

1 Symbols

- a : length of semimajor axis, mean of periapsis and apoapsis
- b : length of semiminor axis
- E : Eccentric anomaly
- e : Eccentricity. In this file, if there are any references to Euler's number, it is denoted as $\exp(1)$ etc..
- h : Angular momentum
- M : Mean anomaly
- n : Average sweep, the average angular velocity per unit time
- ν : True anomaly, angle [periapsis]-[star]-[body]
- p : Semi-latus rectum, length from star to orbit parallel to minor axis
- r : Distance from star
- \mathbf{r} : Position relative to star
- θ : The argument of a body in the standard polar coordinate system
- u : Argument of latitude
- v : Magnitude of velocity
- \mathbf{v} : Velocity
- ω : Longitude of periapsis, or equivalently the argument of periapsis, assuming longitude of ascending node is zero in 2D
- ϵ : Specific orbital energy, sum of potential and kinetic energy

2 Equations

2.1 Orbital states to orbital elements

In 2D, for a star with fixed position (the origin) and mass, orbits have 4 degrees of freedom.

Defining the ellipse requires 3 degrees of freedom:

- Shape: fixed with e
- Size: fixed with a
- Rotation: fixed with ω

For efficient computation over time, the position on the ellipse is represented by M .

The orbital states (\mathbf{r}, \mathbf{v}) , with 4 degrees of freedom, are bijective to the orbital elements. Based on [cart2kep2002.pdf](#)¹, the following procedures convert Cartesian orbital states to the orbital elements in 2D, based on the external parameter μ :

¹<https://web.archive.org/web/20160418175843/https://ccar.colorado.edu/asen5070/handouts/cart2kep2002.pdf>

$$\begin{aligned}
h &= r_x v_y - r_y v_x \\
\epsilon &= \frac{1}{2} \|\mathbf{v}\|^2 - \frac{\mu}{\|\mathbf{r}\|} \\
a &= \frac{-\mu}{2\epsilon} \\
e &= \sqrt{1 - \frac{h^2}{a\mu}} \\
u &= \\
p &= a(1 - e^2) \\
\nu &= \arccos \frac{p - \|\mathbf{r}\|}{\|\mathbf{r}\| e} \\
\nu &= \arctan \frac{\sqrt{\frac{p}{\mu}} (\mathbf{r} \cdot \mathbf{v})}{p - \|\mathbf{r}\|} \\
\omega &= u - \nu \\
E &= 2 \arctan \left(\sqrt{\frac{1-e}{1+e}} \tan \frac{\nu}{2} \right) \\
M &= E - e \sin E
\end{aligned}$$

This gives (e, a, ω, M) as the orbital elements.

2.2 Change of mean anomaly over time

Given mean anomaly M at time t , the new mean anomaly M' at time t' can be found by

$$M' = M + n(t' - t),$$

where the average sweep n can be found with

$$n = \sqrt{\frac{\mu}{a^3}}.$$

2.3 Radius

2.3.1 Testing if orbit is above/below radius

For a frequently queried radius r_q , precompute the (smaller) mean anomaly M_q such that $r(M_q) = r(2\pi - M_q) = r_q$. Then $r(t) > r_q \iff M_q < M(t) < 2\pi - M_q \pmod{2\pi}$.

For a specific r_q , solve $\cos E_q = \frac{1}{e} \left(1 - \frac{r}{a}\right)$ for E_q . Then calculate $M_q = E_q - e \sin E_q$.

Verification: $\cos E_q = \cos(2\pi - E_q)$ are the two eccentric anomalies where the star moves out of/into the queried radius.

$$2\pi - E_q - e \sin(2\pi - E_q) = 2\pi - (E_q - e \sin E_q) = 2\pi - M_q$$

2.4 Bearing

Bearing is the eviov name for argument in the polar coordinate system.

The bearing of the periapsis is ω . The bearing at specific true anomaly ν is $\omega + \nu$.

2.4.1 Approximating the bearing

This is done by direct approximation of true anomaly by Fourier expansion:

$$\theta = \omega + \nu \approx \omega + M + \sum_{i=1}^{\infty} \text{TODO} e^i \sin(iM)$$

2.4.2 Testing if orbit is between bearing

For a frequently queried bearing range (θ_1, θ_2) , precompute the mean anomalies M_1, M_2 such that $\theta(M_1) = \theta_1$ and $\theta(M_2) = \theta_2$. Then $\theta(t) \in (\theta_1, \theta_2) \iff M(t) \in (M_1, M_2)$.

Compute

$$E_q = 2 \arctan \left(\sqrt{\frac{1-e}{1+e}} \tan \frac{\theta_q - \omega}{2} \right)$$

for $q = 1, 2$. Then $M_q = E_q - e \sin E_q$.