Representation Theory of Finite Groups

University of Glasgow MSc Project

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Abstract

In this project we provide an introduction to the representation theory of finite groups, with a focus on character theory. Representation theory is introduced and we go on to explore the properties of characters and character tables, with worked examples throughout. Since its conception in the late 19th century, representation theory has become a crucial area of abstract algebra. It is used extensively in varied areas of mathematics and the physical sciences, and is an active area of research to this day.

Contents

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Glossary of Notation

- $\mathrm{M}_n(F)$ The set of $n \times n$ matrices with entries in the field F.
- I_n The $n \times n$ identity matrix.
- $\mathrm{GL}(V)$ The set of linear automorphisms on a vector space V.
- $\mathrm{GL}_n(F)$ The group of invertible $n \times n$ matrices with entries in the field F.
- $\mathrm{SL}_n(F)$ The subgroup of $\mathrm{GL}_n(F)$ of matrices with determinant 1.
- $X^\dagger$ The conjugate transpose of a matrix X.
- Id_X Identity morphism of a space X.
- C_n Cyclic group of order n.
- D_{2n} Dihedral group of order $2n$.
- S_n Symmetric group of order n.
- A_n Alternating group of order n.
- 1_G The identity element of a group G.
- $\mathbb{T}$ The complex unit circle $\{z \in \mathbb{C} \ : \ |z| = 1\}$.

1 Introduction

1.1 Overview of Project

In this project, we will introduce representation theory for finite groups. We will begin with an explanation of the basic definitions of representations, then introduce FG-modules and examine the implications of Maschke's Theorem and Schur's Lemma. We will then introduce character theory and explore the properties and uses of the Hermitian inner product on characters. In the last section we will introduce character tables and calculate some interesting examples.

Since on a fundamental level representation theory is a marriage of group theory with linear algebra, this project assumes a graduate level familiarity with each. Certain theorems in group theory and linear algebra will be used without proof, however care is taken to reiterate some of the results and terminology from linear algebra and group theory which the reader may not immediately recall.

1.2 An Overview of the History of Representation Theory

This section makes heavy use of the texts *Pioneers of Representation Theory: Frobenius, Burnside, Schur, and Brauer* [1] and *The Genesis of the Abstract Group Concept* [2] to provide a brief history of representation theory.

During the 19th century the notion of an abstract group had not yet been developed. The notion of a group as studied by Cauchy and Galois was primarily considered as theory surrounding permutations, which had importance in large part due to the applicability of permutations to the theory of polynomials. Many of the basic theorems of group theory taught in an introductory class had been worked on, however were not written in the context of an abstract group. For example, Lagrange's Theorem was initially given as a statement on the number of polynomials obtained by permuting the n variables of a polynomial $p(x_1, \ldots, x_n)$, which would later be formalized as the index of the subgroup of permutations constant on p inside S_n, which would in turn be generalized to the modern statement with all groups and subgroups. In 1831 Galois would use the term "group of an equation" and in 1854 Cayley began to use the term "group" in his publications. While these terms were exclusively applied to permutation groups, Cayley had the realization that the concepts could be extended to other areas under study at the time such as matrix multiplication and quaternions, though he was ahead of the curve and there was no rush to generalize among the community.

By 1878 the importance of group theory had begun to be recognized and Cayley published four papers in the field, one of which '*The Theory of Groups*' described an operation mapping a finite set to itself which respected the familiar four axioms of the group operation. In 1879 Frobenius and Stickelberger published the paper '*Ueber Gruppen von vertauschbaren Elementen*' which, among other things, contained a statement of the fundamental theorem of finite abelian groups in modern day group-theoretic terms, introducing the term irreducible.

Frobenius was the father of representation theory. In 1896 he published five papers in group theory. One paper proved Sylow's theorem in terms of abstract groups, as until the conception of abstract groups it had been stated as a result on symmetric groups. This paper is noteworthy as it contains an elegant application of conjugacy classes, which Frobenius later examined in greater detail. His fourth publication that year was a seminal paper on character theory. Character theory has its roots in the ideas of Legendre, Dirichlet and Gauss in 19th century number theory, but Frobenius extended the theory to finite nonabelian groups. This paper contains a homomorphism which in the modern day we would consider a one-dimensional representation, though Frobenius would not define group representations until the following year. He had first defined a character in his letters to Dedekind in 1896 as a complex function on a group G which is constant on elements in the same conjugacy class. In the same writings he went on to generalize the already existing orthogonality relations for finite abelian groups. In 1898 an important application to Frobenius' character theory had already been discovered by Molien in his exploration of polynomial invariants of finite groups.

Frobenius published four papers between 1897 and 1899, two of which introduced representation theory of finite groups. They introduced matrix representations and a collection of the important results which are normally given in an introductory course. It was not immediately clear that characters, as Frobenius initially defined them, could be understood as the trace of a representation. He later found this link by studying matrices with polynomial entries which were associated with the representations of a group. Frobenius used his hitherto developed results in character theory to calculate the characters of S_n. The work done by Frobenius and his doctoral student Schur at this time has been incorporated into modern computational group theory.

An early victory for representation theory was the first proof of Burnside's pq-theorem, which states that for primes p, q and positive integers a_1, a_2 every finite group of order $p^{a_1}q^{a_2}$ is solvable. The proof employing representation theory was the sole proof until a group-theoretic proof could be developed years later [3, page 1]. Representation theory was a crucial tool in the complete classification of finite simple groups [3, page 1]. Richard Brauer had been an early pioneer in the theory of finite simple groups and provided some of the early work towards their classification. The theory he developed "almost entirely" relied on character-theoretic and representation-theoretic proofs [4, page 1].

Representation theory has gone on to be an incredibly useful tool in the physical sciences, notably in solid state physics and molecular physics [5, page 78], and is a staple of quantum mechanics where symmetry groups are used extensively [6, page 187]. Fourier analysis of finite groups depends on representation theory and is essential for modern sound compression methods, it also is used in a variety of probabilistic and statistical methods such as random walks [3, Chapter 11]. It was used in graph theory in the study of expander graphs which have seen widespread engineering application [7, page 3]. Group representation theory is used extremely often in areas of applied maths where group theory is used frequently.

2 Representation Theory

2.1 Group Representations

In this subsection we introduce representation theory in its most simple form. To examine the uses in depth we will need to develop more structure, which we do in later sections. Given a space V with an algebraic structure, an endomorphism is defined to be a homomorphism from V to itself. We denote $\mathrm{End}(V) \coloneqq \mathrm{Hom}(V, V)$ to be the set of all endomorphisms on V. If an endomorphism is an isomorphism, then it is called an automorphism. At its core, representation theory for finite groups seeks to 'represent' a group by mapping group elements to automorphisms of a vector space. We denote the set of all automorphisms on a vector space V by $\mathrm{Aut}(V) \coloneqq \{\phi \in \mathrm{End}(V) \mid \phi \text{ is an isomorphism}\}$.

Definition 2.1. General linear group [3, page 3]
Let V be a vector space. We define

$$\mathrm{GL}(V) \coloneqq \mathrm{Aut}(V)$$

to be the set of invertible linear endomorphisms on V. It is clear from the properties of compostion that $\mathrm{GL}(V)$ is a group under compostion. We call $\mathrm{GL}(V)$ the *general linear group* of vector space V.

Linear maps of finite-dimensional vector spaces can be written as matrices with respect to a chosen basis. For an n-dimensional vector space over a field F, the group of invertible $n \times n$ square matrices $\mathrm{GL}_n(F) \coloneqq \{X \in M_n(F) \ : \ X \text{ is invertible}\}$ is isomorphic to the general linear group $\mathrm{GL}(V)$ as defined above. We will most often be dealing with finite-dimensional vector spaces and so will prefer to view the general linear group in the matrix form.

Definition 2.2. Group representation [3, Definition 3.1.1]
A *group representation* of a group G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ for some vector space V. The degree of ρ is defined to be the dimension of V.

For this project we write 'representation' in lieu of 'group representation' since we only focus on group representations. This is important to note as there are many other kinds of representations for other algebraic structures, for example a Lie algebra representation where the representation map is a Lie algebra homomorphism into $\mathrm{End}(V)$ for some vector space V.

We remark that a representation defines a group action since for a group G, a vector space V, and a representation $\rho : G \to \mathrm{GL}(V)$ we have

(1) $\rho(1_G)v = \mathrm{Id}_V(v) = v$ for all $v \in V$,

(2) $\rho(g_1)\rho(g_2)v = \rho(g_1 g_2)v$ for all $g_1, g_2 \in G$, $v \in V$.

For a representation $\rho : G \to \mathrm{GL}(V)$, and a group element $g \in G$ we will write ρ_g instead of $\rho(g)$.

Definition 2.3. Trivial representation [3, Example 3.1.3]
Any group G can be given the *trivial representation* $\rho^{(\mathrm{triv})} : G \to \mathrm{GL}_1(\mathbb{C})$ such that $\rho^{(\mathrm{triv})}_g = 1$ for all $g \in G$.

Representations will allow us to represent individual group elements by linear maps. If no two group elements share the same linear map then the representation is called 'faithful', or defined more formally:

Definition 2.4. Faithful representation [3, Excercise 8.8]
Let G be a group and V a vector space. A representation $\rho : G \to \mathrm{GL}(V)$ is called *faithful* if it is injective, i.e has trivial kernel: given $g \in G$, $\rho_g = \mathrm{Id}_V$ implies $g = 1_G$.

We provide a concrete example of a faithful representation below. Note that $\mathrm{GL}_1(\mathbb{C})$ is the group $\mathbb{C}\backslash\{0\}$ under usual multiplication.

Example 2.5. [3, Example 3.1.6]
Let $G = \mathbb{Z}/n\mathbb{Z}$. Let $\rho : G \to \mathrm{GL}_1(\mathbb{C})$ be the representation defined by $\rho_m = e^{2\pi im/n}$ for all $m \in G$. We know that $e^{2\pi im/n} = 1$ if and only if $m/n \in \mathbb{Z}$, which implies $m = 0$ and ρ is faithful.

We can determine whether a representation is faithful by examining its image.

Proposition 2.6. [8, Proposition 3.7]
Let G be a finite group and let $\rho : G \to \mathrm{GL}(V)$ be a representation. Then ρ is faithful if and only if its image $\mathrm{Im}(\rho)$ is isomorphic to G.

Proof. [8, Proposition 3.7]
The kernel of ρ is a normal subgroup $\mathrm{Ker}(\rho) \trianglelefteq G$, then by the first isomorphism theorem $G/\mathrm{Ker}(\rho) \cong \mathrm{Im}(\rho)$. Then ρ is faithful if and only if $\mathrm{Ker}(\rho) = \{1_G\}$ implying $G \cong \mathrm{Im}(\rho)$. Conversely, if $G \cong \mathrm{Im}(\rho)$ then $|G| = |\mathrm{Im}(\rho)|$. Also $[G : \mathrm{Ker}(\rho)] = \dfrac{|G|}{|\mathrm{Ker}(\rho)|} = |\mathrm{Im}(\rho)|$ and $|\mathrm{Ker}(\rho)| = 1$, implying ρ is faithful.

From group theory the reader may already be familiar with permutation matrices as a way to represent the elements of the symmetric group S_n. We can provide S_n with a representation where permutation matrices are the automorphisms obtained from the group elements.

Example 2.7. Standard representation of S_n [3, Example 3.1.9]
Let $G = S_n$ and V be an n-dimensional vector space over $\mathbb{C}$ with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be the representation such that $\rho_\sigma(e_i) = e_{\sigma(i)}$ for all permutations $\sigma \in S_n$ and $1 \leqslant i \leqslant n$. The matrix for ρ_σ is given by permuting the rows of the identity matrix I_n by σ. For example when $n = 4$, given a permutation (written in cycle notation) $\sigma = (1432)$, we have

$$\rho_\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that $\rho_\sigma(e_1+e_2+\cdots+e_n) = e_{\sigma(1)}+e_{\sigma(2)}+\cdots+e_{\sigma(n)} = e_1+e_2+\cdots+e_n$ for all $\sigma \in S_n$ (since addition is commutative) and (by scalability of linear ρ_σ), we have $\rho_\sigma(\lambda(e_1+e_2+\cdots+e_n)) = \lambda(e_1+e_2+\cdots+e_n)$ for all $\lambda \in \mathbb{C}$, $\sigma \in S_n$. Hence $\mathbb{C}(e_1+e_2+\cdots+e_n)$ is constant under ρ_σ

for all $\sigma \in S_n$.

In the above example we can see a subspace which is closed under the representation. This provides some motivation for defining a subrepresentation of a representation.

Definition 2.8. G-invariant subspace [3, Definition 3.1.10]
Let G be a group and V a vector space. For a representation $\rho : G \to \mathrm{GL}(V)$, a linear subspace $W \leqslant V$ is said to be *G-invariant* if $\rho_g(w) \in W$ for all $g \in G$, $w \in W$.

Definition 2.9. Subrepresentation [9, page 10]
Let G be a group. For a representation $\rho : G \to \mathrm{GL}(V)$ and a G-invariant subspace $W \leqslant V$, a representation $\rho|_W : G \to \mathrm{GL}(W)$ can be obtained by restricting the endomorphisms obtained from ρ to W with $(\rho|_W)_g(w) \in W$ for all $w \in W,\ g \in G$. We say that $\rho|_W$ is a *subrepresentation* of ρ.

To avoid confusion we emphasize that $\rho|_W$ is a slight abuse of notation. We are not specifically restricting ρ to W since ρ is a function on G. We are instead restricting all of the endomorphisms ρ_g that we obtain from ρ. That is $(\rho|_W)_g \coloneqq \rho_g|_W$.

Given a representation $\rho : G \to \mathrm{GL}(V)$, there are always two subrepresentations: ρ itself, and the zero representation $\rho|_{\{0\}} : G \to \mathrm{GL}(\{0\})$.

In this project, we will see how representations and their subrepresentations can be used to classify groups, and for this purpose it is useful define the term proper subrepresentation.

Definition 2.10. Proper G-invariant subspace/subrepresentation
Let G be a group and let $W \leqslant V$ be a G-invariant subspace. The subspace W is said to be *proper* if $W \neq V$. We call the subrepresentation $\rho|_W : G \to \mathrm{GL}(W)$ of $\rho : G \to \mathrm{GL}(V)$ proper if W is a proper G-invariant subspace.

We can generate new representations by taking the direct sum of representations.

Definition 2.11. Direct sum of representations [3, Definition 3.1.11]
Let G be a group and V_1 and V_2 be vector spaces over field F. Given representations $\rho^{(1)} : G \to \mathrm{GL}(V_1)$ and $\rho^{(2)} : G \to \mathrm{GL}(V_2)$, we can obtain another representation of G which maps to $\mathrm{GL}(V_1 \oplus V_2)$ using the *direct sum*. We write the direct sum of these representations as $\rho^{(1)} \oplus \rho^{(2)} : G \to \mathrm{GL}(V_1 \oplus V_2)$ given by $(\rho^{(1)} \oplus \rho^{(2)})_g(v_1 + v_2) = \rho^{(1)}_g(v_1) + \rho^{(2)}_g(v_2)$ for all $g \in G$, with $(v_1 + v_2) \in V_1 \oplus V_2$ and $v_1 \in V_1, v_2 \in V_2$.

If V_1 is of dimension n_1 and V_2 is of dimension n_2, and both are over a field F such that $\rho^{(1)} : G \to \mathrm{GL}_{n_1}(F)$ and $\rho^{(2)} : G \to \mathrm{GL}_{n_2}(F)$, then

$$\rho^{(1)} \oplus \rho^{(2)} : G \to \mathrm{GL}_{n_1+n_2}(V_1 \oplus V_2)$$

with matrix form

$$(\rho^{(1)} \oplus \rho^{(2)})_g = \begin{pmatrix} \rho_g^{(1)} & 0 \\ 0 & \rho_g^{(2)} \end{pmatrix},$$

which is the $(n_1 + n_2)$ square matrix formed by stacking $\rho_g^{(1)}$ and $\rho_g^{(2)}$ next to each other on the diagonal, with 0 in the other entries.

Similarly, we can take the finite direct sum $\rho \coloneqq \rho^{(1)} \oplus \cdots \oplus \rho^{(m)}$ of m representations $\rho^{(i)} : G \to \mathrm{GL}(V_i)$ for all $1 \leqslant i \leqslant m$, such that $\rho : G \to \mathrm{GL}(V_1 \oplus \cdots \oplus V_m)$ and

$$\rho_g(v_1 + \cdots + v_m) = \rho_g^{(1)}(v_1) + \cdots + \rho_g^{(m)}(v_m) \quad \text{for all } g \in G, v_i \in V_i.$$

Suppose V_i is a finite-dimensional vector space with dimension n_i for all $1 \leqslant i \leqslant m$, then the matrix form of ρ is given by the $n_1 + n_2 + \cdots + n_m$ diagonal square matrix

$$\rho_g = \begin{pmatrix} \rho_g^{(1)} & & 0 \\ & \ddots & \\ 0 & & \rho_g^{(m)} \end{pmatrix} \in \mathrm{GL}_{n_1+\cdots+n_m}(F) \quad \text{for all } g \in G.$$

Since the direct sum of representations uses the direct sum of the underlying vector spaces, it is clear that the degree of the direct sum of representations is the sum of their respective degrees. In the below example, we form a degree 2 representation by direct summing two degree 1 representations.

Example 2.12. [3, Example 3.1.12]

Let $\rho^{(1)} : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_1(\mathbb{C})$ and $\rho^{(2)} : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_1(\mathbb{C})$ be representations, such that $\rho_m^{(1)} = e^{2\pi i m/n}$ and $\rho_m^{(2)} = e^{-2\pi i m/n}$ for all $m \in \mathbb{Z}/n\mathbb{Z}$. Then

$$(\rho^{(1)} \oplus \rho^{(2)})_m = \begin{pmatrix} e^{2\pi i m/n} & 0 \\ 0 & e^{-2\pi i m/n} \end{pmatrix}.$$

Given a group G with a generating set S and a representation ρ of G, we can generate ρ_g for any $g \in G$ by the elements of $\{\rho_s \ : \ s \in S\}$.

Proposition 2.13. [3, page 16]

If a group G is generated by a set S then a representation on G is determined by its values on S.

Proof.

This follows clearly from the properties of a homomorphism. Let $G = \langle S \rangle$, and take a general element $g = \prod_{i \in I_S} s_i$ as the product of elements in S for some indexing set I_S. A representation $\rho : G \to \mathrm{GL}(V)$ gives

$$\rho_g = \prod_{i \in I_S} \rho_{s_i},$$

and we can generate any ρ_g from elements in $\{\rho_s \ : \ s \in S\}$. ■

In the following example we combine our ideas of direct summing of representations and generating representations.

Example 2.14. [3, Example 3.1.14]
S_3 can be generated by two elements: $S_3 = \langle (123), (12) \rangle$.

Let $\rho : S_3 \to \mathrm{GL}_2(\mathbb{C})$ be the representation such that

$$\rho_{(12)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \rho_{(123)} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$

and let $\rho^{(\mathrm{triv})} : S_3 \to \mathrm{GL}_1(\mathbb{C})$ be the trivial representation $\rho_\sigma = 1$ for all $\sigma \in S_3$. Then

$$(\rho \oplus \rho^{(\mathrm{triv})})_{(12)} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\rho \oplus \rho^{(\mathrm{triv})})_{(123)} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2.2 Representation Reducibility

A main goal, when dealing with many areas of abstract algebra, is to find ways to methodically decompose some algebraic structure into a simpler form, showing that it is made up of smaller structures which cannot be decomposed any further. In representation theory we aim to find out if we can decompose a representation into a direct sum of non-zero proper subrepresentations. In this subsection we introduce the reader to this theory.

Definition 2.15. Irreducible representation [3, Definition 3.1.15]
Let G be a group and V a vector space. A non-zero representation $\rho : G \to \mathrm{GL}(V)$ is *irreducible* if it contains no non-zero proper submodules, or equivalently the only G-invariant subspaces of V are $\{0\}$ and V itself. Otherwise ρ is reducible.

Irreducible representations are analogous to prime numbers in number theory. They come together to form more complicated reducible representations.

Proposition 2.16. [**3, Example 3.1.16**]
Let G be a group. A degree 1 representation $\rho : G \to \mathrm{GL}_1(\mathbb{C})$ is clearly irreducible since $V = \mathbb{C}$ has dimension 1 and hence no non-zero proper G-invariant subspaces.

Proposition 2.17. [**3, page 18, 3.1.19**]
Let G be a finite group and V a vector space. Let $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ be a complex degree 2 representation, then ρ is irreducible if and only if there is no common eigenvector $v \in V$ for all automorphisms ρ_g, where $g \in G$.

Proof.
Suppose ρ is reducible. Then there exists a non-zero proper G-invariant linear subspace $W < V$. Then since $\dim(V) = 2$, we have $\dim(W) = 1$. Then there exists a vector $v \in V$ such that $W = \mathbb{C}\{v\}$. Then for all $g \in G$, we have $\rho_g(v) \in W$ and hence $\rho_g(v) = \lambda v$ for some $\lambda \in \mathbb{C}$, and V is a common eigenvector. Conversely, suppose that the automorphisms $\{\rho_g \ : \ g \in G\}$ have a common eigenvector $v \in V$. Then the space defined by $W = \mathbb{C}\{v\}$ is a linear subspace of V with dimension 1. For any element $g \in G$ we have $\rho_g(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. Then $\rho_g(W) = \rho_g(\mathbb{C}\{v\}) = \lambda\mathbb{C}\{v\} = \mathbb{C}\{v\} = W$ and W is a G-invariant subspace. Hence ρ is reducible. ■

Proposition 2.18. [**9, page 11**]
Every irreducible complex representation of $G = D_6 = \langle x, y \mid x^3,\, y^2,\, (yx)^2 \rangle$ has degree less than or equal to 2.

Proof. [**9, page 11**]
Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation. Let $r = x, x^2$ be a rotation, and $s = y, yx, yx^2$ be a reflection. Any choice of rotation and reflection generates the entirety of D_6. Since ρ_r is an automorphism it is invertible and has to have an eigenvalue. Let v be an eigenvector of ρ_r with corresponding eigenvalue $\lambda_v \neq 0$, i.e $\phi_r(v) = \lambda_v v$. Let $W = \mathrm{Span}(v, \rho_s(v)) \leqslant V$. Notice that by $y^2 = (yx)^2 = 1_G$ we have

$$\rho_s \rho_s(v) = \rho_{1_G}(v) = v \in W,$$

also

$$\rho_r\rho_s(v) = \rho_s\rho_{r^{-1}}(v) = \lambda_v^{-1}\rho_s(v) \in W,$$

and both $\rho_r(v) = \lambda_v v$ and $\rho_s(v)$ are in W. Then W is G-invariant, and since $V \neq \{0\}$ is irreducible, $W = V$ and $\dim(V) = 2$. ■

The definition of complete reducibility is motivated by a goal of describing a reducible representation as the direct sum of irreducible subrepresentations, similar to how we can describe a composite natural numbers as a product of prime numbers.

Definition 2.19. Completely reducible [3, Definition 3.1.21]

A representation $\rho : G \to \mathrm{GL}(V)$ is *completely reducible* if $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$, where V_i are G-invariant subspaces and $\rho|_{V_i}$ is irreducible for each $1 \leqslant i \leqslant n$.

Our comparison between irreducible subrepresentations and prime numbers has some limitations. Given a composite number you can always find a unique prime decompostion, but in representaion theory a reducible representation is not always completely reducible.

Another important property of a representation is whether or not it is decomposable.

Definition 2.20. Decomposable representation [3, Definition 3.1.22]

Let G be a group, V a vector space, and $\rho : G \to \mathrm{GL}(V)$ be a non-zero representation. Then we call ρ *decomposable* if $V = V_1 \oplus V_2$ where V_1, V_2 are proper G-invariant subspaces. Otherwise ρ is said to be indecomposable.

In this project we will examine reducibility properties of representations. It is common for there to be many different representations of a finite group G, and representation equivalence will provide us with a way to determine if the same reducibility properties hold for so called 'equivalent' representations.

Definition 2.21. Representation equivalence [3, Definition 3.1.7]

Two representations $\rho^{(1)} : G \to \mathrm{GL}(V)$ and $\rho^{(2)} : G \to \mathrm{GL}(W)$ are said to be *equivalent*, written $\rho^{(1)} \sim \rho^{(2)}$, if there exists a linear isomorphism $T : V \to W$ such that $\rho_g^{(2)} T = T\rho_g^{(1)}$ for all $g \in G$. We can write this as the following commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\rho_g^{(1)}"] \arrow[d, "T"'] & V \arrow[d, "T"] \\ W \arrow[r, "\rho_g^{(2)}"] & W \end{tikzcd}$$

Proposition 2.22.

Representation equivalence is an equivalence relation.

Proof.

Let V_1, V_2, V_3 be vector spaces and let $\rho^{(1)} : G \to \mathrm{GL}(V_1)$, $\rho^{(2)} : G \to \mathrm{GL}(V_2)$, $\rho^{(3)} : G \to \mathrm{GL}(V_3)$ be representations.

- Reflexive: Let Id_V be the identity map $\mathrm{Id}_V(e_i) = e_i$, which is a linear isomorphism. Then $\rho_g^{(1)} \mathrm{Id}_V = \mathrm{Id}_V \rho_g^{(1)}$ for all $g \in G$. Hence $\rho^{(1)} \sim \rho^{(1)}$.
- Symmetric: Suppose $\rho^{(1)} \sim \rho^{(2)}$ with linear isomorphism T, then there exists a T^{-1} which is also an isomorphism and $\rho_g^{(1)} T = T \rho_g^{(2)}$ implies $\rho_g^{(2)} T^{-1} = T^{-1} \rho_g^{(1)}$ implies $\rho^{(2)} \sim \rho^{(1)}$.
- Transitive: Let $\rho^{(1)} \sim \rho^{(2)}$ with isomorphism T_{12} and $\rho^{(2)} \sim \rho^{(3)}$ with T_{23}. Then $T_{12} \circ T_{23}$ is also a linear isomorphism and $\rho_g^{(1)} T_{12} T_{23} = T_{12} \rho_g^{(2)} T_{23} = T_{12} T_{23} \rho_g^{(3)}$ for all $g \in G$ implies $\rho^{(1)} \sim \rho^{(3)}$. ∎

Below we show an example of two equivalent representations.

Example 2.23. [3, Example 3.1.8]
Let $\rho^{(1)} : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ with

$$\rho_m^{(1)} = \begin{pmatrix} \cos(2\pi m/n) & -\sin(2\pi m/n) \\ \sin(2\pi m/n) & \cos(2\pi m/n) \end{pmatrix},$$

the rotation matrix by angle $2\pi m/n$, and let $\rho^{(2)} : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ with

$$\rho_m^{(2)} = \begin{pmatrix} e^{2\pi i m/n} & 0 \\ 0 & e^{-2\pi i m/n} \end{pmatrix}.$$

Let $T = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. Then if $m \in \mathbb{Z}/n\mathbb{Z}$ is an arbitrary element,

$$\begin{aligned} \rho_m^{(2)} T &= \begin{pmatrix} e^{2\pi i m/n} & 0 \\ 0 & e^{-2\pi i m/n} \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} e^{2\pi i m/n} i & -e^{2\pi i m/n} i \\ e^{-2\pi i m/n} & e^{-2\pi i m/n} \end{pmatrix} \\ &= \begin{pmatrix} -\sin(2\pi i m/n) + i\cos(2\pi i m/n) & \sin(2\pi i m/n) - i\cos(2\pi i m/n) \\ \cos(2\pi i m/n) - i\sin(2\pi i m/n) & \cos(2\pi i m/n) - i\sin(2\pi i m/n) \end{pmatrix} \\ &= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos(2\pi m/n) & -\sin(2\pi m/n) \\ \sin(2\pi m/n) & \cos(2\pi m/n) \end{pmatrix} = T\rho_m^{(1)}. \end{aligned}$$

Then we have $\rho^{(1)} \sim \rho^{(2)}$.

Decomposablity, complete reducibility, and reducibility are properties that is invariant under representation equivalence, as we state in the following three propositions.

Proposition 2.24. [3, Lemma 3.1.23]
Let G be a group and $\rho : G \to \mathrm{GL}(V)$ be a representation. Then ρ is decomposable if it is equivalent to a decomposable representation.

Proof. [3, Lemma 3.1.23]
Let G be a group and $\phi : G \to \mathrm{GL}(W)$ be a decomposable representation such that $\phi \sim \rho$. Then there exists a linear isomorphism $T : V \to W$ such that $\rho_g = T^{-1} \phi_g T$ for all $g \in G$.

We know that $W = W_1 \oplus W_2$ where W_1, W_2 are proper G-invariant subspaces. We have the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle T} \\ W & \xrightarrow{\phi_g} & W \end{array}$$

and $T\rho_g = \phi_g T$ for all $g \in G$. Let $V_1 = T^{-1}(W_1)$ and $V_2 = T^{-1}(W_2)$. For $v \in V$, we have $T(v) = w_1 + w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$. Then $v = T^{-1}w_1 + T^{-1}w_2 \in V_1 + V_2$, hence $V = V_1 + V_2$. Let $v \in V_1 \cap V_2$, then $T(v) \in W_1 \cap W_2 = \{0\}$ and since T is injective we have $v = 0$, hence $V = V_1 \oplus V_2$. Now we need only show V_1, V_2 are G-invariant. Let $v \in V_i$, then $T(v) \in W_i$ for each $i = 1, 2$. Then $\rho_g(v) = T^{-1}\phi_g T(v)$ for all $g \in G$. Then $T(v) \in W_i$ implies $\phi_g T(v) \in W_i$ since W_i is G-invariant. Then $\rho_g(v) = T^{-1}\phi_g T(v) \in V_i$ for all $g \in G$, and V_i is G-invariant for each $i = 1, 2$. ■

We cite without proof Propositions 2.25 and 2.26.

Proposition 2.25. [3, Lemma 3.1.24]
Let G be a group, V a vector space, and $\rho : G \to \mathrm{GL}(V)$ a representation. Then ρ is reducible if it is equivalent to a reducible representation.

Proposition 2.26. [3, Lemma 3.1.25]
Let G be a group, and $\rho : G \to \mathrm{GL}(V)$ a representation. Then ρ is completely reducible if it is equivalent to a completely reducible representation.

Let V be an n-dimensional vector space over F, let G be a group, and let $\phi : G \to \mathrm{GL}(V)$ be a representation. Choosing a basis $\mathcal{B} = \{e_1, \dots, e_n\}$ of V, we have that ϕ is equivalent to some matrix representation $\rho = T\phi T^{-1} : G \to \mathrm{GL}_n(F)$ where $T : V \to F^n$ is the linear isomorphism taking the chosen basis elements of V to their indices in the column vector in F^n. This means that so long as the vector space for a given representation is of finite dimension, we can consider automorphisms in matrix form and the properties we wish to examine which are constant across equivalent representations will hold [3, page 21].

Through equivalence, unitary representations allow us to understand the structure of many other representations. We will first recall unitary maps from linear algebra.

Definition 2.27. Unitary map [3, Definition 2.2.4]
Recall that a linear map $L : V \to W$ between inner product spaces V, W is said to be *unitary* if $\langle v_1, v_2 \rangle = \langle L(v_1), L(v_2) \rangle$ for all $v_1, v_2 \in V$. We denote the unitary automorphisms of a vector space V as $\mathrm{U}(V)$ which is a subgroup of $\mathrm{GL}(V)$. For maps over an n-dimensional vector space V over $\mathbb{C}$, we can choose a basis of V and have $\mathrm{U}(V) \cong U_n(\mathbb{C}) \coloneqq \left\{A \in \mathrm{M}_n(\mathbb{C}) \ : \ A^\dagger = A^{-1}\right\}$

For the maps in $\mathrm{GL}_1(\mathbb{C})$, a complex number z is unitary if $\bar{z} = z^{-1}$ then $z\bar{z} = |z|^2 = 1$ and $z \in \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$ is the unit circle. Then $\mathrm{U}_1(\mathbb{C}) = \mathbb{T}$ [3, page 20].

Definition 2.28. Unitary representation [3, Definition 3.2.1]
A representation $\rho : G \to \mathrm{GL}(V)$ where V is an inner product space is said to be *unitary* if

ρ_g is unitary for all $g \in G$. Since $\mathrm{U}_1(\mathbb{C}) = \mathbb{T}$, a one-dimensional unitary representation is a homomorphism $\rho : G \to \mathbb{T}$.

Proposition 2.29. [3, Proposition 3.2.3]
Let G be a group and V be a finite-dimensional vector space. A unitary representation $\rho : G \to \mathrm{GL}(V)$ is either irreducible or decomposable.

Proof. [3, Proposition 3.2.3]
If ρ is irreducible then it is clearly indecomposable. Suppose ρ is reducible. Then there exists some non-zero proper G-invariant subspace $W \leqslant V$. We know that $V = W \oplus W^{\perp}$ for some orthogonal complement $W^{\perp}$. We need to show that $W^{\perp}$ is G-invariant. Let $w^{\perp} \in W^{\perp}$, $w \in W$, and $g \in G$. Since ρ is unitary on the inner product $\langle *, * \rangle$ we have

$$\langle \rho_g(w^{\perp}), w \rangle = \langle \rho_{g^{-1}} \rho_g(w^{\perp}), \rho_{g^{-1}}(w) \rangle = \langle \rho_{1_G}(w^{\perp}), \rho_{g^{-1}}(w) \rangle = \langle w^{\perp}, \rho_{g^{-1}}(w) \rangle = 0,$$

with the equality to 0 given since $\rho_{g^{-1}}(w) \in W$. Then $w^{\perp} \in W^{\perp}$, and $W^{\perp}$ is G-invariant. Hence ρ is decomposable. ∎

Proposition 2.30. [3, Proposition 3.2.4]
Every non-zero complex finite degree representation of a finite group G is equivalent to a unitary representation.

Proof. [3, Proposition 3.2.4]
Let G be a finite group, V an n-dimensional vector space over $\mathbb{C}$, and $\phi : G \to \mathrm{GL}(V)$ be a representation. Choosing a basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ of V, we have that ϕ is equivalent to some matrix representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$. Define the map $(*, *) : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that for $v_1, v_2 \in \mathbb{C}^n$ and $g \in G$ we have

$$(v_1, v_2) \coloneqq \sum_{g \in G} \langle \rho_g(v_1), \rho_g(v_2) \rangle,$$

where $\langle *, * \rangle$ is the standard inner product on $\mathbb{C}^n$. This can easily be verified to be an inner product. Furthermore, given an element $h \in G$ we have

$$(\rho_g(v_1), \rho_g(v_2)) = \sum_{g \in G} \langle \rho_g \rho_h(v_1), \rho_g \rho_h(v_2) \rangle = \sum_{g \in G} \langle \rho_{gh}(v_1), \rho_{gh}(v_2) \rangle.$$

Notice that fixing some $h \in G$, the set $\{gh \ : \ g \in G\}$ is equal to G. Then

$$(\rho_g(v_1), \rho_g(v_2)) = \sum_{gh \in G} \langle \rho_{gh}(v_1), \rho_{gh}(v_2) \rangle = (v_1, v_2)$$

and ϕ is equivalent to the unitary representation ρ. ∎

Since a representation equivalent to an irreducible/decomposable representation is itself irreducible/decomposable, and Proposition 2.30 tells us every non-zero complex finite degree representation of a finite group is equivalent to a unitary representation, Proposition 2.29 provides us with the following Corollary:

Corollary 2.31. [3, Corollary 3.2.5]
Every non-zero complex finite degree representation $\rho : G \to \mathrm{GL}(V)$ of a finite group G is either irreducible or decomposable.

This means that a non-zero complex finite degree representation of a finite group is indecomposabile if and only if it is irreducibile. What if we consider more general representations for arbitrary vector spaces and groups? Irreducibility always trivially implies indecomposability since a decomposable representation has at least two non-zero proper subrepresentations, though for more general representations indecomposability does not necessarily imply irreducibility. We will examine this in an example in the next section.

Theorem 2.32. [3, Theorem 3.2.8]

Every non-zero finite degree complex representation of a finite group is completely reducible.

Proof. [3, Theorem 3.2.8]

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a finite group G. We proceed by induction on the degree of ρ. If $\dim(V) = 1$ then ρ is irreducible since $V = \mathbb{C}$ has no non-zero proper subspaces. We assume true our inductive hypothesis: that ρ is irreducible up to some $\dim(V) = k \in \mathbb{N}$. Then let $\rho : G \to \mathrm{GL}(V)$ for $\dim(V) = k+1$. If ρ is irreducible then it is completely reducible and we are done, if not it is decomposable by Corollary 2.31. Then $V = V_1 \oplus V_2$ for non-zero G-invariant subspaces V_1, V_2. By the inductive hypothesis $\dim(V_1), \dim(V_2) < \dim(V)$ which implies $\rho|_{V_1}, \rho :_{V_2}$ are completely reducible. Then $V_1 = U_1 \oplus \cdots \oplus U_{n_U}$ and $V_2 = W_1 \oplus \cdots \oplus W_{n_W}$ where U_i and W_j are G-invariant and the subrepresentations $\rho|_{U_i}, \rho|_{W_j}$ are irreducible for all $1 \leqslant i \leqslant n_U$, $1 \leqslant j \leqslant n_W$. Then $V = U_1 \oplus \cdots \oplus U_{n_U} \oplus W_1 \oplus \cdots \oplus W_{n_W}$ and ρ is completely reducible. ■

Example 2.33.

By Theorem 2.32, every finite degree complex representation of $\mathbb{Z}/n\mathbb{Z}$ is completely reducible for all $n \in \mathbb{N}$.

3 FG-Modules

In this section we will develop FG-modules, which are closely related to representations. It will often be more concise to describe results about representations in the form of results about FG-modules.

3.1 Basic Definitions

In this subsection we will generalize the results so far in terms of FG-modules.

Definition 3.1. FG-module [8, Definition 4.2]
For a vector space V over a field F, and a group G, we say V is a *FG-module* with respect to an operation $* \cdot * : G \times V \to V$ if the following axioms are satisfied for all $v, v' \in V$, $\alpha \in F$, and $g, g' \in G$:

(1) $g \cdot v \in V$,

(2) $(gg') \cdot v = g \cdot (g' \cdot v)$,

(3) $1_G \cdot v = v$,

(4) $g \cdot (\alpha v) = \alpha(g \cdot v)$,

(5) $g \cdot (v + v') = g \cdot v + g \cdot v'$,

As with group operations we will neglect the '$\cdot$' for ease of reading: $gv \coloneqq g \cdot v$.

Note that axioms (1),(4),(5) imply that given an element $g \in G$, the map $g \cdot * : V \to V$ such that $v \mapsto gv$ is a linear endomorphism. Then choosing a basis for V, we can consider such endomorphisms as matrix transformations as the following definition demonstrates.

Definition 3.2. FG-module with chosen basis [8, Definition 4.3]
Given an FG-module V with an n-dimensional basis $\mathcal{B}$, we denote the matrix of the endomorphism $v \mapsto gv$ with respect to $\mathcal{B}$ as $[g]_{\mathcal{B}}$.

Throughout this project, we will examine the symmetric group S_n heavily.

Definition 3.3. Permutation module [8, Definition 4.10]
Let $G \leqslant S_n$ be a subgroup of the symmetric group on n elements. Let V be the FG-module with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ and action

$$\sigma e_i = e_{\sigma(i)} \quad \text{for all } 1 \leqslant i \leqslant n,\ \sigma \in G.$$

We call V the *permutation module* of G over F. Operating with a permuation element on vector simply permutes the basis elements.

The following theorem demonstrates that a representation can intuitively be seen to be an FG-module.

Theorem 3.4. [8, Theorems 4.4(1)]
Let V be an n-dimensional vector space over F, and $\rho : G \to \mathrm{GL}(V)$ be a representation of a group G. The representation ρ becomes an FG-module by defining multiplication with $gv = \rho_g(v)$ for all $g \in G,\ v \in V$.

Proof.
Given an n-dimensional vector space V over F, and our representation $\rho : G \to \mathrm{GL}(V)$ for all $v, v' \in V,\ \alpha \in F,\ g \in G$ we have:

(1) $\rho_g(v) \in V$,

(2) $\rho_{gg'}(v) = \rho_g\rho_{g'}(v)$,

(3) $\rho_{1_G}v = v$,

(4) $\rho_g(\alpha v) = \alpha(\rho_g v)$,

(5) $\rho_g(v + v') = \rho_g(v) + \rho_g(v')$,

Hence $gv \coloneqq \rho_g(v)$ allows V to become an FG-module. ■

Theorem 3.5. [8, Theorems 4.4(2)]
Given an FG-module V with basis $\mathcal{B}$, and a group G, the function $g \mapsto [g]_{\mathcal{B}}$ is a representation of G.

Proof. [8, Theorems 4.4(2)]
Since $(gg')v = g(g'v)$ for all $g, g' \in G,\ v \in V$, we have $[gg']_{\mathcal{B}} = [g]_{\mathcal{B}}[g']_{\mathcal{B}}$. Also, $[gg^{-1}]_{\mathcal{B}} = [1_G]_{\mathcal{B}} = [g]_{\mathcal{B}}[g]_{\mathcal{B}}^{-1}$ which implies $[g^{-1}]_{\mathcal{B}} = [g]_{\mathcal{B}}^{-1}$. So $g \mapsto [g]_{\mathcal{B}}$ is a homomorphism from G to $\mathrm{GL}_n(F)$. ■

Theorems 3.4 and 3.5 demonstrate the connection between FG-modules and representations. For a group representation $\rho : G \to \mathrm{GL}(V)$ we can describe ρ_g as $g \cdot *$. For the remainder of the project we will occasionally alternate between using FG-modules and representations when showing results. Some of the definitions in this section will be restating the results on representations from the last section in terms of results on FG-modules.

FG-modules can be direct summed analogous to the direct sum of representations. This is inherent from the properies of the vector space direct sum and the module operation. Given the direct sum of vector spaces $V = U \oplus W$, with the bases $\mathcal{B}_U = \{u_1, \dots, u_{n_U}\}$ of U and $\mathcal{B}_W = \{w_1, \dots, w_{n_W}\}$ of W, for an element $v \in V$ and $g \in G$ we have

$$\begin{aligned} gv &= g(\alpha_1 u_1 + \cdots + \alpha u_{n_U} + \beta_1 w_1 + \cdots + \beta_{n_W} w_{n_W}) \\ &= \alpha_1 g u_1 + \cdots + \alpha_{n_U} g u_{n_U} + \beta_1 g w_1 + \cdots + \beta_{n_W} g w_{n_W}) \end{aligned}$$

where $\alpha_i, \beta_i \in F$ for all i. In the below definition we describe the endomorphism $g \mapsto gv$ in matrix form $[g]_{\mathcal{B}_V}$ where $\mathcal{B}_V$ is the basis of V.

Definition 3.6. Direct sum of FG-modules [8, page 66]
Let G be a group and U, W be FG-modules with chosen bases $\mathcal{B}_W$,$\mathcal{B}_U$ respectively. If $V = U \oplus W$ is the direct sum of the vector spaces, then V has basis $\mathcal{B}_V = \mathcal{B}_W \cup \mathcal{B}_U$. Then endomorphisms of V are of the form

$$[g]_{\mathcal{B}_V} = \begin{pmatrix} [g]_{\mathcal{B}_U} & 0 \\ 0 & [g]_{\mathcal{B}_W} \end{pmatrix}.$$

More generally, given FG-modules $W_1, \ldots, W_n$ with $V = W_1 \oplus \cdots \oplus W_n$, and bases $\mathcal{B}_V, \mathcal{B}_{W_1}, \ldots, \mathcal{B}_{W_n}$ respectively, we have $\mathcal{B}_V = \mathcal{B}_{W_1} \cup \cdots \cup \mathcal{B}_{W_n}$ and the diagonal matrix

$$[g]_{\mathcal{B}_V} = \begin{pmatrix} [g]_{\mathcal{B}_{W_1}} & & 0 \\ & \ddots & \\ 0 & & [g]_{\mathcal{B}_{W_n}} \end{pmatrix}.$$

Definition 3.7. Trivial FG-module [8, Definitions 4.8(1)]
The *trivial* FG-module is the 1-dimensional vector space V over F such that $gv = v$ for all $v \in V, g \in G$.

Definition 3.8. Faithful FG-module [8, Definitions 4.8(2)]
An FG-module V is said to be *faithful* if it is injective, i.e it has trivial kernel: $gv = v$ implies $g = 1_G$ for all $v \in V$.

The regular FG-module is an important module which will be employed within proofs later in the project.

Definition 3.9. Regular FG-module [8, Definition 6.5]
Let G be a finite group of order n and $F = \mathbb{C}$ or $\mathbb{R}$. The *regular FG-module* V is the vector space over F obtained using elements of g as a basis, that is

$$V := \left\{ \sum_{i \in I} f_i g_i \ : \ f_i \in F,\, g_i \in G, I \subseteq \{1, \ldots n\} \right\},$$

the set of finite sums of elements of G with coefficients in F. This clearly satisfies the axioms of a vector space with a basis given by elements of the group, and as such it is standard to write e_g for element g of the group.

For a general element $v = \sum_{i \in I} f_i e_{g_i}$, we have the natural module operation induced from the group operation: $vg = \sum_{i \in I} f_i e_{g_i g}$.

We can see this operation as permuting the basis group elements.

Proposition 3.10. [8, Proposition 6.6]
The regular FG-module is faithful.

Proof. [8, Proposition 6.6]

Let $g \in G$, and V the regular FG-module. Then for all $v = \sum_{i\in I} f_i e_{g_i} \in V$, suppose $vg = \sum_{i\in I} f_i e_{g_i g} = \sum_{i\in I} f_i e_{g_i} = v$, then by uniqueness of identity $g = 1_G$. ■

A nice outcome of using FG-modules to describe representations is that equivalent representations give rise to the same underlying FG-module with a different choice of basis, as the following theorem demonstrates.

Theorem 3.11. [8, Theorem 4.12]

Let G be a group, let V be an FG-module with finite basis $\mathcal{B}$ and let $\rho : G \to \mathrm{GL}(V)$ be the representation such that $\rho_g = [g]_{\mathcal{B}}$ for all $g \in G$.

(1) If $\mathcal{B}'$ is another basis of V, and we have another representation $\rho' : G \to \mathrm{GL}(V)$ such that $\rho'_g = [g]_{\mathcal{B}'}$, then $\rho \sim \rho'$.

(2) Conversely, if ρ' is a representation equivalent to ρ then there is a basis $\mathcal{B}'$ of V such that $\rho'_g = [g]_{\mathcal{B}'}$.

Proof. [8, Proposition 6.6]

(1) There exists a change of basis matrix T such that $[g]_{\mathcal{B}} = T[g]_{\mathcal{B}'}T^{-1}$.

(2) Since $\rho \sim \rho'$, there exists T such that $\rho_g = T\rho'_g T^{-1}$ for all $g \in G$. Let $\mathcal{B}'$ be a basis of V such that the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ is T, then $[g]_{\mathcal{B}} = T[g]_{\mathcal{B}'}T^{-1}$ for all $g \in G$, and $\rho'_g = [g]_{\mathcal{B}'}$. ■

Definition 3.12. ***FG*-submodule [8, Definition 5.1]**

Let V be an FG-module. A subspace $W \leqslant V$ is called a *FG-submodule* of V if $gw \in W$ for all $g \in G$, $w \in W$. An FG-submodule $W \leqslant V$ is said to be proper if $W \neq V$.

From the definition it is clear that every FG-module V has at least two FG-submodules, both $\{0\}$ and V.

We would like a way to show that two FG-modules share algebraic structure, since it is possible for the module action to differ between isomorphic vector spaces, a linear map is not enough to determine this. FG-homorphisms are linear maps which preserve the strucure of an FG-module.

Definition 3.13. ***FG*-homomorphism [8, Definition 7.1]**

Let V and W be FG-modules. An *FG-homomorphism* is a a linear function $\theta : V \to W$ such that $\theta(gv) = g\theta(v)$ for all $g \in G, v \in V$. If θ is a bijection then we say it is an FG-isomorphism, and write $V \cong W$.

Proposition 3.14. [8, Fact 7.7]

Let V, W be FG-modules. Then $V \cong W$ if and only if there exists a basis $\mathcal{B}_V$ of V and $\mathcal{B}_W$ of W such that

$$[g]_{\mathcal{B}_V} = [g]_{\mathcal{B}_W}.$$

Proof. [8, Fact 7.7]

Suppose $V \cong W$ with the FG-isomorphism $\theta : V \to W$. Let $\mathcal{B}_V = \{v_1, \dots, v_n\}$ be the chosen basis of V, then $\{\theta(v_1), \dots, \theta(v_n)\}$ is a basis $\mathcal{B}_W$ of W. Since $g\theta(v_i) = \theta(gv_i)$ for all $1 \leqslant i \leqslant n$, we have $[g]_{\mathcal{B}_V} = [g]_{\mathcal{B}_W}$ for all $g \in G$. Conversely, assume $\mathcal{B}_V = \{v_1, \dots, v_n\}$ and $\mathcal{B}_W = \{w_1, \dots, w_n\}$ such that $[g]_{\mathcal{B}_V} = [g]_{\mathcal{B}_W}$ for all $g \in G$. Let $\theta : V \to W$ be the linear map such that $\theta(v_i) = w_i$ for all $1 \leqslant i \leqslant n$. We have $[g]_{\mathcal{B}_V} = [g]_{\mathcal{B}_W}$, implying $\theta(gv_i) = g\theta(v_i)$ for all $g \in G$ and $1 \leqslant i \leqslant n$. Then θ is an isomorphism and $V \cong W$. ■

FG-module projections are a unique kind of FG-homomorphism which will enable us to prove intersting properties of FG-submodules in later subsections.

Definition 3.15. ***FG*-module projection [8, Proposition 7.11]**

Given an FG-module V and a collection of FG-submodules $\{W_i\}_{1 \leqslant i \leqslant n}$ such that $V = W_1 \oplus \cdots \oplus W_n$. Let $v \in V$ be any vector, then we can write $v = w_1 + \cdots + w_n$ where $w_i \in W$ for all $1 \leqslant i \leqslant n$. For each $1 \leqslant i \leqslant n$ we define the *FG-module projection* $\pi_i : V \to W_i$ such that $\pi_i(v) = w_i$. This is a projection of vector spaces since $\pi_i^2(v) = \pi_i(w_i) = w_i$, the image is contained in W_i, and the restriction to W_i is the identity $\pi_i \mid_{W_i} = \mathrm{Id}_{W_i}$.

We validate that the above projection is structure preserving.

Proposition 3.16. [8, Proposition 7.11]

The above projection is an FG-module homorphism.

Proof. [8, Proposition 7.11]

Let $V = W_1 \oplus \cdots W_n$ be an FG-module. We verify that $\pi_i : V \to W_i$ is linear for each $1 \leqslant i \leqslant n$. Given two vectors in V:

$$v = w_1 + \cdots + w_n,\ v' = w_1' + \cdots + w_n', \quad \text{with } w_i, w_i' \in W_i \text{ for all } 1 \leqslant i \leqslant n,$$

and scalars $\alpha, \alpha' \in F$, we have $\pi_i(\alpha v + \alpha' v') = \pi(\alpha w_1 + \alpha' w_1' + \cdots + \alpha w_n + \alpha' w_n') = \alpha w_i + \alpha' w_i' = \alpha \pi_i(v) + \alpha' \pi_i(v')$. It is also structure preserving since given $g \in G$ we have $\pi_i(gv) = \pi_i(gw_1 + \cdots + gw_n) = gw_i = g\pi_i(v)$, since $gw_i \in W_i$ for all $1 \leqslant j \leqslant n$. ■

As with structure preserving maps of other algebraic structures the reader may have studied, the Image and Kernel of an FG-homomorphism are FG-submodules. We will use this basic result in proofs in this section.

Proposition 3.17.

For an FG-homomorphism $\theta : V \to W$, $\mathrm{Im}(\theta)$ is a FG-submodule of W and $\mathrm{Ker}(\theta)$ is an FG-submodule of V.

Proof.

Let $w = \theta(v) \in \mathrm{Im}(\theta)$. Then for all $g \in G$, we have $gw = g\theta(v) = \theta(gv)$ implies $gw \in \mathrm{Im}(\theta)$.
Let $v \in \mathrm{Ker}(\theta)$. Then for all $g \in G$, we have $\theta(gv) = g\theta(v) = 0$ implies $gv \in \mathrm{Ker}(\theta)$. ■

3.2 Maschke's Theorem

In this subsection we examine Maschke's Theorem, and expand on the results on reducibility we developed in Section 2. Maschke's Theorem provides us with the specific conditions for a reducible representation to be decomposable - as mentioned in the last section an indecomposable representation is not neccessarily irreducible.

We will now restate reducibility through the lense of FG-modules.

Definition 3.18. Irreducible FG-module [8, Definition 5.3]
A non-zero FG-module V is called *irreducible* if it has no non-zero proper FG-submodules.

We remark that the zero FG-module $V = \{0\}$ is regarded as neither reducible or irreducible, analogous to $1 \in \mathbb{N}$ being neither composite nor prime.

We also rephrase complete reducibility and decomposability in terms of FG-modules.

Definition 3.19. Completely reducible FG-module [8, Definition 8.6]
Let V be an FG-module. We call V *completely reducible* if $V = U_1 \oplus \cdots \oplus U_k$ where U_i is an irreducible proper FG-submodule of V for all $1 \leqslant i \leqslant k$.

Note that in the definition above there is no requirement for elements in the direct sum $U_1 \oplus \cdots \oplus U_k$ to be non-isomorphic, i.e the same irreducible FG-submodule can appear multiple times.

Definition 3.20. Decomposable FG-module
Let V be an FG-module. We call V *decomposable* if there exist non-zero proper FG-submodules $U, W < V$ such that $V = U \oplus W$. If V is not decomposable then it is called *indecomposable*.

Maschke's Theorem gives the conditions on $\mathrm{Char}(F)$, $|G|$ such that reducibility implies decomposability.

Theorem 3.21. Maschke [8, Theorem 8.1]
Let V be an FG-module where G is a finite group, and F a field of a characteristic $\mathrm{Char}(F)$ such that $\mathrm{Char}(F) \nmid |G|$. If there exists a non-zero proper FG-submodule $W < V$ then there exists an FG-submodule U such that $V = W \oplus U$.

To prove Maschke's Theorem we will require the following key fact about projections.

Theorem 3.22.
Given a projection π of V onto a subspace W, we have $V = \mathrm{Ker}(\pi) \oplus \mathrm{Im}(\pi)$.

Proof.
Let $v \in V$, then $v = v + \pi(v) - \pi(v)$. Clearly $\pi(v) \in \mathrm{Im}(v)$. By linearity $\pi(v - \pi(v)) =$

$\pi(v) - \pi^2(v) = \pi(v) - \pi(v) = 0$, then $v - \pi(v) \in \mathrm{Ker}(\pi)$. Then each $V = \mathrm{Ker}(\pi) + \mathrm{Im}(\pi)$. Now we need to prove that this is a direct sum. Let $v \in \mathrm{Ker}(\pi) \cap \mathrm{Im}(\pi)$, then $\pi(v) = 0$ and there exists some $w \in V$ such that $v = \pi(w)$. Then $\pi(v) = \pi^2(w) = \pi(w) = v = 0$, then $\mathrm{Ker}(\pi) \cap \mathrm{Im}(\pi) = \{0\}$ and $V = \mathrm{Ker}(\pi) \oplus \mathrm{Im}(\pi)$. ■

Now we can prove Maschke's Theorem.

Proof. Maschke [9, page 12]

We have a finite group G and an FG-module V over field F with characteristic such that $\mathrm{Char}(F) \nmid |G|$. Let W be an FG-submodule of V. We define $\pi_W : V \to W$ be the projection onto W as a vector space. Take $\tilde{\pi}_W : V \to W$ such that

$$\tilde{\pi}_W(v) := \frac{1}{|G|} \sum_{g \in G} g\pi_W(g^{-1}v).$$

We will verify that this is an FG-homomorphism, even though π_W on its own is just a projection of vector spaces. Linearity follows from linearity of π_W. Given $v, v' \in V, \alpha \in F$ we have

$$\frac{1}{|G|} \sum_{g \in G} g\pi_W(g^{-1}(v + v')) = \frac{1}{|G|} \sum_{g \in G} g\pi_W(g^{-1}v) + \frac{1}{|G|} \sum_{g \in G} g\pi_W(g^{-1}v'),$$
$$\frac{1}{|G|} \sum_{g \in G} g\pi_W(\alpha g^{-1}v) = \frac{\alpha}{|G|} \sum_{g \in G} g\pi_W(g^{-1}v).$$

Now we demonstrate that given $w \in W$, we have $\tilde{\pi}_W(w) = w$. Clearly $g^{-1}w \in W$ for all $g \in G$, since W is a submodule, which then implies $\pi(g^{-1}w) = g^{-1}w$, then

$$\tilde{\pi}_W(w) = \frac{1}{|G|} \sum_{g \in G} g\pi_W(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} gg^{-1}w = \frac{1}{|G|} \sum_{g \in G} w = w.$$

Also, since π_W has its image in W, $\tilde{\pi}_W$ also has its image in W. Then $\tilde{\pi}_W$ is a projection.

We verify structure presevation, given an element $h \in G$,

$$h\tilde{\pi}_W(v) = \frac{1}{|G|} \sum_{g \in G} hg\pi_W(g^{-1}v) = \frac{1}{|G|} \sum_{g \in G} (hg)\pi_W((hg)^{-1}hv).$$

Now let $g' = hg$. Then summing over all g is the same as summing over all g' so

$$\frac{1}{|G|} \sum_{g \in G})(hg)\pi_W((hg)^{-1}hv) = \frac{1}{|G|} \sum_{g' \in G} g'\pi_W(g'^{-1}hv) = \tilde{\pi}_W(hv)$$

and $\tilde{\pi}_W$ is structure preserving, and hence an FG-homomorphism.

By Proposition 3.17, $\mathrm{Ker}(\tilde{\pi}_W)$ is an FG-submodule. By Theorem 3.22 we have

$$V = \mathrm{Im}(\tilde{\pi}_W) \oplus \mathrm{Ker}(\tilde{\pi}_W) = W \oplus \mathrm{Ker}(\tilde{\pi}_W).$$

■

Notice that the finite G condition is required for $\frac{1}{|G|}$ to be defined, also if $\mathrm{Char}(F) \mid |G|$ then $\frac{1}{|G|}$ is undefined since $|G| \equiv 0$ in F.

Proposition 3.23. [8, Theorem 8.7]
Maschke's Theorem 3.21 implies Theorem 2.32 where the field is $\mathbb{C}$ i.e a $\mathbb{C}G$-module V, where G is finite, is completely reducible.

Proof. [8, Theorem 8.7]
We have that $\mathrm{Char}(\mathbb{C}) = 0$. Let V be an n-dimensional non-zero $\mathbb{C}G$-module for a finite group G. We proceed by induction on $\dim(V)$. Suppose $\dim(V) = 1$, then V is trivially irreducible. Suppose V is completely reducible up to $\dim(V) = k$. Then for $\dim(V) = k+1$, if V is irreducible then the result holds, else there exists some non-zero proper $\mathbb{C}G$-submodule $W < V$. By Maschke's Theorem 3.21 there exists another $\mathbb{C}G$-submodule $U < V$ such that $V = W \oplus U$. Since $\dim(W), \dim(U) \leqslant k < \dim V$, both W and U are completely reducible by the inductive hypothesis, then

$$V = W_1 \oplus \cdots \oplus W_{i_W} \oplus U_1 \oplus \cdots \oplus U_{i_U}$$

for some $i_W, i_U \in \mathbb{N}$, and W_r, U_s are irreducible for all $1 \leqslant r \leqslant i_W$, $1 \leqslant s \leqslant i_U$. ■

While the proof of Maschke's Theorem requires the divisibility condition on $|G|$, a natural question would be if Maschke's Theorem will ever hold for $\mathrm{Char}(F) \mid |G|$. As it turns out the answer is no. We provide Example 3.25 to demonstrate this. We first require the following theorem.

Theorem 3.24.
If V is a finite-dimensional vector space over an algebraically closed field F, and $L : V \to V$ is a linear map, then L has at least one eigenvector.

Proof.
Let V be n-dimensional. Since F is an algebraically closed field, by the Fundamental Theorem of Algebra, the characteristic polynomial $\mathrm{Ch}_L(z)$ has a root λ, then $\mathrm{Ch}_L(\lambda) = \det(L - \lambda \,\mathrm{Id}_v) = 0$. Since this determinant is 0 the map $(L - \lambda \,\mathrm{Id}_V)$ is non-invertible and hence has non-trivial kernel. Then there exists some non-zero $v \in \mathrm{Ker}(L - \lambda \,\mathrm{Id}_V)$ with

$$(L - \lambda \,\mathrm{Id}_V)v = 0.$$

Then $L(v) = \lambda v$. ■

The following example shows a reducible representation which is also indecomposable.

Example 3.25.
We present an example of a group and representation such that $\mathrm{Char}(F) \mid |G|$ and Maschke's Theorem does not hold. For the purpose of contradiction assume Maschke's Theorem holds for FG-modules for fields of all characteristics.

Let $C_3 = \{1, g, g^2\}$ be the cyclic group of order 4 and $\overline{F}_3 = \bigcup_{n\in\mathbb{N}} F_{3^n}$ be the algebraic closure of the finite field of three elements F_3. Then $\mathrm{Char}(\overline{F}_3) = 3 \mid 3 = |C_3|$. We define the representation $\rho : C_3 \to \mathrm{GL}(V)$ with vector space V over $\overline{F}_3$ and basis $\mathcal{B} = \{e_1, e_2, e_3\}$ such that in matrix form we have the following linear maps

$$\rho_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_{g^2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

or more specifically

$$\begin{aligned}
&\rho_1(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) = (\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) \\
&\rho_g(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) = (\alpha_3 e_1 + \alpha_1 e_2 + \alpha_2 e_3), \\
&\rho_{g^2}(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) = (\rho_g)^2(\alpha_1 e_1, \alpha_2 e_2, \alpha_3 e_3) = (\alpha_2 e_1 + \alpha_3 e_2 + \alpha_1 e_3)
\end{aligned}$$

for all $\alpha_1, \alpha_2, \alpha_3 \in \overline{F}_3$.

Let $W = \overline{F}_3\{e_1 + e_2 + e_3\}$. Then for all $\alpha \in \overline{F}_3$, we have $\rho_g(\alpha(e_1 + e_2 + e_3)) = \alpha\rho_g(e_1 + e_2 + e_3) = \alpha(e_1 + e_2 + e_3)$, which implies $\rho_{g^2}(\alpha(e_1+e_2+e_3)) = \alpha(e_1+e_2+e_3)$. So for all $w \in W$, we have $\rho_g(w) = w$ implying $\rho_{g^2}(w) = w$, and W is a C_3-invariant subspace, hence by Maschke's Theorem there exists another C_3-invariant subspace U such that $V = W \oplus U$.

By Theorem 3.24, both W and U have at least one eigenvector for ρ_g, and hence V must have at least two eigenvectors for ρ_g. Then calculating the eigenvectors of ρ_g we have

$$\det \begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} = 0 \text{ implies } -(\lambda^3 + 1) = -(\lambda - 1)^3 = 0 \text{ implies } \lambda = 1.$$

Calculating the corresponding eigenvectors we have

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} v = \lambda v = v \text{ implies } v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

With only one eigenvector we have a contradiction, and Maschke's Theorem does not hold.

In the following example we completely reduce a $\mathbb{C}D_6$-module.

Example 3.26. Completely reducing a $\mathbb{C}D_6$-module [8]

Let $G = D_6 = \langle x, y \mid x^3,\, y^2,\, (yx)^2\rangle$ and V be a $\mathbb{C}G$-module. We can decompose V into a direct sum of irreducible $\mathbb{C}G$-submodules.

We define the rotation $\zeta_3 = e^{2\pi i/3}$ and

$$\begin{aligned}
v_0 &= 1_G + x + x^2, & w_0 &= yv_0, \\
v_1 &= 1_G + \zeta_3^2 x + \zeta_3 x^2, & w_1 &= yv_1, \\
v_2 &= 1_G + \zeta_3 x + \zeta_3^2 x^2, & w_2 &= yv_2.
\end{aligned}$$

Notice that for $i = 0, 1, 2$, we have $v_i x = \zeta_3^i v_i$. Then $U_1 = \mathrm{Span}(v_i)$ and $U_2 = \mathrm{Span}(w_i)$ are $\mathbb{C}\langle x \rangle$-modules. Also,

$$\begin{aligned} v_0 y &= w_0, & w_0 y &= v_0, \\ v_1 y &= w_2, & w_1 y &= v_2, \\ v_2 y &= w_1, & w_2 y &= v_1, \end{aligned}$$

and $U_3 = \mathrm{Span}(v_0, w_0), U_4 = \mathrm{Span}(v_1, w_2), U_5 = \mathrm{Span}(v_2, w_1)$ are all $\mathbb{C}\langle y \rangle$-modules.

The $\mathbb{C}G$-submodules U_4 and U_5 are irreducible, and U_3 contains $W_1 = \mathrm{Span}(v_0 + w_0)$ and $W_2 = \mathrm{Span}(v_0 - w_0)$ as $\mathbb{C}G$-submodules. The elements $v_0, v_1, v_2, w_0, w_1, w_2$ form a basis for V and hence we have $V = W_1 \oplus W_2 \oplus U_4 \oplus U_5$. Note that $U_4 \cong U_5$ by the map $v_1 \mapsto w_1$ and $w_2 \mapsto v_2$.

We will later use character theory to prove Proposition 4.29, which will verify that W_1, W_2, U_4 is a complete list of non-isomorphic irreducble $\mathbb{C}G$-submodules.

Letting $\mathcal{B}_{W_1}, \mathcal{B}_{W_2}, \mathcal{B}_{U_4}$ be the bases of W_1, W_2, U_4 respectively, we have

$$\begin{aligned} [x]_{\mathcal{B}_{W_1}} &= 1, & [y]_{\mathcal{B}_{W_1}} &= 1, \\ [x]_{\mathcal{B}_{W_2}} &= 1, & [y]_{\mathcal{B}_{W_2}} &= -1, \\ [x]_{\mathcal{B}_{U_4}} &= \begin{bmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{bmatrix}, & [y]_{\mathcal{B}_{U_4}} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

3.3 Schur's Lemma

In this subsection we prove Schur's Lemma and explore some of the useful results on reducibility it provides.

Lemma 3.27. Schur [8, Lemma 9.1]
Let V and W be irreducible $\mathbb{C}G$-modules. Then

(1) For a $\mathbb{C}G$-homomorphism $\theta : V \to W$, either θ is a $\mathbb{C}G$-isomorphism or $\theta(v) = 0$ for all $v \in V$.

(2) A $\mathbb{C}G$-isomorphism $\theta : V \to V$ is a scalar multiple of the identity isomorphism, that is $\theta = \alpha \operatorname{Id}_V$ with $\alpha \in \mathbb{C}$.

Proof. Schur [8, Lemma 9.1]

(1) Suppose there exists some $v \in V$ such that $\theta(v) \neq 0$. Then $\operatorname{Im}(\theta) \neq \{0\}$. By Proposition 3.17 we know $\operatorname{Im}(\theta)$ is a $\mathbb{C}G$-submodule of W, but W is irreducible so $\operatorname{Im}(\theta) = W$ and θ is surjective. Also by Proposition 3.17, we know $\operatorname{Ker}(\theta)$ is a $\mathbb{C}G$-submodule of V and since $\operatorname{Ker}(\theta) \neq V$ and V is irreducible, $\operatorname{Ker}(\theta) = \{0\}$ so θ is also injective and hence a $\mathbb{C}G$-isomorphism.

(2) By Theorem 3.24, $\theta : V \to V$ must have at least one eigenvalue $\lambda \in \mathbb{C}$, then $\operatorname{Ker}(\theta - \lambda \operatorname{Id}_V) \neq \{0\}$. By Proposition 3.17, $\operatorname{Ker}(\theta - \lambda \operatorname{Id}_V)$ is a $\mathbb{C}G$-submodule of V, but V is irreducible, so $\operatorname{Ker}(\theta - \lambda \operatorname{Id}_V) = V$ and $(\theta - \lambda \operatorname{Id}_V)v = 0$ for all $v \in V$, then $\theta = \lambda \operatorname{Id}_V$.

■

The following proposition will provide us with another way to check if a module is irreducible.

Proposition 3.28. [8, Proposition 9.2]
For a non-zero $\mathbb{C}G$-module V, if every $\mathbb{C}G$-endomorphism on V is a scalar multiple of Id_V then V is irreducible.

Proof. [8, Proposition 9.2]
Suppose for purpose of contradiction that V is a reducible $\mathbb{C}G$-module where every $\mathbb{C}G$-endomorphism on V is a scalar multiple of the identity. Then there exists a non-zero proper $\mathbb{C}G$-submodule $W < V$, and by Maschke's Theorem there exists a proper FG-submodule $U < V$ such that $V = U \oplus W$.

By Proposition 3.16, the projection $\pi_W : V \to V$ such that $\pi_W(u+w) = w$ for all $w \in W$, $u \in U$ is a $\mathbb{C}G$-homomorphism, but π_W isn't a $\mathbb{C}G$-isomorphism or the zero map, and hence it contradicts Schur's Lemma.

Then by contradiciton V is irreducible. ■

Schur's Lemma allows us to prove some essential properties on the representations of finite abelian groups.

Proposition 3.29. [8, Proposition 9.5]
Let G be a finite abelian group. Every irreducible $\mathbb{C}G$-module V has dimension 1.

Proof.
Let G be a finite abelian group and V an irreducible $\mathbb{C}G$-module with basis $\mathcal{B}$. Then for all $g, g' \in G$, $v \in V, \alpha, \alpha' \in \mathbb{C}$ we have

$$gg' = g'g \text{ implies } g'(gv) = (gg'v), \quad g(\alpha v + \alpha' v') = \alpha(gv) + \alpha'(gv').$$

So the map $\theta_g : V \to V$, such that $v \mapsto gv$ for $v \in V$, is a $\mathbb{C}G$-endomorphism for all $g \in G$. Then by Schur's Lemma (1) either

(1) $gv = 0$ for all $v \in V$,

(2) or θ_g is a $\mathbb{C}G$-automorphism and $gv = \lambda v$ for some $\lambda \in \mathbb{C}$.

Then for a given $g \in G$ there exists $\lambda_g \in G$ such that $gv = \lambda_g v$ for all $v \in G$. Every linear subspace $W < V$ is also an $\mathbb{C}G$-submodule since $gw = \lambda_g w \in W$ for all $w \in W$. Then $\dim(V) = 1$, else we could choose any basis element $e_i \in \mathcal{B}$ an get a $\mathbb{C}G$-submodule $\mathbb{C}\{e_i\} \neq V$. ■

We use the the fundamental theorem of finite abelian groups cited below to prove a useful fact about the representations of fintite abelian groups.

Theorem 3.30. Fundamental theorem of finite abelian groups [10, Theorem 1.2]
Let G be a finite abelian group. Then G is isomorphic to a direct product of cyclic groups $C_{n_1} \oplus \cdots \oplus C_{n_k}$.

Theorem 3.31. [8, Theorem 9.8]
Let G be a finite abelian group. Every irreducible representation of G over $\mathbb{C}$ is equivalent to a degree 1 representation, and there are $|G|$ different non-equivalent representations of this type.

Proof. [8, page 81]
By the fundamental theorem of finite abelian groups $G \cong C_{n_1} \oplus \cdots \oplus C_{n_k}$ for k positive integers $n_1, \ldots, n_k$. For each $1 \leqslant i \leqslant k$, let x_i be the generator of C_{n_i} such that $C_{n_i} = \langle x_i \mid x_i^{n_i} \rangle$, and define $g_i \coloneqq \left(1_{C_{n_1}}, \ldots, x_i, \ldots, 1_{C_{n_k}}\right) \in G$, that is g_i is the element with x_i in the ith index and the respective identities in the other indices. Then it is clear that G is generated by all the g_i elements, that is

$$G = \langle g_1, \ldots, g_k \mid g_i^{n_i}, g_i g_j g_i^{-1} g_j^{-1} \text{ for all } 1 \leqslant i, j \leqslant k \rangle,$$

where the relations are implied by the order n_i of g_i and that the cyclic group is abelian.

Let $\rho : G \to \mathrm{GL}_m(\mathbb{C})$ be an irreducible representation of G. Then by Proposition 3.29 we have $m = 1$ and for each $1 \leqslant i \leqslant k$ there exists $\lambda_i \in \mathbb{C}$ such that

$$\rho_{g_i} = \lambda_i \,\mathrm{Id}_{\mathbb{C}^m}\,.$$

The order of each g_i is n_i, so $\lambda_i^{n_i} = 1$ and λ_i is an n_ith root of unity. Also the values of λ_i determine ρ since for any element $g \in G$ can be written as $g = g_1^{r_1} \cdots g_k^{r_k}$ for some $r_1, \cdots, r_k \in \mathbb{Z}$. Hence

$$\rho_g = \rho_{g_1^{r_1} \cdots g_k^{r_k}} = \lambda_1^{r_1} \cdots \lambda_k^{r_k}.$$

Convesely, any map taking elements of G to the n_ith roots of unity ζ_{n_i}

$$g_1^{r_1} \cdots g_k^{r_k} \mapsto \zeta_{n_1}^{r_1} \cdots \zeta_{n_k}^{r_k}$$

is a representation, and there are $n_1 n_2 \cdots n_k = |G|$ different choices of such representations. ■

When examining character theory in the next section it will be useful to be able to select a basis of an FG-module such that the matrix corresponding to a specific group element is diagonal. The following proposition allows us to do this.

Proposition 3.32. [8, proposition 9.11]
Let G be a finite group and V a $\mathbb{C}G$-module. For an element $g \in G$ there is some basis $\mathcal{B}$ of V such that $[g]_{\mathcal{B}}$ is a diagonal matrix. If g has order n then the entries on the the diagonal of $[g]_{\mathcal{B}}$ are nth roots of unity.

Proof. [8, Proposition 9.11]
Let $C_n = \langle x \mid x^n \rangle$ and U be a non-zero $\mathbb{C}C_n$-module. By Theorem 2.32 U is completely reducible and

$$U = W_1 \oplus \cdots \oplus W_m$$

where W_i is an irreducible $\mathbb{C}C_n$-submodule of U for all $1 \leqslant i \leqslant m$. By Proposition 3.29, each W_i has dimension 1. Let w_i be a vector spanning W_i, and ζ_n be an nth root of unity. Then for each $1 \leqslant i \leqslant m$ there exists some integer k_i such that for all $g \in C_n$ we have $gw_i = \zeta_n^{k_i} w_i$. Then choosing a basis $\mathcal{B} = \{w_1, \ldots, w_m\}$ of U we obtain the diagonal matrix

$$[g]_{\mathcal{B}} = \begin{pmatrix} \zeta_n^{k_1} & & & & \\ & \zeta_n^{k_2} & & 0 & \\ & & \ddots & & \\ & 0 & & \zeta_n^{k_{m-1}} & \\ & & & & \zeta_n^{k_m} \end{pmatrix}.$$

Let G be a finite group. For an element $g \in G$ with order m we have $\langle g \rangle \cong C_m$. Let V be a $\mathbb{C}G$-module, then by restiction V is also a $\mathbb{C}\langle g \rangle$-module, and we can obtain the desired basis through the method above.

4 Character Theory

In this section we develop the character theory of group representations. The idea behind character theory is to encode a compex representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ by a complex valued function $\chi : G \to \mathbb{C}$ called a character. We will examine how the values of a character can be used to determine interesting properties about an underlying representation - for example representations are equivalent if and only if they have the same character. Character theory will allow us to develop character tables in the next section.

4.1 Basic Definitions and Results

We begin with a basic introduction to character theory, detailing the essential definitions and immediate results of characters.

Definition 4.1. Character [8, Definition 13.3]
Let G be a group and V a $\mathbb{C}G$-module with basis $\mathcal{B}$. The *character* of V is the function $\chi : G \to \mathbb{C}$ with

$$\chi(g) = \mathrm{Tr}\,[g]_{\mathcal{B}}.$$

Given our connection between $\mathbb{C}G$-modules and representations, we can see the character of an element $g \in G$ with respect to a matrix representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is $\chi(g) = \mathrm{Tr}(\rho_g)$.

We say that χ is a character of G if χ is the character a $\mathbb{C}G$-module.

It may seem that we need to take choice of basis into account when calculating the character of a $\mathbb{C}G$-module, however the following proposition shows us this is not the case.

Proposition 4.2. [8, Definition 13.3]
Let G be a group, and V a $\mathbb{C}G$-module. The character χ of V is constant under choice of basis, since by for bases $\mathcal{B}, \mathcal{B}'$ of V there exists a matrix T such that $[g]_{\mathcal{B}'} = T[g]_{\mathcal{B}}T^{-1}$. One can verify that for matrices $A, B \in \mathrm{M}_n(\mathbb{C})$ with A invertible, we have $\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}(B)$. Then we have $\mathrm{Tr}([g]_{\mathcal{B}'}) = \mathrm{Tr}(T[g]_{\mathcal{B}}T^{-1}) = \mathrm{Tr}([g]_{\mathcal{B}})$.

By Theorem 3.11 the above proposition can be stated in terms of representations: if χ is the character of a representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ then χ is also the character of all representations equivalent to ρ.

Definition 4.3. Degree of a character [8, Definition 13.7]
Let χ be the character of a $\mathbb{C}G$-module V, the *degree* of χ is the dimension of V.

It is important to note that given a $\mathbb{C}G$-module V with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$, since $1_G v = v$ for all $v \in V$, given the character χ of V, we have that $\chi(1_G) = \mathrm{Tr}([1_G]_{\mathcal{B}}) = \mathrm{Tr}(\mathrm{I}_n) = \dim(V)$.

Definition 4.4. Linear character [9, page 21]
A character χ of a $\mathbb{C}G$-module is called *linear* if it has degree 1.

Notice that the trace is the identity when applied to a scalar (1×1 matrix), so given the linear character χ of a $\mathbb{C}G$-module V with basis $\mathcal{B} = \{e_1\}$, we have $\chi(g) = \mathrm{Tr}([g]_{\mathcal{B}}) = [g]_{\mathcal{B}}$ for all $g \in G$. Alternatively, if χ is the character of a linear representation $\rho : G \to \mathrm{GL}_1(\mathbb{C})$ then $\chi(g) = \mathrm{Tr}(\rho_g) = \rho_g$ for all $g \in G$ and $\chi = \rho$. In the next section we will develop a method to find all linear characters of a group G.

As with other maps we can define the image and the kernel of a character.

Definition 4.5. Kernel and Image of a character

Let $\chi : G \to \mathbb{C}$ be a character. Then the kernel and image of of χ are given by

$$\mathrm{Ker}(\chi) = \{g \in G \ : \ \chi(g) = \chi(1_G)\}, \quad \mathrm{Im}(\chi) = \{\chi(g) \in \mathbb{C} \ : \ g \in G\}$$

respectively.

We can give any $\mathbb{C}G$-module the trivial character.

Definition 4.6. Trivial character [8, Examples 13.8 (3)]

We call the character of the trivial $\mathbb{C}G$-module the *trivial character* which has values $\chi(g) = 1$ for all $g \in G$.

The following proposition informs us what values a character of a finite group is limited to.

Proposition 4.7. [8, Proposition 13.9 (2), (3)]

Let G be a finite group, χ be the character of a $\mathbb{C}G$-module V, and $g \in G$ be an element of order m. Then $\chi(g)$ is the sum of mth roots of unity, and we have $\chi(g^{-1}) = \overline{\chi(g)}$.

Proof. [8, Proposition 13.9 (2), (3)]

By Proposition 3.32, there exists a basis $\mathcal{B}$ of V such that

$$[g]_{\mathcal{B}} = \begin{pmatrix} \zeta_1 & & & & \\ & \zeta_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \zeta_{n-1} & \\ & & & & \zeta_n \end{pmatrix}.$$

where each ζ_i is an mth root of unity for each $1 \leqslant i \leqslant n$. Then $\chi(g) = \zeta_1 + \cdots + \zeta_n$ is the sum of roots of unity. We can choose a basis to obtain the above diagonal matrix for any element $g \in G$, and by Proposition the character is constant under choice of basis.

Now for the second claim, that $\chi(g^{-1}) = \overline{\chi(g)}$. We have

$$\left[g^{-1}\right]_{\mathcal{B}} = \begin{pmatrix} \zeta_1^{-1} & & & & \\ & \zeta_2^{-1} & & 0 & \\ & & \ddots & & \\ & 0 & & \zeta_{n-1}^{-1} & \\ & & & & \zeta_n^{-1} \end{pmatrix},$$

and hence $\chi(g^{-1}) = \zeta_1^{-1} + \cdots \zeta_n^{-1}$. Then since for a comlex root of unity ζ we have $\zeta^{-1} = \overline{\zeta}$, we have $\chi(g^{-1}) = \overline{\zeta_1} + \cdots \overline{\zeta_n} = \overline{\chi(g)}$. ■

We will later be able to use irreducible characters to understand the structure of a group.

Definition 4.8. Irreducible character [8, Definition 13.4]
Let G be a group. A character χ of G is called *irreducible* if χ is the character of an irreducible $\mathbb{C}G$-module, and χ is a reducible character of G if it is the character of a reducible $\mathbb{C}G$-module.

Proposition 4.9. [8, Proposition 13.15]
For a character χ of G, the complex conjugate $\overline{\chi}$ is also a character of G. Further, χ is irreducible if and only if $\overline{\chi}$ is irreducible.

Proof. [8, Proposition 13.15]
Let V be an n-dimensional vector space over $\mathbb{C}$ and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Then for choosing a basis $\mathcal{B}$ of V, let $[g]_{\mathcal{B}} = \rho_g$ and

$$\chi(g) = \mathrm{Tr}([g]_{\mathcal{B}}).$$

For a matrix $A \in \mathrm{M}_n(\mathbb{C})$, let $\overline{A}$ be the conjugate matrix such that the entry A_j^i of A gives the correponding entry $\overline{A}_j^i$ of $\overline{A}$. Notice that since for $z, z' \in \mathbb{C}$ we have $\overline{z} \times \overline{z}' = \overline{z \times z'}$, for $A, A' \in \mathrm{M}_n(\mathbb{C})$ we also have $\overline{A \times A'} = \overline{A} \times \overline{A'}$.

This implies that the representation $\overline{\rho}$ obtained from taking the conjugate of ρ defined is also representation. Then since

$$\mathrm{Tr}(\overline{[g]_{\mathcal{B}}}) = \overline{\mathrm{Tr}([g]_{\mathcal{B}})} = \overline{\chi(g)},$$

the character of $\overline{\rho}$ is $\overline{\chi}$. Clearly if ρ is reducible then $\overline{\rho}$ is reducible, then if χ is irreducible so is $\overline{\chi}$, and vice-versa since taking the conjugate of the conjugate gives the original matrix. ■

A useful quality of characters is that they determine $\mathbb{C}G$-modules up to isomorphism.

Proposition 4.10. [8, Proposition 13.5 (1)]
If V, W are isomorphic $\mathbb{C}G$-modules then they have the same character.

Proof. [8, Proposition 13.5 (1)]
Since $V \cong W$, by Proposition 3.14 there is are bases $\mathcal{B}_V$ of V and $\mathcal{B}_W$ of W such that $[g]_{\mathcal{B}_V} = [g]_{\mathcal{B}_W}$ for all $g \in G$. Then $\mathrm{Tr}([g]_{\mathcal{B}_V}) = \mathrm{Tr}([g]_{\mathcal{B}_W})$. ■

As mentioned in the historical overview, Frobenius first defined characters on conjugacy classes without the notion of a group representation, and later defined representations. We will detail the connection between conjugacy classes and characters below.

Proposition 4.11. [8, Proposition 13.5 (2)]
Let G be a group and let χ be the character of a $\mathbb{C}G$-module V. If $g, g' \in G$ are elements of the same conjugacy class then $\chi(g) = \chi(g')$

Proof. [8, Proposition 13.5 (2)]
Since g and g' are conjugate $g = hg'h^{-1}$ for some $h \in G$. Choosing a basis $\mathcal{B}$ for V we have

$$[g]_{\mathcal{B}} = [h]_{\mathcal{B}}[g']_{\mathcal{B}}[h]_{\mathcal{B}}^{-1}.$$

Then since $\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}(B)$ for $A, B \in \mathrm{M}_n(\mathbb{C})$ and invertible A we get $\mathrm{Tr}([g]_{\mathcal{B}}) = \mathrm{Tr}([g']_{\mathcal{B}})$. ■

Note that for a character χ of a group G, if $g \in G$ is conjugate to g^{-1} then by Proposition 4.11 $\chi(g) = \chi(g^{-1}) = \overline{\chi(g)}$. Then $\chi(g) \in \mathbb{R}$. This motivates the following definition.

Definition 4.12. Real conjugacy classes and characters [8, page 263]
Let G be a finite group. An element $g \in G$ is said to be *real* if g is conjugate to g^{-1}, then we call the conjugacy class g^G real. Equivalently, a conjugacy class is *real* if it contains the inverse of every element in it.

A character χ of G is real if $\chi(g) \in \mathbb{R}$ for all $g \in G$.

The following Theorem will enable us to prove a useful fact about the kernel of a character in Theorem 4.14.

Theorem 4.13. [8, 13.11(1)]
Given a representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$, and the character $\chi : G \to \mathbb{C}$ of ρ, for an element $g \in G$ we have $|\chi(g)| = \chi(1_G)$ if and only if there exists $\alpha \in \mathbb{C}$ such that $\rho_g = \alpha\,\mathrm{I}_n$.

Proof. [8, 13.11(1)]
Let $g \in G$, and let g have order m. We have $\rho_g = \lambda\,\mathrm{I}_n$ for some $\lambda \in \mathbb{C}$. When $\rho_g^m = \rho_{1_G} = \mathrm{I}_n = \lambda^n I_n$ and hence λ is an mth root of unity. Then $\chi(g) = \mathrm{Tr}(\lambda\,\mathrm{I}_n) = n\lambda$ and $|\chi(g)| = n = \chi(1_G)$. Conversely, assume that for all $g \in G$, we have $|\chi(g)| = \chi(1)$. By Proposition 3.32 there is a basis $\mathcal{B}$ of $\mathbb{C}^n$ such that

$$[g]_{\mathcal{B}} = \begin{pmatrix} \zeta_1 & & & & \\ & \zeta_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \zeta_{n-1} & \\ & & & & \zeta_n \end{pmatrix}$$

where ζ_i is an mth root of unity for each $1 \leqslant i \leqslant n$. Then $|\chi(g)| = |\zeta_1 + \cdots + \zeta_n| = \chi(1) = n$. For complex numbers $z_1, \ldots, z_n$ we have

$$|z_1 + \cdots + z_n| \leqslant |z_1| + \cdots + |z_n|$$

with equality if and only if $z_i = z_j$ for each $1 \leqslant i, j \leqslant n$. Since $|\zeta_i| = 1$ for each $1 \leqslant i \leqslant n$ and $|\zeta_1 + \cdots + \zeta_n| = n$ we indeed have $z_i = z_j$ for each $1 \leqslant i, j \leqslant n$. Then

$$[g]_{\mathcal{B}} = \zeta_1 \, \mathrm{I}_n \, .$$

Then for all bases $\mathcal{B}'$ of $\mathbb{C}^n$ we have $[g]_{\mathcal{B}'} = \zeta_1 \, \mathrm{I}_n$, and $\rho_g = \zeta_1 \, \mathrm{I}_n$. ■

Theorem 4.14. [8, 13.11(2)]
Let G be a group, and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ a representation with character $\chi : G \to \mathbb{C}$, Then $\mathrm{Ker}(\chi) = \mathrm{Ker}(\rho)$.

Proof. [8, 13.11(2)]
Let $g \in \mathrm{Ker}(\rho)$, then $\rho_g = \mathrm{I}_n$, so $\chi(g) = n = \chi(1)$ and $g \in \mathrm{Ker}(\chi)$. Conversely, suppose $g \in \mathrm{Ker}(\chi)$, by Theorem 4.13 we have $\rho_g = \alpha \, \mathrm{I}_n$ for some $\alpha \in \mathbb{C}$. Then $\chi(g) = \alpha\chi(1_G)$, hence $\alpha = 1$. Then $\rho_g = \mathrm{I}_n$ and $g \in \mathrm{Ker}(\rho)$.
We have $\mathrm{Ker}(\rho)$ is a normal subgroup of G, hence $\mathrm{Ker}(\chi)$ is also a normal subgroup. ■

The following proposition will allow us to see a connection between characters of a $\mathbb{C}G$-module and the characters of its irreducible submodules.

Proposition 4.15. [8, Proposition 13.18]
Let V be a $\mathbb{C}G$-module with

$$V = W_1 \oplus \cdots \oplus W_n,$$

where W_i is an irreducible $\mathbb{C}G$-submodule for each $1 \leqslant i \leqslant n$. Suppose χ_V is the character of V and χ_{W_i} is the character of W_i for each $1 \leqslant i \leqslant n$. Then $\chi_V = (\chi_{W_1} + \cdots + \chi_{W_n}) : G \to \mathbb{C}$.

Proof.
This follows clearly from Definition 3.6. Given irreducible $\mathbb{C}G$-submodules $W_1, \ldots, W_n$ with $V = W_1 \oplus \cdots \oplus W_n$, and bases $\mathcal{B}_V, \mathcal{B}_{W_1}, \ldots, \mathcal{B}_{W_n}$ respectively, we have

$$\mathrm{Tr}([g]_{\mathcal{B}_V}) = \mathrm{Tr}\begin{pmatrix} [g]_{\mathcal{B}_{W_1}} & & & & \\ & [g]_{\mathcal{B}_{W_2}} & & 0 & \\ & & \ddots & & \\ & 0 & & [g]_{\mathcal{B}_{W_{n-1}}} & \\ & & & & [g]_{\mathcal{B}_{W_n}} \end{pmatrix} = \mathrm{Tr}([g]_{\mathcal{B}_{W_1}}) + \cdots + \mathrm{Tr}([g]_{\mathcal{B}_{W_n}}).$$

■

The permutation character is one of the irreducible characters S_n. In later chapters we will be finding all irreducible characters of S_n as Frobenius did in the early days of representation theory.

Definition 4.16. Permutation character [8, page 129]
Let V be the permutation module of $G \leqslant S_n$ over $\mathbb{C}$ with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Then for

$g \in G$, the matrix $[g]_{\mathcal{B}}$ has diagonal entries

$$([g]_{\mathcal{B}})^i_i = \begin{cases} 0 & \text{if } gi \neq i \\ 1 & \text{if } gi = i \end{cases} \quad \text{for all } 1 \leqslant i \leqslant n.$$

We can then see that the character $\pi : G \to \mathbb{C}$ of the permutation module V is defined by

$$\pi(g) \coloneqq |\{i \ : \ 1 \leqslant i \leqslant n,\ gi = i\}|.$$

We define the set

$$\mathrm{Fix}(g) \coloneqq \{i \ : \ 1 \leqslant i \leqslant n,\ gi = i\}.$$

Proposition 4.17. [8, Proposition 13.24]
Let G be a subgroup of S_n. Then the function $\chi : G \to \mathbb{C}$ such that

$$\chi(g) \coloneqq |\mathrm{Fix}(g)| - 1 \quad g \in G$$

is a character of G.

Proof. [8, Proposition 13.24]
Let V be the permutation module of G over $\mathbb{C}$ with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Let $U = \mathrm{Span}(e_1 + \cdots + e_n)$. Then $gu = u$ for all $g \in G$, $u \in U$. Hence U is a $\mathbb{C}G$-submodule of V. U is ismorphic to the trivial $\mathbb{C}G$-module, and hence the character χ_1 of U is the trivial character. Then by Maschke's Theorem there exists a $\mathbb{C}G$-submodule $W < V$ such that $V = U \oplus W$. Let χ_2 be the character of W. Then given the permutation character $\pi(g) = |\mathrm{Fix}(g)|$ and using Proposition 4.15, we have

$$\pi(g) = \chi_1(g) + \chi_2(g) = 1 + \chi_2(g) = |\mathrm{Fix}(g)| \text{ implies } \chi_2(g) = |\mathrm{Fix}(g)| - 1 \quad \text{for all } g \in G.$$

4.2 Inner Products on Characters

In this subsection we will develop the Hermitian inner products on characters, which will aid us later in classifying groups with character theory.

Definition 4.18. Complex space of class functions [9, page 23]
The set of all functions from a group G to $\mathbb{C}$ which are constant on conjugacy classes forms a vector space denoted

$$\mathcal{C}^G = \{\eta : G \to \mathbb{C} \ : \ \eta(hgh^{-1}) = \eta(g) \ \text{ for all } g, h \in G\},$$

with addition $(\eta + \eta')(g) = \eta(g) + \eta'(g)$ and scalar multiplication $(\alpha\eta)(g) = \alpha(x(g))$ for all $\eta, \eta' \in C(G)$, and $\alpha \in \mathbb{C}$. We call this the *complex space of class functions*.

The vector space C_G clearly contains all characters of a group G by Proposition 4.11. Providing this vector space with an inner product will allow us to prove further useful results about characters.

Definition 4.19. Hermitian inner product on $\mathcal{C}^G$ [8, Definition 14.3]
For functions $\eta, \eta' \in \mathcal{C}^G$, we define the *Hermitian inner product*

$$\langle \eta, \eta' \rangle \coloneqq \frac{1}{|G|} \sum_{g \in G} \eta(g)\overline{\eta'(g)}.$$

This is indeed satisfies the axioms of an inner product since:

- $\overline{\langle \eta, \eta' \rangle} = \frac{1}{|G|} \overline{\sum_{g \in G} \eta(g)\overline{\eta'(g)}} = \frac{1}{|G|} \sum_{g \in G} \overline{\eta(g)}\eta'(g) = \langle \eta', \eta \rangle$ for all $\eta, \eta' \in \mathcal{C}^G$.
- $\langle \alpha\eta + \alpha'\eta', \iota \rangle = \frac{1}{|G|} \sum_{g \in G} (\alpha\eta(g) + \alpha'\eta'(g))\overline{\iota(g)} = \alpha\langle \eta, \iota \rangle + \alpha'\langle \eta', \iota \rangle$ for all $\eta, \eta', \iota \in \mathcal{C}^G$ and $\alpha, \alpha' \in \mathbb{C}$.
- For a complex number $\alpha \in \mathbb{C}$ we get a real number $\alpha\overline{\alpha} \geqslant 0$ with $\alpha\overline{\alpha} = 0$ if and only if $\alpha = 0$. Hence $\langle \eta, \eta \rangle = \frac{1}{|G|} \sum_{g \in G} \eta(g)\overline{\eta(g)} \geqslant 0$ with $\langle \eta, \eta \rangle = 0$ if and only if $\eta = 0$.

The Hermitian inner product has unique properties when restricted to class functions. These properties will aid us in proving further properties of characters.

Proposition 4.20. The inner product is symmetric on characters [8, Proposition 14.5 (1)]
Let G be a group, and let χ, χ' be characters G. Then

$$\langle \chi, \chi' \rangle = \langle \chi', \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi'(g^{-1}).$$

Proof. **[8, Proposition 14.5 (1)]**
By Proposition 4.7 we have $\chi'(g^{-1}) = \overline{\chi'(g)}$ for all $g \in G$, hence

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi'(g)} = \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi'(g^{-1}) = \langle \chi', \chi \rangle$$

since summing over g^{-1} is the same summing over g.

Notice this implies the inner product is real when restricted to characters since $\langle \chi, \chi' \rangle = \langle \chi', \chi \rangle = \overline{\langle \chi, \chi' \rangle}$. ■

To prove Proposition 4.22 we will require the below proposition on the order of conjugacy classes.

Proposition 4.21. **[8, Theorem 12.8]**
For a finite group G, and a conjugacy class $g^G \subseteq G$, we have $\left|g^G\right| = \dfrac{|G|}{|C_G(g)|}$ where C_G is the centralizer of G.

Proof. **[8, Theorem 12.8]**
Let G be a finite group, and $g, h \in G$. Let $*\cdot* : G \times G \to G$ be the map $h \cdot g = hgh^{-1}$. This is a group action of G on itself. Cleary $\mathrm{Orb}(g) = g^G$, and $\mathrm{Stab}(g) = \{h \in G \ : hg = gh\} = C_G(g)$, then by the Orbit-Stabilizer Theorem we have

$$\left|g^G\right| = [G : C_G(g)] = \frac{|G|}{|C_G(g)|}.$$

■

We can rewrite the Hermitian inner product in terms of characters

Proposition 4.22. **[8, Proposition 14.5 (2)]**
Let G be a finite group and χ, χ' be characters of G. Let n be the number of conjugacy classes of G, and let $g_1, \ldots, g_n$ be distinct representatives of the conjugacy classes. Then

$$\langle \chi, \chi' \rangle = \sum_{1 \leqslant i \leqslant n} \frac{1}{|C_G(g_i)|} \chi(g_i)\overline{\chi'(g_i)}.$$

Proof. **[8, Proposition 14.5 (2)]**
Note that since χ is constant on elements of the same conjugacy class,

$$\sum_{g \in g_i^G} \chi(g)\overline{\chi'(g)} = \left|g_i^G\right| \chi(g_i)\overline{\chi'(g_i)}.$$

for all conjugacy classes g_i^G of G. Then by Proposition 4.21 we have $\left|g_i^G\right| = \dfrac{|G|}{|C_G(g_i)|}$ and

$$\begin{aligned}\langle \chi, \chi' \rangle &= \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\chi'(g)} = \frac{1}{|G|}\sum_{1\leqslant i\leqslant n}\sum_{g\in g_i^G}\chi(g)\overline{\chi'(g)} \\ &= \sum_{1\leqslant i\leqslant n}\frac{\left|g_i^G\right|}{|G|}\chi(g_i)\overline{\chi'(g_i)} = \sum_{1\leqslant i\leqslant n}\frac{1}{|C_G(g_i)|}\chi(g_i)\overline{\chi'(g_i)}.\end{aligned}$$

■

In the next few propositions we will prove that the characters of a group G form a basis for the vector space of class functions $\mathcal{C}^G$

Theorem 4.23. Irreducible characters are orthonormal [9, page 24]
Let V and V' be isomorphic irreducible $\mathbb{C}G$-modules with characters χ and χ' repsectively, then $\langle \chi, \chi' \rangle = 1$. If V and V' are non-isomorphic irreducible $\mathbb{C}G$-modules then $\langle \chi, \chi' \rangle = 0$.

Proof. [9, page 28]
Let $\mathcal{B}$ be the basis of V and $\mathcal{B}'$ be the basis of V', and let n, n' be the dimensions of V, V' repsectively. Then since $\chi(g) = \mathrm{Tr}([g]_{\mathcal{B}})$ we have

$$\langle \chi', \chi \rangle = \frac{1}{|G|}\sum_{g\in G}\chi'(g)\chi(g^{-1}) = \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{1\leqslant i\leqslant n'\\ 1\leqslant j\leqslant n}}([g]_{\mathcal{B}'})_i^i\left(\left[g^{-1}\right]_{\mathcal{B}}\right)_j^j.$$

Given a linear map $\phi : V \to V'$, we define $\tilde{\phi} : V \to V'$ such that for $v \in V$

$$\tilde{\phi}(v) \coloneqq \frac{1}{|G|}\sum_{g\in G} g^{-1}\phi(gv).$$

We verify that for $h \in G$

$$h^{-1}\tilde{\phi}(hv) = \frac{1}{|G|}\sum_{g\in G}(gh)^{-1}\phi(ghv) = \frac{1}{|G|}\sum_{g'=gh\in G} g'^{-1}\phi(g'v) = \tilde{\phi}(v) \text{ implies } \tilde{\phi}(hv) = h\tilde{\phi}(v).$$

Thus $\tilde{\phi}$ is a $\mathbb{C}G$-homomorphism. We examine the consequences of $V \ncong V'$ and $V \cong V'$.

(1) Suppose $V \ncong V'$. Then by Schur's lemma $\tilde{\phi} = 0$ for any linear ϕ. Let $\phi = E_{ab}$, the matrix with entries $(E_{ab})_j^i = \delta_a^i\delta_b^j$, that is is 1 for the entry (a, b) and 0 for all others. Then

$$\tilde{E}_{ab} = \frac{1}{|G|}\sum_{g\in G}\left[g^{-1}\right]_{\mathcal{B}'}E_{ab}[g]_{\mathcal{B}} = 0,$$

and for each i, j we get

$$\frac{1}{|G|}\sum_{g\in G}\left(\left[g^{-1}\right]_{\mathcal{B}'}E_{ab}[g]_{\mathcal{B}}\right)_j^i = 0 = \frac{1}{|G|}\sum_{g\in G}\left(\left[g^{-1}\right]_{\mathcal{B}'}\right)_a^i([g]_{\mathcal{B}})_j^b.$$

Now selecting $a = i, b = j$ we have

$$\frac{1}{|G|}\sum_{g\in G}\left(\left[g^{-1}\right]_{\mathcal{B}'}\right)^i_i([g]_{\mathcal{B}})^j_j = 0.$$

Then summing over this and applying the same procedure for all $a = i, b = j$ we get that $\langle \chi', \chi\rangle = 0$.

(2) Suppose $V \cong V'$, then by Proposition 4.10 we have $\chi = \chi'$. As proved above, if $\phi : V \to V'$ is linear then $\tilde{\phi} : V \to V'$ is a $\mathbb{C}G$-homomorphism, more specifically a $\mathbb{C}G$-endomorphism since $V \cong V'$. Notice that since $[g^{-1}]_{\mathcal{B}} = [g]_{\mathcal{B}}^{-1}$,

$$\mathrm{Tr}(\tilde{\phi}) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(\left[g^{-1}\right]_{\mathcal{B}}\phi[g]_{\mathcal{B}}) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(\phi) = \mathrm{Tr}(\phi).$$

Since $\tilde{\phi}$ is a endomorphism, by Schur's Lemma we know that $\tilde{\phi} = \lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{C}$. Then

$$\mathrm{Tr}(\phi) = \lambda \,\mathrm{Tr}(\mathrm{Id}_V) = \lambda n$$

where $n = \dim(V)$, then $\lambda = \frac{1}{n}\mathrm{Tr}(\phi)$. Let $\phi = E_{ab}$, then $\mathrm{Tr}(E_{ab}) = \delta^a_b$ and

$$\tilde{E}_{ab} = \frac{1}{|G|}\sum_{g\in G}\left[g^{-1}\right]_{\mathcal{B}}E_{ab}[g]_{\mathcal{B}} = \frac{1}{n}\delta^a_b \operatorname{Id}_V .$$

For the entry (i, j) we get

$$\frac{1}{|G|}\sum_{g\in G}\left(\left[g^{-1}\right]_{\mathcal{B}}\right)^i_a([g]_{\mathcal{B}})^b_j = \frac{1}{n}\delta^a_b\delta^i_j,$$

and setting $a = i, b = j$ we have

$$\frac{1}{G}\sum_{g\in G}\left(\left[g^{-1}\right]_{\mathcal{B}}\right)^i_i([g]_{\mathcal{B}})^j_j = \frac{1}{n}\delta^i_j.$$

Then summing the above over $1 \leqslant i, j \leqslant n$. We obtain $\langle \chi, \chi\rangle = 1$. ■

Theorem 4.24. [8, Theorem 14.23]
Let G be a group and let $\chi_1, \ldots, \chi_n$ be the irreducible characters of G. Then $\chi_1, \ldots, \chi_n$ are linearly independent vectors in $\mathcal{C}^G$.

Proof. [8, Theorem 14.23]
Let $\alpha_i \in \mathbb{C}$ for each $1 \leqslant i \leqslant n$ such that

$$\alpha_1\chi_1 + \cdots + \alpha_n\chi_n = 0.$$

Then by bilinearity and $\langle \chi_i, \chi_j\rangle = \delta^i_j$ we have

$$\langle \alpha_1\chi_1 + \cdots + \alpha_n\chi_n, \chi_i\rangle = 0 = \alpha_i.$$

Hence $\alpha_i = 0$ for each $1 \leqslant i \leqslant n$. ■

Proposition 4.25. [9, page 24]
The characters irreducible characters of a finite group G form a basis of the class functions $\mathcal{C}^G$.

Proof. [9, page 31]
Let G be a finite group. Let $V_1, \ldots, V_n$ be all the non-ismorphic irreducible $\mathbb{C}G$-modules, with irreducible characters $\chi_1, \ldots, \chi_n$ respectively. Let $X = \{\chi_i \,:\, 1 \leqslant i \leqslant n\}$. We seek to prove that $\operatorname{Span}(X) = \mathcal{C}^G$. We know that $\operatorname{Span}(X) \leqslant \mathcal{C}^G$, so $\mathcal{C}^G = \operatorname{Span}(X) \oplus \operatorname{Span}(X)^{\perp}$. Assume that $f \in \operatorname{Span}(X)^{\perp}$ is orthogonal to each χ_i. Then we need only prove $f = 0$ to imply $\mathcal{C}^G = X$.

Assume $f \in \operatorname{Span}(X)^{\perp}$ and $\langle f, \chi_i \rangle = 0$ for each $1 \leqslant i \leqslant n$. Let V be an irreducible $\mathbb{C}G$-module and define $\psi : V \to V$ such that

$$\psi(v) := \frac{1}{|G|} \sum_{g \in G} \overline{f}(g) gv \quad \text{for all } v \in V.$$

Then for all $h \in G$, since $\overline{f}$ is a class function

$$h^{-1}\psi(hv) = \frac{1}{|G|} \sum_{g \in G} \overline{f}(g) h^{-1}ghv = \frac{1}{|G|} \sum_{g \in G} \overline{f}(h^{-1}gh) h^{-1}ghv = \psi(v).$$

Then ψ is a $\mathbb{C}G$-endomorphism. Then since V is irreducible, by Schur's Lemma we have $\psi = \lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{C}$ and

$$\operatorname{Tr}(\lambda \operatorname{Id}_V) = \operatorname{Tr}(\psi) \text{ implies } k\lambda = \operatorname{Tr}\left(\frac{1}{|G|} \sum_{g \in G} \overline{f}(g) g \right) = \frac{1}{|G|} \sum_{g \in G} \overline{f}(g)\chi(g) = \langle f, \chi \rangle = 0,$$

where $k = \dim(V)$. Then $\lambda = 0$ and $\frac{1}{|G|} \sum_{g \in G} \overline{f}(g) g = 0$. This is true for any irreducible $\mathbb{C}G$-module, and hence any $\mathbb{C}G$-module since Maschke's Theorem implies complete reducibility for $\operatorname{Char}(\mathbb{C}) = 0$ and finite G, together with $\chi(V) = \chi(W_1 \oplus \cdots \oplus W_m) = \chi(W_1) + \cdots + \chi(W_m)$.

Let V be the regular $\mathbb{C}G$-module with basis elements being members of the group. Then

$$\psi(e_{1_G}) = \sum_{g \in G} \overline{f}(g) g e_{1_G} = 0.$$

The elements of G which form the basis are linearly independent, so $\overline{f}(g) = 0$ for all $g \in G$, implying $f = 0$ and $\operatorname{Span}(X)^{\perp} = \{0\}$. Then $\mathcal{C}^G = \operatorname{Span}(X) \oplus \{0\} = \operatorname{Span}(X)$.

Theorem 4.26. [8, Theorem 14.17]
Let V be a $\mathbb{C}G$-module for a finite group G. Let $V_1, \ldots, V_n$ be a complete list of non-zero proper $\mathbb{C}G$-submodules of V, with characters $\chi_1, \ldots, \chi_n$ respectively. Then the character ψ of V is of the form

$$\psi = m_1\chi_1 + \cdots m_n\chi_n$$

where m_i is a non-negative integer for each $1 \leqslant i \leqslant n$. Further, for each $1 \leqslant i \leqslant n$ we have $m_i = \langle \psi, \chi_i \rangle$ for each and $\langle \psi, \psi \rangle = \sum_{1 \leqslant i \leqslant n} m_i^2$

Proof. **[8, page 141]**
The $\mathbb{C}G$-module V is completely reducible by Proposition 3.23, and hence

$$V \cong \bigoplus_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m_i}} V_i = \Big(\bigoplus_{1 \leqslant j \leqslant m_1} V_1 \Big) \oplus \cdots \oplus \Big(\bigoplus_{1 \leqslant j \leqslant m_n} V_n \Big)$$

for a collection of irreducible $\mathbb{C}G$-submodules $V_i \, : \, 1 \leqslant i \leqslant n$ with each V_i appearing in the sum m_i times. Then by Proposition 4.15 we have

$$\psi = m_1\chi_1 + \cdots m_n\chi_n.$$

By Theorem 4.23 we have $\langle \chi_i, \chi_j \rangle = \delta^i_j$, then by bilinearlity we have both

$$\begin{aligned}
\langle \psi, \chi_i \rangle &= \langle m_1\chi_1 + \cdots m_n\chi_n, \chi_i \rangle = m_i \quad \text{for each } 1 \leqslant i \leqslant n, \\
\langle \psi, \psi \rangle &= \langle m_1\chi_1 + \cdots + m_n\chi_n, m_1\chi_1 + \cdots + m_n\chi_n \rangle = m_1^2 + \cdots + m_n^2
\end{aligned}$$

■

Given a finite group G and a $\mathbb{C}G$-module V, we can use the Hermitian inner product on the character of V to easily determine if V is irreducible.

Theorem 4.27. **[8, Theorem 14.20]**
Let G be a finite group and a χ be the character of a $\mathbb{C}G$-module V, then V is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

Proof. **[8, Theorem 14.20]**
Suppose χ is irreducible. Then by Theorem 4.23 we have $\langle \chi, \chi \rangle = 1$.

Conversely, suppose $\langle \chi, \chi \rangle = 1$. By Theorem 4.26 we have $\chi = m_1\chi_1 + \cdots + m_n\chi_n$ where χ_i is the character of an irreducible $\mathbb{C}G$-submodule for each $1 \leqslant i \leqslant n$. Also

$$1 = \langle \chi, \chi \rangle = m_1^2 + \cdots + m_n^2.$$

Since m_i are non-negative integers for each $1 \leqslant i \leqslant n$ we have $m_j = 1$ for some single $1 \leqslant j \leqslant n$ with the rest equaling 0. Then $\chi = \chi_j$ and χ is irreducible. ■

We prove the core property of characters, which validates their usefulness: for finite groups G they classify $\mathbb{C}G$-modules up to isomorphism.

Theorem 4.28. **[8, Theorem 14.21]**
Let G be a finite group. Let V and W be $\mathbb{C}G$-modules, with characters χ_V and χ_W respectively, then V and W are isomorphic if and only if $\chi_V = \chi_W$.

Proof. **[8, Theorem 14.21]**
Suppose $V \cong W$, then $\chi_V = \chi_W$ by Proposition 4.10.

Conversely, suppose $\chi_V = \chi_W$. We can write a complete list of non-isomorphic irreducible $\mathbb{C}G$-submodules $X_1, \ldots, X_n$ with characters $\chi_1, \ldots, \chi_n$ respectively. Both V and W are completely reducible by Proposition 3.23 and hence we can decompose V and W into the direct sum of elements in $\{X_i \,:\, 1 \leqslant i \leqslant n\}$,

$$V \cong \bigoplus_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m_i}} X_i = \Big(\bigoplus_{1\leqslant j\leqslant m_1} X_1\Big) \oplus \cdots \oplus \Big(\bigoplus_{1\leqslant j\leqslant m_n} X_n\Big), \quad \text{and}$$

$$W \cong \bigoplus_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant l_i}} X_i = \Big(\bigoplus_{1\leqslant j\leqslant l_1} X_1\Big) \oplus \cdots \oplus \Big(\bigoplus_{1\leqslant j\leqslant l_n} X_n\Big),$$

where for each $1 \leqslant i \leqslant n$ the submodule X_i appears m_i times in the sum isomorphic to V and l_i times for the sum isomorphic to W. Then by Theorem 4.26 and $\chi_V = \chi_W$ we have

$$\chi_V = m_1\chi_1 + \cdots + m_n\chi_n, \quad \chi_W = l_1\chi_1 + \cdots + l_n\chi_n$$

which implies

$$m_i = \langle \chi_V, \chi_i \rangle = \langle \chi_W, \chi_i \rangle = l_i \quad \text{for each } 1 \leqslant i \leqslant n.$$

Then $V \cong W$. ■

The following proposition will demonstrate that the sum of the degrees of all irreducible characters of a finite group is equal to the order of the group. This can be a useful to quickly check if all irreducible characters of a group have been aquired, which is something we will seek to do in the next section. We use the regular character and the Hermitian inner product in an interesting way in the proof.

Proposition 4.29. [9, page 25]

Let G be a finite group and $\chi_1, \cdots, \chi_n$ be all irreducible characters of G. Then

$$|G| = \sum_{1\leqslant i\leqslant n} \chi_i(1_G)^2.$$

Proof. [9, page 25]

Let χ_{reg} be the character of the regular $\mathbb{C}G$-module. Since the regular $\mathbb{C}G$-module takes vector elements and permutes them by group multiplication on the group element basis, we can see $[g]_{\mathbb{B}}$ as a permutation matrix for all $g \in G$ and the basis $\mathcal{B} = G$. Since the trace of a permutation matrix is the number of fixed points under the permutation, $\chi_{\text{reg}}(g)$ is the number of elements fixed by $g \in G$. Then $\chi_{\text{reg}}(1_G) = |G|$, and since $g \neq 1_G$ fixes no elements $\chi_{\text{reg}}(1_G) = 0$. Then

$$\langle \chi_{\text{reg}}, \chi_i \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_{\text{reg}}(g)\overline{\chi_i(g)} = \chi_i(1_G) \quad \text{for all } 1 \leqslant i \leqslant n.$$

By Theorem 4.26 we know that $\chi_{\text{reg}} = m_1\chi_1 + \cdots + m_n\chi_n$ for some non-negative integers $m_1, \ldots, m_n$, combining this with the above we have

$$\langle \chi_{\text{reg}}, \chi_i \rangle = \chi_i(1_G) = m_i \quad \text{for all } 1 \leqslant i \leqslant n,$$

and $\chi_{\text{reg}} = \chi_1(1_G)\chi_1 + \cdots + \chi_n(1_G)\chi_n$. Then

$$\chi_{\text{reg}}(1_G) = \chi_1(1_G)^2 + \cdots + \chi_n(1_G)^2 = |G|.$$

■

5 Character Tables

We have seen that the characters of a group are constant on elements on the same conjugacy class, and that the values of the the character of a $\mathbb{C}G$-module determine the $\mathbb{C}G$-module up to isomorphism. Then the reader may see the convenience in storing the values of the characters of a finite group evaluated on the conjugacy classes in a matrix. Character tables are such a matrix, and we can use them to classify a finite group. In this section we explore character tables, and explicitly calculate the character tables of several finite groups. We will highlight many methods to aquire the irreducible characters of groups.

5.1 Basic Definitions

We begin by defining the character table and providing examples.

Definition 5.1. Character table [**8, Definition 16.1**]

Let G be a finite group and let V be a $\mathbb{C}G$-module. By Theorem 2.32 V is completely reducible with $V = W_1 \oplus \cdots \oplus W_n$ where W_i is an irreducible non-zero proper $\mathbb{C}G$-submodule with character χ_i for each $1 \leqslant i \leqslant n$. Let m be the number of conjugacy classes of G and $g_1, \ldots, g_m$ be distinct representatives of each of the conjugacy classes. The *character table* is the matrix X such that the entry $X^i_j \coloneqq \chi_i(g_j)$ for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.

We number the irreducible characters and conjugacy classes such that χ_1 is the trivial character and $g_1 = 1_G$.

In the above example we define the number of conjugacy classes as m and the number of non-zero proper $\mathbb{C}G$-submodules as n. As it turns out $m = n$, as we prove below.

Proposition 5.2. [**8, Theorem 15.3**]

A character table is a square matrix.

This proposition is really claiming that the set of conjugacy classes has the the same cardinality as the set of irreducible characters, and hence the matrix is square.

Proof.

We proved already that the irreducible characters of G form a basis for $\mathcal{C}_G$.

A class function can be defined by its value on each conjugacy class, and hence the number of conjugacy classes is equal to $\dim(\mathcal{C}_G)$. ■

It is relatively easy to calculate the character table of cyclic groups.

Example 5.3. Character table of C_5

Since the cyclic group $G = C_5 = \langle x \mid x^5 \rangle$ is abelian, it has $|C_5| = 5$ trivial conjugacy classes

$$(1_G)^G = \{1_G\}, \quad (x)^G = \{x\}, \ \ldots \ , \ (x^4)^G = \left\{x^4\right\}.$$

By Theorem 3.31 there has to be five irreducible characters. Let $\zeta_5 = e^{2\pi i/5}$ be the 5th root of unity. We write out our characters directly

	$(1_G)^G$	$(x)^G$	$(x^2)^G$	$(x^3)^G$	$(x^4)^G$
χ_1	1	1	1	1	1
χ_2	1	ζ_5	ζ_5^2	ζ_5^3	ζ_5^4
χ_3	1	ζ_5^2	ζ_5^4	ζ_5	ζ_5^3
χ_4	1	ζ_5^3	ζ_5	ζ_5^4	ζ_5^2
χ_5	1	ζ_5^4	ζ_5^3	ζ_5^2	ζ_5

The character table of D_6 is harder as finding all irreducible $\mathbb{C}D_6$-modules up to isomorphism can be more complicated.

Example 5.4. Character table of D_6 [8, Example 16.5 (1)]
Let $G = D_6 = \langle x, y \mid x^3, y^2, (yx)^2 \rangle$. In Example 3.26 we calculated some non-isomorphic irreducible $\mathbb{C}G$-modules. For $g \in G$, we define

$$\chi_1(g) := \mathrm{Tr}([g]_{\mathcal{B}_{W_1}}),$$
$$\chi_2(g) := \mathrm{Tr}([g]_{\mathcal{B}_{W_2}}),$$
$$\chi_3(g) := \mathrm{Tr}([g]_{\mathcal{B}_{U_4}}).$$

where $\mathcal{B}_{W_1}, \mathcal{B}_{W_2}, \mathcal{B}_{U_4}$ are the bases of the $\mathbb{C}G$-submodules defined in Example 3.26. By direct calculation, the conjugacy classes of G are

$$(1_G)^G = \{1_G\}, \quad y^G = \{y, yx, yx^2\}, \quad x^G = \{x, x^2\}.$$

Since there are three conjugacy classes we know that we have all irreducible characters already. Therefore the character table of G is:

	$(1_G)^G$	$(x)^G$	$(y)^G$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

We will use the fact that a character table is invertible in later proofs.

Proposition 5.5. [8, Proposition 16.2]
The character table of G is invertible.

Proof. [8, Proposition 16.2]
Columns consist of irreducible characters, then by Theorem 4.24 the columns are linearly independent, then the character table is invertible. ■

5.2 Orthogonality Relations

Entries of character tables are related to one another by orthogonality relations, which are useful for determining specific character tables. In this section we will explore these relations.

Theorem 5.6. Row and column orthogonality relations [8, Theorem 16.4]
Let G be a finite group, let $V = W_1 \oplus \cdots \oplus W_n$ be a $\mathbb{C}G$-module, and W_i an irreducible $\mathbb{C}G$-submodule with character χ_i for each $1 \leqslant i \leqslant n$. Let $g_1, \ldots, g_n$ distinct representatives of all the conjugacy classes of G. We've already seen that by Proposition 4.22, for any $i, j \in \{1, \ldots, n\}$,

$$\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \sum_{1 \leqslant k \leqslant n} \frac{1}{|C_G(g_k)|} \chi_i(g_k)\overline{\chi_j(g_k)} = \delta^i_j.$$

We call this the *row orthogonality relation*. We also have the *column orthogonality relation*

$$\sum_{1 \leqslant k \leqslant n} \chi_k(g_i)\overline{\chi_k(g_j)} = \delta^i_j |C_G(g_i)|.$$

As we have already proved the row orthogonality relation, we proceed to prove the column orthogonality relation.

Proof. Column orthogonality relations [9, page 30]
Let X be the character table of V, with entries $X^i_j = \chi_i(g_j)$. From the row orthogonality relation we have

$$\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \sum_{1 \leqslant k \leqslant n} \frac{1}{|C_G(g_k)|} \chi_i(g_k)\overline{\chi_j(g_k)} = \delta^i_j.$$

Let D be the diagonal matrix with entries $D^i_j = \delta^i_j |C_G(g_i)|$, then $(XD^{-1}X^\dagger)^i_j = \langle \chi_i, \chi_j \rangle$ and $XD^{-1}X^\dagger = \mathrm{I}_n$, where $X^\dagger$ is the conjugate transpose of the character table X. This implies $X^{-1} = D^{-1}X^\dagger$ and $\mathrm{I}_n = X^{-1}X = D^{-1}X^\dagger X$, hence $X^\dagger X = D$ or written with respect to entries,

$$D^i_j = \delta^i_j |C_G(g_i)| = \delta^i_j |C_G(g_j)| = \sum_{1 \leqslant k \leqslant n} (X^\dagger)^i_k X^k_j = \sum_{1 \leqslant k \leqslant n} (\overline{X})^k_i X^k_j = \sum_{1 \leqslant k \leqslant n} \overline{\chi_k(g_i)} \chi_k(g_j).$$

Example 5.7. [8, Examples 16.5 (1)]
Let $G = D_6 = \langle x, y \mid x^3,\, y^2,\, (yx)^2 \rangle$. The calculating some values of the centralizer we have

$$|C_G(1_G)| = |G| = 6, \quad |C_G(x)| = \left|\left\{x, x^2, 1_G\right\}\right| = 3, \quad |C_G(y)| = |\{y, 1_G\}| = 2.$$

Recall from Example 5.4 that the character table of $G = D_6$ is:

	$(1_G)^G$	x^G	y^G
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

Then labeling $g_1 = 1_G,\ g_2 = x$, and $g_3 = y$, we use column orthogonality relations to calculate $\sum\limits_{1 \leqslant k \leqslant 3} \chi_k(g_i)\overline{\chi_k(g_j)}$ for different i and j:

$$
\begin{aligned}
i \neq j \text{ implies } \sum_{1 \leqslant k \leqslant 3} \chi_k(g_i)\overline{\chi_k(g_j)} &= 0, \\
i = j = 1 \text{ implies } \sum_{1 \leqslant k \leqslant 3} \chi_k(g_1)\overline{\chi_k(g_1)} &= 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 2 = 6 = |C_G(g_1)|, \\
i = j = 2 \text{ implies } \sum_{1 \leqslant k \leqslant 3} \chi_k(g_2)\overline{\chi_k(g_2)} &= 1 \cdot 1 + 1 \cdot 1 + -1 \cdot -1 = 3 = |C_G(g_2)|, \\
i = j = 3 \text{ implies } \sum_{1 \leqslant k \leqslant 3} \chi_k(g_3)\overline{\chi_k(g_3)} &= 1 \cdot 1 + 1 \cdot -1 = 2 = |C_G(g_3)|.
\end{aligned}
$$

We can verify the row orthogonality relations:

$$
\begin{aligned}
\langle \chi_1, \chi_1 \rangle &= \frac{1^2}{6} + \frac{1^2}{3} + \frac{1^2}{2} = 1, \\
\langle \chi_2, \chi_2 \rangle &= \frac{1^2}{6} + \frac{1^2}{3} + \frac{(-1)^2}{2} = 1, \\
\langle \chi_3, \chi_3 \rangle &= \frac{2^2}{6} + \frac{(-1)^2}{3} + \frac{0}{2} = 1.
\end{aligned}
$$

Orthogonality relations are a useful tool for deriving more complicated character tables.

5.3 Lifting Characters

For a finite group G, and a normal subgroup $N \neq \{1_G\}$, the quotient group G/N will be smaller than G, and hence characters should be easier to calculate on G/N than G. In this subsection we will explore the process of lifting characters, a method to use the characters of G/N to find the characters of G.

Proposition 5.8. [8, Proposition 17.1]

Let G be a finite group and $N \trianglelefteq G$. Given a character $\tilde{\chi}$ of G/N, define the function χ on G such that

$$\chi(g) = \tilde{\chi}(gN) \quad \text{for all } g \in G.$$

Then χ is a character of G with the same degree as $\tilde{\chi}$.

Proof. [8, Proposition 17.1]

Let $\tilde{\chi}$ be the character of a representation $\tilde{\rho} : G/N \to \mathrm{GL}_n(\mathbb{C})$ of G/N. Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be the function defined by $\rho_g \coloneqq \tilde{\rho}_{gN}$ for every element $g \in G$. We can see ρ is a homomorphism, and hence a representation, with

$$\rho_{g_1}\rho_{g_2} = \tilde{\rho}_{g_1N}\tilde{\rho}_{g_2N} = \tilde{\rho}_{g_1g_2N}.$$

A character χ of ρ gives

$$\chi(g) = \mathrm{Tr}(\rho_g) = \mathrm{Tr}(\tilde{\rho}_{gN}) = \tilde{\chi}(gN) \quad \text{for all } g \in G.$$

Then $\chi(1_G) = \chi(\tilde{N})$. Then χ and $\tilde{\chi}$ have the same degree.

This motivates us to formally define the lift χ of the character $\tilde{\chi}$.

Definition 5.9. Lift of a character [8, Definition 17.2]

Let G be a finite group with normal subgroup $N \trianglelefteq G$, and let $\tilde{\chi}$ be the character of G/N. We call the character χ of a G defined by

$$\chi(g) \coloneqq \tilde{\chi}(gN) \quad \text{for all } g \in G$$

the *lift* of $\tilde{\chi}$ to G.

The character lift has the following useful property.

Theorem 5.10. [8, Theorem 17.3]

Let G be a finite group with normal subgroup $N \trianglelefteq G$. We can associate each character $\tilde{\chi}_j$ of G/N with the corresponding lift χ_j of a G, and form a bijection between the set of characters $\tilde{\chi}$ and the set of characters χ which satisfy $N \leqslant \mathrm{Ker}\,\chi$.

Further, this bijection maps irreducible characters to irreducible characters.

Proof. [8, Theorem 17.3]

Let $\tilde{\chi}$ be the character of a representation $\tilde{\rho} : G \to \mathrm{GL}_m(\mathbb{C})$ and χ its lift. We have

$\tilde{\chi}(N) = \chi(1_G)$. For an element $n \in N$ we have $\chi(n) = \tilde{\chi}(nN) = \tilde{\chi}(N) = \chi(1_G)$. Then $N \leqslant \mathrm{Ker}(\chi)$.

Conversely, let χ be the character of a representation $\rho : G \to \mathrm{GL}_m(\mathbb{C})$ and suppose $N \leqslant \mathrm{Ker}(\chi)$. Given $g_1, g_2 \in G$ such that $g_1N = g_2N$ we have $g_1^{-1}g_2 \in N$, hence $\rho_{g_1^{-1}g_2} = \mathrm{I}_m = \rho_{g_1}^{-1}\rho_{g_2}$ which implies $\rho_{g_1} = \rho_{g_2}$. Then the function $\tilde{\rho} : G/N \to \mathrm{GL}_m(\mathbb{C})$ with

$$\tilde{\rho}_{gN} \coloneqq \rho_g \quad \text{for all } g \in G$$

is will defined. We verify $\tilde{\rho}$ is a homomorphism with

$$\tilde{\rho}_{(g_1N)(g_2N)} = \tilde{\rho}_{g_1g_2N} = \rho_{g_1g_2} = \rho_{g_1}\rho_{g_2} = \tilde{\rho}_{g_1N}\tilde{\rho}_{g_2N} \quad \text{for all } g_1, g_2 \in G,$$

and hence it is a representation. If $\tilde{\chi}$ is the character of $\tilde{\rho}$ then

$$\tilde{\chi}(gN) = \mathrm{Tr}(\tilde{\rho}_{gN}) = \mathrm{Tr}(\rho_g) = \chi(g)$$

and χ is is the lift of $\tilde{\chi}$.

We've established the bijection between the characters $\{\chi : G \to \mathbb{C} \ : \ N \leqslant \mathrm{Ker}(\chi)\}$ and the characters $\{\tilde{\chi} : G/N \to \mathbb{C}\}$.

We now show that irreducible characters are mapped to irreducible characters under this bijection. Let U be any subspace of $\mathbb{C}^m$. Then by the above bijection

$$\rho_g(u) \in U \ \text{ for all } u \in U \text{ if and only if } \tilde{\rho}_{gN}(u) \in U \ \text{ for all } u \in U,$$

and U is a $\mathbb{C}G$-submodule of $\mathbb{C}^m$ if and only if U is a $\mathbb{C}(G/N)$-submodule of $\mathbb{C}^m$. Then ρ is irreducible if and only if $\tilde{\rho}$ is irreducible.

We will calculate the character table of a quotient group of S_4, and lift them to characters of S_4.

Example 5.11. Character lift for S_4 [8, Example 17.4]
Let $G = S_4$ and let

$$N = V_4 \coloneqq \{1_G, (12)(34), (13)(24)(14)(23)\} = (1_G)^G \cup ((12)(24))^G,$$

which is a normal subgroup of G. Let $x \coloneqq (123)N$ and $y \coloneqq (12)N$, then

$$G/N = \langle x, y \mid x^3, y^2, (xy)^2 \rangle \cong D_6.$$

Then by Example 5.4 the character table of G/N is:

	N	$(123)N$	$(12)N$
$\tilde{\chi}_1$	1	1	1
$\tilde{\chi}_2$	1	1	-1
$\tilde{\chi}_3$	2	-1	0

We need to calculate the lift χ of each character $\tilde{\chi}$. Since

$$(12)(34) \in N \implies \chi((12)(34)) = \tilde{\chi}(N),$$
$$(1234)N = (13)N \implies \chi((1234)) = \tilde{\chi}((13)N),$$

the lifts of $\tilde{\chi}_1$,$\tilde{\chi}_2$,$\tilde{\chi}_3$ given by χ_1, χ_2, χ_3 respectively are

	1_{S_4}	(123)	(12)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	2	-1	0	2	0

Furthermore, since $\tilde{\chi}_1$,$\tilde{\chi}_2$,$\tilde{\chi}_3$ are irreducible, so are χ_1, χ_2, χ_3.

5.4 Linear Characters

In this subsection, by exploring the derived subgroups, we will develop a method to aquire all linear characters of a group.

We begin by recalling the definition of the derived subgroup of a group.

Definition 5.12. Derived subgroup [8, Definition 17.7]
Let G be a group, and let $G' \leqslant G$ be the subgroup

$$G' := \{g_1^{-1}g_2^{-1}g_1g_2 \ : \ g_1, g_2 \in G\}.$$

We call G' the *derived subgroup* of G. For simplicity of notation, given elements $g_1, g_2 \in G$ we write $g_1^{-1}g_2^{-1}g_1g_2$ as the commutator bracket $[g_1, g_2]$.

Example 5.13. [8, Examples 17.8 (2)]
Let $G = S_3$. Then For any elements $g_1, g_2 \in G$, the element $[g_1, g_2]$ is an even permutation, hence $G' \leqslant A_3$. For $g_1 = (12)$ and $g_2 = (23)$, we have $[g_1, g_2] = (123)$. Since $\langle(123)\rangle = A_3$, we have that $A_3 \leqslant G'$. Hence $G' = A_3$.

Proposition 5.14. [8, Proposition 17.9]
Let G be a group and G' its derived subgroup. If χ is a linear character of G then $G' \leqslant \mathrm{Ker}(\chi)$.

Proof. [8, Proposition 17.9]
Since χ is linear it is the trace of a one-dimensional representation from G to $\mathrm{GL}_1(\mathbb{C})$, which implies χ itself is a one-dimensional representation and hence a homomorphism. Then for all $g_1, g_2 \in G$ we have

$$\chi(g_1^{-1}g_2^{-1}g_1g_2) = \chi(g_1)^{-1}\chi(g_2)^{-1}\chi(g_1)\chi(g_2) = 1,$$

and $G' \leqslant \mathrm{Ker}(\chi)$.

The following propositions will help us combine character lifts with derived subgroups.

Proposition 5.15. [8, Proposition 17.10 (1)]
The derived subgroup of a group is a normal subgroup.

Proof. [8, Proposition 17.10 (1)]
Let G be a group and G' its derived subgroup. For all $g_1, g_2, h \in G$ we have

$$h^{-1}(g_1g_2)h = (h^{-1}g_1h)(h^{-1}g_2h), \quad h^{-1}g_1^{-1}h = (h^{-1}g_1h)^{-1}.$$

To prove $G' \trianglelefteq G$ we need to prove $h^{-1}[g_1, g_2]h \in G'$ for all $g_1, g_2, h \in G$, which we can see with

$$\begin{aligned} h^{-1}[g_1, g_2]h &= h^{-1}g_1^{-1}g_2^{-1}g_1g_2h \\ &= (h^{-1}g_1h)^{-1}(h^{-1}g_2h)^{-1}(h^{-1}g_1h)(h^{-1}g_2h) \\ &= [h^{-1}g_1h, h^{-1}g_2h] \in G'. \end{aligned}$$

Proposition 5.16. [8, Proposition 17.10 (2)]
Let G be a group, and $G' \trianglelefteq G$ its derived subgroup. Given a normal subgroup $N \trianglelefteq G$, we have $G' \leqslant N$ if and only if G/N is abelian. Further, G/G' is abelian.

Proof. [8, Proposition 17.10 (2)]
For elements $g_1, g_2 \in G$, we have

$$g_1^{-1}g_2^{-1}g_1g_2 \in N \text{ if and only if } g_1g_2N = g_2g_1N \text{ if and only if } (g_1N)(g_2N) = (g_2N)(g_1N).$$

Therefore $G' \leqslant N$ if and only if G/N is abelian. Then by Proposition 5.15 we have that G/G' is abelian. ■

The above propositions imply that G' is the smallest normal subgroup of G with abelian quotient group G/G'. In the below theorem we tie together linear characters and lifts of G.

Theorem 5.17. [8, Theorem 17.11]
Let G be a finite group and G' its derived subgroup. The linear characters $\{\epsilon : G \to \mathbb{C}\}$ are the lifts to G of the irreducible characters of G/G'. Also, there is a bijection between the distinct linear characters of G and the elements of G/G'.

Proof. [8, Theorem 17.11]
Let $n = |G/G'|$. By Proposition 5.16, G/G' is abelian so by Theorem 3.31 we know G/G' has n irreducible characters $\tilde{\chi}_1, \dots, \tilde{\chi}_n$, all of which are degree 1. The lifts $\chi_1, \dots, \chi_n$ of $\tilde{\chi}_1, \dots, \tilde{\chi}_n$ respectively also have degree 1, and by Theorem 5.10 they are the irreducible characters of G with G' in their kernel. Then Proposition 5.14 implies $\chi_1, \dots, \chi_n$ are linear characters of G. ■

We seek to obtain all linear characters of S_n. To do this we will require Theorem 5.20, which we will prove below.

Proposition 5.18. [8, Proposition 12.13]
Let k and n both be positive integers with $k \leqslant n$, and let $A \coloneqq \{a_1, \cdots, a_k\} \subseteq \{1, \dots, n\}$ be a collection of distinct integers from 1 to n. For a k-cycle $\sigma = (a_1a_2 \dots a_k) \in S_n$ and an element $g \in S_n$, we have

$$g\sigma g^{-1} = (g(a_1)\ g(a_2)\ \dots\ g(a_k)),$$

where $g(a_i)$ is the integer g permutes a_i to for each $a_i \in A$.

Proof. [8, Proposition 12.13]
For any element $a_i \in A$ we have

$$(g\sigma g^{-1})ga_i = g\sigma a_i = ga_{(i+1 \bmod k)}.$$

For any $b \notin A$ with $1 \leqslant b \leqslant n$ we have

$$(g\sigma g^{-1})gb = g\sigma b = gb.$$

Then $g(a_1 \dots a_k)g^{-1} = (g(a_1)\ \dots\ g(a_k))$. ■

Definition 5.19. Cycle shape [8, Proposition 12.13]
We can write any permutation $\sigma \in S_n$ as the product of disjoint cycles

$$\sigma = (a_1 \dots a_{k_1})(b_1 \dots b_{k_2}) \cdots (c_1 \dots c_{k_r})$$

with $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_r$. We call the r-tuple $(k_1, \dots, k_r)$ the *cycle shape* of σ.

Theorem 5.20. [8, Theorem 12.15]
Let $G = S_n$. For a permutation $\sigma \in G$, the conjugacy class σ^G is equal to the set of all permutations with the same cycle shape as σ.

Proof. [8, Proposition 12.13]
Let $\sigma \in G$ be a permutation. We can decompose σ into the product of disjoint cycles

$$\sigma = (a_1 \dots a_{k_1})(b_1 \dots b_{k_2}) \cdots (c_1 \dots c_{k_r})$$

with $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_r$. By Proposition 5.18, for an element $g \in G$ we have

$$\begin{aligned} g\sigma g^{-1} &= (g(a_1 \dots a_{k_1})g^{-1})(g(b_1 \dots b_{k_2})g^{-1}) \cdots (g(c_1 \dots c_{k_r})g^{-1}) \\ &= (g(a_1) \ \dots \ g(a_{k_1}))(g(b_1) \ \dots \ g(b_{k_2})) \cdots (g(c_1) \ \dots \ g(c_{k_r})). \end{aligned}$$

From this we can see that $g\sigma g^{-1}$ has the same cycle shape as σ. Suppose two elements $\sigma, \sigma' \in G$ have the same cycle numbers with decompostion

$$\begin{aligned} \sigma &= (a_1 \dots a_{k_1})(b_1 \dots b_{k_2}) \cdots (c_1 \dots c_{k_r}), \\ \sigma' &= (a'_1 \dots a'_{k_1})(b'_1 \dots b'_{k_2}) \cdots (c'_1 \dots c'_{k_r}). \end{aligned}$$

Let $g \in G$ be the permutation sending $a_1 \mapsto a'_1, \dots, c_{k_r} \mapsto c'_{k_r}$. Then by the above $g\sigma g^{-1} = \sigma'$.
■

We now have the tools neccesary to calculate the linear characters of S_n using lifting.

Example 5.21. Linear characters of S_n [8, Example 17.12]
Let $G = S_n$ and let G' be the derived subgroup of G. If $n = 1, 2$ then G is abelian and $G' = \{1_G\} = A_n$. By Example 5.13 for $n = 3$ we have $G' = A_3$. Then we assume $n \geqslant 4$. Since $S_n/A_n \cong C_2$, by Proposition 5.16 we have $G' \leqslant A_n$. Letting $g_1 = (12)$, $g_2 = (23)$, and $g_3 = (12)(34)$ we have

$$[g_1, g_2] = (123), \quad [g_2, g_3] = (14)(23).$$

Because $G' \triangleleft G$, we have $(123)^G, ((14)(23))^G \subset G'$. Hence by Theorem 5.20, G' contains all 3-cycles and elements of cycle shape $(2, 2)$. Since every product of two transpositions is the identity, a 3-cycle, or an element of cycle shape $(2, 2)$, and A_n contains even products of transpositions, we can conclude $A_n \leqslant G'$. Then $G' = A_n$.

Then $G/G' = S_n/A_n = \{A_n, (12)A_n\} \cong C_2$, and the group G/G' has two linear characters

$$\tilde{\chi}_1((12)A_n) = 1, \quad \tilde{\chi}_2((12)A_n) = -1.$$

Then by Theorem 5.17 we find that S_n has two linear characters, precisely the lifts χ_1, χ_2 of $\tilde{\chi}_1, \tilde{\chi}_2$:

$$\chi_1(g) = 1, \quad \chi_2(g) = \begin{cases} 1 & \text{if } g \in A_n \\ -1 & \text{if } g \notin A_n \end{cases} \quad \text{for all } g \in G.$$

Linear characters also allow us to form new characters, by taking their products with other representations, a method we will now examine.

Proposition 5.22. [8, Proposition 17.14]

Let G be a group, let $\chi : G \to \mathbb{C}$ be a character, and let $\epsilon : G \to \mathbb{C}$ be a linear character. Then the product $\chi\epsilon$ defined by

$$\chi\epsilon(g) \coloneqq \chi(g)\epsilon(g) \quad \text{for all } g \in G$$

is a character of G. Furthermore, if χ is irreducible then $\chi\epsilon$ is irreducible.

Proof. [8, Proposition 17.14]

Let χ be the character of a representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$. We define $\rho\epsilon : G \to \mathrm{GL}_n(\mathbb{C})$ with

$$\rho\epsilon_g = \epsilon(g)\rho_g \quad \text{for all } g \in G.$$

Then the matrix of the representation $\rho\epsilon(g)$ is the matrix of $\rho(g)$ multiplied by $\epsilon(g)$. We've already shown that linear charactes are group homomorphisms, then ρ, ϵ are both homomorphisms and $\rho\epsilon$ is also a homomorphism with

$$\rho\epsilon_{(g_1 g_2)} = \rho_{(g_1 g_2)}\epsilon_{(g_1 g_2)} = \rho_{(g_1)}\epsilon_{(g_1)}\rho_{(g_2)}\epsilon_{(g_2)} = \rho\epsilon_{(g_1)}\rho\epsilon_{(g_2)} \quad \text{for all } g_1, g_2 \in G.$$

Note that since $\epsilon(g)$ is a scalar, $\mathrm{Tr}(\rho\epsilon_g) = \epsilon(g)\,\mathrm{Tr}(\rho_g) = \epsilon(g)\chi(g)$. Then $\rho\epsilon$ is a representation of G with character $\chi\epsilon$.

Now we show that if χ is irreducible then $\chi\epsilon$ is irreducible. We know that $\epsilon(g)$ is a root of unity for all $g \in G$. Hence $\epsilon(g)\overline{\epsilon(g)} = 1$. Then

$$\begin{aligned} \langle \chi\epsilon, \chi\epsilon \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\epsilon(g)\overline{\chi(g)\epsilon(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} = \langle \chi, \chi \rangle. \end{aligned}$$

Then by Theorem 4.27,

$$\chi \text{ is irreducible if and only if } \langle \chi, \chi \rangle = 1 \text{ if and only if } \langle \chi\epsilon, \chi\epsilon \rangle = 1 \text{ if and only if } \chi\epsilon \text{ is irreducible.}$$

We can now finally calculate the full character table of S_4.

Example 5.23. Character table of S_4 [8, Example 18.1]

Let $G = S_4$. By direct calculation, the conjugacy classes of G are

- $(1_G)^G = \{1_G\}$,
- $(12)^G = \{(12), (13), (14), (23), (24), (34)\}$,
- $(123)^G = \{(123), (132), (124), (142), (134), (143), (234), (243)\}$,
- $((12)(34))^G = \{(12)(34), (13)(24), (14)(23)\}$,
- $(1234)^G = \{(1234), (1342), (1423), (1243), (1432), (1324)\}$.

In Example 5.11 we found three irreducible characters χ_1, χ_2, χ_3, and in Proposition 4.17 we found the character $\chi_4(g) = |\mathrm{Fix}(g)| - 1$ for all $g \in G$. All of which have the following values on conjugacy classes:

	1_G	(123)	(12)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	2	-1	0	2	0
χ_4	3	0	1	-1	-1

Since the number of conjugacy classes is equal to the number of irreducible characters, we only require one more irreducible character to complete the character table. In Example 5.21 we found linear irreducible characters, written here as χ_1, χ_2. We showed in Proposition 5.22 that the product of an irreducible character and a linear character is also a character. Hence the product $\chi_4\chi_2$ is also a character, with values

	1_G	(123)	(12)	(12)(34)	(1234)
$\chi_4\chi_2$	3	0	-1	-1	1

Noting the orders of the values of the centralizers we have

	1_G	(123)	(12)	(12)(34)	(1234)
$\lvert C_G \rvert$	24	3	4	8	4

Then calculating the inner product, by Proposition 4.22

$$\langle \chi_4\chi_2, \chi_4\chi_2 \rangle = \frac{3^2}{24} + \frac{(-1)^2}{4} + \frac{(-1)^2}{8} + \frac{1}{4} = 1.$$

Then by Theorem 4.27 we have $\chi_5 = \chi_4\chi_2$ is irreducible and the character table of G is

	1_G	(123)	(12)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	2	-1	0	2	0
χ_4	3	0	1	-1	-1
χ_5	3	0	-1	-1	1

5.5 Further Examples

In this subsection we will work calculate the character tables of A_4 and $\mathrm{SL}_2(F_3)$ for the finite field F_3.

Example 5.24. Character table of A_4 [8, Example 18.2]

Recall $G = A_4$ is the group of even permutations on 4 elements, we write elements in cycle notation.

By direct calculation, there are $n = 4$ conjugacy classes:

- $g_1^G = (1_G)^G = \{1_G\}$,
- $g_2^G = ((12)(34))^G = \{(12)(34), (13)(24), (14)(23)\}$,
- $g_3^G = (123)^G = \{(123), (134), (142), (243)\}$,
- $g_4^G = (132)^G = \{(132), (234), (143), (124)\}$.

Then since $|C_G(g_i)| = \dfrac{|G|}{|g_i^G|}$ we have

	1_G	$(12)(34)$	(123)	(132)
$\lvert C_G \rvert$	12	4	3	3

Let χ_4 be the character found in Proposition 4.17 defined by $\chi_4(g) = |\mathrm{Fix}(g)| - 1$ for all $g \in G$. Then calculating the values of χ_4 on conjugacy classes we have

	1_G	$(12)(34)$	(123)	(132)
χ_4	3	-1	0	0

Hence

$$\langle \chi_4, \chi_4 \rangle = \frac{3^2}{12} + \frac{(-1)^2}{4} = 1$$

and χ_4 is an irreducible character by Theorem 4.27, with degree $\chi_4(1_G) = 3$. Since there are four conjugacy classes, by the column orthogonality relation, the sum of the square of degrees gives us

$$\sum_{1 \leqslant k \leqslant 4} \chi_k(1_G)\overline{\chi_k(1_G)} = |C_G(1_G)| = 12.$$

The squares of the degrees of the remaining three characters must sum to $12 - 3^2 = 3$, hence all three remaining characters are linear characters. Let G' be the derived subgroup of G. Then by Theorem 5.17 we have $|G/G'| = 3$. By direct calculation we can show

$$G' = V_4 = \{1_G, (12)(34), (13)(24), (14)(23)\},$$

which implies $G/G' = \{G', G'(123), G'(132)\} \cong C_3$. Letting ζ_3 be the 3rd root of unity we write the character table of C_3:

	G'	$G'(123)$	$G'(132)$
$\tilde{\chi}_1$	1	1	1
$\tilde{\chi}_2$	1	ζ_3	ζ_3^2
$\tilde{\chi}_3$	1	ζ_3^2	ζ_3

Then we can lift the characters of G/G' to G and we are left with the compete character table of G:

	1_G	(12)(34)	(123)	(132)
χ_1	1	1	1	1
χ_2	1	1	ζ_3	ζ_3^2
χ_3	1	1	ζ_3^2	ζ_3
χ_4	3	-1	0	0

To calculate the character table of $\mathrm{SL}_2(F_3)$ we require the following three propositions.

Proposition 5.25. [8, Theorem 23.1]
The number of real irreducible characters of a finite group G is equal to the number of real conjugacy classes.

Proof. [8, Theorem 23.1]
Let G be a finite group and V a $\mathbb{C}G$-module. Let X be the character table of V and $\overline{X}$ its complex conjugate. By Proposition 4.9, the complex conjugate character table $\overline{X}$ contains the same characters as X. Then $PX = \tilde{X}$ for some permutation matrix P. For each conjugacy class g^G, the entries in the column in X corresponding to g are the complex conjugates of the entries corresponding to the column of g^{-1}. Then $\overline{X}$ can be obtained by permuting the columns of X, and there exists a permutation matrix Q such that $XQ = \overline{X}$. Then by Propostion 5.5 X is invertible and

$$Q = X^{-1}\overline{X} = X^{-1}P\overline{X}.$$

Then $\mathrm{Tr}(Q) = \mathrm{Tr}(P)$, and since trace of a permutation matrix is equal to the number of points fixed by the permuation matrix, we have that the number of irreducible charaters equal to $\mathrm{Tr}(P)$, and the number of real conjugacy classes is equal to $\mathrm{Tr}(Q)$. These numbers have been shown to be equal.

In the interest of brevity we cite without proof the following two propositions. These are neccessary for the derivation of the character table of $\mathrm{SL}_2(F_3)$.

Proposition 5.26. [8, Corollary 22.27]
Let G be a finite group. Let p be a prime number and $g \in G$ be an element with order p^n for some positive integer n. Then if χ is a character of G with $\chi(g) \in \mathbb{Z}$ then

$$\chi(g) \equiv \chi(1_G) \mod p$$

Proposition 5.27. [8, Excercise 5 Chapter 13]
Let χ be an irreducible character of a group G, and let z be an element in the center $Z(G) := \{h \in G \ : \ gh = hg \ \text{ for all } g \in G\}$ of order m. Then there exists an mth root of unity ζ_m such that

$$\chi(zg) = \zeta_m \chi(g) \quad \text{ for all } g \in G.$$

In general, for a field F, the special linear group $\mathrm{SL}_n(F)$ is the subgroup of $\mathrm{GL}_n(F)$ of matrices with deteminant $1 \in F$. In the below example we calculate the character table of $\mathrm{SL}_2(F_3)$ where $F_3 = \{-1, 0, 1\}$ is the finite field of three elements.

Example 5.28. $\mathrm{SL}_2(F_3)$ **[8, page 439]**

Let F_3 be the finite field of 3 elements. By direct calculation $G = \mathrm{SL}_2(F_3)$ has 24 elements and 7 conjugacy classes:

- $g_1^G = (1_G)^G = \{1_G\}$
- $g_2^G = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^G = \left\{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$,
- $g_3^G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^G = \left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\right\}$,
- $g_4^G = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^G = \left\{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\right\}$,
- $g_5^G = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}^G = \left\{\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\right\}$,
- $g_6^G = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}^G = \left\{\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right\}$,
- $g_7^G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^G = \left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\right\}$,

where 1_G is the identity matrix I_2. Then there are 7 characters $\chi_1, \ldots, \chi_7$, where χ_1 is the trivial character. Notice that the vector space $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ has four 1-dimensional subspaces

$$U_1 = \mathrm{Span}((0,1)),\ U_2 = \mathrm{Span}((1,1)),\ U_3 = \mathrm{Span}((2,1)),\ U_4 = \mathrm{Span}((1,0)),$$

which G permutes. Then there is a homomorphism $\phi : G \to S_4$ with $\mathrm{Ker}(\phi) = \{\mathrm{I}_2, -\mathrm{I}_2\}$. Hence by the First Isomorphism Theorem $G/\mathrm{Ker}(\phi) \cong \mathrm{Im}(\phi)$, which is a subgroup of S_4 of order 12. Hence $G/\mathrm{Ker}(\phi) \cong A_4$. Then we can lift the characters of A_4 from Example 5.24 to give characters

	1_G	g_2	g_3	g_4	g_5	g_6	g_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	ζ_3	ζ_3^2	ζ_3^2	ζ_3	1
χ_3	1	1	ζ_3^2	ζ_3	ζ_3	ζ_3^2	1
χ_4	3	3	0	0	0	0	-1

where ζ_3 is the 3rd root of unity. By Proposition 4.29 we have $\chi_5(1_G)^2 + \chi_6(1_G)^2 + \chi_7(1_G)^2 = 12$ therefore all the remaining characters have degree 2. Noting the centralizers

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
$\lvert C_G\rvert$	24	24	6	6	6	6	4

we can use the column orthogonality relation to find the value of the remaining characters on g_2,g_7 and find $\chi_5(g_2) = \chi_6(g_2) = \chi_7(g_2) = -2$ and $\chi_5(g_7) = \chi_6(g_7) = \chi_7(g_7) = 0$. There are three real conjugacy classes, so by Proposition 5.25 there are three real characters. Then only one of our remaining three characters is real, say χ_5. Then $\chi_5(g_3) = x \in \mathbb{R}$ and by Proposition 5.26 we have $x \neq 0$. Then the products $\chi_5\chi_2$ and $\chi_5\chi_3$ are irreducible by Proposition 5.22 and both have degree 2. Then $\chi_6 = \chi_5\chi_2$ and $\chi_7 = \chi_5\chi_3$ are the last irreducible characters in the table, with results $\chi_6(g_3) = x\zeta_3$ and $\chi_7(g_3) = x\zeta_3^2$. By the row orthogonality relation we have

$$\sum_{1\leqslant i\leqslant 7} \chi_i(g_3)\overline{\chi_i(g_3)} = 1+1+1+0+3x\overline{x} = 6 \implies x\overline{x} = 1 \implies x = 1, -1.$$

Then by Proposition 5.26 we have $\chi_5(g_3) \equiv \chi_5(1_G) \mod 3$ which implies $x = -1$. Then by Proposition 5.27 we have $\chi_i(g_6) = -\chi_i(g_3)$ for $i = 5, 6, 7$. Furthermore we know $\chi_i(g_4) = \overline{\chi_i(g_3)}$ and $\chi_i(g_5) = \overline{\chi_i(g_6)}$ for all $1 \leqslant i \leqslant 7$. Then our character table is:

	1_G	g_2	g_3	g_4	g_5	g_6	g_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	ζ_3	ζ_3^2	ζ_3^2	ζ_3	1
χ_3	1	1	ζ_3^2	ζ_3	ζ_3	ζ_3^2	1
χ_4	3	3	0	0	0	0	-1
χ_5	2	-2	-1	-1	1	1	0
χ_6	2	-2	$-\zeta_3$	$-\zeta_3^2$	ζ_3^2	ζ_3	0
χ_7	2	-2	$-\zeta_3^2$	$-\zeta_3$	ζ_3	ζ_3^2	0

6 Conclusion

We have introduced representation theory and character theory for finite groups to the reader and examined multiple interesting examples.

Representation theory is a broad field and there is far more theory beyond what we have examined. The reader may be interested in continuing their study of representation theory of finite groups with the following areas:

- Induced characters and modules [8, Chapter 21].
- The proof of Burnside's theorem [8, Chapter 31].
- The theory of modular characters [11, Chapter 18].
- Linear representations of compact groups [11, Chapter 4].
- Young diagrams and Frobenius' character formula [12, Chapter 4].

By studying induced characters the reader will be equipped to find more complicated character tables such as that of the group $\mathrm{GL}_2(F_q)$ where q is a positive integer power of some prime p and F_q is the finite field of q elements.

Finite group representation theory is still under research today. In the wake of the classification of simple finite groups, 21st century algebraists have sought to understand the representation theory of simple finite groups over fields of arbitrary characteristic [13, page 1]. As we highlighted in the introduction, representation theory is currently seen in a wide range of mathematical contexts and it is sure to be researched well into the future.

7 References

1. Curtis, C. *Pioneers of Representation Theory: Frobenius, Burnside, Schur, and Brauer* ISBN: 0821890026 (American Mathematical Society, 1999).
2. Wussing, H. *The Genesis of the Abstract Group Concept* ISBN: 0486458687 (Dover, 2007).
3. Steinberg, B. *Represesenation Theory of Finite Groups* ISBN: 9781461407751 (Springer, 2012).
4. Gorenstein, D. *The Classification of Finite Simple Groups I. Simple Groups and Local Analysis* (American Mathematical Society Bulletin, 1979).
5. Jones, H. F. *Groups, Representations and Physics* ISBN: 0750305045 (Institute of Physics, 1998).
6. Jeevanjee, N. *An Introduction to Tensors and Group Theory for Physicists* ISBN: 9783319147932 (Springer, 2015).
7. Davidoff, G., Sarnak, P. & Valette, A. *Elementary Number Theory, Group Theory, and Ramanujan Graphs* ISBN: 9780521531436 (Cambridge University Press, 2003).
8. James, G. & Liebeck, M. *Representations and Characters of Groups* ISBN: 052100392X (Cambridge University Press, 2001).
9. Chua, D. *Part II Represesenation Theory Notes* `https://dec41.user.srcf.net/notes/II_L/representation_theory.pdf`. (Cambridge University Class, 2015).
10. Grillet, P. A. *Abstract Algebra* ISBN: 139780387715674 (Springer, 2007).
11. Serre, J. *Linear Representations of Finite Groups* ISBN: 0387901906 (Springer, 1977).
12. Fulton, W. & Harris, J. *Representation Theory: A First Course* ISBN: 9780387974958 (Springer, 1991).
13. Misra, K. C., Nakano, D. K. & Parshall, B. J. *Recent Developments in Lie Algebras, Groups and Representation Theory* ISBN: 2012026116 (American Mathematical Society, 2012).