MSc Project: Representations of Finite Groups

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2.1 The Linear Algebra Rep and the General Linear Group

Definition 2.1 (Endomorphisms and Automorphisms)

Given a space V with an algebraic structure, an endomorphism is defined to be a homomorphism from V to itself. We denote $\mathrm{End}(V) := \mathrm{Hom}(V, V)$ to be the set of all endomorphisms on V. An automorphism is an endomorphism which is also an isomorphism, and we denote $\mathrm{Aut}(V) := \{\phi \in \mathrm{End}(V) \mid \phi \text{ is an isomorphism}\}$ as the set of all automorphisms on V.

In this project, we focus on vector space endomorphisms - linear maps from a space to itself.

Definition 2.2 (General linear group [1])

Let V be a vector space. We define

$$\mathrm{GL}(V) := \mathrm{Aut}(V)$$

to be the set of invertable linear endomorphisms over V. We prove this is a group under composition.

Proof

Associativity: Composition is always associative.
Existance of inverse elements: ϕ an isomorphism $\iff$ ϕ invertible. Hence every element of $\mathrm{Aut}(V)$ has an inverse.
Closedness: The composition of linear maps is linear, and the composition of bijective maps is bijective. Therefore $\mathrm{Aut}(V)$ is closed under composition.
Existence of identity: The identity map is linear and bijective, hence in $\mathrm{Aut}(V)$. ■

Proposition 2.3

If V is an n-dimensional vector space over $\mathbb{C}$ then there is a group isomorphism

$$\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C}) := \{A \in \mathrm{M}_n(\mathbb{C}) \mid A \text{ is invertable}\},$$

the group of invertable $n \times n$ matrices.

Proof

Let V be an n-dimensional vector space over $\mathbb{C}$ and fix a basis $e = \{e_1, \dots, e_n\}$. Recall that the result of a linear transformation is entirely determined by its result on basis elements (once a basis is chosen). Then for $L : V \to V$, we can write the result of L on basis element e_k as $L(e_k) = \alpha_k^1 e_1 + \cdots + \alpha_k^n e_n$. Let $\phi : \mathrm{Aut}(V) \to \mathrm{M}_n(\mathbb{C})$ such that

$$\phi(L) = \begin{bmatrix} \alpha_1^1 & \alpha_2^1 & \dots & \alpha_n^1 \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^n & \alpha_2^n & \dots & \alpha_n^n \end{bmatrix}.$$

Any matrix has a corresponding linear map which sends the kth basis vector to another vector with the basis coefficients made up of the scalars in the kth column. Also, since any linear map is determined by the vectors that the basis elements are mapped to, there is a unique linear map with columns as the coefficients. Then ϕ is a bijection.

Now we show that ϕ is an homomorphism. Given $L_1(e_k) = \alpha_k^1 e_1 + \cdots + \alpha_k^n e_n$ and $L_2(e_k) = \beta_k^1 e_1 + \cdots + \beta_k^n e_n$, we have

$$\begin{aligned} L_2 \circ L_1(e_k) &= L_2(\alpha_k^1 e_1 + \cdots + \alpha_k^n e_n) = \alpha_k^1 L_2(e_1) + \cdots + \alpha_k^n L_2(e_n) \\ &= \alpha_k^1(\beta_1^1 e_1 + \cdots + \beta_1^n e_n) + \cdots + \alpha_k^n(\beta_n^1 e_1 + \cdots + \beta_n^n e_n) \\ &= e_1(\alpha_k^1 \beta_1^1 + \cdots + \alpha_k^n \beta_n^1) + \cdots e_n(\alpha_k^1 \beta_1^n + \cdots + \alpha_k^n \beta_n^n). \end{aligned}$$

Therefore

$$\begin{aligned} \phi(L_2 \circ L_1) &= \begin{bmatrix} (\alpha_1^1\beta_1^1 + \cdots + \alpha_1^n\beta_n^1) & (\alpha_2^1\beta_1^1 + \cdots + \alpha_2^n\beta_n^1) & \dots & (\alpha_n^1\beta_1^1 + \cdots + \alpha_n^n\beta_n^1) \\ (\alpha_1^1\beta_1^2 + \cdots + \alpha_1^n\beta_n^2) & (\alpha_2^1\beta_1^2 + \cdots + \alpha_2^n\beta_n^2) & \cdots & (\alpha_n^1\beta_1^2 + \cdots + \alpha_n^n\beta_n^2) \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha_1^1\beta_1^n + \cdots + \alpha_1^n\beta_n^n) & (\alpha_2^1\beta_1^n + \cdots + \alpha_2^n\beta_n^n) & \dots & (\alpha_n^1\beta_1^n + \cdots + \alpha_n^n\beta_n^n) \end{bmatrix} \\ &= \begin{bmatrix} \beta_1^1 & \beta_2^1 & \dots & \beta_n^1 \\ \beta_1^2 & \beta_2^2 & \cdots & \beta_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^n & \beta_2^n & \dots & \beta_n^n \end{bmatrix} \times \begin{bmatrix} \alpha_1^1 & \alpha_2^1 & \dots & \alpha_n^1 \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^n & \alpha_2^n & \dots & \alpha_n^n \end{bmatrix} = \phi(L_2) \times \phi(L_1). \blacksquare \end{aligned}$$

Definition 2.4 (Unitary map)
A linear map $L : V \to W$ between inner product spaces V, W is said to be unitary if and only if $\langle v_1, v_2 \rangle = \langle L(v_1), L(v_2) \rangle \; \forall v_1, v_2 \in V$. We denote the unitary maps of a vector space V as $\mathrm{U}(V)$, and for maps over an n-dimensional vector space over $\mathbb{C}$, we have $U(V) \cong U_n(\mathbb{C}) := \{A \in M_n(\mathbb{C}) \mid A^* = A^{-1}\}$ where A^* is the standard conjugate transpose.

We will denote the invertable elements of a ring R as R^*, then $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$. Then $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$ and not the dual space of $\mathbb{C}$ as is traditional.

Example 2.5 ([1])
For the maps $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$, a complex number z is unitary if $\bar{z} = z^1 \implies z\bar{z} = |z|^2 = 1 \implies z \in \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle. Then $U_1(\mathbb{C}) = \mathbb{T}$

Theorem 2.6 (Caley-Hamilton [1])
Let A be a matrix with characteristic polynomial $p_A(x)$. Then $p_A(A) = 0$.

Definition 2.7 (Minimal polynomial [1])
For an endomorphism $A \in \mathrm{End}(V)$, the minimal polynomial of A, $m_A(x)$ is the smallest degree monic polynomial $f(x)$ such that $f(A) = 0$

Theorem 2.8 ([1])
A matrix $A \in M_n(\mathbb{C})$ is diagonalizable if and only if all factors of $m_A(x)$ have multiplicity 1.

Theorem 2.9 (Spectral theorem [1])
For a self adjoint $A \in M_n(\mathbb{C})$, there exists a unitary matrix $U \in U_n(\mathbb{C})$ such that U^*AU is diagonal. The eigenvalues of A are real.

TODO prove

2.2 Group Representations

We will see that there are two main ways to define a group representation:

Definition 2.10 (Group representation [1])
A representation of a group G is a group homomorphism $\phi : G \to \mathrm{GL}(V)$ for some finite dimensional vector space V. The degree of ϕ is defined to be the dimension of V.

Definition 2.11 (Trivial representation [1])
Any group can be given the trivial representation $\phi : G \to \mathrm{GL}_1(\mathbb{C})$ such that $\phi(g) = 1 \ \forall g \in G$.

Definition 2.12 (Zero representation [1])
Any group can be given the zero representation $\phi : G \to \mathrm{GL}_1(\mathbb{C})$ such that $\phi(g) = 0 \ \forall g \in G$.

Example 2.13 ([1])
$\phi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ such that $\phi([m]) = e^{2\pi i m/n} \ \forall [m] \in \mathbb{Z}/n\mathbb{Z}$ is a representation.

Definition 2.14 (Representation equivalence [1])
Two representations $\phi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ are said to be equivalent $\phi \sim \psi$ if there exists a linear isomorphism $T : V \to W$ such that $\psi(g)T = T\phi(g) \ \forall g \in G$, and we have the following commutative diagram

$$\begin{tikzcd} V \arrow[r, "\phi(g)"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\psi(g)"] & W \end{tikzcd}$$

Example 2.15 ([1])
Let $\phi : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ with

$$\phi([m]) = \begin{bmatrix} \cos(2\pi m/n) & -\sin(2\pi m/n) \\ \sin(2\pi m/n) & \cos(2\pi m/n) \end{bmatrix},$$

the rotation matrix by angle $2\pi m/n$, and let $\psi : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ with

$$\psi([m]) = \begin{bmatrix} e^{2\pi i m/n} & 0 \\ 0 & e^{-2\pi i m/n} \end{bmatrix}.$$

We have $\phi \sim \psi$.

Proof ([1])
Let $T = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$. Then

$$\begin{aligned} \psi([m])T &= \begin{bmatrix} e^{2\pi i m/n} & 0 \\ 0 & e^{-2\pi i m/n} \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} e^{2\pi i m/n} i & -e^{2\pi i m/n} i \\ e^{-2\pi i m/n} & e^{-2\pi i m/n} \end{bmatrix} \\ &= \begin{bmatrix} -\sin(2\pi i m/n) + i\cos(2\pi i m/n) & \sin(2\pi i m/n) - i\cos(2\pi i m/n) \\ \cos(2\pi i m/n) - i\sin(2\pi i m/n) & \cos(2\pi i m/n) - i\sin(2\pi i m/n) \end{bmatrix} \\ &= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \cos(2\pi m/n) & -\sin(2\pi m/n) \\ \sin(2\pi m/n) & \cos(2\pi m/n) \end{bmatrix} = T\phi([m]). \blacksquare \end{aligned}$$

We will now use the notation $\phi_g := \phi(g)$ for a representation ϕ, to allow us to write the corresponding linear map acting on a vector more clearly.

Definition 2.16 (Symmetric Group)

Recall that the symmetric group S_n is the group of all bijections from a set of n elements to itself, with the group operation of composition of bijections. The group is of order $n!$ since there are $n!$ permutations of n elements.

We write elements of S_n in cycle notation: for example when $n = 6$ $\sigma = (2\ 1\ 3)(4)(5\ 6)$ is the element which sends the 3rd element to the 1st 1st to the 2nd 2nd to the 3rd 4th to 4th and 5th to 6th (and vice versa). Then we can write $\sigma(1, 2, 3, 4, 5, 6) = (\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5), \sigma(6)) = (3, 1, 2, 4, 6, 5)$.

Example 2.17 (Standard representation of S_n [1])

Let $\phi : S_n \to \mathrm{GL}_n(\mathbb{C})$ such that $\phi_\sigma(e_i) = e_{\sigma(i)}$ $\forall \sigma \in S_n$ $1 \leqslant i \leqslant n$. The matrix for ϕ_σ is given by permuting the columns of I by σ for example when $n = 4$ $\sigma = (1\ 4\ 3\ 2)$ gives

$$\phi_\sigma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that $\phi_\sigma(e_1 + e_2 + \cdots + e_n) = e_{\sigma(1)} + e_{\sigma(2)} + \cdots + e_{\sigma(n)} = e_1 + e_2 + \cdots + e_n$ $\forall \sigma \in S_n$ since addition is commutative. Then by scalability of linear ϕ_σ, we have $\phi_\sigma(\alpha(e_1+e_2+\cdots+e_n)) = \alpha(e_1+e_2+\cdots+e_n)$ $\forall \alpha \in \mathbb{C}$, $\sigma \in S_n$. Hence $\mathbb{C}(e_1 + e_2 + \cdots + e_n)$ is invariant under ϕ_σ $\forall \sigma \in S_n$.

Definition 2.18 (G-invariant subspace [1])

For a representation $\phi : G \to \mathrm{GL}(V)$, a (linear) subspace $W \leqslant V$ is said to be G-invariant if and only if $\phi_g(w) \in W$ $\forall g \in G$, $w \in W$.

Definition 2.19 (Direct sum of representations [1])

Given representations $\phi^1 : G \to \mathrm{GL}(V_1)$ and $\phi^2 : G \to \mathrm{GL}(V_2)$, we can find another representation $\phi^1 \oplus \phi^2 : G \to \mathrm{GL}(V_1 \oplus V_2)$ given by $(\phi^1 \oplus \phi^2)_g(v_1, v_2) = (\phi^1_g(v_1), \phi^2_g(v_2))$ $\forall g \in G$, $v_1 \in V_1, v_2 \in V_2$.

If V_1 is of dimension n_1 and V_2 is of dimension n_2, and both are over $\mathbb{C}$ such that $\phi^1 : G \to \mathrm{GL}_{n_1}(\mathbb{C})$ and $\phi^2 : G \to \mathrm{GL}_{n_2}(\mathbb{C})$, then

$$\phi^1 \oplus \phi^2 : G \to \mathrm{GL}_{n_1+n_2}(V_1 \oplus V_2)$$

with matrix form

$$(\phi^1 \oplus \phi^2)_g = \begin{bmatrix} \phi^1_g & 0 \\ 0 & \phi^2_g \end{bmatrix},$$

which is the $(n_1 + n_2)$ square matrix formed by stacking ϕ_g and ψ_g next to each other on the diagonal, with 0 in the other entries.

Example 2.20 ([1])

Let $\phi^1 : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ and $\phi^2 : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ such that $\phi^1_{[m]} = e^{2\pi i m/n}$ and $\phi^2_{[m]} = e^{-2\pi i m/n}$. Then

$$(\phi^1 \oplus \phi^2)_{[m]} = \begin{bmatrix} e^{2\pi i m/n} & 0 \\ 0 & e^{-2\pi i m/n} \end{bmatrix}.$$

Lemma 2.21 ([1])

If a group G is generated by a set S then a representation on G is determined by its values on S, since representations are a homeomorphism.

Proof

For $G = \langle S = \{s_1, s_2, \dots\}\rangle$, and $x = \prod_{i \in I_S} s_i$ a product of elements in S, a representation $\phi : G \to \mathrm{GL}(V)$ gives $\phi_x = \prod_{i \in I_S} \phi_{s_i}$.

Example 2.22 ([1])

S_3 can be generated by two elements: $S_3 = \langle (1\ 2\ 3), (1\ 2)\rangle$.

Let $\phi : S_3 \to \mathrm{GL}_2(\mathbb{C})$ be the representation such that

$$\phi_{(1\ 2)} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \phi_{(1\ 2\ 3)} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix},$$

and let $\psi : S_3 \to \mathbb{C}^*$ be the trivial representation $\phi_\sigma = 1\ \forall \sigma \in S_3$. Then

$$(\phi \oplus \psi)_{(1\ 2)} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (\phi \oplus \psi)_{(1\ 2\ 3)} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition 2.23 (Subrepresentation [1])

For a representation $\phi : G \to \mathrm{GL}(V)$ and a G-invariant subspace $W \leqslant V$, a representation $\phi|_W : G \to \mathrm{GL}(W)$ can be obtained by restricting ϕ to W with $(\phi|_W)_g(w) = \phi_g(w) \in W\ \forall w \in W,\ g \in G$. We say that $\phi|_W$ is a subrepresentation of ϕ.

TODO explain external direct sum bit

Definition 2.24 (Irreducible representation [1])

A non-zero representation $\phi : G \to \mathrm{GL}(V)$ of a group G is irreducible if and only if the only G-invariant subspaces of V are $\{0\}$ and V.

Irreducible representations are analagous to prime numbers in number theory, or simple groups in group theory.

Lemma 2.25 ([1])

Any degree 1 representation $\phi : G \to \mathbb{C}^*$ is irreducible since $\mathbb{C}$ has no proper subspaces.

Proposition 2.26 ([1])

For a degree 2 representation $\phi : G \to \mathrm{GL}(V)$, ϕ is irreducible if and only if there is no common eigenvector $v\ \forall \phi_g,\ g \in G$.

Definition 2.27 (Completely reducible [1])

A representation $\phi : G \to \mathrm{GL}(V)$ is completely reducible if and only if $V = V_1 \oplus V_2 + \cdots + V_n$, where V_i are G-invariant subspaces and $\phi|_{V_i}$ are irreducible $\forall 1 \leqslant i \leqslant n$.

Proposition 2.28 ([1])

The following are equivalent:

1. $\phi : G \to \mathrm{GL}(V)$ is completely reducible.
2. $\phi \sim \phi^1 \oplus \phi^2 \oplus \cdots \oplus \phi^n$ where ϕ^i is irreducible $\forall 1 \leqslant i \leqslant n$.

Definition 2.29 (Decomposable representation [1])

Let $\phi : G \to \mathrm{GL}(V)$ be a non-zero representation. ϕ is decomposable if and only if $V = V_1 \oplus V_2$ where V_1, V_2 are non-zero G-invariant subspaces. Otherwise ϕ is said to be indecomposable.

Lemma 2.30 ([1])
If $\phi : G \to \mathrm{GL}(V)$ is equivalent to a decomposable representation then ϕ is decomposable.

Lemma 2.31 ([1])
If $\phi : G \to \mathrm{GL}(V)$ is equivalent to an indecomposable representation then ϕ is indecomposable.

Lemma 2.32 ([1])
If $\phi : G \to \mathrm{GL}(V)$ is equivalent to a completely reducible representation then ϕ is completely reducible.

3 Maschke's Theorem

Definition 3.1 (Unitary representation [1])

A representation $\phi : G \to \mathrm{GL}(V)$ where V is an inner product space is said to be unitary if and only if ϕ_g is unitary $\forall g \in G$. Since $U_1(\mathbb{C}) = \mathbb{T}$, a one dimensional unitary representation is a homomorphism $\phi : G \to \mathbb{T}$.

Example 3.2 ([1])

Let $\phi : \mathbb{R} \to \mathbb{T}$ such that $\phi_t = e^{2\pi i t}$. Then $\phi_{t+s} = e^{2\pi i(t+s)} = \phi_t \phi_s$, hence ϕ is a representation.

Proposition 3.3 ([1])

A unitary representation $\phi : G \to \mathrm{GL}(V)$ is either irreducible or decomposable.

Proposition 3.4 ([1])

Every representation of a finite group G is equivalent to a unitary representation.

Corollary 3.5 ([1])

Every non-zero representation $\phi : G \to \mathrm{GL}(V)$ of a finite group is either irreducible or decomposable.

Proposition 3.6 ([1])

Every irreducible representation is indecomposable, though the contrary is not true in general.

Theorem 3.7 (Maschke [1])

Every representation of a finite group is completely reducible.

Proof ([1])

Let $\phi : G \to \mathrm{GL}(V)$ be a representation of a finite group G. We proceed by induction on the degree of ϕ. If $\dim V = 1$ then ϕ is irreducible since V has no non-zero proper subspaces. We assume true our inductive hypothesis that ϕ is irreducible if $\dim V = k \in \mathbb{N}$.

4 Character Theory

5 Orthogonality Relations

6 Character Tables

7 Conclusion

8 References

1. Steinberg, B. *Represesenation Theory of Finite Groups* ISBN: 9781461407751 (Springer, 2012).