FLUID MECHANICS

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ABSTRACT. Everything about Fluid Mechanics

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Part 1. On Landau and Lifshitz's Fluid Mechanics

[1]

1. IDEAL FLUIDS

Chapter 1 of Landau and Lifshitz (1987) [1].
Use the **Eulerian** or spatial velocity, the velocity at time t of the particle currently in position $\mathbf{x}$ [9] which is a time-dependent velocity field:
Let N be the spatial manifold, with spacetime manifold $M = \mathbb{R} \times N$, $\dim N = n$

$$\begin{tikzcd} TN & u(t,x) \\ J \times N \arrow[u, "u"] & (t,x) \arrow[u, mapsto, "u"] \end{tikzcd}$$

cf. Section 11.2 "Geometric setting of ideal continuum motion" of Holm, Schmah, Stoics [9]

Definition 1. Let spacetime manifold M admit a time-foliation $M = \mathbb{R} \times N$, where N represents spatial points. Let domain $\mathscr{D} \subseteq N$ represent positions of material particles of system in its **reference configuration.** Coordinate function (a^i) on $\mathscr{D}$ represent **particle labels**.

- **configuration** := diffeomorphism $g : \mathscr{D} \to \mathscr{D}$, $g \in \text{Diff}(\mathscr{D})$ space of diffeomorphisms from $\mathscr{D}$ to $\mathscr{D}$
- **fluid motion** $g_t \equiv g(t) \in \text{Diff}(\mathscr{D})$

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Definition 2.

$$x : \mathscr{D}\times\mathbb{R}\to\mathscr{D}$$
$$x(a,t) := g_t(a) = g(t)\cdot a \in \mathscr{D} \tag{1}$$

describes path in $\mathscr{D}$ by a particle labeled $a\in\mathscr{D}$

Definition 3. Lagrangian or material velocity - keep particle labels a fixed.

$$U(a,t) := \frac{\partial}{\partial t} g_t\cdot a = \frac{\partial}{\partial t}x(a,t) \tag{2}$$

$U(a,t)$ is velocity of particle with label a at time t.
Eulerian or spatial velocity u
if $x = x(a,t) = g_t(a)$

$$u(x,t) := U(a,t) = U(g_t^{-1}(x),t) \tag{3}$$

$u(x,t)$ is velocity at time t of particle currently in position x

Now $u \Longrightarrow u_t \in \mathfrak{X}(\mathscr{D})$
$u_t(x) := u(x,t)$
$U_t(a) := U(a,t)$ though this isn't really a vector field.

$$U_t = u_t\circ g_t$$

$$\begin{tikzcd} & \mathfrak{X}(\mathscr{D}) \\ \mathscr{D}\subseteq N \arrow[ur, "U_t"] \arrow[r, "g(t)=g_t"] & \mathscr{D}\subseteq N \arrow[u] \arrow[l, bend left, "g_t^{-1}=g_{-t}"] \end{tikzcd} \qquad \begin{tikzcd} & u_t(x):=u(x,t)=U(a,t)=:U_t(a) \\ a \arrow[ur, mapsto, "U_t"] \arrow[r, mapsto, "g(t)=g_t"] & x(a,t)=x \arrow[u, mapsto] \arrow[l, mapsto, bend left, "g_t^{-1}=g_{-t}"] \end{tikzcd}$$

Definition 4. Given path $g(t)\in\mathrm{Diff}(\mathscr{D})$,

Lagrangian velocity field $\begin{aligned} &U_t : \mathscr{D}\to\mathfrak{X}(\mathscr{D}) \\ &U_t \equiv \dot{g}(t) \equiv \frac{\partial g(t)}{\partial t}\end{aligned}$

$$\dot{g}(t)\cdot a := \dot{g}(t)(a) = U_t(a) = \frac{\partial g_t}{\partial t}\cdot a$$

$$\begin{tikzcd} g:\mathbb{R}\to\mathrm{Diff}(\mathscr{D}) \\ g(t)\in\mathrm{Diff}(\mathscr{D}) \arrow[d, mapsto, "\frac{\partial}{\partial t}"'] \\ \dot{g}(t)\equiv\frac{\partial g(t)}{\partial t}\equiv U_t \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Diff}(\mathscr{D}) \arrow[d, mapsto, "\frac{\partial}{\partial t}"'] \\ T_g\mathrm{Diff}(\mathscr{D}) \end{tikzcd}$$

$$\begin{tikzcd} a\in\mathscr{D} \arrow[r, mapsto, "\dot{g}(t)\equiv\frac{\partial g(t)}{\partial t}\equiv U_t"] \arrow[dr, mapsto, "g(t)=g_t"'] & U_t(a)\in T_{g_t(a)}\mathscr{D} \\ & g(t)\cdot a = x(a,t)\in\mathscr{D} \arrow[u, mapsto, "u_t"] \end{tikzcd}$$

Thus

$$U_t = u_t\circ g_t$$

$$\begin{aligned} T_g\mathrm{Diff}(\mathscr{D}) &= \{u\circ g | u\in\mathfrak{X}(\mathscr{D})\} = \\ &= \{\text{ smooth } U:\mathscr{D}\to T\mathscr{D} | U(a)\in T_{g(a)}\mathscr{D} \quad \forall a\in\mathscr{D}\} \end{aligned} \tag{4}$$

Theorem 1 ((Tangent lift of right translation))**.** *Let $\varphi\in Diff(\mathscr{D})$*
Let R_φ be right translation map
$R_\varphi : Diff(\mathscr{D})\to Diff(\mathscr{D})$
$R_\varphi : g\mapsto g\circ\varphi$
tangent life of R_φ is map $TR_\varphi : TDiff(\mathscr{D})\to TDiff(\mathscr{D})$

$$TR_\varphi(U) = TR_\varphi\left(\left.\frac{d}{dt}\right|_{t_0} g_t\right) := \left.\frac{d}{dt}\right|_{t_0}(g_t\circ\varphi) = U\circ\varphi \tag{5}$$

since $\forall a\in\mathscr{D}$

$$\left.\frac{d}{dt}\right|_{t_0}(g_t\circ\varphi)(a) = \left.\frac{d}{dt}\right|_{t_0}(g_t\circ\varphi(a)) = \left(\left.\frac{d}{dt}\right|_{t_0} g_t\right)\cdot\varphi(a) = U\circ\varphi(a) \tag{6}$$

$U\varphi \equiv TR_\varphi(U)$
Eulerian velocity *corresponding to flow $g(t)$ is $u_t = \dot{g}(t)g^{-1}(t)$*

$$\begin{tikzcd} T\mathrm{Diff}(\mathscr{D}) \arrow[r, "TR_\varphi"] & T\mathrm{Diff}(\mathscr{D}) \\ \mathrm{Diff}(\mathscr{D}) \arrow[u] \arrow[r, "R_\varphi"] & \mathrm{Diff}(\mathscr{D}) \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \left.\frac{d}{dt}\right|_{t_0} g_t = U \arrow[r, mapsto, "TR_\varphi"] & U\circ\varphi = \left.\frac{d}{dt}\right|_{t_0}(g_t\circ\varphi) \\ g \arrow[u, mapsto] \arrow[r, mapsto, "R_\varphi"] & g\circ\varphi \arrow[u, mapsto] \end{tikzcd}$$

cf. Section 5.2 "Abstract Lie groups and Lie algebras" of Holm, Schmah, Stoics [9]
The mass of fluid in some volume $V_0\subset N$ is $\int_{V^0}\rho\,\mathrm{vol}^n$, where ρ is fluid density, $\rho\in C^\infty(N)$.
The total mass of fluid flowing out of volume V_0 is

$$\frac{d}{dt}\int_{V_0}\rho\,\mathrm{vol}^n = \int_{V_0}\mathscr{L}_{\frac{\partial}{\partial t}+\mathbf{u}}(\rho\,\mathrm{vol}^n) = \int_{V_0}\frac{\partial}{\partial t}\rho\,\mathrm{vol}^n + \int_{V_0}\mathscr{L}_u\rho\,\mathrm{vol}^n$$
$$\int_{V_0}\mathscr{L}_u\rho\,\mathrm{vol}^n = \int_{V_0} di_{\mathbf{u}}\rho\,\mathrm{vol}^n + i_{\mathbf{u}}d\rho\,\mathrm{vol}^n = \int_{V_0} di_{\mathbf{u}}\rho\,\mathrm{vol}^n + 0 = \int_{V_0} di_{\mathbf{u}}\rho\,\mathrm{vol}^n = \int_{\partial V_0} i_{\mathbf{u}}\rho\,\mathrm{vol}^n$$

Now

$$i_u\mathrm{vol}^n = i_u\frac{\sqrt{g}}{n!}\varepsilon_{i_1\dots i_n}dx^{i_1}\wedge\dots\wedge dx^{i_n}$$
$$i_u dx^{i_1}\wedge\dots\wedge dx^{i_n} = u^{i_1}dx^{i_2}\wedge\dots\wedge dx^{i_n} - dx^{i_1}\wedge u^{i_2}dx^{i_3}\wedge\dots\wedge dx^{i_n} + \dots + (-1)^{n+1}dx^{i_1}\wedge\dots\wedge dx^{i_{n-1}}u^{i_n} = \varepsilon^{i_1\dots i_n}_{j_1\dots j_n}u^{j_1}dx^{j_2}\wedge\dots\wedge dx^{j_n}$$
$$\Longrightarrow i_u\mathrm{vol}^n = \frac{\sqrt{g}}{(n-1)!}\varepsilon_{j_1\dots j_n}u^{j_1}dx^{j_2}\wedge\dots\wedge dx^{j_n}$$

If $\sqrt{g}=1$, $n=2$,

$$i_u \text{vol}^2 = (u^1 dx^2 - u^2 dx^1) = u\cdot n_1 dx^2 + u\cdot n_2 dx^1 = u\cdot n dS$$

with $n_1 = e_1$ and $n_2 = -e_2$.
Now

$$di_u \rho \text{vol}^n =$$

$$= \frac{\partial(\sqrt{g}\rho u^{j_1})}{\partial x^k}\frac{\varepsilon_{j_1\dots j_n}}{(n-1)!}dx^k\wedge dx^{j_2}\wedge\dots\wedge dx^{j_n} = \frac{\partial(\sqrt{g}\rho u^k)}{\partial x^k}\frac{\varepsilon_{j_1\dots j_n}}{n!}dx^{j_1}\wedge\dots\wedge dx^{j_n} = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\rho u^k)}{\partial x^k}\text{vol}^n =$$

$$= \frac{\partial(\rho u^k)}{\partial x^k}\text{vol}^n + \rho u^k \frac{\partial \ln\sqrt{g}}{\partial x^k}\text{vol}^n = \text{div}(\rho u)\text{vol}^n + \rho u^k\frac{\partial\ln\sqrt{g}}{\partial x^k}\text{vol}^n$$

Now if $\sqrt{g}=1$, then

$$\frac{d}{dt}\int_{V_0}\rho\text{vol}^n = \int_{V_0}\frac{\partial\rho}{\partial t}\text{vol}^n + \int_{V_0} di_u\rho\text{vol}^n = \int_{V_0}\frac{\partial\rho}{\partial t}\text{vol}^n + \int_{V_0}\text{div}(\rho u)\text{vol}^n \Longrightarrow \frac{\partial\rho}{\partial t} + \text{div}(\rho u) = 0$$

which is the so-called mass continuity equation. $j=\rho u$ is the mass flux density.
cf. Sec.1.2. Euler's equation [1],
The Cauchy-stress tensor T is a symmetric $(0,2)$ tensor, so that $T\in\Gamma(T^*M\otimes T^*M)$.
Let the normal field n be normal to the surface of V_0.
Let's do this:

$$T \xrightarrow{(n,-)} T(n,-)\in T^*M = \Omega^1(M) \xrightarrow{*} *T(n,-)\in\Omega^{m-1}(M)$$

So applying the Hodge operator,

$$pg_{ij}n^i dx^j \overset{*}{\mapsto} \frac{\sqrt{g}}{(n-1)!}pg_{ij}n^i\varepsilon^j_{\ j_2\dots j_n}dx^{j_2}\wedge\dots\wedge dx^{j_n} = pi_n\text{vol}^n$$

$-\int *T(n,-)$
The force on V_0 due to the Cauchy-stress tensor is

$$\int_{\partial V_0} *T(n,-)$$

Taking, for the special case of the perfect fluid, $T=-pg$, then

$$-\int_{\partial V_0} pi_n\text{vol}^n = -\int_{V_0} d(pi_n\text{vol}^n) = -\int \text{div}(pn)\text{vol}^n = -\int \text{grad}p\text{vol}^n$$

using Stoke's law.
EY : 20150720 come to think about it, we should probably treat T as $T_{ij}dx^i\otimes dx^j$ and only do stuff on dx^i to retain $\otimes dx^j$ to get a direction out, a 1-formed valued $m-1$ form, $\dim M = m$
This is probably the correct way to think about it:
Given the Cauchy stress tensor $T=T^{ij}e_i\otimes e_j$, which is a $(2,0)$-rank tensor, T is a section of the $TM\otimes TM$ bundle, i.e. $\Gamma(TM\otimes TM)$, so that $T\in\Gamma(TM\otimes TM)$.
We want to do this:

$$\Gamma(TM\otimes TM)\xrightarrow{(\sharp,-)}\Omega^1(M,TM)\xrightarrow{(*,-)}\Omega^{n-1}(M,TM)$$

$$T = T^{ij}e_i\otimes e_j = T^{ij}e_j\otimes e_i \xmapsto{(\sharp,-)} T^i_{\ j}e^j\otimes e_i \xmapsto{(*,-)} T^i_{\ j}\tfrac{\sqrt{g}}{(n-1)!}\varepsilon^j_{\ j_2\dots j_n}dx^{j_2}\wedge\dots\wedge dx^{j_n}\otimes e_i = T^i_{\ j}dS^j\otimes e_i$$

Thus, the force on V_0 due to the Cauchy-stress tensor is

$$\int_{\partial V_0} T^i_{\ j}dS^j\otimes e_i$$

and so for the case of $T=-pg$,

$$-\int_{\partial V_0} pg^i_{\ j}dS^j\otimes e_i$$

For the time rate of change of momentum, see my other pdf entitled "Aspects of Geometry in Propulsion." There, we find the rate of change of momentum

$$\int_{V_0}\rho\left(\frac{\partial u^i}{\partial t} + u^j\frac{\partial u^i}{\partial x^j}\right)\text{vol}^n\otimes e_i$$

Euler's equation is then

$$\rho\left(\frac{\partial u^i}{\partial t} + u^j\frac{\partial u^i}{\partial x^j}\right) = -\frac{\partial\rho}{\partial x^j}g^{ij} \tag{7}$$

The energy flux.

Review of Thermodynamics. Let Σ be the manifold of equilibrium (and non-equilibrium) states of the system.

Proposition 1 (First Law: Energy Conservation)**.**

$$dU = Q - dW = Q - pdV \tag{8}$$

with $U,p,V\in C^\infty(\Sigma)$, and $dU,Q,dW,dV\in\Omega^1(\Sigma)$, and where U is internal energy, p is pressure, V is the volume of the system.

Proposition 2 (Second Law)**.**

$$Q = TdS \tag{9}$$

where $Q,dS\in\Omega^1(\Sigma)$, and $S\in C^\infty(\Sigma)$ and with

$$dS\geq 0 \tag{10}$$

describing irreversibility.

Definition 5 (Enthalpy)**.**

$$H = U + pV \tag{11}$$

where H is the **enthalpy**, $H\in C^\infty(\Sigma)$.

Now, for M being the molar mass, and defining per unit mass quantities as we go,

$$dH = dU + Vdp + pdV = Q + Vdp = TdS + Vdp$$

$$\xrightarrow{1/M}\frac{dH}{M} = dh = T\frac{dS}{M} + \frac{V}{M}dp = Tds + \frac{dp}{\rho}$$

where $\rho = M/V$.
Now the total energy in a fluid occupying region V_0 is

$$\int_{V_0}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right)\text{vol}^n$$

where $\int_{V_0}\frac{1}{2}\rho v^2\text{vol}^n$ is the total kinetic energy of fluid and ε internal energy per unit mass.
The time rate of change of the energy is

$$\frac{d}{dt}\int_{V_0}(\frac{1}{2}\rho v^2+\rho\varepsilon)\text{vol}^n = \int_{V_0}\mathscr{L}_{\frac{\partial}{\partial t}+v}(\frac{1}{2}\rho v^2+\rho\varepsilon)\text{vol}^n = \int_{V_0}\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2+\rho\varepsilon)\text{vol}^n + \mathscr{L}_v((\frac{1}{2}\rho v^2+\rho\varepsilon)\text{vol}^n) =$$

$$= \int_{V_0}\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2+\rho\varepsilon)\text{vol}^n + di_v((\frac{1}{2}\rho v^2+\rho\varepsilon)\text{vol}^n)$$

So-called momentum flux. Sec. 1.7 of Landau and Lifshitz [1].
Let the total momentum of a fluid in volume $V_0 \subset N$, V_0 a submanifold of spatial manifold N, with $\text{dim}V_0 = \text{dim}N = n$ be P:

$$P = \int_{V_0} \rho \text{vol}^n u^i \otimes e_i \in \mathfrak{X}(M)$$

with

$$\rho \text{vol}^n u^i \otimes e_i \in \Omega^n(M;TM)$$

Now

$$\dot{P} := \frac{d}{dt}P = \int_{V_0} \mathscr{L}_{\frac{\partial}{\partial t}+u}(\rho \text{vol}^n u^i \otimes e_i)$$

which can be shown (see Yeung (2015) "Aspects of Geometry in Propulsion") to be

$$\dot{P} = \int_{V_0} \frac{\partial(\rho u^i)}{\partial t}\text{vol}^n \otimes e_i + \int_{\partial V^0} \rho u^i i_u \text{vol}^n \otimes e_i$$

Now

$$\rho u^i i_u \text{vol}^n = \rho u^i \frac{\sqrt{g}}{(n-1)!}\varepsilon_{ki_2\dots i_n} u^k dx^{i_2}\wedge\dots\wedge dx^{i_n} = \rho u^i u^k \frac{\sqrt{g}}{(n-1)!}\varepsilon_{ki_2\dots i_n} dx^{i_2}\wedge\dots\wedge dx^{i_n} = \rho u^i u^k dS_k$$

So starting from equating the time rate of change of momentum $\dot{P}$ with the external forces on it, $\int_{\partial V_0} T^i_{\ j} dS^j \otimes e_i$, then, for the special case of the perfect fluid, $T = -pg$

$$\dot{P} = \int_{\partial V_0} T^i_{\ j} dS^j$$

$$\implies \int_{V_0} \frac{\partial \rho u^i}{\partial t}\text{vol}^n \otimes e_i + \int_{\partial V^0} \rho u^i u^k dS_k \otimes e_i = -\int_{\partial V^0} pg^{ij} dS_j \otimes e_i$$

Now move the boundary term on the left hand side, $\int_{\partial V^0} \rho u^i u^j dS_j \otimes e_i$ over to the right hand side:

$$\int_{V_0} \frac{\partial \rho u^i}{\partial t}\text{vol}^n \otimes e_i = -\int_{\partial V^0} \rho u^i u^k dS_k \otimes e_i - \int_{\partial V^0} pg^{ij} dS_j \otimes e_i = -\int_{\partial V_0} (\rho u^i u^j + pg^{ij}) dS_j \otimes e_i$$

Then the momentum flux tensor $\Pi \in \Gamma(TM \otimes TM)$, in this case, takes the form

$$\Pi^{ij} = \rho u^i u^j + pg^{ij}$$

Drag force in potential flow past a body. Sec. 1.11 of Landau and Lifshitz [1].
The dictionary: for $v = v^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(M)$,
$v^\flat = v_i dx^i = g_{ij} v^j dx^i \in \Omega^1(M)$

$$*v^\flat = \frac{\sqrt{g}}{(n-1)!} v_i \varepsilon^i_{\ j_2\dots j_n} dx^{j_2}\wedge\dots\wedge dx^{j_n}$$

$$d*v^\flat = \frac{\partial}{\partial x^k}(\sqrt{g}v_i)\frac{\varepsilon^i_{\ j_2\dots j_n}}{(n-1)!} dx^k \wedge dx^{j_2}\wedge\dots\wedge dx^{j_n} = \text{div}(v)\text{vol}^n + \frac{\partial \ln\sqrt{g}}{\partial x^i} v^i \text{vol}^n$$

$$dv^\flat = \frac{\partial v_j}{\partial x^j} dx^i \wedge dx^j$$

$$*dv^\flat = \frac{\partial v_j}{\partial x^i}\frac{\sqrt{g}}{(n-2)!}\varepsilon^{ij}_{\ \ k_3\dots k_n} dx^{k_3}\wedge\dots\wedge dx^{k_n}$$

if $\sqrt{g} = 1$, $n = 3$,

$$(*dv^\flat)^\sharp = \text{curl}(v)$$

If $dv^\flat = 0$, then $v^\flat = d\phi$ (i.e. $v^\flat$ is an exact form).

2. Viscous Fluids

Chapter 2 of Landau and Lifshitz [1]
Introduce a viscous stress tensor $\sigma' \in \Gamma(TM \otimes TM)$. Then do the following transformations:

$$\sigma' \xrightarrow{(\sharp,-)} (\sigma')^i_{\ j} e^j \otimes e_i \xrightarrow{(*,-)} (\sigma')^{ij} dS_j \otimes e_i$$

So the external force on the fluid in region V_0 is

$$\int_{\partial V_0} (T^{ij} + (\sigma')^{ij}) dS_j \otimes e_i$$

$$\begin{tikzcd} \sigma \arrow[r, "c"] \arrow[d, "F"] & \tau \arrow[d, "F"] \\ F(\sigma) \arrow[r, "F(c)"] & F(\tau) \end{tikzcd} \qquad \begin{tikzcd} \rho \arrow[r, "c_1"] \arrow[d, "F"] \arrow[rr, bend left, "c_2 \circ c_1"] & \sigma \arrow[r, "g"] \arrow[d, "F"] & \tau \arrow[d, "F"] \\ F(\rho) \arrow[r, "F(c_1)"] \arrow[rr, bend right, "F(c_2\circ c_1) = F(c_2)\circ F(c_1)"'] & F(\sigma) \arrow[r, "F(c_2)"] & F(\tau) \end{tikzcd}$$

3. Navier-Stokes Equations

I am following closely Chorin and Marsden, Sec. 1.3 "The Navier-Stokes Equations", Ch. 1 The Equations of Motion [6].
"On the left hand side,"

$$\frac{d}{dt}\int_{B(t)} \rho u \text{vol}^n =: \frac{d}{dt}P := \frac{d}{dt}\int_{B(t)} m \otimes u = \int_{B(t)} (\mathscr{L}_{\frac{\partial}{\partial t}+u} m) \otimes u + \int_{B(t)} m \otimes \mathscr{L}_{\frac{\partial}{\partial t}+u} u =$$
$$= 0 + \int_{B(t)} m \otimes \left(\frac{\partial u}{\partial t} + u^i \frac{\partial u^j}{\partial x^i}\frac{\partial}{\partial x^j} \right) = \int_{B(t)} m \otimes \left(\left(\frac{\partial u^j}{\partial t} + u^i \frac{\partial u^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \right)$$

assuming mass conservation $\mathscr{L}_{\frac{\partial}{\partial t}+u} m = 0$.
"On the right hand side" are the physical forces on the fluid. Consider first only the forces on its surface $\partial B(t)$:

$$\int T^{ij} dS_j \otimes \frac{\partial}{\partial x^i}$$

with T being the stress tensor. For the case of a fluid,

$$T = -gp + \sigma \in \Gamma(\otimes^2 TM)$$

where σ is usually called the *viscous stress tensor*.
For deformation tensor E, σ can appear in two forms for its constitutive relation with E (this constitutive relation is assumed to be linear, E and σ are symmetric rank-2 tensors, due to angular momentum conservation, and fluid is assumed to be isotropic),

$$\sigma = \lambda(\text{tr}E)1 + 2\mu E$$

$$\sigma = 2\mu(E - \frac{1}{d}\text{tr}(E)1) + \eta\text{tr}(E)1 \in \Gamma(\otimes^2 TM)$$

with μ being the 1st. coefficient of viscosity
$\eta = \lambda + \frac{2}{d}\mu$ being the 2nd. coefficient of viscosity.
Now if we took the exterior derivative d of each of the following:

$$-g^{ij}pdS_j = -pg^{ij}\frac{\sqrt{g}}{(n-1)!}\varepsilon_{ji_2\dots i_n} dx^{i_2}\wedge\dots\wedge dx^{i_n} \xrightarrow{d} -\frac{1}{\sqrt{g}}\frac{\partial(pg^{ik}\sqrt{g})}{\partial x^k}\text{vol}^n$$

Consider $E = \frac{1}{2}\left(g_{kj}\frac{\partial u^k}{\partial x^i} + g_{ik}\frac{\partial u^k}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k}u^k\right)dx^i \otimes dx^j \in \Gamma(\otimes^2 T^*M)$, then

$$E^\sharp = \frac{1}{2}\left(g_{kj}\frac{\partial u^k}{\partial x^i} + g_{ik}\frac{\partial u^k}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k}u^k\right)g^{il}\frac{\partial}{\partial x^l}\otimes g^{jm}\frac{\partial}{\partial x^m} = \frac{1}{2}\left(g^{il}\frac{\partial u^m}{\partial x^i} + g^{jm}\frac{\partial u^l}{\partial x^j} + g^{il}g^{jm}\frac{\partial g_{ij}}{\partial x^k}u^k\right)\frac{\partial}{\partial x^l}\otimes\frac{\partial}{\partial x^m}$$

$$\Longrightarrow \frac{\partial (E^\sharp)^{ij}}{\partial x^j} =$$
$$= \frac{1}{2}\left(\frac{\partial g^{ki}}{\partial x^j}\frac{\partial u^j}{\partial x^k} + g^{ki}\frac{\partial^2 u^j}{\partial x^j \partial x^k} + \frac{\partial g^{kj}}{\partial x^j}\frac{\partial u^i}{\partial x^k} + g^{kj}\frac{\partial^2 u^i}{\partial x^j\partial x^k} + \frac{\partial}{\partial x^j}(g^{li}g^{mj})\frac{\partial g_{lm}}{\partial x^k}u^k + g^{li}g^{mj}\frac{\partial^2 g_{lm}}{\partial x^j \partial x^k}u^k + g^{li}g^{mj}\frac{\partial g_{lm}}{\partial x^k}\frac{\partial u^k}{\partial x^j}\right)$$

If $g^{ij} = \delta^{ij}$ (i.e. Cartesian coordinates)

$$(E^\sharp)^{ij} = \frac{1}{2}\left(\frac{\partial u^j}{\partial x^i} + \frac{\partial u^i}{\partial x^j}\right)$$
$$\Longrightarrow \mathrm{tr}(E^\sharp) = \frac{\partial u^i}{\partial x^i}$$

$$\frac{\partial (E^\sharp)^{ij}}{\partial x^j} = \frac{1}{2}\left(\frac{\partial^2 u^j}{\partial x^j \partial x^i} + \frac{\partial^2 u^i}{\partial x^j \partial x^j}\right) := \frac{1}{2}\left(\frac{\partial}{\partial x^i}(\mathrm{div}u) + \Delta u^i\right)$$

So then, if λ, μ don't depend on position, for

$$\sigma^{ij}dS_j = \lambda(\mathrm{tr}E)\delta^{ij} + 2\mu E^{ij} \xrightarrow{d} \left(\frac{\partial(\lambda(\mathrm{tr}E)\delta^{ij}\sqrt{g})}{\partial x^j} + 2\frac{\partial(\mu E^{ij}\sqrt{g})}{\partial x^j}\right)\mathrm{vol}^n\frac{1}{\sqrt{g}} =$$
$$= \left(\lambda\frac{\partial(\mathrm{tr}E\sqrt{g})}{\partial x^i} + 2\mu\frac{\partial E^{ij}\sqrt{g}}{\partial x^j}\right)\mathrm{vol}^n\frac{1}{\sqrt{g}}$$

then with $g^{ij} = \delta^{ij}$,

$$d\sigma^{ij}dS_j = \left(\lambda\frac{\partial}{\partial x^i}\mathrm{div}u + \mu\left(\frac{\partial}{\partial x^i}(\mathrm{div}u) + \Delta u^i\right)\right)\mathrm{vol}^n$$

Thus, we reproduce, in Cartesian coordinates, the *Navier-Stokes equations* for *compressible, viscous* fluid flow:

$$\boxed{\rho\left(\frac{\partial u^i}{\partial t} + u^j\frac{\partial u^i}{\partial x^j}\right) = -\frac{\partial p}{\partial x^i} + (\lambda+\mu)\frac{\partial}{\partial x^i}\mathrm{div}u + \mu\Delta u^i} \qquad (12)$$

The case of *incompressible homogeneous* flow is when $\rho = \rho_0$, a constant, and $\mathrm{div}u = 0$ (i.e. "no volume expansion"), so that the Navier-Stokes equations for incompressible flow is

$$\rho_0\left(\frac{\partial u^i}{\partial t} + u^j\frac{\partial u^i}{\partial x^j}\right) = -\frac{\partial p}{\partial x^i} + \mu\Delta u^i$$

Set L *characteristic length*
U *characteristic velocity*
Then

$$(u')^i = \frac{u^i}{U} \qquad \frac{\partial u^i}{\partial t} = \frac{U}{L/U}\frac{\partial (u^i)'}{\partial t'} = \frac{U^2}{L}\frac{\partial (u^i)'}{\partial t'}$$
$$(x')^j = \frac{x^j}{L} \qquad \frac{\partial u^i}{\partial x^j} = \frac{U}{L}\frac{\partial (u')^i}{\partial (x')^j}$$
$$t' = \frac{t}{L/U} \qquad \frac{\partial u^i}{\partial t} + u^j\frac{\partial u^i}{\partial x^j} = \frac{U^2}{L}\frac{\partial (u^i)'}{\partial t'} + U(u')^j\frac{U}{L}\frac{\partial (u')^i}{\partial (x')^j} = \frac{U^2}{L}\left(\frac{\partial (u^i)'}{\partial t'} + (u')^j\frac{\partial (u')^i}{\partial (x')^j}\right)$$

$$\mathrm{div}u = \frac{L}{U}\mathrm{div}u' \qquad -\frac{\partial p}{\partial x^i} = \frac{-1}{L}\frac{\partial p}{\partial (x')^i} \xrightarrow{\cdot\frac{L}{\rho U^2}} -\frac{1}{\rho U^2}\frac{\partial p}{\partial (x')^i}$$
$$\Delta(u')^i = \frac{L^2}{U}\Delta u^i \qquad (\lambda+\mu)\frac{\partial}{\partial x^i}\mathrm{div}u = (\lambda+\mu)\frac{U}{L^2}\frac{\partial}{\partial (x')^i}\mathrm{div}u' \xrightarrow{\frac{L}{\rho U^2}} (\lambda+\mu)\frac{1}{\rho LU}\frac{\partial}{\partial (x')^i}\mathrm{div}u'$$
$$\mu\Delta u^i = \mu\frac{U}{L^2}\Delta(u')^i \xrightarrow{\frac{L}{\rho U^2}} \frac{\mu L}{\rho U^2}\Delta u^i = \frac{\mu}{\rho LU}\Delta(u')^i$$

Define $R := \frac{\rho LU}{\mu}$, the **Reynolds number**. R is a dimensionless quantity. $\frac{(\lambda+\mu)}{\rho LU}$ is a dimensionless quantity.
Bhatia, Norgard, Pascucci, Bremer have an insightful survey on the so-called Helmholtz-Hodge decomposition and expands upon it and what's out there already in the literature [7].
Let domain be the smooth submanifold $\mathscr{D} \subset N$, N is the spatial manifold. $\mathrm{dim}a = \mathrm{dim}N = n$; spacetime manifold $M = \mathbb{R}\times N$.
$a = (a^i) \in \mathscr{D}$ is a particle label.
It'd be instructive to compare expressions between Chorin and Marsden [6] and Landau and Lifshitz [1]:

$$\sigma = 2\mu(E - \frac{1}{d}\mathrm{tr}(E)1) + \eta\,\mathrm{tr}(E)1 \in \Gamma(\otimes^2 TM)$$
$$E = \frac{1}{2}(g_{kj}\frac{\partial u^k}{\partial x^i} + g_{ik}\frac{\partial u^k}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k}u^k)dx^i\otimes dx^j \in \Gamma(\otimes^2 T^*M)$$
$$E^\sharp = \frac{1}{2}\left(\frac{\partial u^j}{\partial x^i} + \frac{\partial u^i}{\partial x^j}\right)\frac{\partial}{\partial x^i}\otimes\frac{\partial x^j}{\in}\Gamma(\otimes^2 TM) \iff \sigma'_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right) + \eta\delta_{ik}\frac{\partial v_l}{\partial x_l}$$
$$\mathrm{tr}E^\sharp = \frac{\partial u^i}{\partial x^i}$$

$$\begin{aligned} \mu &\equiv \text{1st. coefficient of viscosity} \\ \eta = \lambda + \frac{2}{d}\mu &\equiv \text{2nd. viscosity} \end{aligned} \iff \begin{aligned} \eta &\equiv \text{1st. coefficient of viscosity} \\ \zeta &\equiv \text{second viscosity} \end{aligned}$$

$$\rho\left(\frac{\partial u^i}{\partial t} + u^j\frac{\partial u^i}{\partial x^j}\right) = -\frac{\partial p}{\partial x^i} + (\lambda+\mu)\frac{\partial}{\partial x^i}\mathrm{div}u + \mu\Delta u^i =$$
$$= -\frac{\partial p}{\partial x^i} + \mu\Delta u^i + \left(\eta + (1-\frac{2}{d})\mu\right)\frac{\partial}{\partial x^i}\mathrm{div}u \iff \rho\left[\frac{\partial v}{\partial t} + (v\cdot\mathrm{grad})v\right] = -\mathrm{grad}p + \eta\Delta v + \left(\zeta + \frac{\eta}{3}\right)\mathrm{grad}\,\mathrm{div}v$$

3.1. **Material Derivative.** Consider $u^j\frac{\partial u^i}{\partial x^j}$ or $(u\cdot\mathrm{grad})u$. Landau and Lifshitz on pp. 48, Sec. 15 has a useful table of the Equations of Motion in Curvilinear Coordinates [1]. Recall the metric g for cylindrical and spherical coordinates, for $g \in \otimes^2 T^*M$ and $g^{-1} \in \otimes^2 TM$

$$g = dr^2 + r^2 d\phi^2 + dz^2 \qquad g = d\rho^2 + \rho^2 d\theta^2 + \rho^2\sin\theta d\phi^2$$
$$g^{-1} = (\frac{\partial}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial}{\partial \phi})^2 + (\frac{\partial}{\partial z})^2 \qquad g^{-1} = (\frac{\partial}{\partial \rho})^2 + \frac{1}{\rho^2}(\frac{\partial}{\partial \theta})^2 + \frac{1}{\rho^2\sin\theta}(\frac{\partial}{\partial \phi})^2$$

4. Energy transport and Energy dissipation

Chorin and Marsden, pp. 10 [6].

Definition 6. Fluid is *incompressible* if $\frac{d}{dt}\int \mathrm{vol}^n = 0$

Now

$$\frac{d}{dt}\int \mathrm{vol}^n = \int \mathscr{L}_{\frac{\partial}{\partial t}+u}\mathrm{vol}^n = \int \frac{\partial}{\partial t}\mathrm{vol}^n + \mathscr{L}_u\mathrm{vol}^n = \int 0 + di_u\mathrm{vol}^n + i_u d\mathrm{vol}^n = \int d\frac{\sqrt{g}}{n}\epsilon_{i_1 i_2 \dots i_n}u^{i_1}dx^{i_2}\wedge\dots\wedge dx^{i_n} + 0 =$$
$$= \int \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}u^k)}{\partial x^k}\mathrm{vol}^n$$

If $\frac{d}{dt}\int \mathrm{vol}^n = 0$ i.e. fluid is incompressible, $\mathrm{div}u := \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}u^k)}{\partial x^k} = 0$

Let

$$E_{\mathrm{KE}} := \frac{1}{2}\int_{B(t)} \rho |u|^2 \mathrm{vol}^n$$

Then

$$\frac{d}{dt}E_{\mathrm{KE}} = \frac{d}{dt}\left[\frac{1}{2}\int_{B(t)} \rho |u|^2 \mathrm{vol}^n\right] = \frac{1}{2}\int_{B(t)} \mathscr{L}_{\frac{\partial}{\partial t}+u}(m|u|^2) = \frac{1}{2}\int_{B(t)} (\mathscr{L}_{\frac{\partial}{\partial t}+u}m)|u|^2 + m\mathscr{L}_{\frac{\partial}{\partial t}+u}|u|^2 = \frac{1}{2}\int_{B(t)} 0 + m\mathscr{L}_{\frac{\partial}{\partial t}+u}|u|^2$$

Now

$$\frac{\partial}{\partial t}|u|^2 = u^i\frac{\partial}{\partial t}(g_{ij}u^j) + u_i\frac{\partial u^i}{\partial t}$$

If g_{ij} is time-independent,

$$\frac{\partial}{\partial t}|u|^2 = 2u_i\frac{\partial u^i}{\partial t}$$

Also,

$$\begin{aligned}\mathscr{L}_u|u|^2 &= u^k\frac{\partial}{\partial x^k}(g_{ij}u^iu^i) = 2u^ku_i\frac{\partial u^i}{\partial x^k} + u^ku^ju^i\frac{\partial g_{ij}}{\partial x^k} = 2u_i(u^k\frac{\partial u^i}{\partial x^k} + \frac{1}{2}u^l\frac{\partial g_{jk}}{\partial x^l}u^jg^{ik}) = \\ &= 2u_i\left(u^k\frac{\partial u^i}{\partial x^k} + \Gamma^i_{jk}u^ju^k\right) = 2u\cdot\nabla_u u\end{aligned}$$

and all (I claim; I worked it out) that was required was that g be metric-compatible, i.e. $\nabla g = 0$.
Thus,

$$\frac{d}{dt}E_{\mathrm{KE}} = \int_{B(t)} mu_i\left(\frac{\partial u^i}{\partial t} + \nabla_u u^i\right)$$

Consider the "right hand side" (RHS), the work done on the fluid system by the stresses on the surface boundary, $T\in\Gamma(\otimes^2T^*M)$, and body forces per mass, b. I propose that this work takes this form:

$$\int_{\partial B} u_iT^{ij}dS_j + \int_B mb^iu_i \equiv \int_{\partial B} T(u,dS) + \int_B m\langle u,b\rangle \tag{13}$$

Now

$$\begin{aligned}\int_{\partial B} u_iT^{ij}dS_j &= \int_B \frac{1}{\sqrt{g}}\frac{\partial(u_iT^{ij}\sqrt{g})}{\partial x^j}\mathrm{vol}^n \equiv \int_B \mathrm{div}(u_iT^{ij})\mathrm{vol}^n = \int_B\left(u_i\frac{\partial T^{ij}}{\partial x^j} + T^{ij}\frac{1}{\sqrt{g}}\frac{\partial(u_i\sqrt{g})}{\partial x^j}\right)\mathrm{vol}^n \equiv \\ &\equiv \int_B\left(u_i\frac{\partial T^{ij}}{\partial x^j} + T^{ij}\mathrm{div}u_i\right)\mathrm{vol}^n\end{aligned}$$

Suppose $T^{ij} = -pg^{ij}$. Then

$$\begin{aligned}\int_B \mathrm{div}(u_iT^{ij})\mathrm{vol}^n &\equiv \int_B \frac{-1}{\sqrt{g}}\frac{\partial(u_ipg^{ij}\sqrt{g})}{\partial x^j}\mathrm{vol}^n = -\int_B\frac{1}{\sqrt{g}}\frac{\partial(u^jp\sqrt{g})}{\partial x^j}\mathrm{vol}^n = -\int_B\left(u^j\frac{\partial p}{\partial x^j} + \frac{p}{\sqrt{g}}\frac{\partial(u^j\sqrt{g})}{\partial x^j}\right)\mathrm{vol}^n = \\ &= -\int_B\left(u^j\frac{\partial p}{\partial x^j} + p\,\mathrm{div}u\right)\mathrm{vol}^n\end{aligned}$$

For $T^{ij} = -pg^{ij}$, this result is the most general case. If the fluid is incompressible, then $\mathrm{div}u = 0$, so the contribution to the work done on the fluid is $-\int_B u^j\frac{\partial p}{\partial x^j}\mathrm{vol}^n$.
Thus, energy conservation, for an incompressible fluid is

$$\rho u_i\left(\frac{\partial u^i}{\partial t} + \nabla_u u^i\right) = -u^j\frac{\partial p}{\partial x^j} + \rho u_ib^i$$

Part 2. Notes on Sabersky

5. Shear: shear stress

cf. Chapter 1 Introduction
In Appendix A Forms in Continuum Mechanics, Subsection A.g. Some Typical Computations Using Forms, Frankel (2004) [3] defines the so-called **rate of deformation** tensor, which Frankel describes as measuring how the flow deforms the physical system in consideration.

Consider curve $\gamma:\mathbb{R}\to N$ and generating velocity vector field $u = \dot{\gamma}\in\mathfrak{X}(N)$.
$\gamma(t)\in N$

Now $\gamma(0) = x(0)$ and so we recall the idea of ψ_t, a local 1-parameter group $(\psi_t)_{t\in I\subset\mathbb{R}}$ of local diffeomorphisms s.t. $\psi_t(x(0)) = x(t)\in N$.
$\gamma(t) = x(t)$
Now recall the definition of the Lie derivative of a tensor (e.g. Def. 2.2.6 on pp. 58, Chapter 2 Lie Groups and Vector Bundles of Jost (2011) [4]:

$$\mathscr{L}_ug := \frac{d}{dt}(\psi_t^*g)|_{t=0} = \lim_{t\to 0}\frac{\psi_t^*g' - g}{t} = \lim_{t\to 0}\frac{\left(g'_{ij}\frac{\partial x^i}{\partial a^k}\frac{\partial x^k}{\partial a^l} - g_{kl}\right)}{t}da^k\otimes da^l$$

If $\mathscr{L}_ug = 0$, then metric g doesn't change along integral curves of u, so u generates cont. 1-parameter family of symmetries ψ_t for g. [1]
Otherwise, $\mathscr{L}_ug\neq 0$.
Then u is *not* a Killing vector. ψ_t are not symmetric diffeomorphisms (i.e. isometries).
Let's compute $\mathscr{L}_ug$.
Calin and Chang (2005) [5] does a great job in calculating out *explicitly* with *explicit* calculation steps in all their proofs of theorems, lemmas, claims. For instance, in Chapter 5 Conservation Theorems, Section 5.3 The Energy-momentum tensor, Subsection 5.3.4 Divergence of the energy-momentum tensor, pp. 83, Lemma 5.33 of Calin and Chang (2005) [5], it gives the explicit calculation of $\mathscr{L}_ug$, exactly what we need:

$$\begin{aligned}\mathscr{L}_ug &= (\mathscr{L}_ug_{ij}dx^i)\otimes dx^j + g_{ij}dx^i\otimes\mathscr{L}_udx^j = (i_ud(g_{ij}dx^i) + d(g_{ij}u^i))\otimes dx^j + g_{ij}dx^i\otimes du^j = \\ &= (i_u\left(\frac{\partial g_{ij}}{\partial x^k}dx^k\wedge dx^i\right) + \frac{\partial(g_{ij}u^i)}{\partial x^k}dx^k)\otimes dx^j + g_{ij}dx^i\otimes\frac{\partial u^j}{\partial x^k}dx^k = \\ &= \left(u^k\frac{\partial g_{ij}}{\partial x^k}dx^i - u^i\frac{\partial g_{ij}}{\partial x^k}dx^k + u^i\frac{\partial g_{ij}}{\partial x^k}dx^k + g_{ij}\frac{\partial u^i}{\partial x^k}dx^k\right)dx^j + g_{ij}\frac{\partial u^j}{\partial x^k}dx^i\otimes dx^k = \left(u^k\frac{\partial g_{ij}}{\partial x^k} + g_{kj}\frac{\partial u^k}{\partial x^i} + g_{ik}\frac{\partial u^k}{\partial x^j}\right)dx^i\otimes dx^j\end{aligned}$$

This calculation is completely equivalent to using, besides the product rule on symmetric tensors, the $\mathbb{K}$-linearity property of $\mathscr{L}$,

$$\mathscr{L}_ug = (\mathscr{L}_ug_{ij})dx^i\otimes dx^j + g_{ij}((\mathscr{L}_udx^i\otimes dx^j) + dx^i\otimes\mathscr{L}_udx^j) = (u^k\frac{\partial g_{ij}}{\partial x^k} + g_{kj}\frac{\partial u^k}{\partial x^i} + g_{ik}\frac{\partial u^k}{\partial x^j})dx^i\otimes dx^j \tag{14}$$

Now, by theorem, for any smooth manifold, $\exists!$ Levi-Civita connection that's torsion-free (which can always be constructed to be torsion-free) and metric-compatible (which has to be defined, but otherwise, we've got not enough structure). Then on any coordinate chart, since by requiring metric compatibility, then $\nabla g = 0$, so

$$\begin{gathered}\nabla g = 0\Longrightarrow \\ \frac{\partial g_{ij}}{\partial x^k} - g_{lj}\Gamma^l{}_{ik} - g_{il}\Gamma^l{}_{jk} = \frac{\partial g_{ij}}{\partial x^k} - g_{lj}\Gamma^l{}_{ki} - g_{il}\Gamma^l{}_{kj} = 0\end{gathered}$$

using torsion free ∇ (so $\Gamma^l{}_{ki} = \Gamma^l{}_{ik}$ i.e. index symmetry) property.
Then from our expression for $\mathscr{L}_ug$, Eq. 14,

$$\begin{aligned}u^k\frac{\partial g_{ij}}{\partial x^k} + g_{kj}\frac{\partial u^k}{\partial x^i} + g_{ik}\frac{\partial u^k}{\partial x^j} &= u^k\frac{\partial g_{ij}}{\partial x^k} + g_{kj}((\nabla_iu)^k - \Gamma^k{}_{ji}u^j) + g_{ik}((\nabla_ju)^k - \Gamma^k{}_{ij}u^i) = \\ &= \underbrace{u^k\frac{\partial g_{ij}}{\partial x^k} - g_{kj}\Gamma^k{}_{ji}u^j - g_{ik}\Gamma^k{}_{ij}u^i}_{=\nabla_ug=\nabla g(u)=0} + g_{kj}(\nabla_iu)^k + g_{ik}(\nabla_ju)^k\end{aligned}$$

[1] Fionn Fitzmaurice. "Differential Geometry." https://www.maths.tcd.ie/~fionn/dg/dg.pdf

Thus,

$$\boxed{\mathscr{L}_u g = (g_{kj}(\nabla_i u)^k + g_{ik}(\nabla_j u)^k) dx^i \otimes dx^j} \tag{15}$$

for the Levi-Civita connection on any metric manifold (N,g).
Note that wikipedia calls this the "strain rate" tensor in its "Viscous stress tensor" article.
The viscous stress tensor, ε_{ij} or $\varepsilon = \varepsilon_{ij} dx^i \otimes dx^j \in \Gamma(\otimes^2 T^*M)$, then is related, for the case of Newtonian fluids, to $\mathscr{L}_u g$ in the following manner:

$$\varepsilon_{ij} = \mu_{ij}^{\ \ kl} (\frac{1}{2}\mathscr{L}_u g)_{kl} \text{ or } \varepsilon = (\mu_{ij}(\frac{1}{2}\mathscr{L}_u g)) dx^i \otimes dx^j = \mu(\frac{1}{2}\mathscr{L}_u g)$$

where $\otimes^2 T^*M \xrightarrow{\mu} \otimes^2 T^*M$.
Examples, i.e. specific cases.
Suppose u is all in $\frac{\partial}{\partial x}$ direction (1 direction).

$$u = u^x \frac{\partial}{\partial x} \equiv u \frac{\partial}{\partial x}$$

$$g_{kj}(\nabla_i u)^k = g_{kj}\left(\frac{\partial u^k}{\partial x^i} + \Gamma^k_{\ ji} u^j \right) = g_{xj}\left(\frac{\partial u}{\partial x^i} \right) + g_{kj}\Gamma^k_{\ xi} u$$

$$\mathscr{L}_u g = (g_{kj}(\nabla_i u)^k + g_{ik}(\nabla_j u)^k) dx^i \otimes dx^j = (g_{xj}\left(\frac{\partial u}{\partial x^i}\right) + g_{kj}\Gamma^k_{\ xi} u + g_{xi}\left(\frac{\partial u}{\partial x^j}\right) + g - ki\Gamma^k_{\ xj} u) dx^i \otimes dx^j =$$

$$= 2\left(g_{xj}\left(\frac{\partial u}{\partial x^i}\right) + g_{ki}\Gamma^k_{\ xj} u \right) dx^i \otimes dx^j$$

if $u = u(y)$, $\Gamma^k_{\ xj} = 0$, $g = 1$

$$\mathscr{L}_u g = 2\left(\frac{\partial u}{\partial y}\right) dy \otimes dx$$

$$\mu\left(\frac{1}{2}\mathscr{L}_u g\right)\left(\frac{\partial}{\partial y}\right) = \mu \frac{\partial u}{\partial y} dx \xmapsto{\sharp} \mu \frac{\partial u}{\partial y} = \tau_x = \tau$$

6. Compressible Fluids-One-Dimensional Flow

6.1. thermodynamic Preliminaries.

6.2. The Energy Equation.

Since

$$h_0 = h_1 + \frac{u_1^2}{2}$$

then

$$\frac{C_p}{MN}\tau_0 = \frac{C_p \tau_1}{MN} + \frac{u_1^2}{2} = \frac{C_p \tau_2}{MN} + \frac{u_2^2}{2} \text{ or } \tau_0 = \tau + \frac{u^2}{2\frac{C_p}{MN}}$$

Now the speed of sound at a particular point along the flow, 1, is

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{\gamma \frac{\tau_1}{M}}$$

and so

$$\implies \tau_0 = \tau + \frac{\mathfrak{M}^2 \gamma \tau}{2\frac{C_p}{M}} = \tau\left(1 + \frac{\mathfrak{M}^2(\gamma - 1)}{2}\right)$$

If $\mathfrak{M} = 1$, when $\tau_1 = \tau^*$,

$$\tau_0 = \tau^*\left(1 + \frac{\gamma - 1}{2}\right) = \tau^*\left(\frac{\gamma+1}{2}\right) \text{ or } \tau^* = \frac{2\tau_0}{\gamma + 1}$$

$$a^* = \sqrt{\gamma R T^*} = \sqrt{\frac{\gamma \tau^*}{M}} = \sqrt{\frac{2\gamma\tau_0}{M(\gamma+1)}} = \left(\frac{2\gamma R T_0}{\gamma+1}\right)^{1/2} \implies u^* \equiv a^* = \left(\frac{2\gamma R T_0}{\gamma+1}\right)^{1/2} = \sqrt{\frac{2\gamma\tau_0}{M(\gamma+1)}} \tag{16}$$

If $u > u^*$,

$$\mathfrak{M} := \frac{u}{a} = \frac{u}{\sqrt{\frac{\gamma\tau}{M}}} > \sqrt{\frac{2\tau_0}{\tau(\gamma+1)}} = \sqrt{\frac{2}{\gamma+1}(1 + \frac{\mathfrak{M}^2(\gamma-1)}{2})}$$

$$\implies \mathfrak{M}^2 > \frac{2}{\gamma+1} + \mathfrak{M}^2\frac{(\gamma-1)}{\gamma+1} \text{ or } \mathfrak{M}^2\left(\frac{2}{\gamma+1}\right) > \frac{2}{\gamma+1}$$
$$\implies \mathfrak{M} > 1$$

So if $u > u^*$, then $\mathfrak{M} > 1$. Likewise,
if $u < u^*$, then $\mathfrak{M} < 1$
Thus, the name $u^* \equiv a^*$ *critical velocity*

6.3. Normal Shock Waves.

Use moving reference frame in which shock is stationary, and resulting steady.
Continuity:

$$\rho_1 v_1 A = \rho_2 v_2 A \implies \rho_1 v_1 = \rho_2 v_2$$

momentum equation: using

$$\Pi^{ij} = \rho u^i u^j + p g^{ij}$$

then

$$\rho_1 v_1^2 + p_1 = \rho_2 v_2^2 + p_2$$

energy equation:

$$h_0 = h_1 + \frac{v_1^2}{2} = h_2 + \frac{v_2^2}{2} \text{ or } \frac{C_p \tau_1}{MN} + \frac{v_1^2}{2} = \frac{C_p\tau_2}{MN} + \frac{v_2^2}{2} \implies \frac{C_p R T_1}{N} + \frac{v_1^2}{2} = \frac{C_p R}{N} T_0$$

Also, note that I end up using this heat capacity *for the ideal gas* relation all the time in (rocket) propulsion:

$$C_p = \gamma C_v = \frac{\gamma N}{\gamma - 1} \text{ since } C_p = C_v + N \text{ or } \gamma = 1 + N/C_V$$

With mass continuity, momentum conservation, and the energy equation (Bernoulli invariant), then, using Python's sympy to do the algebra, detailed in file `fluid.py`

```
# mass conservation
massconsEq = Eq(rho_1*u_1,rho_2*u_2)

# momentum flux conservation
momconsEq  = Eq(rho_1*u_1**2+p_1,rho_2*u_2**2+p_2)

# energy equation or Bernoulli invariant
Bernoulli_invariant_1to2_Eq = Eq( C_p*R*T_1/N + u_1**2/2, C_p*R*T_2/N + u_2**2/2)

# stagnation enthalpy relation
stagh1Eq = Eq( C_p*R*T_0/N , C_p*R*T_1/N + u_1**2/2)
stagh2Eq = Eq( C_p*R*T_0/N , C_p*R*T_2/N + u_2**2/2)

# ideal gas law
ideal_gas1Eq = Eq(p_1,rho_1*R*T_1)
ideal_gas2Eq = Eq(p_2,rho_2*R*T_2)

# This reproduces Eq. (9.26) of Sabersky, Acosta, Hauptmann, Gates pp. 357, Sec. 9.6., Normal Shock Waves !!!
(momconsEq.subs(p_1,ideal_gas1Eq.rhs).subs(p_2,ideal_gas2Eq.rhs).subs(rho_2,solve(massconsEq,rho_2)[0])/rho_1).simplify()
# R*T_1 - R*T_2*u_1/u_2 + u_1**2 - u_1*u_2

# This reproduces Eq. (9.27) of Sabersky, Acosta, Hauptmann, Gates pp. 357, Sec. 9.6., Normal Shock
 Waves, where stagnation temperature relation was substituted into momentum and continuity equation
PrandtlEq1d = momconsEq.subs(p_1,ideal_gas1Eq.rhs).subs(p_2,ideal_gas2Eq.rhs).
subs(rho_2,solve(massconsEq,rho_2)[0]).subs(T_1,solve(stagh1Eq,T_1)[0]).subs(T_2,solve(stagh2Eq,T_2)[0])

PrandtlEq1d = PrandtlEq1d.subs(C_p, gamma*N/(gamma-1) )

solve(PrandtlEq1d,u_1**2)[0]
# (2*R*T_0*gamma*u_1 - 2*R*T_0*gamma*u_2 + gamma*u_1*u_2**2 + u_1*u_2**2)/(u_2*(gamma + 1))
# Thus, writing this out on paper, we get the desired result, Prandtl's equation (9.28) on pp. 357 of
# Sabersky, Acosta, Hauptmann, Gates
```

and thus

$$u_1u_2 = \frac{2RT_0\gamma}{\gamma+1} = (a^*)^2$$

for, the critical velocity was derived from only the energy equation (or Bernoulli invariant) and Mach *definition*, and was shown, explicitly, that if the velocity u at a point is greater than this critical velocity $u^* \equiv a^*$, then the flow is supersonic (cf. Eq. 16):

$$u^* \equiv a^* = \left(\frac{2\gamma RT_0}{\gamma+1} \right)^{1/2} = \sqrt{ \frac{2\gamma\tau_0}{M(\gamma+1)} }$$

So if $u_1 > a^*$, then $u_2 < a^*$, and so approaching flow is supersonic and downstream flow is subsonic.
Second law of thermodynamics forbids the other way.
Note that

$$u_1u_2 = \frac{2RT_0\gamma}{\gamma+1} \text{ or } \mathfrak{M}_1\mathfrak{M}_2 = \frac{2}{\gamma+1}\frac{T_0}{\sqrt{T_1T_2}}$$

(9.16) from Sabersky,

(17) $$\boxed{\mathfrak{M}_2^2 = \frac{\mathfrak{M}_1^2(\gamma-1)+2}{2\gamma\mathfrak{M}_1^2-\gamma+1}}$$

Although preceding results obtained for constant-area duct, they're valid in gradually varying duct
shock region very thin; order of a few molecular mean free path lengths; area change across the shock is then usually negligible for practical purposes
Preceding, viscosity effects near wall ignored; since velocity must still be 0 at wall, supersonic flow must revert to subsonic in near-wall region.[2]
Now recall that for $U = U(\tau, V)$ (in general, $U = U(\tau)$ for perfect ideal gas),

$$\tau d\sigma = dU + pdV = \left(\frac{\partial U}{\partial \tau}\right)_V d\tau + \left(\frac{\partial U}{\partial V}\right)_\tau dV + pdV = C_V d\tau + \left(\left(\frac{\partial U}{\partial V}\right)_\tau + p \right) dV$$

If $U = U(\tau)$,

$$\tau d\sigma = C_V d\tau + pdV \text{ or } d\sigma = \frac{C_V}{\tau}d\tau + \frac{p}{\tau}dV = \frac{C_V}{\tau}d\tau + \frac{N}{V}dV$$

$$\Longrightarrow \int_\gamma d\sigma = \sigma_2 - \sigma_1 = C_V \ln\left(\frac{\tau_2}{\tau_1}\right) + N\ln\frac{V_2}{V_1} = C_V\ln\left(\frac{\tau_2}{\tau_1}\right) + N\ln\left(\frac{\rho_1}{\rho_2}\right)$$

From momentum conservation, and ideal gas law and Mach definition,

$$\rho_1u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \text{ or } p_1 - p_2 = \rho_2u_2^2 - \rho_1u_1^2 = \frac{p_2}{RT_2}u_2^2 - \frac{p_1}{RT_1}u_1^2 = \gamma(p_2\mathfrak{M}_2^2 - p_1\mathfrak{M}_1^2) \text{ or } \frac{p_1}{p_2} = \frac{1+\gamma\mathfrak{M}_2^2}{1+\gamma\mathfrak{M}_1^2}$$

For real shocks, $\mathfrak{M}_1 > \mathfrak{M}_2$, so $p_1 < p_2$
Now

$$\frac{\rho_1}{\rho_2} = \frac{p_1T_2}{p_2T_1}$$

$$\frac{T_1}{T_2} = \frac{1+\frac{\gamma-1}{2}\mathfrak{M}_2^2}{1+\frac{\gamma-1}{2}\mathfrak{M}_1^2}$$

$$\frac{\rho_1}{\rho_2} = \frac{(1+\gamma\mathfrak{M}_2^2)}{(1+\gamma\mathfrak{M}_1^2)}\frac{(1+\frac{\gamma-1}{2}\mathfrak{M}_2^2)}{(1+\frac{\gamma-1}{2}\mathfrak{M}_1^2)}$$

Shock waves are highly irreversible, since very large velocity and temperature gradients occur through shock itself; hence frictional, or dissipative effects must be present.[2]

$$\sigma_2 - \sigma_1 = \frac{N}{\gamma-1}\ln\left[\frac{1+\frac{\gamma-1}{2}\mathfrak{M}_1^2}{1+\frac{\gamma-1}{2}\mathfrak{M}_2^2}\right] + N\ln\left[\frac{1+\gamma\mathfrak{M}_2^2}{1+\gamma\mathfrak{M}_1^2}\frac{(1+\frac{\gamma-1}{2}\mathfrak{M}_2^2)}{(1+\frac{\gamma-1}{2}\mathfrak{M}_1^2)}\right]$$

From Sabersky (9.29),(9.16),(9.31) [2]

$$\Longrightarrow s_2 - s_1 = \frac{R\gamma}{\gamma-1}\ln\left[\frac{2}{(\gamma+1)\mathfrak{M}_1^2} + \frac{\gamma-1}{\gamma+1}\right] + \frac{R}{\gamma-1}\ln\left[\frac{2\gamma\mathfrak{M}_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\right]$$

Part 3. Fluid Mechanics (revisited)

7. Conservation

(18) $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

$$\frac{d}{dt}m \equiv \dot{m} = \frac{d}{dt}\int \rho \,\mathrm{vol}^n = \int \left(\frac{\partial\rho}{\partial t} + \mathscr{L}_u\rho \right)\mathrm{vol}^n = \int \left(\frac{\partial \rho}{\partial t} + (\mathbf{d}i_u + i_u\mathbf{d})\rho \right)\mathrm{vol}^n = \int \frac{\partial\rho}{\partial t} + \mathbf{d}i_u\rho\,\mathrm{vol}^n$$

Now

$$\mathrm{vol}^n = \frac{\sqrt{g}}{n!}\varepsilon_{i_1\dots i_n}dx^{i_1}\wedge\dots\wedge dx^{i_n}$$

$$i_{\mathbf{u}}\mathrm{vol}^n = \frac{\sqrt{g}}{(n-1)!}\varepsilon_{i_1\dots i_n}u^{i_1}dx^{i_2}\wedge\dots\wedge dx^{i_n}$$

$$\xrightarrow{\mathbf{d}} \frac{\varepsilon_{i_1\dots i_n}}{(n-1)!}\frac{\partial(\sqrt{g}u^{i_1}\rho)}{\partial x^j}dx^j\wedge dx^{i_2}\wedge\dots\wedge dx^{i_n} = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}u^j\rho)}{\partial x^j}\mathrm{vol}^n$$

$$\Longrightarrow \frac{\partial\rho}{\partial t} + \mathrm{div}\, j = 0 \text{ or } \frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}u^j\rho)}{\partial x^j} = 0$$

As a sanity check, consider a change of coordinates from cylindrical to Cartesian coordinates.
Consider $g = g_{ij}dx^i\otimes dx^j \in T^*M\otimes T^*M \equiv \otimes^2T^*M$.
For smooth (embedding or diffeomorphism) $F : N \to M$,
in our particular case, $F(r,\phi,z) = (x,y,z) = \begin{pmatrix} r\cos\phi \\ r\sin\phi \\ z\end{pmatrix}$
Now the pullback is $F^*g \in \otimes^2T^*N$

$$F^*g(X,Y) = g(F_*X, F_*Y) = g\left(\frac{\partial y^j}{\partial x^i}X^i\frac{\partial}{\partial y^j}, \frac{\partial y^k}{\partial x^l}Y^l\frac{\partial}{\partial y^k}\right) = \frac{\partial y^j}{\partial x^i}\frac{\partial y^k}{\partial x^l}X^iY^lg\left(\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\right) = \frac{\partial y^j}{\partial x^i}\frac{\partial y^k}{\partial x^l}X^iY^lg_{jk}$$

$$\Longrightarrow (F^*g)_{ij} = \frac{\partial y^l}{\partial x^i}\frac{\partial y^m}{\partial x^j}g_{lm}$$

If $g_{jk} = \delta_{jk}$ (usual Euclidean metric),

$$(F^*g)_{ij} = \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j} = \left(\frac{\partial y^i}{\partial x^k}\right)^T\frac{\partial y^k}{\partial x^j} = (D_xy)^TD_xy$$

(F^*g) is simply the $\text{Jacobian}^T \cdot \text{Jacobian}$.
For cylindrical coordinates,

$$D_xy = \begin{bmatrix} c\phi & -rs\phi & \\ s\phi & rc\phi & \\ & & 1\end{bmatrix}$$

$$\Longrightarrow F^*g = \begin{bmatrix} c\phi & s\phi & \\ -rs\phi & rc\phi & \\ & & 1\end{bmatrix}\begin{bmatrix} c\phi & -rs\phi & \\ s\phi & rc\phi & \\ & & 1\end{bmatrix} = \begin{bmatrix} 1 & & \\ & r^2 & \\ & & 1\end{bmatrix}$$

$$\sqrt{\det(F^*g)} = \sqrt{r^2} = r$$

So

$$\mathrm{div}\, j = \frac{1}{r}\frac{\partial(ru^r\rho)}{\partial r} + \frac{1}{r}\frac{\partial(ru^\varphi\rho)}{\partial\varphi} + \frac{1}{r}\frac{\partial(ru^z\rho)}{\partial z} = \frac{1}{r}\frac{(\partial ru^r\rho)}{\partial r} + \frac{\partial(u^\varphi\rho)}{\partial\varphi} + \frac{\partial(u^z\rho)}{\partial z}$$

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