Artificial Intelligence — Probabilistic Modeling Worksheet

Prof. Emma Tosch

Due in class on 22 April 2022.

Name:

Collaborated With:

Graded By:

Description. This is a supplemental worksheet. Each question is worth 1 point. The first three questions are warm-ups. The remaining questions exercise your probabilistic modeling skills as part of a single task.

Instructions. You may collaborate with others. Post to Teams (*(Non-Programming) Assignments Help*) or come to student hours with questions. Answer neatly on a separate worksheet (or feel free to type up your solutions using the LaTeX file linked on the website). Please list everyone you collaborated with.

Grading. **Worksheets are due at the start of class on Friday, April 22.** Students may optionally grade another's worksheet, but only for students they did not collaborate with. The purpose of this exercise is to have students engage with alternative solutions and develop a deeper understanding of the material. *Students will not be given the solution key. You may grade collaboratively. All incorrectly marked items must include an explanation.* **Graded solutions are due in class on Wednesday, April 25.** I will release solutions to the worksheet on Friday, April 27.

The first three questions are taken from or inspired by problems in *Introduction to Probability* by Bertsekas and Tsitsiklis.

1. 70% of Vermonters ski and 60% of Vermonters mountain bike, while 50% of Vermonters do both. What is the probability that a randomly selected Vermonter neither skis nor mountain bikes?
   $P((S \cup M)^c) = 1 - P(S \cup M) = 1 - (P(S) + P(M) - P(S \cap M) = 1 - 0.7 - 0.6 + 0.5 = 0.2$

2. A six-sided die is loaded in a way such that each even face is twice as likely as each odd face. Given the parity of a face, all faces are equally likely. What is the probability that a single roll of the die is less than 4?
   Let p represent the probability of seeing an arbitrary odd face. We can solve for p by setting $1 = 3 * p + 3 * 2p$. Then $p = 1/9$ and $P(roll < 4) = P(1) + P(2) + P(3) = 1/9 + 2/9 + 1/9 = 4/9$.

3. Given a roll of two fair-sided dice, what is the probability that at least one die roll is a 6, given that the two dice land on different numbers?
   The easiest way to reason about this is over the sample space induced by the condition. There are 36-6=30 outcomes where $D_1 \neq D_2$. There are 5 ways that each of D_1 and D_2 can be 6, so the probability is $10/30 = 1/3$.

The next set of questions all use the same setup and are inspired by the embaressing number of hours I've sent playing the game FTL. Suppose you are playing a space ship battle simulation where there are three types of weapons:

weapon	charge time	effect (shields up)	effect (shields down)
ion	13s	disables shields for 40s	disables system for 40s
laser	11s	disables shields for 2s	damages system, 2 hull damage
missile	10s	damages system, 2 hull damage	

Both ships can dodge volleys some percentage of time. Your dodge probability is known, but the enemy ship can dodge volleys with some unknown probability p and each ship can take some amount of hull damage, known to all parties. Each ship can repair its damage. Each system on each ship can be repaired and it takes the same amount of time to repair a system, given the ship. Repair does not start until the time step after a ship has been hit. Repair time for your ship is known, but the repair time for the enemy ship r is unknown.

All ships have one shield. Your ship has one ion weapon and one laser weapon, while the enemy ship has one missile weapon. Weapons can be fired at any point *after* their charge time. Weapon effects are instantaneous (i.e., they go into effect at the same time that they make contact). It takes one second for a volley to hit its target. Concurrent events happen in random order, distributed uniformly over the set of events. Assume all encounters start at time $t = 0$ and weapons begin charging at the next time step.

4. Describe what happens during the interval $t = [0, 15]$ if both ships shoot their weapons as soon as they are ready. Assume both ships target the other's shield system and it takes 10s for you to repair any damaged systems. You can dodge volleys 5% of the time. What is the state of your ship in terms of hull damage and system liveness at the end of the interval?
During this period, the enemy can only fire its missiles once and you will not have enough time to repair. Thus there is a 95% chance that your shields are damaged at the end of the interval and the expected damage to your hull is 1.9.

5. Under the same scenario, what is the state of the enemy ship in terms of hull damage and system liveness? *Tip: you can check your work by ensuring that the sum of the likelihoods of all possible outcomes is one.*
Since you are shooting the laser first, there will be no hull damage. Therefore, we focus on analysing the likelihood of system liveness.

You will shoot the laser weapon at $t = 12$ and the ion cannon at $t = 14$. Then there are four possible outcomes for the weapons hitting:

(a) Both volleys hit: laser at $t = 13$ turning off their shields for $t = [13, 14]$ and ion arriving at $t = 15$. Since the shields will be returning at this time, there is a 50% chance that the ion weapon hits first. However, since we are targeting the shields, no matter what they will be disabled. Thus this event happens with probability $(1-p)^2$.

(b) The first volley hits, but the second does not. The shields will be fully recovered. This scenario happens with probability $(1 - p)p$.

(c) The first volley misses, but the second hits. Thus the shields will be disabled with probability $p(1 - p)$.

(d) Both volleys miss with probability p^2.

Thus there are two possible outcomes for the enemy ship: the shields are disabled with probability $1 - p$ and live with probability p.

6. Assume the same setup as the previous question, but now assume you are targeting the enemy's weapon system. What is the state of the enemy ship in terms of hull damage and system liveness?
Same analysis as the above, except that in scenario (a), there are two possible outcomes: the ion weapon hitting first and disabling the weapons system or the ion weapon hitting

second and disabling the shields. Thus, there are three possible outcomes for the enemy ship:

(a) Only weapons disabled with probability $0.5(1-p)^2$.
(Enemy fails to dodge twice: $(1-p)^2$; ion hits weapons *before* shields are up: 0.5.)

(b) Only shields disabled with probability $0.5(1-p)^2 + p(1-p) = 0.5(1-p^2)$.
(Happens in two cases: (1) when the enemy fails to dodge twice, but the shields return before the ion weapon hits, causing the ion weapon to hit shields instead (first term) and (2) when the enemy dodges first, but does not dodge in the second volley (second term).)

(c) Fully live system with probability $p^2 + (1-p)p = p$.
(Happens when the enemy dodges twice (first term) or when the enemy is hit the first time, but dodges the second time (second term).)

7. Now let's consider the interval $t = [0, 25]$. Assume the enemy fires missiles as soon as they can on autofire and they target your shields every time, and that you can dodge 5% of volleys. What is the state of your ship in terms of hull damage and system liveness at the end of the interval?

Assuming you are *not* targeting enemy weapons. During this period, the enemy can fire its missiles twice. There are four possible scenarios:

(a) Hit both times. Shields will be down with probability 0.95^2. Expected hull damage is $4 * (0.95)^2$.

(b) The first volley hits, but the second misses. This gives you time to repair and thus shields will be live with probability 0.95*0.05. Expected hull damage is $2(0.95 * 0.05)$.

(c) The first volley misses, but the second hits. You do not have time to repair and thus the shields will be disabled with probability 0.05*0.95. Expected hull damage is $2(0.05 * 0.95)$.

(d) Both volleys miss. Then the shields will be live with probability $(0.05)^2$. and the expected hull damage is 0.

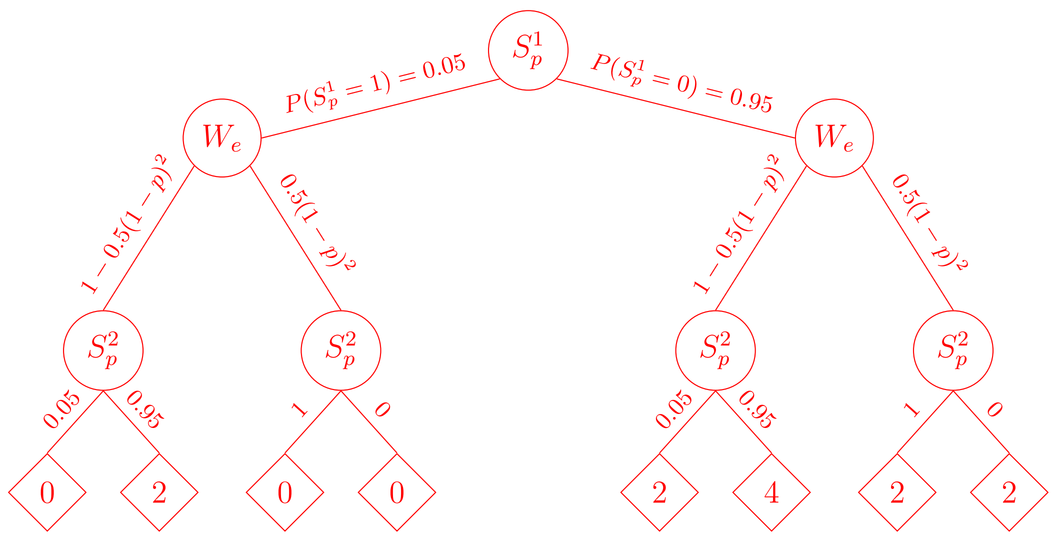
Thus, your shields are down with probability 0.95 and up with probability 0.05. Your expected hull damage is 3.8. Note that in the above analysis, we treated the event as a a 2-tuple, but since the total damage is the sum of all accrued damage, and since each volley hitting is an iid random variable, we could define the total hull damage to be the sum of all independent trials and leverage the linearity of expectation to derive the total expected hull damage.

Assuming you *are* targeting enemy weapons. During this period, the enemy can fire its missiles twice. Let S_p^i denote your shields being down after the i^{th} enemy volley and D_p^i

denote whether you dodged that volley. Then

$$S_p^i = \begin{cases} 0, D_p^i = 0 \\ 1, D_p^i = 1 \end{cases}$$

and we are given $P(D_p^i = 1) = 0.05$. Let W_e denote whether the enemy's weapons system is live. We know from Question 6 that $P(W_e = 0 \mid \text{this scenario}) = 0.5(1-p)^2$ — we can use this probability directly because the enemy ship's weapons being down is independent of our shields being down after the first volley. Conversely, our shields being down after the second volley is *not* independent of the enemy's weapons being down. Finally, we know it takes 10s to repair, so shields being up or down after the second volley only depends on our dodge probability. Thus we have:



Let H_p denote the player's dull damage. Noting that we can eliminate any paths where either the hull damage is 0 or there occurs an event with probability 0,

$$\begin{aligned} E[H_p] &= \sum_{h \in \{0,2,4\}} h * P(S_p^2 \mid W_e, S_p^1) P(W_e \mid S_p^1) P(S_p^1) \\ &= \quad 2 * \big(0.95 * (1 - 0.5(1-p)^2) * 0.05 \\ &\qquad\quad + 0.05 * (1 - 0.5(1-p)^2) * 0.95 \\ &\qquad\quad + 1 * (0.5(1-p)^2) * 0.95\big) \\ &\quad + 4 * 0.95 * (1 - 0.5(1-p)^2) * 0.95 \\ &= -0.95p^2 + 1.9p + 2.85 \end{aligned}$$

We can similarly add up the probabilities that our shields are down, or we could just note that our shields being down only depends on our dodge and the enemy's weapons being up, but not our first dodge, due to our repair time, i.e.,

$$\begin{aligned}
P(S_p^2 = 0) &= \sum_{w \in \{0,1\}} \sum_{s^1 \in \{0,1\}} P(S_p^2 = 0 \mid W_e = w, S_p^1 = s^1) P(W_e = w \mid S_p^1 = s^1) P(S_p^1 = s^1) \\
&= \sum_{w \in \{0,1\}} \sum_{s^1 \in \{0,1\}} P(S_p^2 = 0 \mid W_e = w) P(W_e = w) P(S_p^1 = s^1) \\
&= \sum_{w \in \{0,1\}} P(S_p^2 = 0 \mid W_e = w) P(W_e = w) \sum_{s^1 \in \{0,1\}} P(S_p^1 = s^1) \\
&= \sum_{w \in \{0,1\}} P(S_p^2 = 0 \mid W_e = w) P(W_e = w) \\
&= 0.95 * (1 - 0.5(1-p)^2) \\
&= 0.475 + 0.95p - 0.475p^2
\end{aligned}$$

8. Again let's consider the interval $t = [0, 25]$, but this time assume you shoot your laser weapon immediately after shooting your ion weapon and you target the enemy's weapon system. Assume the enemy continues to target your shields and that it takes more time than the interval for the enemy to repair their systems. What is the state of the enemy ship in terms of hull damage and system liveness at the end of the interval?

   We shoot the ion weapon at $t = 14$ and the laser at $t = 15$. There is not enough time for either weapon to charge before the end of the time interval. There are no concurrent events, so we only need to reason about the success of the volleys:

   (a) Both hit with probability $(1-p)^2$. Both the shields and the weapons will be disabled. Expected damage is $2(1-p)^2$.
   (b) Only the ion weapon hits. Only the shields are disabled with probability $(1-p)p$.
   (c) Only the laser hits. The shields will have recovered by the end of the interval. No damage and all systems live with probability $p(1-p)$.
   (d) Both miss. All systems live with probability p^2.

   Thus the expected hull damage is $2(1-p)^2$, the weapons systems will be disabled with probability $(1-p)^2$, the shields will be disabled with probability $1 - p^2$, and all systems will be live with probability p.

   Moving forward, we will always play the opening strategy of shooting the ion first (when it is ready) and then shooting the laser at the next time step.

9. Now assume the same scenario as above, but this time it's possible for the enemy to repair their hit system. Let's model your belief about the enemy's repair capabilities.

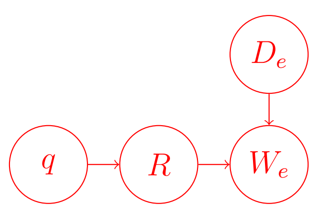
We can model the length of time R it takes to repair a system as a geometric distribution $Geom(R = r) = (1 - q)^r q$. The parameter q corresponds to the probability of "success," which in this case means "finishing repairs." We can start by saying our best guess is that the enemy will take about as much time to repair their ship as it takes for us to repair ours. The expected value of this distribution is $\frac{1-q}{q}$. If we set the expected value to the length of time it takes us to repair, what should parameter q be set to?
E[R] = 10 = (1-q)/q. Then q = 1/11.

10. Note that q is itself a quantity we don't know. We can additionally model our uncertainty over q using the Beta distribution, which has two parameters α and β ($q \sim \text{Beta}(\alpha, \beta)$). Recall that distributions can often be described in terms of mean, median, and mode, which for the Beta are:

mean	median	mode
$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha-1/3}{\alpha+\beta-2/3}$	$\frac{\alpha-1}{\alpha+\beta-2}$

Pick one of these quantities to set q to and solve for α and β. Justify your choice of quantity.
The mode corresponds to the highest probability probability and the distribution represents our belief over the parameter. Therefore if we are saying that we believe q=1/11 is the most likely value, then we should set it equal to the mode of the Beta, so $\alpha = 2$ and $\beta = 11$.

11. Assume the shields are down. Let W_e be a Bernoulli (i.e., 0/1) random variable denoting whether the enemy's weapon system is live. Let D_e be a Bernoulli random variable corresponding to whether or not the enemy dodges. Draw a Bayes net for the variables W_e, R, q, and D_e. You do not need to account for interleaving/concurrent events at this time.



12. Using the Bayes Net above, write out an expression using the conditional probabilities encoded in the Bayes Net for $P(\neg W_e)$, given that the shields are down. *Hint: one of these sums will be an integral.*

$$P(\neg W_e) = 1 - \sum_{r=0}^{\infty} \sum_{d \in \{0,1\}} P(W_e \mid R = r, D_e = d) P(D_e = d) \int_0^1 dq \; P(R = r \mid q) P(q)$$

13. The Beta distribution for $q \sim Beta(\alpha, \beta)$ is defined to be $P(q) = q^{\alpha-1}(1-q)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$, where Γ is known as the gamma function, which is an extension of factorial to the complex numbers. One of its many definitions is $\Gamma(x) = (x-1)!$ when $x \in \mathbb{Z}$. Write out an expression for $P(q)$, simplified as much as you can.
$P(q) = q^{2-1}(1-q)^{11-1}\frac{\Gamma(13)}{\Gamma(2)\Gamma(11)} = q(1-q)^{10}\frac{12!}{1!10!} = q(1-q)^{10}(12*11) = 131q(1-q)^{10}$

14. Recall the definition of the geometric distribution in question 9. Write out an expression for $P(R = r)$ and simplify as much as you can. *Hint: you will need to integrate over q and will recognize this expression from an earlier question. You may use Wolfram Alpha to get a closed form.*

$$P(R = r) = \int_0^1 dq \; P(R = r \mid q)P(q) = 131q^2(1-q)^{10+r} = \frac{262\Gamma(r+11)}{\Gamma(r+14)}$$

15. Compare $P(R = 10)$ with $P(R = 10 \mid q = 1/11)$. Are they the same or different? Explain the result you see.
$P(R = 10) \approx 0.02, \quad P(R = 10 \mid q = 1/11) = 0.03$
The former is a marginal probability, while the latter is the conditional probability. The conditional probability distribution limits the sample space. Put another way, the marginal probability is incorporating other hypothetical values of q.

16. Recall that in Question 9, we chose our initial q so that $E[R] = 10$. Compute $E[R]$ with the distribution that models uncertainty over q. Are they the same or different? Explain the result you see. *You may use Wolfram Alpha.*
$E[R] \approx 11$. This differs because the earlier quantity we were computing was actually $E[R \mid q = 1/11]$.

17. The state of the weapons system W at time $t = 25$ depends on whether the ship was actually hit (D) and whether repairs have finished in time R. Fill out a conditional probability table for W_e at time $t = 25$ for a subset of $R = \{8, 9, 10\}$. You do not need to reason about concurrent/interleaving events. Is this table complete? Why or why not? Explain your reasoning about the values you've derived and write an expression for $P(W_e \mid D, R)$.

W_e	$P(W_e)$	D	R
1	1	0	8
0	0	0	9
0	0	0	10
1	1	1	*

First line corresponds to "probability that the weapons system is damaged, given that the enemy ship did not dodge and it took 8 seconds to repair." The laser arrives at $t = 16$ and repairs begin at $t = 17$ and finish at $t = 25$. There is no randomness inherent in W_e, so given the conditioning set, behavior is deterministic at a cutoff value of R and for values of D:

$$P(W_e \mid D, R) = \begin{cases} 1, D = 1 \vee R \leq 8, \\ 0, o/w \end{cases}$$

18. You now have all the pieces to solve for the expression you derived in question 12. Solve for $P(\neg W_e)$ here.
Recall that graphical model of Question 11 describes the factorization of joint probability distribution $P(W_e, D_2, R, q)$. We can find $P(W_e)$ by marginalizing over R, D, and q:

$$P(W_e) = \sum_{r=0}^{\infty} \sum_{d \in \{0,1\}} P(W_e \mid R = r, D = d) P(D = d) \int_0^1 dq \; P(R = r \mid q) P(q)$$

We then note that $\int_0^1 dq \; P(R = r \mid q)P(q) = P(R)$, which we computed in Question 14 ($P(R) = \frac{262\Gamma(r+11)}{\Gamma(r+14}$):

$$P(W_e) = \sum_{r=0}^{\infty} \sum_{d \in \{0,1\}} P(W_e \mid R = r, D = d) P(D = d) \frac{262\Gamma(r + 11)}{\Gamma(r + 14)}$$

Now we can use the piecewise function of Question 12 to break the infinite sum over r into two summands:

$$\begin{aligned} P(W_e) = &\sum_{r=0}^{8} \left(\sum_{d \in \{0,1\}} P(W_e \mid R = r, D = d) P(D = d) \frac{262\Gamma(r + 11)}{\Gamma(r + 14)} \right) \\ &+ \sum_{r=9}^{\infty} \left(\sum_{d \in \{0,1\}} P(W_e \mid R = r, D = d) P(D = d) \frac{262\Gamma(r + 11)}{\Gamma(r + 14)} \right) \end{aligned}$$

We know from Question 17 that $P(W_e \mid R, D) = 0$ when $D = 0$ and $R > 8$, so we no longer need to sum over values of d in the second summand and can simply set $D = 1$

there:

$$P(W_e) = \sum_{r=0}^{8}\left(\sum_{d\in\{0,1\}} P(W_e \mid R=r, D=d)P(D=d)\frac{262\Gamma(r+11)}{\Gamma(r+14)}\right) + \sum_{r=9}^{\infty}\left(P(W_e \mid R=r, D=1)P(D=1)\frac{262\Gamma(r+11)}{\Gamma(r+14)}\right)$$

Now we can separate the sum over dodging and factor out our expression for $P(R)$:

$$P(W_e) = \sum_{r=0}^{8}\left(\frac{262\Gamma(r+11)}{\Gamma(r+14)}\left(P(W_e \mid R=r, D=0)P(D=0) + P(W_e \mid R=r, D=1)P(D=1)\right)\right) + \sum_{r=9}^{\infty}\left(P(W_e \mid R=r, D=1)P(D=1)\frac{262\Gamma(r+11)}{\Gamma(r+14)}\right)$$

We can now substitute terms using $P(D=1)=p$ and the terms from Question 17:

$$P(W_e) = \sum_{r=0}^{8}\left(\frac{262\Gamma(r+11)}{\Gamma(r+14)}\left(1*(1-p)+1*p\right)\right) + \sum_{r=9}^{\infty}\left((1*p)\frac{262\Gamma(r+11)}{\Gamma(r+14)}\right)$$

We can simplify this to:

$$P(W_e) = \sum_{r=0}^{8}\left(\frac{262\Gamma(r+11)}{\Gamma(r+14)}\right) + p\sum_{r=9}^{\infty}\left(\frac{262\Gamma(r+11)}{\Gamma(r+14)}\right)$$

Plugging into Wolfram Alpha we obtain:

$$P(W_e) = \frac{262}{385} + \frac{131}{420}p$$

or $0.68 + 0.31p$. And thus $P(\neg W_e) = \frac{123}{385} - \frac{131}{420}p \approx 0.32 - 0.31p$.

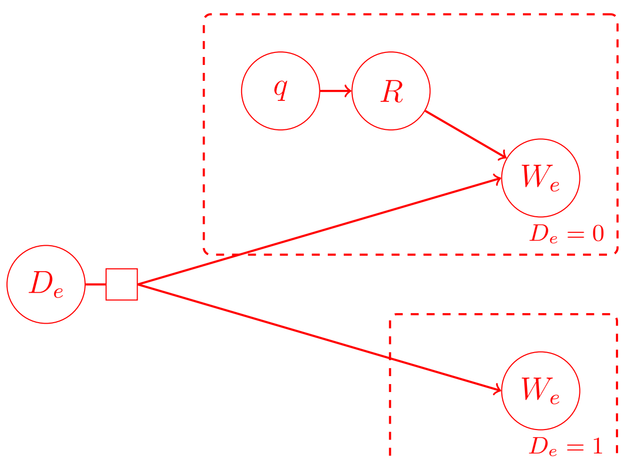
Sanity Check. We can check our work by solving this in another way. This time, instead of separating out our sum by repair time first, we will separate it by dodge:

$$P(W_e) = \sum_{r=0}^{\infty} P(W_e \mid R=r, D=0)P(D=0)\frac{262\Gamma(r+11)}{\Gamma(r+14)} + \sum_{r=0}^{\infty} P(W_e \mid R=r, D=1)P(D=1)\frac{262\Gamma(r+11)}{\Gamma(r+14)}$$

Now we notice something odd: the second term is the case where we are conditioning on dodging. If we condition on the enemy dodging, we definitely *don't* hit their ship, which means they would have nothing to repair. This would lead us conclude that $P(W_e \mid D = 1, R) = P(W_e \mid D = 1)$ (this is captured by the disjunction in the conditional probability table of Question 17). We break apart the sum capturing the enemy not dodging to get and rewrite to get:

$$\begin{aligned}
P(W_e) = \quad & \sum_{r=0}^{8} \underbrace{P(W_e \mid R = r, D = 0)}_{1} \underbrace{P(D = 0)}_{1-p} \frac{262\Gamma(r + 11)}{\Gamma(r + 14)} \\
& + \sum_{r=9}^{\infty} \underbrace{P(W_e \mid R = r, D = 1)}_{0} \underbrace{P(D = 1)}_{p} \frac{262\Gamma(r + 11)}{\Gamma(r + 14)} \\
& + \underbrace{P(W_e \mid D = 1)}_{1} \underbrace{P(D = 1)}_{p} \underbrace{\sum_{r=0}^{\infty} \frac{262\Gamma(r + 11)}{\Gamma(r + 14)}}_{1} \\
= \quad & (1 - p)\frac{262}{385} + 0 + p \\
= \quad & \frac{262}{385} - \frac{123}{385}p \\
\approx \quad & 0.68 + 0.32p
\end{aligned}$$

Note that these numbers are *slightly* different. What's going on here? In our sanity check, we rewrote $P(W_e \mid D = 1, R)$ as $P(W_e \mid D = 1)$. This leveraged our domain knowledge, but is *not* an assertion that is supported by the graphical model in Question 11. Recall that the *lack* of edges in a Bayes Net is an independence assertion, but the presence is *not*. Also recall that a variable is *not* independent of another if, for at least one value, they are correlated. Bayes Nets cannot capture what's known as *context-sensitive independence* (CSI), which is what we are seeing here. One way to capture CSI is with *gates*:



That said, since the difference is quite small, we will just use the first formulation moving forward. Some of the upcoming scenarios will implicitly leverage CSI, so you may notice discrepancies between these solutions and your own; they should have minimal effect on the end result.

19. Assuming you target the enemy's weapons in your first volley of the laser, what is the expected hull damage to your ship after the enemy has had the chance to fire twice? *Note that we are switching form reasoning about explicit time steps to reasoning over time in a more coarse-grained way, over events. You may still want to sketch out explicit time steps to help understand the problem.*
Expected damage from the first hit is 1.9. They will only be able to do damage in the second hit if the weapons are live, which depends on whether the shields are up. Note that $P(\neg W_e)$ we computed in the previous question assumed shields are down, thus this probability is actually $P(\neg W_e \mid \neg S_e)$.

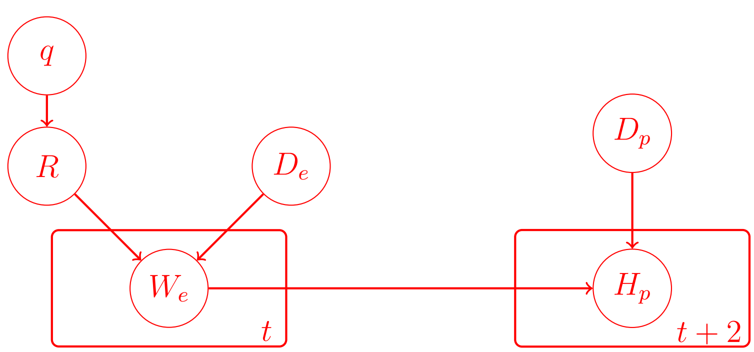
The conditional probability table we computed in Question 17 was calculated for $t = 25$. Since this window includes the enemy being able to fire twice, we can reuse $P(W_e)$ (although later we will re-define what we mean by "after" to generalize to questions about events). Thus, the probability that their weapons are still live for the second hit is the probability that $P(W_e \mid \neg S_e)P(\neg S_e) = (0.68 + 0.31p)(1 - p)$. Thus the expected damage after the second volley is $1.9 + 2 * 0.95 * (0.68 + 0.31p)(1 - p) = 1.9 + 1.9(0.68 + 0.31p)(1 - p)$.

20. Now assume that you target the enemy's piloting in your first volley. When piloting is hit, ships lose dodge capabilities until the system is repaired. What is the expected hull damage to your ship after the enemy has had a chance to fire twice? For what value of p, if any, would you trade off targeting weapons for piloting?
Since we targeted piloting in our first hit, $P(\neg W_e) = 0$, thus the expected damage to the hull is 2*1.9 = 3.8. Given no other information, you should always target weapons.

21. Now compute your expected hull damage when you have had the chance to fire the laser twice, both times at the weapons system, and the enemy has had the chance to hit you three times. Assume that the enemy is targeting your shields.

$$E[\text{damage on first hit}] = 2 * 0.95 = 1.9$$

The enemy's second hit happens have you have had the chance to fire the ion weapon and the laser once.



Recall that the value we computed for $P(W_e)$ implicitly conditioned on $S_e = 0$. We now need to explicitly factor this into our reasoning:

$$E[\text{damage on second hit}] = \sum_{D_p}\sum_{W_e}\sum_{S_e} 2 * P(H_p \mid D_p, W_e)P(D_p)P(W_e \mid S_e)P(S_e)$$

As alluded to in Question 19, we will now be more precise about the interval for repair time. Recall that in Question 18 we computed $P(W_e)$ at $t = 25$, after we had had the chance to fire the lasers at the weapons system twice. Since in this scenario we have also had the chance to fire at the weapons systems twice, the only difference in this computation is the repair interval (because we truncated at $t = 25$, we used 8s as our cutoff for repair time). We will use the interval of 12s to bound whether the weapons have been repaired "in time:"

$$\begin{aligned}
P(W_e \mid \neg S_e) &= \sum_{r=0}^{12} \frac{262\Gamma(r+11)}{\Gamma(r+14)} + p\sum_{r=13}^{\infty} \frac{262\Gamma(r+11)}{\Gamma(r+14)} \\
&= \frac{1703}{2200} + p\frac{131}{600} \\
&\approx 0.77 + 0.23p \quad \textit{Note: we rounded 131/600 up to ensure a proper distribution}
\end{aligned}$$

Since our hull damage is not 0 only when the enemy actually fires their weapons and you do not dodge, we only consider the case where $W_e = 1$ and $D_p = 0$:

$$\begin{aligned}
& 2 * \underbrace{P(H_p \mid D_p = 0, W_e = 1)}_{1}\underbrace{P(D_p = 0)}_{0.95}\underbrace{P(W_e = 1 \mid S_e = 0)}_{0.77+0.23p}\underbrace{P(S_e = 0)}_{1-p} \\
+ \quad & 2 * \underbrace{P(H_p \mid D_p = 0, W_e = 1)}_{1}\underbrace{P(D_p = 0)}_{0.95}\underbrace{P(W_e = 1 \mid S_e = 1)}_{1}\underbrace{P(S_e = 1)}_{p} \\
= \quad & 2 * 0.95(0.77 + 0.23p)(1 - p) + 2 * 0.95p \\
= \quad & -0.437p^2 + 0.874p + 1.463
\end{aligned}$$

Computing expected damage for the third hit depends on how you modeled the problem:

Is system damage is independent of charge time? Assuming independence, the enemy can fire weapons as soon as the system is live (either during the same time step or in the next one, depending on how you modeled it). If you did not assume independence, your timing calculation will have an offset of the amount of time to recharge the weapons. This means that you may have had more than two opportunities to fire your lasers. This leads to the next modeling assumption.

What constitutes being within the specified window? Your setup will also depends on how you interpreted the window "enemy has had the chance to fire three times" and "you have had the chance to fire twice." For example, we know that the enemy can fire their non-damaged missiles every 11s and you can fire your lasers every 12s. If it takes more than 12s for the enemy to repair their system, then you will have had an additional chance to fire your lasers. Are you interpreting "chance to fire twice" as a minimum? If so, you will need to account for hitting their weapons systems before they have had a chance to repair, effectively resetting their repair time.

Can we more or less treat each hit independently? Consider the interval where you've fired the lasers twice. If you don't hit the first time, reasoning about the enemy's systems state is unchanged from whether you've fired at all. If you do hit the first time *and* you hit the second time, then the enemy's systems state is again independent of your first volley. The only case where enemy system state matters after your first volley is when you hit in the first volley and miss in the second. Do we even need to reason about this case at all? Consider the definition of $P(R = r)$ from Question 11:

$$P(R = r) = \int_0^1 dq\ P(R = r \mid q)P(q) = 131q^2(1-q)^{10+r} = \frac{262\Gamma(r+11)}{\Gamma(r+14)}$$

We can fire every 12s, so we should see what the probability of the enemy repairing their system within that 12s is:

$$\sum_{r=0}^{12} \frac{262\Gamma(r+11)}{\Gamma(r+14)} = \frac{1703}{2200} \approx 0.77$$

Let's consider the following possible outcomes:

(a) We missed the ion. Then weapons are up and the expected damage from this hit is 1.9. This situation depends solely on their dodge probability p.

(b) The ion weapon hit but our first laser missed. This may appear to be the same scenario as the expected damage on second hit, but recall that the quantity we computed there included reasoning over shields being up or down, which we have already accounted for in outcome 1. We break down our reasoning into the case where there dodge and the case where they do not dodge:

   i. The enemy dodges our second laser volley. Let t_i represent the player's i^{th} event. Then our expected hull damage is:

$$\begin{aligned} 2* &\underbrace{P(S_e = 0 \mid D_{e_{t_1}} = 0)}_{1}\underbrace{P(D_{e_{t_1}} = 0)}_{1-p} \\ &\times \underbrace{P(W_{e_{t_2}} = 1 \mid S_e = 0, D_{e_{t_2}} = 1)}_{1}\underbrace{P(D_{e_{t_2}} = 1)}_{p} \\ &\times \underbrace{P(W_{e_{t_3}} = 1 \mid S_e = 0, W_{e_{t_2}} = 1, D_{e_{t_3}} = 1)}_{1}\underbrace{P(D_{e_{t_3}} = 1)}_{p} \\ &\times \underbrace{P(H_p \mid W_{e_{t_3}} = 1, D_p = 0)}_{1}\underbrace{P(D_p = 0)}_{0.95} \end{aligned}$$

Note that we do not need to compute the case where we dodge, since the hull damage is 0. Thus, the expected hull damage is $1.9(1-p)p^2$.

ii. The enemy does not dodge our second laser. Now we need to consider repair time. We are going to use a cutoff window of the period before our next volley, 12s, i.e. weapons will only be live if they can be repaired within 12s.

$$\begin{aligned}
&2 * \underbrace{P(S_e = 0 \mid D_{e_{t_1}} = 0)}_{1} \underbrace{P(D_{e_{t_1}} = 0)}_{1-p} \\
&\quad \times \underbrace{P(W_{e_{t_2}} = 1 \mid S_e = 0, D_{e_{t_2}} = 1)}_{1} \underbrace{P(D_{e_{t_2}} = 1)}_{p} \\
&\quad \times \underbrace{P(W_{e_{t_3}} = 1 \mid S_e = 0, W_{e_{t_2}} = 1, D_{e_{t_3}} = 0, R \leq 12)}_{1} \underbrace{P(D_{e_{t_3}} = 0)}_{1-p} \underbrace{\sum_{r=0}^{12} P(R = r)}_{.77} \\
&\quad \times \underbrace{P(H_p \mid W_{e_{t_3}} = 1, D_p = 0)}_{1} \underbrace{P(D_p = 0)}_{0.95}
\end{aligned}$$

Thus the expected damage in this scenario is $1.46p(1-p)^2$

(c) The first volleys of both the ion and the laser hit. If our third volley hits, then the expected damage is 0, so we only need to reason about the case where the the third volley does not hit. Since the enemy is still repairing their system from the previous volley, we consider two periods of our weapons cycling.

$$\begin{aligned}
&2 * \underbrace{P(S_e = 0 \mid D_{e_{t_1}} = 0)}_{1} \underbrace{P(D_{e_{t_1}} = 0)}_{1-p} \\
&\quad \times \underbrace{P(W_{e_{t_2}} = 0 \mid S_e = 0, D_{e_{t_2}} = 0)}_{1} \underbrace{P(D_{e_{t_2}} = 0)}_{1-p} \\
&\quad \times \underbrace{P(W_{e_{t_3}} = 1 \mid S_e = 0, W_{e_{t_2}} = 0, D_{e_{t_3}} = 1, R > 24)}_{1} \underbrace{P(D_{e_{t_3}} = 1)}_{p} \underbrace{\sum_{r=25}^{\infty} P(R = r)}_{.10} \\
&\quad \times \underbrace{P(H_p \mid W_{e_{t_3}} = 1, D_p = 0)}_{1} \underbrace{P(D_p = 0)}_{0.95}
\end{aligned}$$

Thus the expected damage under this scenario is $0.19p(1-p)^2$ (an exceedingly small value in comparison with the other quantities; it would be fine to make an assumption that it wouldn't matter and drop the term).

$E[\text{damage on third hit}] = 1.9p + 1.9(1-p)p^2 + 1.46p(1-p)^2 + 0.19p(1-p)^2 = 3.55p - 1.4p^2 - 0.25p^3$

Thus, the expected damage after three hits is

$$\begin{aligned}
(3.55p - 1.4p^2 - 0.25p^3) + (-0.437p^2 + 0.874p + 1.463) + 1.9 \\
= -0.25p^3 - 1.837p^2 + 4.424p + 3.363
\end{aligned}$$

As a sanity check, we can plug in the extreme values for p and find that damage depends on the dodge enemy probability, ranging from 3.363 to 5.7: when $p = 1$, the enemy always dodges, so the expected damage should be 1.9*3, which is equal to 5.7!

22. Now compute your expected hull damage when both you and the enemy you have fired three times, where you first first the ion weapon, then the lasers twice. You target the piloting system in the first laser volley and the weapons system in the second laser volley. Assume that the enemy is targeting your shields.

After the enemy's first hit. Again, we cannot prevent this hit, so expected damage is 1.9.

After the enemy's second hit. Again, we have had the chance to fire once. However, we are targeting piloting, so it has no effect on the weapons system, as above. Therefore, the expected damage is 1.9.

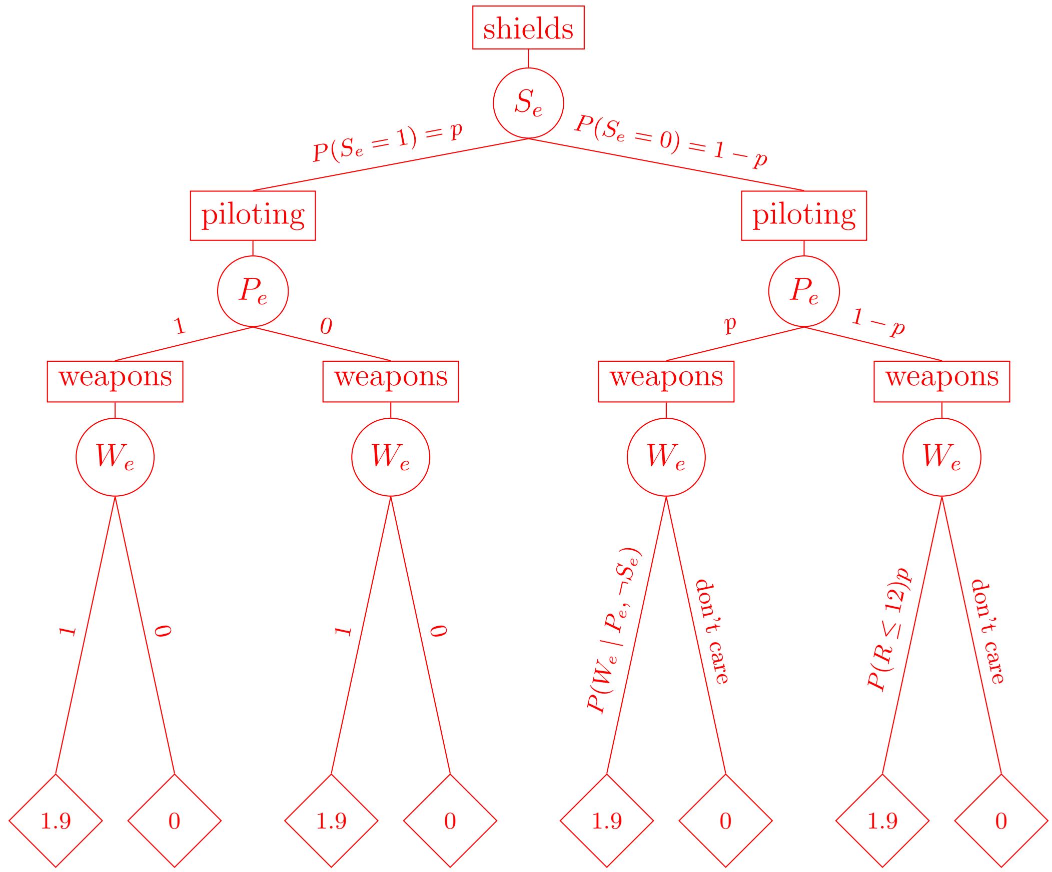
After the enemy's third hit. We have now had the chance to fire three times. If we hit the piloting system in our first volley, then we are guaranteed to hit their weapons system *if they have not repaired their piloting system in time.* If we missed the piloting system in our first volley, then we we have the usual scenario. First it might be helpful to draw out the decision tree and mark the payoff nodes for the state of the enemy ship and the reward/immediate cost to hull damage for your ship after you both have completed your three volleys. A few things to note about this tree:

(a) All decision nodes have one choice. This is because in this problem, you are given a fixed policy to follow, i.e., you do not yet have the ability to choose which room to target.

(b) Each state node is a random variable corresponding to whether the system is live (S_e for shields, P_e for piloting, W_e for weapons).

(c) The reward nodes are for the scenario where the state takes on each value emanating from the node, *even if that set of states on systems is not possible — this is what the probabilities encode!*.

(d) The reward nodes represent the expected hull damage just for this volley. Note that this question only asks for the expected hull damage, not the probability of a system being up. Therefore, we only need to model the *enemy*'s ship state, not our ship state, as we did in Question 7.



Note that we have also already computed $P(R \leq 12) = \sum_{r=0}^{12} \frac{262\Gamma(r+11)}{\Gamma(r+14)} = \frac{1703}{2200} = 0.77$. Now we can compute the expected hull damage after three volleys. We drop all terms that are equal to 0, either due to having 0 reward or because one of the events along the path to the leaf in the tree is 0:

$$
\begin{aligned}
&1.9 * \underbrace{P(W_e \mid P_e, S_e)}_{1} \underbrace{P(P_e \mid S_e)}_{1} \underbrace{P(S_e)}_{p} + \\
&1.9 * \underbrace{P(W_e \mid P_e, \neg S_e)}_{0.77+0.23p} \underbrace{P(P_e \mid \neg S_e)}_{p} \underbrace{P(\neg S_e)}_{1-p} + \\
&1.9 \underbrace{P(W_e \mid \neg P_e, \neg S_e)}_{0.77p} \underbrace{P(\neg P_e \mid \neg S_e)}_{1-p} \underbrace{P(\neg S_e)}_{1-p} \\
&= 1.9p + 1.9((0.77 + 0.23p) * p * (1-p)) + 1.9 * 0.77p * (1-p)^2 \\
&= 4.826p - 3.952p^2 + 1.026p^3
\end{aligned}
$$

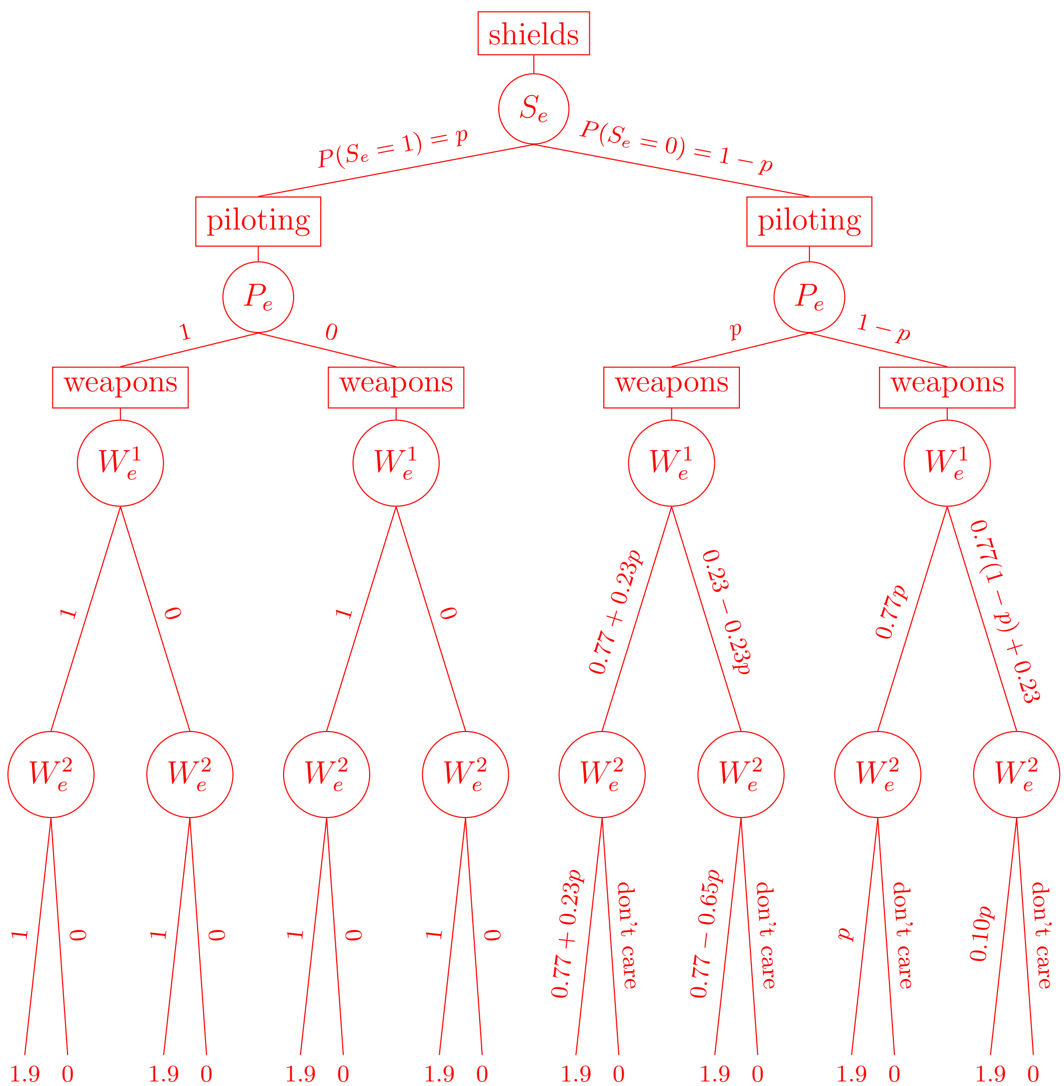
Thus, your expected hull damage after three volleys under this strategy is $3.8 + 4.826p - 3.952p^2 + 1.026p^3$. Again, if the enemy always dodges, this comes out to 5.7. Also note that if the enemy never dodges, your expected hull damage is about 3.8, which is — as expected – more than your expected hull damage when you target the enemy weapons both times.

23. Assume you play for 1000 time steps. For what value of p should you target the piloting system first?

The idea with this question was to pick an arbitrarily large number of time steps to give you many changes to hit their systems. However, the only thing that differs is that in the first volley of the lasers, we must choose between targeting the piloting and targeting the weapons. We can start by asking how we might set up the tradeoff for the quantities we have computed so far. We'd look for a value of p such that the payoff for (piloting, weapons) and (weapons, weapons) is equal:

$$3.8 + 4.826p - 3.952p^2 + 1.026p^3 = -0.25p^3 - 1.837p^2 + 4.424p + 3.363$$

We can see that the only suitable solution for this equation is when $p = 1$ — this makes sense because we have not yet had the chance to see the long term effect of hitting piloting first (see Question 20). What happens if we expand the decision tree of the previous question to another step?



Regarding $P(W_e^1 \mid \neg P_e, \neg S_e)$, or weapons being back online after piloting has been hit, can only be true when the enemy has repaired piloting in time *and then* they dodge, thus $P(W_e^1 \mid \neg P_e, \neg S_e) = 0.77p$. Conversely, weapons can only be down at this point either when (1) they have repaired piloting in time (0.77) *and* they don't dodge $(1-p)$ or when (2) they have not repaired piloting in time (0.23). Thus, $P(W_e^1 \mid \neg P_e, \neg S_e) = 0.77(1-p)+0.23$. We now need to consider three cases:

(a) $P(W_e^2 \mid \neg W_e^1, P_e, \neg S_e)$, or weapons being up after they were down and piloting was not hit. Weapons will only be up under two conditions:

   i. The enemy dodged the most recent volley (p) *and* they finished repairing their prior weapons hit (i.e., $\neg W_e^1$). "Finished repairing in time" corresponds to less than 24s to repair piloting ($P(R \leq 24) = \sum_0^{24} \frac{262\Gamma(r+11)}{\Gamma(r+14)} = \frac{3275}{3663} \approx 0.89$). However, we are conditioning weapons not being up from the previous time period, so we actually need to subtract out the probability of repairing within 12s, which is ≈ 0.12.

   ii. The enemy did not dodge, but finished repairing in time $(0.77(1-p))$.

   Thus, we add together the probability of the first condition $(0.12p)$ and the probability of the second condition $(0.77(1-p))$ to obtain $0.77 - 0.65p$.

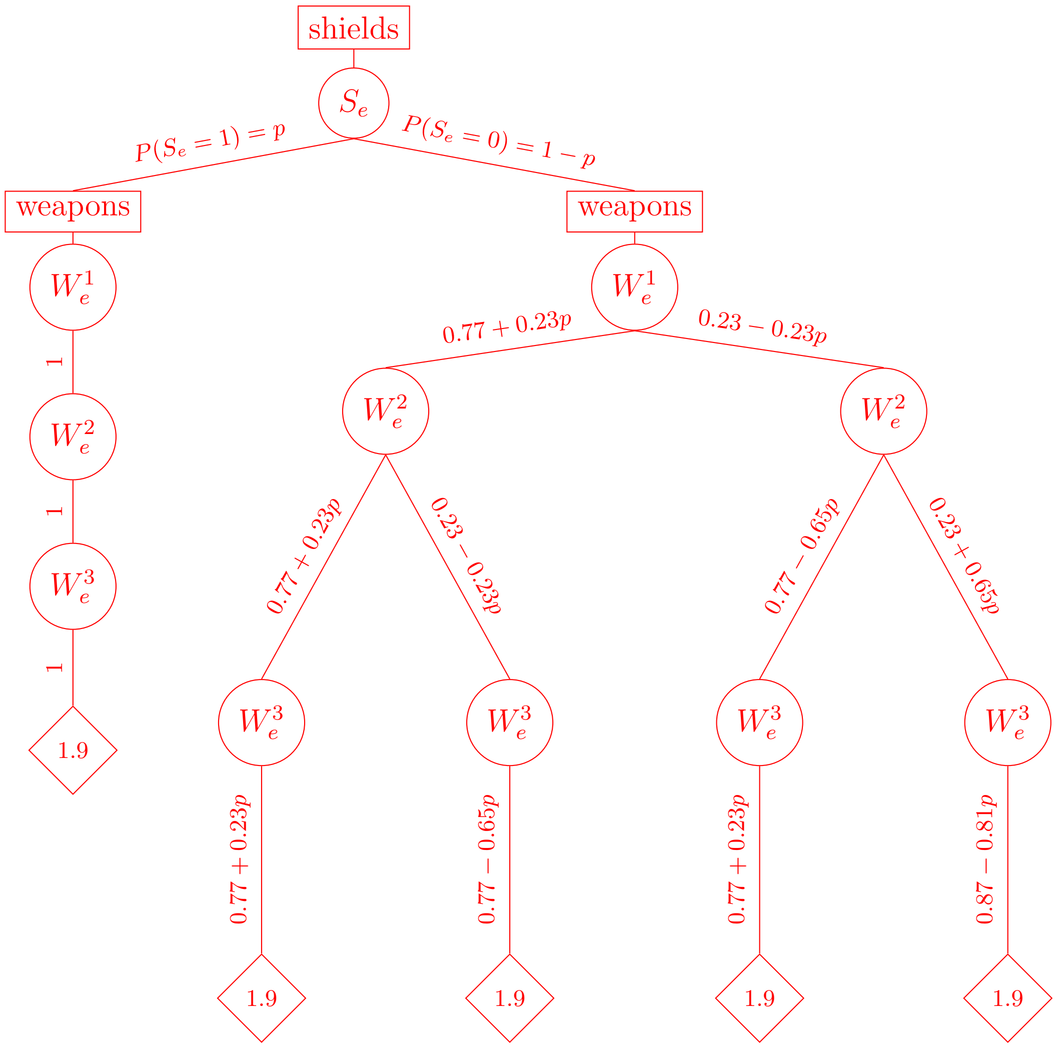
(b) $P(W_e^2 \mid W_e^1, \neg P_e, \neg S_e)$, or weapons continuing to be up, given that piloting was hit. This can only happen when piloting was repaired in time *and* the enemy dodged. Because this event conditions on weapons being up, given that piloting was hit, it is *already* conditioning on piloting being repaired within the window (i.e., weapons up after action 3 is independent of action 1, given action 2: $P(W_e^2 \mid W_e^1, \neg P_e, \neg S_e) = P(W_e^2 \mid W_e^1, \neg S_e)$). Thus this branch only depends on enemy dodge probability.

(c) $P(W_e^2 \mid \neg W_e^1, \neg P_e, \neg S_e)$, or weapons being up given they were previously down and piloting was hit. This can only happen if it takes more than 24s repair piloting ($P(R > 12) = \sum_{25}^{\infty} \frac{262\Gamma(r+11)}{\Gamma(r+14)} = \frac{131}{1332} \approx 0.10$) *and* the enemy dodges (p).

One of the things we notice about this tree is that its further expansion will have repeated structure; we have thus far ignored the interleaving of firing the ion, since the shields are down for 40s. Since we get to fire the ion at the shields in less than this interval, the probability that the shields are still down 40s after our initial firing is greater than the dodge probability (due to having additional opportunities to hit them). For the purpose of our comparison, we will assume that the above structure is repeated. We can now set up a back-of-the-envelope calculation using this assumption since it favors targeting piloting first. After firing the laser three times, first a piloting, we have an expected hull damage of $1.9 - 2.23649p + 6.17196p^2 - 5.63445p^3 + 1.69898p^4$, which we add to the expected hull damage from the previous calculation:

$$5.7 + 2.58951p + 2.21996p^2 - 4.60845p^3 + 1.69898p^4$$

Now let's consider a similar tree for only targeting weapons:



A note on how we calculated $P(W_e^3 \mid \neg W_e^2, \neg W_e^1, \neg S_e)$: either we were hit and repaired in time $(0.77(1-p)$ or we were not hit and finished repairing from a previous hit. Our model does not differentiate between the two previous hits; it just conditions on the system being down. Therefore, we need to model these events here: the case where we were hit in W_e^2 and haven't finish repairing in time $((1-p)P(R > 24) \approx (1-p)0.1)$ or and the case where we were not hit, but were still repairing from W_e^1 $(pP(R > 37) \approx 0.06p)$. If we add these events up, we get $0.87 - 0.81p$.

We can then compute our expected hull damage: $1.47305 + 0.481661p - 0.0663366p^2 + 0.395485p^3 - 0.383861p^4$, which we add to our previously computed damage:

$$4.83605 + 4.90566p - 1.90334p^2 + 0.145485p^3 - 0.383861p^4$$

and compare the two equations:

$$\begin{aligned}5.7 + 2.58951p + 2.21996p^2 - 4.60845p^3 + 1.69898p^4 \\ = 4.83605 + 4.90566p - 1.90334p^2 + 0.145485p^3 - 0.383861p^4\end{aligned}$$

Note that if we plug in $p = 1$, they are equivalent (as expected). If $p = 0$, targeting weapons still has lower expected hull damage. In fact, the difference is growing (5.7-4.8 ¿ 3.8 - 3.4). Intuitively though we would expect that at some point hitting piloting would have some kind of positive effect. We did not model shields having an effect, so that could make the difference. Repair time could make a significant difference – our repair time of 10s means that the enemy is likely to be able to repair before we can really make use of the effect of our damage. Additionally, our strategy only targeted piloting once; if we continually target piloting until we get a hit and then alternate between weapons and piloting, we might have a better outcome.

I suspect some students will run simulations. Some will try to model the problem. I'm primarily looking for engagement with the question and an indication that students can break down the problem and reason through the pieces in at least a somewhat rigorous way.

24. What is a possible strategy implied by this analysis? Express your strategy in LTL. You will need to define the propositions over which your LTL strategy holds.
There could be a variety of solutions of varying complexity, but I am generally looking for something like:

$\mathcal{U}$	{ enemy ship systems } ∪ { our weapons }
Live	predicate indicating a system is live
Ready	predicate indicating that a weapon is ready to be fired
Target	predicate indicating you have targeted a system and fired at it with a particular weapon
Is	system name

$$\begin{aligned}
r_p &\triangleq \exists r\,(Is(r, \text{“piloting”}) \wedge Live(r)) \\
r_s &\triangleq \exists r\,(Is(r, \text{“shields”} \wedge Live(r)) \\
w_l &\triangleq \exists w\,(Is(w, \text{“laser”}) \wedge Ready(w)) \\
w_i &\triangleq \exists w\,(Is(w, \text{“ion”}) \wedge Ready(w)) \\
t_{lp} &\triangleq \exists r \exists w\,(Is(r, \text{“piloting”}) \wedge Is(w, \text{“laser”}) \wedge Target(r, w)) \\
t_{lw} &\triangleq \exists r \exists w\,(Is(r, \text{“weapons”}) \wedge Is(w, \text{“laser”}) \wedge Target(r, w)) \\
t_{is} &\triangleq \exists r \exists w\,(Is(r, \text{“shields”}) \wedge Is(w, \text{“ion”}) \wedge Target(r, w))
\end{aligned}$$

If the shields are up, target them with the ion weapon when they are ready. If shields are down and piloting is up, target piloting with the laser. If shields are down and piloting is down, target the weapons.

$$G\left((r_s \wedge w_i \wedge Xt_{is}) \vee (\neg r_s \rightarrow ((r_p \wedge w_l \wedge Xt_{lp}) \vee (r_w \wedge w_l \wedge Xt_{lw})))\right)$$

Note that students may simply define propositions without expressing them in FOL.

25. Now suppose you have the ability to hack an arbitrary system on the enemy ship and the enemy ship has no defenses. However, your hacking comes at a cost – it reduces your ability to dodge. What technique or operation is hacking equivalent to and how might you use it to decide whether or not to expend resources in order to hack?
Hacking is equivalent to the `do` operator. You could analyze the expected payoff in terms of your hull damage for each possible system to hack and use that to select the room to hack.