Chasing the blend

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Abstract

We model the mathematical process whereby new mathematical theories are produced, involving shared and individual creativity. Here we provide rational reconstructions of some developments from mathematical history; our longer-term goal is to support machine and human mathematical creativity.

Introduction

To be written by Alan.

Background

Blending in Mathematics

Alison?

Image Schemas

Marco

Blending and the infinite

Marco, Ewen, Alan, Felix

Naturals and Integers

Potential and actual infinity

Prime Ideals as a blend

Introduction

One of the most fundamental concepts of modern mathematics, which is the basis of commutative algebra and a seminal ingredient of the language of schemes in modern algebraic geometry is the one of prime ideal (Grothendieck and Diedonné, 1971; Eisenbud, 1995).

In this section, we will recover the concept of prime ideal of a commutative ring with unity as a sort of partial (or weaken) blending (i.e. a blend for just some axioms of the input theories) between the concepts of an ideal of a commutative ring with unity (enriched with the collection of all the ideals of the corresponding ring) and the concept of a prime number of the integers.

In fact, in order to obtain the desired space it is enough to consider a more general version of the prime numbers (in our case a partial version), namely, a monoid $(Z, *, 1)$ with an "special" divisibility relation $\lfloor$. Besides, the generic space would capture just the syntactic correspondences that we wish to identity in the blending space, since the blend would be basically the union of the collection of axioms given on each space, doing the corresponding identifications.

Our approach to blending is the one adopted by Goguen in terms of colimits (Goguen, 1999, 2001, 2005).

We present the conceptual spaces from the standard "pure" mathematical point of view doing concurrently the corresponding translation into the setting of the Common Algebraic Specification Language (CASL) (Bidoit and Mosses, 2004).

The first conceptual space

Let $(R, +.*, 0, 1)$ be a commutative ring with unity, i.e. R is a set with two binary operations, $+$ and $*$, and two special elements $0, 1 \in R$ satisfying the following axioms:

1. $(\forall a \in R)(a + 0 = 0 + a = a)$
2. $(\forall a \in R)(\exists b \in R)(a + b = b + a = 0)$
3. $(\forall a, b, c \in R)((a + b) + c = a + (b + c)))$
4. $(\forall a, b \in R)(a + b = b + a)$
5. $(\forall a \in R)(a * 1 = 1 * a = a)$
6. $(\forall a, b, c \in R)((a * b) * c = a * (b * c)))$
7. $(\forall a, b \in R)(a * b = b * a)$
8. $(\forall a, b, c \in R)(a * (b + c) = a * b + a * c)$

Now, R can be understood as the sort containing the elements of the corresponing commutative ring with unity. An ideal I is a subset of R satisfying the following axiom:

$$(\forall i, j \in I)(\forall r \in R)(i + (-j) \in I \wedge r * i \in I).$$

Let us define a unary relation (predicate) $isideal$ on the set (sort) of subsets of R, $P(R)$, as follows: $isideal(I)$ if and only if I is an ideal of R.

Now, we define

$$\mathrm{Spec}_I R = \{A \subseteq R : isideal(A)\}.$$

Here, $\mathrm{Spec}_I R$ is considered as a subsort of the sort $P(R)$.

There is one natural operation on $\mathrm{Spec}_I R$, let us say $\cdot_\iota$, inherited in a natural way from the corresponding operations $+$ and $\cdot$ on S:

Let $I, J \in \mathrm{Spec}_I R$, then we define

$$I \cdot_\iota J := \{i_1 \cdot j_1 + ... + i_n \cdot j_n : n \in \mathbb{N} \wedge i_k \in I \wedge j_k \in J\}.$$

With this operation $\mathrm{Spec}_I R$ forms a commutative monoid (i.e. it holds commutativity, associativity and there exists a neutral element (in this case the ring)). However, this fact is irrelevant in our case for the blending process. As a matter of fact, the only property that we want to keep into the blend is the one saying that this operation has a neutral element 1_ι, which can be seen as an additional notation for the ring, but respect to this operation $\cdot_\iota$ instead of being the sort of elements of the ring, i.e., R.

On the other side, we want to see the contention relation $\subseteq$ as a binary relation over the sort $\mathrm{Spec}_I R$.

Summarizing, our first conceptual space consists of sorts $R, \mathrm{Spec}_I R$ and $P(R)$; operations $+_R, *_R, 0_R, 1_R, 1_\iota$ and $\cdot_\iota$; and the relations $\subseteq$ and $isideal$.

Here we add all the corresponding axioms defining R as a commutative ring, the explicit former definition of $isideal$, $\mathrm{Spec}_I R$ and $\cdot_\iota$; and the axiom guaranteeing that 1_ι is the neutral element for $\cdot_\iota$.

Let us denote this space by $\mathbb{I}$.

The second conceptual space

Let $\mathbb{Z}$ be the set of the integer numbers. Here, we can choose any axiomatization of them, since for the (partial) blending we just take into account only the fact that $(\mathbb{Z}, *, 1)$ is a commutative monoid. Or even simpler, we only use the fact that 1 is the neutral element with the operation $*$. One can, for example, take the simple characterization of $\mathbb{Z}$, given by Martin Brandenburg ('Possible axioms for Integers', 2010), as the only ordered commutative ring with unity satisfying the following "bi-inductive" property:

$$\forall M \subseteq \mathbb{Z}\, [0 \in M \wedge (\forall n \in \mathbb{Z}(n \in M \rightarrow n \pm 1 \in M)) \\ \rightarrow M = \mathbb{Z}].$$

We define also an upside down divisibility relation $\lfloor$ defined as

$$e \lfloor g := g | e,$$

We re-write the classical divisibility relation on this way in order to obtain the right primality condition on the blend. Let us define a unary relation $isprime$ on $\mathbb{Z}$ as follows: for all $p \in \mathbb{Z}$, $isprime(p)$ holds if $p \neq 1$ and the following (primality) condition holds:

$$(\forall a, b \in \mathbb{Z})(ab \lfloor p) \rightarrow (a \lfloor p \vee b \lfloor p).$$

Besides, we define the set (sort) of the prime numbers as

$$Prime = \{p \in \mathbb{Z}/isprime(p)\}$$

Now, it is an elementary fact to see that this condition is an equivalent form of the standard definition of prime number given in the classical number theory books (see for example Apostol, 1976. In the CASL language, we consider $\mathbb{Z}$ as the sort of the integer numbers, $*$ as a binary operation , $prime$ as a predicate and $\lfloor$ as a binary relation, any of them defined over the sort $\mathbb{Z}$.

We denote this conceptual space by $\mathbb{P}$.

The Generic Space

The generic space consists of a set (sort) G with a binary operation $*_G$, a neutral element S and a binary relation $\leq_G$.

Let us denote this space by $\mathbb{G}$.

The Blending Morphisms

Now, let us define the morphisms from the generic space into the two corresponding conceptual spaces. Let $\varphi : \mathbb{G} \rightarrow \mathbb{I}$ be the morphism induced by the following syntactic correspondences $\varphi(G) = \mathrm{Spec}_I R, \varphi(*_G) = *_\iota, \varphi(S) = 1_\iota$ and $\varphi(\leq_G) = \subseteq$.

Furthermore, let $\delta := \mathbb{G} \rightarrow \mathbb{P}$ be the morphism induced by the syntactic correspondences $\delta(G) = \mathbb{Z}, \delta(*_G) = *, \delta(S) = 1$ and $\delta(\leq_G) = \lfloor$.

The Axiomatization of the Blending

In the every-day research of the working mathematician it happens frequently that one starts to develop general theoretical frameworks by combining just some aspects of two particular theories but without considering the whole theories. For example, the development of differential geometry was obtained combining just some aspect of general and algebraic topology and some aspects of real analysis (Velez and Cadavid, 2005). The same happens with the methods use in analytic number theory which are a fusion of some components of elementary number theory and some of the real analysis techniques (Apostol, 1976).

Therefore, it is more natural in the daily mathematical research to obtain new concepts as "partial" combinations of two former ones, i.e., as combinations (blends) of just some axioms of the corresponding two theories.

Thus, in our case, a partial blend will give us the desired concept. For example, from the properties defining the integers we transfer into the blend only the fact that $\mathbb{Z}$ is a set with a binary operation $*$ having 1 as neutral element.

So, after using the same symbols for denoting the ring as a sort of elements or as the neutral element for product of ideals $\cdot_G$, the blend has the form

$$(S, +_S, *_S, 0_S, 1_S, G = \mathrm{Spec}_I S, isprime, Prime, \cdot_G, S = 1_G, \subseteq)$$

with all the corresponding axioms of the first conceptual space plus the translated version of the axiom defining the primality predicate after doing the corresponding symbolic identifications i.e., an element $P \in G$ (i.e., an ideal of S) satisfied the predicate $isprime$ if and only if

$$P \neq S \wedge (\forall X, Y \in G = \mathrm{Spec}_I S). \\ (X \cdot_\alpha Y \subseteq P \rightarrow (X \subseteq P \vee Y \subseteq P)).$$

Now, it is an elementary exercise to see that this definition is equivalent to the fact that P is a prime ideal of S, i.e. to the condition

$$P \neq S \wedge (\forall a, b \in S)(ab \in B \rightarrow (a \in P \vee b \in P)).$$

Therefore, the predicate $isprime$ turns out to be the predicate characterizing the primality of ideals of S and the set (sort) $Prime$ turns out to be the set of prime ideals of S.

Besides, we just consider the fact that the up-side down divisibility relation is a binary relation without taking into account the formal definition into the blend.

In conclusion, the blending space consists of the axioms assuring that S is a commutative ring with unity, G is the set of ideals of S, $isprime$ is the predicate specifying primality for ideals of S and $Prime$ is the collection of all prime ideals of S.

Implementation for the Principal Ideal Domain Case

On this section we present an implementation done in Hets (Mossakowski, Maeder, and Codescu, 2014), in the language of CASL for the case of a principal ideal domain (PID). The resulting blending space contains two equivalent definitions of the containing relation for ideals. One of them is the trivial one in terms of elements and the other one is given in terms of product of ideals. It is an elementary exercise to see this equivalence in the PID case.

```
library ideal_blend

logic CASL
%% Prime Ideals Over Principal Ideal Domains as a Blend

spec IdealsOfRing =
     sort  RingElt
           %% sort of Ring Elements
     sort  SubSetOfRing
           %% sort of parts of this ring
     pred  IsIdeal : SubSetOfRing
           %% when a subset is an ideal
     op    0 : RingElt
     op    1 : RingElt
     op    __∗__ : RingElt × RingElt → RingElt
     op    __+__ : RingElt × RingElt → RingElt
     pred  __isIn__ : RingElt × SubSetOfRing
     sort  Ideal = {I : SubSetOfRing • IsIdeal(I)}
     op    R : Ideal
           %% the Ring as an ideal
     op    __∗∗__ : Ideal × Ideal → Ideal, unit R
           %%Definition of the predicate of contention
     pred  __issubsetOf__ : Ideal × Ideal
     ∀ A, B : Ideal
     • A issubsetOf B ⇔ ∀ a : RingElt • a isIn A ⇒ a isIn B

     %% axiomatization of a commutative Ring with unity
     ∀ x : RingElt; y : RingElt • x + y = y + x
     ∀ x : RingElt; y : RingElt; z : RingElt
     • (x + y) + z = x + (y + z)
     ∀ x : RingElt • x + 0 = x ∧ 0 + x = x
     ∀ x : RingElt • ∃ x' : RingElt • x' + x = 0
     ∀ x : RingElt; y : RingElt • x ∗ y = y ∗ x
     ∀ x : RingElt; y : RingElt; z : RingElt • (x ∗ y) ∗ z = x ∗ (y ∗ z)
     ∀ x : RingElt • x ∗ 1 = x ∧ 1 ∗ x = x
     ∀ x, y, z : RingElt • (x + y) ∗ z = (x ∗ z) + (y ∗ z)
     ∀ x, y, z : RingElt • z ∗ (x + y) = (z ∗ x) + (z ∗ y)

     %%axioms for Ideal
     ∀ I : SubSetOfRing
     • IsIdeal(I)
       ⇔ ∀ a, b, c : RingElt
         • ((a isIn I ⇒ a isIn R) ∧ 0 isIn I)
           ∧ (a isIn I ∧ c isIn R ⇒ c ∗ a isIn I)
           ∧ (a isIn I ∧ b isIn I ∧ c isIn R ∧ b + c = 0
              ⇒ a + c isIn I)

     %%axioms for PID−s
     %%every ideal is generated by an element
     pred  __generates__ : RingElt × Ideal
     ∀ A : Ideal • ∃ a : RingElt • a generates A
     ∀ a : RingElt
     • ∀ A : Ideal
       • a generates A
         ⇔ ∀ c : RingElt • c isIn A ⇒ ∃ d : RingElt • c = a ∗ d

     %% Definition of the product of ideals without subindexes
     ∀ A, B : Ideal
     • ∀ a, b : RingElt • a isIn A ∧ b isIn B ⇒ a ∗ b isIn A ∗∗ B
     • ∀ D : Ideal
       • (∀ a, b : RingElt • a isIn A ∧ b isIn B ⇒ a ∗ b isIn A ∗∗ B)
         ⇒ A ∗∗ B issubsetOf D
end


%%%axioms defining a very simple version of the integers,
%%%considered with an operation ∗ with neutral element,
%%% a binary relation || (upside−down divisibility relation)
%%% and a primality axiom.
spec SimpleInt =
     sort  SimpleElem
     ops   1 : SimpleElem;
           __x__ : SimpleElem × SimpleElem → SimpleElem,
           unit 1
     preds __||__ : SimpleElem × SimpleElem;
           IsPrime : SimpleElem
           %Def_upsidedownDivisilityRelation%
     ∀ x, y : SimpleElem • x || y ⇔ ∃ c : SimpleElem • x = y x c

     %% subsort of primes
     sort  SimplePrime = {p : SimpleElem • IsPrime(p)}
     ∀ p : SimpleElem
     • IsPrime(p)
       ⇔ (∀ a, b : SimpleElem • a x b || p ⇒ a || p ∨ b || p) ∧ ¬ p = 1
     %Def_primality%
end


%%% Generic space
spec Gen =
     sort  Generic
     ops   S : Generic;
           __gpr__ : Generic × Generic → Generic, unit S
     pred  gcont : Generic × Generic
end

view I1 :
     Gen to IdealsOfRing =
```

$Generic \mapsto Ideal$, $S \mapsto R$, $\_\_gpr\_\_ \mapsto \_\_{**}\_\_$,
$gcont \mapsto \_\_issubsetOf\_\_$
end

view I2 :
GEN **to** SIMPLEINT =
$Generic \mapsto SimpleElem$, $S \mapsto 1$, $\_\_gpr\_\_ \mapsto \_\_x\_\_$,
$gcont \mapsto \_\_||\_\_$
end

spec COLIMIT =
combine *I1*, *I2*
end

After computing the corresponding colimit in HETS and after interpreting "RingEl" as the sort containing the elements of the ring S, the theory defining the blend corresponds to the axioms defining a PID (S), the set of all its ideals (Generic) and the set all its prime ideals (SimplePrime):

library *ideal_colim*

logic CASL.SULFOL=

spec SPEC =
sorts *Generic*, *RingElt*, *SimplePrime*, *SubSetOfRing*
sorts *SimplePrime* < *Generic*;
Generic < *SubSetOfRing*
op $0 : RingElt$
op $1 : RingElt$
op $S : Generic$
op $\_\_*\_\_ : RingElt \times RingElt \to RingElt$
op $\_\_+\_\_ : RingElt \times RingElt \to RingElt$
op $\_\_x\_\_ : Generic \times Generic \to Generic$
pred $IsIdeal : SubSetOfRing$
pred $IsPrime : Generic$
pred $\_\_generates\_\_ : RingElt \times Generic$
pred $\_\_isIn\_\_ : RingElt \times SubSetOfRing$
pred $gcont : Generic \times Generic$
$\forall I : SubSetOfRing \bullet I \in Generic \Leftrightarrow IsIdeal(I)$ %(Ax1)%
$\forall x : Generic$
$\bullet\ x\ x\ S = x$
%(ga_right_unit___**__)%
$\forall x : Generic$
$\bullet\ S\ x\ x = x$
%(ga_left_unit___**__)%
$\forall A, B : Generic$
$\bullet\ gcont(A, B) \Leftrightarrow \forall a : RingElt \bullet a\ isIn\ A \Rightarrow a\ isIn\ B$ %(Ax4)%
$\forall x, y : RingElt \bullet x + y = y + x$ %(Ax5)%
$\forall x, y, z : RingElt \bullet (x + y) + z = x + (y + z)$ %(Ax6)%
$\forall x : RingElt \bullet x + 0 = x \wedge 0 + x = x$ %(Ax7)%
$\forall x : RingElt \bullet \exists x' : RingElt \bullet x' + x = 0$ %(Ax8)%
$\forall x, y : RingElt \bullet x * y = y * x$ %(Ax9)%
$\forall x, y, z : RingElt \bullet (x * y) * z = x * (y * z)$ %(Ax10)%
$\forall x : RingElt \bullet x * 1 = x \wedge 1 * x = x$ %(Ax11)%
$\forall x, y, z : RingElt \bullet (x + y) * z = (x * z) + (y * z)$ %(Ax12)%
$\forall x, y, z : RingElt \bullet z * (x + y) = (z * x) + (z * y)$ %(Ax13)%
$\forall I : SubSetOfRing$
$\bullet\ IsIdeal(I)$
$\Leftrightarrow \forall a, b, c : RingElt$
$\bullet\ ((a\ isIn\ I \Rightarrow a\ isIn\ S) \wedge 0\ isIn\ I)$
$\wedge\ (a\ isIn\ I \wedge c\ isIn\ S \Rightarrow c * a\ isIn\ I)$
$\wedge\ (a\ isIn\ I \wedge b\ isIn\ I \wedge c\ isIn\ S \wedge b + c = 0$
$\Rightarrow a + c\ isIn\ I)$
%(Ax14)%
$\forall A : Generic \bullet \exists a : RingElt \bullet a\ generates\ A$ %(Ax15)%
$\forall a : RingElt; A : Generic$
$\bullet\ a\ generates\ A$
$\Leftrightarrow \forall c : RingElt \bullet c\ isIn\ A \Rightarrow \exists d : RingElt \bullet c = a * d$
%(Ax16)%
$\forall A, B : Generic; a, b : RingElt$
$\bullet\ a\ isIn\ A \wedge b\ isIn\ B \Rightarrow a * b\ isIn\ A\ x\ B$ %(Ax17)%
$\forall A, B, D : Generic$
$\bullet\ (\forall a, b : RingElt \bullet a\ isIn\ A \wedge b\ isIn\ B \Rightarrow a * b\ isIn\ A\ x\ B)$
$\Rightarrow gcont(A\ x\ B, D)$
%(Ax18)%
$\forall x, y : Generic$
$\bullet\ gcont(x, y) \Leftrightarrow \exists c : Generic \bullet x = y\ x\ c$ %(Ax3)%
$\forall p : Generic \bullet p \in SimplePrime \Leftrightarrow IsPrime(p)$ %(Ax4_19)%
$\forall p : Generic$
$\bullet\ IsPrime(p)$
$\Leftrightarrow (\forall a, b : Generic$
$\bullet\ gcont(a\ x\ b, p) \Rightarrow gcont(a, p) \vee gcont(b, p))$
$\wedge \neg\ p = S$
%(Ax5_20)%
end

Related Example

— especially Galois.

Galois Theory

Danny, Joe, Felix, Ewen

Issues raised

Alan, Ewen, Felix

Evaluation and Outlook

Joe et al.

Conclusions

(and references) – everyone!

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