Brownian motion

The `brownian_motion_1D` and `brownian_motion_2D` functions, written with the `cumsum` command and without `for` loops, are used to generate a one-dimensional and two-dimensional Brownian motion, respectively. These Wiener processes are characterized by normal-centered increments with variance h, where h is the time increment, generated by the command `randn(1,n)*sqrt(h)`. We consider a time interval $T = 1000$, divided into $n = 1000$ increments of value $h = 1$. Figure 1 shows, for example, two trajectories $W(t)$ of a one-dimensional Wiener process. Figure 2, on the other hand, shows two examples of a two-dimensional Brownian motion trajectory, this time as a function of the X and Y spatial coordinates.

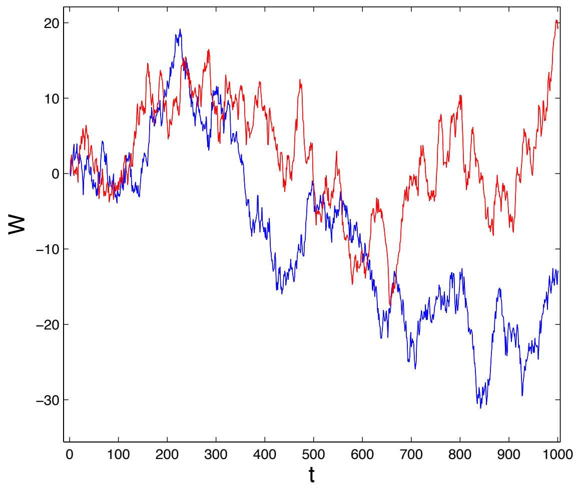

FIGURE 1 – Two examples of trajectories as a function of the time t of a Wiener process $W(t)$ in one dimension.

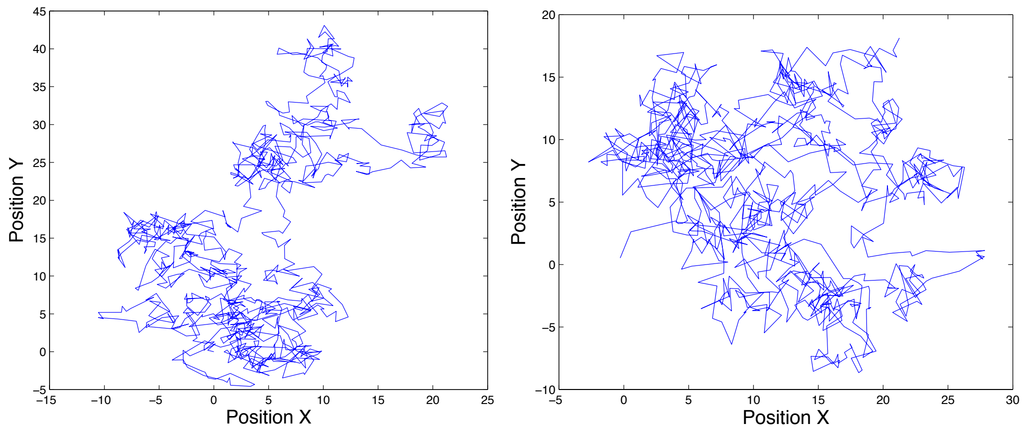

FIGURE 2 – Two examples of trajectories of a two dimensional Wiener process in the plane XY.

Given N (number of steps), M (number of trajectories) and T (maximum of the time interval), we generate a matrix `W_all` containing M trajectories of the Brownian motion in one dimension on the interval $[0, T]$ with a discretization step $h = T/N$. Figure 3 shows $M = 10, 100, 1000$ trajectories over the interval $[0, 10]$ with $N = 1000$ steps.

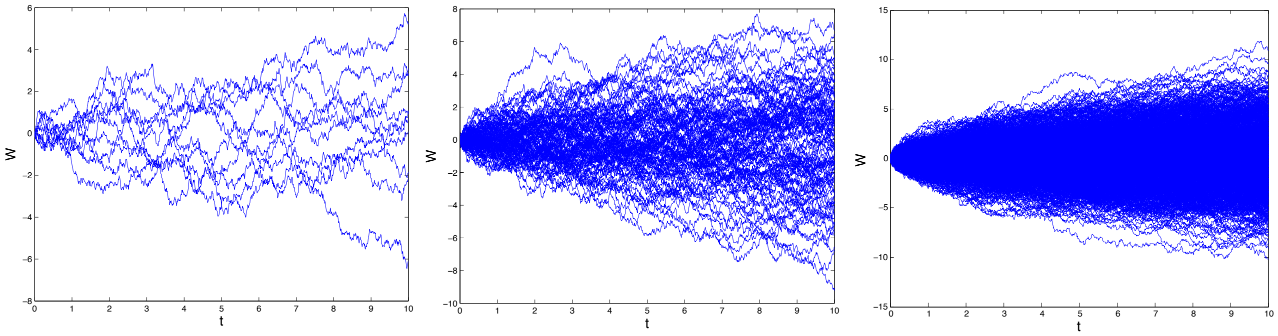

FIGURE 3 – $M = 10, 100, 1000$ (from left to right) trajectories of a one-dimensional Wiener process over the time interval $[0, 10]$ with $N = 1000$ discretisation steps.

We simulate $M = 1000$ trajectories over the interval $[0, 10]$. Figure 4 shows the mean and the variance over time of these trajectories. In contrast, figure 5 shows the expectation values $\mathbb{E}[W(t)]$, $\mathbb{E}[W(t)^2]$ et $\mathbb{E}[W(t)^4]$ obtained numerically as a function of time. The first moment corresponds exactly to the average. In the presence of a zero mean, the variance is equivalent to the moment $\mathbb{E}[W(t)^2]$. The red lines in each panel of Figure 5 show that the equalities $\mathbb{E}[W(t)] = 0$, $\mathbb{E}[W(t)^2] = t$, and $\mathbb{E}[W(t)^4] = 3t^2$ are satisfied.

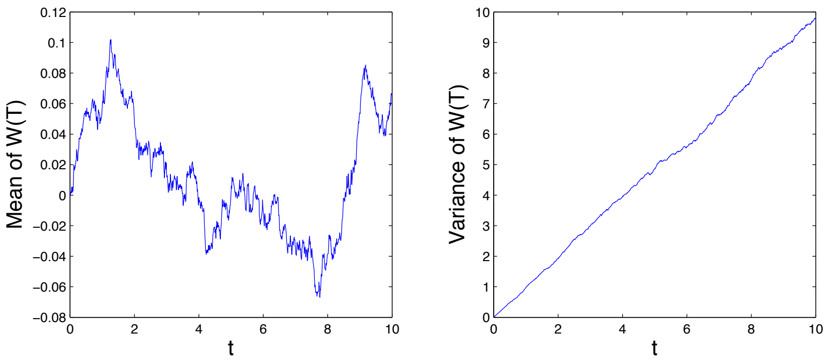

FIGURE 4 – Mean and variance of $M = 1000$ trajectories of a Brownian motion in one dimension.

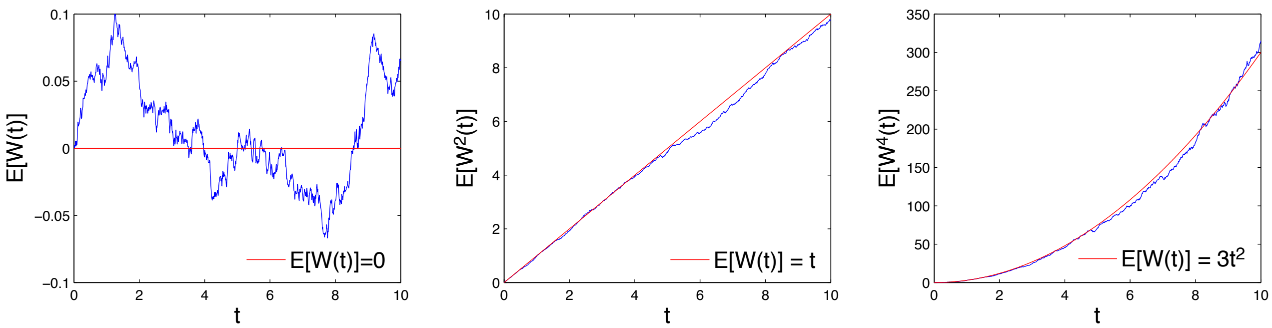

FIGURE 5 – Expectation values $\mathbb{E}[W(t)]$, $\mathbb{E}[W(t)^2]$ and $\mathbb{E}[W(t)^4]$ calculated numerically and compared with the curves (in red) expected theoretically.