Music Machine Learning

V – Bayesian inference

Master ATIAM - Informatique

Philippe Esling (esling@ircam.fr)

Maître de conférences – UPMC

Equipe représentations musicales (IRCAM, Paris)

UPMC SORBONNE UNIVERSITÉS master ATIAM ircam Centre Pompidou

Learning

Using machine learning to model creativity ?

- **Supervised learning** is inferring a function from labeled training data
- **Unsupervised learning** is trying to find hidden structure in unlabeled data

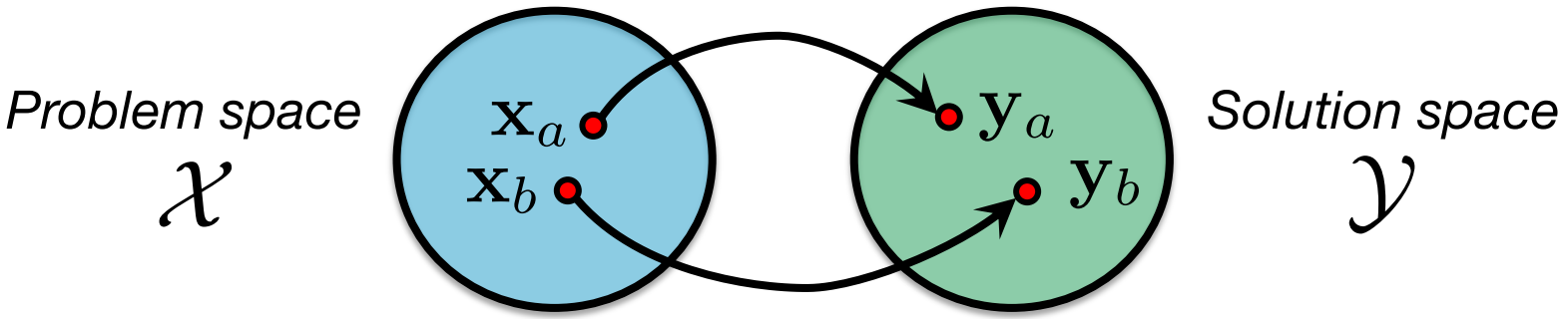
We can define any problem as $\hat{\mathbf{y}} = f_\theta(\mathbf{x})$

So we first need to define what **function** could *approximate* this process

$$f_\theta \in \mathcal{F} \qquad \theta \in \Theta$$

Then we need to **evaluate the « quality »** of this approximation

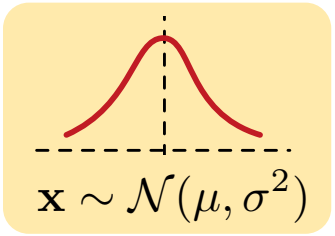
$$\mathcal{L}(\hat{\mathbf{y}}, \mathbf{y} \mid \theta, f_\theta)$$



How can we model uncertain (or creative) problems ?

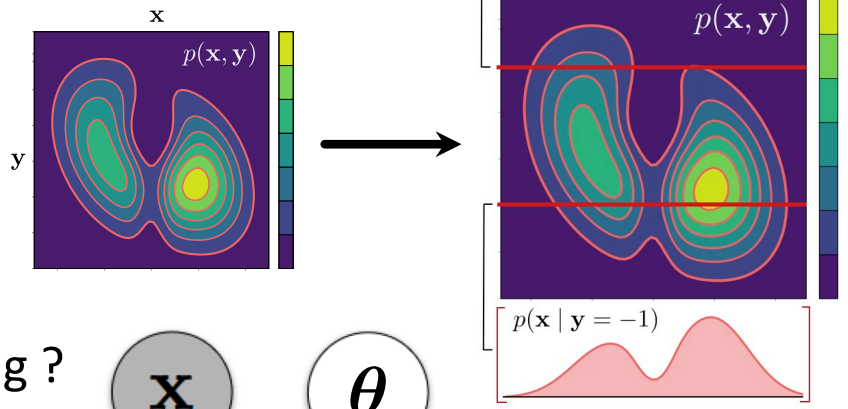
Probabilistic machine learning

Consider data $\mathbf{x} \in \mathbb{R}^n$ following a distribution $\mathbf{x} \sim p(\mathbf{x})$



Sampling new individuals

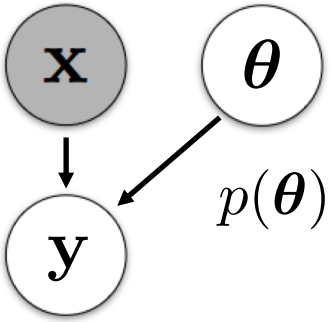
$$\mathbf{x}_1 \rightarrow -0.01$$
$$\mathbf{x}_2 \rightarrow 3.72$$
$$\mathbf{x}_3 \rightarrow 0.046$$



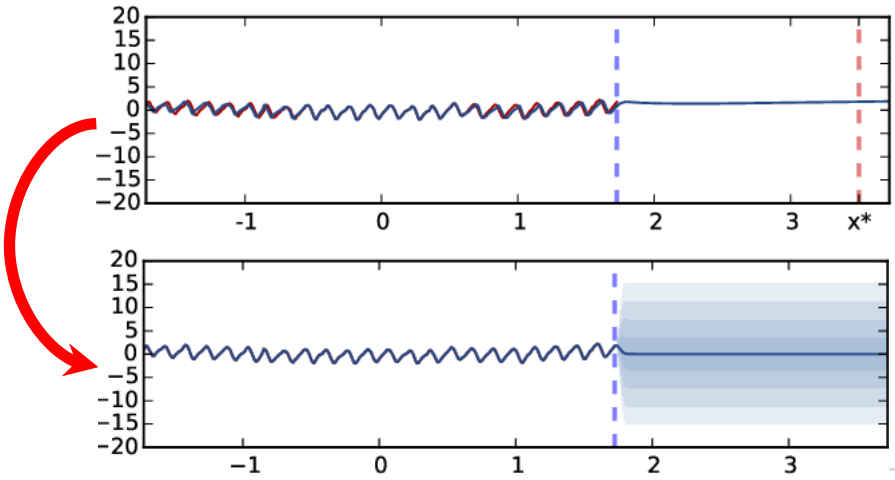
Now how to define (*discriminative*) learning ?

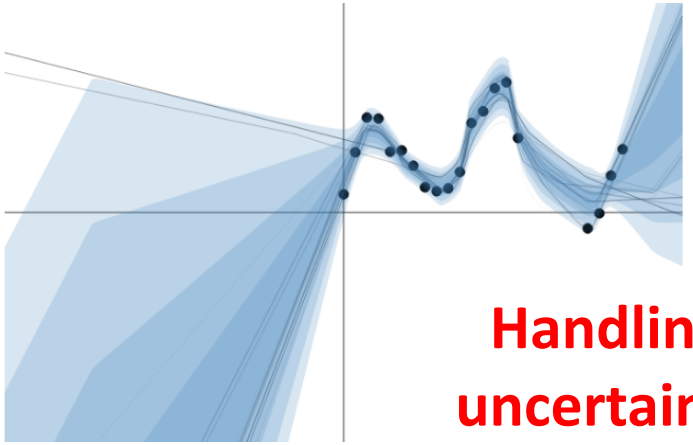
Looking for the answer $\mathbf{y}$ *given* the input $\mathbf{x}$

$$p(\mathbf{y}|\mathbf{x}) = \int p(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}) d\boldsymbol{\theta}$$



Why ?





Handling uncertainty

2. Conditional probability

Definition: $\mathcal{P}(a|b) = \dfrac{\mathcal{P}(a,b)}{\mathcal{P}(b)}$

$$\mathcal{P}(a|b)\mathcal{P}(b) = \mathcal{P}(a,b) = \mathcal{P}(b,a) = \mathcal{P}(b|a)\mathcal{P}(a)$$

$$\mathcal{P}(a|b) = \frac{\mathcal{P}(b|a)\mathcal{P}(a)}{\mathcal{P}(b)}$$

a = class

b = evidence

Let's say we have a classification problem in which

Easy

Hard — $\mathcal{P}(c|e) = \dfrac{\mathcal{P}(e|c)\mathcal{P}(c)}{\mathcal{P}(e)}$

Bayes' rule

Definition: $\mathcal{P}(a|b) = \dfrac{\mathcal{P}(a,b)}{\mathcal{P}(b)}$

$$\mathcal{P}(a|b)\mathcal{P}(b) = \mathcal{P}(a,b) = \mathcal{P}(b,a) = \mathcal{P}(b|a)\mathcal{P}(a)$$

Bayes' rule $\mathcal{P}(a|b) = \dfrac{\mathcal{P}(b|a)\mathcal{P}(a)}{\mathcal{P}(b)}$

a = class

b = evidence

Let's say we have a classification problem in which

Easy

Hard — $\mathcal{P}(c|e) = \dfrac{\mathcal{P}(e|c)\mathcal{P}(c)}{\mathcal{P}(e)}$ **Bayes' rule**

Bayesian classification

Now starting from Bayes' rule $\mathcal{P}(c|e) = \frac{\mathcal{P}(e|c)\mathcal{P}(c)}{\mathcal{P}(e)}$

- I have several classes, I have to decide given some evidence
- If I have the probability of the evidence given each class
- And the a priori probability of each class
- Then I'm done … **as the denominator is the same for all classes**
- But usually there is more than one piece of evidence

$$\mathcal{P}(c_i|e) = \frac{\mathcal{P}(e_1, ...e_n|c_i)\mathcal{P}(c_i)}{d}$$

- So what if these pieces of evidence are independent (given the class)

$$\mathcal{P}(c_i|e) = \frac{\mathcal{P}(e_1|c_i)\mathcal{P}(e_2|c_i)...\mathcal{P}(e_n|c_i)\mathcal{P}(c_i)}{d}$$

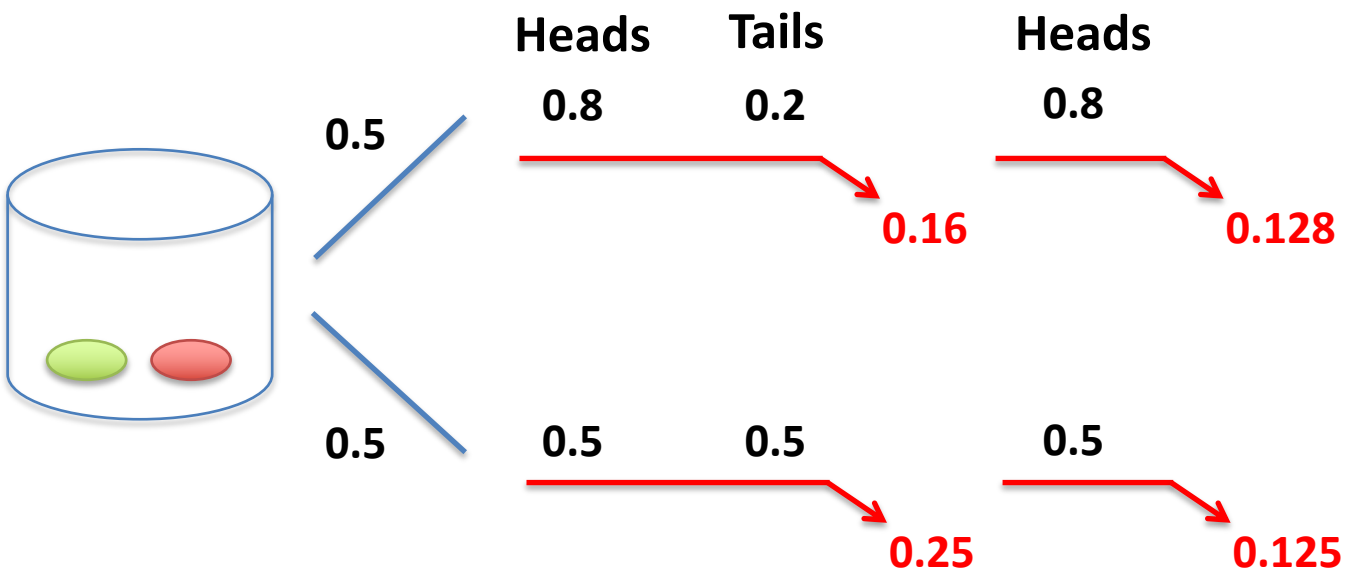
- So we just need to go through every class and select the biggest one

Bayesian classification

- Let's say I have two coins, one is fair and one is rigged

$$\mathcal{P}(Heads) = 0.5 \quad \mathcal{P}(Tails) = 0.5$$

$$\mathcal{P}(Heads) = 0.8 \quad \mathcal{P}(Tails) = 0.2$$



$$\mathcal{P}(c_i|e) = \frac{\mathcal{P}(e_1|c_i)\mathcal{P}(e_2|c_i)...\mathcal{P}(e_n|c_i)\mathcal{P}(c_i)}{d}$$

Bayesian inference

Make inferences about unknown quantities using available information.

Inference = make probability statements

Unknowns = parameters, functions of parameters, states or latent variables, "future" outcomes, classes labels

Information = data-based / non data-based

- theories of behavior and "subjective views"
- there is an underlying structure
- parameters are finite or in some range

Bayesian inference

Likelihood function, *probability of the data given the model*

Prior probability, the probability of the model on previous knowledge

$$\mathcal{P}(model|data) = \frac{\mathcal{P}(data|model)\mathcal{P}(model)}{\mathcal{P}(data)}$$

Posterior probability, *the probability of the model (class) given the data*

Probability of the data, *normalizing constant ensuring the probabilities sum to 1*

Posterior ∝ "Likelihood" × Prior

- Likelihood contains all information relevant for inference.
- Same likelihood function = same inferences about unknowns.

Bayesian Inference **= Find the explanation with highest posterior probability**

Bayesian inference

- Goal is to represent the **belief** of learning agents
- Bayesian inference of classes y_i can be interpreted as

Update the prior beliefs with a new information x

Likelihood

$$\text{Prior} \quad p(y_i) \cdot \frac{p(x|y_i)}{\sum_j p(x|y_j)p(y_j)} = p(y_i|x) \quad \text{Posterior}$$

Evidence ($p(x)$)

- Priors are usually unknown
  - Can be removed by **assuming equal belief** in all models at the start
- **Posterior is prior influenced by the likelihood according to new evidence** (information)

Bayesian inference

- How to choose the best class given the data?
- Choose the **Maximum A Posteriori (MAP)** class

$$\hat{y} = \underset{y_i}{\operatorname{argmax}}\ p(y_i|x)$$

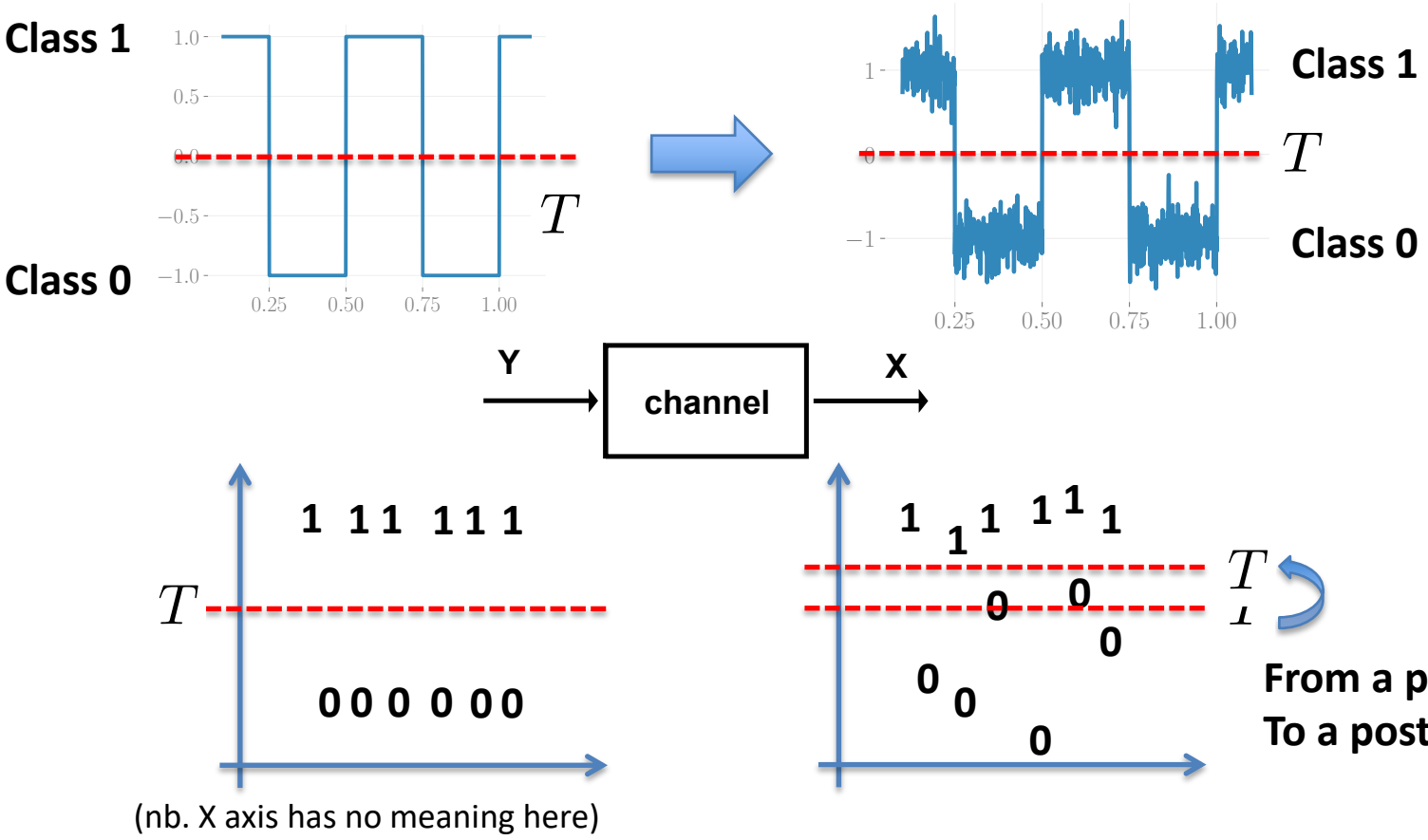
- Intuitive: Choose the most probable class given the observation(s)
- We do not know $p(y_i|x)$
- But we know $p(x|y_i)$
- If we apply Bayes' rule in $\hat{y} = \underset{y_i}{\operatorname{argmax}}\ p(y_i|x)$

We obtain $\underset{y_i}{\operatorname{argmax}} \frac{p(x|y_i)p(y_i)}{\sum_j p(x|y_j)p(y_j)}$

We have a similar denominator ($p(x)$) for all y_i

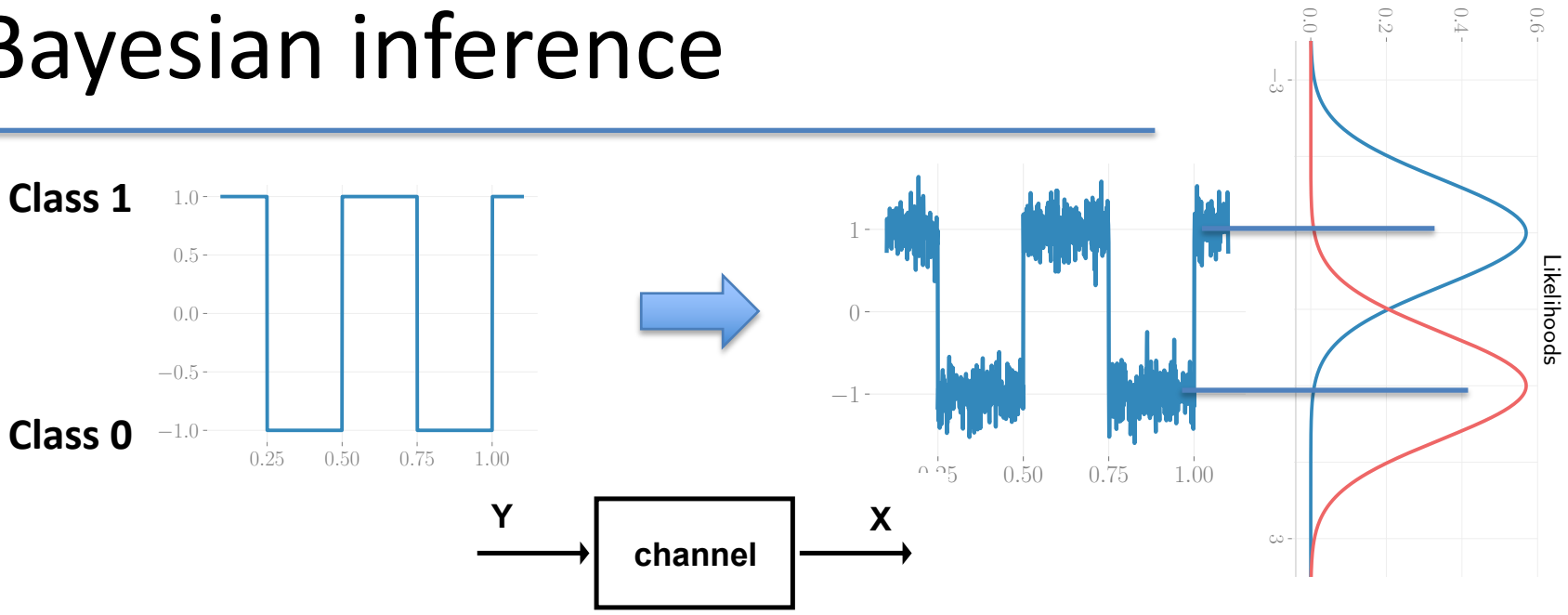
So we need to solve for $\hat{y} = \underset{y_i}{\operatorname{argmax}}\ p(x|y_i)p(y_i)$

Bayesian inference



Intuitively $$Y = \begin{cases} 0, & \text{if } x > T \\ 1, & \text{if } x > T \end{cases}$$

Bayesian inference



- **What do we need** to lay a Bayesian inference ?
  - (A priori) **class probabilities**

    Without other knowledge **an equal belief**

    $$p_y(0) = p_y(1) = 1/2$$
  - **Class-conditionnal probabilities**

    Here the class depends on **the noise added by the channel**

    By the central limit theorem we can **assume that noise is Gaussian**

    $$\epsilon \sim \mathcal{N}(\mu, \sigma^2)$$

Bayesian inference

- **Gaussian probability density** function $p(x) = \mathcal{N}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
- Since we assumed that the noise is Gaussian and **additive**

$$X = Y + \varepsilon, \qquad \varepsilon \sim N(0, \sigma^2)$$

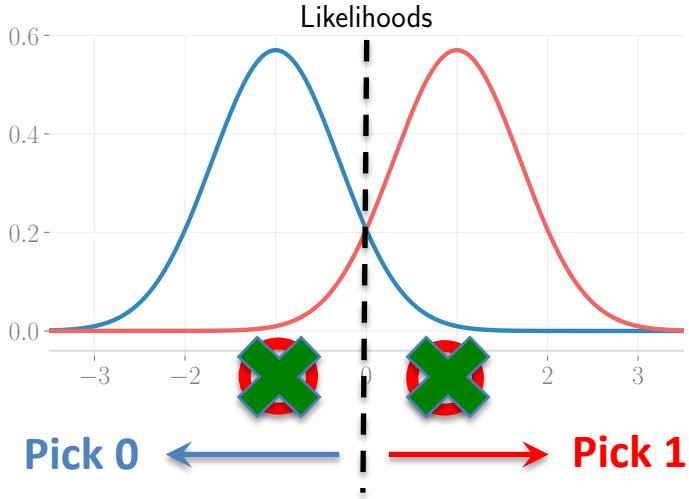
- So X corresponds to the input (Y) plus Gaussian noise

Probability of input given classes

Class 0 $p_{X|Y}(x|0) = \mathcal{N}(x$ [illegible] $\sigma^2)$

Class 1 $p_{X|Y}(x|1) = \mathcal{N}(x$ [illegible] $\sigma^2)$

$$p_y(0) = p_y(1) = 1/2$$



Real-world

$$p_{X|Y}(x|0) = \mathcal{N}(x \mid \mu_0, \sigma^2)$$
$$p_{X|Y}(x|1) = \mathcal{N}(x \mid \mu_1, \sigma^2)$$

Bayesian inference

- What happens for the general case $p_{X|Y}(x|0) = \mathcal{N}(x \mid \mu_0, \sigma^2)$ $p_{X|Y}(x|1) = \mathcal{N}(x \mid \mu_1, \sigma^2)$
- To compute the Bayesian Decision Rule (BDR), we can use **log probabilities**

$$i^* = \underset{i}{\operatorname{argmax}} \left[\log p_{X|Y}(x|i) + \log p_Y(i)\right]$$

- And note that the priors are equal for everybody so

$$i^* = \underset{i}{\operatorname{argmax}} \log p_{X|Y}(x|i)$$

- If we develop this equation

$$\begin{aligned} i^* &= \underset{i}{\operatorname{argmax}} \log p_{X|Y}(x|i) \\ &= \arg\max_i \log\left\{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu_i)^2}{2\sigma^2}}\right\} \\ &= \arg\max_i \left\{-\frac{1}{2}\log(2\pi\sigma^2) - \frac{(x-\mu_i)^2}{2\sigma^2}\right\} \\ &= \arg\min_i \frac{(x-\mu_i)^2}{2\sigma^2} \end{aligned}$$

Bayesian decision theory

- If we consider that both distributions have the same variance

$$i^* = \arg\min_i \frac{(x-\mu_i)^2}{2\sigma^2}$$
$$= \arg\min_i (x^2 - 2x\mu_i + \mu_i^2)$$
$$= \arg\min_i (-2x\mu_i + \mu_i^2)$$

- So the optimal decision will be

$$-2x\mu_0 + \mu_0^2 < -2x\mu_1 + \mu_1^2$$
$$2x(\mu_1 - \mu_0) < \mu_1^2 - \mu_0^2$$

$$\boxed{x < \frac{\mu_1 + \mu_0}{2}}$$

- All this work to find this … but we did find back the intuition
- And we had to **make lots of assumptions**
  - **Uniform class probabilities, additive noise, gaussianity**

Bayesian decision rule

- Let's pump up the BDR for **multivariate Gaussian**

$$i^*(x) = \arg\max_i \left[\log P_{X|Y}(x \mid i) + \log P_Y(i)\right]$$

covariance

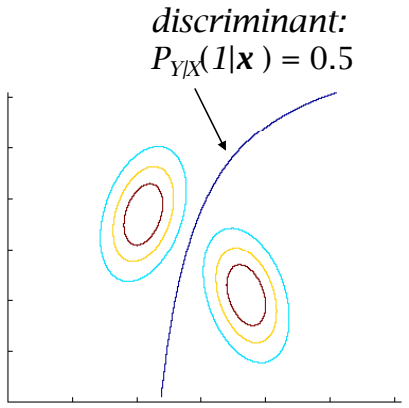
$$P_{X|Y}(x \mid i) = \frac{1}{\sqrt{(2\pi)^d \mid \Sigma_i \mid}} \exp\left\{-\frac{1}{2}(x-\mu_i)^T \Sigma_i^{-1}(x-\mu_i)\right\}$$

- Thanks to the *log* the BDR becomes

$$i^*(x) = \arg\max_i \left[-\frac{1}{2}(x-\mu_i)^T \Sigma_i^{-1}(x-\mu_i) - \frac{1}{2}\log(2\pi)^d \left|\Sigma_i\right| + \log P_Y(i)\right]$$

$$d_i(x,y) = (x-y)^T \Sigma_i^{-1}(x-y) \qquad \alpha_i = \log(2\pi)^d \left|\Sigma_i\right| - 2\log P_Y(i)$$

So the **final BDR** is $i^*(x) = \arg\min\left[d_i(x,\mu_i) + \alpha_i\right]$



- The optimal rule is to **assign x to the closest class**
- Mesured with the Mahalanobis distance (d)
- To which a **constant** is added to account for class prior

Summary of BDR

- The Bayesian Decision Rule is the **optimal one**
- The models reflect a causal interpretation of the problem
- Natural decomposition of the problem into
  - « what we knew » **(prior)**
  - « what the data tells us » **(observation)**
- No need for heuristics to combine these two informations
- However BDR optimal **only if models are correct**
- **How do we estimate the parameters of our distribution ?**

Maximum Likelihood

- So we have the optimal (and geometric) solution

$$i^*(x) = \arg\max_i [\underbrace{\mu_i^T \Sigma^{-1} x}_{w_i^T} - \underbrace{\frac{1}{2}\mu_i^T \Sigma^{-1} \mu_i + 2 \log P_Y(i)}_{w_{i0}}]$$

- But we still don't know the parameters $\mu, \Sigma, P_Y(i)$
- We have to **estimate** these values from a **training set**

(ex. use the average value as estimate for the mean)

We rely on the **Maximum Likelihood (ML)** principle

1. Choose a **parametric model** for probabilities (function of parameters)
2. Assemble a **training dataset**
3. Find the **parameters that maximize the probabilities**

Maximum Likelihood

1. **Choose a parametric model** for all probabilities
   - We usually denote parameters by Θ and the class-conditional distributions by
   $$p_{X|Y}(x|i,\Theta)$$
   - Θ is not a random variable but a parameter (probabilities are function of it)

2. Assemble a **collection of datasets** $X^{(i)} = \{x_1^{(i)}, \cdots, x_n^{(i)}\}$
   - Set of examples **independently drawn** from class i (cf. sampling schemes)

3. Select the **parameters that maximize the probability of data** (for that i)

$$\Theta_i = \underset{\Theta}{\operatorname{argmax}}\ p_{X|Y}(X^{(i)}|i,\Theta)$$

$$= \underset{\Theta}{\operatorname{argmax}}\ \log p_{X|Y}(X^{(i)}|i,\Theta)$$

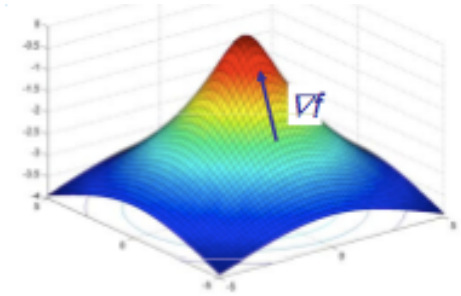
How do we solve this ?

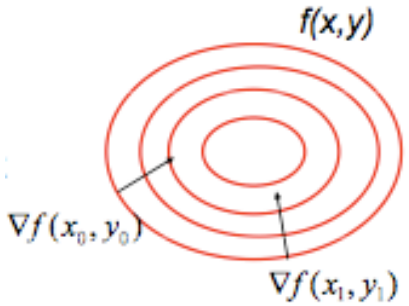
Maximum Likelihood

- The gradient:
  - in higher dimensions, the generalization of the derivative is the gradient. The gradient of a function f(x) at z is:

$$\nabla f(z) = \left(\frac{\partial f}{\partial x_0}(z), \ldots, \frac{\partial f}{\partial x_{n-1}}(z) \right)^T$$

  - It has a nice geometric interpretation:
    - It points in the direction of *maximum growth* of the function
    - *Perpendicular* to the contour where the function is constant





Maximum Likelihood

- The Hessian:
  - extension of the 2nd-order derivative is the Hessian Matrix:

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_0^2} & \frac{\partial^2 f}{\partial x_0\, \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_0\, \partial x_{n-1}} \\ \frac{\partial^2 f}{\partial x_1\, \partial x_0} & \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1\, \partial x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_{n-1}\, \partial x_0} & \frac{\partial^2 f}{\partial x_{n-1}\, \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_{n-1}^2} \end{bmatrix}$$

  - In an ML setup we have a *maximum* when Hessian is negative definite or

$$x^T \nabla^2 f(x) x \leq 0$$

Maximum Likelihood

Summary of Maximum Likelihood

1. Choose a parametric model for probabilities $p_X(X, \Theta)$
2. Assemble a dataset of independent examples $X = \{x_1, \cdots, x_n\}$
3. Select parameters that maximize the probability of the data
   - Given the dataset, we need to solve

$$\Theta^* = \underset{\Theta}{\operatorname{argmax}}\ p_X(X|\Theta)$$

$$= \underset{\Theta}{\operatorname{argmax}}\ \log p_X(X|\Theta)$$

   - The solutions are the parameters such that

$$\nabla_\Theta p_X(X, \Theta) = 0$$

$$\Theta^t \nabla^2_\Theta p_X(X, \Theta)\Theta \leq 0, \forall \Theta \in \mathbb{R}^n$$

Frequentist vs. Bayesian

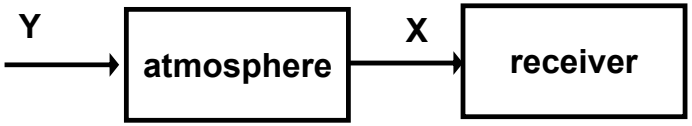
- But wait … why not **using Bayes to estimate the parameters** ?!
- In fact we can ! There is just a **difference in interpretation**
- **Frequentist** (ML) view
  - Probabilities are relative frequencies
  - Makes sense with lot of observations
  - But in most cases we do not have lots of observations
  - And the probabilities are mostly not objective
- **Bayesian** view
  - Probabilities are *subjective* (not equal to relative count)
  - Probabilities are **degrees of belief** on the outcome
  - Equates to « frequentist over alternative realities »

Frequentist vs. Bayesian

- **Optimal estimate**
  - Under ML there is one « best » estimate (optimization-wise)
  - Under Bayes there is no « best » estimate
  - In the probabilistic framework, « best » has no real sense
- **Predictions**
  - We do not really care about the parameters themselves
  - Only in the fact that they build models …
  - Models can be used to make predictions
  - Unlike ML, Bayes uses **all information in training to predict**

Example

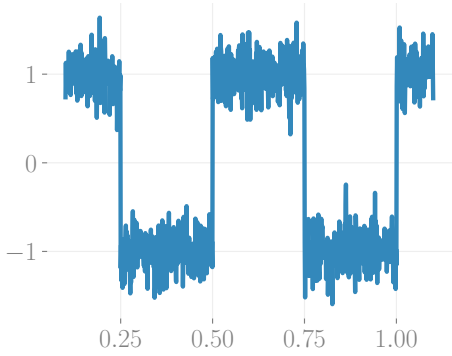
Communications problem

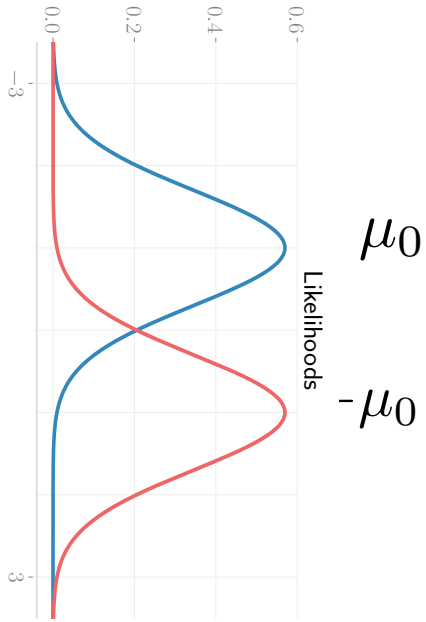


Two states

Y = 0 : transmit signal $s = \mu_0$

Y = 1 : transmit signal $s = -\mu_0$





Noise model

$$X = Y + \epsilon \qquad \epsilon \sim \mathcal{N}(0, \sigma^2)$$

Example

- **Calibration mode**
  - Ask the system to **transmit a single class** and measure X
  - Compute the **ML estimate of the mean of X**

$$\mu = \frac{1}{n}\sum_i X_i$$

- Result: the **estimate is different than** μ_0
- We need to combine **two forms of information**

Our **prior** was $\mu \sim \mathcal{N}(\mu_0, \sigma^2)$

Our **data-driven estimate** is $X \sim \mathcal{N}(\hat{\mu}, \sigma^2)$

We could stop there but think about **erratic jitter** on one calibration

Bayesian solution

- Gaussian **likelihood** (observations)

$$p_X(X|\mu) = \mathcal{N}(X|\mu, \sigma^2)$$

The **calibration data** we receive — The **mean we estimated** from it

- Gaussian **prior** (what we knew)

$$p_\mu(\mu) = \mathcal{N}(\mu|\mu_0, \sigma_0^2)$$

The **mean we used** up to now

μ_0, σ_0^2 are known **hyper-parameters**

- We **need to compute** the **posterior distribution** for μ

$$p_\mu(\mu|X) = \frac{p_X(X|\mu)p_\mu(\mu)}{p_X(X)}$$

The « real » (optimal) **mean we will use** afterwards

Conjugate prior

- We should **sweat over the mathematics** … but note that
  - The prior is Gaussian $p_\mu(\mu) = \mathcal{N}(\mu|\mu_0, \sigma_0^2)$
  - The posterior is Gaussian $p_X(X|\mu) = \mathcal{N}(X|\mu, \sigma^2)$
- Whenever the **posterior is in the same family as the prior**

  $p_\mu(\mu)$ is a **conjugate prior** for the likelihood $p_X(X|\mu)$

  Posterior $p_\mu(\mu|X)$ is the **reproducing density**
- A number of likelihoods have **conjugate priors**

Likelihood	Conjugate prior
Bernoulli	Beta
Poisson	Gamma
Exponential	Gamma
Normal (known σ^2)	Gamma

- And **development of posterior is straightforward**