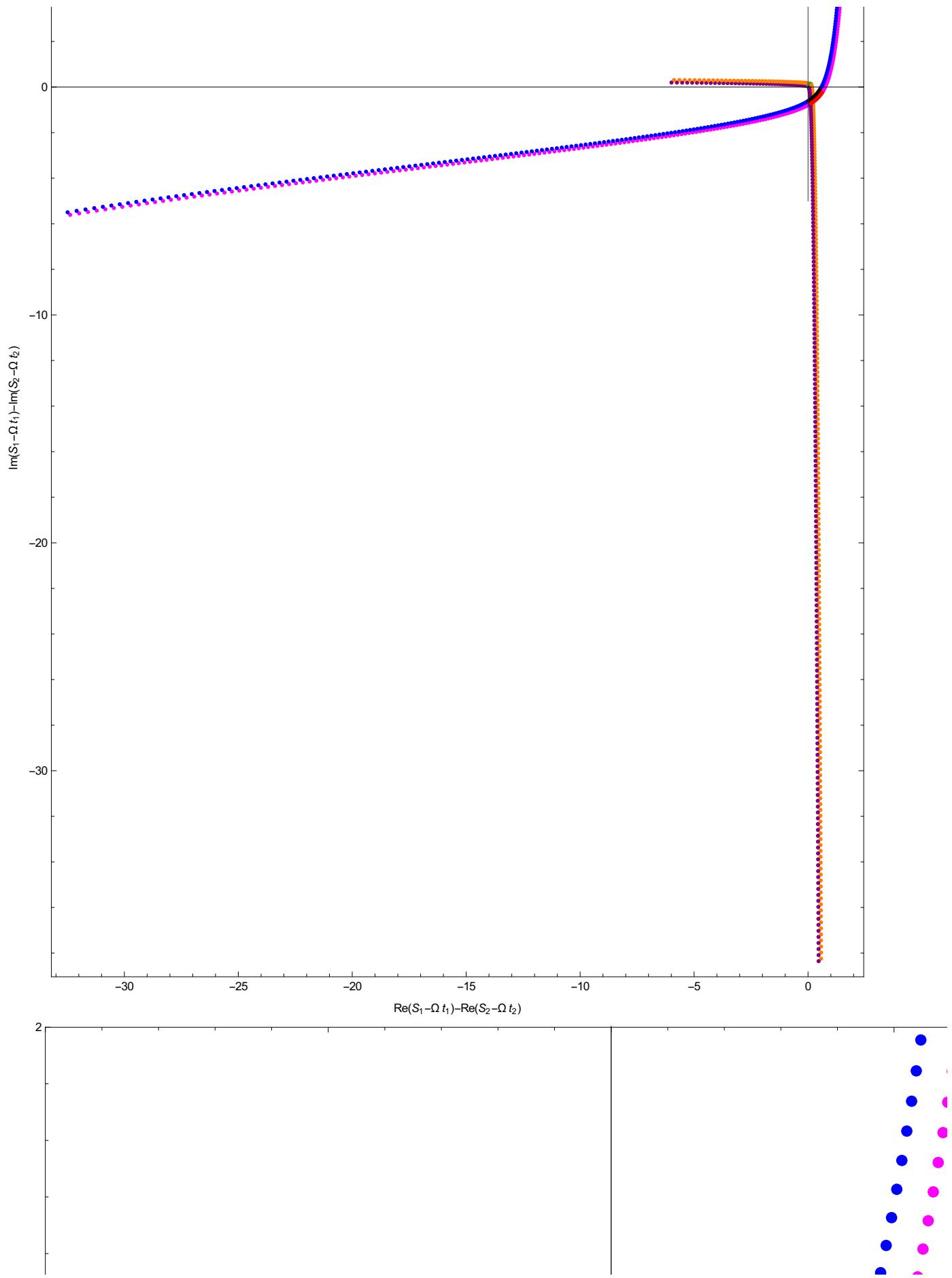
               Tooltip[Point[ReIm[ω t]], {Ω/ω, index, n, ω {t, τ},  $\frac{1}{\pi} \text{Floor}[\omega \text{Re}[t - \tau], \pi]$ }]]]  

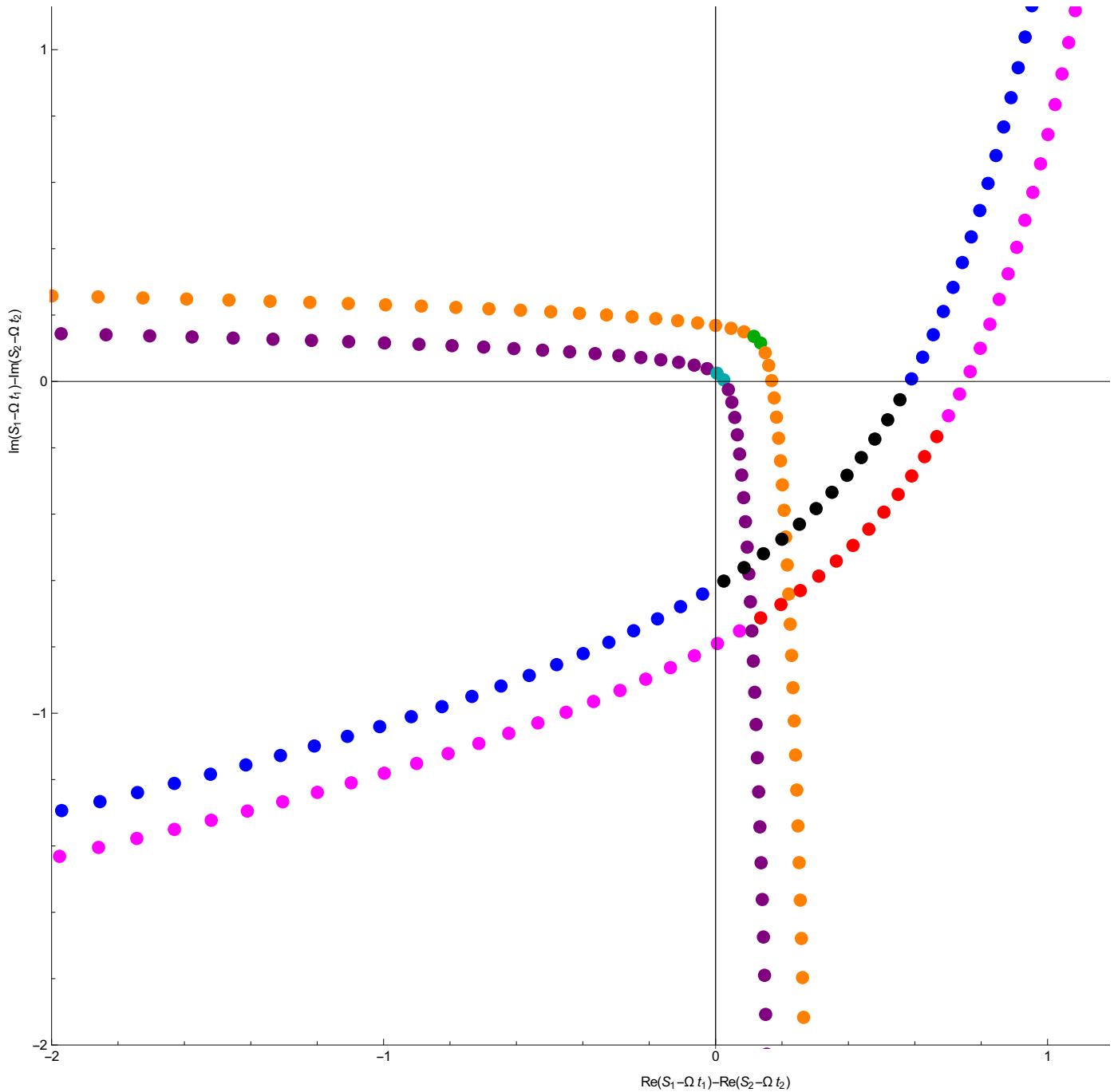
              ] @* Apply[Sequence] @* Flatten @* List  

              , selection[index, Ω]
            }, {Ω, Keys[selection[index]]}], {index, Keys[selection]}]
    ]
  , Frame → True, Axes → True
  , ImageSize → 900
  , FrameLabel → {"Re(ωt)", "Im(ωt)"}
]

```







Saddle-point per-pair spectrum

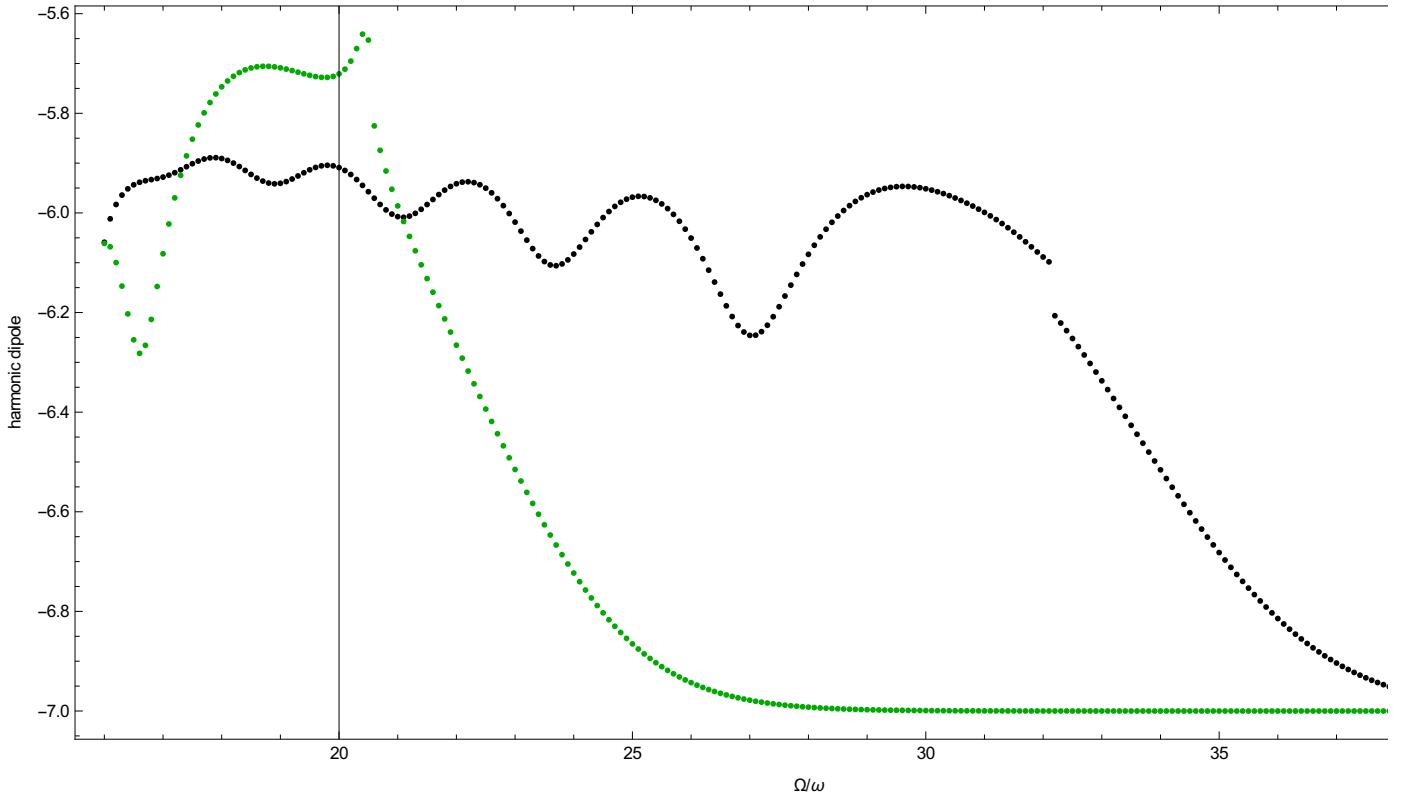
```

Block[{ω, Ip, κ, U, γ, selection,
       classifierFunction, sortingFunction, keyColour, transitions},
{ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω/F} //.
parameters;

classifierFunction = Function[{t, τ, Ω}, Which[
  And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == -1], "A",
  (*And[1.65<ω Re[τ]<6.3, Floor[ω Re[t-τ],π]==0], "B",*)
  And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == 0], "C",
  (*And[6.3<ω Re[τ]<8.9, Floor[ω Re[t-τ],π]==-1], "D",*)
  True, "Discard"
]];
sortingFunction =
  Function[list, SortBy[list, Function[Re[ω #[[1]] - Floor[ω Re[#[[1]] - #[[2]]], 2 π]]]]];
(*keyColour=⟨|"A"→⟨|1→Black,2→Blue|⟩,"B"→⟨|1→Red,2→Magenta|⟩,
  "C"→⟨|1→Darker[Green],2→Orange|⟩,"D"→⟨|1→Darker[Cyan],2→Purple|⟩|⟩;*)
keyColour = ⟨|"A" → Black, "B" → Red, "C" → Darker[Green], "D" → Purple|⟩;

selection = ClassifyQuantumOrbits[saddlePoints,
  classifierFunction, sortingFunction, DiscardedLabels → {"Discard"}];
transitions = FindStokesTransitions[S, selection];
Show[
Graphics[
Table[Table[{{
  keyColour[index] /. {Missing[_] → Gray},
  Tooltip[
    Point[
      {Ω/ω, Log10[10^-7 + Norm[
        SPAdipole[S, prefactor, Ω, selection[index, Ω], transitions[index]
        ]]
      ]}]
    ], {Ω/ω, index, selection[index, Ω]}]
  }, {Ω, Keys[selection[index]]}], {index, Keys[selection]}]
]
, Frame → True, Axes → True
, ImageSize → 800
, AspectRatio → 1/2
, FrameLabel → {"Ω/ω", "harmonic dipole"}
]
]

```



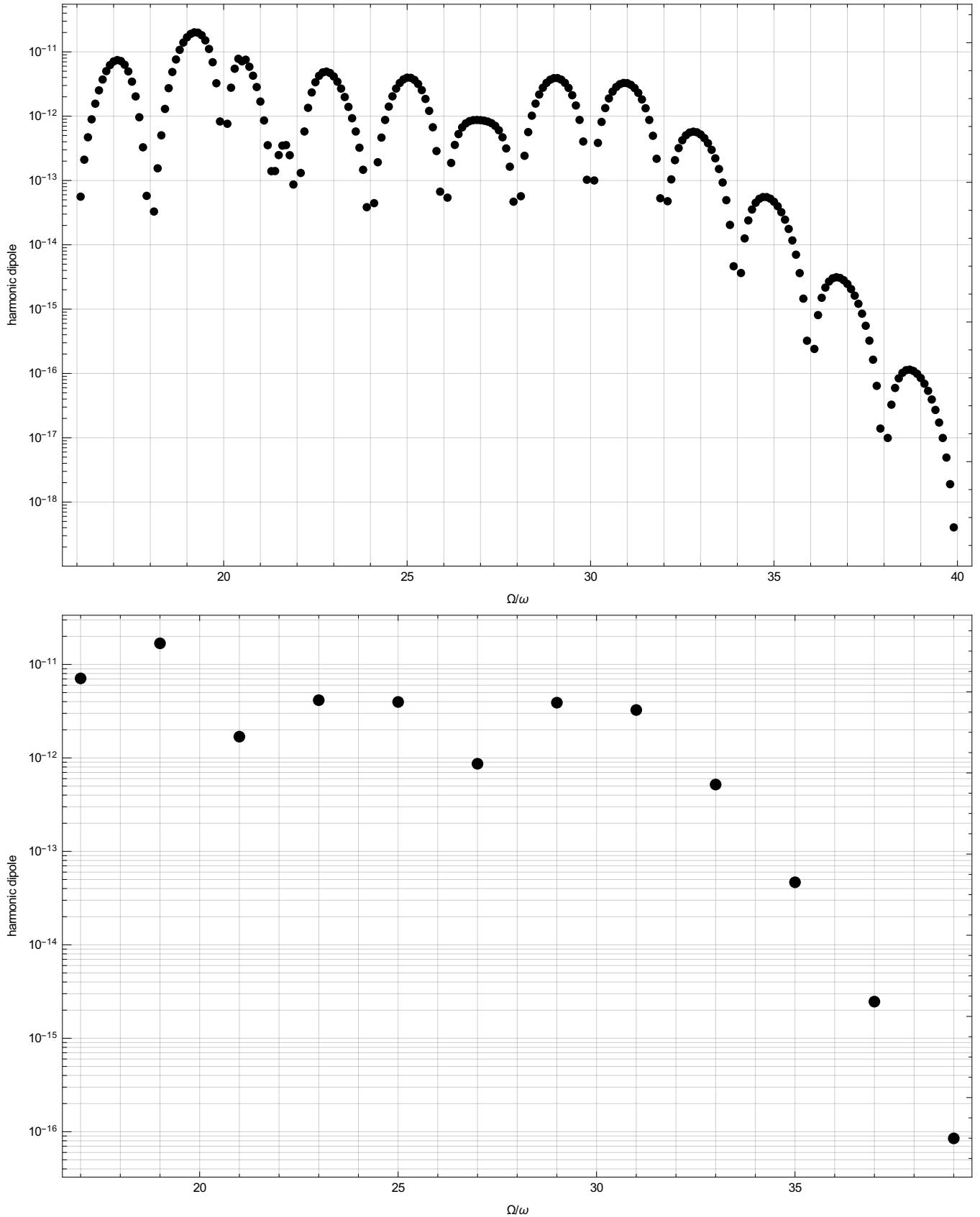
Saddle-point total spectrum

```

Block[{ω, Ip, κ, U, γ, selection,
       classifierFunction, sortingFunction, keyColour, transitions},
      {ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω/F} //.
        parameters;
      classifierFunction = Function[{t, τ, Ω}, Which[
        And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == -1], "A",
        And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == 0], "B",
        And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == 0], "C",
        And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == -1], "D",
        True, "Discard"
      ]];
      sortingFunction =
        Function[list, SortBy[list, Function[Re[ω #[[1]] - Floor[ω Re[#[[1]] - #[[2]]], 2 π]]]]];
      (*keyColour = <|"A" → <|1 → Black, 2 → Blue|>, "B" → <|1 → Red, 2 → Magenta|>,
         "C" → <|1 → Darker[Green], 2 → Orange|>, "D" → <|1 → Darker[Cyan], 2 → Purple|>|>;*)
      keyColour = <|"A" → Black, "B" → Red, "C" → Darker[Green], "D" → Purple|>;
      selection = ClassifyQuantumOrbits[saddlePoints,
        classifierFunction, sortingFunction, DiscardedLabels → {"Discard"}];
      transitions = FindStokesTransitions[S, selection];
    
```

```

Column[{
  ListLogPlot[
    DeleteCases[
      KeyValueMap[
        {#1 / ω, Norm[Total[#2]]^2} &
        , Query[Transpose][
          MapIndexed[(*calculate the dipole for each label and harmonic energy*)
            With[{saddles = #1, index = #2[[1, 1]], Ω = #2[[2, 1]]},
              SPAdipole[S, prefactor, Ω,
                If[Ω < transitions[[index, 2]], saddles, selection[index, Ω][[transitions[[index, 3]]]]]
                ]
              ] &, selection, {2}]]
        ], {n_, d_ /; Abs[d] < 10^-20}]
      , Frame → True
      , GridLines → All
      , PlotStyle → Black
      , ImageSize → 700
      , FrameLabel → {"Ω/ω", "harmonic dipole"}
    ]
  ]
  , ListLogPlot[
    KeyValueMap[
      {#1 / ω, Norm[Total[#2]]^2} &
      , Query[Transpose][
        MapIndexed[(*calculate the dipole for each label and harmonic energy*)
          With[{saddles = #1, index = #2[[1, 1]], Ω = #2[[2, 1]]},
            SPAdipole[S, prefactor, Ω, selection[index, Ω], transitions[index]
            ]
            ] &, KeySelect[Abs[# - Round[#, ω]] < 0.05 ω && OddQ[Round[# / ω]] &] /@
              selection[[All(*,11;;-1;;20*)]], {2}]]
      ]
    ]
    , Frame → True
    , GridLines → All
    , PlotStyle → Black
    , ImageSize → 700
    , FrameLabel → {"Ω/ω", "harmonic dipole"}
  ]
]
}
]
]
]
```



Uniform Approximation per-pair spectrum

```

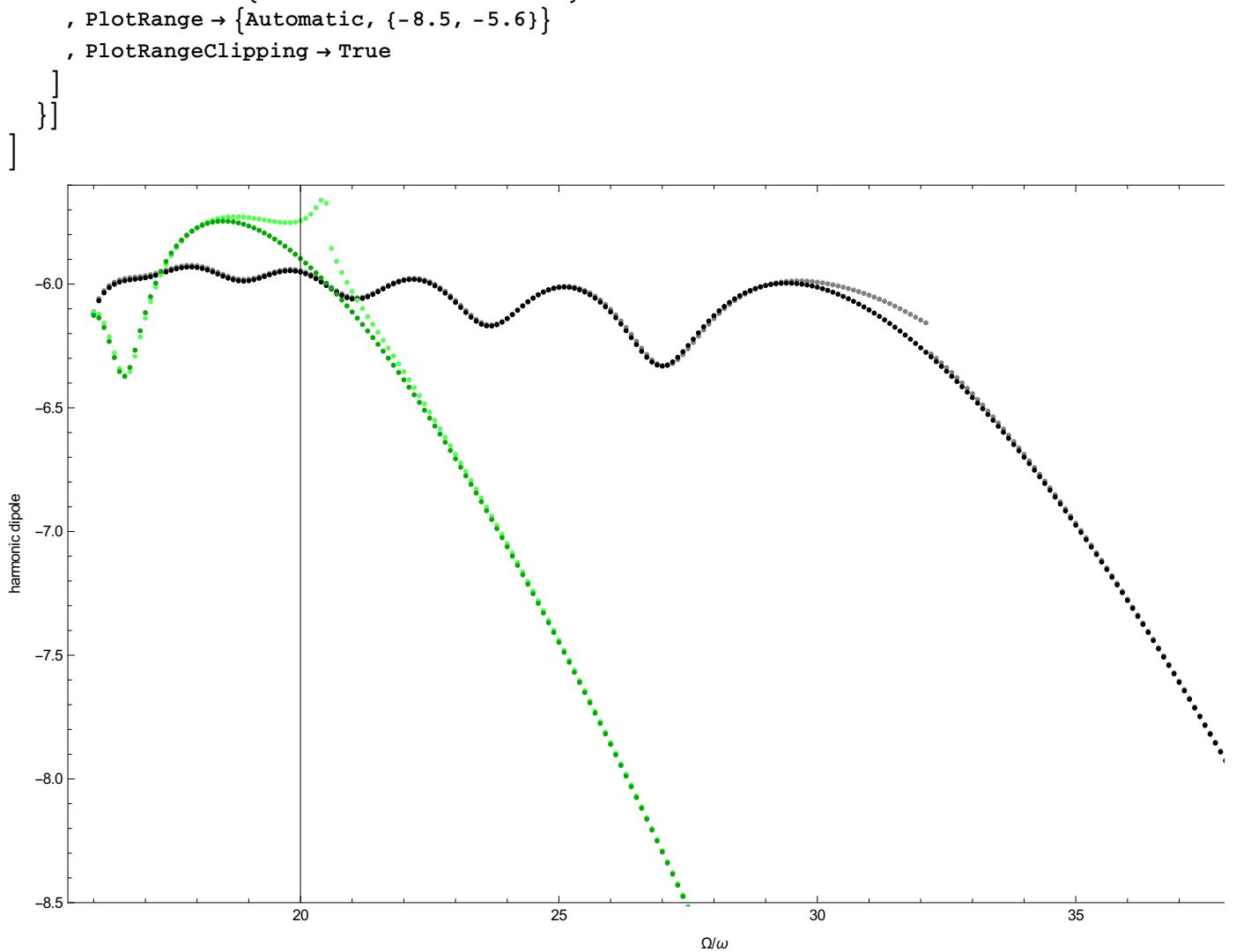
Block[{ω, Ip, κ, U, γ, selection,
       classifierFunction, sortingFunction, keyColour, transitions},
{ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω/F} //.
parameters;

classifierFunction = Function[{t, τ, Ω}, Which[
  And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == -1], "A",
  (*And[1.65<ω Re[τ]<6.3, Floor[ω Re[t-τ],π]==0], "B",*)
  And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == 0], "C",
  (*And[6.3<ω Re[τ]<8.9, Floor[ω Re[t-τ],π]==-1], "D",*)
  True, "Discard"
]];
sortingFunction =
  Function[list, SortBy[list, Function[Re[ω #[[1]] - Floor[ω Re[#[[1]] - #[[2]]], 2 π]]]]];
keyColour = <|"A" → Black, "B" → Red, "C" → Darker[Green], "D" → Purple|>;

selection = ClassifyQuantumOrbits[saddlePoints,
  classifierFunction, sortingFunction, DiscardedLabels → {"Discard"}];
transitions = FindStokesTransitions[S, selection];

Column[{Show[
  Graphics[
    Table[Table[{keyColour[index] /. {Missing[_] → Gray} /.
      {Black → Gray, Darker[Green] → Lighter[Green]},
    Tooltip[
      Point[
        {Ω/ω, Log10[(*10^-8.5+*)Norm[
          SPAdipole[S, prefactor, Ω, selection[index, Ω], transitions[index]]]
        ]}]
      ], {Ω/ω, index, selection[index, Ω]}],
    keyColour[index] /. {Missing[_] → Gray},
    Tooltip[
      Point[
        {Ω/ω, Log10[(*10^-8.5+*)Norm[
          UAAdipole[S, prefactor, Ω, selection[index, Ω], transitions[index]]]
        ]}]
      ], {Ω/ω, index, selection[index, Ω]}]
    }
  , {Ω, Keys[selection[index]]}], {index, Keys[selection]}]
]
, Frame → True, Axes → True
, ImageSize → 800
, AspectRatio → 1/1.8
, FrameLabel → {"Ω/ω", "harmonic dipole"}]

```



Uniform Approximation total spectrum

```

Block[{ω, Ip, κ, U, γ, selection,
  classifierFunction, sortingFunction, keyColour, transitions},
{ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω/F} //.
  parameters;
classifierFunction = Function[{t, τ, Ω}, Which[
  And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == -1], "A",
  And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == 0], "B",
  And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == 0], "C",
  And[6.3 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == -1], "D",
  True, "Discard"
  ]];
sortingFunction =

```

```

Function[list, SortBy[list, Function[Re[w #[[1]] - Floor[w Re#[[1]] - #[[2]]], 2 π]]]];
(*keyColour= <|"A"→<|1→Black,2→Blue|>,"B"→<|1→Red,2→Magenta|>,
 "C"→<|1→Darker[Green],2→Orange|>,"D"→<|1→Darker[Cyan],2→Purple|>>|>;*)
keyColour = <|"A" → Black, "B" → Red, "C" → Darker[Green], "D" → Purple|>;
selection = ClassifyQuantumOrbits[saddlePoints,
 classifierFunction, sortingFunction, DiscardedLabels → {"Discard"}];
transitions = FindStokesTransitions[S, selection];
Column[{  

  ListLogPlot[  

    DeleteCases[  

      KeyValueMap[  

        {#1 / w, Norm[Total[#2]]^2} &  

       , Query[Transpose][  

         MapIndexed[(*calculate the dipole for each label and harmonic energy*)  

          With[{saddles = #1, index = #2[[1, 1]], Ω = #2[[2, 1]]},  

            UAdipole[S, prefactor, Ω,  

              selection[index, Ω]  

              , transitions[index]]  

            ] &, selection, {2}]]  

      ], {n_, d_ /; Abs[d] < 10^-20}]  

     , Frame → True  

     , GridLines → All  

     , PlotStyle → Black  

     , ImageSize → 700  

     , FrameLabel → {"Ω/w", "harmonic dipole"}  

    ]  

   , ListLogPlot[  

     KeyValueMap[  

       {#1 / w, Norm[Total[#2]]^2 + 10^-20} &  

      , Query[Transpose][  

        MapIndexed[(*calculate the dipole for each label and harmonic energy*)  

          With[{saddles = #1, index = #2[[1, 1]], Ω = #2[[2, 1]]},  

            UAdipole[S, prefactor, Ω,  

              selection[index, Ω]  

              , transitions[index]]  

            ] &, KeySelect[Abs[# - Round[#, w]] < 0.05 w && OddQ[Round[#/w]] &] /@ selection, {2}]]  

    ]  

   , Frame → True  

   , GridLines → All  

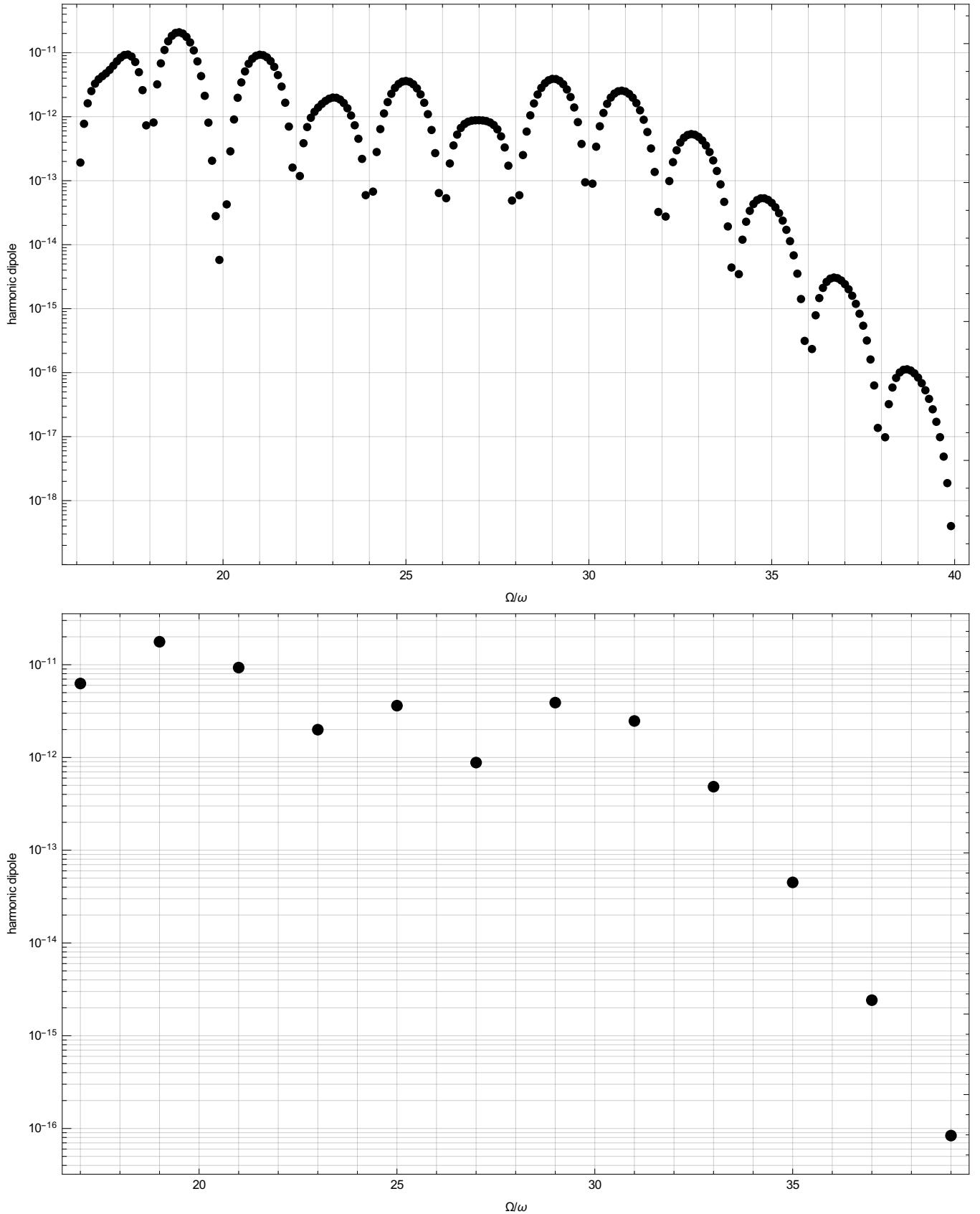
   , PlotStyle → Black  

   , ImageSize → 700  

   , FrameLabel → {"Ω/w", "harmonic dipole"}  

  ]  

}]
]
]
```



Testing the degrading of UAdipole to SPAdipole when there's too few or too many solutions

The classifierFunction below has been explicitly chosen so that the spectrum gets damaged: the 'keeper' member of the A pair has been cut off by asking for $\text{Im}(\omega t) < 0.5$, and the lower $\text{Re}(\omega t)$ limit of the C pair has been chosen too low so that there is an extra saddle intruding upon the set.

```

Block[{ω, Ip, κ, U, γ, selection, classifierFunction,
       sortingFunction, keyColourPoints, keyColourSpectrum, transitions},
      {ω, Ip, κ, U, γ} = {ω, Ip, Sqrt[2 Ip], F^2/(4 ω^2), κ ω/F} //.
        parameters;
      classifierFunction = Function[{t, τ, Ω}, Which[
        And[1.65 < ω Re[τ] < 6.3, Floor[ω Re[t - τ], π]/π == -1, Im[ω t] < 0.5], "A",
        (*And[1.65<ω Re[τ]<6.3, Floor[ω Re[t-τ],π]==0], "B", *)
        And[5.654 < ω Re[τ] < 8.9, Floor[ω Re[t - τ], π]/π == 0, Im[ω t] > -0.4], "C",
        (*And[6.3<ω Re[τ]<8.9, Floor[ω Re[t-τ],π]==-1], "D", *)
        True, "Discard"
      }];
      sortingFunction =
        Function[list, SortBy[list, Function[Re[ω #[[1]] - Floor[ω Re#[[1]] - #[[2]]], 2 π]]]];
      (*keyColour=⟨|"A"→⟨|1→Black,2→Blue|⟩,"B"→⟨|1→Red,2→Magenta|⟩,
      "C"→⟨|1→Darker[Green],2→Orange,3→Darker[Red]|⟩,"D"→⟨|1→Darker[Cyan],2→Purple|⟩|>;*)
      keyColourPoints = ⟨|"A" → ⟨|1 → Black, 2 → Blue|⟩, "B" → ⟨|1 → Red, 2 → Magenta|⟩, "C" →
        ⟨|1 → Darker[Green], 2 → Orange, 3 → Darker[Red]|⟩, "D" → ⟨|1 → Darker[Cyan], 2 → Purple|⟩|>;
      keyColourSpectrum = ⟨|"A" → Black, "B" → Red, "C" → Darker[Green], "D" → Purple|⟩;
      selection = ClassifyQuantumOrbits[saddlePoints,
        classifierFunction, sortingFunction, DiscardedLabels → {"Discard"}];
      transitions = FindStokesTransitions[S, selection];

      Column[{
        transitions,
        Show[
          Graphics[
            Table[Table[{{
              KeyValueMap[
                Function[{n, t, τ},
                  {keyColourPoints[[index, n]] /. Missing[_] → Gray,
                   Tooltip[Point[ReIm[ω t]], {Ω/ω, index, n, ω {t, τ}, 1/π Floor[ω Re[t - τ], π]}]
                  ] @* Apply[Sequence] @* Flatten @* List
                  , selection[[index, Ω]]
                }, {Ω, Keys[selection[[index]]]}], {index, Keys[selection]}]
            ]]
        ]
      ]
    ]
  
```

```

    , Frame → True, Axes → True
    , ImageSize → 800
    , FrameLabel → {"Re(ωt)", "Im(ωt)"}
]
, Show[
Graphics[
Table[Table[{{
keyColourSpectrum[index] /. {Missing[_] → Gray}
(*/.{Black→Gray,Darker[Green]→Lighter[Green]}*),
Tooltip[
Point[
{Ω / ω, Log10[10-7 + Norm[
SPAdipole[S, prefactor, Ω, selection[index, Ω], transitions[index]]
]]}
], {Ω / ω, index, ω selection[index, Ω], "SPA"}},
keyColourSpectrum[index] /. {Missing[_] → Gray},
Tooltip[
Point[
{Ω / ω, Log10[10-7 + Norm[
UAdipole[S, prefactor, Ω, selection[index, Ω], transitions[index]]
]]}
], {Ω / ω, index, ω selection[index, Ω], "UA"}]
}
, {Ω, Keys[selection[index]]}], {index, Reverse@Keys[selection]}]
]
, Frame → True, Axes → True
, ImageSize → 800
, AspectRatio → 1 / 1.8
, FrameLabel → {"Ω/ω", "harmonic dipole"}
, PlotRange → {Automatic, {-7.1, -5.3}}
, PlotRangeClipping → True
]
]
]
]
```

FindStokesTransitions::saddleno :

FindStokesTransitions called with 55 of 241 saddle-point sets of length different from 2, with set length structure {{2, 186}, {1, 55}}. Excluding those sets from the calculation.

FindStokesTransitions::saddleno :

FindStokesTransitions called with 219 of 241 saddle-point sets of length different from 2, with set length structure {{3, 33}, {2, 22}, {1, 186}}. Excluding those sets from the calculation.

SPAdipole::wrongno : SPAdipole called with a Stokes transition but with an input

association of length 3 at harmonic energy $\Omega=0.912`$. Reverting to unstructured evaluation.

UAdipole::saddleno : UAdipole called with 3 time pairs at $\Omega=0.912`$. Reverting to the saddle-point approximation for this set.

SPAdipole::wrongno : SPAdipole called with a Stokes transition but with an input association of length 3 at harmonic energy $\Omega=0.9177000000000001`$. Reverting to unstructured evaluation.

UAdipole::saddleno :

UAdipole called with 3 time pairs at $\Omega=0.9177000000000001`$. Reverting to the saddle-point approximation for this set.

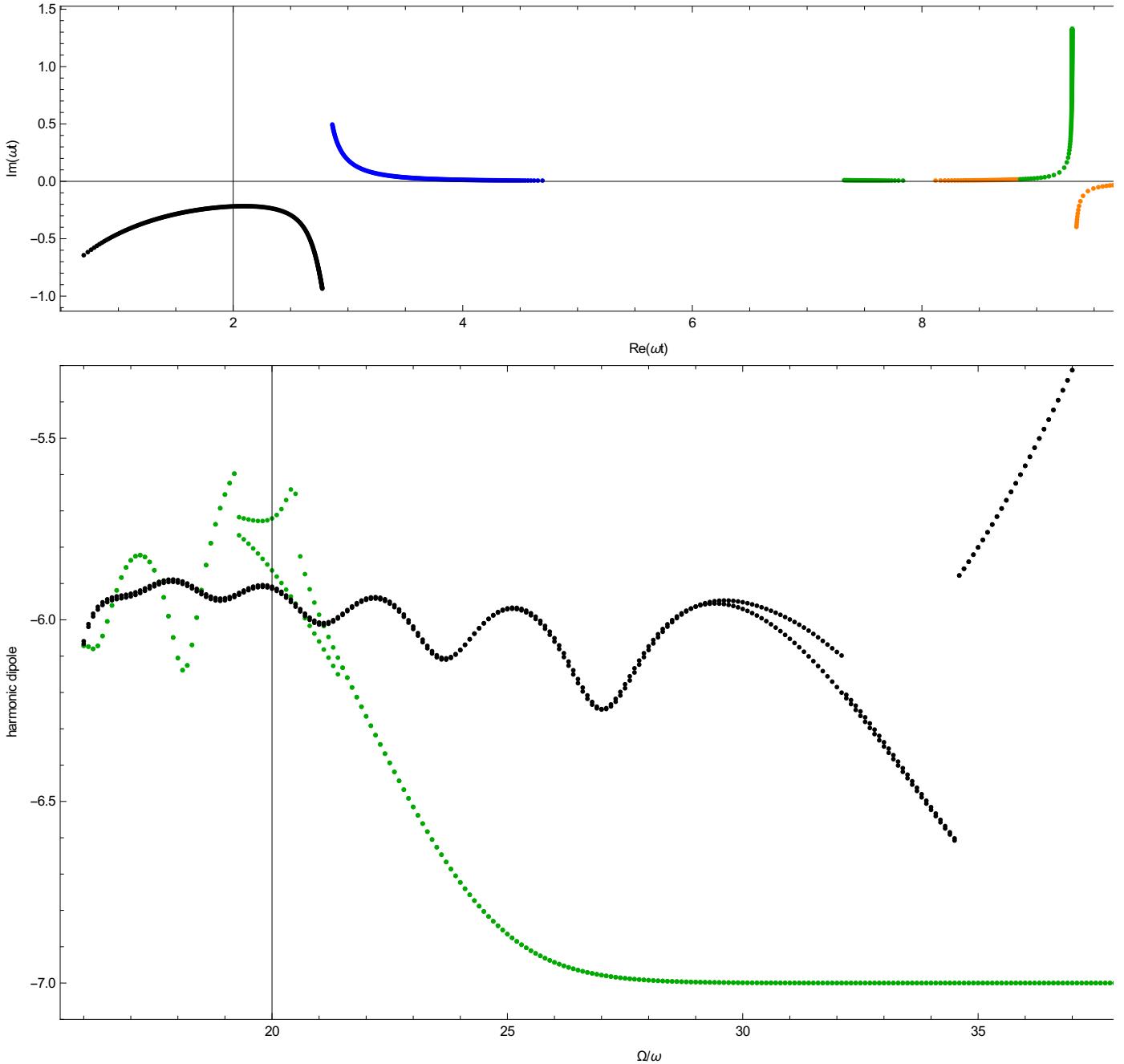
SPAdipole::wrongno : SPAdipole called with a Stokes transition but with an input association of length 3 at harmonic energy $\Omega=0.9234`$. Reverting to unstructured evaluation.

General::stop : Further output of `SPAdipole::wrongno` will be suppressed during this calculation. >>

`UAdipole::saddleno` : `UAdipole` called with 3 time pairs at $\Omega=0.9234^\circ$. Reverting to the saddle-point approximation for this set.

General::stop : Further output of `UAdipole::saddleno` will be suppressed during this calculation. >>

$\langle |A \rightarrow \{1.767, 1.8354, 2\}, C \rightarrow \{1.1628, 1.1742, 1\}| \rangle$



The spectrum for this saddle-point classification is very ugly, but it does exactly what it needs to be doing.

- For $\Omega \lesssim 19.5 \omega$, the C set gets a third saddle from what ought to be B, and UAdipole surrenders the calculation to SPAdipole, so the two match at low Ω for the green curve.
- For $\Omega \lesssim 19.5 \omega$ on the C set, there is a third saddle contributing to SPAdipole, so it's displaying a bigger dipole with more contributions (and also more interference). This saddle gets dropped after $\Omega = 19.4 \omega$, which gives the discontinuity there.
- For $\Omega \gtrsim 35 \omega$, the wanted root (A,1) is no longer present. The UAdipole call surrenders the calculation to SPA dipole and the two match on the black curve after $\Omega = 34.6 \omega$.

- For $\Omega \gtrsim 35 \omega$ on the SPA branch it's getting a malformed input so it ignores the Stokes transition and evaluates the saddle it's been given, which is the divergent one at $\text{Im}(t) < 0$, $\text{Im}(S(t)) < 0$, and this gives the exponential growth after $\Omega = 34.6 \omega$
- For $\Omega \gtrsim 21.5 \omega$, the C set loses the $\text{Im}(t) < 0$ root, which is the unwanted $\text{Im}(S(t)) < 0$ one, so everything is mostly fine. Here the SPAdipole call has a single saddle so it issues a warning that it's reverting to unstructured evaluation, but this does not change the outcome.
- For $\Omega \gtrsim 21.5 \omega$ on the C set, the UAdipole call has a single saddle so it reverts to unstructured SPAdipole, which is a good approximation here, but this causes a slight discontinuity at $\Omega = 21.5 \omega$; both curves match after that.