MSc Project: Representations of Finite Groups

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Abstract

In this project we provide an introduction to representation theory of finite groups, and go on to show its usefulness in the classification of simple finite groups.

Contents

1 Glossary of Notation

- $\mathrm{M}_n(F)$ The set of $n \times n$ matrices with entries in field F.
- I_n The $n \times n$ identity matrix.
- $\mathrm{GL}_n(F)$ The set of invertible $n \times n$ matrices with entries in field F.
- $\mathrm{GL}(V)$ The set of linear automorphisms on vector space V.
- Id_X Identity morphism of a space X.
- C_n Cyclic group of order n.
- D_{2n} Dihedral group of order $2n$.
- S_n Symmetric group of order n.
- 1_X The identity of a space X with algebraic structure.

2 Introduction

3 Background

3.1 Group Theory

Definition 3.1. Conjugacy class [1]

For elements $g, h \in G$. g is conjugate to h if $\exists x \in G$ such that $h = x^{-1}gx$. The conjugacy class of g is the equivalence class $g^G = \{x^{-1}gx \mid x \in G\}$.

Proposition 3.2.

Given two conjugacy classes g^G, h^G in G, either $g^G = h^G$ or $g^G \cap^G = \varnothing$.

Proof.

TODO

Definition 3.3. Distinct conjugacy classes

If $g^G \cap h^G = \varnothing$ then we say g^G and h^G are distinct.

Definition 3.4. Representatives of conjugacy classes [1]

Given distinct conjugacy classes $g_1^G, \dots, g_n^G$ such that $G = g_1^G \cup \cdots \cup g_n^G$, we call $g_1, \dots, g_n$ representatives of the conjugacy classes of G.

3.2 The Linear Algebra Recap and the General Linear Group

Theorem 3.5.
Given a projection π of V onto a subspace W, we have $V = \mathrm{Ker}(\pi) \oplus \mathrm{Im}(\pi)$.

Definition 3.6. Trace of a matrix
The trace of a matrix $A \in \mathrm{M}_n(F)$ is the sum of the diagonal elements $\mathrm{Tr}(A) = \sum_{i=1}^{n} A_i^i$.

Theorem 3.7.
If V is a finite dimensional vector space over an algebraically closed field F, and $L : V \to V$ is a linear map, then L has at least one eigenvector.

Proposition 3.8.
Let $A, B \in \mathrm{M}_n(F)$. Then

- $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$,
- $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Also, if T is invertable then $\mathrm{Tr}(T^{-1}AT) = \mathrm{Tr}(A)$.

Definition 3.9. Endomorphisms and Automorphisms
Given a space V with an algebraic structure, an endomorphism is defined to be a homomorphism from V to itself. We denote $\mathrm{End}(V) \coloneqq \mathrm{Hom}(V, V)$ to be the set of all endomorphisms on V. An automorphism is an endomorphism which is also an isomorphism, and we denote $\mathrm{Aut}(V) \coloneqq \{\phi \in \mathrm{End}(V) \mid \phi \text{ is an isomorphism}\}$ as the set of all automorphisms on V.

In this project, we focus on vector space endomorphisms - linear maps from a space to itself.

Definition 3.10. Projection [1]
Linear map π from V to a subspace W is called a projection if and only if it satisfies $\pi^2 = \pi$, $\mathrm{Im}(\pi) = W$, $\pi|_W = \mathrm{Id}_W$.

Definition 3.11. General linear group [2]
Let V be a vector space. We define

$$\mathrm{GL}(V) \coloneqq \mathrm{Aut}(V)$$

to be the set of invertable linear endomorphisms over V. We prove this is a group under composition.

Proof.
Associativity: Composition is always associative.
Existance of inverse elements: ϕ an isomorphism $\iff$ ϕ invertible. Hence every element of $\mathrm{Aut}(V)$ has an inverse.
Closedness: The composition of linear maps is linear, and the composition of bijective maps is bijective. Therefore $\mathrm{Aut}(V)$ is closed under composition.
Existence of identity: The identity map is linear and bijective, hence in $\mathrm{Aut}(V)$. ■

Proposition 3.12.

If V is an n-dimensional vector space over $\mathbb{C}$ then there is a group isomorphism

$$\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C}) := \{A \in \mathrm{M}_n(\mathbb{C}) \mid A \text{ is invertable}\},$$

the group of invertable $n \times n$ matrices.

Proof.

Let V be an n-dimensional vector space over $\mathbb{C}$ and fix a basis $e = \{e_1, \ldots, e_n\}$. Recall that the result of a linear transformation is entirely determined by its result on basis elements (once a basis is chosen). Then for $L : V \to V$, we can write the result of L on basis element e_k as $L(e_k) = \alpha_k^1 e_1 + \cdots + \alpha_k^n e_n$. Let $\phi : \mathrm{Aut}(V) \to \mathrm{M}_n(\mathbb{C})$ such that

$$\phi(L) = \begin{pmatrix} \alpha_1^1 & \alpha_2^1 & \ldots & \alpha_n^1 \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^n & \alpha_2^n & \ldots & \alpha_n^n \end{pmatrix}.$$

Any matrix has a corresponding linear map which sends the kth basis vector to another vector with the basis coefficients made up of the scalars in the kth column. Also, since any linear map is determined by the vectors that the basis elements are mapped to, there is a unique linear map with columns as the coefficients. Then ϕ is a bijection.

Now we show that ϕ is an homomorphism. Given $L_1(e_k) = \alpha_k^1 e_1 + \cdots + \alpha_k^n e_n$ and $L_2(e_k) = \beta_k^1 e_1 + \cdots + \beta_k^n e_n$, we have

$$\begin{aligned} L_2 \circ L_1(e_k) &= L_2(\alpha_k^1 e_1 + \cdots + \alpha_k^n e_n) = \alpha_k^1 L_2(e_1) + \cdots + \alpha_k^n L_2(e_n) \\ &= \alpha_k^1(\beta_1^1 e_1 + \cdots + \beta_1^n e_n) + \cdots + \alpha_k^n(\beta_n^1 e_1 + \cdots + \beta_n^n e_n) \\ &= e_1(\alpha_k^1 \beta_1^1 + \cdots + \alpha_k^n \beta_n^1) + \cdots e_n(\alpha_k^1 \beta_1^n + \cdots + \alpha_k^n \beta_n^n). \end{aligned}$$

Therefore

$$\begin{aligned} \phi(L_2 \circ L_1) &= \begin{pmatrix} (\alpha_1^1\beta_1^1 + \cdots + \alpha_1^n\beta_n^1) & (\alpha_2^1\beta_1^1 + \cdots + \alpha_2^n\beta_n^1) & \ldots & (\alpha_n^1\beta_1^1 + \cdots + \alpha_n^n\beta_n^1) \\ (\alpha_1^1\beta_1^2 + \cdots + \alpha_1^n\beta_n^2) & (\alpha_2^1\beta_1^2 + \cdots + \alpha_2^n\beta_n^2) & \cdots & (\alpha_n^1\beta_1^2 + \cdots + \alpha_n^n\beta_n^2) \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha_1^1\beta_1^n + \cdots + \alpha_1^n\beta_n^n) & (\alpha_2^1\beta_1^n + \cdots + \alpha_2^n\beta_n^n) & \ldots & (\alpha_n^1\beta_1^n + \cdots + \alpha_n^n\beta_n^n) \end{pmatrix} \\ &= \begin{pmatrix} \beta_1^1 & \beta_2^1 & \ldots & \beta_n^1 \\ \beta_1^2 & \beta_2^2 & \cdots & \beta_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^n & \beta_2^n & \ldots & \beta_n^n \end{pmatrix} \times \begin{pmatrix} \alpha_1^1 & \alpha_2^1 & \ldots & \alpha_n^1 \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^n & \alpha_2^n & \ldots & \alpha_n^n \end{pmatrix} = \phi(L_2) \times \phi(L_1). \blacksquare \end{aligned}$$

Definition 3.13. Unitary map

A linear map $L : V \to W$ between inner product spaces V, W is said to be unitary if and only if $\langle v_1, v_2 \rangle = \langle L(v_1), L(v_2) \rangle \ \forall v_1, v_2 \in V$. We denote the unitary maps of a vector space V as $\mathrm{U}(V)$, and for maps over an n-dimensional vector space over $\mathbb{C}$, we have $U(V) \cong U_n(\mathbb{C}) := \{A \in \mathrm{M}_n(\mathbb{C}) \mid A^* = A^{-1}\}$ where A^* is the standard conjugate transpose.

We will denote the invertable elements of a ring R as R^*, then $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$. Then $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and not the dual space of $\mathbb{C}$ as is traditional.

Example 3.14. [2]

For the maps $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$, a complex number z is unitary if $\bar{z} = z^1 \implies z\bar{z} = |z|^2 = 1 \implies z \in \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle. Then $U_1(\mathbb{C}) = \mathbb{T}$

Theorem 3.15. Caley-Hamilton [2]

Let A be a matrix with characteristic polynomial $p_A(x)$. Then $p_A(A) = 0$.

Definition 3.16. Minimal polynomial [2]

For an endomorphism $A \in \mathrm{End}(V)$, the minimal polynomial of A, $m_A(x)$ is the smallest degree monic polynomial $f(x)$ such that $f(A) = 0$

Theorem 3.17. [2]

A matrix $A \in \mathrm{M}_n(\mathbb{C})$ is diagonalizable if and only if all factors of $m_A(x)$ have multiplicity 1.

Theorem 3.18. Spectral theorem [2]

For a self adjoint $A \in \mathrm{M}_n(\mathbb{C})$, there exists a unitary matrix $U \in U_n(\mathbb{C})$ such that U^*AU is diagonal. The eigenvalues of A are real.

TODO prove

4 Representation Theory

4.1 Group Representations

Definition 4.1. Group representation [2]

A representation of a group G is a group homomorphism $\phi : G \to \mathrm{GL}(V)$ for some finite dimensional vector space V. The degree of ϕ is defined to be the dimension of V.

Remark 4.2.

Recall the group action on a set X is a map $* \cdot * : G \times X \to X$ such that

(1) $1 \cdot x = x \ \forall x \in X$,

(2) $g \cdot (g' \cdot x) = gg' \cdot x \ \forall g, g' \in G, \, x \in X$.

We can then regard a representation as a form of group action since for a representation $\phi : G \to \mathrm{GL}(V)$ we satisfy the axioms of a group action

(1) $\phi(1)v = Iv = v \ \forall v \in V$, where 1 and I are the identities of G and $GL(V)$ respectively,

(2) $\phi(g)\phi(g')v = \phi(gg') \ \forall g, g' \in G, \, v \in V$.

Definition 4.3. Trivial representation [2]

Any group can be given the trivial representation $\phi : G \to \mathrm{GL}_1(\mathbb{C})$ such that $\phi(g) = 1 \ \forall g \in G$.

Definition 4.4. Zero representation [2]

Any group can be given the zero representation $\phi : G \to \mathrm{GL}_1(\mathbb{C})$ such that $\phi(g) = 0 \ \forall g \in G$.

Example 4.5. [2]

$\phi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ such that $\phi([m]) = e^{2\pi i m/n} \ \forall [m] \in \mathbb{Z}/n\mathbb{Z}$ is a representation.

Definition 4.6. Representation equivalence [2]

Two representations $\phi : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(W)$ are said to be equivalent $\phi \sim \psi$ if there exists a linear isomorphism $T : V \to W$ such that $\psi(g)T = T\phi(g) \ \forall g \in G$, and we have the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi(g)} & V \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ W & \xrightarrow{\psi(g)} & W \end{array}$$

Proposition 4.7. [1]

Representation equivalence is an equivalence relation.

Proof.

Let $\phi^1 : G \to \mathrm{GL}(V_1)$, $\phi^2 : G \to \mathrm{GL}(V_2)$, and $\phi^3 : G \to \mathrm{GL}(V_3)$ be representations.

- Reflexive: Let Id be the identity map $\mathrm{Id}(e_i) = e_i$, which is a linear isomorphism. Then $\phi_g \,\mathrm{Id} = \mathrm{Id}\,\phi_g \forall g \in G$ and $\phi \sim \phi$.

- Symmetric: Suppose $\phi^1 \sim \phi^2$ with linear isomorphism T, then $\exists T^{-1}$ which is also an isomorphism and $\phi^1_g T = T\phi^2_g \implies \phi^2_g T^{-1} = T^{-1}\phi^1_g \implies \phi^2 \sim \phi^1$.
- Transitive: Let $\phi^1 \sim \phi^2$ with isomorphism T_{12} and $\phi^2 \sim \phi^3$ with T_{23}. Then $T_{12} \circ T_{23}$ is also a linear isomorphism and $\phi^1_g T_{12}T_{23} = T_{12}\phi^2_g T_{23} = T_{12}T_{23}\phi^3_g \ \forall g \in G \implies \phi^1 \sim \phi^3$. ■

Example 4.8. [2]
Let $\phi : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ with

$$\phi([m]) = \begin{pmatrix} \cos(2\pi m/n) & -\sin(2\pi m/n) \\ \sin(2\pi m/n) & \cos(2\pi m/n) \end{pmatrix},$$

the rotation matrix by angle $2\pi m/n$, and let $\psi : \mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ with

$$\psi([m]) = \begin{pmatrix} e^{2\pi im/n} & 0 \\ 0 & e^{-2\pi im/n} \end{pmatrix}.$$

We have $\phi \sim \psi$.

Proof. [2]
Let $T = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. Then

$$\begin{aligned}
\psi([m])T &= \begin{pmatrix} e^{2\pi im/n} & 0 \\ 0 & e^{-2\pi im/n} \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} e^{2\pi im/n}i & -e^{2\pi im/n}i \\ e^{-2\pi im/n} & e^{-2\pi im/n} \end{pmatrix} \\
&= \begin{pmatrix} -\sin(2\pi im/n) + i\cos(2\pi im/n) & \sin(2\pi im/n) - i\cos(2\pi im/n) \\ \cos(2\pi im/n) - i\sin(2\pi im/n) & \cos(2\pi im/n) - i\sin(2\pi im/n) \end{pmatrix} \\
&= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \cos(2\pi m/n) & -\sin(2\pi m/n) \\ \sin(2\pi m/n) & \cos(2\pi m/n) \end{pmatrix} = T\phi([m]). \ ■
\end{aligned}$$

Definition 4.9. Symmetric Group
Recall that the symmetric group S_n is the group of all bijections from a set of n elements to itself, with the group operation of composition of bijections. The group is of order $n!$ since there are $n!$ permutations of n elements.

We write elements of S_n in cycle notation: for example when $n = 6$ $\sigma = (2\ 1\ 3)(4)(5\ 6)$ is the element which sends the 3rd element to the 1st 1st to the 2nd 2nd to the 3rd 4th to 4th and 5th to 6th (and vice versa). Then we can write $\sigma(1,2,3,4,5,6) = (\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5),\sigma(6)) = (3,1,2,4,6,5)$.

Example 4.10. Standard representation of S_n [2]
Let $\phi : S_n \to \mathrm{GL}_n(\mathbb{C})$ such that $\phi_\sigma(e_i) = e_{\sigma(i)}\ \forall\sigma \in S_n$ $1 \leqslant i \leqslant n$. The matrix for ϕ_σ is given by permuting the columns of I by σ for example when $n = 4$ $\sigma = (1\ 4\ 3\ 2)$ gives

$$\phi_\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Notice that $\phi_\sigma(e_1+e_2+\cdots+e_n) = e_{\sigma(1)}+e_{\sigma(2)}+\cdots+e_{\sigma(n)} = e_1+e_2+\cdots+e_n$ $\forall\sigma\in S_n$ since addition is commutative. Then by scalability of linear ϕ_σ, we have $\phi_\sigma(\alpha(e_1+e_2+\cdots+e_n)) = \alpha(e_1+e_2+\cdots+e_n)$ $\forall\alpha\in\mathbb{C},\ \sigma\in S_n$. Hence $\mathbb{C}(e_1+e_2+\cdots+e_n)$ is invariant under ϕ_σ $\forall\sigma\in S_n$.

Definition 4.11. G-invariant subspace [2]

For a representation $\phi : G \to \mathrm{GL}(V)$, a (linear) subspace $W \leqslant V$ is said to be G-invariant if and only if $\phi_g(w) \in W$ $\forall g\in G,\ w\in W$.

Definition 4.12. Subrepresentation [2]

For a representation $\phi : G \to \mathrm{GL}(V)$ and a G-invariant subspace $W \leqslant V$, a representation $\phi|_W : G \to \mathrm{GL}(W)$ can be obtained by restricting ϕ to W with $(\phi|_W)_g(w) = \phi_g(w) \in W$ $\forall w\in W,\ g\in G$. We say that $\phi|_W$ is a subrepresentation of ϕ.

Definition 4.13. Proper subrepresentation

A subrepresentation $\phi^W : G \to \mathrm{GL}(W)$ of $\phi^V : G \to \mathrm{GL}(V)$ is said to be proper if $W \neq (0), V$.

Definition 4.14. Direct sum of representations [2]

Given representations $\phi^1 : G \to \mathrm{GL}(V_1)$ and $\phi^2 : G \to \mathrm{GL}(V_2)$, we can find another representation $\phi^1\oplus\phi^2 : G \to \mathrm{GL}(V_1\oplus V_2)$ given by $(\phi^1\oplus\phi^2)_g(v_1,v_2) = (\phi^1_g(v_1), \phi^2_g(v_2))$ $\forall g\in G,\ v_1\in V_1, v_2\in V_2$.

If V_1 is of dimension n_1 and V_2 is of dimension n_2, and both are over $\mathbb{C}$ such that $\phi^1 : G \to \mathrm{GL}_{n_1}(\mathbb{C})$ and $\phi^2 : G\to\mathrm{GL}_{n_2}(\mathbb{C})$, then

$$\phi^1\oplus\phi^2 : G \to \mathrm{GL}_{n_1+n_2}(V_1\oplus V_2)$$

with matrix form

$$(\phi^1\oplus\phi^2)_g = \begin{pmatrix}\phi^1_g & 0\\ 0 & \phi^2_g\end{pmatrix},$$

which is the (n_1+n_2) square matrix formed by stacking ϕ_g and ψ_g next to each other on the diagonal, with 0 in the other entries.

Example 4.15. [2]

Let $\phi^1 : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ and $\phi^2 : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ such that $\phi^1_{[m]} = e^{2\pi i m/n}$ and $\phi^2_{[m]} = e^{-2\pi i m/n}$. Then

$$(\phi^1\oplus\phi^2)_{[m]} = \begin{pmatrix}e^{2\pi i m/n} & 0\\ 0 & e^{-2\pi i m/n}\end{pmatrix}.$$

Lemma 4.16. [2]

If a group G is generated by a set S then a representation on G is determined by its values on S, since representations are a homeomorphism.

Proof.

For $G = \langle S = \{s_1, s_2, \dots\}\rangle$, and $x = \prod_{i\in I_S} s_i$ a product of elements in S, a representation $\phi : G \to \mathrm{GL}(V)$ gives $\phi_x = \prod_{i\in I_S}\phi_{s_i}$. ■

Example 4.17. **[2]**

S_3 can be generated by two elements: $S_3 = \langle (1\ 2\ 3), (1\ 2) \rangle$.

Let $\phi : S_3 \to \mathrm{GL}_2(\mathbb{C})$ be the representation such that

$$\phi_{(1\ 2)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \phi_{(1\ 2\ 3)} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$

and let $\psi : S_3 \to \mathbb{C}^*$ be the trivial representation $\phi_\sigma = 1\ \forall \sigma \in S_3$. Then

$$(\phi \oplus \psi)_{(1\ 2)} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\phi \oplus \psi)_{(1\ 2\ 3)} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Definition 4.18. Faithful representation [**2**]

A representation $\phi : G \to \mathrm{GL}(V)$ is faithful if and only if it is injective: $\phi_g = I \implies g = 1$ where I is the identity of $\mathrm{GL}(V)$ and 1 is the identity of G.

4.2 Maschke's Theorem and Reducibility

Definition 4.19. Irreducible representation [2]
A non-zero representation $\phi : G \to \mathrm{GL}(V)$ is irreducible if and only if the only G-invariant subspaces of V are $\{0\}$ and V.

Irreducible representations are analagous to prime numbers in number theory, or simple groups in group theory.

Lemma 4.20. [2]
Any degree 1 representation $\phi : G \to \mathbb{C}^*$ is irreducible since $\mathbb{C}$ has no proper subspaces.

Proposition 4.21. [2]
For a degree 2 representation $\phi : G \to \mathrm{GL}(V)$, ϕ is irreducible if and only if there is no common eigenvector $v\ \forall \phi_g,\ g \in G$.

Definition 4.22. Completely reducible [2]
A representation $\phi : G \to \mathrm{GL}(V)$ is completely reducible if and only if $V = V_1 \oplus V_2 + \cdots + V_n$, where V_i are G-invariant subspaces and $\phi|_{V_i}$ are irreducible $\forall 1 \leqslant i \leqslant n$.

Proposition 4.23. [2]
The following are equivalent:

(1) $\phi : G \to \mathrm{GL}(V)$ is completely reducible.

(2) $\phi \sim \phi^1 \oplus \phi^2 \oplus \cdots \oplus \phi^n$ where ϕ^i is irreducible $\forall 1 \leqslant i \leqslant n$.

Definition 4.24. Decomposable representation [2]
Let $\phi : G \to \mathrm{GL}(V)$ be a non-zero representation. ϕ is decomposable if and only if $V = V_1 \oplus V_2$ where V_1, V_2 are non-zero G-invariant subspaces. Otherwise ϕ is said to be indecomposable.

Lemma 4.25. [2]
If $\phi : G \to \mathrm{GL}(V)$ is equivalent to a decomposable representation then ϕ is decomposable.

Lemma 4.26. [2]
If $\phi : G \to \mathrm{GL}(V)$ is equivalent to an indecomposable representation then ϕ is indecomposable.

Lemma 4.27. [2]
If $\phi : G \to \mathrm{GL}(V)$ is equivalent to a completely reducible representation then ϕ is completely reducible.

Definition 4.28. Unitary representation [2]
A representation $\phi : G \to \mathrm{GL}(V)$ where V is an inner product space is said to be unitary if and only if ϕ_g is unitary $\forall g \in G$. Since $U_1(\mathbb{C}) = \mathbb{T}$, a one dimensional unitary representation is a homomorphism $\phi : G \to \mathbb{T}$.

Example 4.29. [2]
Let $\phi : \mathbb{R} \to \mathbb{T}$ such that $\phi_t = e^{2\pi i t}$. Then $\phi_{t+s} = e^{2\pi i(t+s)} = \phi_t \phi_s$, hence ϕ is a representation.

Proposition 4.30. [2]

A unitary representation $\phi : G \to \mathrm{GL}(V)$ is either irreducible or decomposable.

Proposition 4.31. [2]

Every representation of a finite group G is equivalent to a unitary representation.

Corollary 4.32. [2]

Every non-zero representation $\phi : G \to \mathrm{GL}(V)$ of a finite group is either irreducible or decomposable.

Proposition 4.33. [2]

Every irreducible representation is indecomposable, though the contrary is not true in general.

Theorem 4.34. Maschke [2]

Every representation of a finite group is completely reducible.

Proof. [2]

Let $\phi : G \to \mathrm{GL}(V)$ be a representation of a finite group G. We proceed by induction on the degree of ϕ. If $\dim V = 1$ then ϕ is irreducible since V has no non-zero proper subspaces. We assume true our inductive hypothesis that ϕ is irreducible for some $\dim V = k \in \mathbb{N}$. Then let $\phi : G \to \mathrm{GL}(V)$ for $\dim V = k + 1$. If ϕ is irreducible then it is completely reducible, if not it is decomposable by Corollary 4.32. Then $V = V_1 \oplus V_2$ with $0 \neq V_1, V_2$ are G-invariant subspaces, and by the inductive hypothesis $\dim V_1, \dim V_2 < \dim V \implies \phi|_{V_1}, \phi|_{V_2}$ are completely reducible. Then $V_1 = U_1 \oplus \cdots \oplus U_{n_U}$ and $V_2 = W_1 \oplus \cdots \oplus W_{n_W}$ where U_i and W_j are G-invariant and the subrepresentations $\phi|_{U_i}, \phi|_{W_j}$ are irreducible $\forall\, 1 \leqslant i \leqslant n_U,\ 1 \leqslant j \leqslant n_W$. Then $V = U_1 \oplus \cdots \oplus U_{n_U} \oplus W_1 \oplus \cdots \oplus W_{n_W}$ and ϕ is completely reducible. ■

Example 4.35.

By Maschke's theorem, every representation of $\mathbb{Z}/n\mathbb{Z}$ is completely reducible $\forall n \in \mathbb{N}$.

5 FG-Modules

5.1 Basic Definitions

We will generalize the results so far in terms of FG-modules.

Definition 5.1. ***FG*-module [1]**
For a vector space V over a field F, and a group G, we say V is an FG-module with respect to a multiplication operation $g \cdot v$ for $v \in V,\ g \in G$ if the following axioms are satisfied:

(1) $g \cdot v \in V$,

(2) $(gg') \cdot v = g \cdot (g' \cdot v)$,

(3) $1 \cdot v = v$ (where 1 is the identity element of G),

(4) $g \cdot (\alpha v) = \alpha(g \cdot v)$,

(5) $g \cdot (v + v') = g \cdot v + g \cdot v'$,

$\forall\, v, v' \in V,\ g, g' \in G$. As with group operations we will neglect the '$\cdot$' for ease of reading: $gv \coloneqq g \cdot v$.

Note that axioms (1),(4),(5) imply that $g\ \cdot : V \to G$ such that $v \mapsto gv$ is a linear endomorphism [1].

Definition 5.2. ***FG*-module with chosen basis [1]**
Given an FG-module V with a finite n dimensional basis $\mathcal{B}$, we denote the matrix of the endomorphism $v \mapsto gv$ with respect to $\mathcal{B}$ as $[g]_{\mathcal{B}}$.

Theorem 5.3. [1]
Let V be an n dimensional vector space over F, and $\phi : G \to \mathrm{GL}(V)$ be a group representation. V becomes an FG-module by defining multiplication with $gv = \phi_g(v)\ \forall g \in G,\ v \in V$.

We can see that given a basis $\mathcal{B}$ we have $\phi_g = [g]_{\mathcal{B}}$.

Proof. [1]
Given an n dimensional vector space V over F, and a representation $\phi : G \to \mathrm{GL}(V)$ we have

(1) $\phi_g(v) \in V$,

(2) $\phi_{gg'}(v) = \phi_g \phi_{g'}(v)$,

(3) $\phi_1 v = v$ (since ϕ is a homomorphism it maps identity to identity),

(4) $\phi_g(\alpha v) = \alpha(\phi_g v)$,

(5) $\phi_g(v + v') = \phi_g(v) + \phi_g(v')$,

$\forall v, v' \in V,\ \alpha \in F,\ g \in G$. Hence $gv \coloneqq \phi_g(v)$ allows $V \cong F^n$ becomes FG-module. ■

Theorem 5.4. [1]
Given and FG-module V with basis $\mathcal{B}$, the function $g \mapsto [g]_{\mathcal{B}}$ is a representation of of G.

Proof. [1]

Given an FG-module with basis $\mathcal{B}$. Since $(gg')v = g(g'v)\ \forall g, g' \in G,\ v \in V$, we have $[gg']_{\mathcal{B}} = [g]_{\mathcal{B}}[g']_{\mathcal{B}}$, then $[gg^{-1}]_{\mathcal{B}} = [1]_{\mathcal{B}} = [g]_{\mathcal{B}}[g']_{\mathcal{B}}$, and so $g \mapsto [g]_{\mathcal{B}}$ is a homomorphism from G to $\mathrm{GL}_n(F)$. ■

Definition 5.5. Direct sum of FG-modules

Our direct sum of representations in Definition 4.14 extends to FG-modules, that is if we have $V = U \oplus W$ for FG-modules V, U, W with chosen bases $\mathcal{B}_V, \mathcal{B}_W, \mathcal{B}_U$ respectively, then

$$[g]_{\mathcal{B}_V} = \begin{pmatrix} [g]_{\mathcal{B}_U} & 0 \\ 0 & [g]_{\mathcal{B}_W} \end{pmatrix}.$$

More generally, given FG-modules $V, W_1, \dots, W_n$ with $V = W_1 \oplus \cdots \oplus W_n$, and bases $\mathcal{B}_V, \mathcal{B}_{W_1}, \dots, \mathcal{B}_{W_n}$ respectively, we have

$$\begin{pmatrix} [g]_{\mathcal{B}_V} = [g]_{\mathcal{B}_{W_1}} & 0 & \dots & 0 \\ 0 & [g]_{\mathcal{B}_{W_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & [g]_{\mathcal{B}_{W_n}} \end{pmatrix}.$$

Definition 5.6. Trivial FG-module [1]

The trivial FG-module is the 1-dimensional vector space V over F such that $gv = v\ \forall\, v \in V, g \in G$.

Definition 5.7. Faithful FG-module [1]

An FG-module V is said to be faithful if and only if it is injective: $gv = v \implies g = 1\ \forall v \in V$.

Definition 5.8. Regular FG-module [1]

Let G be a finite group of order n and $F = \mathbb{C}$ or $\mathbb{R}$. The regular FG-module V is the vector space over F obtained using elements of g as a basis, that is

$$V = \left\{ \sum_{i \in I} f_i g_i \mid f_i \in F,\ g_i \in G, I \subseteq \{1, \dots n\} \right\},$$

the set of finite sums of elements of G with coefficients in F. It has the natural multiplication $vg = \sum\limits_{i \in I} f_i g_i g$. TODO verify this is a vector space

Notice that the regular FG-module has dimension $\#G$.

Definition 5.9. Regular representation [1]

Let $G = \{g_1 = 1_G, g_2, \dots, g_n\}$ be a finite group of order n. The representation of the regular FG-module V with basis $\mathcal{B} = G$ given by $g \mapsto [g]_{\mathcal{B}}$ is called the regular representation of G over F.

Proposition 5.10. [1]

The regular representation is faithful.

Proof.

Let $g \in G$, then $\forall v = \sum\limits_{i \in I} f_i g_i \in V$, suppose $vg = \sum\limits_{i \in I} f_i g_i g = \sum\limits_{i \in I} f_i g_i = v$, then an identity basis term in a sum has $f 1_G g = f 1_G$, so $g = 1_G$.

Theorem 5.11. **[1]**
Let V be an FG-module with basis $\mathcal{B}$ and let ϕ be a representation $\phi : G \to \mathrm{GL}(V)$ such that $\phi_g = [g]_{\mathcal{B}}$.

(1) If $\mathcal{B}'$ is another basis of V, and we have another representation $\psi : G \to \mathrm{GL}(V)$ such that $\psi_g = [g]_{\mathcal{B}'}$, then $\phi \sim \psi$.

(2) Conversely, if ψ is a representation equivalent to ϕ then there is a basis $\mathcal{B}'$ of V such that $\psi_g = [g]_{\mathcal{B}'}$.

Proof. **[1]**

(1) There exists a change of basis matrix T such that $[g]_{\mathcal{B}} = T^{-1}[g]_{\mathcal{B}'}T$.

(2) Since $\phi \sim \psi$, $\exists T$ such that $\phi_g = T^{-1}\psi_g T \ \forall g \in G$. Let $\mathcal{B}'$ be a basis of V such that the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$ is T, then $[g]_{\mathcal{B}} = T^{-1}[g]_{\mathcal{B}'}T \ \forall g \in G$, and $\psi_g = [g]_{\mathcal{B}'}$.

■

We will now examine reducibility through the lense of FG-modules.

Definition 5.12. ***FG*-submodule** **[1]**
Let V be an FG-module. A subspace $W \leqslant V$ is an FG-submodule of V is $gw \in W \ \forall \, g \in G$, $w \in W$.

Definition 5.13. Proper submodule **[1]**
Every FG-module V has at least two FG-submodules: (0) and V. An FG-submodule $W < V$ is said to be proper if $W \neq (0), V$.

Definition 5.14. **[1]**
A nonzero FG-module V is irreducible if and only if it has no proper FG-submodules.

We remark that the zero FG-module $V = (0)$ is regarded as neither reducible or irreducible, analogous to $1 \in \mathbb{N}$ being neither composite nor prime. (ASK)

Definition 5.15. ***FG*-homomorphism** **[1]**
Let V and W be FG-modules. An FG-homomorphism is a a linear function $\lambda : V \to W$ such that $\lambda(gv) = g\lambda(v) \ \forall g \in G, v \in V$. If λ is a bijection then we say it is an FG-isomorphism, and $V \cong W$.

Proposition 5.16. **[1]**
Let V, W be FG-modules. Then $V \cong W$ if and only if there exists a basis $\mathcal{B}_V$ of V and $\mathcal{B}_W$ of W such that

$$[g]_{\mathcal{B}_V} = [g]_{\mathcal{B}_W}.$$

Proof.
TODO

Definition 5.17. ***FG*-module projection** **[1]**
Given an FG-module V and a collection of FG-submodules $\{W_i\}_{1 \leqslant i \leqslant n}$ such that $V = W_1 \oplus \cdots \oplus W_n$,

for any vector $v = w_1 + \cdots + w_n$ where $v \in V$, $w_i \in W$ we define the projection $\pi_i : V \to W_i$ such that $\pi_i(v) = w_i$. This is a projection since $\pi_i^2(v) = \pi_i(w_i) = w_i$, $\mathrm{Im}(\pi_i) = W_i$, $\pi_i \mid_{W_i} = \mathrm{Id}_{W_i}$.

Proposition 5.18. [1]

The above projection is an FG-module homorphism.

Proof.

$\pi_i : V \to W_i$ is linear since given $v, v' \in V$ and scalars $\alpha, \alpha' \in F$ we have $\pi_i(\alpha v + \alpha' v') = \pi(\alpha w_1 + \alpha' w_1' + \cdots + \alpha w_n + \alpha' w_n') = \alpha w_i + \alpha' w_i' = \alpha \pi_i(v) + \alpha' \pi_i(v')$ where $w_j, w_j' \in W_j$ $\forall 1 \leqslant j \leqslant n$. It's structure preserving since given $\pi_i(gv) = \pi_i(gw_1 + \cdots + gw_n) = gw_i = g\pi_i(v)$.

Definition 5.19. Image and kernel of FG-homomorphism

Like with other homomorphisms, given a FG-homomorphism $\phi : V \to W$

$$\mathrm{Im}(\phi) = \{\phi(v) \in W \mid v \in V\}, \quad \mathrm{Ker}(\phi) = \{v \in V \mid \phi(v) = 0 \in W\}.$$

Proposition 5.20.

For a FG-homomorphism $\phi : V \to W$, $\mathrm{Im}(\phi)$ is a FG-submodule of W and $\mathrm{Ker}(\phi)$ is an FG-submodule of V.

Proof.

Let $w = \phi(v) \in \mathrm{Im}(\phi)$. Then $\forall\, g \in G$, $gw = g\phi(v) = \phi(gv) \implies gw \in \mathrm{Im}(\phi)$. Let $v \in \mathrm{Ker}(\phi)$. Then $\forall\, g \in G$, $\phi(gv) = g\phi(v) = 0 \implies gv \in \mathrm{Ker}(\phi)$.

5.2 Maschke's Theorem for FG-Modules

In the last section we covered Maschke's theorem in the case of a vector space over a field F with $\operatorname{Char} F = 0$. We use a more general case of Maschke's theorem in terms of FG-modules.

Theorem 5.21. Maschke [1]

Let V be an FG-module where G is a finite group, and F a field of a characteristic such that $\operatorname{Char}(F) \nmid \#G$. If there exists an FG-submodule $W < V$ then there exists an FG-submodule U such that $V = W \oplus U$.

Definition 5.22. Completely reducible FG-module [1]

Let V be an FG-module. V is said to be completely reducible if and only if $V = W_1 \oplus \cdots \oplus W_k$ where U_i is an irreducible FG-submodule of $V \; \forall \, 1 \leqslant i \leqslant k$.

Proposition 5.23.

Maschke's theorem 5.21 implies our earlier description Maschke's theorem 4.34 where the field is $\mathbb{C}$.

Proof. [1]

We have that $\operatorname{Char} \mathbb{C} = 0$. Let V be an n-dimensional non-zero FG-module, with finite G, and $F = \mathbb{C}$. We proceed by induction on $\dim V$. Suppose $\dim V = 1$, then V is trivially irreducible. Suppose V is completely reducible up to $\dim V = k$. Then for $\dim V = k + 1$, if V is irreducible then the result holds, else $\exists W < V$ such that $W \neq \{0\}, V$. By Maschke's theorem 5.21 $\exists U < V$ with $U \neq \{0\}, V$ such that $V = W \oplus U$. Since $\dim W, \dim U \leqslant k < \dim V$, both W and U are completely reducible by the inductive hypothesis, then

$$V = W_1 \oplus \cdots \oplus W_{i_W} \oplus U_1 \oplus \cdots \oplus U_{i_U}$$

where $i_W, i_U \in \mathbb{N}$ and W_j, U_j are irreducible $\forall j$. ∎

With these results out the way, we proove Maschke's theorem for FG-modules.

Proof. Maschke [3]

We have a finite group G and an FG-module V over field F with characteristic such that $\operatorname{Char}(F) \nmid \#G$. Let W be an FG-submodule of V. We define $\pi_W : V \to W$ be the projection onto W as a vector space. Take $\tilde{\pi}_W : V \to W$ such that

$$\tilde{\pi}_W(v) \coloneqq \frac{1}{\#G} \sum_{g \in G} g \pi_W(g^{-1} v).$$

We will verify that this is an FG-homomorphism, even though π_W on its own is just a projection of vector spaces. Linearity follows from linearity of π_W with

$$\frac{1}{\#G} \sum_{g \in G} g \pi_W(g^{-1}(v + v')) = \frac{1}{\#G} \sum_{g \in G} g \pi_W(g^{-1} v) + \frac{1}{\#G} \sum_{g \in G} g \pi_W(g^{-1} v'),$$
$$\frac{1}{\#G} \sum_{g \in G} g \pi_W(\alpha g^{-1} v) = \alpha \frac{1}{\#G} \sum_{g \in G} g \pi_W(g^{-1} v).$$

Now we demonstrate that $\tilde{\pi}_W(w) = w$. Clearly $g^{-1}w \in W$ since W is a submodule, which implies $\pi(g^{-1}w) = g^{-1}w$, then

$$\tilde{\pi}_W(w) = \frac{1}{\#G}\sum_{g\in G} g\pi_W(g^{-1}w) = \frac{1}{\#G}\sum_{g\in G} gg^{-1}w = \frac{1}{\#G}\sum_{g\in G} w = w.$$

Also, since π_W has its image in W, $\tilde{\pi}_W$ also has its image in W. Then $\tilde{\pi}_W$ is a projection.

We verify structure presevation, given an element $h \in G$,

$$h\tilde{\pi}_W(v) = \frac{1}{\#G}\sum_{g\in G} hg\pi_W(g^{-1}v) = \frac{1}{\#G}\sum_{g\in G}(hg)\pi_W((hg)^{-1}hv).$$

Now let $g' = hg$. Then summing over all g is the same as summing over all g' so

$$\frac{1}{\#G}\sum_{g\in G})(hg)\pi_W((hg)^{-1}hv) = \frac{1}{\#G}\sum_{g'\in G} g'\pi_W(g'^{-1}hv) = \tilde{\pi}_W(hv)$$

and $\tilde{\pi}_W$ is structure preserving, and hence an FG-homomorphism.

By Proposition 5.20, $\mathrm{Ker}(\tilde{\pi}_W)$ is an FG-submodule. By Theorem 3.5 we have

$$V = \mathrm{Im}(\tilde{\pi}_W) \oplus \mathrm{Ker}(\tilde{\pi}_W) = W \oplus \mathrm{Ker}(\tilde{\pi}_W). \blacksquare$$

Notice that the finite G condition is required for $\frac{1}{\#G}$ to be defined, also if $\mathrm{Char}(F) \mid \#G$ then $\frac{1}{\#G}$ is undefined since $\#G \equiv 0$ in F.

While this proof requires this condition on $\#G$, a natural question would be if Maschke's theorem will ever hold for $\mathrm{Char}(F) \mid \#G$. As it turns out the answer is no.

Example 5.24.

We present an example of a group and representation such that $\mathrm{Char}(F) \mid \#G$ and Maschke's theorem does not hold. For the purpose of contradiction assume Maschke's theorem holds for FG-modules for fields of all characteristics.

Let $C_3 = \{1, g, g^2\}$ be the cyclic group and $\overline{\mathbb{F}}_3 = \bigcup_{n\in\mathbb{N}} \mathbb{F}_{3^n}$ be the algebraic closure of the finite field of three elements. Then $\mathrm{Char}(\overline{\mathbb{F}}_3) = 3 \mid 3 = \#C_3$. We define the representation $\phi : C_3 \to \mathrm{GL}(V)$ with vector space V over $\overline{\mathbb{F}}_3$ and basis $\mathcal{B} = \{e_1, e_2, e_3\}$ such that in matrix form we have the following linear maps

$$\phi_1 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \phi_g = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \quad \phi_{g^2} = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},$$

or more specifically

$$\begin{aligned}
\phi_1(\alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3) &= (\alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3)\\
\phi_g(\alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3) &= (\alpha_3e_1 + \alpha_1e_2 + \alpha_2e_3),\\
\phi_{g^2}(\alpha_1e_1 + \alpha_2e_2 + \alpha_3e_3) &= (\phi_g)^2(\alpha_1e_1, \alpha_2e_2, \alpha_3e_3) = (\alpha_2e_1 + \alpha_3e_2 + \alpha_1e_3)
\end{aligned}$$

$\forall\, \alpha_i \in \overline{\mathbb{F}}_3$.

Let $W = \overline{\mathbb{F}}_3\{e_1+e_2+e_3\}$. Then $\forall \alpha \in \overline{\mathbb{F}}_3$, $\phi_g(\alpha(e_1+e_2+e_3)) = \alpha\phi_g(e_1+e_2+e_3) = \alpha(e_1+e_2+e_3) \implies \phi_{g^2}(\alpha(e_1+e_2+e_3)) = \alpha(e_1+e_2+e_3)$. So $\forall w \in W$, $\phi_g(w) = w \implies \phi_{g^2}(w) = w$, and W is a C_3-invariant subspace, hence by Maschke's theorem there exists another C_3-invariant subspace U such that $V = W \oplus U$.

By theorem 3.7, both W and U have at least one eigenvector for ϕ_g, and hence V must have at least two eigenvectors for ϕ_g. Then calculating the eigenvectors of ϕ_g we have

$$\det\begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} = 0 \implies -(\lambda^3+1) = -(\lambda-1)^3 = 0 \implies \lambda = 1.$$

TODO finish with eigenvectors

5.3 Schur's Lemma

Lemma 5.25. Schur [1]

Let V, W be irreducible $\mathbb{C}G$-modules.

(1) For a $\mathbb{C}G$-homomorphism $\phi : V \to W$, either ϕ is a $\mathbb{C}G$-isomorphism or $\phi(v) = 0 \ \forall v \in V$.

(2) For a $\mathbb{C}G$-isomorphism $\phi : V \to V$, ϕ is a scalar multiple of the identity isomorphism $\phi = \alpha \operatorname{Id}_V$.

Proof. [1]

(1) Suppose $\exists\, v \in V$ such that $\phi(v) \neq 0$. Then $\operatorname{Im}(\phi) \neq \{0\}$. By proposition 5.20 we know $\operatorname{Im}(\phi)$ is a $\mathbb{C}G$-submodule of W, but W is irreducible so $\operatorname{Im}(\phi) = W$ and ϕ is surjective. Also by proposition 5.20 $\operatorname{Ker}(\phi)$ is a $\mathbb{C}G$-submodule of V and since $\operatorname{Ker}(\phi) \neq V$ and V is irreducible, $\operatorname{Ker}(\phi) = \{0\}$ so ϕ is injective and hence a $\mathbb{C}G$-isomorphism.

(2) By theorem 3.7 we have $\phi : V \to V$ must have at least one eigenvalue $\lambda \in \mathbb{C}$, then $\operatorname{Ker}(\phi - \lambda \operatorname{Id}_V) \neq \{0\}$. $\operatorname{Ker}(\phi - \lambda \operatorname{Id}_V)$ is a $\mathbb{C}G$-submodule of V, but V is irreducible, so $\operatorname{Ker}(\phi - \lambda \operatorname{Id}_V) = V$ and $(\phi - \lambda \operatorname{Id}_V)v = 0 \ \forall v \in V$, then $\phi = \lambda \operatorname{Id}_V$. ■

Proposition 5.26. [1]

For a $\mathbb{C}G$-module V, if every $\mathbb{C}G$-endomorphism on V is a scalar multiple of Id_V then V is irreducible.

Proof. [1]

Suppose for purpose of contradiction that V is a reducible $\mathbb{C}G$-module where every $\mathbb{C}G$ endomorphism is a scalar multiple of the identity. Then there exists a proper $\mathbb{C}G$-submodule $W < V$, and by Maschke's theorem there exists a proper FG-submodule $U < V$ such that $V = U \oplus W$.
Then by Proposition 5.18, $\pi_W : V \to V$ such that $\pi(u + w) = w \ \forall w \in W, u \in U$ is a $\mathbb{C}G$-homomorphism. But π_W isn't a scalar multiple of 0.

Then by contradiciton V is irreducible. ■

Theorem 5.27. Fundamental theorem of finite abelian groups [4]

Let G be a finite abelian group. Then G is isomorphic to a direct product of cyclic groups $C_{p_1}^{n_1} \oplus \cdots C_{p_m}^{n_m}$ each of which has an order equal to a prime power.

Proof.

TODO

Proposition 5.28. [1]

If G is finite abelian then every irreducible $\mathbb{C}G$-module V has dimension 1.

Proof.

Let G be a finite abelian and V and irreducible $\mathbb{C}G$-module with basis $\mathcal{B}$. $\forall g, g' \in G$, $v \in V, \alpha, \alpha' \in \mathbb{C}$ we have

$$gg' = g'g \implies g'(gv) = (gg'v), \quad g(\alpha v + \alpha' v') = \alpha(gv) + \alpha'(gv').$$

So the map $\phi_g : V \to V$ such that $v \mapsto gv \; \forall v \in V$ is a $\mathbb{C}G$-endomorphism for any $g \in G$. Then by Schur's lemma (1) either

(1) $gv = 0 = 0v \, \forall v \in V$,

(2) or ϕ_g is a $\mathbb{C}G$-automorphism and $gv = \lambda v$ for some $\lambda \in \mathbb{C}$.

Then $\exists \lambda_g \in \mathbb{C}$ for a given g such that $gv = \lambda_g v \; \forall v \in G$. Every subvectorspace $W < V$ is also an $\mathbb{C}G$-submodule since $gw = \lambda_g w \in W \; \forall w \in W$. Then $\dim(V) = 1$, else we could choose any basis element $e_i \in \mathcal{B}$ an get a $\mathbb{C}G$-submodule $\mathbb{C}\{e_i\} \neq V$. ■

6 Character Theory

6.1 Basic Definitions and Results

Definition 6.1. Character [1]
Let V be a $\mathbb{C}G$-module with basis $\mathcal{B}$. The character of V is the function $\chi : G \to \mathbb{C}$ with

$$\chi(g) = \mathrm{Tr}\,[g]_{\mathcal{B}}.$$

Given our connection between FG-modules and representations, we can see the character of an element $g \in G$ with respect to a representation $\phi : G \to \mathrm{GL}(V)$ is $\chi(g) = \mathrm{Tr}(\phi_g)$.

Proposition 6.2. [1]
χ is invariant under choice and change of basis, since $[g]_{\mathcal{B}'} = T^{-1}[g]_{\mathcal{B}}T$ by theorem 5.11 $\implies$ $\mathrm{Tr}([g]_{\mathcal{B}'}) = \mathrm{Tr}(T^{-1}[g]_{\mathcal{B}}T) = \mathrm{Tr}([g]_{\mathcal{B}})$ by proposition 3.8.

Definition 6.3. Irreducible character [1]
We say that χ is a character of G if χ is the character a $\mathbb{C}G$-module. χ is called an irreducible character of G if χ is the character of an irreducible $\mathbb{C}G$-module, and χ is reducible if it is the character of a reducible $\mathbb{C}G$-module.

Definition 6.4. Trivial character [1]
The trivial character of a group G is the character χ such that $\chi(g) = 1 \in \mathbb{C}\ \forall g \in G$.

Proposition 6.5. [1]
If V, W are isomorphic $\mathbb{C}G$-modules then they have the same character.

Proof. [1]
Since $V \cong W$, by Proposition 5.16 there is are bases $\mathcal{B}_V$ of V and $\mathcal{B}_W$ of W such that $[g]_{\mathcal{B}_V} = [g]_{\mathcal{B}_W}\ \forall g \in G$. Then $\mathrm{Tr}([g]_{\mathcal{B}_V}) = \mathrm{Tr}([g]_{\mathcal{B}_W})$. ■

Definition 6.6. Degree of a character [1]
Let χ be the character of a $\mathbb{C}G$-module V, the degree of χ is the dimension of V.

Proposition 6.7. [2]
Let V be a $\mathbb{C}G$-module, and $g \in G$ an element with order n. Then

(1) $\chi(1) = \dim(V)$,

(2) $\chi(g)$ is the sum of all nth roots of unity,

(3) $\chi(g^{-1}) = \bar{\chi}(g)$ the complex conjugate of $\chi(g)$,

(4) TODO check if we need conjugate stuff

Proof. [1]

(1) Let $\mathcal{B}$ be a chosen basis of V, and $n = \dim(V)$. Then $\chi(1) = \mathrm{Tr}[1]_{\mathcal{B}} = \mathrm{Tr}[1]_{\mathcal{B}} = \mathrm{Tr}(I) = n$.

(2) TODO rest

Definition 6.8. Kernel of a character [1]
Let $\chi : G \to F$ be a character. Then the kernel of χ is given by $\mathrm{Ker}(\chi) = \{g \in G \mid \chi(g) = \chi(1)\}$.

Definition 6.9. Faithful character [1]
A character χ is faithful if it has trivial kernel $\mathrm{Ker}(\chi) = \{1\}$.

Theorem 6.10. [1]
Given a representation $\phi : G \to \mathrm{GL}_n(\mathbb{C})$, and a character $\chi : G \to \mathbb{C}$, we have $|\chi(g)| = \chi(1) \iff \exists\, \alpha \in \mathbb{C}$ such that $\phi_g = \alpha I$.

Proof. [1]
TODO

Theorem 6.11. [1]
Given a representation $\phi : G \to \mathrm{GL}_n(\mathbb{C})$, and a character $\chi : G \to \mathbb{C}$, $\mathrm{Ker}(\chi) = \mathrm{Ker}(\phi)$. ■

Proof. [1]
Choosing a basis $\mathcal{B}$ of V, For $g \in \mathrm{Ker}(\phi)$, then $\phi_g = I \implies \chi(g) = n = \chi(1) \implies g \in \mathrm{Ker}(\chi)$. For $g \in \mathrm{Ker}(\chi)$, by theorem 6.10 we have $\phi_g = \alpha I$ for some $\alpha \in \mathbb{C}$. Then $\chi(g) = \alpha\chi(1)$, hence $\alpha = 1$. Then $\phi_g = I$ and $g \in \mathrm{Ker}(\phi)$. ■

Corollary 6.12.
Since the kernel of a homomorphism is a normal subgroup, $\mathrm{Ker}(\phi) = \mathrm{Ker}(\chi) \triangleleft G$.

Proposition 6.13. [1]
For a character χ of G, $\overline{\chi}$ is also a character of G. χ is irreducible $\iff$ $\overline{\chi}$ is irreducible.

Proof.
Let V be n-dimensional over $\mathbb{C}$ and let $\phi : G \to \mathrm{GL}(V)$ be a representation of G. Then for choosing a basis $\mathcal{B}$ of V, let $[g]_{\mathcal{B}} = \phi_g$ and

$$\chi(g) = \mathrm{Tr}([g]_{\mathcal{B}}).$$

For a matrix $A \in \mathrm{M}_n(\mathbb{C})$, let $\overline{A}$ be the matrix such that $A^i_j \in A \implies \overline{A}^i_j \in \overline{A}$, that is each entry in $\overline{A}$ is the complex conjugate of the corresponding entry in A. Notice that since for $z, z' \in \mathbb{C}$ we have $\overline{z} \times \overline{z}' = \overline{z \times z'}$, for $A, A' \in \mathrm{M}_n(\mathbb{C})$ we also have $\overline{A \times A'} = \overline{A} \times \overline{A}'$.

This implies that the function $\overline{\phi} : G \to \mathrm{GL}(V)$ obtained from taking the conjugate of ϕ defined by $\overline{\phi}_g = \overline{(\phi_g)}$ is a representation. Then since

$$\mathrm{Tr}(\overline{[g]_{\mathcal{B}}}) = \overline{\mathrm{Tr}([g]_{\mathcal{B}})} = \overline{\chi(g)},$$

the character of $\overline{\phi}$ is $\overline{\chi}$. Clearly if ϕ is reducible then $\overline{\phi}$ is reducible, then if χ is irreducible so is $\overline{\chi}$. ■

6.2 The Regular Character

Definition 6.14. Regular character [1]

Let V be the regular $\mathbb{C}G$-module with basis $\mathcal{B} = G$. We write the regular character as χ_{reg}.

Proposition 6.15. [1]

Let V be a $\mathbb{C}G$-module, which is completely reducible with

$$V = W_1 \oplus \cdots \oplus W_n,$$

where W_i is irreducible for all $1 \leqslant i \leqslant n$. Then $\chi(V) = \chi(W_1 \oplus \cdots \oplus W_n) = \chi(W_1) + \cdots + \chi(W_n)$.

Proof.

This follows clearly from Definition 5.5. Given FG-modules $V, W_1, \ldots, W_n$ with $V = W_1 \oplus \cdots \oplus W_n$, and bases $\mathcal{B}_V, \mathcal{B}_{W_1}, \ldots, \mathcal{B}_{W_n}$ respectively, we have

$$\mathrm{Tr}([g]_{\mathcal{B}_V}) = \mathrm{Tr}\begin{pmatrix} [g]_{\mathcal{B}_{W_1}} & 0 & \ldots & 0 \\ 0 & [g]_{\mathcal{B}_{W_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & [g]_{\mathcal{B}_{W_n}} \end{pmatrix} = \mathrm{Tr}([g]_{\mathcal{B}_{W_1}}) + \cdots + \mathrm{Tr}([g]_{\mathcal{B}_{W_n}}). \blacksquare$$

6.3 Inner products of Characters

7 Character Tables

Definition 7.1. Character table [1]

Let $\chi_1 \dots, \chi_n$ be irreducible characters of a group G. Let $g_1, \dots, g_n$ be representatives of the conjugacy classes of G. The character table is the $n \times n$ matrix X such that the entry $X^i_j = \chi_i(g_j)$.

We number the irreducible characters and conjugacy classes such that χ_1 is the trivial character and $g_1 = 1_G$.

Proposition 7.2. [1]

The character table of G is invertible

8 Orthogonality Relations

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9 Conclusion

10 References

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