Chasing the blend

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Abstract

We model the mathematical process whereby new mathematical theories are produced, involving shared and individual creativity. Here we provide rational reconstructions of some developments from mathematical history; our longer-term goal is to support machine and human mathematical creativity.

Introduction

To be written by Alan.

Background

Blending in Mathematics

Alison?

Image Schemas

Marco

Blending and the infinite

Marco, Ewen, Alan, Felix

Naturals and Integers

Potential and actual infinity

Integers and Ideals

Danny, Felix, Alan

A Simple Example – the Integers

In order to demonstrate the machinery involved in blending mathematical theories, we consider combining a theory of natural numbers with the concept of the inverse of a function to obtain the integers. Let us assume an simple axiomatisation of the natural numbers (without order axioms) as shown in Figure 1, and call this theory $\mathbb{N}$. Now let us also define a simple theory which introduces the concept of a function with an inverse as shown in Figure 2, and call this theory $\mathbb{F}$.

```
spec NAT =
     sort  Nat
     ops   zero : Nat;
           s : Nat → Nat;
           __+__ : Nat × Nat → Nat
     ∀ x, y, z : Nat
     • s(x) = y ∧ s(x) = z ⇒ y = z
     • s(x) = s(y) ⇒ x = y
     • ∃ a : Nat • s(x) = a
     • ¬ s(x) = zero
     • s(x) + y = s(x + y)
     • zero + y = y
end
```

Figure 1: A theory of the natural numbers without order

```
spec FUNC =
     sort  X
     op    f : X → X
     op    finv : X → X
     ∀ x : X
     • f(finv(x)) = x
     • finv(f(x)) = x
end
```

Figure 2: A theory with a function and its inverse defined

Identifying a Generic Space In order to incorporate the notion of blending here we want to be able to identify a "generic" component of each theory and compute the pushout as discusses in §**??**. We can use the HDTP system **GustKS2006**; **Schmidt2010** to discover a common theory and signature morphism between symbols in the two theories $\mathbb{N}$ and $\mathbb{F}$. The Generic theory contains a sort N and a function $func$, and the morphisms from the Generic theory to $\mathbb{N}$ and $\mathbb{F}$ are:

$$s \quad \leftarrow_{g_{\mathbb{N}}} \quad func \quad \rightarrow_{g_{\mathbb{F}}} \quad f \qquad (1)$$

$$Nat \quad \leftarrow_{g_{\mathbb{N}}} \quad N \quad \rightarrow_{g_{\mathbb{F}}} \quad X \qquad (2)$$

Here the successor function is identified in the mapping with the function in the theory $\mathbb{F}$, and g_K is the label for the set of symbol mappings determined by the signature morphism from the Generic space the theory K.

Computing the Colimit The HETS system**MossakowskiEA06** can then be exploited to find a new theory by computing the colimit:

$$
\begin{tikzcd}
 & \text{Gen} \arrow[dl, "g_{\mathbb{N}}"'] \arrow[dr, "g_{\mathbb{F}}"] & \\
\mathbb{N} \arrow[dr, "b_{\mathbb{N}}"'] & & \mathbb{F} \arrow[dl, "b_{\mathbb{F}}"] \\
 & \text{Blend} &
\end{tikzcd}
$$

This generates the theory shown in 3.

spec SPEC =
sort Z
op $\_\_+\_\_ : Z \times Z \to Z$
op $p : Z \to Z$
op $s : Z \to Z$
op $zero : Z$
$\forall\, x, y, z : Z \bullet s(x) = y \wedge s(x) = z \Rightarrow y = z$ %(Ax1)%
$\forall\, x, y : Z \bullet s(x) = s(y) \Rightarrow x = y$ %(Ax2)%
$\forall\, x : Z \bullet \exists\, a : Z \bullet s(x) = a$ %(Ax3)%
$\forall\, x : Z \bullet \neg\, s(x) = zero$ %(Ax4)%
$\forall\, x, y : Z \bullet s(x) + y = s(x + y)$ %(Ax5)%
$\forall\, y : Z \bullet zero + y = y$ %(Ax6)%
$\forall\, x : Z \bullet s(p(x)) = x$ %(Ax1_7)%
$\forall\, x : Z \bullet p(s(x)) = x$ %(Ax2_8)%

Figure 3: An inconsistent version of the integers (without order)

Removal of Inconsistencies This theory is automatically determined to be inconsistent due to the axioms

$$\forall x : \mathbb{Z}.\ \neg s(x) = 0 \tag{3}$$

$$s(p(x)) = x \tag{4}$$

removal of the limiting axiom (3) results in a theory which is very similar to what we understand to be the integers as shown in Figure 4.

Running the Blend Running the blend refers to discovering axioms or definitions which make the blend incomplete. In the example of the version in Figure 4, the definition of plus needs to be extended to understand how to calculate with the predecessor function:

$$p(x) + y = p(x + y)$$

from which theorems such as

$$p(x) + s(y) = x + y$$

can be discovered.

Related Example

— especially Galois.

spec SPEC =
sort Z
op $\_\_+\_\_ : Z \times Z \to Z$
op $p : Z \to Z$
op $s : Z \to Z$
op $zero : Z$
$\forall\, x, y, z : Z \bullet s(x) = y \wedge s(x) = z \Rightarrow y = z$ %(Ax1)%
$\forall\, x, y : Z \bullet s(x) = s(y) \Rightarrow x = y$ %(Ax2)%
$\forall\, x : Z \bullet \exists\, a : Z \bullet s(x) = a$ %(Ax3)%
$\forall\, x, y : Z \bullet s(x) + y = s(x + y)$ %(Ax4)%
$\forall\, y : Z \bullet zero + y = y$ %(Ax5)%
$\forall\, x : Z \bullet s(p(x)) = x$ %(Ax1_7)%
$\forall\, x : Z \bullet p(s(x)) = x$ %(Ax2_8)%
end

Figure 4: A consistent version of the integers (without order)

Galois Theory

Danny, Joe, Felix, Ewen

Issues raised

Alan, Ewen, Felix

Evaluation and Outlook

Joe et al.

Conclusions

(and references) – everyone!