RATIONAL EDGED CUBOIDS WITH EQUAL VOLUMES AND EQUAL SURFACES.

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1. In the BULLETIN, May, 1909, p. 401, Professor Kasner proposed the problem to find two cuboids (rectangular parallelopipeds) with equal volumes and equal surfaces, and in which the dimensions are all integral.

The problem is to find two distinct triples of integers such that

$$xyz=x'y'z', \quad xy+xz+yz=x'y'+x'z'+y'z'. \tag{1}$$

I shall prove that each integer must exceed unity and that the volume xyz must be the product of five or more primes. Among the solutions which I obtain, the simplest are

$$p,\ p,\ \tfrac{1}{2}(p+1)p^2; \quad p^2,\ p^2,\ \tfrac{1}{2}(p+1) \qquad (p \text{ odd}), \tag{2}$$

$$p,\ (p+1)q,\ (p+1)(p+1-q); \quad p+1,\ (p+1)p,\ q(p+1-q) \quad (1<q<p), \tag{3}$$

in which $q=r$ and $q=p+1-r$ give the same sets. While for (3) the volume contains at least six primes, for (2) there may be only five primes, for example,

$$3,\ 3,\ 18;\ \ 9,\ 9,\ 2. \qquad 5,\ 5,\ 75,\ \ 25,\ 25,\ 3. \tag{2'}$$

The simplest examples under (3) are

$$3,\ 8,\ 8;\ \ 4,\ 12,\ 4. \qquad 4,\ 10,\ 15;\ \ 5,\ 20,\ 6. \tag{3'}$$

Generalizations of (2) and (3) are given by (9)-(10) and (16)-(17).

The only solutions in which $xyz<200$ are the first solutions under (2′) and (3′).

In §§7-9, I show that for three rational edged cuboids with equal volumes and equal surfaces, the volume is the product of six or more primes, that no two of the triples are of the type $p_1,\ p_2,\ p_3p_4p_5p_6$, and that the cases in which two of the triples are $p_1,\ p_2,\ p_3p_4p_5p_6$ and $p_3,\ p_1p_2,\ p_4p_5p_6$ are excluded.

2. Theorem. *Two triples are identical if they have an element in common.*

If $x=x'$, equations (1) reduce to $yz=y'z'$, $y+z=y'+z$.

3. Theorem. *In any solution each integer exceeds unity.*

Two sets with equal products and having an element unity must be of the form

(4) $$1,\ f_1f_2f_3,\ f_4f_5f_6;\ f_1f_4,\ f_2f_5,\ f_3f_6.$$

Denote the corresponding surfaces by $2S_1$ and $2S_2$. Then

$$S_1=f_1f_2f_3+f_4f_5f_6+f_1\dots f_6,\qquad S_2=f_1f_2f_4f_5+f_1f_3f_4f_6+f_2f_3f_5f_6.$$

First, let each $f_i>1$. Since the substitution (12) (45) leaves the triples (4) unaltered, we may set $f_4\geqq f_5$.

Now $(f_i-1)(f_j-1)\overline{\overline{>}}1$, so that $f_if_j\geqq f_i+f_j$. Thus

$$f_1\dots f_6\overline{\overline{>}}f_1f_2f_4f_6(f_3+f_5)\overline{\overline{>}}f_3f_4f_6(f_1+f_2)+f_1f_2f_5(f_4+f_6).$$

Also $f_3f_4f_6f_2\geqq f_2f_3f_5f_6$. Hence $f_1\dots f_6>S_2$, $S_1>S_2$.

Second, let a single f_i be unity. We may set $f_3=1$, $f_5\geqq f_4$. Then

$$f_1\dots f_6\geqq f_1f_2f_5(f_4+f_6)\geqq f_1f_2f_4f_5+f_5f_6(f_1+f_2)\geqq S_2,$$

since $f_1f_5f_6\geqq f_1f_4f_6$. Hence $S_1>S_2$.

Third, let at least two f_i equal unity. If two such f's belong to the same product in (4_1), we may set $f_2=f_3=1$. Then

$$S_1=f_1+f_4f_5f_6+f_1f_4f_5f_6>f_5f_6+f_1f_4(f_5+f_6),\quad S_1>S_2,$$

if $f_5>1$, $f_6>1$. If $f_5=1$ or $f_6=1$, the triples have the common element 1 and are identical by §2. Next, let the two f's which equal unity belong to different products in (4_1). A case like $f_3=f_6=1$ is excluded by §2. Hence we may set $f_3=1$, $f_4=1$. Thus

$$S_1-S_2=f_2f_5(f_1f_6-f_1-f_6)+f_1f_2+f_5f_6-f_1f_6.$$

If $f_2>1$ or $f_5>1$, $S_1-S_2>(f_2f_5-1)(f_1f_6-f_1-f_6)$, so that $S_1-S_2>0$ unless $f_1=1$ or $f_6=1$. In the latter case the triples (4) have a common element 1. The same is true in the remaining case $f_2=f_5=1$.

4. Theorem. *The volume must contain at least five prime factors.*

By §3, each element exceeds 1. Hence there occur four or more primes. If there are just four primes, the triples without a common element (§2) may be designated

(5) $$p_1,\ p_2,\ p_3p_4;\quad p_3,\ p_4,\ p_1p_2,$$

p_4 being the greatest p, and $p_2\geqq p_1$. Then $S_1+p_1p_2p_3p_4=S_2+p_1p_2p_3p_4$ may be written

$$p_1p_2(p_3-1)(p_4-1)=p_3p_4(p_1-1)(p_2-1).$$

Hence $p_4=p_2$, so that the triples (5) are identical by §2.

5. For five primes, each triple must be of the form (1, 1, 3) or (1, 2, 2), the notation indicating the number of prime factors of each element. Consider first

$$(6) \qquad p_1,\ p_2,\ p_3p_4p_5;\quad p_3,\ p_1p_2,\ p_4p_5,$$

where $p_1 \geqq p_2$. From $S_1+p_1\ldots p_5=S_2+p_1\ldots p_5$, we get

$$p_1p_2(p_3-1)(p_4p_5-1)=p_3p_4p_5(p_1-1)(p_2-1).$$

Hence p_1 divides $p_3p_4p_5$. But $p_1 \neq p_3$ by §2. Since (6) is unaltered by the interchange of p_4 and p_5, we may set $p_1=p_4$. Thus

$$(7) \qquad p_2(p_3-1)(p_1p_5-1)=p_3p_5(p_1-1)(p_2-1) \quad (p_1 \geqq p_2).$$

If p_2 does not divide p_1-1, then $p_2=p_5$, and (7) becomes

$$(8) \qquad p_3(p_1+p_2-2)=p_1p_2-1,\ \ (p_3-1)^2=(p_1-p_3)(p_2-p_3).$$

Let g be the greatest common divisor of $p_1-p_3=ga^2$, $p_2-p_3=gb^2$. Then $p_3-1=gab$,

(9) $p_1=1+gab+ga^2$, $p_2=1+gab+gb^2$, $p_3=1+gab$ (a, b relatively prime)

For these values we have the solution

$$(10) \qquad p_1,\ p_2,\ p_1p_2p_3;\ p_3,\ p_1p_2,\ p_1p_2.$$

For $a=b$, then $a=1$, $p_3=\frac{1}{2}(p_1+1)$ and (10) becomes solution (2). For the remaining cases we may set $a>b$. Examples when the p's are all primes are (2′) and

$a=2$, $b=1$, $g=2$ or 6, p_1, p_2, $p_3=13$, 7, 5 or 37, 19, 13;
$a=3$, $b=2$, $g=10$, $p_1=151$, $p_2=101$, $p_3=61$.

Next, let p_2 divide p_1-1. Then (7) is equivalent to

$$(11) \qquad p_1-1=cp_2,\ p_3-1=kp_5,\ p_1p_5-1=lp_3,\ lk=c(p_2-1).$$

From the second and third we eliminate p_3 and see that $l=-1+mp_5$, $m=p_1-lk$. Hence by the fourth and first, $m=1+c$. Hence (11_3) may be replaced by

$$(12) \qquad l=-1+(1+c)p_5.$$

Let g be the greatest common divisor of c and k. Then, by (11_4),

$$(13) \qquad c=g\gamma,\ k=g\mu,\ l=\lambda\gamma,\ p_2-1=\lambda\mu \quad (\lambda \text{ and } \mu \text{ relatively prime}).$$

Set $\lambda-gp_5=\rho$. Then (12) becomes $\lambda\rho=p_5-1$. We may thus eliminate p_5 and λ:

$$(14) \quad p_5=1+\gamma\rho,\ p_2=1+\lambda\mu,\ p_1=1+g\gamma p_2,\ p_3=1+g\mu p_5 \quad (\lambda=g+\rho+g\rho\gamma).$$

For any positive integers g, ρ, γ, μ, of which the last two are relatively prime, formulae (14) *give values of the p_i leading to a solution**

$$(15) \qquad p_1,\ p_2,\ p_1p_3p_5;\ p_3,\ p_1p_2,\ p_1p_5.$$

For $\gamma=\mu=1$, $p_2=(1+g)(1+\rho)$ is composite. For $\gamma=1$, $\mu=2$, $\rho=1$, the least value of g giving prime p's is $g=10$ and $p_5=2$, $p_2=43$, $p_1=431$, $p_3=41$.

6. For two triples of type (1, 2, 2), it suffices to consider

$$(16) \qquad p_1,\ p_2p_3,\ p_4p_5;\ p_2,\ p_1p_4,\ p_3p_5 \quad (p_2>p_1;\ p_5\neq p_1,\ p_2;\ p_3\neq p_4).$$

Then $S_1=S_2$ gives $p_2p_3p_5(p_4-1)\equiv 0 \pmod{p_1}$. If $p_1=p_3$, $S_1=S_2$ gives

$$p_1p_2+p_4p_5+p_2p_4p_5=p_2p_4+p_3p_5+p_1p_4p_5.$$

Thus $p_4p_5(p_1-1)\equiv 0 \pmod{p_2}$, so that $p_2=p_4$. The middle terms in (16) would then be equal. This case is thus excluded by §2. Hence $p_4\equiv 1 \pmod{p_1}$.

Next, $S_1=S_2$ gives $p_1p_4p_5(p_3-1)\equiv 0 \pmod{p_2}$. First, let p_3-1 be prime to p_2, so that $p_4=p_2$. Removing the factor p_2 from $S_1=S_2$, we get

$$p_1(p_2-p_3-p_5)=p_3p_5(p_2-p_1-1).$$

Since $p_2\equiv 1 \pmod{p_1}$, $p_2-p_1-1=cp_1$, where c is an integer $\geqq 0$. Thus

$$(17) \qquad p_4=p_2=(c+1)p_1+1=p_3+p_5+cp_3p_5.$$

For $c=0$, we set $p_1=p$, $p_3=q$ and obtain solution (3); note that p occurs in the second triple if and only if $q=1$ or p.

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*To show that these triples are distinct, it suffices to prove that p_1 does not occur in the second. But $p_1<p_1p_2$, $p_1<p_1p_5$. If $p_1=p_3$, then $\gamma p_2=\mu p_5$. But γ and μ are relatively prime, and γ does not divide p_5 by (14_1).

For $c=1$, $2(p_1+1)=(p_3+1)(p_5+1)$. If $p_3=2$, then $p_1=3k-1$, $p_5=2k-1$. The lowest values of k leading to prime values for each p_i are $k=2, 4, 10$:

$$p_1,\ p_5,\ p_2=5,\ 3,\ 11;\ 11,\ 7,\ 23;\ 29,\ 19,\ 59.$$

The first gives the solution 5, 22, 33; 6, 11, 55. If p_3 is odd, then

$$(18)\qquad p_3=2k-1,\ p_1=k(p_5+1)-1,\ p_4=p_2=2k(p_5+1)-1.$$

The p_i are all primes for $k=2$, $p_5=5, 11$; $k=3$, $p_5=7, 13$:

$$p_5,\ p_3,\ p_1,\ p_2=5,\ 3,\ 11,\ 23;\ 11,\ 3,\ 23,\ 47;\ 7,\ 5,\ 23,\ 47;\ 13,\ 3,\ 41,\ 83.$$

For $c=2$, we have $p_3=3$, $p_5=3k+1$, $p_1=7k+3$, or

$$p_3=3l+1,\ p_1=l+p_5+2lp_5,\ \text{or}\ p_3=3l+2,\ p_5=3m+1,\ p_1=6lm+3l+5m+2.$$

But for c even, the p's in (17) are not all primes if $p_1\neq 2$.

In view of the variety of solutions obtained, further cases are not considered.

7. Do there exist three distinct cuboids with rational edges having equal volumes and equal surfaces? We show that there is no solution in which the volume is the product of fewer than six primes. In view of §4, we consider the case of three triples of five primes p_i. Not all three are of the type (1, 1, 3). Let two be of type (1, 1, 3) and one of type (1, 2, 2); it suffices to consider

$$(19)\qquad p_1,\ p_2,\ p_3p_4p_5;\ p_3,\ p_4,\ p_1p_2p_5;\ p_5,\ p_1p_2,\ p_3p_4,\ \text{or}\ p_5,\ p_1p_3,\ p_2p_4.$$

In the first alternative, we reduce the S_i modulo p_1 and get

$$p_2p_3p_4p_5\equiv p_3p_4\equiv p_3p_4p_5 \pmod{p_1}.$$

Since p_3p_5 is not divisible by p_1, $p_5\equiv 1$, $p_2\equiv 1 \pmod{p_1}$. Similarly $p_1\equiv 1 \pmod{p_2}$, contrary to the former. Likewise (19_2) gives $p_3\equiv 1 \pmod{p_1}$, $p_1\equiv 1 \pmod{p_3}$ and is excluded.

Next, let one triple be of the type (1, 1, 3) and two of the type (1, 2, 2); it suffices to treat

(20) $\quad (p_1,\ p_2p_3,\ p_4p_5;\ p_2,\ p_1p_4,\ p_3p_5;\ p_3,\ p_4,\ p_1p_2p_5,$ or $p_4,\ p_5,\ p_1p_2p_3.$

But, in either case, we find that $p_4\equiv 1 \pmod{p_1}$, $p_1\equiv 1 \pmod{p_4}$.

Hence all three triples must be of type (1, 2, 2). Two of them may be taken to be the first two in (20), with $p_3\neq p_4$, $p_2\neq p_5$, $p_1\neq p_5$. Since these are unaltered by (12)(34), the third may be restricted to one of the three:

$$p_3,\ p_1p_5,\ p_2p_4;\ p_5,\ p_1p_2,\ p_3p_4;\ p_5,\ p_1p_3,\ p_2p_4.$$

The first case gives $p_3\equiv 1 \pmod{p_2}$, $p_2\equiv 1 \pmod{p_3}$. The second case gives $p_2\equiv 1 \pmod{p_1}$, $p_1\equiv 1 \pmod{p_2}$. In the third case we may set $p_2>p_1$; then

$$p_1p_4p_5\equiv p_1p_2p_4\equiv p_2p_4p_5 \pmod{p_3},\quad p_1p_4p_5\equiv p_1p_3p_4p_5\equiv p_1p_3p_5 \pmod{p_2}.$$

If $p_3=p_5$, then $p_1p_2p_4$ is a multiple of p_3, whereas $p_1\neq p_5$, $p_2\neq p_5$, $p_4\neq p_3$. Hence $p_4p_5(p_2-p_1)\equiv 0$, gives $p_2-p_1=cp_3$, $c>0$. Similarly, $p_1p_4p_5(p_3-1)\equiv 0 \pmod{p_2}$ gives $p_3\equiv 1 \pmod{p_2}$, since $p_4=p_2$ would require $p_1p_3p_5\equiv 0 \pmod{p_2}$. Hence $-p_1\equiv c \pmod{p_2}$, so that $c\geqq p_2-p_1$. The former equation $p_2-p_1=cp_3$ is thus impossible in view of $p_3>1$. *There is no set of three triples involving fewer than six primes.*

8. Consider three triples involving six primes. If all are of type (1, 1, 4), they may be taken to be

(21) $\quad p_1,\ p_2,\ p_3p_4p_5p_6;\ p_3,\ p_4,\ p_1p_2p_5p_6;\ p_5,\ p_6,\ p_1p_2p_3p_4,$

with $p_3p_4>p_5p_6$. By $S_2=S_3$, $p_3p_4=p_5p_6+\delta p_1p_2$, $\delta>0$. But $p_1p_2\equiv p_5p_6 \pmod{p_3p_4}$ by $S_1=S_3$. Hence $0\equiv p_5p_6(1+\delta) \pmod{p_3p_4}$. Thus $\delta=-1+\epsilon p_3p_4$, $\epsilon>0$,

$$p_3p_4+p_1p_2=p_5p_6+\epsilon p_1p_2p_3p_4.$$

But $p_1p_3\cdot p_2p_4\geqq(p_1+p_3)(p_2+p_4)>p_1p_2+p_3p_4$. Hence this case is excluded.

Let two of the triples be the first two in (21) and the third of type (1, 2, 3). Since the former are unaltered or interchanged by (12), (34), (56), (13)(24), the third may be assumed to be $p_5,\ p_1p_2,\ p_3p_4p_6$; or $p_5,\ p_1p_3,\ p_2p_4p_6$; or $p_5,\ p_1p_6,\ p_2p_3p_4$. The first yields $p_2\equiv 1 \pmod{p_1}$, $p_1\equiv 1 \pmod{p_2}$; the second, $p_3\equiv 1 \pmod{p_1}$, $p_1\equiv 1 \pmod{p_3}$. For the third case, $S_1=S_2$ gives

(22) $\quad p_1p_2=p_3p_4+\delta p_5p_6, \quad \delta+p_1p_3p_4+p_2p_3p_4=p_1p_2p_3+p_1p_2p_4,$

while $S_1=S_3$ gives $p_5p_6\equiv p_2 \pmod{p_3p_4}$. Thus $\delta\equiv p_1 \pmod{p_3p_4}$, by (22_1). Then (22_2) gives $p_1\equiv p_1p_2(p_3+p_4)$, mod p_3p_4. Hence

(23) $$1=p_2(p_3+p_4)+lp_3p_4 \quad (l \text{ negative integer}).$$

But $p_1p_3p_4p_5p_6\equiv p_3p_4\equiv p_1p_5p_6 \pmod{p_2}$. Thus $p_1p_5p_6\equiv 1$, $p_3p_4\equiv 1 \pmod{p_2}$. Hence in (23), $1\equiv l$, $l=1-mp_2$, $m>0$. Thus (23) may be given the form

$$p_2[(m-1)p_3p_4-p_3-p_4]+p_2p_3p_4=p_3p_4-1.$$

But if $m>1$, the quantity in brackets is positive and the equation impossible. Hence $m=1$, and (23) becomes

(24) $$p_2(p_3p_4-p_3-p_4)=p_3p_4-1.$$

Since $p_2\geqq 2$, the first member exceeds the second if $(p_3-2)(p_4-2)>3$. Since (34) leaves each triple unaltered, we may set $p_3\geqq p_4$. Hence, there remains the cases

$$p_3=5,\ p_4=3;\ p_3=p_4=3;\ p_4=2.$$

For $p_3=5$, $p_4=3$, we have $p_2=2$, $S_2=15+16p_1p_5p_6$, $S_3=30p_5+30p_1p_6+p_1p_5p_6$. Hence $S_2=S_3$ gives $(p_1p_6-2)(p_5-2)=3$, $p_1p_6=5$ or 3, which is impossible. For $p_3=p_4=3$, (24) gives $3p_2=8$. Hence must $p_4=2$, $p_2(p_3-2)=2p_3-1$. Thus $p_2\geqq 3$, and the first member exceeds the second if $p_3>5$. Hence $p_3=5$, $p_2=3$, or $p_3=3$, $p_2=5$. For the first, $2S_2=2S_3$ becomes $(2p_5-3)(2p_1p_6-3)=7$, whence $p_1p_6=5$ or 2. For the second, $4S_2=4S_3$ becomes $(4p_1p_6-5)(4p_5-5)=21$, whence $p_1p_6=3$ or 2.

Next, let two of the triples be the first two in (21) and the third of type (2, 2, 2). The latter may be assumed to be p_1p_2, p_3p_4, p_5p_6; p_1p_2, p_3p_5, p_4p_6; p_1p_3, p_2p_4, p_5p_6; or p_1p_3, p_2p_5, p_4p_6. The first two and the last two are excluded by the argument excluding the first and second cases, respectively, at the beginning of the preceding paragraph.

Hence at most one of the three triples is of the type (1, 1, 4).

9. Let two triples be p_1, p_2, $p_3p_4p_5p_6$ and p_3, p_1p_2, $p_4p_5p_6$. Adding $p_1\ldots p_6$ to S_1 and S_2 and equating the sums, we get

(25) $$p_1p_2(p_3-1)(p_4p_5p_6-1)=p_3p_4p_5p_6(p_1-1)(p_2-1).$$

We may set $p_2 \geqq p_1$, $p_2 \neq p_3$. Hence p_2 divides $p_4p_5p_6$, so that we may set $p_4 = p_2$. Since $p_4p_5p_6 - 1 > p_5p_6(p_2-1)$, we have $p_1(p_3-1) < p_3(p_1-1)$, $p_1 > p_3$. Hence $p_2 > p_3$. If the third triple is of type (1, 2, 3), it may be taken to be p_5, p_1p_3, $p_2^2p_6$; p_5, p_1p_6, $p_2^2p_3$; p_5, p_2p_3, $p_1p_2p_6$; p_5, p_2^2, $p_1p_3p_6$; p_5, p_2p_6, $p_1p_2p_3$; or p_5, p_3p_6, $p_1p_2^2$. Now S_1 is a multiple of p_2, so that S_3 must be. Hence for the first case, $p_1p_3p_5$ is a multiple of p_2, whence $p_1 = p_2$. Then S_1 is a multiple of p_2^2, so that S_3 must be; hence $p_3p_5 \equiv 0 \pmod{p_2}$, which is impossible. For the second case, $p_1p_5p_6 \equiv 0 \pmod{p_2}$. If $p_1 = p_2$, $S_1 \equiv 0 \pmod{p_2^2}$, so that by S_3, $p_6 = p_2$. In any event, $p_6 = p_2$ and the second and third triples have $p_1p_2 = p_1p_6$ in common. For the third case, $S_1 = S_2$ gives $p_1p_3 = p_1 + \varepsilon\, p_5p_6$, whence $\varepsilon = \delta\, p_1$; while $S_1 = S_3$ gives $p_1p_2 \equiv p_1p_2p_5p_6$, $p_5p_6 \equiv 1 \pmod{p_3}$, $p_5p_6 > p_3$. For the fourth case, $S_1 \equiv S_3 \bmod p_2$ gives $p_1p_3p_6p_5 \equiv 0 \pmod{p_2}$. If $p_1 = p_2$ the second and third triples would have a common element. Hence $p_2 = p_6$. By (25), p_5p_6 must divide $p_1(p_3-1)$. But $p_2 > p_3 - 1$. Hence $p_2 = p_6$ requires $p_1 = p_2$. For the fifth case, $p_1 \neq p_6$ by p_2p_6 and p_1p_2. Removing the factor p_2 from the S's, we see that $p_2p_3p_5p_6 \equiv p_3p_6p_6 \equiv p_5p_6 \pmod{p_1}$, $p_3 \equiv 1 \pmod{p_1}$, contrary to $p_3 < p_1$. For the sixth case, $S_1 \equiv S_3 \pmod{p_2}$ gives $p_3p_5p_6 \equiv 0 \pmod{p_2}$, $p_6 = p_2$. As in the fourth case, $p_1 = p_2$. Then S_1, but not S_3, is a multiple of p_2^2.

If the third triple is of type (2, 2, 2), it may be taken to be

$$p_2^2,\ p_1p_3,\ p_5p_6;\ p_2^2,\ p_1p_5,\ p_3p_6;\ p_2p_3,\ p_2p_5,\ p_1p_6;\ \text{or}\ p_2p_5,\ p_2p_6,\ p_1p_3.$$

For the first two cases, $p_1 \neq p_2$ and $S_1 \equiv S_3 \pmod{p_2}$ gives $p_6 = p_2$, contrary to the above. The third case is excluded by $p_3 \equiv 1 \pmod{p_5p_6}$, $p_5p_6 \equiv 1 \pmod{p_3}$; the fourth by $p_1 \equiv 1 \pmod{p_3}$, $p_3 \equiv 1 \pmod{p_1}$.

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NOTE ON SOME POLYNOMIALS RELATED TO LEGENDRE'S COEFFICIENTS.*

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The object of this note is to point out some properties of a class of functions which contains Legendre's coefficients as a special case. It will be seen that the former possess some interesting properties belonging to the latter.

1. Consider the definite integral†

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†Cf. Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 173.