Algorithm

Defn: It is combination of sequence of finite steps to solve a particular problem.

Ex: addition of 2 numbers

```
add ()
{
  ① Take 2-num (a, b)
  ② c = a+b;
  ③ printf (c);
}
```


Properties of Algorithm :-

1. It should terminate after finite time.
2. It should produce atleast 1 output
3. It is independent of the programming language.
4. Every statement in the algorithm should be unambiguous.
   (For random generator, the algorithm might be non-deterministic i.e. produce diff outputs on same i/p)

Steps required to design algorithm :-

1. Problem definition. (knowing problem clearly)
2. Design algo [select one of the existing algorithms]
   ex: Divide & Conquer ....
3. Draw flowchart.
4. Testing
5. Coding / Implementation.

⑥ Analysis (finding time and space complexity.
↓
CPU time main memory space.

CH-1 Analysis of Algorithms

If the problem contains more than 1 solution Best one will be ~~defi~~ decided by analysis based on the ~~time~~ 2 factors
- Time
- Space.

Time Complexity :

$$T(P) = C(P) + R(P)$$

- $T(P)$ → Time complexity
- Program
- $C(P)$ → Compile Time → Compiler (S/W) → depends on Lang & Compiler
- $R(P)$ → Run Time → processor → H(W) → Type of H/W.

Java compiler can be written in C Lang.
C program executes in less time as compared to Java.
Earlier, Lang (Java) compiler cannot be written in Java But now it can be.
Bootstraping :- using others (ex. Java C for Compliers) for your use if you are incapable of it & later

Types of Analysis

postpone

Silent

A Posteriary analysis

① It is dependent of Lang of compiler and type of Hardware

② Exact

③ Time complexity changes from computer to computer H/w (Ex: i3 – slow, i7 – fast)

A Priori Analys

① It is independent of Lang. of compiler & type of H/w.

② Approximate.

③ Time complexity is constant in every computer.

Apriori Analysis :-

It is a determination of order of magnitude of a statement.

order of mag. → No of times a stmt will execute by processor

Ex-1

```
main ()
{
  x = y+z;  — 1
}
```

Time complexity = O(1)

Big-O notation

Ex-2

```
main ()
{
① x = y+z;  —— 1
```

```
for (i=1; i<=n; i++)
{
② x=y+z; ———————— n
}
}
```

Time complexity = $n+1 = O(n)$

↑ Larger of $n, 1$

Ex3.

```
main()
{
① x= y+z; ———————— 1
  for (i=1 to n)
  {
② x= y+z; ———————— n
  }
  for (i=1 to n)
  {
    for (j=1 to n)
    {
③ x= y+z; ——— n²
    }
  }
}
```

if change 'j' for loop like this

```
for (j=1 to n/2)
{
}
```

Time complexity = $n \cdot \frac{n}{2}$

$= \frac{n^2}{2} = O(n^2)$

Time complexity = $n^2+n+1 = O(n^2)$

Time complexity means finding the largest loop. If no loop in the program, then constant time complexity (1).

↓ where CPU spending max. time

$n=8 \quad 8/2 = 4/2 = 2/2 = 1$

$n=5 \quad 5/2 = 2.5/2 = 1.25/2 = 0.625$

$n=6 \quad 6/2 = 3/2 = 1.5/2 = 0.75$

Ex 4:

Divide

```
main()
{
    while (n >= 1)
    {
        n = n/2;
    }
}
```

$\longrightarrow \log_2 n$

$64 \geq 1$ → n

$32 \geq 1$ → $n/2$

$16 \geq 1$ → $n/4 = n/2^2$

$8 \geq 1$ → $n/8 = n/2^3$

$4 \geq 1$ → $n/16 = n/2^4$

$2 \geq 1$ → $n/32 = n/2^5$

$1 \geq 1$ → $n/64 = n/2^6$

$\log_2 n$

$16 \Rightarrow 5 = (4+1)$

$32 \Rightarrow 6$

$64 \Rightarrow 7$

$\Rightarrow \frac{n}{2^k} = 1$

$n = 2^k$

$\boxed{k = \log_2 n}$

If condition changed to $(n > 2)$ then $\frac{n}{2^k} = 2$

$\Rightarrow 2^{k+1} = n$

$k = \log_2 n - 1$

If $n = n/2$ changed to $n = n/3$ then $\frac{n}{3^k} = 1$

$\Rightarrow k = \log_3 n$

If inside while loop, $\left.\begin{matrix} n = n/2; \\ n = n/3; \end{matrix}\right\} \Rightarrow n = n/6$

Ex-5

Multiply

```
main()
{
  i = 1;
  while (i <= n)
  {
    i = 2 * i;
  }
}
```

Termination condtn is $i > n$

$n = 16$
$i = 1$
2
4
⋮
$2^k = 16$

$\log_2 2^k = \log_2 2^4$

$\Rightarrow k = 4 = \log_2 n$

Time Complexity $= \log_2 n + 1$

$= O(\log_2 n)$

- If $i = 2 * i$ is replaced by $i = \underline{20} * i$, then time complexity $= O(\log_{\underline{20}} n)$
- If i value initialized to 10, no change in time complexity, $\log_2 n - 10 = O(\log_2 n)$
- If inside while, $\left.\begin{matrix} i = 2 * i; \\ i = 3 * i; \end{matrix}\right\} i = 6 * i;$
- If inside while, $\left.\begin{matrix} i = 2 * i; \\ i = 3 * i; \\ i = 5 * i; \\ i = i/10; \end{matrix}\right\} i = 3 * i; \Rightarrow \log_3 n$

Ex 6. Subtraction

```
main()
{
  while (n >= 1)
  {
    n = n - 1;
  }
}
```

$n \to n+1 = O(n)$

n, $n-1$, $\vdots$, $\frac{1}{0}$

```
while(n > 1)
{
  n = n - 10;
}
```

$n-10$, $n-20$, $\vdots$

$n - 10 \cdot k = 1$

$k = \frac{n-1}{10} = O(n)$

```
while (n > 15)
{
  n = n - 50;
}
```

$n - 50k = 15$

$k = \frac{n-15}{50} = O(n)$

```
while (n > 1)
{
  n = n - 10;
  n = n - 20;
}
```

$\Big\}\ n = n - 30;$

Ex 7:

Addition

```
main()
{
    i = 0;
    while (i < n)
    {
        i = i + 1;
    }
}
```

$n = 100$

0, 1, 2, 3, ..., 100 } $n + 1 = O(n)$

```
i = 1;
while (i < n)
{
    i = i + 10
}
```

1, 1+10, 1+2*10, ..., 1+k*10 = n

$k = \frac{n-1}{10} = O(n)$

```
while (i < n)
{
    i = i + 10
    i = i + 20
}
```

$i = i + 30 \Rightarrow n/30 = O(n)$

```
while (i < n)
{
    i = i + 10
    i = i + 20
    i = i - 5
}
```

$i = i + 25 \Rightarrow n/25 = O(n)$

Ex 8

```
main()
{
    while (n>2)
    {
        n = √n
    }
}
```

Time Complexity = $\log_2(\log_2 n)$

$n = 16$

$\sqrt{16} = 4$ $\quad n^{1/2}$

$\sqrt{4} = 2$ $\quad n^{1/4}$

$n^{1/2^k}$ $\quad n^{1/2^k}$

$n^{1/2^k} = 2$

$\Rightarrow \frac{1}{2^k} = \log_n 2$

$2^k = \frac{1}{\log_n 2}$

$\Rightarrow k = \log_2\left(\frac{1}{\log_n 2}\right)$

$= \log_2(\log_2 n)$

-
```
while (n > 20)
{
    n = √n
}
```

$\Rightarrow n^{1/2^k} = 20$

$\frac{1}{2^k}\log_{20} n = 1$

$\Rightarrow 2^k = \log_{20} n$

$\log_2 2^k = \log_2 \log_{20} n$

$\boxed{k = \log_2 \log_{20} n}$

$\log_2$ due to $\sqrt{}$; $\log_{20}$ due to while condtn

-
```
while (n > 125)
{
    n = n^(1/23)
}
```

$\Rightarrow \log_{23} \log_{125} n$

-
```
while (n > 125)
{
    n = n^(1/5)
    n = n^(1/3)
}
```
} $n = n^{1/4}$

$\Rightarrow \log_{15} \log_{125} n$

Ex 9.

```
main ()
{  i=2
   while (i<n)
   {
      i = i^2
   }
}
```

$\Rightarrow O(\log_2 \log_2 n)$

2
2^2
2^4
:
2^{2^k}

$2^{2^k} = n$

$\log_2 2^{2^k} = \log_2 n$

$\Rightarrow 2^k = \log_2 n$

$\Rightarrow \log_2 2^k = k = \log_2 \log_2 n$

-
```
i=2
while (i<n)
{
   i = i^23
}
```

2
2^{23}
$(2^{23})^{23} = 2^{23^2}$
:
2^{23^k}

$2^{23^k} = n$

$\log_2 2^{23^k} = \log_2 n$

$23^k = \log_2 n \Rightarrow \boxed{k = \log_{23} \log_2 n}$

-
```
i=25
while (i<n)
{
   i = i^2
   i = i^3
}
```

$i = i^6 \Rightarrow \log_6 \log_{25} n$

```
while (n > 5)
{
    n = n - 2
    n = n / 2
    n = √n
}
```

If simplification not possible, remove the unnecessary & take that stmt which has max impact.
Here, $n = \sqrt{n}$

Ex 10:

```
main()
{
    for (i = 10; i ≤ n^3; i = i^15 * i)        → log_15 (n^3/10)
    {
        for (j = n^5; j > 10; j = j^1/16)      → log_16 (5 log_10 n)
        {
            for (k = 1; k ≤ n; k = k + 10)     → n/10
            {
                x = x + y
            }
        }
    }
```

10	10	n^5
~~10 * k~~	15 * 10	$n^{5/16}$
~~10 * k * k~~,	10 * 15 * 15	$n^{5/16^2}$
	⋮	⋮
	$10 * 15^k = n^3$	$n^{5/16^k} = 10$

1
1 + 10
1 + 2·10
⋮
1 + k·10 = n + 1
k = (n + 1 − 1)/10

$\Rightarrow 15^k = n^3/10$

$\Rightarrow k = \log_{15}\left(\frac{n^2}{10}\right)$

$\log_{n^{16^k}} n^{\frac{5}{16^k}} = \log_n 10$

$\Rightarrow \frac{5}{16^k} = \log_n 10$

$\Rightarrow 16^k = 5 \log_{10} n$

$k = \log_{16}(5 \log_{10} n)$

Time complexity $= \log_{15}\left(\frac{n^3}{10}\right) * \frac{\log_{16}(5\log_{10} n)}{} * \frac{n}{10}$

$\Downarrow$

$\log_{16} \log_{10} n^5$

Ex 11:

```
main ()
{
   for (i = 10; i ≤ n^2; i = i^4)  ⟶ log_4(log_10 n^2)
   {
      for (j = 1 ; j ≤ n ; j++)  ⟶ n(n+1)/2
      {
         for (k = 1 ; k ≤ j ; k = k++)
         {
            x = y + 2
         }
      }
   }
}
```

Dependency (So, individual cannot be taken)

$j=1 \quad j=2 \quad j=3 \qquad j=n$

$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

10
10^4
10^{4^2}
:
$10^{4^k} = n^2$

$\log_{10} 10^{4^k} = \log_{10} n^2$

$\Rightarrow 4^k = \log_{10} n^2$

$k = \log_4(\log_{10} n^2)$

Time Complexity =

$\log_4(\log_{10} n^2) \cdot \frac{n(n+1)}{2}$

EX12

```
main()
{
  for(i=1; i<=n; i++)
  {
    for(j=1; j<=i^2; j++)
    {
      for(k=n^3; k>5; k=k^(1/6))
      {
        x=y+z;
      }
    }
  }
}
```

$\Rightarrow \frac{n(n+1)(2n+1)}{6}$

$\Rightarrow \log_6 \log_5 n^3$

$i=1$	$i=2$	$i=3$	$\dots$	$i=n$
$j=1$	$j=4$	$j=9$		$j=n^2$

$\Rightarrow 1+4+9+\dots = \frac{n(n+1)(2n+1)}{6}$

$\times 5$

$5^{1/6}$

$5^{1/6^2}$

$\vdots$

$5^{1/6^k} = n^3$

Time complexity $= \frac{n(n+1)(2n+1)}{6} \cdot \log_6 \log_5 n^3$

$= n^3 \cdot \log_6 \log_5 n^3$

$\Rightarrow \log_5 5^{1/6^k} = \log_5 n^3$

$\Rightarrow \frac{1}{6^k} = \log_5 n^3 \Rightarrow \log_6 6^{-k} = \log_6 \log_5 n^3$

$\Rightarrow k = -\log_6 \log_5 n^3$

Ex 12.

```
main()
{
    for (i=1; i<=n; i++)
    {
        for (i=1; i<=n^2; i++)
        {
            for (i=1; i<=n^3; i++)
            {
                x = y+z;
            }
        }
    }
```

n^3

$= O(n^3)$

$i = 1, 2, 3, \ldots, n^3, n^3+1, n^3+2, n^3+3$

```
for (i=1; i<=n; i++)  ⟶ 1
{
    for (j=1; i<=n^2; i++)  ⟶ n^2
    {
        for (k=1; k<=n^3; k++)  ⟶ n^3
        {
            x = y+z;
        }
    }
}
```

$i = 1, 2, \ldots, n^2, n^2+1, n^2+2$

$\Rightarrow n^3 \cdot n^2$

$= O(n^5)$

```
for (i=1; i≤n; i++)
{
    for (i=1; i≤n^4; i++)
    {
        for (i=1; i≤n^3; i++)
        {
            x = y + z
        }
    }
}
```

i = 1 1 1 ... n^3 n^3+1 n^3+2 ... $\leq n^4$

n^3 n^3+1 n^3+2 ... n^8 n^3+1 n^3+2 ...

Infinite Loop

So, it is not an algorithm

Ex 14

```
main()
{
    for(i=1; i≤n; i++)  ⟶ n
    {
        for(j=1; j≤n; j++)  ⟶ n
        {
            if(n%j==0)   n²   j=1 k=1 to n   j=n k=1 to n
            {
                for(k=1; k≤n; k++) ⟶ 2
                {
                    x = y + z;   2n² — Total.
                }
            }
        }
    }
}
```

n is prime

n^2 : j=1 (k=1 to n) (n), j=2 — j=n (k=1 to n) (n)

j=1 n ; j=n n

$\Rightarrow n \cdot (2n) \Rightarrow n^2 + 2n$ (out of n^2 times, $2n$)

$\Rightarrow O(n^2)$

Time complexity = $O(n^2)$

- for (i=1; i≤n; i++) ⟶ n

{

for (j=1; j≤n; j++) ⟶ n

{

if (n%2 == 0) ⟹ n^2

{

for (k=0; k≤n; k++)

{

x=y+2; ⟶ 0

}

}

}

Time complexity

$= O(n^2)$

n — prime

- for (i=1; i≤n; i++) ⟶ n

{

for (j=1; j≤n; j++) ⟶ n

{

if (j%2 == 0)

{

for (k=0; k≤n; k++) ⟶ n/2

{

x=y+2;

}

}

}

$n \cdot n \cdot \frac{n}{2} = \frac{n^3}{3} = O(n^3)$

Ex15

```
main()
{
    for(i=1; i^2 <= n; i++)          ---> sqrt(n)
    {
        for(j=1; j^5 <= n^10; j++)   ---> n^2
        {
            for(k=1; k^25 <= n^100; k++)  ---> n^4
            {
                x=y+z;
            }
        }
    }
}
```

1^{25}

2^{25}

3^{25}

$\vdots$

$k^{25} = n^{100}$

$k = n^{100/25}$

$= n^4$

$j^5 = n^{10}$

$j = n^{10/5}$

$= n^2$

$i^2 = n$

$i = \sqrt{n}$

Time Complexity $= \sqrt{n} \cdot n^2 \cdot n^4 = n^{13/2}$

$= O(n^{13/2})$

Ex 16.

```
A(n)
{
  if (n < 2)
    return;
  else
    return (A (√n))
}
```

n

$\sqrt{n} = n^{1/2}$

$n^{1/4}$

$n^{1/8}$

$n^{1/2^k} = 2$

$\Rightarrow \frac{1}{2^k} \log_2 n = 1$

$\Rightarrow 2^k = \log_2 n$

$$\boxed{k = \log_2 \log_2 n}$$

- For every recursive program, there exists an equivalent non-recursive program also.

Ex 17.

```
main()
{
  p=1;
  for (i=1; i<=n; i=2*i)
    p++;
}
```

```
main()
{
  p=1, q=1
  for (k=1; k<=n; k++)  ---> n
  {
    p=1;
    for (i=1; i<=n; i=2*i)  ---> log n (base 2)
      p++;   p = log n (base 2)
    for (j=1; j<=p; j=2*j)  ---> log (log n) (base 2)
      q++;
  }
}
```

i = 1, 2, 4, 8, ...

Time complexity = $n \cdot (\log_2 n + \log_2(\log_2 n))$ Take larger

$= n \cdot \log_2 n$

q-value = $n \cdot \log_2(\log_2 n)$, p-value = $n \log_2 n$

Ex18.

```
main()
{
  x=1;
  for (i=1; i<=n; i=i+10) ⟶ n/10.
  {
    for (j=1; j<=n; j=2*j) ⟶ log2 n
    {
      x=x+n;
    }
  }
}
```

Time complexity = $\frac{n}{10} \cdot \log_2 n$.

$= O(n \log_2 n)$

x-value = $1 + \left(\frac{n}{10} \cdot \log_2 n\right) \cdot n$

$= 1 + \frac{n^2}{10} \log_2 n$

$= 1 + n^2 \log_2 n = O(n^2 \log_2 n)$

Asymptotic Notation

1. Big - oh (O)
2. Omega (Ω)
3. Theta (Θ)

Let $f(n)$ && $g(n)$ be 2 positive functions

Big - oh notation : → order of

$$f(n) = O(g(n))$$

iff

$$f(n) \leq c \cdot g(n), \forall n, n \geq n_0$$

such that $\exists$ 2 - tive constants c & n_0

where $c > 0$ && $n_0 \geq 1$

Ex 1: $f(n) = n^2 + n + 1$

$g(n) = n^2$

Prove $f(n) = O(g(n))$

$$f(n) \leq c \cdot g(n)$$

$$\Rightarrow n^2 + n + 1 \leq c \cdot n^2$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$$n^2 \quad n^2 \quad n^2$$

$$3n^2$$

$$\Rightarrow n^2 + n + 1 \leq c \cdot n^2, \forall n, n \geq n_0$$

$$\Downarrow \qquad \Downarrow$$

$$3 \qquad 1$$

$\Rightarrow n^2+n+1 = O(n^2)$

If $c=2$, $n_0=2$.

$n^2+n+1 \leq n^2$ (X)

$n^2+n+1 = O(n^2)$ ✓

(C-hidden)

$a = O(b)$ means b is asymptotically greater than a.

↓ constant use

$a \leq b$ means b is mathematically greater than a

Ex 2: $f(n) = n+10$

$g(n) = n-10$ Prove $f(n) = O(g(n))$

$\Rightarrow n+10 \leq c(n-10)$

$c=2$

$n+10 \leq 2(n-10)$

$2n-n \geq 30$

$n \geq 30$

$n_0 = 30$ and $c=2$

$\Rightarrow (n+10) = O(n-10)$

$100 = O(2)$

$100 \leq c \cdot 2$

$\Downarrow$

51

$c=2$

$n+10 \leq 2n-20$

$30 \leq 2n-n$

$n \geq 30$

41

$2(31-10)$

2×21

42 ✓

50

$2(40-10)$

60 ✓

Ex 3. $f(n) = n^2$

$g(n) = n$

Prove $f(n) = O(g(n))$

$n^2 \leq c \cdot n$

$\Downarrow$

n

(but it is not a constant)

~~$n \geq 10$~~

~~$100 \leq 11 \cdot 10$~~ ~~$c = 11$~~

$n = 20$

~~$400 \leq 11 \cdot 20$~~

$\Rightarrow n^2 \neq O(n)$

Ex 4. $f(n) = n + 10$

$g(n) = n + 10$ ——

$\Rightarrow (n-10) \leq c \cdot (n+10)$

$\Downarrow$

1

$n - 10 \leq n + 10 \quad \forall n, n_0 \geq 11$

(as fncts should be +ive)

$\Rightarrow n - 10 = O(n+10)$

$n+10$ is mathematically as well as asymptotically greater.

Omega Notation:

$$f(n) = \Omega(g(n))$$

iff

$$f(n) \geq c \cdot g(n) \quad \forall\ n,\ n \geq n_0$$

such that $\exists$ 2 +ive constants c & n_0

where $c > 0$ && $n_0 \geq 1$

Ex1 $f(n) = n^2$

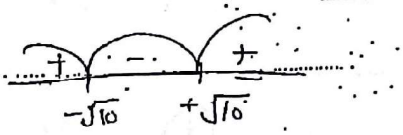
$g(n) = n^2 + 10$

Prove $f(n) = \Omega(g(n))$

$$n^2 \geq c \cdot (n^2 + 10) \qquad c = 1/2$$

$$n^2 \geq \frac{n^2}{2} + 5$$

$$\frac{n^2}{2} \geq 5 \Rightarrow$$



$$n^2 - 10 \geq 0 \Rightarrow (n+\sqrt{10})(n-\sqrt{10}) \geq 0$$

$$\Rightarrow n \geq \sqrt{10}$$

So, $c = 1/2$ and $n \geq 4$.

$$\Rightarrow n^2 = \Omega(n^2 + 10)$$

Ex2. $f(n) = n+10, \quad g(n) = n-10$

Prove $f(n) = \Omega(g(n))$

$$\Rightarrow n+10 \geq c(n-10) \qquad c = 1 \ \&\ n \geq 11.$$

already true

Ex 2: $f(n) = n$, $g(n) = n^2$ Prove $f(n) = \Omega(g(n))$

$\Rightarrow n \geq c \cdot n^2$

$\downarrow$

$1/n$ (which is not a constant)

so, not possible.

~~f(n)~~ $n \neq \Omega(n^2)$

Theta (Θ) - notation :-

$f(n) = \Theta(g(n))$

iff

① $f(n) \leq c_1 g(n)$

&&

② $f(n) \geq c_2 \cdot g(n)$

$\forall n, n \geq n_0$.

Ex-1: $f(n) = n^2$, $g(n) = n^2 + 10$

$f(n) = O(g(n))$

$n^2 \leq c_1 \cdot (n^2 + 10)$

$c_1 = 1, n_0 = 1$

$f(n) = \Omega(g(n))$

$n^2 \geq c_2 \cdot (n^2 + 10)$

$c_2 = \frac{1}{2}, n_0 = 4$

So, $c_1 = 1$, $c_2 = 1/2$ and $n_0 = 4$.

$\Rightarrow n^2 = \theta(n^2+10)$ ✓

Ex2:

$f(n) = n+10$

$g(n) = n-10$

$f(n) = O(g(n))$

$\Rightarrow n+10 \leq c_1(n-10)$

$c_1 = 2$; $n_0 = 30$

$f(n) = \Omega(g(n))$

$n+10 \geq c_2(n-10)$

$c_2 = 1$, $n_0 = 11$

$\Rightarrow c_1 = 2$, $c_2 = 1$, $n_0 = 30$

$\Rightarrow (n+10) = \theta(n-10)$ ✓

Ex3:

$f(n) = n$

$g(n) = n$

$n \leq c_1 n$

$\Downarrow$

1

$n = O(n)$

$n \geq c_2 \cdot n$

$\Downarrow$

1

$0 = \Omega(n)$

$\Rightarrow n = \theta(n)$ ✓

Ex 4: $f(n) = n$

$g(n) = n^2$

$n \leq c_1 n^2$

$\Downarrow$

1

$n = O(n^2)$ ✓

$n \geq c_2 \cdot n^2$

$\Downarrow$

$1/n$

$n \neq \Omega(n^2)$

So, $n \neq \Theta(n^2)$

$100 = \Theta(1)$ ✓

$c_1 = 1$

(Omega ✓)

$c_1 = 100$

(Big-oh ✓)

Big-oh Notation ($\leq$) {There exists c}

$n^2 = O(n^2) \Rightarrow$ Tighest UB

$= O(n) \Rightarrow$ Upper Bound

$= O(n^3)$

$= O(n^{10})$ } NTUB

$A = O(B)$

$\Downarrow$

UB

T NT

Small-oh-notation ($<$) {For all c}

~~$n^2 = O(n^2)$~~ X

$O(n^3)$ ✓ $\Rightarrow$ UB

$O(n^{10})$ ✓ $\Downarrow$ NT

$A = o(B)$

$\Downarrow$

NTUB

Omega Notation ($\geq$) {There exists c}

$n^3 = \Omega(n)$

$\Omega(n^2)$ } — Lower Bound

$\Omega(n^3) \Rightarrow$ TLB

Small - omega notation (>)

$n^3 = \omega(n) \Longrightarrow$ NTLB

$= \omega(n^2)$

$= \omega(n^3)$ ×

$A = \{ \omega(B)$

⇓

NTLB

Theta Notation

$n^3 = O(n^3) \Rightarrow$ TUB

&

$n^3 = \Omega(n^3) \Rightarrow$ TLB

⇕

$n^3 = \Theta(n^3)$

⇕

$A = \Theta(B)$

⇓

TUB

TLB

If there is an algorithm A, time complexity of algorithm is T(A) then if $T(A) = O(n^3)$, it means that upper bound for time complexity is n^3.

Problem P

(algorithms)

A B

Time Complexity: T(A) T(B)

$T(A) = O(T(B))$

↓

smaller or equal to T(B)

$\Rightarrow$ A is better algorithm

$T(A) = T(B)$ (n^3 vs n^3) ✓

$T(A) = \theta(T(B))$ (n^3 vs n^2+10) ⟶ constant diff. may be there.

Complexity Classes :- (Imp)

(1) Constant :- $O(1)$

$O(\log(\log n))$

(2) Logarithmic :- $O(\log n)$

$O(\sqrt{n})$

(3) Linear :- $O(n)$

$O(n \log n)$

(4) Quadratic :- $O(n^2)$

(5) Cubic :- $O(n^3)$

(6) Polynomial :- $O(n^c)$ where c is constant, $c > 0$

(7) Exponential :- $O(c^n)$ where c is constant, $c > 1$

increasing order ↓

(8) $c^n < n^n$ ⟹ $c=2$ ⟹ $2^n < n^n$

$2^n = O(n^n)$ ✓

$o(A)$ satisfied $\Longrightarrow$ $O(A)$ satisfied.

$O(A)$ satisfied $\not\Longrightarrow$ $o(A)$ satisfied.

- $2^n > n^{\log n}$ Taking log $n \log 2 > \log n . \log n$.

(9)
- $n > \log n$
- $n > (\log n)^2$

To check:

① $\sqrt{n} . \sqrt{n} > (\log n) . (\log n)$ $\{\sqrt{n} > \log n\}$

Proved.

② $n > (\log n)^2$

Taking log, $\log n > 2 \log(\log n)$ $\{\log n > \log \log n\}$

(2 ↓ const.)

Proved.

⋮

- $n > (\log n)^{1000}$

Imp
- $n < (\log n)^{\log n}$

To check,

$\log n . 1 < \log n . \log(\log n)$

$\log(\log n) > 1$.

(10) • $n! < n^n$ $\quad \underbrace{n.(n-1)\ldots 1}_{n\text{-times}} \;\Big|\; \underbrace{n.n.n\ldots n}_{n\text{ times}}$

• $n! > 2^n$ $\quad \underbrace{n.(n-1)\ldots 1}_{n\text{-times}} \;\Big|\; \underbrace{2.2.2\ldots 2}_{n\text{ times}}$

$2^n < n! < n^n$

(11) $\log_2 n = \theta(\log_3 n)$

$\log_2 n > \log_3 n$

$\Downarrow$

$\frac{\log_2 n}{\log_2 3} = \frac{\log_2 n}{1.5}$

Difference is constant.

$\Rightarrow$ They are asymptotically equal.

(12) $2^n < 3^n$

$\Downarrow$

$(2 * 1.5)^n$

$\Downarrow$

$2^n \cdot 1.5^n$ → greater by function so not asymptotically equal.

$2^n = \theta(3^n)$

↓

small-o

~~$n^2 < n^3$~~ Take log

$2 \log n \;|\; 3 \log n$

If ignore constant, it seems

Simplify first, then apply log.

n^2	n^3
~~x^x~~	$n \cdot x^x$
1	n

Take Log

$\log 1 = 0 < \log n$ ✓

Ques 1: Check the foll. stmts are true/false?

(a) $250\, n \log n = O\left(\frac{n \log n}{250}\right)$ ✓ $c = 250 \times 250$

(b) $\sqrt{\log n} = O(\log \log n)$ X — Ω, ω ✓

(c) If $0 < x < y$ then $n^x = O(n^y)$ ✓ — θ, Ω X (Function Difference)

(d) $2^n \neq O(n^c)$ where c is const, $c > 1$ ✓
Exponential Polynomial.

$\sqrt{n} > \log n$

$\sqrt{\log n} > \log \log n$

$0 < x < y$ $\quad n^x \quad n^y$ $\quad x=2, y=3$

$n^2 < n^3$ ✓

$2^n \quad n^c \rightarrow n \log 2 > c \log n$

Ques 2: Check the foll. statements true / false ?

(a) $(n+a)^b = O(n^b)$ where a & b are +ive const. ✓
(b) $2^{n+5} = O(2^n)$ ✓
(c) $2^{2n} = O(2^n)$ ✗
(d) $f(n) = O((f(n))^2)$ ✗ < Depends on Function > Fails for Decreasing Function

$(n+a)^b$ $\equiv$ n^b

2^{n+5} $2^5 \cdot 2^n$

$2^n \cdot 2^5$

$(2^2)^n$ 2^n

$= 2^n \cdot 2^n$ 2^n

$2^n > 1$

$(n+1)^2$ n^2

n^2+n+1 $3n^2$

$n < O(n^2)$ ✓

n^2 $O(n^4)$ ✓

$\sqrt{n} > \sqrt{n^2-n}$

Function Diff

If Θ is possible $\Rightarrow$ O, ω not possible

$f(n) > 1 \longrightarrow$ Increasing Fnctn

$f(n) < 1 \longrightarrow$ Decreasing "

Ques 3: Check the foll. statements are true/false.

O, o ✓

(a) $n^5 \, 64^{\log_8 n} = \theta(n^{10})$ ✗

(b) $32^{\log_2 n} \; 64^{\log_8 n} = O(n^{10})$ ✓ (O or ✓)

(c) ~~log~~ $32^{\log_2 n} \; 64^{\log_2 n} = \Omega(n^3)$ ✓ (Ω ✓)

(d) $\frac{4^n}{2^n} = \theta(2^n)$ ✓ (O, Ω ✗)

$n^5 \; 64^{\log_8 n} \Rightarrow n^5 \cdot 8^{\log_8 n^2}$ — $n^5 \cdot n^5$ — n^5 — n^5

~~$\log_8 n^2$~~ ~~$\log_8 n^2$~~ ~~$\log_8 n^2$~~ ~~$\log x = \log n^2$~~ ~~$\log 8$~~

$32^{\log_2 n} \; 64^{\log_8 n}$ — n^{10}

$2^{\log_2 n^5} \cdot 8^{\log_8 n^2}$ — n^{10}

$n^5 \cdot n^2$ — n^{10}

n^7 — n^{10}

$2^{\log_2 n^5} \cdot 2^{\log_2 n^6}$ | n^3

$n^5 \cdot n^6$ | n^3

n^{11} | n^3

$\frac{4^n}{2^n}$ | 2^n

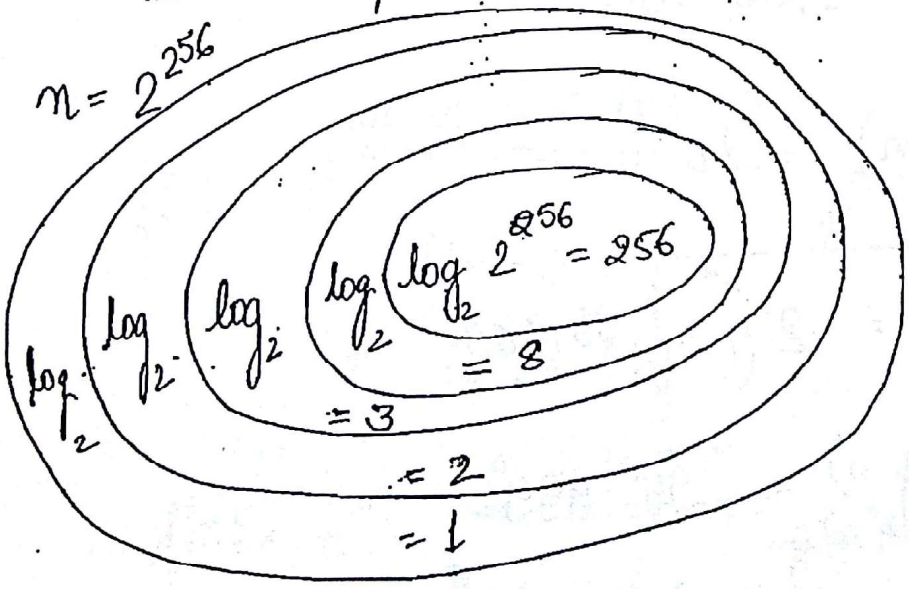
$\frac{2^{2n}}{2^n} = 2^n$ | 2^n

Note: $a^{\log_b n} = n^{\log_b a}$

Ques 4: $f(n) = \log^*(\log n)$

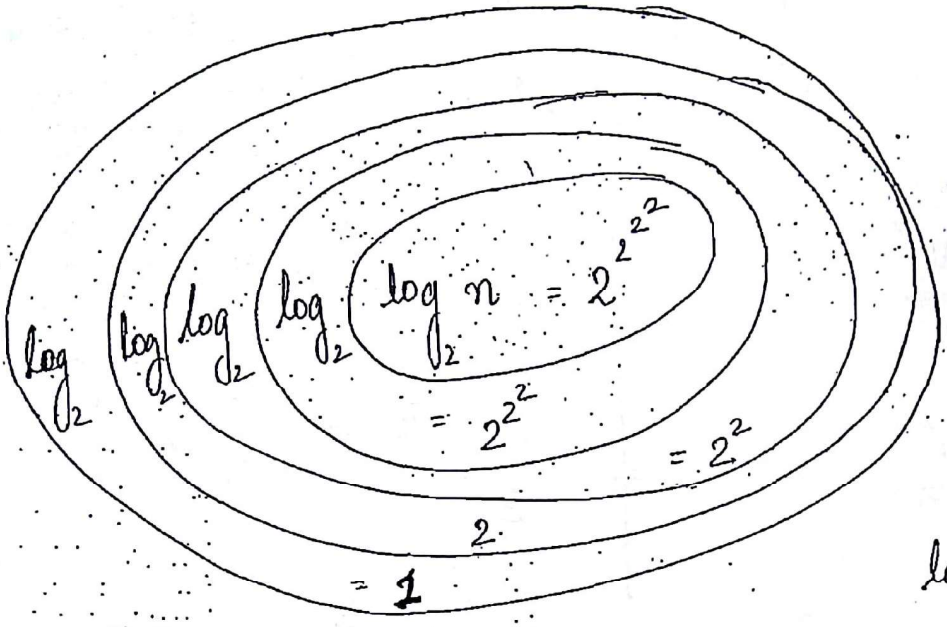
$g(n) = \log(\log^* n)$

Relation b/w these two?

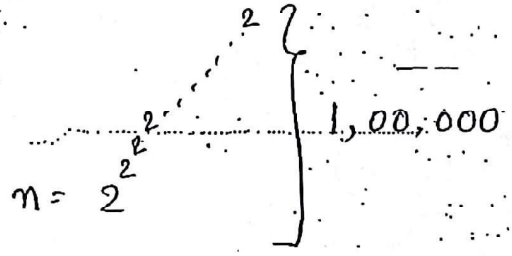


$\log^* n = 5$

$$n = 2^{2^{2^{2}}}$$



$\log^* n = 5$



$$\log_2^* n = 1,00,000$$

$$\log_2 (\log_2^* n) = 16$$

$$\log_2 n = 2^{2^{\cdots^{2}}} \Big\} 99,999$$

$$\log_2^* (\log_2 n) = 99,999$$

$$\log^* (\log^* n) = S2 \left(\log (\log^* n)\right)$$

$\Rightarrow \boxed{f(n) = \Omega(g(n))}$

Ques 5:

(a) $$f(n) = \begin{cases} n^3 & 0 < n < 10000 \\ n^5 & n \geq 10000 \end{cases}$$

$$g(n) = \begin{cases} n^7, & 0 < n < 100 \\ n^4, & n \geq 100. \end{cases}$$

Relation b/w these two?

$0 < n < 100$

$f(n) = n^3$ $\qquad$ $g(n) = n^7$

$f(n) < g(n)$

$f(n) = O(g(n))$

$100 \leq n < 10000$

$f(n) = n^3$ $\qquad$ $g(n) = n^4$

$f(n) < g(n)$

$\Rightarrow f(n) = O(g(n))$

$n \geq 10000$

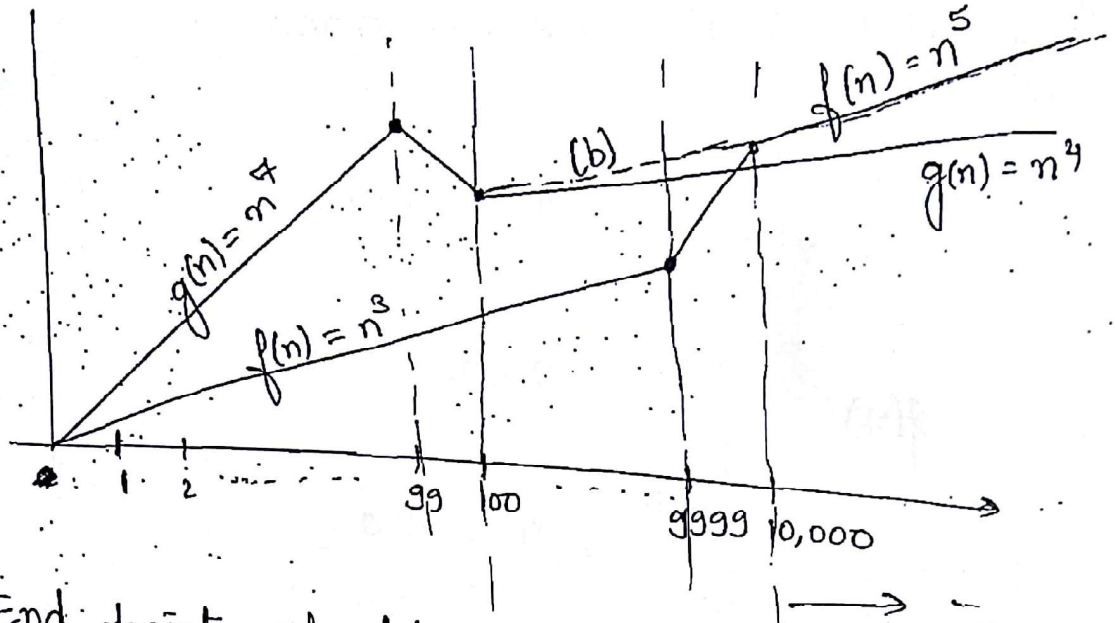
$f(n) = n^5$ $\qquad$ $g(n) = n^4$

$f(n) > g(n)$

$\Rightarrow f(n) = \Omega(g(n))$

Finally,

$\Rightarrow f(n) = \Omega(g(n))$, $n > 10{,}000$.



End point should be infinity.

So, $f(n) = \Omega(g(n))$, $n_0 = 10{,}000$

(b) $f(n)$ same

$$g(n) = \begin{cases} n^7 & , \quad 0 < n < 100 \\ n^5 & , \quad n \geq 100 \end{cases}.$$

$f(n) = \Theta(g(n))$ $\quad n_0 = 10000$

OR

$f(n) = O(g(n))$, $n_0 = 1$

$\downarrow$

$\leq$

Ques 6: $f(n) = n^{2+\sin n}$

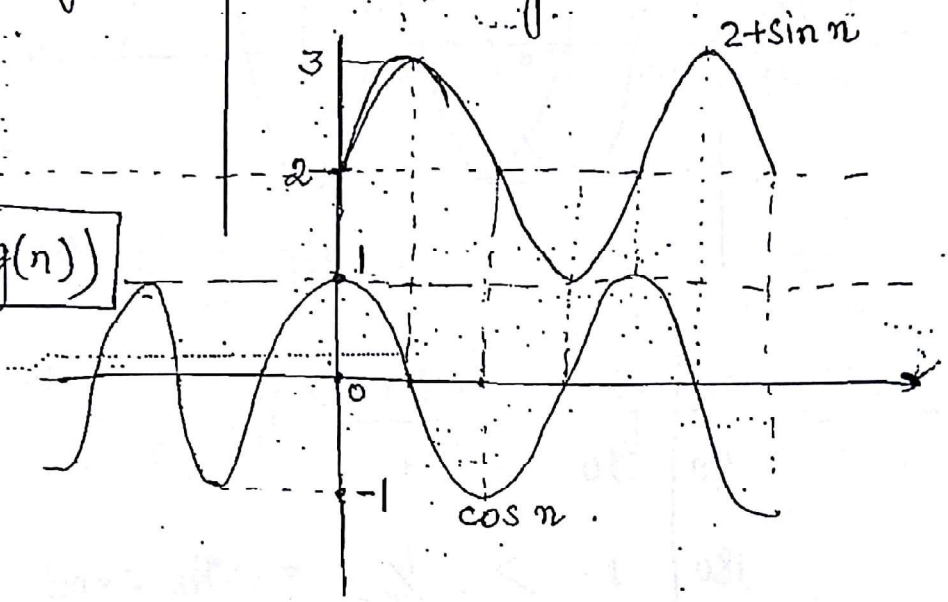
$g(n) = n^{\cos n}$.

Relation b/w them?

$n^{2+\sin n}$ $\quad$ $n^{\cos n}$

$(2+\sin n)\log n > \cos n \log n$

$f(n) = \Omega(g(n))$

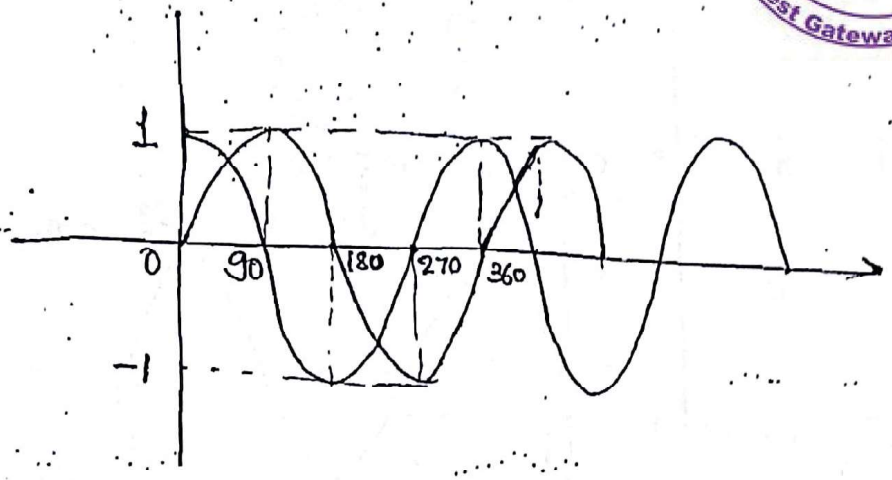


n	$f(n)$		$g(n)$
90	90^3	$>$	90^0
180	180^2	$>$	$\frac{1}{180}$
270	270	$>$	1
360	360^2	$>$	360

Ques 7: $f(n) = n^{\sin n}$

$g(n) = n^{\cos n}$

Relation b/w them?



n	$f(n)$		$g(n)$
90	90	>	1
180	1	>	1/180
270	1/270	<	1
360	1	<	360

The same pattern will repeat for next 360°.

So, there is no relation

Ques 8: Check the foll. statements are true/false?

(a) $\frac{1}{n} = O(1)$ ✓

(b) $\frac{1}{n} = \Theta(1)$ ✗

Good (c) If $f(n) = O(g(n))$

then

$2^{f(n)} = O(2^{g(n)})$ ✗ (Same is applicable for Ω)

(d) $1000 = \theta(1)$ ✓

$\frac{1}{n} \leq c_1 \cdot 1$

$\frac{1}{n}$ 1

Imp: n is no. of i/ps. So it cannot be fraction ✓

$f(n) = O(g(n))$

$f(n) \leq c_1 \cdot g(n)$

$2^{f(n)} \leq c_1 2^{g(n)}$ ✗

n+10 $\quad$ n-10

n $\quad$ n^2

2^n $\quad$ 2^{n^2} ✓

$f(n) = 2n$, $g(n) = n$

$\Rightarrow f(n) = O(g(n))$

then

$2^{2n} \leq 2^n$ {which is false}

$2^n \cdot 2^n \leq c \cdot 2^n$ (✗)

$\downarrow$

2^n

Properties of Asymptotic Notation

① Reflexive property :-

- $f(n) = O(f(n))$
- $f(n) = \Omega(f(n))$
- $f(n) = \Theta(f(n))$

o, ω fails.

② Symmetric property :-

If $f(n) = O(g(n))$ then $g(n) = O(f(n))$ ✗

$n^2 = O(n^3) \implies n^3 = O(n^2)$ ✗

O, ω, Θ, Ω fails ✗

Θ pass ✓

③ Transitive property :-

If $f(n) = O(g(n))$ && $g(n) = O(h(n))$

then

$f(n) = O(h(n))$ ✓

O, Ω, Θ, o, ω pass.

④ If $f(n) = O(g(n))$

then

$h(n) \cdot f(n) = O(h(n) \cdot g(n))$

⑤ If $f(n) = O(g(n))$

&&

$d(n) = O(e(n))$

then

(i) $f(n) + d(n) = \max(g(n), e(n)) = O(g(n) + e(n))$

(ii) $f(n) \cdot d(n) = O(g(n) \cdot e(n))$

Ques 1: Let $f(n)$, $g(n)$ & $h(n)$ be 3 positive fnctns which are defined as follows:-

i) $f(n) = O(g(n))$ & $g(n) \neq O(f(n))$

ii) $g(n) = O(h(n))$ & $h(n) = O(g(n))$

then

T/F

a) $f(n) = \Omega(h(n))$ ✗

b) $f(n) \cdot h(n) = \Theta(g(n) \cdot h(n))$ ✗

c) $g(n) \cdot h(n) = O(g(n) \cdot g(n))$ ✓

d) $f(n) + h(n) = O(g(n))$ ✓

$f(n) < g(n) = h(n)$

Ques 10 : If $T_1(n) = O(f(n))$ && $T_2(n) = O(f(n))$ then

If $T_1(n) = \Theta(f(n))$ && $T_2(n) = \Theta(f(n))$ all a, b, c, d (True)

T/F

(a) $T_1(n) + T_2(n) = O(f(n))$ ✓

(b) $T_1(n) = O(T_2(n))$ ... ×

(c) $T_1(n) = \Omega(T_2(n))$ ×

(d) $T_1(n) = \Theta(T_2(n))$ ×

} No relation b/w T_1 & T_2

Divide & Conquer

BASICS

1. Recursion
2. Recurrence relation
3. Recurrence relation solving.

Recursion ① function calling itself.

② Recursion is nothing but solving big problems in terms of smaller ones.

Ex: $f(6) = 6 * f(5)$ → 720

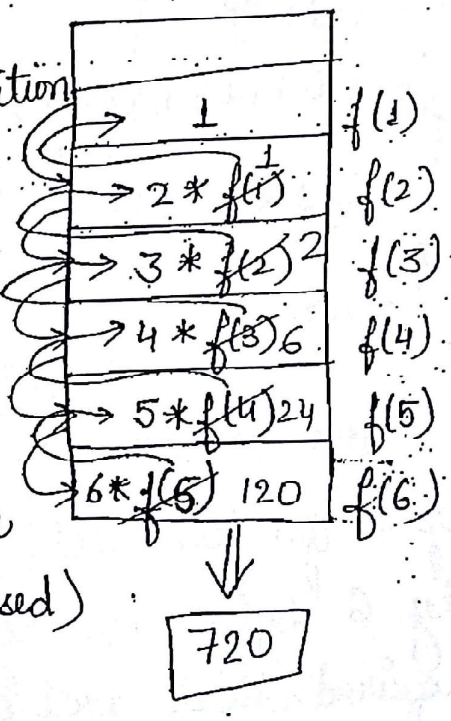
$5 * f(4)$ → 120

$4 * f(3)$ → 24

$3 * f(2)$ → 6

$2 * f(1)$ → 2

$f(1)$ → 1



(3) Every recursive program should have termination condition otherwise program will go to infinite loop * error message of Stack Overflow

(4) To execute the recursive program, stack data structure is used. (Queue cannot be used)
↓
FIFO

(5) In recursion, from 1 function call to another, parameter value will change but not no. of parameters and names of parameters.

(6) For every recursive program, equivalent non-recursive program exists / is possible.

```
main()
{
    int n=6;
    printf("%d", f(n));
}
```

n: 6 (1000)

6 → Non-recursive

6 → Recursive

Non-recursive

```
f(int n)
{
    int i,s;
    s=1;
    for(i=1;i<=n;i++)
    {
        s=s*i;
    }
    return s;
}
```

n: 6 (2000) | i: 1 | s: 1

int – 2 bytes

Only 1 function call, only 6 bytes memory required which will be updated on each iteration

Recursive

```
f(int n)
{
    if(n==1)
        return 1;
    else
    {
        b=f(n-1);
        c=n*b;
    }
    return c;
}
```

n: 6 | b: | c:

For every function call, 6 bytes of memory allocated for n, b, c ⇒ 36 bytes of memory

(4) Comparing recursion & non-recursion, recursion will take more stack space. (but path will take stack space) because of more fnctn calls.

Note:-

(8) Recursion & Non-recursive program does not affect time complexity. Logic affects time complexity.

↳ Recursive programs are beneficial to programmer and non-recursive programs are beneficial to computer.

6/12/16

Ex1

```
fact (n)
{
  if (n ≤ 1)
    return 1;
  else
    return n * fact(n-1);
}
```

$$fact(n) = \begin{cases} 1, & n \leq 1 \\ n * fact(n-1), & n > 1 \end{cases}$$

recurrence relation (for value)

For a recursive program, more than 1 recurrence relations exists (for time, value etc)

Ex2 Write a recursive program & recurrence relation to find power of (a,b), a, b > 0.
(Best & Worst & Average case same)

```
power (a, b)
{
  if (b == 1)
    return a;
  else if (a == 1)
    return 1;
  else
    return a * power (a, b-1);
}
```

pow(5,3) = 125
⇓ 25
5 * pow(5,2)
⇓
5 * pow(5,1)
⇓
5

Recurrence Relation :

$$power(a,b) = \begin{cases} a & , b = 1 \\ 1 & , a = 1 \\ a * power(a, b-1) & , b > 1 \end{cases}$$

Time complexity = $O(b) = O(n)$

Ex 3 Write a recursive program & recurrence relation to find gcd(m, n) where m, n > 0 & integers

```
gcd (m, n)
{
  if(m == 0)
    return n;
  if (n == 0)
    return m;
  if (m % n == 0)
    return n;
  if (n % m == 0)
    return m;
  else
  {
if-else not reqd, it will automatically reverse if after 1 fn call.
    return gcd (n%m, m);
  }
}
```

m	n
24	32

greatest common divisor

2, 4, 8, ... 12

12, 16 → 6, 8 → 3, 4

2*2*1 ... 2*1 ... 1

(m % j == 0 && n % j == 0)

Termination Condition

GCD(20, 5) = 5

GCD(13, 17) = 1

GCD(250, 50) = 50

GCD(10, 0) = 10

GCD(0, 10) = 10

GCD(0, 0) = ∞

Procedure

13) 17 (1
13
4) 13 (3
12
1) 4 (4
4
X

GCD

(24, 32) ⇒ 24) 32 (1
24
8

(8, 24) 8) 24 (3
24
0

(0, 8) ✓

Recursive Program

```
gcd(m,n)
{
  if (m==0 && n==0)
      return (inf); // printf("Error");
  if (m==0)
      return n;
  if (n==0)
      return m;
  else
      return gcd(n%m, m);
}
```

Recursence Relation :-

$$gcd(m,n) = \begin{cases} \infty, & m=0 \&\& n=0 \\ n, & m=0 \&\& n!=0 \\ m, & n=0 \&\& m!=0 \\ gcd(m\%n, n), & \text{otherwise} \end{cases}$$

Time complexity = ~~log n~~ $O(\log_2(\max(m,n)))$

due to cont. division

Base can be anything as only constant diff. (division is not by a const number)

GCD (23, 53)

$$23\overline{)53}\,(2$$
$$\underline{46}$$
$$7$$

GCD (7, 23)

$$7\overline{)23}\,(3$$
$$\underline{21}$$
$$2$$

GCD (2, 7)

$$2\overline{)7}\,(3$$
$$\underline{6}$$
$$1$$

GCD (1, 2)

$$1\overline{)2}\,(2$$
$$\underline{2}$$
$$\times 0$$

GCD (0, 1) = 1

Best Case for gcd :-

Nos. m & n are multiple of each other

$O(1)$ Ex: gcd (400, 20000)

Worst Case for gcd :-

Nos. m & n are prime.

$O(\log_2 n)$ if $m < n$

Average case for gcd :-

b/w best & worst case.

Best Case, worst & average case are determined by else condition of recursion.

Ex4 Write a recursive program & recurrence relation to find nth fibonacci no.

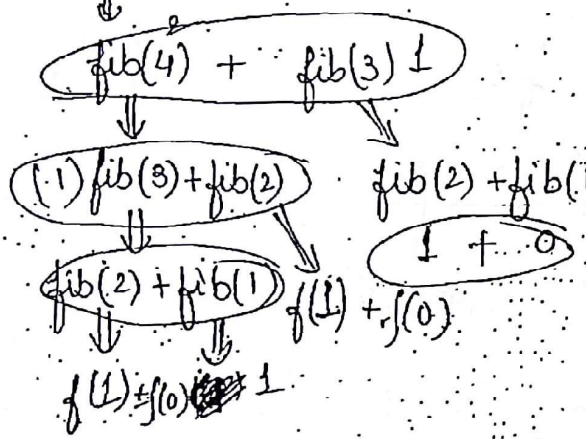
0 1 2 3 4 5 6 7

0, 1, 1, 2, 3, 5, 8, 13, - - - -

```
fibonacci (n)
{
    if (n == 0)
        return 0;
    if (n == 1)
        return 1;
    else
        return fibonacci(n-1) + fibonacci(n-2);
}
```

if (n == 0 || n == 1)
    return n;

Ex: fib(5)



Recurrence Relation :

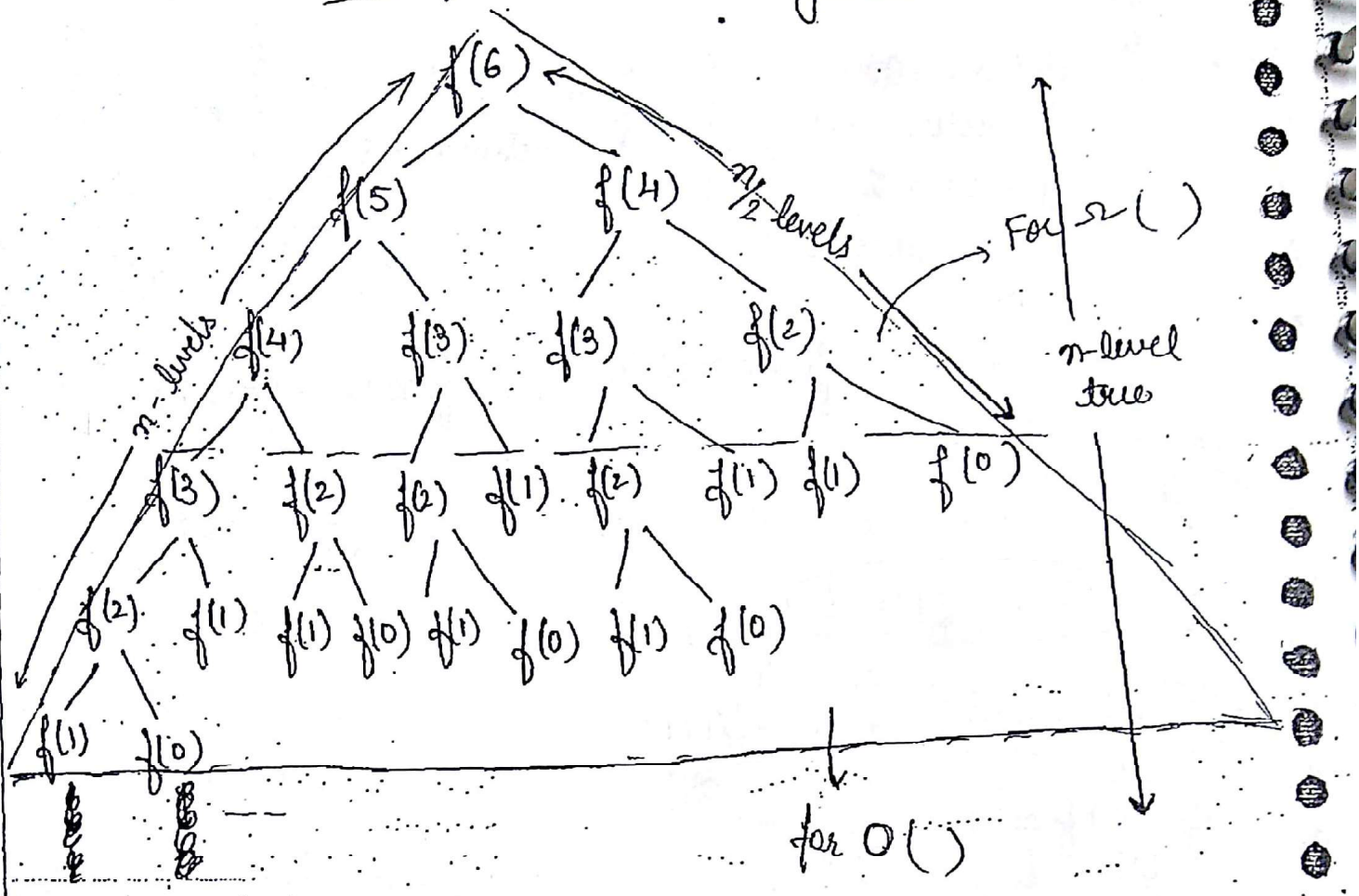
$$fib(n) = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ fib(n-1) + fib(n-2), & \text{otherwise} \end{cases}$$

~~Time Complexity = O(1) Best Case~~

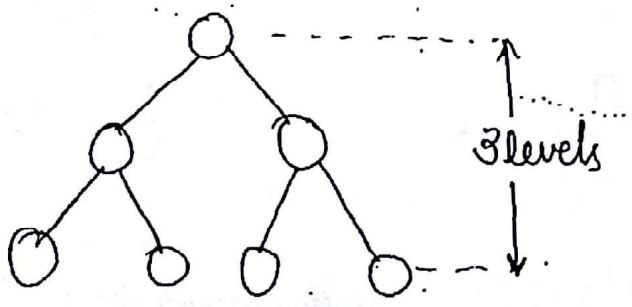
~~O(n)~~ ~~worst Case~~

~~Best~~

For fibonacci (Binary Tree)



Complete Binary Tree



No. of Nodes = $2^3 - 1$

$= 2^n - 1$ (max)

Min. nodes = n. (Unary Tree)

fib(n) : n-level Binary Tree but not complete

⇓

n-level Complete Binary Tree (Upper Bound)

⇓

$(2^n - 1)$ nodes.

⇓

2^n - nodes (ignore -1)

⇓

$O(2^n) \Leftarrow 2^n$ function calls

$T(fib(n)) = O(2^n)$ {Ulppu Bound} ⇒ Not worst case (more than actual ans)

$T(fib(n)) = \Omega(2^{n/2})$ {Lower Bound} ⇒ Not best case (less than actual ans)

$$2^{n/2} \leq T(fib(n)) \leq 2^n$$

If in else,

- return $\underbrace{fib(n-1)}_{max} + fib(n-2) + fib(n-3)$

will give n-level Ternary tree

$\Rightarrow T(f(n)) = O(3^n)$

$T(f(n)) = \Omega(3^{n/3})$

- return $\underbrace{fib(n-1)}_{n\,(max)} \overset{\text{can be any operator}}{(+)} \underbrace{fib(n/2)}_{\log n\,(min)}$ Binary Tree

$T(f(n)) = O(2^n)$

$T(f(n)) = \Omega(2^{\log n})$

- return $\underbrace{fib(n/3)}_{\log_3 n\,(min)} + \underbrace{fib(n/2)}_{\log_2 n\,(max)}$ Binary Tree

$T(f(n)) = O(2^{\log_2 n}) = O(n)$

$T(f(n)) = \Omega(2^{\log_3 n})$

- return $\underbrace{fib(n-1)}_{n} + \underbrace{fib(n-1)}_{n}$

$T(f(n)) = O(2^n)$

$T(f(n)) = \Omega(2^n)$

$T(f(n)) = \Theta(2^n)$

Exact

No Gap in Binary Tree.

In fibonacci function, there is no best case & worst case. (always left side will be n-level tree & right side will be $n/2$ - level tree)

Recurrence Relation Solving

1. Substitution method
2. recursive tree method
3. Master Theorem

SUBSTITUTION METHOD

Note: Substituting the given function repeatedly until given function is eliminated.

Ex 1.

$$T(n) = \begin{cases} 1 & ; n = 1 \\ T(n-1) + n & ; n > 1 \end{cases}$$

→ if stopping condtn not given, take any constant

~~Find~~ Solve the recurrence relation?

$$T(n) = T(n-1) + n$$

$$\Downarrow$$

$$T(n-2) + n-1 + n$$

$$\Downarrow$$

$$T(n-3) + (n-2) + (n-1) + n$$

$$\vdots \quad (n-1) \text{ times (to get } T(1) \text{ which is termination condtn)}$$

$$T(n-(n-1)) + (n-(n-2)) + \ldots + (n-1) + n$$

$$T(1) + 2 + 3 + 4 + \ldots + (n-1) + n$$

$$\Downarrow$$

$$1 + 2 + 3 \ldots + (n-1) + n$$

$$T(n) = \frac{n(n+1)}{2} = O(n^2)$$
$$= \Omega(n^2)$$
$$= \theta(n^2)$$

Ex2 $T(n) = \begin{cases} 1, & n = 1 \\ T(n-1) + \log(n), & \text{if } n > 1 \end{cases}$

$T(n) = T(n-1) + \log n$

$\Downarrow$

$T(n-2) + \log(n-1) + \log n$

$\Downarrow$

$T(n-3) + \log(n-2) + \log(n-1) + \log n$

$\vdots$

$(n-1)$ times

$\vdots$

$T(n-(n-1)) + \log(n-(n-2)) + \ldots + \log(n-2) + \log(n-1) + \log n$

$T(n) = 1 + \log 2 + \log 3 + \log 4 + \ldots + \log(n-1) + \log n$

$= 1 + \log(2.3.4\ldots(n-1).n)$

$= 1 + \log(n!)$

$2^n < n! < n^n$

$= O(\log n^n) = \underline{O(n \log n)}$

$= \Omega(\log 2^n) = \underline{\Omega(n)}$

Ex3: $T(n) = \begin{cases} 1 & , n = 0 \\ T(n-2) + n^2 & , n > 0 \end{cases}$

$T(1) = T(1) + 3^2$

$T(n) = T(n-2) + n^2$

$\Downarrow$

$T(n-4) + (n-2)^2 + n^2$

$\Downarrow$

$T(n-6) + (n-4)^2 + (n-2)^2 + n^2$

$\vdots$ (n times)

$\Downarrow$

$n-(n$

$T(n-n) + (n-(n-2))^2 + \dots + (n-2)^2 + n^2$

$T(0) + 2^2 + 4^2 + \dots + (n-2)^2 + n^2$

$T(n) = 1 + 2^2 + 4^2 + \dots (n-2)^2 + n^2$

$= 1 + (2.1)^2 + (2.2)^2 + (2.3)^2 + \dots + \left(2.\frac{n-2}{2}\right)^2 + \left(2.\frac{n}{2}\right)^2$

$= 1 + 4\left\{1^2 + 2^2 + 3^2 + \dots \left(\frac{n}{2}\right)^2\right\}$

$= 1 + 4 \cdot \frac{1}{6} \cdot \frac{n}{2}\left(\frac{n}{2}+1\right)(n+1)$

$= O(n^3) = \Omega(n^3) = \Theta(n^3)$

(Upper Bound) (Lower Bound)

No Best & worst Case

Ex 4: $T(n) = \begin{cases} 1 & , n = 1 \\ 2T(n/2) + n & , n > 1 \end{cases}$

$T(n) = 2T(n/2) + n$

$\Downarrow$

$2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n = 4T\left(\frac{n}{4}\right) + n + n$

$\Downarrow$

$2\left(2\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + \frac{n}{2}\right) + n = 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + n + n$

$\vdots$ $\log n$ times $= 2^3 T\left(\frac{n}{8}\right) + 2^2 \cdot \frac{n}{2^2} + n + n$

$\frac{n}{2^k} = 1$

$2^k = n$

$k = \log n$

~~$2T(1) + 2$~~

$= 2^{\log n} T(1) + \underbrace{n + n + n + \ldots + n}_{\log n \text{ times}}$

$= 2^{\log_2 n} + n \log_2 n$

$= n + n \log n = O(n \log n)$ $\rightarrow c = 2$

$= \Omega(n \log n) = \Theta(n \log n)$

$\rightarrow c = 1$

Ex5. $T(n) = \begin{cases} 1, & n = 1 \\ T(n/2) + c, & n > 1 \end{cases}$ ← same as

Variations :-

$T(n) = T\left(\frac{n}{2}\right) + c$

$T(n) = \left[T\left(\frac{n}{2}\right)\right] + c$

$T(n) = T\left(\frac{n}{2} - 10\right) + c$

$= T\left(\frac{n}{4}\right) + c + c$

$= T\left(\frac{n}{8}\right) + c + c + c$

$\frac{n}{2^k} = 1$

$2^k = n$

$\log K = \log n$

$K = \log n$ times

$= \{ T(1) + \underbrace{c + c + c + \ldots + c}_{\log n \text{ times}} \}$

$= 1 + c \cdot \log n$

$= O(\log n) = \Omega(\log n) = \Theta(\log n)$

Ex6. $T(n) = \begin{cases} 1, & n = 1 \\ T(n/2) + n, & n > 1 \end{cases}$

$$T(n) = T\left(\frac{n}{2}\right) + n$$

$$\Downarrow$$

$$T\left(\frac{n}{4}\right) + \frac{n}{2} + n$$

$$\Downarrow$$

$$T\left(\frac{n}{8}\right) + \frac{n}{4} + \frac{n}{2} + n$$

$\log n$ times

$$\frac{n}{2^k} = 1 \qquad 2^{\log_2 n}$$

$$k = \log n$$

$$T(1) + \frac{n}{2^{\log_2 n - 1}} + \cdots + \frac{n}{8} + \frac{n}{4} + \frac{n}{2} + \frac{n}{2^0} \qquad \downarrow \frac{n}{n} \times (i)$$

$$= 1 + n\left(\frac{(1/2)^{\log_2 n + 1}}{-1/2 + 1}\right) = 1 + n\left(\frac{1 - \left(\frac{1}{2}\right)^{\log_2 n}}{1 - 1/2}\right) \quad a\left(\frac{r^n - 1}{r - 1}\right)$$

~~$= 1 + n.(\frac{1}{n} - 1) = \Theta(n^2)$~~

~~$= \Omega(n^2)$~~

~~$= \Theta(n^2)$~~

$$= 1 + 2n\left(1 - \frac{1}{n}\right) = 1 + 2n\frac{(n-1)}{n} = 1 + 2(n-1)$$

$\frac{1}{n}$ can be ignored as $\frac{1}{n} \to 0$ when $n \to \infty$

$= O(n)$

$= \Omega(n)$

$= \Theta(n)$

Decreasing G.P. series : $|r| < 1$

$$\frac{1}{2^0} + \left(\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{\log_2 n}}\right) \quad n \to \infty.$$

(1) + (1) = 2

Ex 7. $T(n) = \begin{cases} 2 & , n = 2 \\ 7T\left(\frac{n}{2}\right) + n^2 & , n > 2 \end{cases}$

$$T(n) = 7T\left(\frac{n}{2}\right) + n^2$$

$$7\left[7T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right] + n^2 = 7^2 \cdot T\left(\frac{n}{2^2}\right) + 7\cdot\left(\frac{n}{2}\right)^2 + n^2$$

$7\left(\frac{1}{2}\right)^2 = \frac{7}{4}$

$$= 7^2 \cdot \left[7T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2\right] + 7\cdot\left(\frac{n}{2}\right)^2 + n^2$$

$7\left(\frac{1}{2}\right)^4 = \frac{7}{16}$

$$= 7^3 \cdot T\left(\frac{n}{2^3}\right) + 7^2 \cdot \left(\frac{n}{4}\right)^2 + 7\left(\frac{n}{2}\right)^2 + n^2$$

$\frac{n}{2^k} = 2$; $\log n - 1$

$n = 2^{k+1}$

$k+1 = \log n$

$$= 7^{\log_2 n - 1} \cdot T(2) + 7^{\log_2 n - 2} \cdot \left(\frac{n}{2^{\log_2 n - 2}}\right)^2 + \cdots + 7\left(\frac{n}{2}\right)^2 + n^2$$

$\frac{2}{2^{\log_2 n - 1}}$

$$= 2\cdot 7^{\log n - 1} + n^2\left(1 + 7\left(\frac{1}{2}\right)^2 + 7^2\left(\frac{1}{2}\right)^4 + \cdots + 7^{\log_2 n - 2}\left(\frac{1}{2^{\log_2 n}}\right)\right)$$

$$= 2\cdot 7^{\log n - 1} + n^2\left[1\cdot\frac{\left(\frac{7}{4}\right)^{\log_2 n - 1} - 1}{\frac{7}{4} - 1}\right]$$

$$= 2.7^{\log_2 n - 1} + n^2 \cdot \frac{4}{3}\left[\left(\frac{7}{4}\right)^{\log_2 n - 1} - 1\right]$$

$$= 7^{\log_2 n} + n^2 \cdot \left[\left(\frac{7}{4}\right)^{\log n}\right] \quad \{\text{After eliminating constants}\}$$

$$= 7^{\log_2 n} + n^2 \frac{7^{\log n}}{2^{\log_2 n^2}} = 7^{\log_2 n} + \frac{n^2 \, 7^{\log_2 n}}{n^2}$$

$$= 2.7^{\log_2 n} = O(7^{\log_2 n})$$

$$= O(2^{\log n \cdot \log_2 7}) = O\left(n^{\log_2 7}\right)$$

$$= O(n^{2.81})$$

Imp ×8

$$T(n) = \begin{cases} 2 & ; \quad n = 2 \\ \sqrt{n}\,T(\sqrt{n}) + n & , \quad n > 2 \end{cases}$$

$(n)^{1/2} + (n^{1/2})$

$$T(n) = \sqrt{n}\,T(\sqrt{n}) + n$$

$$= n^{1/2}\left((n)^{1/4} \cdot T(n^{1/4}) + n^{1/2}\right) + n$$

$n^{\frac{1}{2}} \cdot n^{\frac{1}{2}}$

$n^{\frac{1+1}{2}}$

$\frac{1}{8} + \frac{1}{4}$

$\frac{1+2}{8}$

$$= n^{1/2^3} \cdot T(n^{1/4}) + n + n$$

$\frac{1}{2} \cdot \frac{1}{2}$

$\sqrt{n} \cdot \sqrt{n} = n$

$$= n^{1/2^3}\left(n^{1/8} T(n^{1/8}) + n^{1/4}\right) + n + n$$

$n^{\frac{1}{2}} \cdot n^{\frac{1}{4}}$

$\frac{1}{2} + \frac{1}{4}$

$\frac{2+1}{4} = 3/4$

$$= T(n^{1/8}) + n$$

$\frac{3}{4} \times \frac{1}{8}$ $\frac{6+1}{8}$ $7/8$

$$= n^{1/2}\left[n^{1/2^2} T\left(n^{1/2^2}\right) + n^{1/2}\right] + n$$

$\frac{1}{2}+\frac{1}{4}$

$$= n^{\frac{1}{2}+\frac{1}{2^2}} T\left(n^{1/2^2}\right) + n + n$$

$$\Downarrow$$

$\frac{3}{2^2}+\frac{1}{2^2}$

$$n^{3/2^2}\left[n^{1/2^3} T\left(n^{1/2^3}\right) + n^{1/2^2}\right] + n + n$$

$\frac{6+1}{2^3}$

$$= n^{7/2^3} T\left(n^{1/2^3}\right) + n + n + n$$

$$\vdots$$

$$\frac{\log \log n}{2}$$

$$\vdots$$

$$= n^{\frac{2^k-1}{2^k}} T\left(n^{1/2^k}\right) + n \log \log n$$

$n^{\frac{1}{2^k}} = 2$

$\log_2 n^{\frac{1}{2^k}} = \log_2 2$

$\frac{1}{2^k}\log_2 n = \log_2 2^k$

$\Rightarrow \log_2 n = k$

$k = \log \log n$

$2^{\log_2 n}$

$$= n^{\frac{2^{\log_2 n}-1}{2^{\log_2 n}}} T\left(n^{\frac{1}{2^{\log_2 n}}}\right) + n \cdot \frac{\log n}{2}$$

$$= n T(2) + \frac{n \log n}{2}$$

$$= 2n + \frac{n \log n}{2} = O(n \log n)$$

$$= \frac{n}{n^{2\log_2 \log n}} T(2) + n \cdot \log \log n \qquad 2^{\log_2 (\log n)}$$

$$\boxed{a^{\log_b n} = n^{\log_b a}}$$

$$= \frac{n}{2^{\log_2 \log n}} \cdot 2 + n \cdot \log_2 \log_2 n$$

$$= n + n \log \log n = O(n \log_2 \log_2 n)$$

Ex 9: used in $\rightarrow$ $\boxed{T(n) = 2T(n-1) + n}$

AGP

$$T(n) = 1 \cdot \left(\tfrac{1}{2}\right)^1 + 2\left(\tfrac{1}{2}\right)^2 + 3\left(\tfrac{1}{2}\right)^3 + \cdots + (n-1)\left(\tfrac{1}{2}\right)^{n-1} + n\left(\tfrac{1}{2}\right)^n$$

$$\Rightarrow T(n) = 1 \cdot \left(\tfrac{1}{2}\right)^1 + 2\left(\tfrac{1}{2}\right)^2 + 3\left(\tfrac{1}{2}\right)^3 + \cdots (n-1)\left(\tfrac{1}{2}\right)^{n-1} + n\left(\tfrac{1}{2}\right)^n$$

$$\tfrac{1}{2}T(n) = \quad 1 \cdot \left(\tfrac{1}{2}\right)^2 + 2\left(\tfrac{1}{2}\right)^3 + \cdots + (n-2)\left(\tfrac{1}{2}\right)^{n-1} + (n-1)\left(\tfrac{1}{2}\right)^n + n\left(\tfrac{1}{2}\right)^{n+1}$$

$$\left(1 - \tfrac{1}{2}\right)T(n) = \left[\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \cdots + \left(\tfrac{1}{2}\right)^{n-1} + \left(\tfrac{1}{2}\right)^n\right] - n\left(\tfrac{1}{2}\right)^{n+1}$$

$$\frac{1}{2}T(n) = \frac{1}{2} \cdot \frac{1 - (1/2)^n}{1 - 1/2} - n \cdot \left(\frac{1}{2}\right)^{n+1}$$

$$= \frac{1}{2} \cdot 2 \cdot \left(1 - \frac{1}{2^n}\right) - \frac{n}{2^{n+1}}$$

$$T(n) = 2\left[1 - \frac{n}{2^{n+1}}\right] = 2\left[1 - \frac{n}{2^{n+1}}\right]$$

$$= \frac{(n+2)}{2^{n+1}} = 2\left[1 - n \cdot \underbrace{\left(\frac{1}{2^{n+1}}\right)}_{\text{decreasing}}\right]$$

Divide & Conquer (DAC)

Divide :- the given problem into some sub-problems

Conquer the subproblems by calling recursively until we will get subproblem solution.

Combine the subproblem solution to get final problem solution.

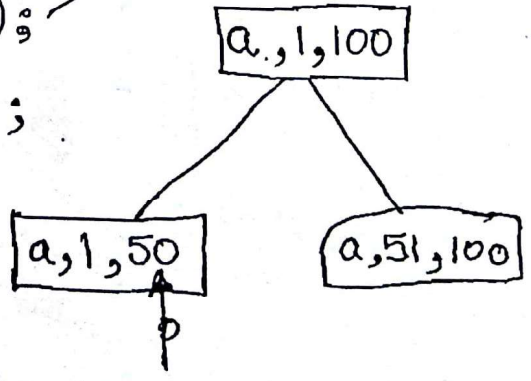
Quick Sort, Binary Search ⇒ Partially divide & Conquer
(No combine.)

Ex: Sorting of array :-

Divide and Conquer Abstract Algorithm

DAC (a, i, j) $\xrightarrow{\text{For } n}$ T(n)

```
{
    if (small (a,i,j))  ⟹ O(1)
        return (solution (a, i, j));
    else
        p = divide (a, i, j);   → O(f1(n))
        L = DAC (a, i, p);      → T(n/2)
        R = DAC (a, p, j);      → T(n/2)
        C = Combine (L, R);     ← O(f2(n))
        return C;
}
```



Code of small(), solution(), divide() & combine() is dependent on programmer and varies problem by problem.

- If some problem can be solved using Divide & Conquer algorithm, it should be possible to define the above 4 functions.

Time Complexity of any problem solved using DAC :-

If $T(n)$ is the time taken to solve the entire DAC algorithm for the problem for n elements.

$T(n/2) \longrightarrow$ for $n/2$ elements

Assume 2 subproblems with size $n/2$ each.

$$T(n) = \begin{cases} O(1) & \text{, for small problem} \\ O(f_1(n)) + 2T(n/2) + O(f_2(n)) & \text{, for big problem} \end{cases}$$

$$= 2T(n/2) + \underbrace{O(f_1(n)) + O(f_2(n))}$$

$$= 2T(n/2) + O(f(n))$$

- 2 : no. of subproblems
- $T(n/2)$: Time taken to solve a subproblem; $n/2$: size of subproblem
- $O(f(n))$: Time taken in divide & combine

$$= aT(n/b) + O(f(n)) \quad \{\text{In general}\}$$

where $a \geq 1$, $b > 1$; $f(n)$: Time Taken

For Quick Sort $T(n) = 2T(n/2) + n = O(n \log n)$

no. of subproblems; size of subproblem

Binary Search $T(n) = T(n/2) + c$

1 subproblem; size of subproblem.

Matrix Multiplication $T(n) = 8T(n/2) + n^2$

Time for divide & combine

APPLICATIONS of Divide & Conquer

1) Finding max. & min. element
2) Power of an element
3) Binary Search
4) Merge Sort
5) Quick Sort
6) Selection procedure
7) Strassen's Matrix multiplication

1) Finding Max & Min

i/p: array of n elements.

o/p: max & min element

Ex: A [25 (1), 79 (2), 88 (3), 16 (4), 32 (5), 11 (6), 17 (7)].

~~max = 25 79~~

~~min = 25~~

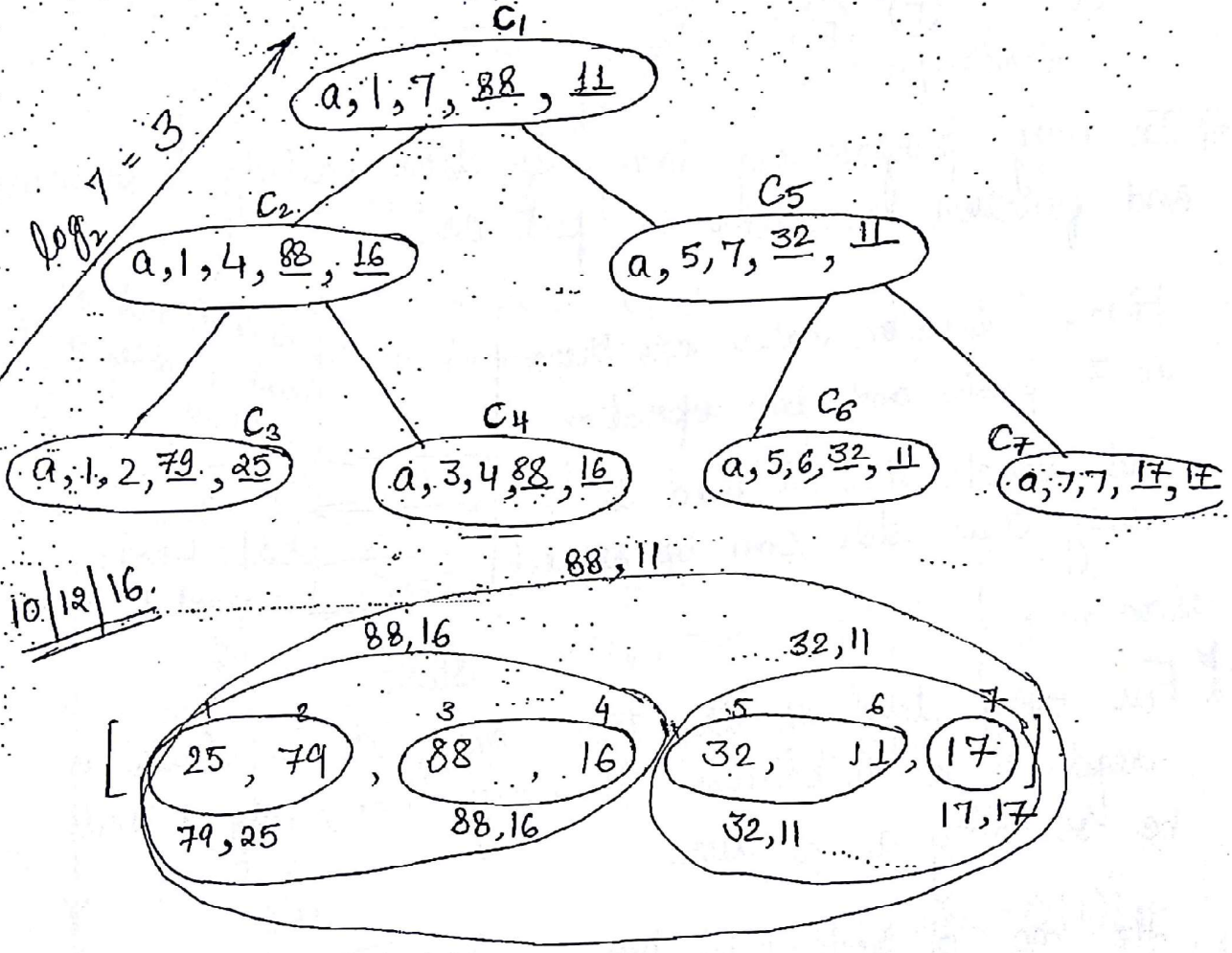
Define small problem,

small

⇓

1 element ⇒ 0 comparisons

2 elements ⇒ 1 comparison



10 | 12 | 16.

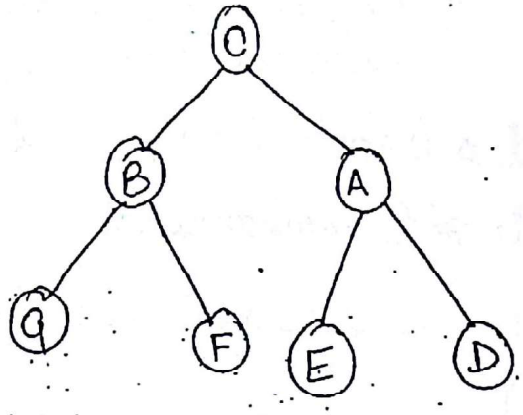
Every node in the tree is a function call.

Preorder :- Root - Left - Right

Inorder :- Left - Root - Right

Postorder :- Left - Right - Root.

} on the basis of position of root



Pre-order :

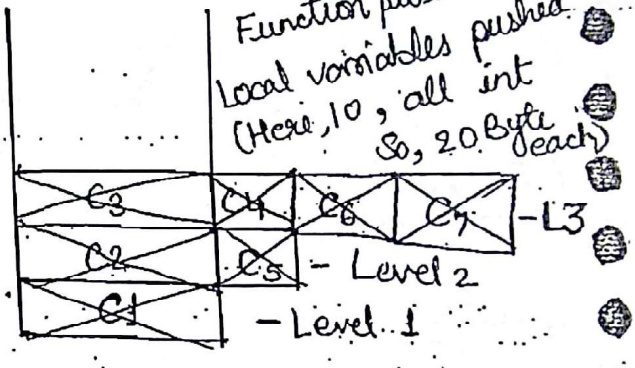
CBGFAED.

Post order :-

GFBEDAC

In every programming lang, function calling is preorder and function execution is post-order.

Here, 7 function calls are there so 7 push and pop operations. But stack size utilized is only 3 as slots can be reused.



Stack

* For every level of the tree, one slot in stack is reqd. It is so because at every level, only 1 will be running at a time.

The no. of levels in tree = $\log_2 n$ where n is the no. of records in the array = No. of slots in stack space.

Space complexity = total space utilized in running the program

$$= \underbrace{\log n}_{\text{Stack space}} + \underbrace{n}_{\text{input size}} = O(n)$$

Return , break & exit :-

Return → out of fnctn

break → out of Block

exit → out of entire program

Program Code :-

DAC minmax (a, i, j) → T(n) → T(n) comparisons

{

small

1-element

if (i == j)

{ max = min = a[i]; } 0-comparisons

return (max, min);

}

2-element

if (i == (j-1))

{

if (a[i] > a[j])

max = a[i]; min = a[j]; } 1 comparison

else

max = a[j]; min = a[i];

return (max, min);

}

else

{

Divide {O(1)}

mid = ⌊(i+j)/2⌋ → 0 comp.

Conquer

(max_1, min_1) = DAC minmax (a, i, mid); → $T(n/2)$

(max_2, min_2) = DAC minmax (a, mid+1, j); → $T(n/2)$

if (max1 > max2)

max = max1;

else

max = max2;

$T(n/2)$ comp.

```
if (min1 < min2)
    min = min1;
else
    min = min2;
return (max, min);
}
```

Combine {O(1)}.
2 comparisons.

$$T(n) = 2T\left(\frac{n}{2}\right) + O(1) + O(1)$$

Let $T(n)$ = time complexity of ~~this~~ above algorithm for n elements.

Recurrence Relation =

$$T(n) = \begin{cases} O(1) & , \ n = 1 \text{ or } n = 2 \\ 2T\left(\frac{n}{2}\right) + O(1) + O(1) & , \ n > 2 \end{cases}$$

$2T\left(\frac{n}{2}\right)$ ↓ Conquer, $O(1)$ ↓ Divide, $O(1)$ ↓ Combine

$$T(n) = 2T\left(\frac{n}{2}\right) + c$$

$$2\left\{2T\left(\frac{n}{2^2}\right) + c\right\} = 2^2 \cdot T\left(\frac{n}{2^2}\right) + 2c + c$$

$$= 2^2\left\{2T\left(\frac{n}{2^3}\right) + c\right\} + 2c + c$$

$$= 2^3 \cdot T\left(\frac{n}{2^3}\right) + 2^2 c + 2c + c$$

$$\left(\text{Keep stopping at } n=2\right) \frac{n}{2^k} = 1$$

$$\Rightarrow k = \log_2 n$$

$$2^k T\left(\frac{n}{2^k}\right) + 2^{k-1}c + \ldots + 2^2 c + 2c + c$$

$$= n \cdot T(1) + 2^{\log n - 1} \cdot c + \ldots + 2^2 c + 2^1 c + 2^0 c$$

$$= n + c\left\{\frac{2^{\log_2 n} - 1}{2 - 1}\right\}$$

$$= n + c(n-1) = O(n) = \Omega(n) = \Theta(n)$$

// Time Complexity

Let $T(n)$ = No. of comparisons for the above algorithm on 'n' elements.

Recurrence Relation =

$$T(n) = \begin{cases} 0 & \text{if } n=1 \\ 1 & \text{if } n=2 \\ 0 + 2T(n/2) + 2, & n>2 \end{cases}$$

$$T(n) = 2T(n/2) + 2$$

$$= 2\left\{2T\left(\frac{n}{2^2}\right) + 2\right\} + 2$$

$$= 2^2 T\left(\frac{n}{2^2}\right) + 2^2 + 2$$

$$= 2^2\left\{2T\left(\frac{n}{2^3}\right)+2\right\} + 2^2 + 2$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 2^3 + 2^2 + 2$$

$$\frac{n}{2^k} = 2 \Rightarrow 2^{k+1} = n$$

$$k = \log_2 n - 1$$

Stack Space

$$= 2^k T\left(\frac{n}{2^k}\right) + 2^k + \ldots\ldots 2^3 + 2^2 + 2^1$$

$$= \frac{n}{2} T(2) + 2^{\log_2 n - 1} + \ldots + 2^3 + 2^2 + 2$$

$$= \frac{n}{2} \cdot 1 + 2\left(\frac{2^{\log_2 n - 1} - 1}{2-1}\right)$$

$$= \frac{n}{2} + 2\left(\frac{n}{2} - 1\right) = \frac{n}{2} + n - 2$$

$$= \frac{3n}{2} - 2$$

{ Comparisons ⇓ Exact no. reqd. }

- Better algorithm than this not available for max & min.
- Best Case / Worst Case same here (as without comparison we cannot know if element is max/min).

2) Power of an element

i/p : 2 +ive integers a & n.
where $a \geq 2$, $n \geq 1$

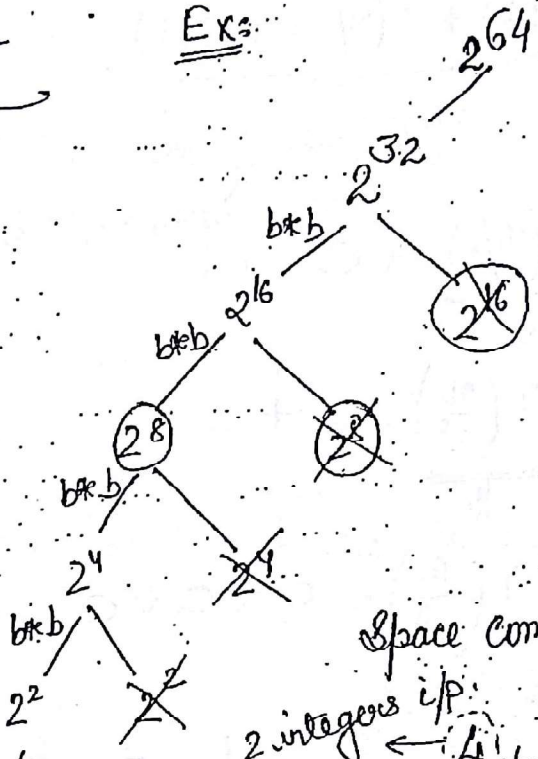
o/p : Find a^n

$a^n = \underbrace{a^{n/2} \cdot a^{n/2}}_{\text{Divide}}$ (b . b)

b

Small:
$n = 1$
return a

Ex:



Space complexity :=
2 integers i/p ← 4 + log n
= O(log n)

Algorithm :-

DACPower (a, n) ⟹ T(n)
{
if (n == 1) → 0 mult.
return a;
else
{
For n is odd → if (n%2 == 0)
return power;
else
return a * power;

mid = n/2 ; → 0 mult. Divide, O(1)
b = DACPower (a, mid) ; → T(n/2), Conquer
power = b * b ; → T(n/2) mult.
return power; → O(1), 1 mult. combine
}
}

Let $T(n)$ = time complexity of above algorithm to find a^n

$$T(n) = \begin{cases} O(1) & , n = 1 \\ \underbrace{O(1) + T(n/2) + O(1)}_{T(n/2) + c} & , n > 1 \end{cases}$$

$c \rightarrow$ Divide time + Combine Time

$$T(n) = T(n/2) + c$$

$$= T\left(\frac{n}{2^2}\right) + c + c$$

$$= T\left(\frac{n}{2^3}\right) + c + c + c \quad \text{---}$$

$$\frac{n}{2^k} = 1 \Rightarrow k = \log_2 n \text{ (Stack Space)}$$

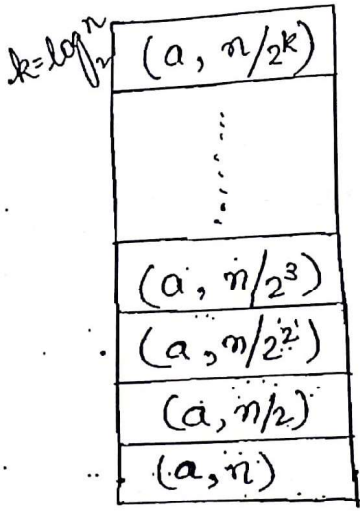
$$= T\left(\frac{n}{2^k}\right) + k \cdot c$$

$$= \underbrace{T(1)}_{= O(1)} + \underbrace{c \cdot \log_2 n}_{\Downarrow} = O(\log_2 n) \begin{cases} \text{Best/Worst/} \\ \text{Avg Case} \end{cases}$$

$= O(1)$ which is Small problem Cost

division & combine cost at all levels ($\log_2 n$)

$= \Theta(\log_2 n)$



Total space = i/p + extra

$= 2(2) + 10 \times \log_2 n$

$= 4 + 10 \log_2 n$

Let $T(n)$ = No. of multiplications

Recurrence Relation

$$T(n) = \begin{cases} 0 + T(n/2) + 1 & ; n > 1 \\ 0 & ; n = 1 \end{cases}$$

$T(n) = T(n/2) + 1$

$\Downarrow$

$T(n/4) + 1 + 1$

$\Downarrow$

$T\left(\frac{n}{2^3}\right) + 1 + 1 + 1$

$\vdots$

$\frac{n}{2^k} = 1 \Rightarrow k = \log_2 n$

$= T\left(\frac{n}{2^k}\right) + k$

$= 0 + \log_2 n = \log_2 n$

3) Binary Search

Linear Search:

i/p: an array & an element x

o/p: position of x

Best Case	Worst Case	Avg Case
$O(1)$	$O(n)$	$\frac{(n+1)}{2} = O(n)$

For all elements:
$1+2+3+\dots n = \frac{n(n+1)}{2}$

$= \frac{n+1}{2}$

No stack reqd. so better in comparison to Binary Search in terms of space.

Binary Search: (There is no combine, so, Binary Search is partial appl^n of Divide & Conquer)

i/p: a **sorted** array of n elements & element x

o/p: position of x

i/p : (10 (1) 20 (2) 30 (3) 40 (4) 50 (5) 60 (6) ..... 70 (7))

Program:

```
BS(a, i, j, x)
{
    if (i == j)
    {
        if (a[i] == x)
            return i;
        else
            return -1;
    }
```

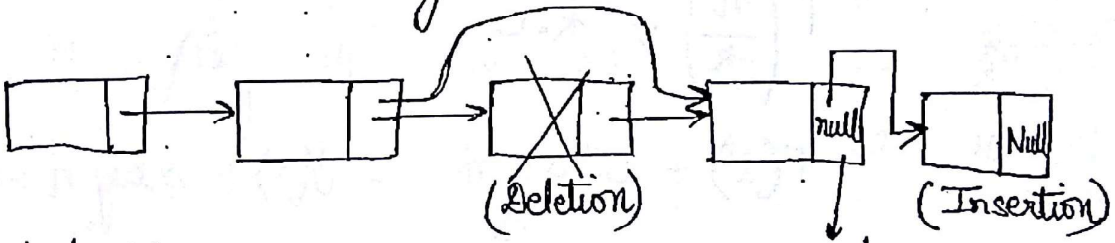
```
else
{
    mid = ⌊(i+j)/2⌋;          ⟹ O(1)
    if (x == a[mid])          ⟹ O(1)
        return mid;
    if (x < a[mid])           ⟹ O(1)
    {
        j = mid - 1;
        pos = (BS(a, i, j, x); ⟹ T(n/2)
    }
    else { if "else" is removed, it will go on both sides
    {                                ⟹ Linear Search }
        i = mid + 1;
        pos = BS(a, i, j, x); ⟹ T(n/2)
    }
    return pos;
}
```

O(1) = C

Best Case stops here

Linked List & Array :- Linked List → dynamic allocation

Size is not fixed; easily expandible, element can be deleted/inserted at any position.



In Linked List, random access is not possible (only drawback)

Array → size specified, random access possible, expansion not allowed.

Let $T(n)$ = Time complexity of above algorithm on n-elements, then

Recurrence Relation :=

$$T(n) = \begin{cases} O(1) & , n = 1 \\ O(1) + T\left(\frac{n}{2}\right) & , n > 1 \end{cases}$$

$$T(n) = T\left(\frac{n}{2}\right) + c \quad \leftarrow \text{time to decide which side (L/R) to go}$$

$$= T\left(\frac{n}{4}\right) + c + c$$

$$T\left(\frac{n}{2^3}\right) + c + c + c$$

$$\frac{n}{2^k} = 1 \Rightarrow k = \log_2 n$$

$$= T\left(\frac{n}{2^k}\right) + k \cdot c$$

$$= T(1) + c \cdot \log_2 n = O(1) + c \log_2 n = O(\log_2 n)$$

Best Case for Binary Search : $O(1)$ Avg./Worst Case

When middle element is search element.

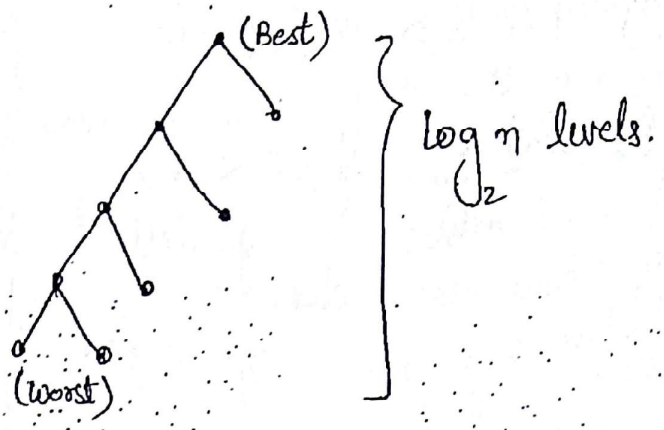
Avg case Proof :-

$1+2+3+\ldots+\log_2 n$

$= \frac{\log n(\log n+1)}{2}$

$Avg = \frac{\log n(\log n+1)}{2 \cdot \log n}$

$= \frac{\log n+1}{2} = O(\log_2 n)$



Main() —array passed→ BS(a, i, j, x)

(all values are not passed, its base address only passed ⇒ Call by Reference)

Binary Search is not possible in LinkedList.

Ques:

A	-150	-120	-110	-100	-80	-50	-20	-10	0	5
	1	2	3	4	5	6	7	8	9	10

8	10	12	14	20	25	30	50	80
11	12	13	14	15	16	17	18	19

100	130	150	180	200	250	value
20	21	22	23	24	25	pos

i/p: a sorted array of n- distinct elements

o/p: find any elements $a[i]$ such that $a[i] == i$

① LS $\Rightarrow$ $O(n)$

② BS : if ($p < V$) go left
else go right
} $O(\log_2 n)$

mid
<12 V: 12 >12

12 p: 13 14

<12 cannot be equal to 12.

>12 can be 14. Go Right.

- If sorted array not given,

(maybe) mid

Value: 12 .... 12-0 14 (may be)

pos: 12 13 14

< Not possible >

- If sorted array but not distinct elements :-

$\leftarrow$ mid $\rightarrow$

value : 12 15 15 15

pos : 12 13 14 15

Both sides possible $\Rightarrow$ Binary Search not possible.

Imp Ques: i/p : an array of n elements in which until some place all are integers & afterward all are ∞.

o/p : find position of 1st ∞. (array is not sorted in asc/desc order yet Binary Search possible)

⇓

Best algo, worst case

A	-150	+120	+110	-50	-10	6	8	90	∞	∞	∞
	1	2	3	4	5	6	7	8	9	10	11

←

∞	∞	∞	∞	∞	∞	∞	∞	∞
12	13	14	15	16	17	18	19	20

∞	∞	∞	∞	∞
21	22	23	24	25

① LS ⇒ O(n) ✓

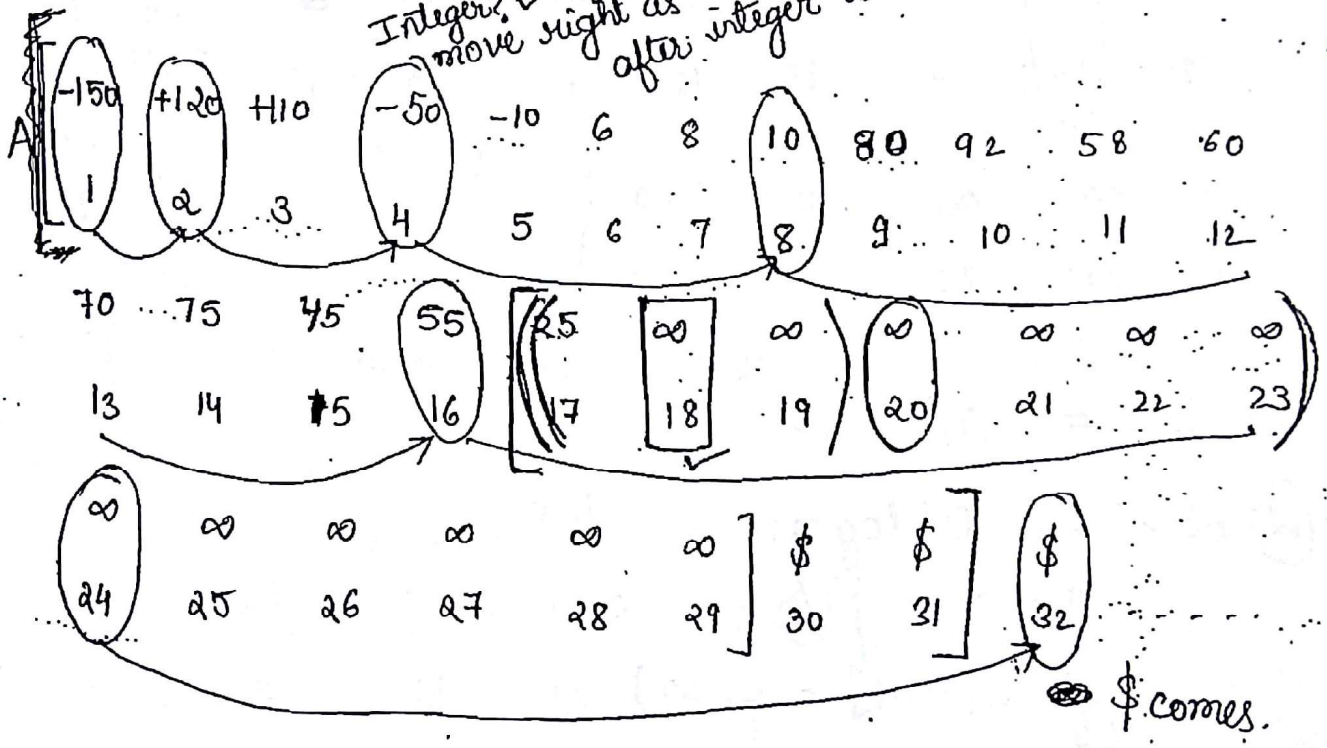
② BS ✓ ⇒ O(log n)

```
mid = (i+j)/2;
if (a[mid] == ∞)
{
    if (a[mid-1] != ∞)
        return mid;
    else
    {
        j = mid - 1;
    return BS(a, i, j);
    }
}
else
{
```

```
        i = mid + 1;
        return BS(a, i, j);
    }
```

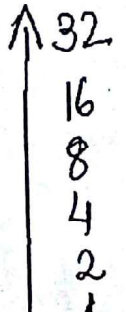
Ques: i/p: an array of n elements in which until some place all are integers & afterwards all are ∞.
[n is unknown & afterwards all are $]

o/p: find position of first ∞.



n is unkown means n cannot be used in algorithm.

Instead of dividing array into halves, multiply by 2.



Ques: i/p : a sorted array of n elements.

o/p : find any 2 elements a & b such that $a+b > 1000$.

100	200	300	400	500	600	700
1	2	3	4	5	6	7

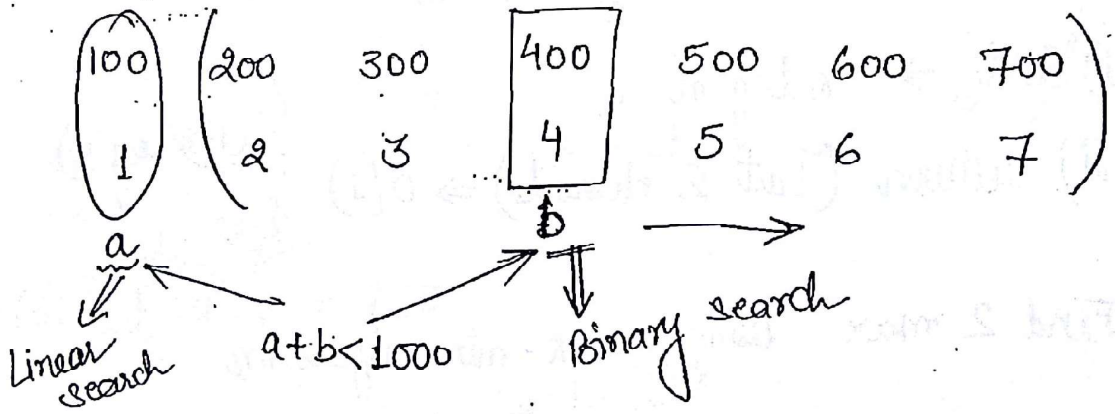
① Linear Search ✓ (n-times)

```
for (i = 1; i <= 7; i++)
{
    for (j = i; j <= 7; j++)
    {
        if (a[i] + a[j] > 1000)
            break;
    }
}
```

Sum of (n-1) nos :-

$$\frac{(n-1)n}{2} = O(n^2)$$

② Binary Search ✓ (n-times)



$O(n \log n)$

③ Check for last 2 elements :

if possible ⇒ they are a & b.

otherwise, no pair possible → O(1) ✓

Ques same as prev., only the i/p array is not sorted.

1) Linear Search ✓ $O(n^2)$

2) Binary Search ✗

✓ But if we sort the array and then apply Binary search

For sort $\Rightarrow n \log n$

For Binary search $\Rightarrow n \log n$

Total $= n \log n + n \log n$

$= O(n \log_2 n)$

(3) i) Sort $\Rightarrow n \log n$

ii) return (Last 2 elements) $\Rightarrow O(1)$ } $O(n \log n)$

(4) Find 2 max. using max-min algorithm

Best

i) 1st max $\Rightarrow O(n)$

ii) 2nd max $\Rightarrow O(n)$ } $O(n)$

iii) Then sum (1st max, 2nd max) > 1000 (Ans)

else no pair possible.

To calculate max & min, use 2 passes of Bubble

Sort

After 1st iteration, max element at last place $\Rightarrow n$

After 2nd iteration, 2nd max at 2nd last place $\Rightarrow n-1$

$$\text{Total} = 2n-1$$
$$= O(n)$$

Ques: i/p: sorted array of n-elements

o/p: find any 2 elements such that $a+b=1000$

① Linear search $O(n^2)$ ~

```
for(i=1 to n)
{
  for(j=i to n)
  {
    if(a[i]+a[j]==1000)
      break;
  }
}
```

② Binary search $O(n \log n)$ ✓

100	300	400	700	800	900
1	2	3	4	5	6

a →

③ Greedy Algo

Let $a+b==950$

$a+b=950$

100	200	300	450	500	600	700
1	2	3	4	5	6	7
			a	b		

a → a → a ... b

$a+b = 1000 > 950$ (dec b)

$a+b = 900 < 950$

Check $a+b = 800 < 950$ (inc. smaller i.e. a)

(i) a – 1st b → last

(ii) while (a+b != 1000)

```
{
    if (a+b < 1000)
        a = a+1;
    else
        b = b-1;
}
```

⇒ O(n) // Best

If the array is not sorted,

Sort it first ⇒ $n \log n$

Then $n + n \log n = O(n \log n)$

If a+b+c = 1000. then fix a, apply greedy on b&c. and iterate on a.

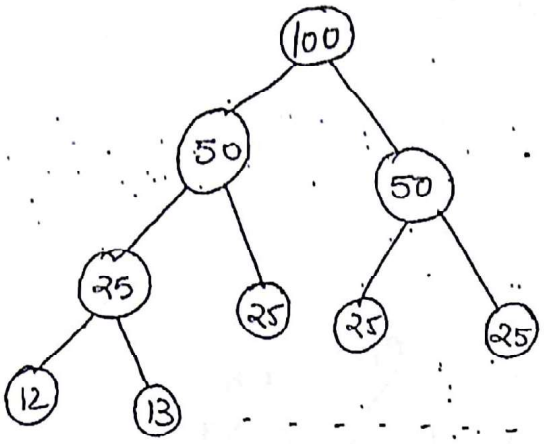
⇒ $O(n^2)$ ✓

Linear Search ⇒ $O(n^3)$

Binary Search ⇒ $O(n^2 \log n)$

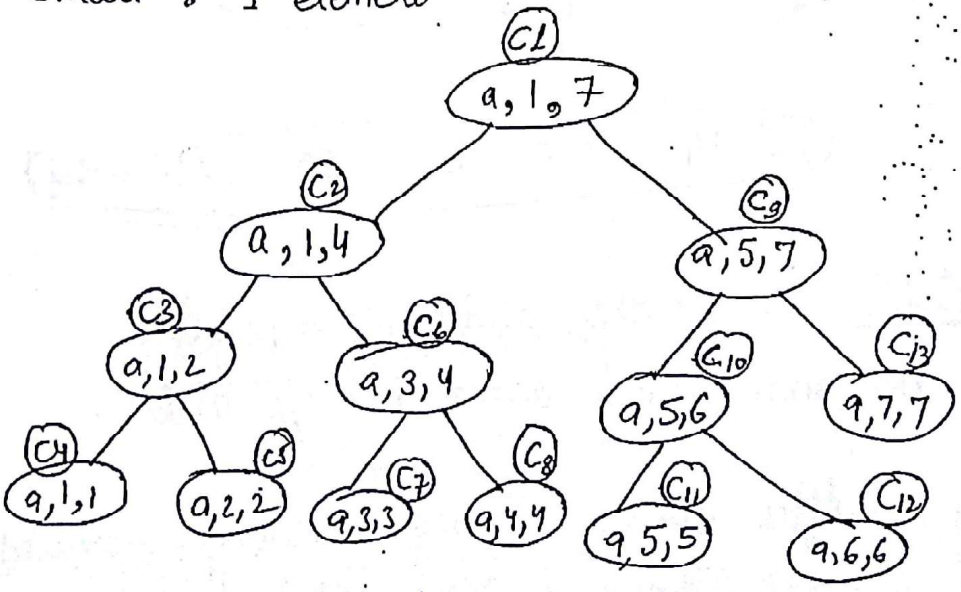
4) Merge Sort

Note: Merging 2 sorted subarrays.

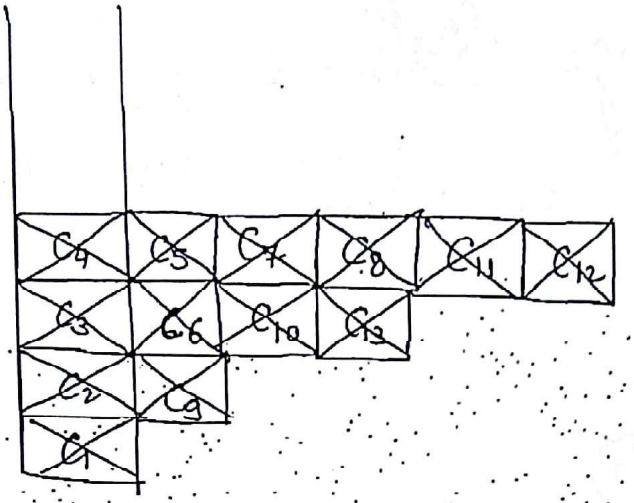


Ex: i/p A = [75 15 90 19 65 28 31]
1 2 3 4 5 6 7

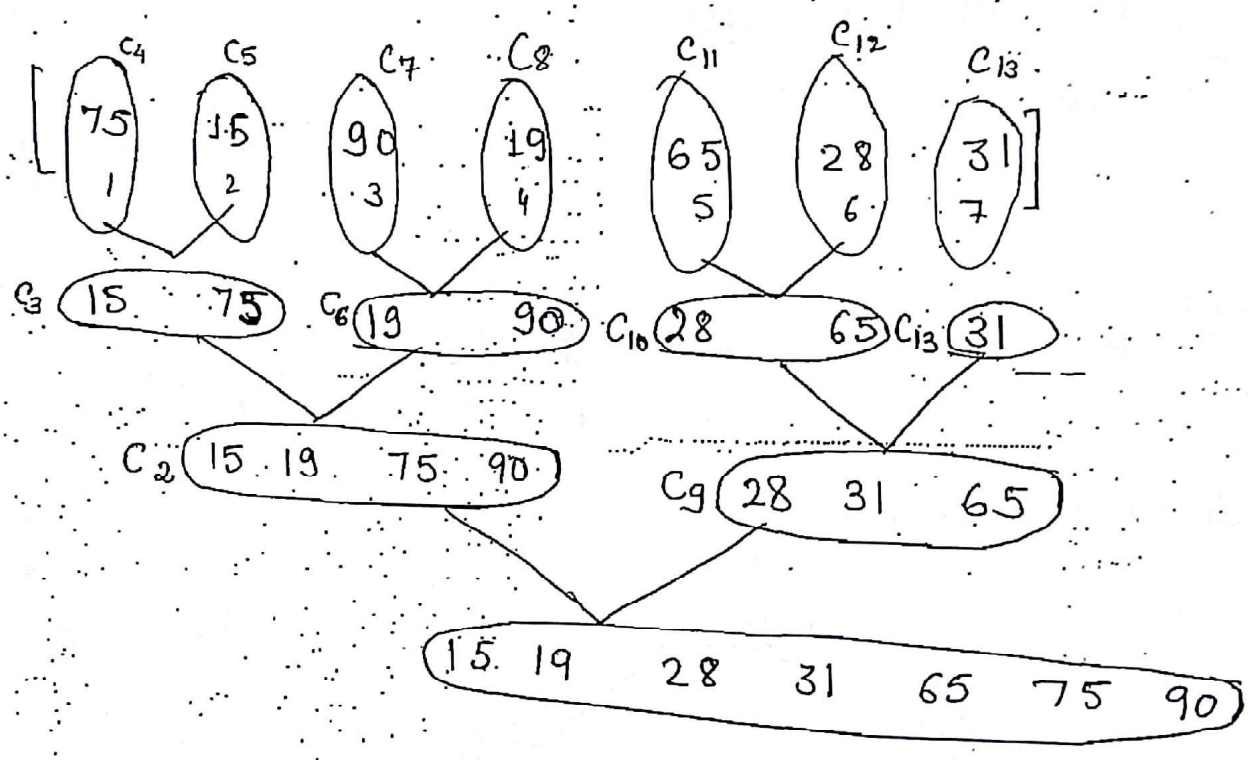
Small : 1 element



Stack :-



Stack Space = 4

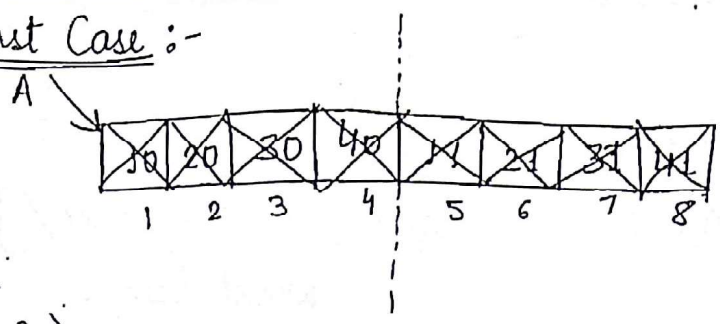


Merge sort is stable i.e. the order of repeated elements should be maintained same as i/p order.

- Stable Sorting :- After sorting & before sorting, repeated elements order is not changed. Merge sort is stable but quick sort is not stable.

Merge Algorithm :-

Worst Case :-



10, 11 ⇒ 10
20, 11 ⇒ 11
20, 21 ⇒ 20
30, 21 ⇒ 21
30, 31 ⇒ 30
40, 31 ⇒ 31
40, 41 ⇒ 40
41 (No comparison)

Comparisons

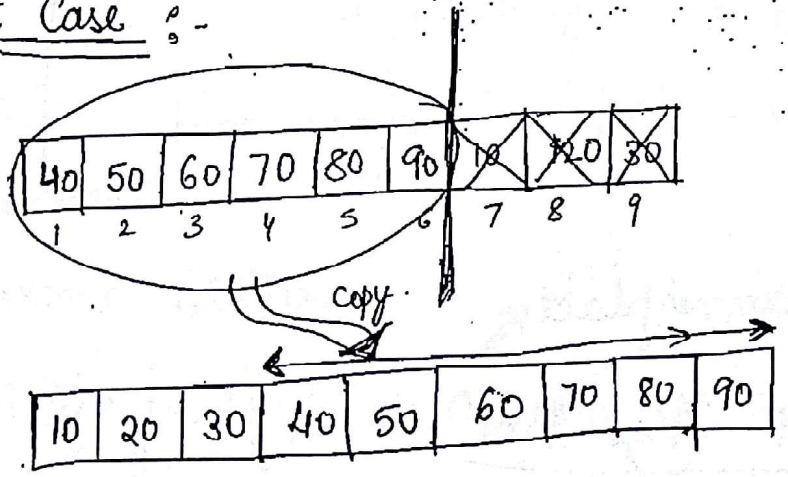
4, 4 ⇒ 4+4−1
m, n ⇒ m+n−1
n/2, n/2 ⇒ n/2 + n/2 − 1

Moves

4, 4 ⇒ 8
m, n ⇒ m+n
n/2, n/2 ⇒ n/2 + n/2 = n

Moves > Comparisons
⇒ Time complexity depends on moves

Best Case :-



40, 10 ⇒ 10
40, 20 ⇒ 20
40, 30, ⇒ 30
40
50
60
70

moves

4, 4 ⇒ 4 + 4
m, n ⇒ m + n
n/2, n/2 ⇒ n/2 + n/2 = n

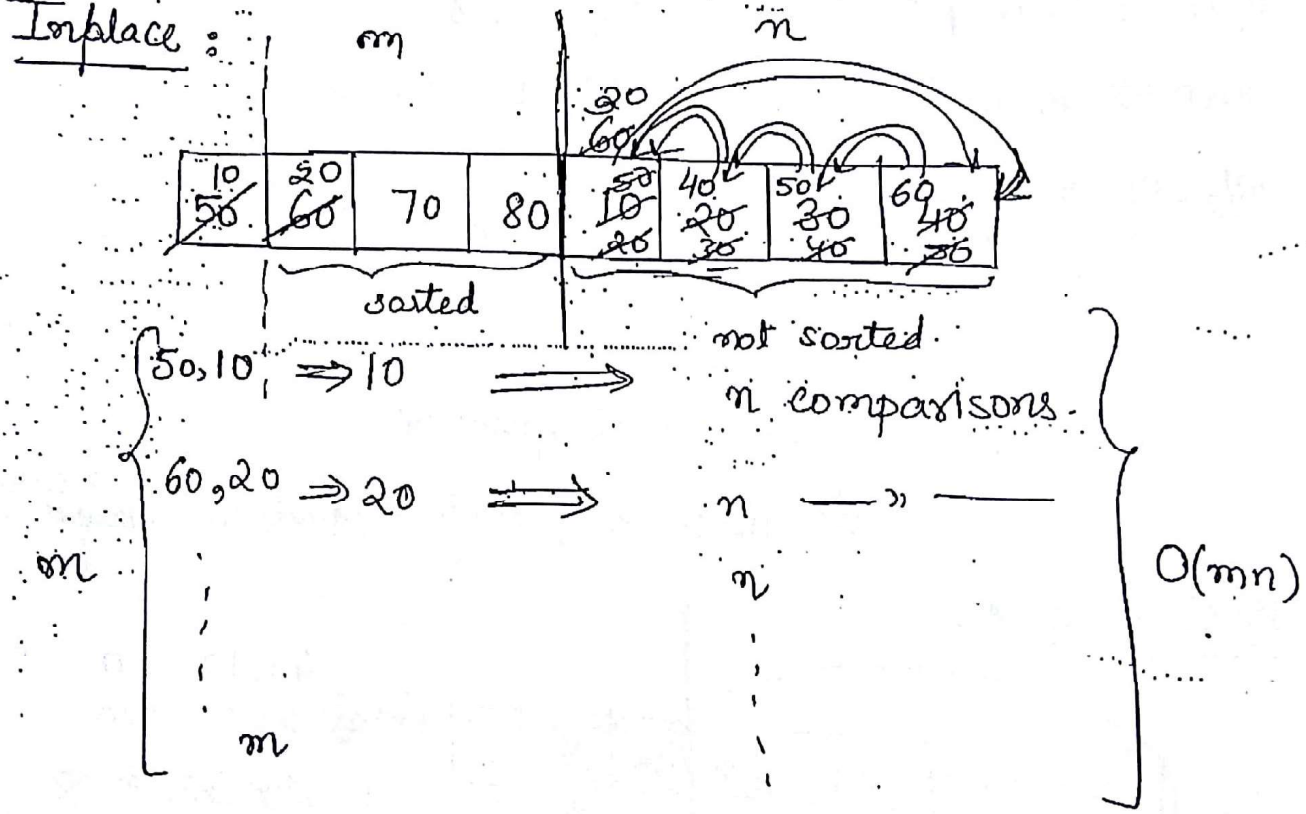
Comparisons

4, 4 ⇒ 4
m, n ⇒ min (m, n)
n/2, n/2 ⇒ n/2

Note :-

- Merging 2 sorted subarrays each of size m & n will take $O(m+n)$ [Best Case / Avg case / Worst case].
- Merge algorithm is requires another extra array for o/p and is called outplace merge algo.
- If space is saved by inplace merge algo,
  Time complexity = $m * n$. and not $m+n$

Inplace :



Quick sort is inplace.

Merge Sort Algorithm :-

$\Rightarrow T(n)$

```
Merge Sort (a, i, j)
{
    if (i == j)
        return a[i];
    else
    {
        mid = ⌊(i+j)/2⌋        → O(1)
        Merge Sort (a, i, mid);      → T(n/2)
        Merge Sort (a, mid+1, j);    → T(n/2)
        Merge (a, i, mid, mid+1, j)
        return a;
    }
}
```

If it is inplace

$\Rightarrow m * n = \frac{n}{2} \cdot \frac{n}{2}$

$= n^2/4 = O(n^2)$

It is outplace

So, $m+n = \frac{n}{2} + \frac{n}{2}$

$= n$

Since new 'b' array used, data copied back from b to a which is again $n/2 + n/2 = n$

So, no change $\Rightarrow O(n)$.

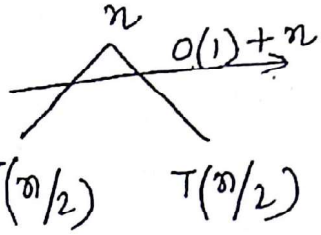
Let $T(n)$ = Time complexity of above algorithm on n-elements.

Recurrence Relation

Combine ↗ Divide ↗

$$T(n) = \begin{cases} 2T(n/2) + O(n) + O(1), & n > 1 \\ O(1), & n = 1 \end{cases}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

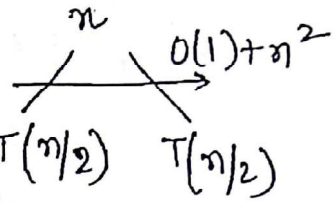


Outplace

$$= 2\left\{2T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right\} + n$$

$$= 2^2 T\left(\frac{n}{2^2}\right) + n + n$$

$$= 2^2\left\{2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right\} + n + n$$



Inplace

$$k = \log n$$

$$= 2^k T\left(\frac{n}{2^k}\right) + n + \dots + n + n$$

$$= nO(1) + n.k$$

$$= n + n\log n = O(n\log_2 n)$$ {Best / Avg / Worst case}

n: For every leaf node a cost. Total leaf = n

$n\log n$: All levels cost. Total levels = $\log n$. Cost/level = n.

Space complexity :-

i/p + extra

i/p: n

extra: Stack ($\log n$), B-array (outplace) (n).

$n + \log n + n$

$= O(n)$

Important Points :-

① Merge sort is outplace because in merge sort we take 1 extra array B.

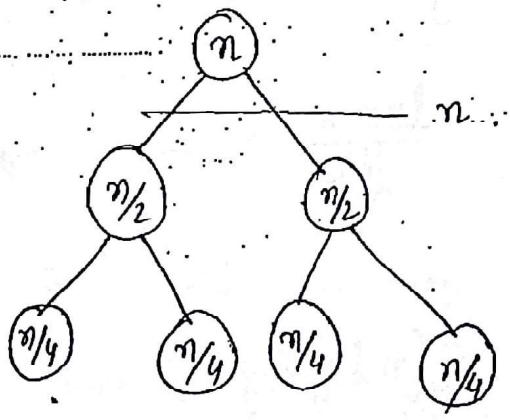
② Merge sort is not advisable for smaller size arrays. { Insertion sort is suitable for smaller arrays }

Ques 1 i/p : $\log n$ sorted subarrays each of size $\frac{n}{\log n}$

o/p : find single sorted array of n element.

~~$= \log n \cdot O(1) + \log n \cdot n$~~

~~$= \log n + n \log n$~~

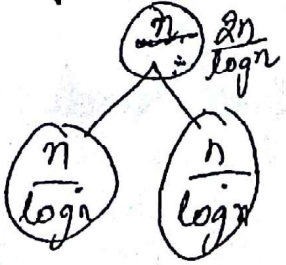


Try

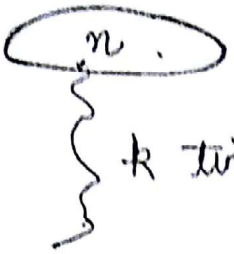
$\frac{2^k \cdot n}{\log n} = 2n$

$\Rightarrow 2^k = \frac{\log n}{\log n}$

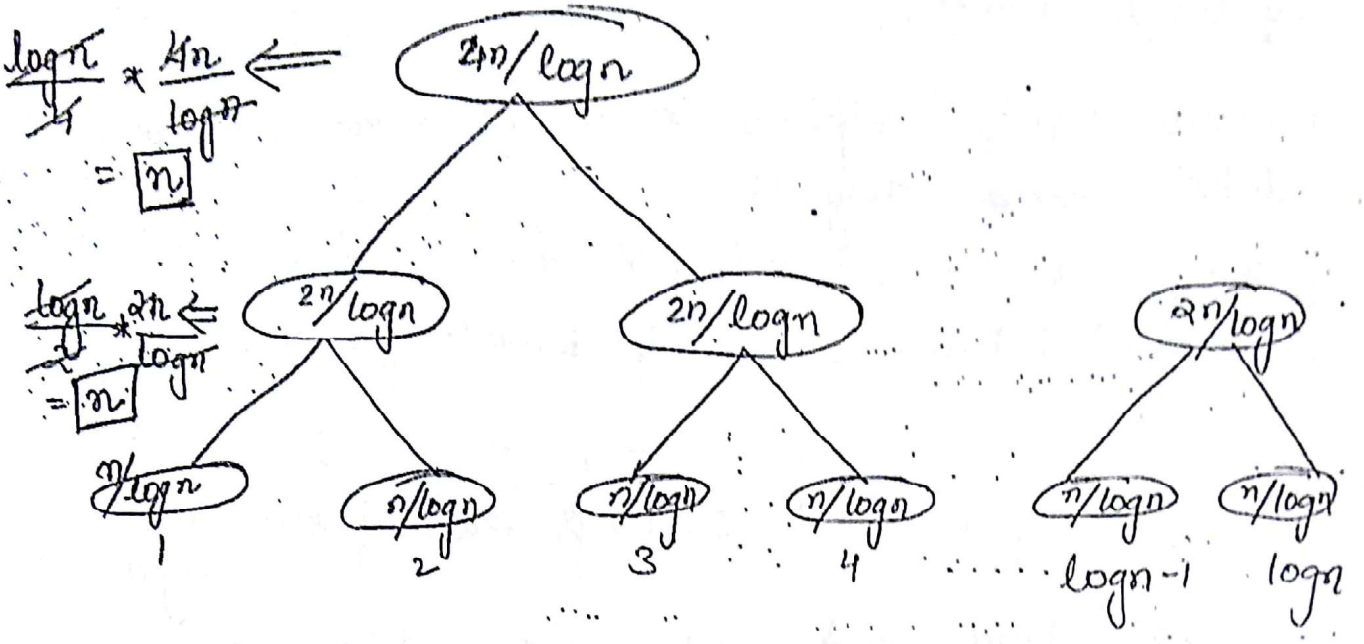
$\Rightarrow k = \log\left(\frac{\log n}{\log n}\right)$



Correct $\frac{\log n}{2^k} * \frac{2^k n}{\log n} = \boxed{n}$ ⟸



$\frac{\log n}{1} * \frac{4n}{\log n} = \boxed{n}$ ⟸



$\frac{\log n}{2} * \frac{2n}{\log n} = \boxed{n}$ ⟸

Cost of each level = n

$$\frac{\log n}{2^k} = 1 \Rightarrow 2^k = \log n$$

$$k = \log \log n$$

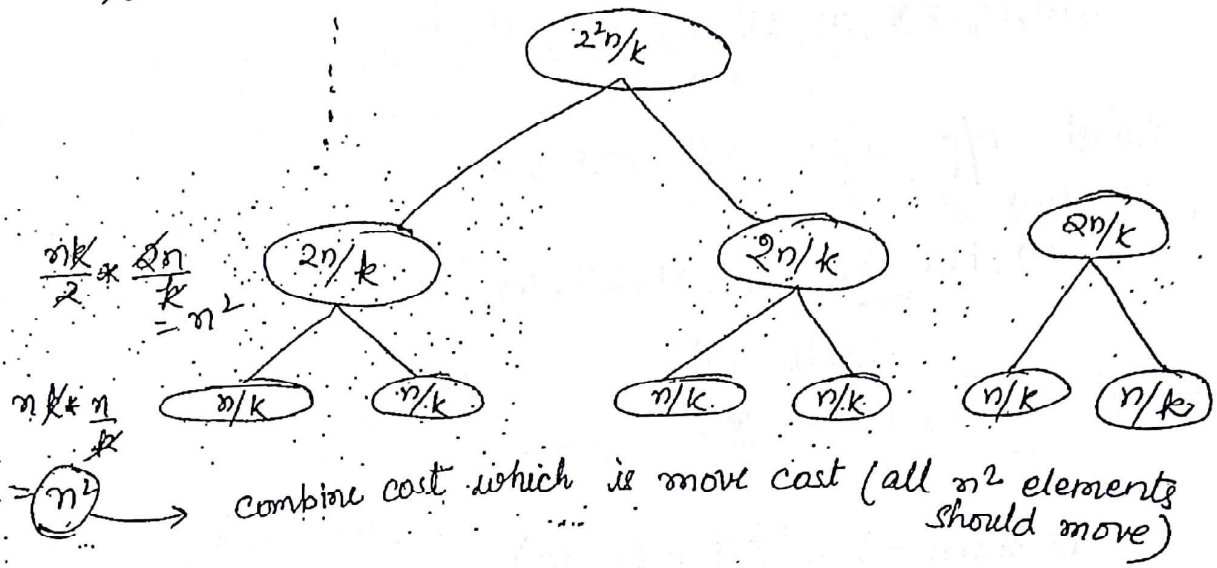
$\Rightarrow O(n \log \log n)$

If in Ques, 1 sorted array given ⇒ Binary Search
more than 1 sorted array ⇒ Merge Sort.

Ques 2 i/p : nk sorted sub-arrays each of size $\frac{n}{k}$

o/p : find single sorted array.

$\frac{nk}{2^p} * \frac{2^p n}{k} = n^2$



→ combine cost which is move cost (all n^2 elements should move)

$$\frac{nk}{2^p} = 1 \Rightarrow nk = 2^p$$

$$\Rightarrow p = \log nk.$$

$O(n^2 \log nk)$

Ques3: Consider the foll. array.

25 10 15 7 99 88 64 21 15 11

what will be o/p after 2nd pass of the straight 2-way merge sort algorithm {without dividing start} combining.

after 8th pass: 7,10,15,21,25,64,88,99

11,15

after 2nd pass: 7,10,15,25 | 21,64,88,99 | 11,15 | 11,15

1st pass o/p: 10,25 | 7,15 | 88,99 | 21,64 | 11,15

25 10 15 7 99 88 64 21 15 11

o/p after 2nd pass :

7, 10, 15_a, 25, 21, 64, 88, 99, 11, 15_b

Final o/p after 4th pass :-

7, 10, 11, 15_a, 15_b, 21, 25, 64, 88, 99

stable sort

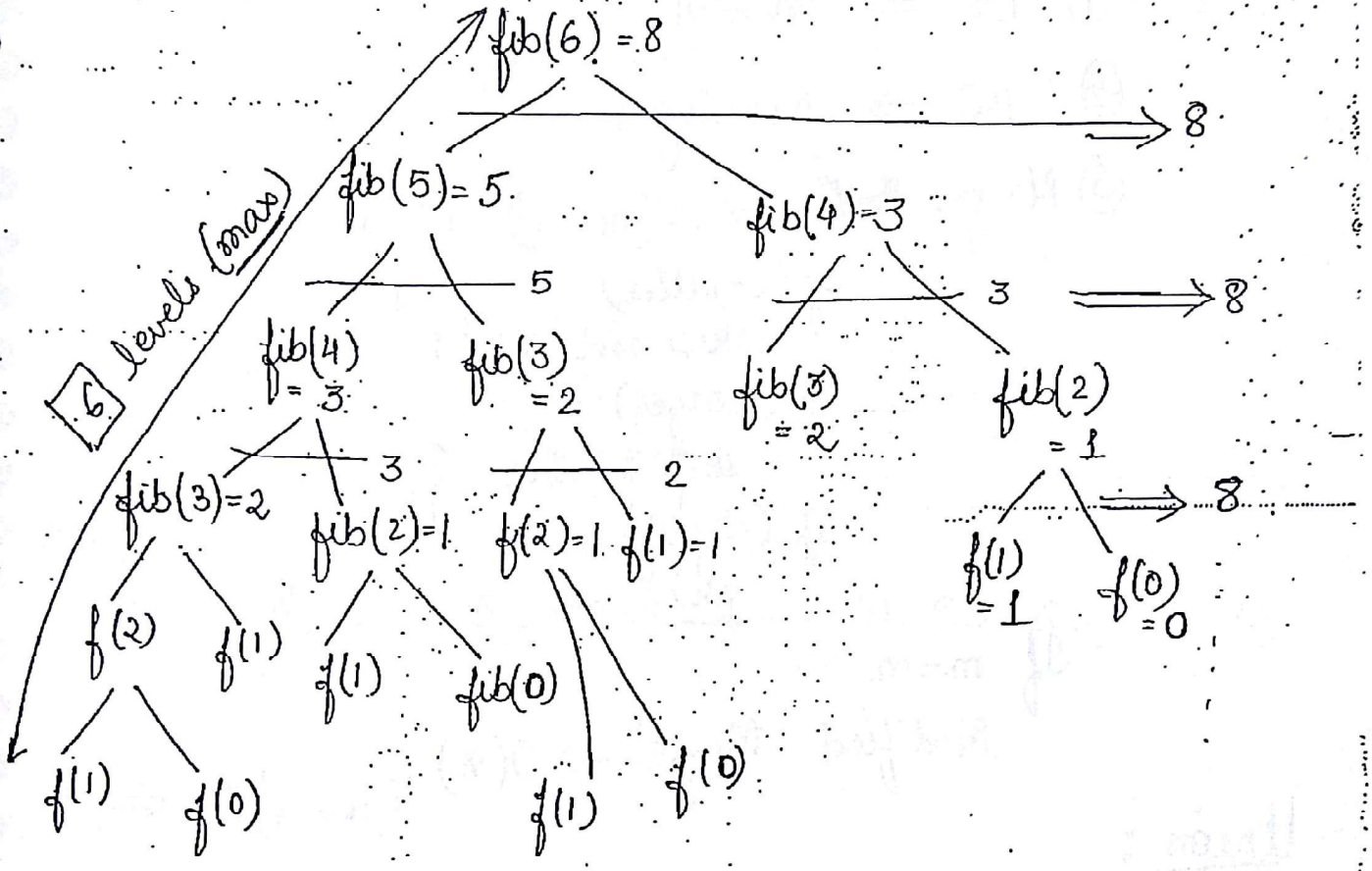
#Levels $= \lceil \log_2 10 \rceil = 4$

$O(n \log n) = O(10 \log_2 10)$

Ques4-

n	0	1	2	3	4	5	6	7	8
$fib(n)$	0	1	1	2	3	5	8	13	21

Using Fibonacci - merge sort ; Time taken to sort $f(n)$ elements.



Total cost $= O(6 \cdot fib(6)) = O(k \cdot fib(k))$

Ex5: i/p : 2 sorted arrays A – m
B – n

o/p : find intersection & union of A & B.

Intersection :-

A : 10 20 30 40 50 60 70.

B : 5 8 10 15 20 25 50 65 70 80

A ∩ B : ① LS ⇒ $m * n$.

② BS ⇒ $m * \log n$

③ Merge ⇒ $O(m+n)$

if (smaller)
skip and inc i
if (larger)
skip & inc. j
if (equal)
print

If $m = n$.
Modified Merge ⇒ $O(n)$.

Union :

A ∪ B : ① LS ⇒ (i) Add A.
(ii) Add B also if not in A
to check this L.S.

⇒ Time = $m + n * m$
$= O(m * n)$

② BS ⇒ Time = $m + n * \log m$
$= O(n \log m)$

③ Modified merge ⇒

```
if(A[i] > B[j])
{
    print (B[j]);
    j = j+1;
}
if(A[i] < B[j])
{
    print (A[i]);
    i = i+1;
}
if (A[i] == B[j])
{
    print (A[i]);
    i = i+1;
    j = j+1;
}
```

A ∪ B =

⇒ {5, 8, 10, 15, 20, 25, 30, 40, 50, 60, 65, 70, 80}

⇒ $O(m+n)$

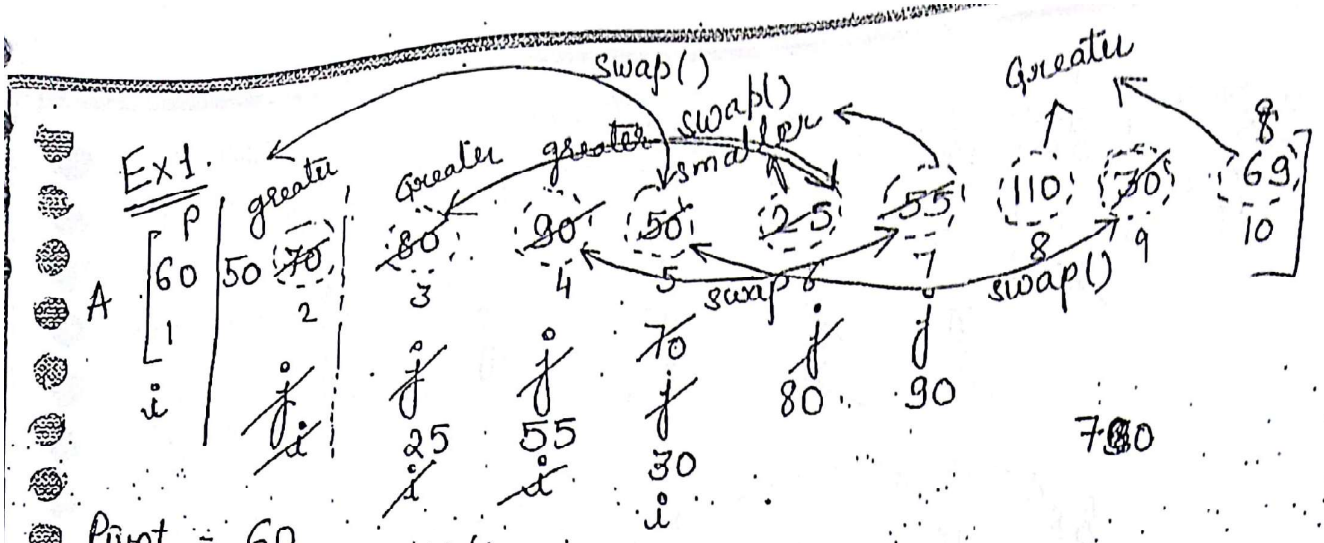
QUICK SORT (Tony Hoare)

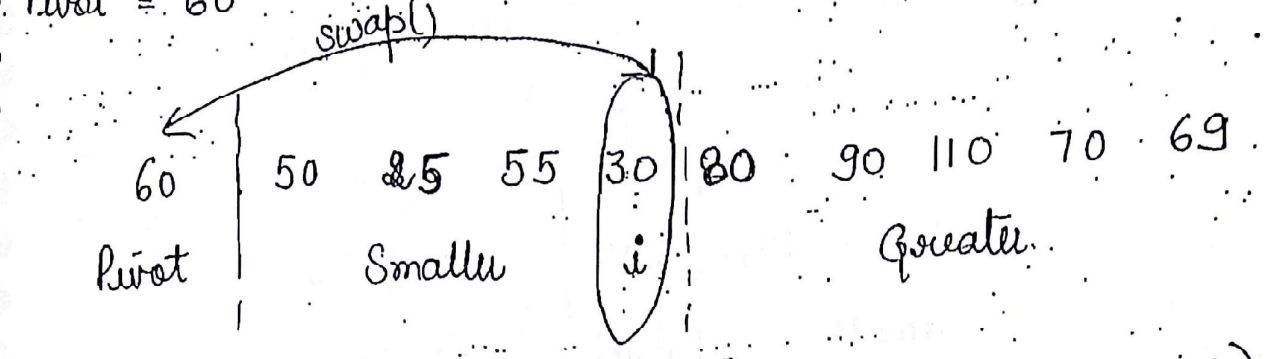
(Fastest Sorting Algorithm)

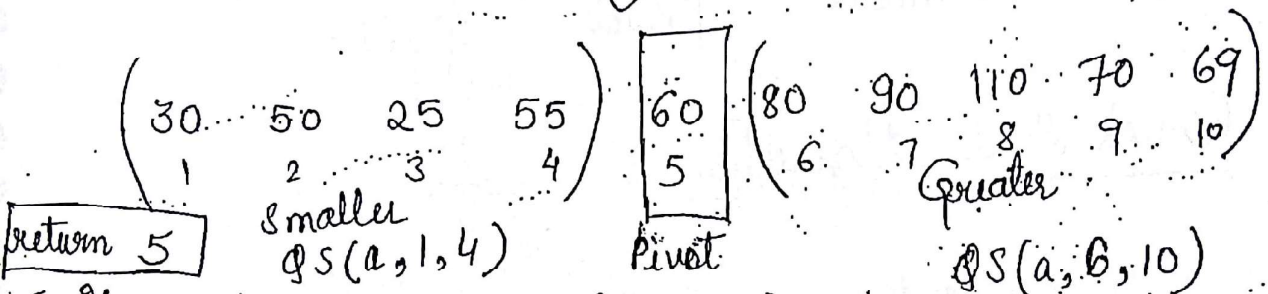
1. DAC application
2. Inplace (no extra space reqd.)
3. not stable
4. Most practically used sorting algo.

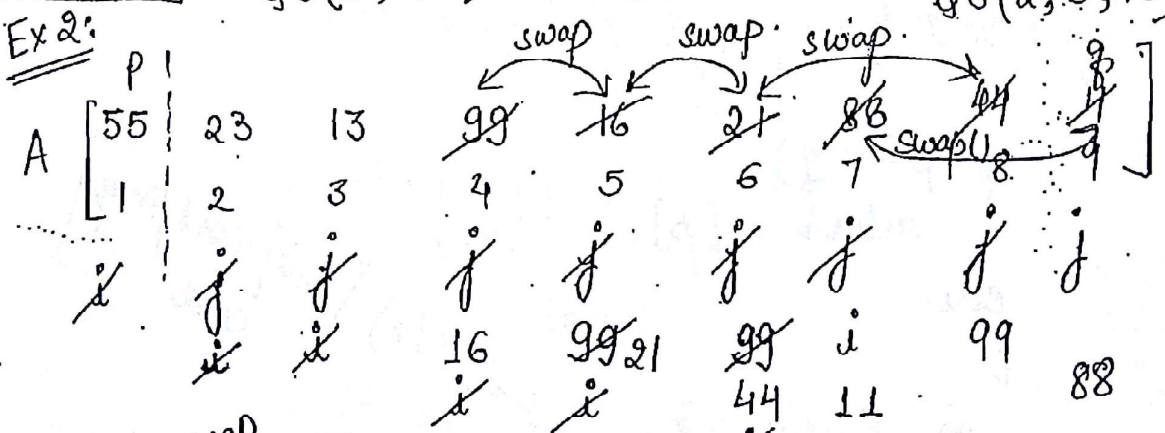
Partition :-

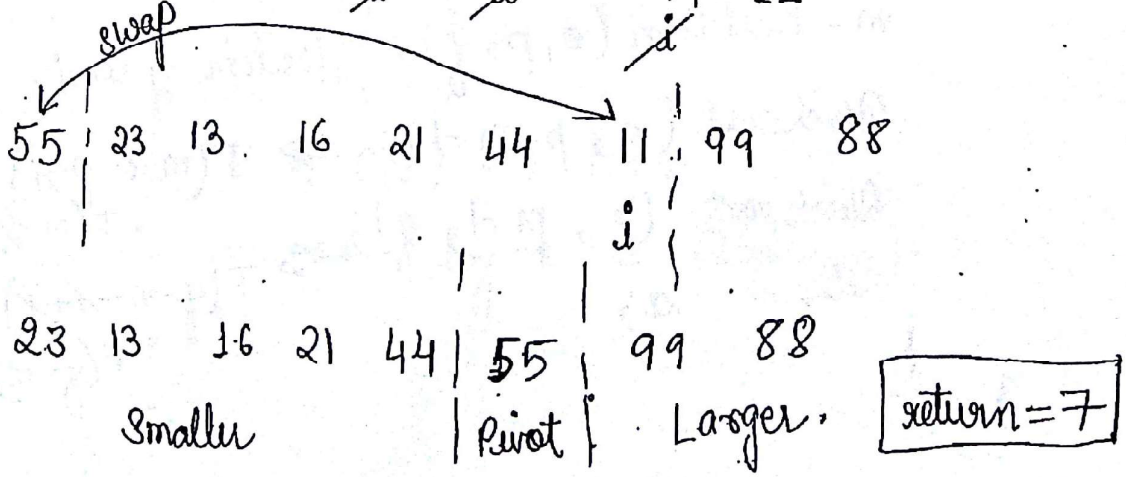

```
Partition (a, p, q)
{
    x = a[p];
    i = p;
    for (j = i+1; j <= q; j++)
    {
        if (a[j] <= x)
        {
            i = i+1;
            swap (a[i], a[j]);
        }
    }
    swap (a[i], a[p]);
    return i;
}
```


Ex 1:
swap()
swap()
swap()
swap()
Greater
Greater
greater
greater
smaller
A
P
i
j
Pivot = 60


swap()
60 50 25 55 30 80 90 110 70 69
Pivot
Smaller
i
Greater


30 50 25 55
60
80 90 110 70 69
Smaller
QS(a,1,4)
Pivot
Greater
QS(a,6,10)
return 5


Ex 2:
swap
swap
swap
swap()
A
P
i
j


swap
55 23 13 16 21 44 11 99 88
i
11 23 13 16 21 44 55 99 88
Smaller
Pivot
Larger
return = 7

Ex 3.

	p					q
A =	88	19	120	15	140	7
	1	2	3	4	5	6

i, j, j, j, j, j

i, i, i

88 19 15 7 140 120

i

7 19 15 | 88 | 140 120

Small | Pivot | Greater

Quick Sort Algorithm :-

```
Quicksort ( a , p , q)  ⇒ T(n)
{
    if (p == q)
        return a[p];
    else
    {
      m = Partition (a, p, q);  // Position of Pivot.   ⇒ O(n) [best/worst/Avg case]
      Quicksort ( a , p , m-1);  ⇒ T(m-1-p+1) = T(m-p)
      Quicksort ( a , m+1 , q);  ⇒ T(q-m-1+1) = T(q-m)
      return a;
    }
}
```

Let $T(n)$ be the time complexity of above program on an array of n elements

Recurrence Relation

$$T(n) = \begin{cases} O(1) & n = 1 \\ O(n) + 2T(n/2) + O & n > 1 \end{cases}$$

$$= 2T\left(\frac{n}{2}\right) + n$$

$\Rightarrow$

$$= 2\left\{2T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right\} + n$$

$$= 2^2 T\left(\frac{n}{2^2}\right) + n + n$$

$$2^2\left\{2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right\} + n + n$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + n + n + n$$

$$k = \log n$$

$$= 2^k T\left(\frac{n}{2^k}\right) + n.k$$

$$= n.O(1) + n.\log n$$

$$= n + n\log n = O(n\log n)$$

☆ When 1st element chosen as pivot, always, then it is Quick Sort. When element chosen randomly, it is randomized Quick Sort.

$$T(n) = \begin{cases} O(1) & , n = 1 \\ O(n) + T(m-p) + T(q-m) & , n > 1 \end{cases}$$

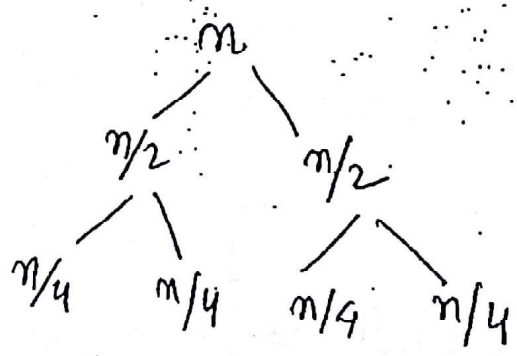
Best Case or Avg Case — Worst Case

$$T(n) = n + T(n/2) + T(n/2)$$

$$= 2T(n/2) + n$$

$$= O(n \log n)$$

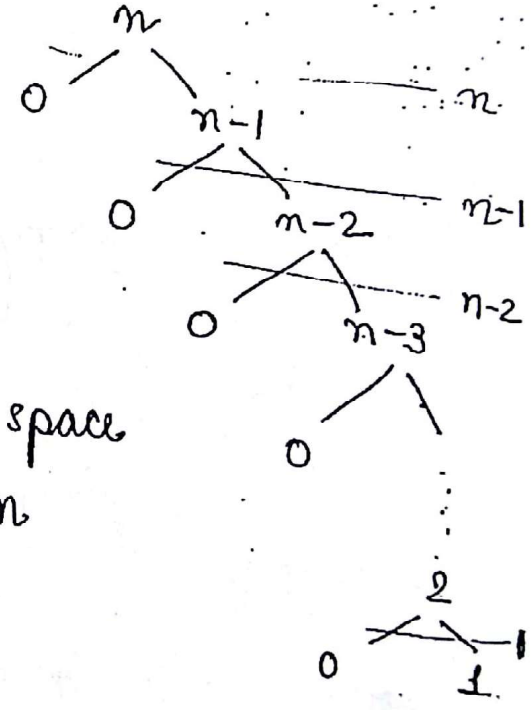
It will give Balanced tree



Stack Space = $\log n$.

$$T(n) = n + \underbrace{T(0)}_{\text{1 Pivot}} + T(n-1)$$

$$= n + T(n-1)$$



Stack space = n

$$= n + n-1 + n-2 + \ldots + 1$$

$$= \frac{n(n+1)}{2}$$

Example:

① 40 10 50 60 30 20 70
1 2 3 4 5 6 7

o/p (10 30 20) 40 (50 60 70)

② 70 10 50 60 30 20 40
1 2 3 4 5 6 7

o/p: (10 50 60 30 20 40) 70 ( )

③ 20 70 50 60 30 10 40.

o/p: (10) 20 (70 50 60 30 40)

Stack Space (worst Case) = n $\xrightarrow[\text{Programming (Equivalent non-recursive program)}]{\text{Using Better}}$ $\log n$.

Average Case :-

Lucky (best)
unlucky (worst)

Avg case (LULU ..... LU
or
ULUL ..... UL)

Avg (LULU ... LU) $\Rightarrow T(n) = 2T(n/2) + n$

$= 2\left[T\left(\frac{n}{2}-1\right) + \frac{n}{2}\right] + n$

$= 2T\left(\frac{n}{2}-1\right) + 2n$

$\approx 2T(n/2) + n = O(n \log n)$

Avg (ULULU...UL) $\Rightarrow T(n) = T(n-1) + n$

$$= 2T\left(\frac{n-1}{2}\right) + n + n$$

$$= 2T\left(\frac{n-1}{2}\right) + 2n$$

$$\approx 2T\left(\frac{n}{2}\right) + n$$

$$= O(n \log n) \quad \{\text{Same as Best Case}\}$$

Ques: Quick Sort algorithm is applied on 2 i/ps :-

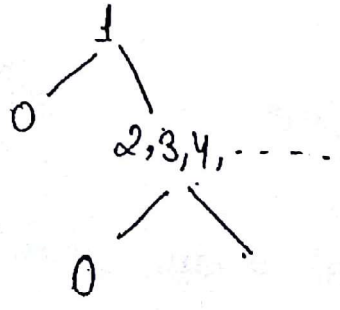
i) 1, 2, 3, 4, ... $n-1, n$

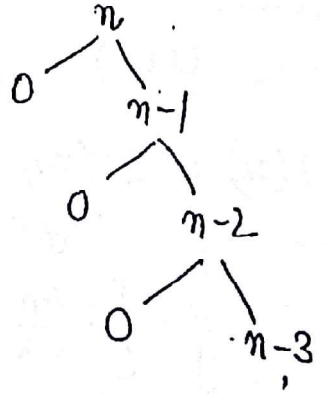
ii) $n, n-1, n-2, \ldots\ldots 3, 2, 1$

Let C_1 & C_2 be the no. of comparisons made for the i/p (i) & (ii) respectively.

What is the relation b/w them

a) $C_1 = C_2$, b) $C_1 < C_2$, c) $C_1 > C_2$, d) not comparable.

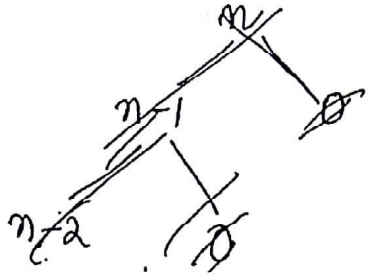




$C_1 =$ [illegible] $n-1 + \ldots\ldots + 1$

$= \frac{n(n+1)}{2}$

$= O(n^2)$



$$C_2 = \frac{n(n-1)}{2} = O(n^2)$$

i) Taking example of some array :-

	10	20	30	40	50	60	70
	1	2	3	4	5	6	7
i	i	j	j	j	j	j	j

$\Rightarrow$ 6 comparisons, 1 swap.

( )(10)(20 30 40 50 60 70)

$\Downarrow$ $\Rightarrow$ 5 comparisons,

( )(20)(30 40 50 60 70) 1 swap

$\Downarrow$ $\Rightarrow$ 4, 1

( )(30)(40 50 60 70)

$\Downarrow$ $\Rightarrow$ 3, 1

( )(40)(50 60 70)

$\Downarrow$ $\Rightarrow$ 2, 1

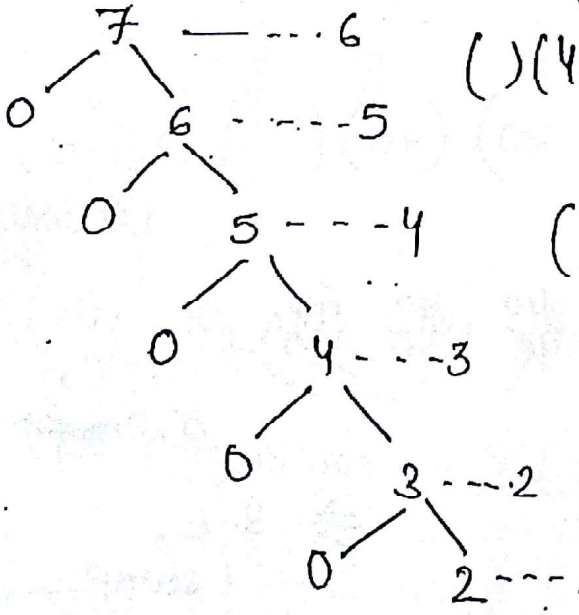
( )(50)(60 70)

$\Downarrow$ $\Rightarrow$ 1, 1

( )(60)(70)

$\Downarrow$

small problem.



Recurrence Relation :-

$$T(n) = n-1 + T(n-1)$$

$$= T(n-2) + n-2 + n-1$$

$$= T(n-3) + n-3 + n-2 + n-1$$

$$= 1 + T(1) + 2 + 3 + \dots + n-1$$

$$= \frac{n(n-1)}{2} = O(n^2)$$

Swaps = $1+1+1+\dots$ $(n-1)$ times

$= n-1$

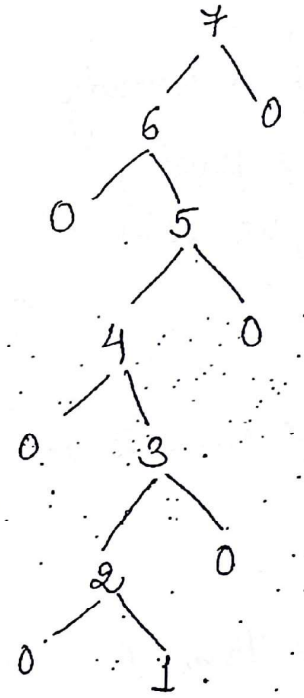
ii) 70, 60, 50, 40, 30, 20, 10 ⇒ 6-comparisons, 7 swaps

(10 | 60 50 40 30 20)(70)( ) ⇒ 5-c, 1 swaps

( )(10)(60 50 40 30 20) ⇒ 4-c, 5 swaps

(20 50 40 30)(60)( ) ⇒ 3-c, 1 swap

( )(20)(50 40 30) ⇒ 2-c, 3 swaps



No. of comparisons $(C_2) = \sum(n-1)$

$= \frac{n(n-1)}{2}$

$= O(n^2)$

A) $(C_1 = = C_2)$

Total swaps $\Rightarrow \approx n+1+n+1+\cdots\cdots$

$= \frac{n}{2} \cdot n + \frac{n}{2} \cdot 1$

$= \frac{n^2}{2} + \frac{n}{2} = O(n^2)$

For swaps, (B) $C_1 < C_2$

In partition algorithm, swaps might change but no. of comparisons remain same (always).

iii) swap

10_f ~~10_a~~ ~~10_b~~ ~~10_c~~ ~~10_d~~ ~~10_e~~ ~~10_f~~ 10_a

1 2 3 4 5 6.

~~i~~ ~~j~~ ~~j~~ ~~j~~ ~~j~~ j

i i ~~i~~ ~~i~~ i

Unbalanced Tree $(10_f \; 10_b \; 10_c \; 10_d \; 10_e)(10_a)( )$ // Repeated elements order changed.

Total comparisons $= \sum(n-1) = \frac{n(n-1)}{2} = O(n^2)$

Total swaps $= \frac{n(n-1)}{2} = O(n^2)$

Quick sort worst case is when array is already sorted. (prev. ques.) and 1st element is pivot element. If middle element is pivot then it'll be best case.

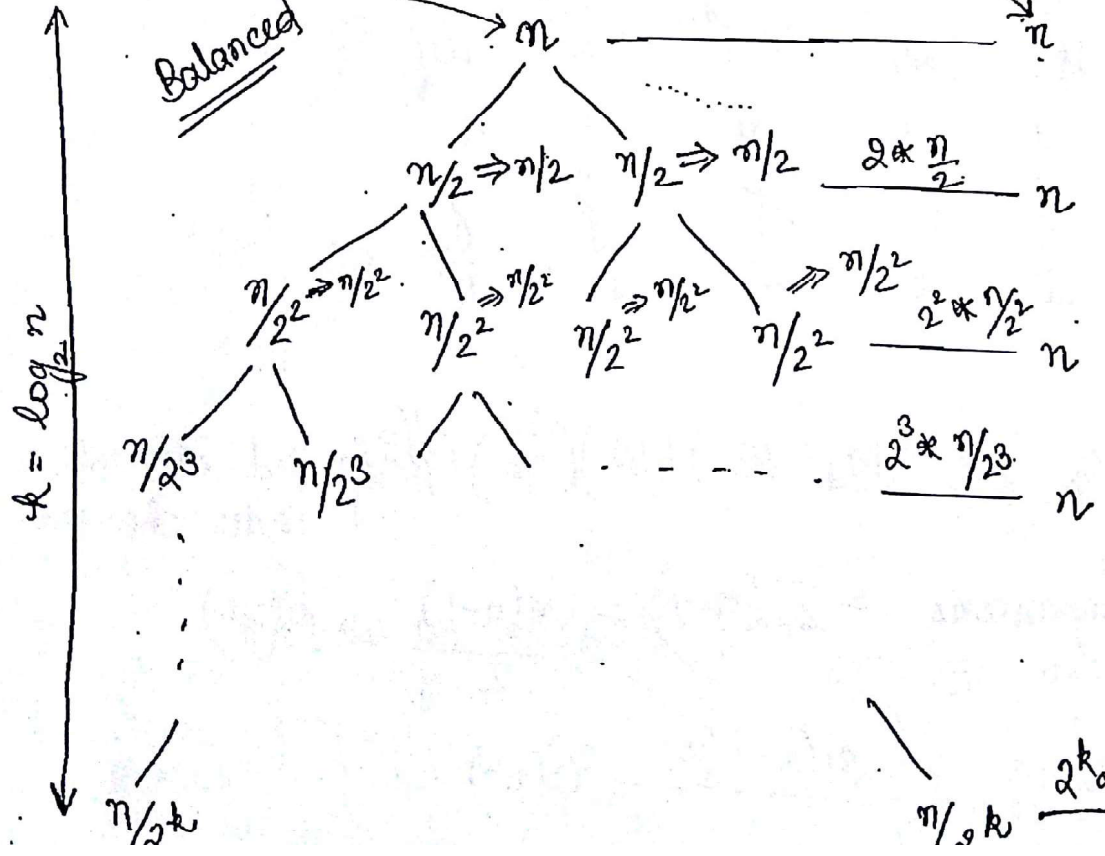
Insertion sort is prefered when max. elements are already sorted.

When recurrence relation has more than 1 function call, Recursive tree method is used.

Ex: $T(n) = n + T(n/5) + T(4n/5)$

Recursive Tree Method

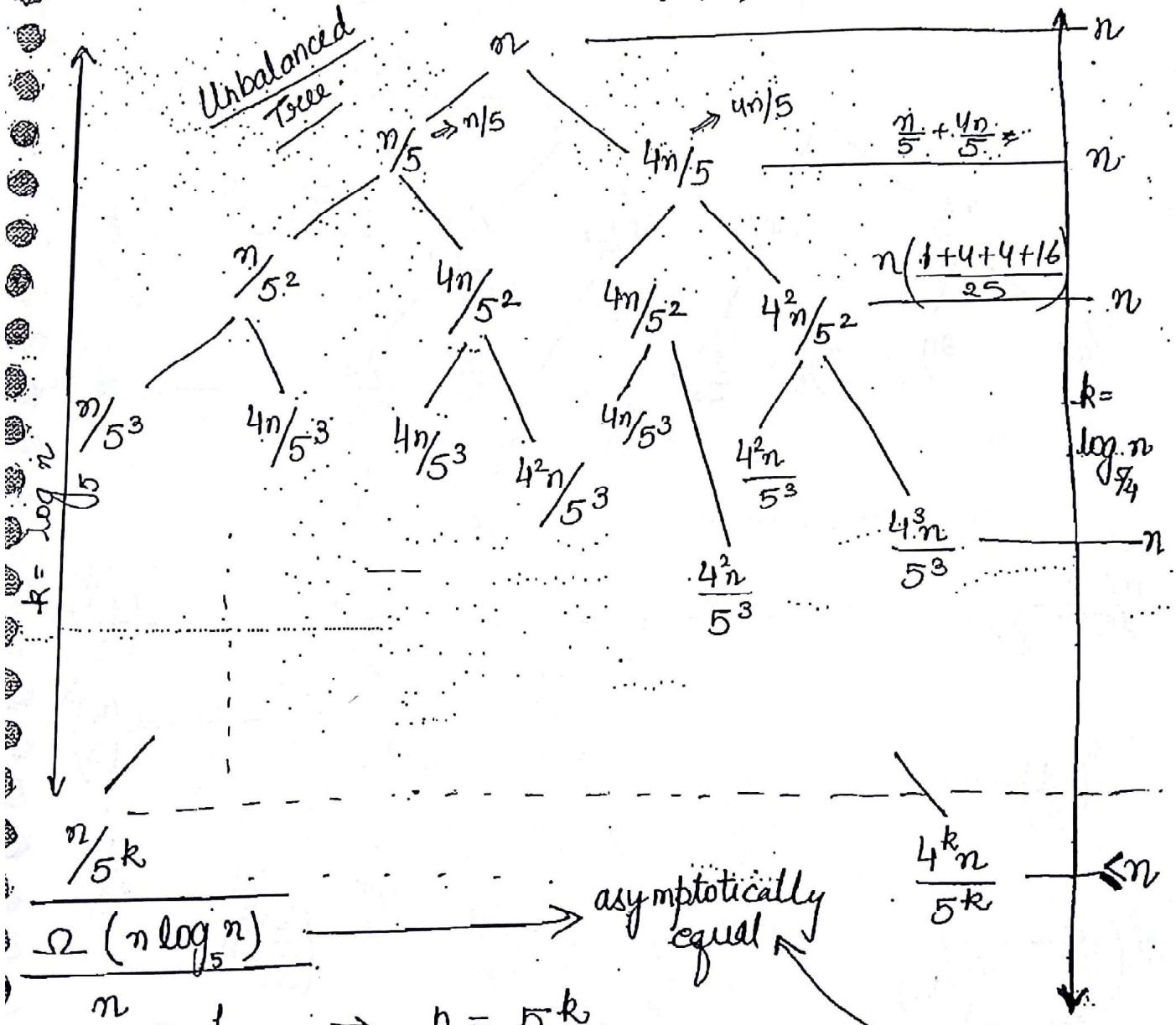
(1) $T(n) = n + T(n/2) + T(n/2)$



$\frac{n}{25} + \frac{4n}{25} + \frac{4n}{25} + \frac{16n}{25}$

$T(n) = \theta(n \log_2 n) = O(n \log_2 n) = \Omega(n \log_2 n)$

② $T(n) = n + T(n/5) + T(4n/5)$



$\frac{n}{5^k} = 1 \Rightarrow n = 5^k$

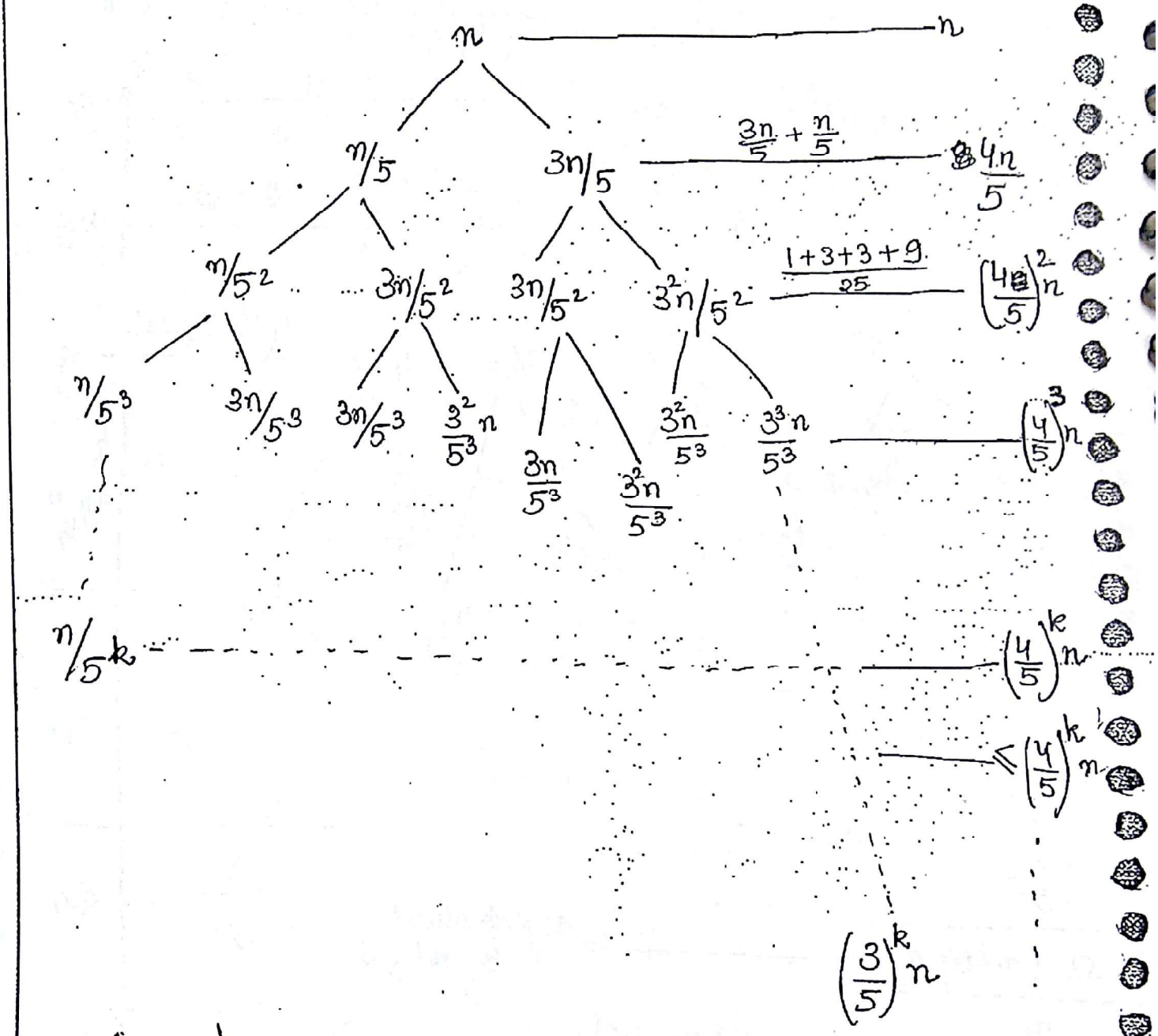
$k = \log_5 n$

$\frac{n}{(5/4)^k} = 1 \Rightarrow n = \left(\frac{5}{4}\right)^k$

$\Rightarrow k = \log_{5/4} n$

$\Rightarrow T(n) = \theta(n \log n)$

③ $T(n) = n + T\left(\frac{n}{5}\right) + T\left(\frac{3n}{5}\right)$



$\frac{n}{5^k} = 1$

$\Rightarrow k = \log_5 n$

$\Rightarrow n + \frac{4n}{5} + \left(\frac{4}{5}\right)^2 n + \cdots \left(\frac{4}{5}\right)^{\log_5 n} n$

$\Rightarrow n\left(1 + \frac{4}{5} + \left(\frac{4}{5}\right)^2 + \cdots \left(\frac{4}{5}\right)^{\log n}\right)$

$= 2n \rightarrow \Omega(n)$