CS 295A/395D: Artificial Intelligence

More on Markov Chains & Intro to MDPs

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The University of Vermont

Agenda

- Review Markov chains
- Important properties of Markov Chains
- Introduce Markov Decision Processes (MDPs)

Review: Markov Chains

A first-order Markov chain is defined by the model $\langle S, T, v_0 \rangle$ such that:

$S = \{s_1, \dots, s_n\}$ is the set of *n* states

T is a $n \times n$ matrix of transition probabilities such that:

$$p_{ij} = P(X_t = s_j \mid X_{t-1} = s_i)$$

$v_0 = \langle P(X_1 = s_1), P(X_1 = s_2), \dots, P(X_1 = s_n) \rangle$

Review: Markov Property

The *Markov property* for a k^th-order Markov chain is:

$$P(X_t \mid X_{t-1}, \dots, X_{t-k}, \dots, X_1) = P(X_t \mid X_{t-1}, \dots, X_{t-k})$$

i.e., X_t is independent of the sequence that produced it, given its *k* predecessors.

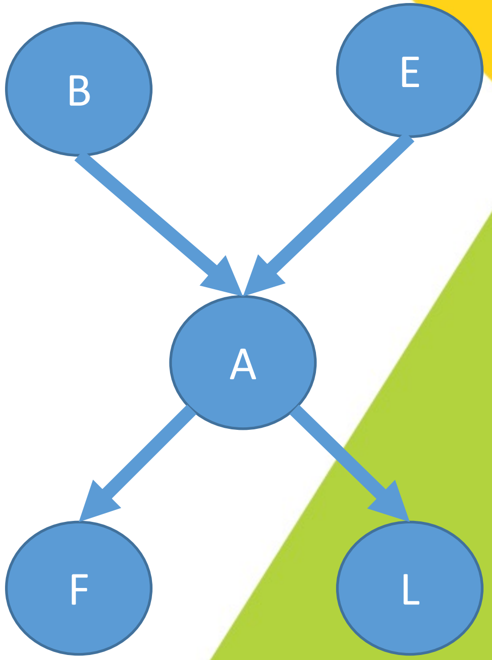
Can be thought of as:

- *Memoryless-ness* (statistics)
- Temporal locality (computer science)

Review: Graphical representation vs. Bayes Nets

Bayes Net

- Encodes independence relations
- Nodes represent conditional probability distributions



Bayes Net

Review: Graphical representation vs. Bayes Nets

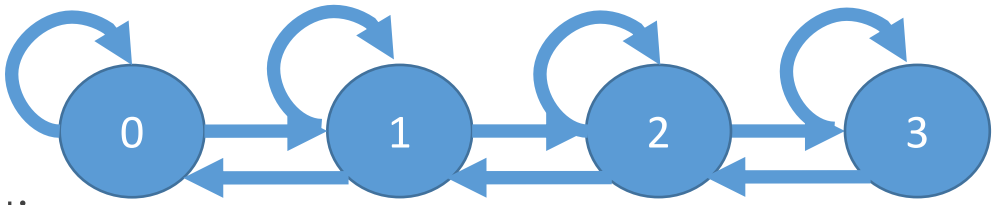
Bayes Net

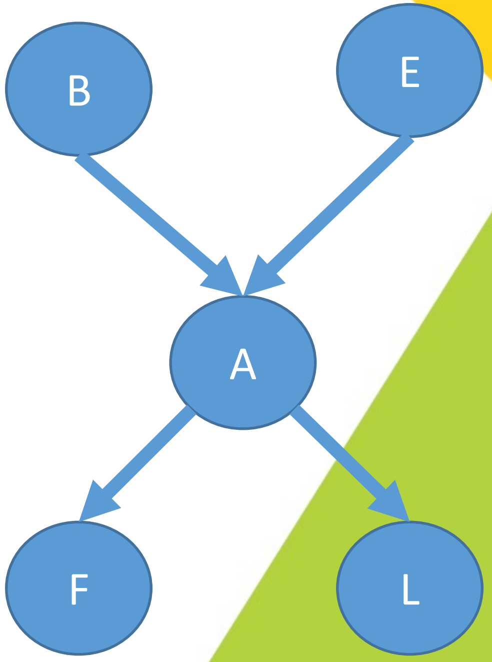
- Encodes independence relations
- Nodes represent conditional probability distributions

State transition diagram

- Encodes transition probabilities
- Nodes encode conditional probability distributions

State transition diagram





Bayes Net

Review: Graphical representation vs. Bayes Nets

Bayes Net

- Encodes independence relations
- Nodes represent conditional probability distributions

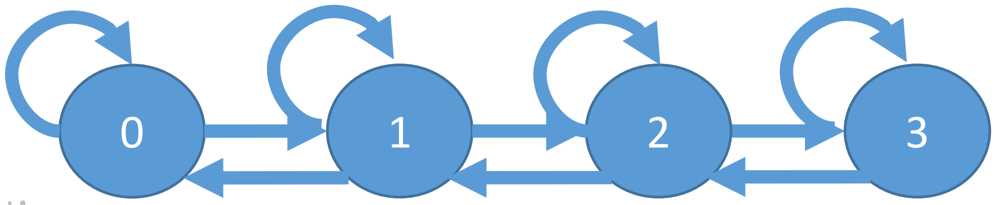
State transition diagram

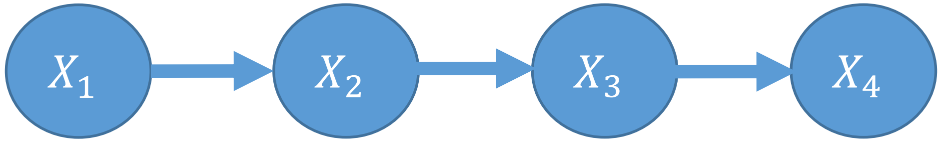
- Encodes transition probabilities
- Nodes encode conditional probability distributions

Markov Chain

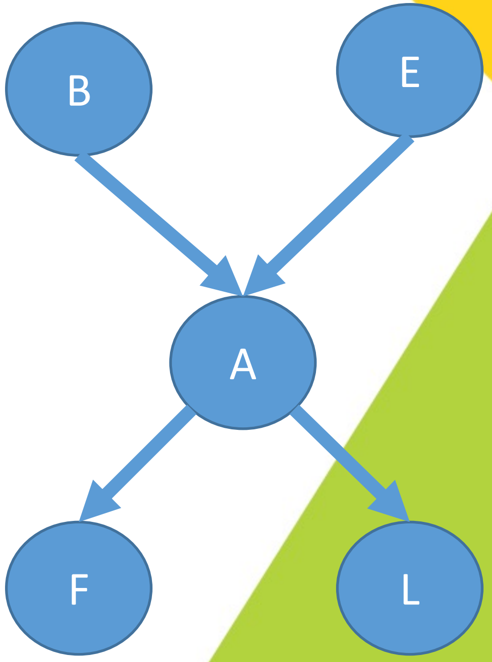
- Represents the sequence of states
- Encodes independence relations

State transition diagram





Markov chain

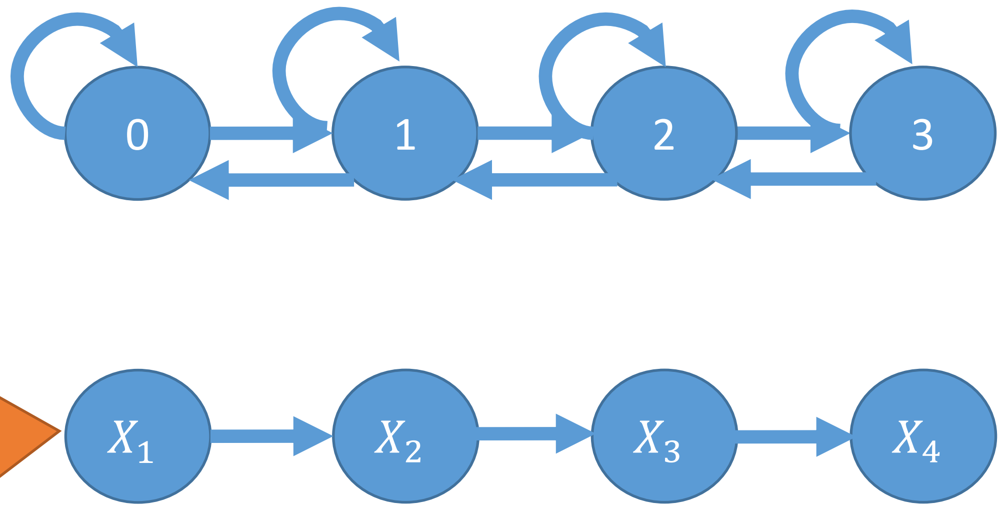


Bayes Net

k^{th}-order Markov Chain

Specifically, a first order Markov chain (i.e., k=1)

State transition diagram

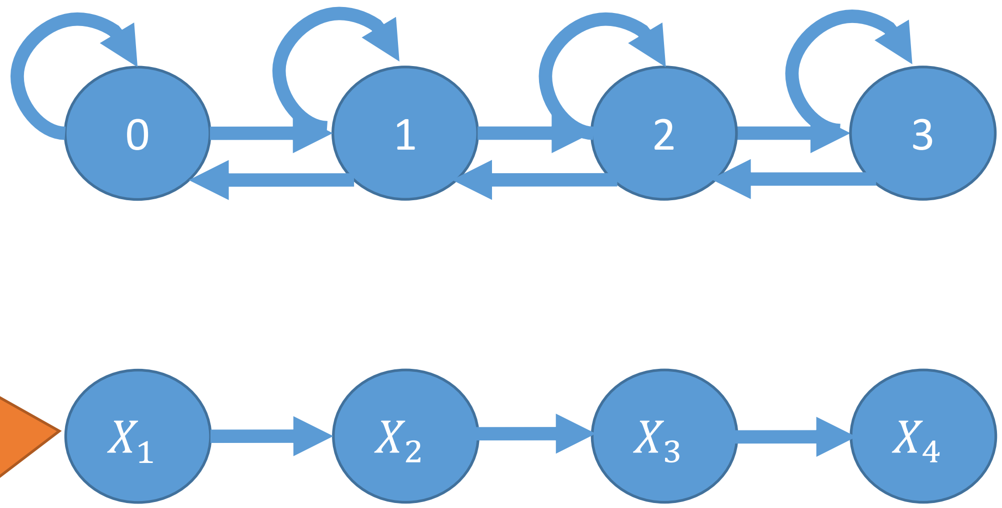


Markov chain

k^{th}-order Markov Chain

What might a 2nd order (k=2) Markov chain look like?

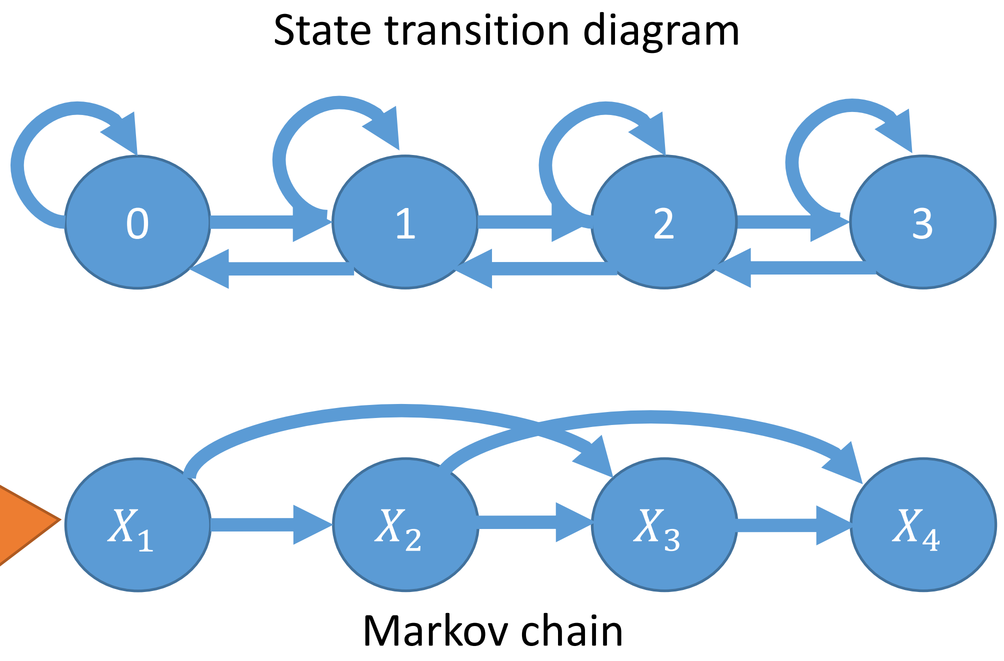
State transition diagram



Markov chain

kth-order Markov Chain

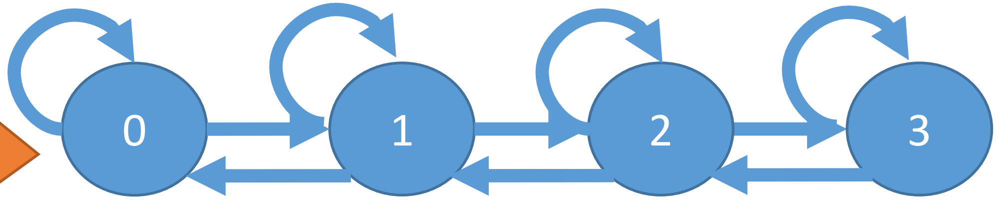
Graphically, add back edges to the chain

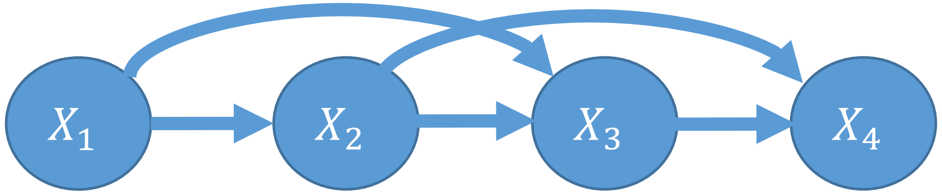


k^{th}-order Markov Chain

What about the state transition diagram?

State transition diagram

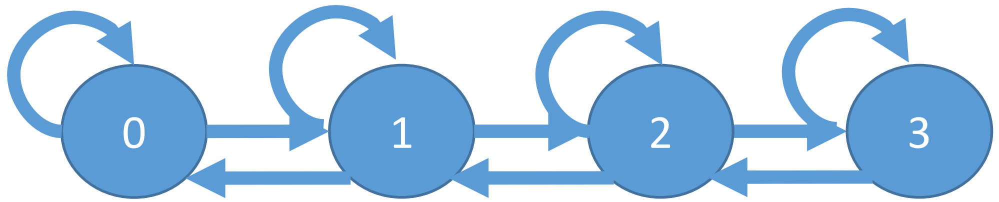




Markov chain

k^{th}-order Markov Chain

State transition diagram



What about the state transition diagram?

k^{th}-order Markov Chain

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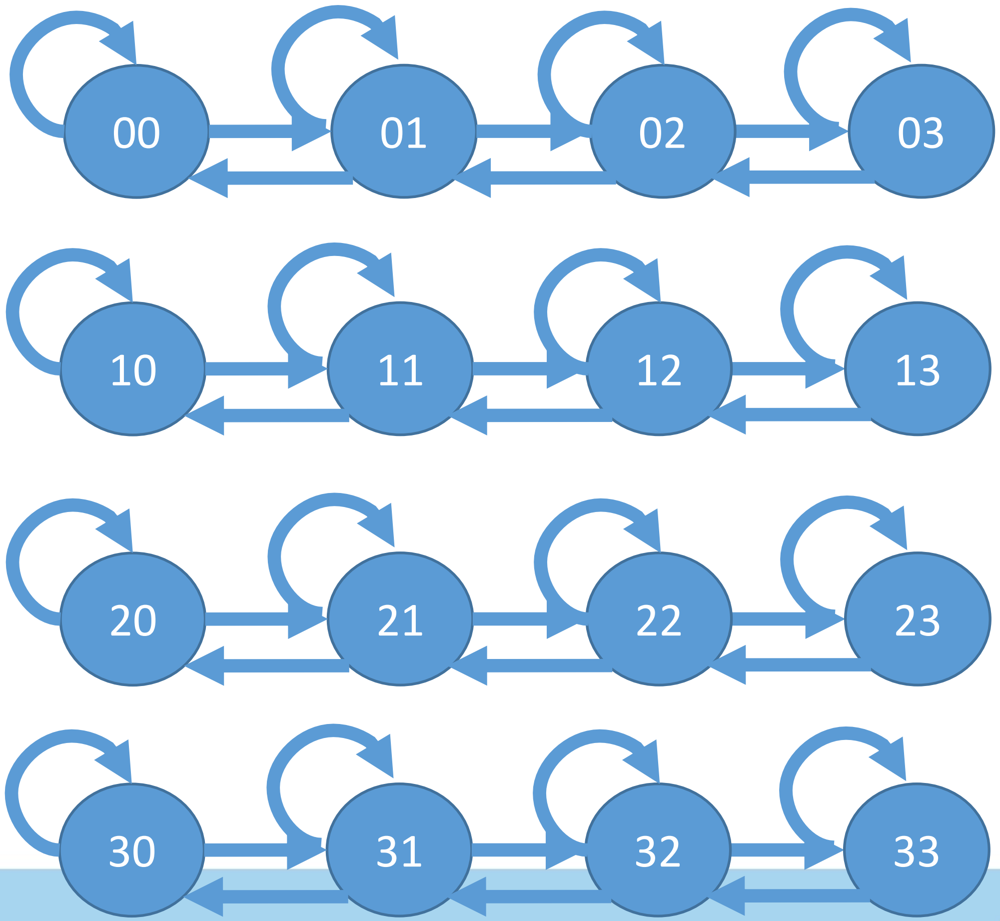

k^{th}-order Markov Chain

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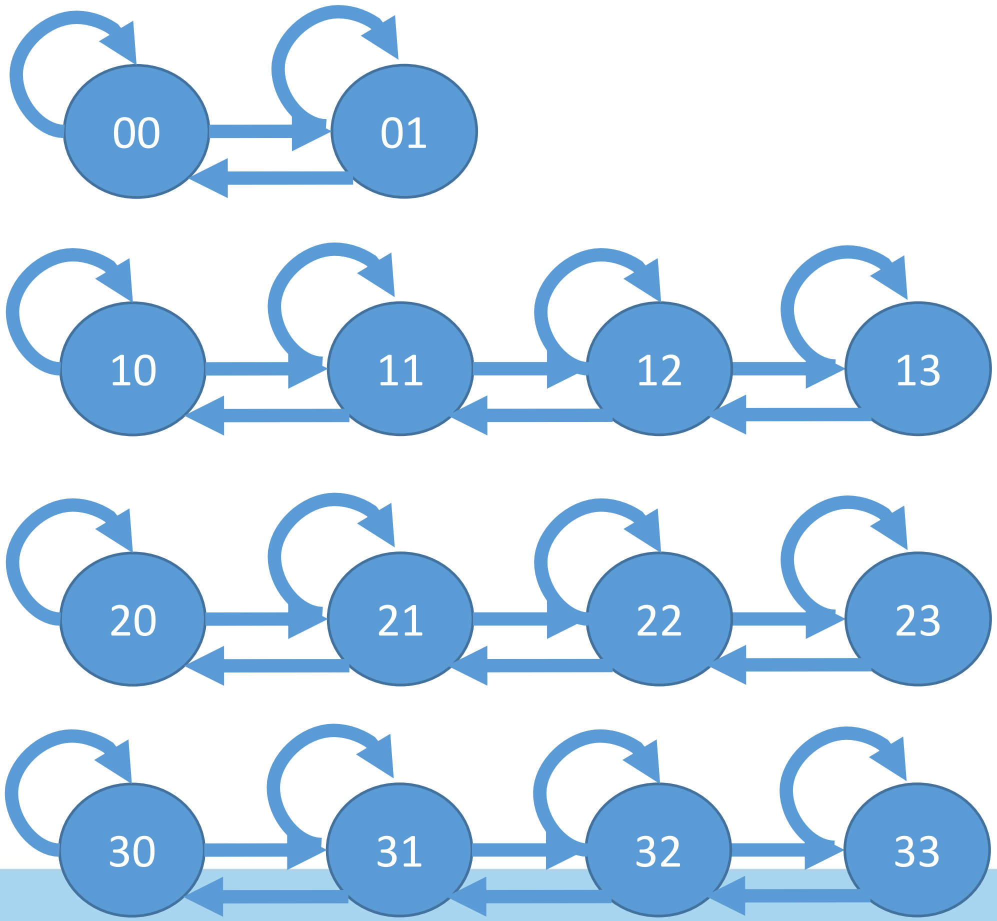

k^{th}-order Markov Chain

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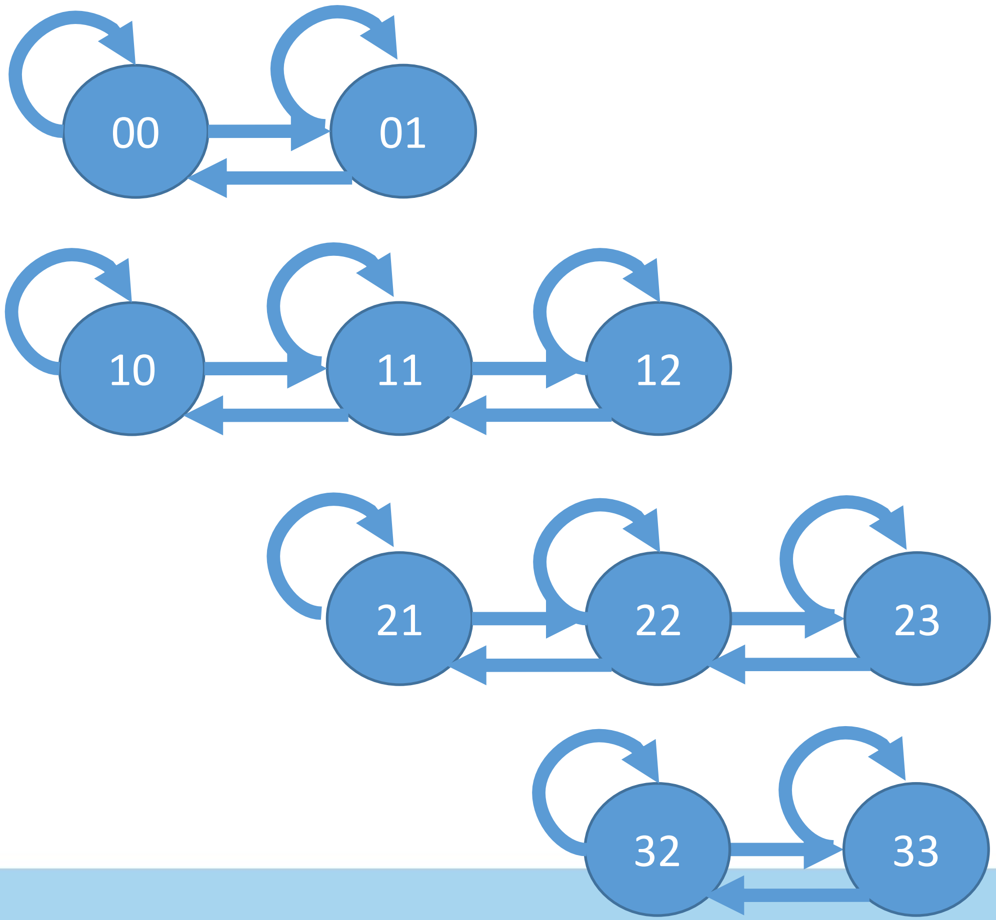

k^{th}-order Markov Chain

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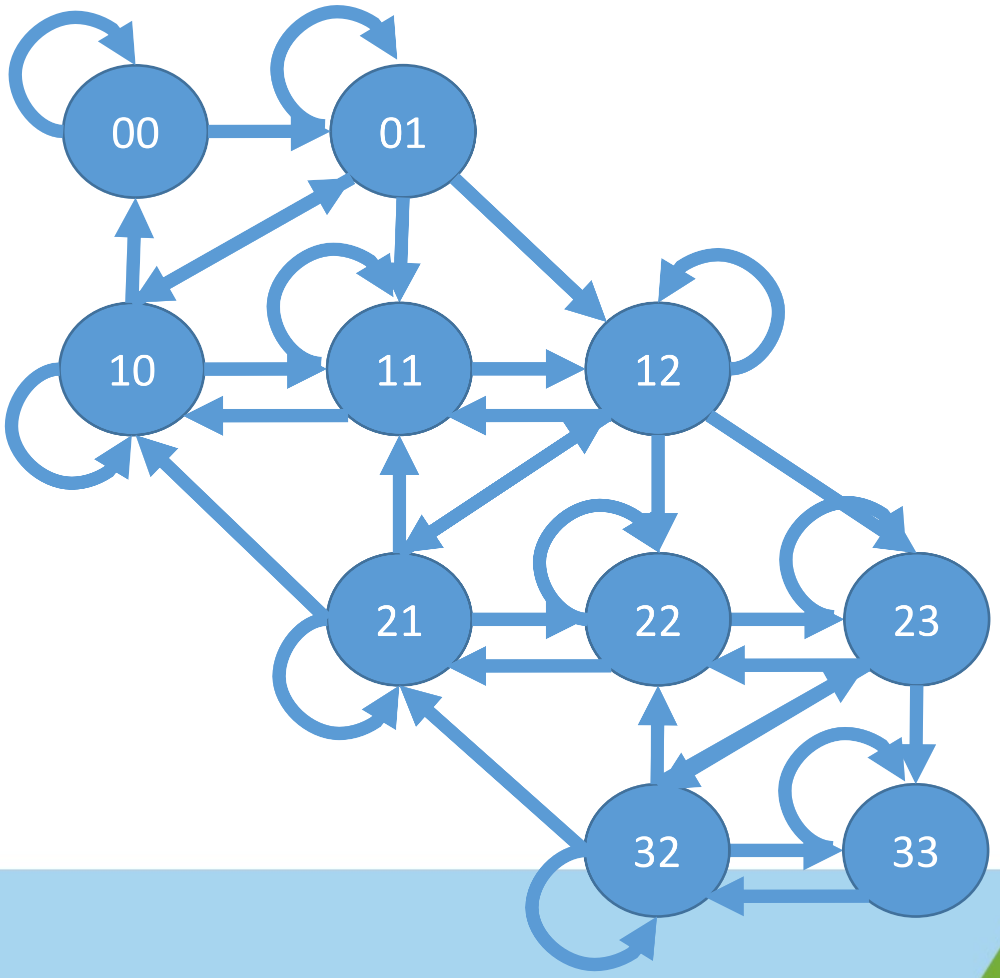
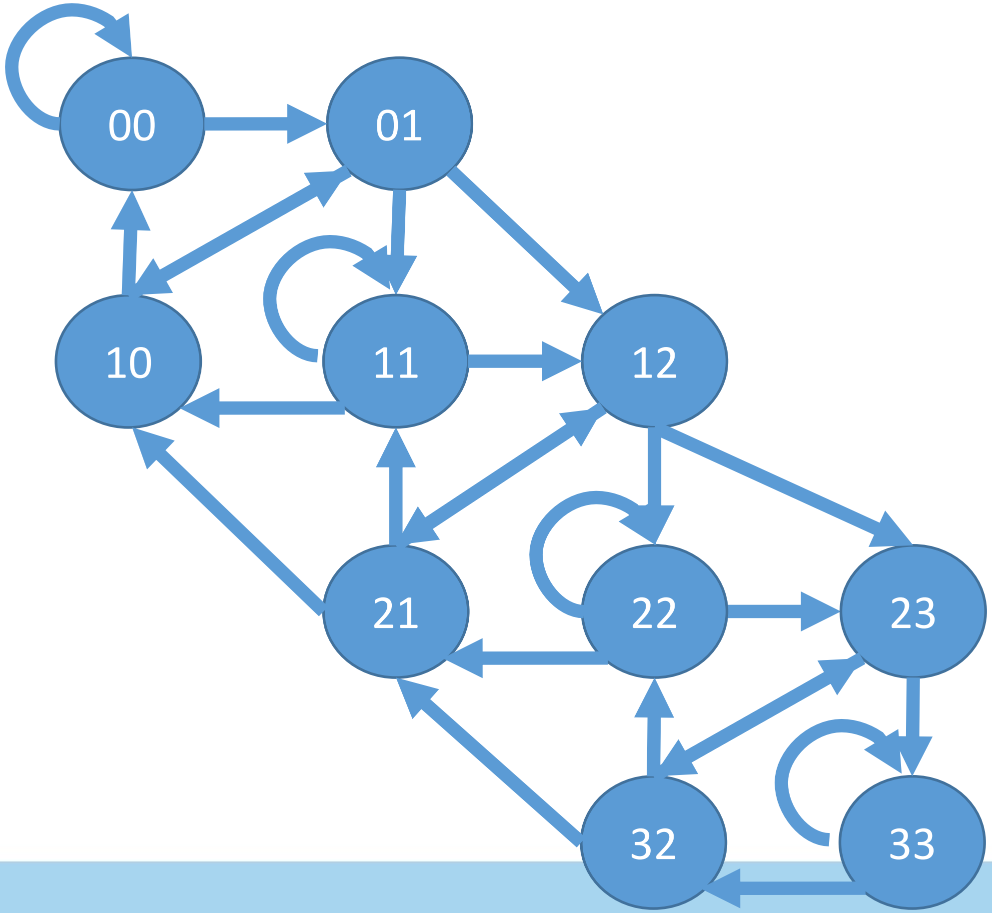


k^{th}-order Markov Chain

What about the state transition diagram?



Recall: Markov Property

The *Markov property* for a k[th]-order Markov chain is:

$$P(X_t \mid X_{t-1}, \dots, X_{t-k}, \dots, X_1) = P(X_t \mid X_{t-1}, \dots, X_{t-k})$$

i.e., X_t is independent of the sequence that produced it, given its *k* predecessors.

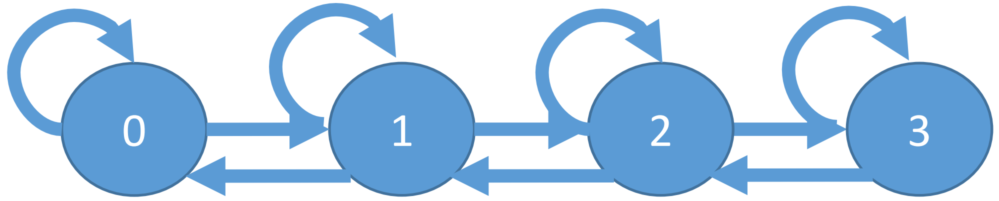
Thus, a second-order Markov chain has the property:

$$P(X_t \mid X_{t-1}, \dots, X_{t-k}, \dots, X_1) = \boxed{P(X_t \mid X_{t-1}, X_{t-2})}$$

Can we get these probabilities from our original transition diagram?

- Have: $P(X_t \mid X_{t-1})$
- Want: $P(X_t \mid X_{t-1}, X_{t-2})$

State transition diagram



Thus, a second-order Markov chain has the property:

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k^{th}-order Markov Chain

A first-order Markov chain is defined by the model $\langle S, T, v_0 \rangle$ such that:

$S = \{s_1, \dots, s_n\}$ is the set of n states

T is a $n \times n$ matrix of transition probabilities such that:

$$p_{ij} = P\left(X_t = s_j \mid X_{t-1} = s_i \right)$$

$$v_0 = \langle P(X_1 = s_1), P(X_1 = s_2), \dots, P(X_1 = s_n) \rangle$$

k^{th}-order Markov Chain

A **first-order** Markov chain is defined by the model $\langle S, T, v_0 \rangle$ such that:

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k^{th}-order Markov Chain

A k^{th}-order Markov chain is defined by the model $\langle S, T, v_0 \rangle$ such that:

$\boldsymbol{S = \{s_1, \dots, s_n\}}$ **is the set of *n* states**

T is a $n \times n$ matrix of transition probabilities such that:

$$p_{ij} = P\left(X_t = s_j \mid X_{t-1} = s_i \right)$$

State set remains the same!

$$v_0 = \langle P(X_1 = s_1), P(X_1 = s_2), \dots, P(X_1 = s_n) \rangle$$

kth-order Markov Chain

A kth-order Markov chain is defined by the model $\langle S, T, v_0 \rangle$ such that:

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T is a $n \times n$ matrix of transition probabilities such that:

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$$v_0 = \langle P(X_1 = s_1), P(X_1 = s_2), \ldots, P(X_1 = s_n) \rangle$$

Previously: just encoded edges from the transition diagram as a matrix

k^th-order Markov Chain

A k^th-order Markov chain is defined by the model $\langle S, T, v_0 \rangle$ such that:

$S = \{s_1, \dots, s_n\}$ is the set of *n* states

T is a $n \times n$ matrix of transition probabilities such that:

$$p_{ij} = P(X_t = s_j \mid X_{t-1} = s_i)$$

$$v_0 = \langle P(X_1 = s_1), P(X_1 = s_2), \dots, P(X_1 = s_n) \rangle$$

Recall:

s_i = state we are coming from

s_j = state we are going to

k^th-order Markov Chain

A k^th-order Markov chain is defined by the model $\langle S, T, v_0 \rangle$ such that:

$S = \{s_1, \ldots, s_n\}$ is the set of *n* states

T is a $n \times n$ matrix of transition probabilities such that:

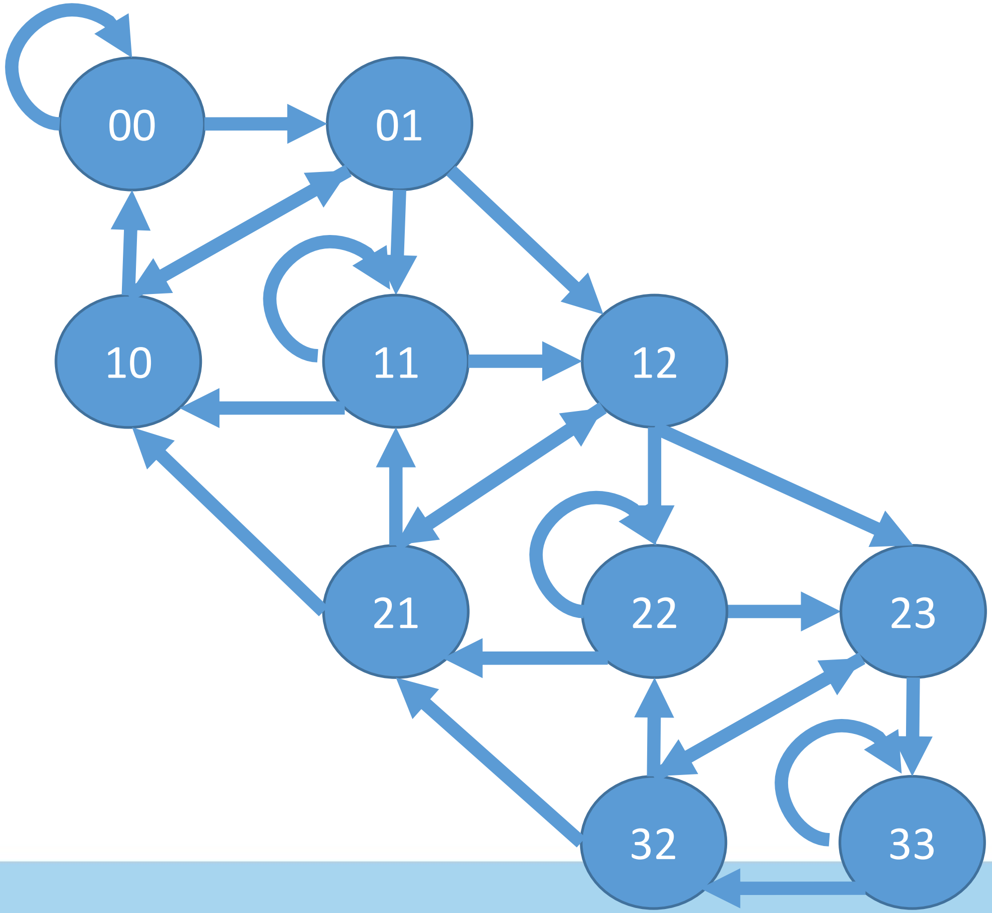
$$p_{ij} = P\left(X_t = s_j \mid X_{t-1} = s_i \right)$$

Now: what is i?

$$v_0 = \langle P(X_1 = s_1), P(X_1 = s_2), \ldots, P(X_1 = s_n) \rangle$$

k^{th}-order Markov Chain

What about the state transition diagram?



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$$v_0 = \langle P(X_1 = s_1), P(X_1 = s_2), \dots, P(X_1 = s_n) \rangle$$

Now: what is i?
i matches a state in the transition diagram,
<u>not</u> a state from S

k^th-order Markov Chain

A k^th-order Markov chain is defined by the model $\langle S, T, v_0 \rangle$ such that:

$S = \{s_1, \dots, s_n\}$ is the set of *n* states

T is a $n^k \times n$ matrix of transition probabilities such that:

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k^{th}-order Markov Chain

A k^{th}-order Markov chain is defined by the model $\langle S, T, v_0 \rangle$ such that:

$S = \{s_1, \dots, s_n\}$ is the set of *n* states

$\boldsymbol{T}$ is a $\boldsymbol{n^k \times n}$ matrix of transition probabilities such that:

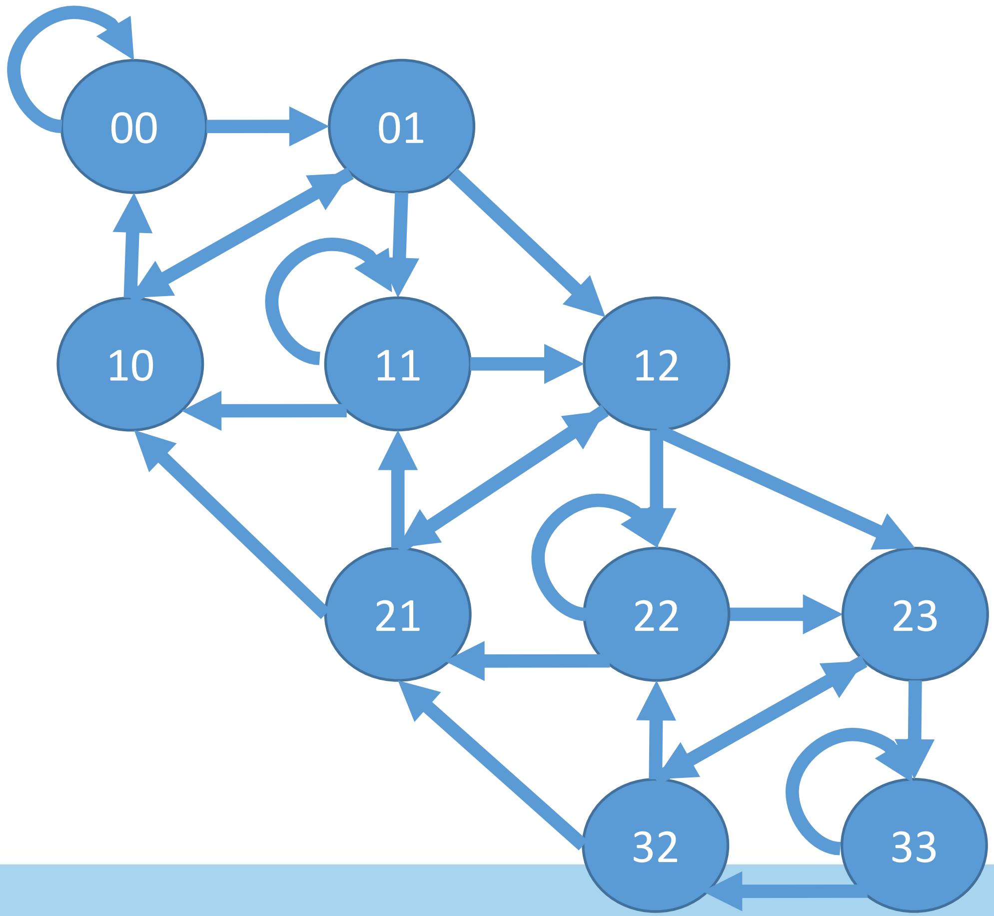
… and let $\boldsymbol{v_i = \langle s^1, \dots, s^k \rangle}$

$$p_{ij} = P\big(X_t = s_j \mid X_{t-1} = s_i \big)$$

$$v_0 = \langle P(X_1 = s_1), P(X_1 = s_2), \dots, P(X_1 = s_n) \rangle$$

k^{th}-order Markov Chain

What about the state transition diagram?



k[th]-order Markov Chain

A k^th^-order Markov chain is defined by the model $\langle S, T, v_0 \rangle$ such that:

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T is a $n^k \times n$ matrix of transition probabilities such that:

… and let $v_i = \langle s^1, \dots, s^k \rangle$

$$\boldsymbol{p_{ij} = P\left(X_t = s_j \mid X_{t-1} = v_i(k), \dots, X_{t-k} = v_i(1) \right)}$$

$$v_0 = \langle P(X_1 = s_1), P(X_1 = s_2), \dots, P(X_1 = s_n) \rangle$$

k^{th}-order Markov Chain

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$$p_{ij} = P\left(X_t = s_j \mid X_{t-1} = v_i(k), \dots ,\ X_{t-2} = v_i(1) \right)$$

$$\boldsymbol{v_0 = \langle P(X_2 = s_1, X_1 = s_1), P(X_2 = s_2, X_1 = s_1), \dots, P(X_2 = s_n, X_1 = s_n) \rangle}$$

Markov Chains via Linear Algebra

Assume first order. Then we can compute

$$P(X_t) = v_0 T^t$$

Produces a vector v_t of size 1 x n giving the probability that we are in each state of S after t steps.

What does this look like for an arbitrary order chain?

First order spreadsheet demo 1

Steady State Distributions

Some Markov chains have the property that:

$$v = Tv$$

v is the *steady state distribution* for the Markov Chain

- Can solve for v directly

Most Markov chains have a unique steady state distribution that the chain converges to, regardless of starting distribution.

First order spreadsheet demo 2

Property: Irreducibility

A Markov Chain is irreducible iff in the transition graph there exists a path from every state to every other state, i.e., you can't get stuck in a group of nodes.

First order spreadsheet demo 3

Property: Aperiodicity

An irreducible Markov chain with transition matrix T is called periodic if there exists some $t > 1$ such that there exists a state s which can only be visited at time steps $\{t, 2t, 3t, \ldots\}$, that is with period t.

If T is not periodic (i.e., $t=1$), then the chain is *aperiodic*.

Property: Uniqueness of steady state distribution

A Markov chain that is irreducible and aperiodic has a unique steady state distribution.

This is the property that allows *Markov Chain Monte Carlo* sampling.

Markov Decision Processes (MDPs)

Markov Chains + decisions

Two new pieces of information:

- Actions to take (choices/decisions)
- Rewards for entering a new state *due to an action*