1 Hermite Curves

A **Hermite curve** is a curve defined by parametric cubic equations. Note that this does NOT mean that the curve itself is a portion of the graph of a cubic equation; it just means that the x and the y components of the curve are defined by cubic equations.

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$
$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_x$$

Each Hermite curve is defined by four sets of points:

- Two endpoints: (x_0, y_0) and (x_1, y_1)
- Two rate-of-change points that say how fast the curve is changing at the endpoints: (dx_0, dy_0) and (dx_1, dy_1).

The problem here is finding the coefficients a, b, c, d from this set of points.

We will derive the method here for the x components only. For simplicity, we let the function be

$$f(t) = at^3 + bt^2 + ct + d$$

When $t = 0$, $f(0) = x_0 = d$ and $f'(0) = dx_0 = c$.
When $t = 1$, $f(1) = x_1 = a + b + c + d$ and $f'(1) = dx_1 = 3a + 2b + c$. So:

$$\begin{bmatrix} x_0 \\ x_1 \\ dx_0 \\ dx_1 \end{bmatrix} = \begin{bmatrix} d \\ a+b+c+d \\ c \\ 3a+2b+c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_0 \\ x_1 \\ dx_0 \\ dy_0 \end{bmatrix}$$

where the latter matrix is the inverse of the former matrix. And that gives the coefficients of the parametric equation; so now you just have to graph the parametric, and you're done!

2 Bezier Curves

A **Bezier curve** is a curve with a certain degree. An n-degree Bezier curve is defined by $n + 1$ points:

- Two endpoints, P_0 and P_n

- $n-2$ "tugging" points, $P_1, P_2, \ldots, P_{n-1}$

The parametric equation for the Bezier curve is (boldface letter represent matrix representations):

$$\mathbf{B}_n(t) = t^n\mathbf{P}_n + \sum_{k=0}^{n-1}(1-t)^{n-k}\mathbf{P}_k$$

The first few low-degree Bezier curves look like:

$$\begin{aligned}
\text{Linear} &: \mathbf{B}_1(t) = (1-t)\mathbf{P}_0 + t\mathbf{P}_1 \\
\text{Quadratic} &: \mathbf{B}_2(t) = (1-t)^2\mathbf{P}_0 + (1-t)\mathbf{P}_1 + t^2\mathbf{P}_2 \\
\text{Cubic} &: \mathbf{B}_3(t) = (1-t)^3\mathbf{P}_0 + (1-t)^2\mathbf{P}_1 + (1-t)\mathbf{P}_2 + t^3\mathbf{P}_3
\end{aligned}$$