This file documents the symbols, formulae and equations used in this module.

1 Symbols

- a: length of semimajor axis, mean of periapsis and apoapsis
- b: length of semiminor axis
- E: Eccentric anomaly
- e: Eccentricity
- h: Angular momentum
- M: Mean anomaly
- n: Average sweep, the average angular velocity per unit time
- ν: True anomaly, angle [periapsis]-[star]-[body]
- p: Semi-latus rectum, length from star to orbit parallel to minor axis
- r: Distance from star
- $\mathbf{r}$: Position relative to star
- u: Argument of latitude
- v: Magnitude of velocity
- $\mathbf{v}$: Velocity
- ω: Longitude of periapsis, or equivalently the argument of periapsis, assuming longitutde of ascending node is zero in 2D
- ϵ: Specific orbital energy, sum of potential and kinetic energy

2 Equations

2.1 Orbital states to orbital elements

In 2D, for a star with fixed position (the origin) and mass, orbits have 4 degrees of freedom.

Defining the ellipse requires 3 degrees of freedom:

- Shape: fixed with e
- Size: fixed with a
- Rotation: fixed with ω

For efficient computation over time, the position on the ellipse is represented by M.

The orbital states $(\mathbf{r}, \mathbf{v})$, with 4 degrees of freedom, are bijective to the orbital elements. Based on cart2kep2002.pdf [1], the following procedures convert Cartesian orbital states to the orbital elements in 2D, based on the external parameter μ:

[1] https://web.archive.org/web/20160418175843/https://ccar.colorado.edu/asen5070/handouts/cart2kep2002.pdf

$$
\begin{aligned}
h &= r_x v_y - r_y v_x \\
\epsilon &= \frac{1}{2}\left\|\mathbf{v}\right\|^2 - \frac{\mu}{\left\|\mathbf{r}\right\|} \\
a &= \frac{-\mu}{2\epsilon} \\
e &= \sqrt{1 - \frac{h^2}{a\mu}} \\
u &= \\
p &= a(1-e^2) \\
\nu &= \arccos\frac{p - \left\|\mathbf{r}\right\|}{\left\|\mathbf{r}\right\| e} \\
\nu &= \arctan\frac{\sqrt{\frac{p}{\mu}}(\mathbf{r}\cdot\mathbf{v})}{p - \left\|\mathbf{r}\right\|} \\
\omega &= u - \nu \\
E &= 2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\nu}{2}\right) \\
M &= E - e\sin E
\end{aligned}
$$

This gives (e, a, ω, M) as the orbital elements.

2.2 Change of mean anomaly over time

Given mean anomaly M at time t, the new mean anomaly M' at time t' can be found by

$$M' = M + n(t' - t),$$

where the average sweep n can be found with

$$n = \sqrt{\frac{\mu}{a^3}}.$$

2.3 Testing if orbit is above/below radius

For a frequently queried radius r_q, precompute the (smaller) mean anomaly M_q such that $r(M_q) = r(2\pi - M_q) = r_q$. Then $r > M \iff M_q < M < 2\pi - M_q \pmod{2\pi}$.

For a specific r_q, solve $\cos E_q = \frac{1}{e}\left(1 - \frac{r}{a}\right)$ for E_q. Then calculate $M_q = E_q - e\sin E_q$.

Verification: $\cos E_q = \cos(2\pi - E_q)$ are the two eccentric anomalies where the star moves out of/into the queried radius.

$$2\pi - E_q - e\sin(2\pi - E_q) = 2\pi - (E_q - e\sin E_q) = 2\pi - M_q$$