MSRDT: An R package for Multi-State Reliability Demonstration Tests

Suiyao Chen

suiyaochen@usf.edu

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Abstract

The MSRDT R package introduces a novel approach to address the limitations of traditional reliability demonstration tests (RDTs). This package enables the design, simulation, and analysis of multi-state RDTs that account for varying reliability requirements across different time periods and failure modes. By integrating Bayesian analysis, historical data, and expert knowledge, MSRDT empowers users to make more informed decisions, potentially reducing required sample sizes. Through case studies and simulations, we showcase the effectiveness of the proposed strategies in meeting modern reliability assessment needs. This package represents a significant advancement in reliability engineering, offering a versatile solution to enhance customer satisfaction, ensure product safety, and optimize resource allocation in the face of evolving reliability standards. The package is available through the Comprehensive R Archieve Network and GitHub.com.

Keyword: R, MSRDT, BRDT, Simulation, Reliability, Sample size

Highlights

- This paper documents an R package for multi-state reliability demonstration tests, which is the method proposed in [1].
- The package implements two types of MSRDT, multiple periods (MP) and multiple failure modes (MFM).
- For MP, two different scenarios with criteria on cumulative periods (Cum) or separate periods (Sep) are implemented respectively.

Software Availability

`MSRDT` is available on the world wide web at
`https://cran.r-project.org/web/packages/MSRDT/index.html` [6; 7] and
`https://github.com/ericchen12377/MSRDT`
License: MIT
System Requirements: R version 3.3.0 or higher

Contents

1 Introduction

The ever-evolving landscape of reliability assurance and quality assessment demands innovative approaches to address the multifaceted challenges faced by industries. In response to this need, the Multi-State Reliability Demonstration Tests (`MSRDT`) R package emerges as a powerful toolkit, enabling researchers and practitioners to navigate the intricacies of product reliability. In this introduction, we delve into the reliability demonstrate test (RDT) designs and showcase the practices applying `MSRDT`.

Reliability is vital for product quality and customer satisfaction. It's the probability of a product performing its function over time. Manufacturers conduct Reliability Demonstration Tests (RDTs) to prove their products meet customer requirements. These tests involve decisions on the number of units, test duration, and maximum allowed failures, balancing budget and time constraints. Controlling Consumer's Risk (CR) is crucial in RDTs to ensure products that pass the test meet reliability standards when used. Lower CR enhances customer satisfaction and trust, leading to more repeat purchases, revenue, and competitiveness in the market. Reliability directly impacts a product's quality, safety, and market success.

Reliability demonstration tests (RDTs) play a critical role in assuring product quality and maintaining a competitive edge in the global market. However, traditional binary RDTs, which focus on pass or fail outcomes at a single time point, may no longer suffice due to diverse consumer demands. The multi-state reliability demonstration tests (MSRDTs) are capable of assessing reliability across various time periods or failure modes. MSRDTs utilize a Bayesian approach, allowing the integration of prior knowledge and potentially reducing the required sample size. These tests can simultaneously address multiple objectives and explicitly demonstrate compliance with critical requirements, ensuring high customer satisfaction.

Different categories of RDTs have been studied in the literature based on different types of reliability data, such as failure counts data [8; 2; 3], failure time data [9; 14] and degradation data [20]. Failure counts data reports the number of failures that occur during a fixed test period. The RDTs based on failure counts data [17, pp. 208-210] are also called binomial RDTs (BRDTs) since failure counts are modeled with binomial distributions. To meet the ever-increasing demands of customers, MSRDTs serve as more versatile RDTs with more tailored plans for testing multiple reliability requirements, which can better serve the customers with enriched information on product reliability.

In the MSRDT package, two categories of reliability demonstration tests are implemented, over multiple time periods and for multiple failure modes, both of which are referred to as multi-state RDTs (MSRDTs). For the convenience of comparison, the methods for conventional BRDTs [4; 5] are also implemented. Bayesian analysis is used for quantifying the CR associated with various test plans. The Bayesian method offers more flexibility on incorporating prior information of product reliability from either subject matter expert knowledge or historical data [19]. The impacts of different test strategies and different prior elicitations on the minimum test sample size (i.e. the number of test units required) are illustrated to provide more insights on guiding decisions on demonstration test plans.

The remaining of the paper is organized as follows. In the next section, the mathematical background for all implemented methods in MSRDT are discussed, including BRDT, MSRDT over multiple time periods and over multiple failure modes. Then, the comprehenive package design will be introduced to discuss the details on how each method is implemented. Finally, examples will be given for each type of RDT in the packages. The paper will conclude with brief summary, overview of how the package has been utilized for various research topics and discussion of future work.

2 Mathematical Background

2.1 Binomial RDTs

For one-shot products where testing is destructive, Binomial Reliability Demonstration Tests (BRDTs) are commonly used to gather failure count data after a predefined test period [11, pp.759-768]. Let π denote the probability of failure over the test period, and R denote the minimum acceptable reliability at the end of the test duration. In Bayesian analysis, for a given number of test units n and a maximum allowable failure count c, we calculate the Consumer's Risk (CR) as the probability that the product will fail to meet the reliability requirement after passing the test:

$$\begin{aligned}
\mathrm{CR}_{\mathrm{binomial}} &= P(\text{Failure probability fails to meet the reliability requirement}|\text{Test is passed}) \\
&= P(\pi > 1-R|y \le c) \\
&= 1 - P(\pi \le 1-R|y \le c) \\
&= 1 - \frac{\int_0^{1-R}[\sum_{y=0}^{c}\binom{n}{y}\pi^y(1-\pi)^{n-y}]p(\pi)d\pi}{\int_0^{1}[\sum_{y=0}^{c}\binom{n}{y}\pi^y(1-\pi)^{n-y}]p(\pi)d\pi}. \qquad (1)
\end{aligned}$$

Here, $p(\pi)$ represents the prior distribution of π, which can be informed by expert knowledge or historical data, and y denotes the number of failures observed during the test period. The BRDT is determined by choosing the (n, c) combination that yields a $\mathrm{CR}_{\mathrm{binomial}}$ value within the acceptable range β. For a fixed c, increasing the test sample size n also increases $\mathrm{CR}_{\mathrm{binomial}}$ [13]. We denote the minimum required test sample size as n_{b}.

In Bayesian analysis, we approximate $\mathrm{CR}_{\mathrm{binomial}}$ using Monte Carlo integration [16, pp.71-131], where a large number of samples of π of size $M = 15000$ are generated from the specified prior distribution $p(\pi|x)$, and $\mathrm{CR}_{\mathrm{binomial}}$ is calculated approximately by

$$\mathrm{CR}_{\mathrm{binomial}} \approx 1 - \frac{\sum_{j=1}^{M}[\sum_{y=0}^{c}\binom{n}{y}(\pi^{(j)})^y(1-\pi^{(j)})^{n-y}]I(\pi^{(j)} \le 1-R)}{\sum_{j=1}^{M}[\sum_{y=0}^{c}\binom{n}{y}(\pi^{(j)})^y(1-\pi^{(j)})^{n-y}]}, \qquad (2)$$

where $\pi^{(j)}$ is the jth generated sample of failure probability for the specified prior distribution.

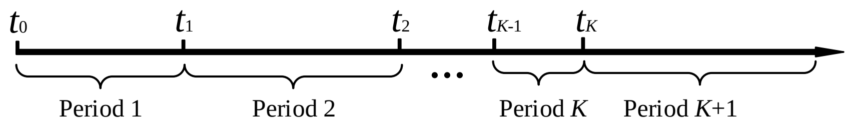

Figure 1: Illustration of the multiple time periods in a K-period MSDRT between $(t_0, t_K]$

2.2 MSRDTs over Multiple Time Periods

Consider a finite testing period with the start time at t_0 and end time at t_K. The testing time duration $(t_0, t_K]$ is exclusively partitioned into K non-overlapping time periods, $(t_{i-1}, t_i], i = 1, ..., K$, as illustrated in Figure 1. Let π_i and y_i denote the probability of failure and the number of observed failures within the i^{th} time period $(t_{i-1}, t_i]$, respectively. Then the number of units that survive the entire test duration (right-censored at the end of the test duration t_K) can be expressed as $n - \sum_{i=1}^{K} y_i$, where n is the total number of test units. The probability of surviving the test is given by $\pi_{K+1} = 1 - \sum_{i=1}^{K} \pi_i$. The objective of a MSRDT over multiple time periods is to simultaneously demonstrate the product reliability at multiple time points satisfying a set of lower reliability requirements, $R_i, i = 1, \cdots, K$, with the assurance level controlled at $(1-\beta)$. Here, R_i is the minimum acceptable reliability in the first i cumulative time periods, $(t_0, t_i]$, β is the maximum acceptable consumer's risk and assurance level can be explained as the minimum level of probability that the test can be passed [10, pp.343-346].

2.2.1 Passing Criteria on Cumulative Periods

Scenario I: The MSRDT will be passed if the cumulative number of observed failures $\sum_{k=1}^{i} y_k$ at each cumulative time period $(t_0, t_i]$ is no more than its corresponding cumulative maximum number of allowable failures $\sum_{k=1}^{i} c_k$ for all cumulative time periods $(t_0, t_i]$, at $i = 1, ..., K$. For example, consider a two-period MSRDT with tests conducted at the end of the second and fifth year. For 100 test units, the MSRDT will be passed if the number of observed failures in first two years do not exceed 1 and the number of observed failures at the end of the fifth years do not exceed 5.

2.2.2 Passing Criteria on Separate Periods

Scenario II: The MSRDT will be passed if the number of observed failures y_i at each non-overlapping time period $(t_{i-1}, t_i]$ is no greater than its corresponding maximum number of allowable failures c_i for all time periods $(t_{i-1}, t_i]$, at $i = 1, ..., K$. For such two-period test, the MSRDT will be passed if the number of observed failures in first two years do not exceed 1 and the number of observed failures in the next three years do not exceed 4. It is noticed that the major difference between the two scenarios is that scenario II plans the tests for non-overlapping time periods while scenario I considers the cumulative time-periods instead.

For each acceptance criterion, the design of MSRDT over multiple time periods aims to determine (i) the minimum sample size, denoted by n_{I} and n_{II} for scenarios I and II, respectively, and (ii) the cumulative maximum number of allowable failures at time t_i, $\sum_{k=1}^{i} c_k$, for scenario I and the maximum number of allowable failures within i^{th} time period, c_i, $i = 1, ..., K$ for scenario II. For either scenario, the MSRDT is selected by choosing the test plans which control the CR at or below β.

To illustrate the proposed MSRDTs over multiple time periods and further investigate the difference between two scenarios of acceptance criteria, the MSRDTs over two time periods (i.e., $K = 2$) are considered without loss of generality. Let R_1 and R_2 denote the minimum acceptable reliabilities over the time periods $(t_0, t_1]$ and $(t_0, t_2]$ with $R_2 < R_1$. The probabilities of failure for each time period meet the requirements if $\pi_1 \leq 1 - R_1$ and $\pi_1 + \pi_2 \leq 1 - R_2$. For acceptance criterion in scenario I, the test plan of MSRDT is to determine $\{n_{\text{I}}, c_1, c_1 + c_2\}$, and the probability of accepting the test plan for any given (π_1, π_2), denoted by $H_{\rm I}(n, c_1, c_2)$, can be explicitly written as

$$H_{\rm I}(n, c_1, c_2) = \sum_{y_1=0}^{c_1} \sum_{y_2=0}^{c_1+c_2-y_1} [(\frac{n!}{y_1!y_2!(n-y_1-y_2)!})\pi_1^{y_1}\pi_2^{y_2}(1-\pi_1-\pi_2)^{n-y_1-y_2}]$$

and the corresponding $\text{CR}_{\rm I}$ is controlled at or below β by

$$\text{CR}_{\rm I} = 1 - \frac{\int_0^{1-R_1}\int_0^{1-R_2-\pi_1} H_{\rm I}(n, c_1, c_2)p(\pi_1, \pi_2)d\pi_2 d\pi_1}{\int_0^1\int_0^1 H_{\rm I}(n, c_1, c_2)p(\pi_1, \pi_2)d\pi_2 d\pi_1} \leq \beta, \tag{3}$$

where $p(\pi_1, \pi_2)$ denotes the joint prior distribution of $(\pi_1, \pi_2, 1-\pi_1-\pi_2)$.

For the acceptance criterion in scenario II, the MSRDT plan can be determined by specifying $\{n_{\rm II}, c_1, c_2\}$, and the probability of accepting the test plan for any combination of (π_1, π_2), denoted by $H_{\rm II}(n, c_1, c_2)$ is given by

$$H_{\rm II}(n, c_1, c_2) = \sum_{y_1=0}^{c_1} \sum_{y_2=0}^{c_2} [(\frac{n!}{y_1!y_2!(n-y_1-y_2)!})\pi_1^{y_1}\pi_2^{y_2}(1-\pi_1-\pi_2)^{n-y_1-y_2}],$$

and the corresponding $\text{CR}_{\rm II}$ is controlled by

$$\text{CR}_{\rm II} = 1 - \frac{\int_0^{1-R_1}\int_0^{1-R_2-\pi_1} H_{\rm II}(n, c_1, c_2)p(\pi_1, \pi_2)d\pi_2 d\pi_1}{\int_0^1\int_0^1 H_{\rm II}(n, c_1, c_2)p(\pi_1, \pi_2)d\pi_2 d\pi_1} \leq \beta. \tag{4}$$

2.3 MSRDTs for Multiple Failure Modes

Assume a product has J independent failure modes. For each test unit, it will either have failed in mode $j, j = 1, ..., J$ or remain working by the end of the testing period. Let π_j and y_j denote the probability of failure and the number of observed failures in failure mode j within the test period (or an equivalent mission time period), respectively. Then, $\pi_{J+1} = 1 - \sum_{j=1}^J \pi_j$ and $n - \sum_{j=1}^J y_j$ denote the probability of success and the number of survived units by the end of the test. The MSRDTs for multiple failure modes aim to demonstrate at an assurance level at $(1-\beta)$ that the product reliability will meet multiple minimum reliability requirements for each of the different failure modes, denoted by $R_j, j = 1, \cdots, J$. Here, β is the CR on having a product that has passed the demonstration test but fails to meet all reliability requirements for different failure modes. Note that all failure modes are defined in the same test period. For any specified reliability requirements R_j's and the maximum acceptable CR controlled at or below β, the MSRDTs for multiple failure modes are designed to determine the minimum sample size $n_{\rm m}$ as well as the maximum number of allowable failures c_j in the jth failure mode for $j = 1, ..., J$.

Without loss of generality, considering two failure modes with $J = 2$ for illustrating the proposed MSRDT strategy. Let R_1 and R_2 denote the minimum acceptable reliabilities for failure modes 1 and 2, respectively. The test is passed if the number of observed failures y_j is less or equal to the maximum number of allowable failures c_j for both failure modes, and the test plan is to determine the choice on $\{n_{\rm m}, c_1, c_2\}$. For independent failure modes, the acceptance probability $H_{\rm m}(n, c_1, c_2)$ for certain (π_1, π_2) values can be written as

$$H_{\rm m}(n, c_1, c_2) = \sum_{y_1=0}^{c_1} [(\frac{n!}{y_1!(n-y_1)!})\pi_1^{y_1}(1-\pi_1)^{n-y_1}] \sum_{y_2=0}^{c_2} [(\frac{n!}{y_2!(n-y_2)!})\pi_2^{y_2}(1-\pi_2)^{n-y_2}]$$

and the corresponding CR, denoted by $\text{CR}_{\rm m}$, is calculated by

$$\text{CR}_{\rm m} = 1 - \frac{\int_0^{(1-R_1)}\int_0^{(1-R_2)} H_{\rm m}(n, c_1, c_2)p(\pi_1, \pi_2)d\pi_2 d\pi_1}{\int_0^1\int_0^1 H_{\rm m}(n, c_1, c_2)p(\pi_1, \pi_2)d\pi_2 d\pi_1}, \tag{5}$$

where $p(\pi_1, \pi_2)$ is the joint prior distribution of (π_1, π_2). For independent failure modes, there is $p(\pi_1, \pi_2) = p(\pi_1)p(\pi_2)$. The minimum sample size is determined by controlling the $\mathrm{CR_m}$ obtained in Eq. (5) to be at or below β.

3 Package Design

For each RDT method, to get the optimal n of testing samples, there are four major steps in the calculation given the bound of CR. First is to simulate the prior distributions of failure probability. For BRDT, the Beta distribution simulator is implemented in `pi_MCSim_beta.R`. For MSRDT, the Dirichlet distribution simulator is implemented in `pi_MCSim_dirichlet.R`.

Second step is to compute the acceptance probability of test plan given specific failure probabilities, which are the cumulative probabilities of the failure distribution. For BRDT, it's implemented in `bcore.R`. For MSRDTs, the multiple time periods Scenario I with passing criteria on cumulative periods is implemented in `MPCum_core.R`. Scenario II with passing criteria on separate periods is implemented in `MPSep_core.R`. Multiple Failure Modes is implemented in `MFM_core.R`.

Third step is to compute the CR, which the the probability of passing the test when reliability requirement is not satisfied. The utility functions for binary check are used to check whether the failure probability satisfies the reliability requirement. For BRDT, the utility function is `bIndicator.R`. For MSRDT, multiple time periods is implemented in `MP_Indicator.R`. Multiple Failure Modes is implemented in `MFM_Indicator.R`. To compute the CR, BRDT has the function `bconsumerrik.R`. For MSRDT, Scenario I has `MPCum_consumerrisk.R` and Scenario II has `MPSep_consumerrisk.R`. Multiple failure mode has `MFM_consumerrisk.R`.

The final step is to compute the optimal n, given the inputs of test criteria and requirements. For BRDT, it's implemented in `boptimal_n.R`. For MSRDT, Scenario I has `MPCum_optimal_n.R` and Scenario II has `MPSep_optimal_n.R`. Multiple failure mode has `MFM_optimal_n.R`.

For each type of RDTs, following the four steps and setting the proper criteria, it will be straightforward to simulate the testing plan and obtain the optimal number of testing samples n.

4 Examples

4.1 BRDT

The following code shows a quick example to generate CR for BRDT and get optimal n.

```
###Generate the prior distribution of failure probability
##Beta is conjugate prior to binomial distribution
#Get the non-informative prior Beta(1, 1)
pi <- pi_MCSim_beta(M = 5000, seed = 10, a = 1, b = 1)

#Get the consumer's risk
n = 10
R = 0.8
c = 2
b_CR <- bconsumerrisk(n = n, c = c, pi = pi, R = R)
print(b_CR)
#>           [,1]
#> [1,] 0.3330482

##As n increases, CR decreases
#Get the optimal test sample size
thres_CR = 0.05 #CR < 0.05
```

```
b_n <- boptimal_n(c = c, pi = pi, R = R, thres_CR = thres_CR)
print(b_n)
#> [1] 24
```

Table 1 shows an example of BRDT plans with different choices of prior distributions of π. The mean and standard deviation (i.e., the square of variance) values are provided to give some intuitions about the center and the spread of the prior distributions. For example, $\pi \sim \text{Beta}(1, 1)$ is centered at 0.5 but has large standard deviation at 0.2893. While $\pi \sim \text{Beta}(2, 18)$ has the mean failure probability of 0.1 but much smaller standard deviation (0.0647) around its mean. The minimum acceptable reliability from the consumers requirement was set at $R = 0.8$ and the maximum tolerable CR is chosen to be $\beta = 0.05$. When no historical data or prior information is available, a non-informative prior $\pi \sim \text{Beta}(1, 1)$ can be used. For any assumed prior distribution of π, manufacturers can choose a test plan determined by (n_{b}, c) using the minimum sample size n_{b} for any chosen maximum number of allowable failures c. For instance, when $c = 0$ and a non-informative prior $\pi \sim \text{Beta}(1, 1)$ is assumed, the minimum sample size which can ensure the CR calculated in Eq. (2) is no more than $\beta = 0.05$ is calculated to be $n_{\text{b}} = 13$. Hence, at least 13 units need to be tested if the test can only be passed when no failure is observed. However, as larger maximum number of allowable failures being set for passing the test, the CR increases as it becomes easier to pass the test for a given sample size n. Hence, to control the CR at or below $\beta = 0.05$, more test units need to be tested to pass the test due to more allowable failures.

Table 1: Minimum sample sizes required by BRDTs with different choices on c and prior distributions of π

$\pi \sim$ Beta	$(1, 1)$	$(2, 18)$	$(4, 16)$	$(10, 15)$	$(10, 10)$
Mean(π)	0.5	0.1	0.2	0.4	0.5
SD(π)	0.2893	0.0647	0.0873	0.0965	0.1086
c			n_{b}		
0	13	4	18	45	53
1	18	7	23	51	58
2	24	11	28	57	64
3	29	15	34	62	69
4	34	19	39	68	74
5	39	22	44	74	80
6	44	26	49	80	85
Settings: $M = 15000, R = 0.8, \beta = 0.05$					

To illustrate, the following code can generate the first column of n_b when $\pi \sim \text{Beta}(1, 1)$ in Table 1.

```
###Generate the prior distribution of failure probability
##Beta is conjugate prior to binomial distribution
#Get the non-informative prior Beta(1, 1)
pi <- pi_MCSim_beta(M = 15000, seed = 10, a = 1, b = 1)
#Set the reliability requirement
R = 0.8
#Set the CR threshold
thres_CR = 0.05 #CR < 0.05
#loop through options of c
for (c in 0:6){
b_n <- boptimal_n(c = c, pi = pi, R = R, thres_CR = thres_CR)
print(b_n)
}
#>  [1] 13
     [1] 18
```

```
[1] 24
[1] 29
[1] 34
[1] 39
[1] 44
```

4.2 MSRDT-MPCum&MPSep

The following code shows a quick example to compute CR and get optimal n for MSRDT with multiple time periods and considering passing criteria for cumulative periods.

```
###Generate the prior distribution of failure probability
##Dirichlet is conjugate prior to multinomial distribution
#Get the non-informative prior Dirichlet(1, 1, 1)
pi <- pi_MCSim_dirichlet(M = 5000, seed = 10, par = c(1, 1, 1))

#Get the consumer's risk
n = 10
cvec = c(1, 1)
Rvec = c(0.8, 0.7)
MPCum_CR <- MPCum_consumerrisk(n = n, cvec = cvec, pivec = pi, Rvec = Rvec)
print(MPCum_CR)
#> [1] 0.3383538

##As n increases, CR decreases
#Get the optimal test sample size
thres_CR = 0.05 #CR <= 0.05
MPCum_n <- MPCum_optimal_n(cvec = cvec, pivec = pi, Rvec = Rvec, thres_CR = thres_CR)
print(MPCum_n)
#> [1] 20
```

The following code shows a quick example to compute CR and get optimal n for MSRDT with multiple time periods and considering passing criteria for separate periods.

```
###Generate the prior distribution of failure probability
##Dirichlet is conjugate prior to multinomial distribution
#Get the non-informative prior Dirichlet(1, 1, 1)
pi <- pi_MCSim_dirichlet(M = 5000, seed = 10, par = c(1, 1, 1))

#Get the consumer's risk
n = 10
cvec = c(1, 1)
Rvec = c(0.8, 0.7)
MPSep_CR <- MPSep_consumerrisk(n = n, cvec = cvec, pivec = pi, Rvec = Rvec)
print(MPSep_CR)
#> [1] 0.3002541

##As n increases, CR decreases
#Get the optimal test sample size
thres_CR = 0.05 #CR <= 0.05
MPSep_n <- MPSep_optimal_n(cvec = cvec, pivec = pi, Rvec = Rvec, thres_CR = thres_CR)
print(MPSep_n)
#> [1] 19
```

A case study is shown below for illustrating the proposed MSRDT strategies for a two-period test. The reliability requirements are set as $R_1 = 0.8$ and $R_2 = 0.6$ over the time periods $(t_0,\, t_1]$ and $(t_0,\, t_2]$ with $t_2 < 2t_1$, which indicates longer time interval of $(t_0,\, t_1]$ than $(t_1,\, t_2]$. Hence, a higher reliability requirement R_1 is desired for the early cumulative time period $(t_0,\, t_1]$ because the customers are averse to early failures. The CR is controlled at $\beta = 0.05$, indicating that the probability of accepting the test when the actual reliability requirements are not met is controlled at or below 0.05. To evaluate the complex integration in either Eq. (3) or Eq. (4), Monte Carlo sampling is performed with the sample size of $M = 15000$ to maintain the evaluation accuracy. The Dirichlet distribution, denoted by Dirichlet$(\alpha_1, \alpha_2, \alpha_3)$, is used as the prior distribution for $(\pi_1, \pi_2, 1 - \pi_1 - \pi_2)$, where $\alpha_1, \alpha_2, \alpha_3$ are hyper-parameters to be elicited based on the prior knowledge. The Dirichlet distribution is a family of continuous multivariate probability distribution parametrized by the vector of positive hyper-parameters $\alpha_i, i = 1, \cdots, K$ for K categories of outcomes. The advantage of using Dirichlet distribution is two folded. First of all, it is the conjugate prior for the multinomial distribution, and hence can facilitate a convenience of updating knowledge as new data are observed because the posterior distribution of the failure probabilities also follow a Dirichlet distribution. Second, the hyper-parameters in the Dirichlet distribution are associated with more intuitive practical implications as they are directly connected with the failure probabilities for each category of outcomes based on the prior knowledge in the form of $\alpha_i / \sum_{i=1}^{K} \alpha_i$. A few different settings of hyper-parameters will be explored later to investigate the impact of prior knowledge on the performance of the proposed test plan.

Table 2: Comparison between scenarios I & II and BRDT, with non-informative prior

Scenario I			Scenario II			BRDT	
c_1	$c_1 + c_2$	n_{I}	c_1	c_2	n_{II}	c	n_{b}
0	0	12	0	0	12	0	5
0	1	13	0	1	13	1	8
1	1	15	1	0	17		
0	2	14	0	2	14	2	11
1	2	17	1	1	18		
2	2	19	2	0	22		
0	5	20	0	5	20	5	18
1	5	22	1	4	21		
2	5	24	2	3	23		
3	5	26	3	2	28		
4	5	28	4	1	33		
5	5	30	5	0	37		
0	6	22	0	6	22	6	20
1	6	24	1	5	23		
2	6	26	2	4	24		
3	6	28	3	3	28		
4	6	30	4	2	33		
5	6	32	5	1	37		
6	6	34	6	0	42		
Settings: $p(\pi_1, \pi_2) \sim$ Dirichlet$(1, 1, 1)$ $R_1 = 0.8, R_2 = 0.6, M = 15000, \beta = 0.05$							

When no prior information is available, a non-informative prior distribution, given by $(\pi_1, \pi_2, 1 - \pi_1 - \pi_2) \sim$ Dirichlet$(1, 1, 1)$ can be used for indicating the lack of prior knowledge. The selected test plans under the acceptance criteria of two scenarios with different choices on the maximum number of allowable failures are illustrated in Table 2. The test plans are grouped based on the total number of failures allowed during the entire test duration. Several features are observed. First of all, under both scenarios I & II, given a fixed choice of c_2, the minimum sample size n_{I} or n_{II} increases as c_1 increases.

Similarly, given a fixed c_1, n_{I} and n_{II} also increase with c_2. As for a given fixed number of test units, allowing more failures (i.e. increasing c) can make it easier to pass the test and thus increase the CR. Hence, it requires to test more units to control the CR at a predetermined maximum acceptable level.

To illustrate, the following code can generate n_{I} and n_{II} in in Table 2.

```
###Generate the prior distribution of failure probability
##Dirichlet is conjugate prior to multinomial distribution
#Get the non-informative prior Dirichlet(1, 1, 1)
pi <- pi_MCSim_dirichlet(M = 15000, seed = 10, par = c(1, 1, 1))
#Set the reliability requirement
Rvec = c(0.8, 0.6)
#Set the CR threshold
thres_CR = 0.05 #CR < 0.05


## Scenario I MPCum
#if c1+c2=0
c1<-0
c2<-0
cvec=c(0, 0)
MPCum_n <- MPCum_optimal_n(cvec = cvec, pivec = pi, Rvec = Rvec, thres_CR = thres_CR)
print(MPCum_n)
#> [1] 12

#if c1+c2=2
c1<-0
c2<-0
for (c1 in 0:2){
    cvec=c(c1,2-c1)
    MPCum_n <- MPCum_optimal_n(cvec = cvec, pivec = pi, Rvec = Rvec, thres_CR = thres_CR)
    print(MPCum_n)
}
#> [1] 14
[1] 17
[1] 19

#if c1+c2=6
c1<-0
c2<-0
for (c1 in 0:6){
    cvec=c(c1,6-c1)
    MPCum_n <- MPCum_optimal_n(cvec = cvec, pivec = pi, Rvec = Rvec, thres_CR = thres_CR)
    print(MPCum_n)
}
#> [1] 22
[1] 24
[1] 26
[1] 28
[1] 30
[1] 32
[1] 34
```

```
## Scenario II MPSep
#if c1+c2=0
c1<-0
c2<-0
cvec=c(0, 0)
MPSep_n <- MPSep_optimal_n(cvec = cvec, pivec = pi, Rvec = Rvec, thres_CR = thres_CR)
print(MPSep_n)
#> [1] 12

#if c1+c2=2
c1<-0
c2<-0
for (c1 in 0:2){
    cvec=c(c1,2-c1)
    MPSep_n <- MPSep_optimal_n(cvec = cvec, pivec = pi, Rvec = Rvec, thres_CR = thres_CR)
    print(MPCum_n)
}
#> [1] 14
[1] 18
[1] 22

#if c1+c2=6
c1<-0
c2<-0
for (c1 in 0:6){
    cvec=c(c1,6-c1)
    MPSep_n <- MPSep_optimal_n(cvec = cvec, pivec = pi, Rvec = Rvec, thres_CR = thres_CR)
    print(MPSep_n)
}
#> [1] 22
[1] 23
[1] 24
[1] 28
[1] 33
[1] 37
[1] 42
```

4.3 MFM

The following code shows a quick example to compute CR and get optimal n for MSRDT with multiple failure modes, with each mode following binary distribution.

```
###Generate the prior distribution of failure probability
##Beta is conjugate prior to binomial distribution
#Get the non-informative prior Beta(1, 1)
pi1 <- pi_MCSim_beta(M = 1000, seed = 10, a = 1, b = 1)
pi2 <- pi_MCSim_beta(M = 1000, seed = 10, a = 1, b = 1)
#Get the consumer's risk
n = 10
cvec = c(1, 1)
```

```
Rvec = c(0.8, 0.7)
MFM_CR <- MFM_consumerrisk(n = 10, cvec = cvec, pivec = cbind(pi1, pi2), Rvec = Rvec)
print(MFM_CR)
#> [1] 0.07429376

#Get the optimal test sample size
thres_CR = 0.05 #CR <= 0.05
MFM_n <- MFM_optimal_n(cvec = cvec, pivec = cbind(pi1, pi2), Rvec = Rvec, thres_CR = thres_CR)
print(MFM_n)
#> [1] 12
```

Simulation case studies are conducted for exploring different reliability requirements, maximum numbers of allowable failures for different failure modes, as well as different prior elicitations and their impacts on the required minimum sample size for the MSRDTs for two failure modes. In Table 3, identical minimum reliability requirements are assumed for the two failure modes, where $R_1 = R_2 = 0.8$ indicates that the customers have the same expectation on reliability for both failure modes. The $\mathrm{CR_m}$ is still controlled at $\beta = 0.05$ and the sample size for Monte Carlo sampling is chosen as $M = 15000$ to maintain the simulation accuracy. Beta distributions are used for specifying the prior distributions for both π_1 and π_2 for the two failure modes.

Table 3: Multiple failure modes with the same reliability requirements for different prior distributions

Beta		π_1 / π_2	$(1,1)$ / $(1,1)$	$(2,18)$ / $(2,18)$	$(4,16)$ / $(4,16)$	$(10,15)$ / $(10,15)$	$(2,18)$ / $(4,16)$	$(2,18)$ / $(10,15)$	$(4,16)$ / $(10,15)$
c_1	c_2		n_{m}						
0	0		16	7	22	70	18	48	60
0	1		20	9	25	75	23	55	66
1	0		20	9	25	71	19	45	57
0	2		25	12	29	81	29	61	73
1	1		22	11	28	75	24	52	62
2	0		25	12	30	74	20	45	56
0	5		40	22	43	103	44	81	94
1	4		35	19	39	93	39	70	79
2	3		31	18	36	84	34	62	72
3	2		31	19	36	80	30	56	67
4	1		35	20	40	82	27	50	61
5	0		40	23	45	88	27	45	57
0	6		45	25	47	112	49	88	102
1	5		40	22	43	100	44	76	85
2	4		35	21	40	90	39	68	77
3	3		34	21	39	84	34	61	72
4	2		35	21	40	83	31	55	66
5	1		40	23	45	86	29	50	62
6	0		45	27	50	94	30	46	59
Settings: $M = 15000, R_1 = 0.8, R_2 = 0.8, \beta = 0.05$									

When two failure modes have the same reliability requirements at $R_1 = R_2 = 0.8$, Table 3 summarizes the minimum sample size with different choices of the maximum allowable failures and different prior settings. When no prior information is available, a non-informative prior distribution of $\mathrm{Beta}(1, 1)$ is assigned for both π_1 and π_2. Similar patterns can be observed as for the MSRDTs over multiple time periods. When c_1 is fixed, the minimum sample size n_m increases as c_2 increases; when c_2 is fixed, n_m increases with c_1. This is intuitive as having more allowable failures makes it easier to pass the test and thus increases the CR. To control a reasonable CR, a larger number of test units need to be tested by allowing more failures to be observed during the test. When $c_1 + c_2$ is fixed, the minimum sample size n_m exhibits a symmetric pattern under the non-informative prior setting due to the identical reliability requirements for both failure modes. For example, when $c_1 + c_2 = 6$, the minimum sample sizes for $c_1 = 0, c_2 = 6$ and $c_1 = 6, c_2 = 0$ are identical. In addition, when c_1 and c_2 become more similar in size (e.g., $c_1 = 2, c_2 = 4$ compared to $c_1 = 0, c_2 = 6$), it requires smaller minimum sample size to remain the same assurance level for demonstrating the requirements on both failure modes. This makes sense as when the maximum number of allowable failures is considerably larger for one failure mode given the same reliability requirement, it requires to test more units for demonstrating the requirement for this failure mode, which then inflates the overall minimum sample size needed in the MSRDT for demonstrating reliability requirements for both failure modes.

To illustrate, the following code can generate the first column of n_b when $\pi_1 \sim \text{Beta}(1, 1)$ and $\pi_2 \sim \text{Beta}(1, 1)$ in Table 3.

```
###Generate the prior distribution of failure probability
##Beta is conjugate prior to binomial distribution
#Get the non-informative prior Beta(1, 1)
pi1 <- pi_MCSim_beta(M = 15000, seed = 10, a = 1, b = 1)
pi2 <- pi_MCSim_beta(M = 15000, seed = 10, a = 1, b = 1)
#Set the reliability requirement
Rvec = c(0.8, 0.8)
#Set the CR threshold
thres_CR = 0.05 #CR < 0.05

#if c1+c2=0
c1<-0
c2<-0
cvec=c(0,0)
MFM_n <- MFM_optimal_n(cvec = cvec, pivec = cbind(pi1, pi2), Rvec = Rvec, thres_CR = thres_CR)
print(MFM_n)
#> [1] 16

#if c1+c2=2
c1<-0
c2<-0
for (c1 in 0:2){
    cvec=c(c1,2-c1)
    MFM_n <- MFM_optimal_n(cvec = cvec, pivec = cbind(pi1, pi2), Rvec = Rvec, thres_CR = thres_CR)
    print(MFM_n)
}
#> [1] 25
[1] 22
[1] 25

#if c1+c2=6
c1<-0
c2<-0
for (c1 in 0:6){
    cvec=c(c1,6-c1)
    MFM_n <- MFM_optimal_n(cvec = cvec, pivec = cbind(pi1, pi2), Rvec = Rvec, thres_CR = thres_CR)
    print(MFM_n)
}
#> [1] 45
[1] 40
```

```
[1] 35
[1] 34
[1] 35
[1] 40
[1] 45
```

5 Conclusions

Conventional binomial RDTs, which focus on demonstrating a single reliability requirement within a single test period, have limited use when multiple reliability requirements need to be met. This paper proposes two types of RDTs for demonstrating reliabilities over multiple time periods and for multiple failure modes. These RDTs with multiple reliability requirements are all referred to as multi-state RDTs (MSRDTs).

In the MSRDTs over multiple time periods, every time period of interest is treated as a state, and the joint distribution of failure counts over the non-overlapping time periods can be modeled by a multinomial distribution. Two different test strategies are proposed for demonstrating multiple requirements over different time periods. One strategy uses the cumulative failure counts at the end of all periods as the criteria for passing the test; while the other uses separate failure counts over non-overlapping time intervals as the criteria for passing the test. Simulation studies were conducted for comparing the two strategies by considering two-period MSRDTs. It was founded that the strategy based on cumulative failure counts (scenario I) is generally preferred for cases that allow fewer total failure counts over all time periods or when a larger maximum number of allowable failures is allowed for the early cumulative time period. The strategy using separate failure counts (scenario II) is only preferred for requiring smaller minimum sample size when a smaller maximum number of allowable failures is allowed for the early separate time period.

In the MSRDTs for multiple failure modes, each failure mode is treated as a state and all reliability requirements for the multiple failure modes that may be associated with different consequences in varied levels of severity and/or costs of repair/replacement can be simultaneously demonstrated. The required minimum sample size is usually determined mainly by the failure mode that has the highest reliability requirement and/or least stringent criterion for passing the test (i.e. allowing a larger maximum number of allowable failures for a particular failure mode).

The impacts of incorporating different prior distributions are also explored for both categories of MSRDTs. The patterns are consistent regardless of which test strategy is considered. When the prior knowledge supports higher reliabilities than the requirements to be demonstrated, fewer units can be tested compared to using the non-informative priors for demonstrating the same reliability requirements. However, if the historical data supports lower reliabilities than what are required to be demonstrated, then more units need to be tested to override the effects of the prior distribution for demonstrating higher reliabilities than what has been indicated from existing data. For future work, it is expected to develop thorough mathematical justifications with theoritical formulations and derivations to validate the discussed patterns using both non-informative and informative prior distributions.

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