ON WALTER WYSS'S NO PERFECT CUBOID PAPER.

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Abstract. The perfect cuboid problem is an old famous unsolved problem in mathematics concerning the existence or non-existence of a rectangular parallelepiped whose edges, face diagonals, and space diagonal are of integer lengths. Recently Walter Wyss has published a paper claiming a solution of this problem. The purpose of this paper is to check out Walter Wyss's result.

1. Introduction.

Actually Walter Wyss's paper is a series of three papers (three versions) each updating the previous one. The version 1 is entitled "No perfect cuboid" (see [1]), the version 2 is entitled "On perfect cuboids" (see [2]), and the version 3 is again entitled "No perfect cuboid" (see [3]). There is also the version 4 in ArXiv (see [4]), which looks pretty the same as the version 3. Here we consider the version 4 of Walter Wyss's paper which is the most recent by now. In this paper Walter Wyss considers leaning boxes (slanted cuboids) which are parallelepipeds with four rectangular faces and two faces being parallelograms (see Fig. 1.1). Such cuboids have three edges, four face diagonals and two space diagonals. Regular cuboids correspond to the case $\theta = \pi/2$. Slanted cuboids and their equations are used in proving the "No perfect cuboid" claim for the rectangular case $\theta = \pi/2$.

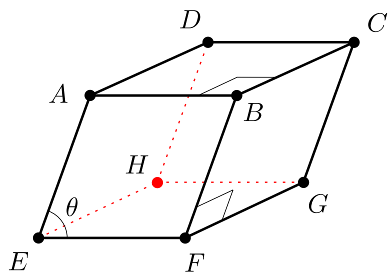

Fig. 1.1

2. Perfect and non-perfect parallelograms.

The study of slanted cuboids in [4] is based on the study of perfect parallelograms which was carried out in the other paper [5] by Walter Wyss (see also [6]).

Definition 2.1. A parallelogram is called perfect if its sides and diagonals are of integer lengths.

Let's consider the parallelogram $ABFE$ in Fig 1.1. and denote their sides and diagonals by $u_1 = |AE|$, $u_2 = |EF|$, $u_3 = |AF|$, $u_4 = |EB|$. Then, applying cosine

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theorem, we get the following equations

$$\begin{aligned} u_3^2 &= u_1^2 + u_2^2 - u_1\,u_2\,\cos(\theta),\\ u_4^2 &= u_1^2 + u_2^2 - u_1\,u_2\,\cos(\pi-\theta), \end{aligned} \tag{2.1}$$

Since $\cos(\pi-\theta) = -\cos(\theta)$, from (2.1) we derive the equation

$$u_3^2 + u_4^2 = 2\,u_1^2 + 2\,u_2^2. \tag{2.2}$$

Theorem 2.1. *Four positive real numbers u_1, u_2, u_3, u_4 represent lengths of sides and diagonals of some parallelogram if and only if they obey the quadratic equation* (2.2) *and the following inequalities:*

$$|u_1 - u_2| < u_3 < u_1 + u_2. \tag{2.3}$$

Let's denote through A, B, E, F a quadruple of points on a plane such that $|AE| = |BF|$. Using the notations $u_1 = |AE|$, $u_2 = |EF|$, $u_3 = |AF|$, $u_4 = |EB|$, we see that the inequalities (2.3) mean that the points A, E, F constitute a non-degenerate triangle AEF. If $ABFE$ is a parallelogram, the triangle AEF is non-degenerate, hence the inequalities (2.3) are fulfilled. As we see above, the equation (2.2) in this case is also fulfilled. So the necessity in Theorem 2.1 is established.

Let's proceed to the sufficiency. Squaring the inequalities (2.3), we get

$$u_1^2 + u_2^2 - 2\,u_1\,u_2 < u_3^2 < u_1^2 + u_2^2 + 2\,u_1\,u_2. \tag{2.4}$$

Then, applying the equation (2.2) to (2.4), we derive the inequalities

$$u_1^2 + u_2^2 - 2\,u_1\,u_2 < u_4^2 < u_1^2 + u_2^2 + 2\,u_1\,u_2. \tag{2.5}$$

Since u_1, u_2, u_3, u_3 are positive, the inequalities (2.5) are equivalent to

$$|u_1 - u_2| < u_4 < u_1 + u_2. \tag{2.6}$$

Due to the notations $u_1 = |AE|$, $u_2 = |EF|$, $u_3 = |AF|$, $u_4 = |EB|$ and since $|AE| = |BF|$, the inequalities (2.6) mean that the points B, F, E constitute another non-degenerate triangle BFE with $|BF| = |AE|$. The cosine of the angle at the node F in this triangle is calculated as follows:

$$\cos(\hat{F}) = \frac{u_4^2 - u_1^2 - u^2}{2\,u_1\,u_2}. \tag{2.7}$$

Similarly for the cosine of the angle at the node E in the triangle AEF we have:

$$\cos(\hat{E}) = \frac{u_3^2 - u_1^2 - u^2}{2\,u_1\,u_2}. \tag{2.8}$$

Applying the equation (2.2) to (2.7) and (2.8), we easily derive

$$\cos(\hat{F}) = -\cos(\hat{E}). \tag{2.9}$$

The cosine equality (2.9) means that

$$\hat{F} = \pi - \hat{E}. \tag{2.10}$$

Let's draw a triangle AEF using the values of its sides $u_1 = |AE|$, $u_2 = |EF|$, $u_3 = |AF|$ and relying on the inequality (2.3). Since $|BF| = |AE| = u_1$ is fixed, on the plane there are exactly two locations of the point B relative to the triangle AEF at which the equality (2.10) holds. They are symmetric to each other with respect to the line EF (see Fig. 2.1). Only for one of these two locations the points A, B, E, F form a parallelogram. Choosing this location, we find that there is a parallelogram the lengths of whose sides and diagonals coincide with the numbers u_1, u_2, u_3, u_4 obeying the equation (2.2) and the inequalities (2.3). The sufficiency in Theorem 2.1 is also established.

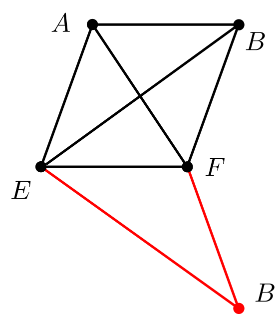

Fig. 2.1

Now let's recall that the inequalities (2.3) are equivalent to the inequalities (2.6) modulo the equation (2.2). Therefore Theorem 2.1 can be reformulated as follows.

Theorem 2.2. *Four positive real numbers u_1, u_2, u_3, u_4 represent lengths of sides and diagonals of some parallelogram if and only if they obey the quadratic equation (2.2) and the following inequalities:*

$$|u_1 - u_2| < u_4 < u_1 + u_2. \tag{2.11}$$

There are two more statements equivalent to Theorem 2.1.

Theorem 2.3. *Four positive real numbers u_1, u_2, u_3, u_4 represent lengths of sides and diagonals of some parallelogram if and only if they obey the quadratic equation (2.2) and the following inequalities:*

$$u_3 < u_1 + u_2, \qquad\qquad u_4 < u_1 + u_2. \tag{2.12}$$

Theorem 2.4. *Four positive real numbers u_1, u_2, u_3, u_4 represent lengths of sides and diagonals of some parallelogram if and only if they obey the quadratic equation (2.2) and the following inequalities:*

$$|u_1 - u_2| < u_3, \qquad\qquad |u_1 - u_2| < u_4. \tag{2.13}$$

According to Definition 2.1, perfect parallelograms are those for which u_1, u_2, u_3, u_4 are positive integers.

Definition 2.2. A parallelogram is called rational if its sides and diagonals are of rational lengths.

Rational parallelograms are equivalent to perfect ones since we can bring the quotients representing the rational numbers u_1, u_2, u_3, u_4 to the common denominator and then obtain a perfect parallelogram by multiplying u_1, u_2, u_3, u_4 by this common denominator.

3. Rational slanted cuboids.

Definition 3.1. A slanted cuboid (leaning box) is called perfect if its edges, its face diagonals, and its space diagonals are of integer lengths.

Definition 3.2. A slanted cuboid (leaning box) is called rational if its edges, its face diagonals, and its space diagonals are of rational lengths.

In [4] Walter Wyss considers rational leaning boxes. Rational leaning boxes are equivalent to perfect ones for the same reasons as in the case of rational and perfect parallelograms (see above). Following Walter Wyss in [4], let's consider the rational leaning box shown in Fig. 1.1. The edges AD, BC, FG, EH of this leaning box are perpendicular to the face parallelogram $ABFE$. Using an appropriate scaling factor, we can bring their length to the unity

$$|AD| = |BC| = |FG| = |EH| = 1. \tag{3.1}$$

Apart from (3.1) we use the notations

$$\begin{aligned} |AE| = |BF| = |CG| = |DH| = u_1,\\ |AB| = |EF| = |DC| = |HG| = u_2,\\ |AH| = |ED| = |BG| = |FC| = v_1,\\ |FH| = |EG| = |AC| = |BD| = v_2, \end{aligned} \tag{3.2}$$

$$\begin{aligned} |AF| = |DG| = u_3,\\ |EB| = |HC| = u_4,\\ |AG| = |DF| = v_3,\\ |EC| = |BH| = v_4, \end{aligned} \tag{3.3}$$

From (3.1), (3.2), and (3.3) we derive the slanted cuboid equations

$$\begin{aligned} 1 + u_1^2 = v_1^2,\\ 1 + u_2^2 = v_2^2,\\ 1 + u_3^2 = v_3^2,\\ 1 + u_4^2 = v_4^2, \end{aligned} \tag{3.4}$$

$$2\,u_1^2 + 2\,u_2^2 = u_3^2 + u_4^2. \tag{3.5}$$

These equation coincide with the equations (8)–(12) in [4].

The equations (3.4) are Pythagoras equations for rectangular triangles with rational sides. They can be solved in a parametric form:

$$u_k = \frac{1 - s_k^2}{2\,s_k}, \qquad\qquad v_k = \frac{1 + s_k^2}{2\,s_k}. \tag{3.6}$$

Here $k = 1, \ldots, 4$ and s_1, s_2, s_3, s_4 are rational numbers obeying the inequalities

$$0 < s_k < 1, \ \text{ for } \ k = 1, \ldots, 4. \tag{3.7}$$

Let's denote through ψ_1, ψ_2, ψ_3, ψ_4 the angles opposite to the sides of the unit length in the rectangular triangles associoated with the equations (3.4). Then

$$\sin(\psi_k) = \frac{1}{v_k}, \qquad\qquad \cos(\psi_k) = \frac{u_k}{v_k}, \tag{3.8}$$

where $k = 1, \ldots, 4$. Since u_k and v_k are rational numbers, sines and cosines in (3.8) are also rational numbers.

Definition 3.3. An angle ψ is called a Heron angle if both $\sin(\psi)$ and $cos(\psi)$ are rational numbers.

Definition 3.4. An angle ψ is called an Euler angle if $\tan(\psi)$ is a rational number.

These definitions can be found in Appendix A of the paper [4]. According to Definition 3.3, the angles ψ_1, ψ_2, ψ_3, ψ_4 in (3.8) are Heron angles.

The equation (3.5) differs from the equations (3.4). It coincides with the parallelogram equation (2.2). Substituting (3.6) into (3.5) and simplifying, we derive

$$\begin{gathered} s_4^4\,s_1^2\,s_2^2\,s_3^2+s_3^4\,s_1^2\,s_2^2\,s_4^2-2\,s_2^4\,s_1^2\,s_3^2\,s_4^2-2\,s_1^4\,s_2^2\,s_3^2\,s_4^2+\\ +\,4\,s_1^2\,s_2^2\,s_3^2\,s_4^2-2\,s_2^2\,s_3^2\,s_4^2-2\,s_1^2\,s_3^2\,s_4^2+s_1^2\,s_2^2\,s_4^2+s_1^2\,s_2^2\,s_3^2=0. \end{gathered} \tag{3.9}$$

The parallelogram equation (3.5) should be complemented with parallelogram inequalities. The most simple form of them are given by Theorem 2.3. Substituting (3.6) into (2.12) and simplifying, we derive

$$\begin{aligned} &s_1\,s_2^2\,s_3+s_1^2\,s_2\,s_3-s_1\,s_2\,s_3^2+s_1\,s_2-s_2\,s_3-s_1\,s_3<0,\\ &s_1\,s_2^2\,s_4+s_1^2\,s_2\,s_4-s_1\,s_2\,s_4^2+s_1\,s_2-s_2\,s_4-s_1\,s_4<0. \end{aligned} \tag{3.10}$$

Theorem 3.1. *Each rational slanted cuboid (leaning box) corresponds to some quadruple of rational numbers s_1, s_2, s_3, s_4 obeying the polynomial equation* (3.9) *and the polynomial inequalities* (3.7) *and* (3.10).

4. Parallelogram parametrization.

Let's return back to the parallelogram $ABFE$ in Fig. 1.1. Its sides and diagonals are rational numbers $|AE|=u_1$, $|EF|=u_2$, $|AF|=u_3$, $|EB|=u_4$. For such a parallelogram Walter Wyss introduces two parameters:

$$m=\frac{2\,u_2+u_3-u_4}{2\,u_1+u_3+u_4}, \tag{4.1}$$

$$n=\frac{2\,u_2-u_3+u_4}{2\,u_1+u_3+u_4} \tag{4.2}$$

(see (D.8) and (D.9) in Appendix D of [4]). Both parameters range within

$$0<m<1, \qquad\qquad 0<n<1, \tag{4.3}$$

provided the parallelogram equation (3.5) and the parallelogram inequalities (2.3), (2.11), (2.12), (2.13) are fulfilled.

Indeed, if $m\leqslant 0$, then $u_4\geqslant 2\,u_2+u_3$. Combining this inequality with the inequality $u_4<u_1+u_2$ from (2.11), we derive the inequality $u_3<u_2-u_1$ which contradicts the inequality $|u_1-u_2|<u_3$ from (2.3). If $m\geqslant 1$, then we have $2\,u_2+u_3-u_4\geqslant 2\,u_1+u_3+u_4$. This inequality reduces to $u_4\leqslant u_2-u_1$ which contradicts the inequality $|u_1-u_2|<u_4$ from (2.11). Thus, the inequalities for m in (4.3) are proved. The inequalities for n in (4.3) can be proved similarly.

If $u_1\geqslant u_2$ and $u_3\to u_1-u_2$, then from (3.5) we derive $u_4\to u_1+u_2$. Under these conditions $m\to 0$. Conversely, if $u_2\geqslant u_1$ and $u_3\to u_1+u_2$, then from (3.5) we derive $u_4\to u_2-u_1$. Under these conditions $m\to 1$. This means that all values from the range $0 < m < 1$ are taken by the expression (4.1). Similarly one can prove that all values from the range $0 < n < 1$ are taken by the expression (4.2).

Now let's combine (4.1) with the equation (3.5) and consider

$$\begin{cases} 2\,u_1^2 + 2\,u_2^2 = u_3^2 + u_4^2, \\ \dfrac{2\,u_2 + u_3 - u_4}{2\,u_1 + u_3 + u_4} = m \end{cases} \tag{4.4}$$

as a system of two equations with u_3 and u_4 treated as unknowns. Resolving the equations (4.4) with respect to u_3 and u_4, we get

$$\begin{aligned} u_3 &= \frac{2\,m - m^2 + 1}{m^2 + 1}\,u_1 + \frac{2\,m + m^2 - 1}{m^2 + 1}\,u_2, \\ u_4 &= \frac{1 - m^2 - 2\,m}{m^2 + 1}\,u_1 + \frac{2\,m - m^2 + 1}{m^2 + 1}\,u_2. \end{aligned} \tag{4.5}$$

The formulas (4.5) coincide with the formulas (D.19) and (D.20) in Appendix D of [4]. They can be derived with the use of the following Maple[1] code:

```
restart;
Eq_0:=2*u1^2+2*u2^2-u3^2-u4^2=0:
Eq_m:=m=(2*u2+u3-u4)/(2*u1+u3+u4):
sss:=solve({Eq_0,Eq_m},{u3,u4}):
assign(sss):
u3:=collect(u3,[u1,u2]);
u4:=collect(u4,[u1,u2]);
```

The formulas (4.5) are understood as a rational parametric solution of the parallelogram equation (3.5) with three parameters

$$u_1 > 0, \qquad\qquad u_2 > 0, \qquad\qquad 0 < m < 1.$$

Another parametric solution of the equation (3.5) is obtained with the use of the formula (4.2). Combining it with (3.5), we write

$$\begin{cases} 2\,u_1^2 + 2\,u_2^2 = u_3^2 + u_4^2, \\ \dfrac{2\,u_2 - u_3 + u_4}{2\,u_1 + u_3 + u_4} = n \end{cases} \tag{4.6}$$

and treat (4.6) as a system of equations for unknowns u_3 and u_4. Resolving the equations (4.6) with respect to u_3 and u_4, we get

$$\begin{aligned} u_3 &= \frac{1 - 2\,n - n^2}{n^2 + 1}\,u_1 + \frac{1 + 2\,n - n^2}{n^2 + 1}\,u_2, \\ u_4 &= \frac{1 - n^2 + 2\,n}{n^2 + 1}\,u_1 + \frac{2\,n + n^2 - 1}{n^2 + 1}\,u_2. \end{aligned} \tag{4.7}$$

[1] Maple is a trademark of Waterloo Maple Inc.

The formulas (4.7) coincide with the formulas (D.25) and (D.26) in Appendix D of [4]. They can be derived with the use of the following Maple code:

```
restart;
Eq_0:=2*u1^2+2*u2^2-u3^2-u4^2=0:
Eq_n:=n=(2*u2-u3+u4)/(2*u1+u3+u4):
sss:=solve({Eq_0,Eq_n},{u3,u4}):
assign(sss):
u3:=collect(u3,[u1,u2]);
u4:=collect(u4,[u1,u2]);
```

The formulas (4.7) provide a rational parametric solution of the parallelogram equation (3.5) with three parameters

$$u_1 > 0, \qquad\qquad u_2 > 0, \qquad\qquad 0 < n < 1.$$

In Appemndix A of his paper [4] Walter Wyss introduces the term generator for an angle. Here is the definition of this term.

Definition 4.1. For an arbitrary angle α its generator $m = m(\alpha)$ is defined by the formula $m = \tan(\alpha/2)$.

Any angle $= -\pi < \alpha < \pi$ is uniquely defined by its generator. From the formulas

$$\cos(\alpha) = \frac{1 - \tan^2(\alpha/2)}{1 + \tan^2(\alpha/2)}, \quad \sin(\alpha) = \frac{2\,\tan(\alpha/2)}{1 + \tan^2(\alpha/2)}, \quad \tan(\alpha/2) = \frac{\sin(\alpha)}{1 + \cos(\alpha)},$$

which are elementary, we can derive the following theorems.

Theorem 4.1. *An angle $\alpha \neq \pm\pi$ is a Heron angle if and only if its generator is a rational number.*

Theorem 4.2. *If an angle α is an Euler angle, then $2\,\alpha$, $2\,\alpha - \pi$, and $\pi - 2\,\alpha$ are Heron angles.*

Using the rational parameters m and n from (4.1) and (4.2) as generators, for each rational parallelogram Walter Wyss defines two Heron angles $0 < \alpha < \pi/2$ and $0 < \beta < \pi/2$ such that

$$\tan(\alpha/2) = m, \qquad\qquad \tan(\beta/2) = n. \tag{4.8}$$

Then he introduces two Euler angles

$$\sigma = \frac{\alpha + \beta}{2}, \qquad\qquad \delta = \frac{\alpha - \beta}{2} \tag{4.9}$$

(see (D.29), (D.30), and (D.39) in Appendix D of [4]). The latter two angles obey the inequalities

$$0 < \sigma < \frac{\pi}{2}, \qquad\qquad -\frac{\pi}{4} < \delta < \frac{\pi}{4}$$

(see (D.52) and (D.53) in Appendix D of [4]).

The functions ω_+ and ω_- from Appendix C of [4] are just notations:

$$\omega_+(x)=\cos(x)+\sin(x), \qquad\qquad \omega_-(x)=\cos(x)-\sin(x). \tag{4.10}$$

Various formulas using these functions in Appendix D of [4] can be verified with the use of a rational parametrization, e. g. with the use of (4.7). Substituting (4.7) into the formula (4.1), upon simplifying we get

$$m=\frac{u_2-n\,u_1}{u_1+n\,u_2}. \tag{4.11}$$

The Maple code responsible for this operation is

```
m:=(2*u2+u3-u4)/(2*u1+u3+u4):
m:=normal(m);
```

This code continues the above code on page 7. Therefore the `restart` instruction is not issued in it.

Now we need to code the values of sine, cosine, and tangent functions. This is done according to (4.8), (4.9), (4.10), and (4.11):

```
unprotect(sin,cos,tan):
sin:=subsop(3=NULL,eval(sin)):
cos:=subsop(3=NULL,eval(cos)):
tan:=subsop(3=NULL,eval(tan)):
tan(alpha/2):=m:
tan(beta/2):=n:
sin(alpha/2):=m*cos(alpha/2):
sin(beta/2):=n*cos(beta/2):
sin(alpha):=normal(2*m/(1+m^2)):
cos(alpha):=normal((1-m^2)/(1+m^2)):
sin(beta):=2*n/(1+n^2):
cos(beta):=(1-n^2)/(1+n^2):
omega_plus:=proc(x) sin(x)+cos(x) end proc:
omega_minus:=proc(x) cos(x)-sin(x) end proc:
```

This code continues the previous code and therefore, again, the `restart` instruction is not issued in it. Upon running this code we can proceed to verifying formulas in Appendix D of [4]. In the case of (D.31) we use the following code:

```
Expr_1:=omega_plus(alpha)*u1-omega_minus(alpha)*u2-u3:
Expr_2:=omega_minus(alpha)*u1+omega_plus(alpha)*u2-u4:
Expr_1:=normal(Expr_1):
Expr_2:=normal(Expr_2):
Expr_1,Expr_2;
```

The output of this code should look like `0,0` confirming that both expressions `Expr_1` and `Expr_2` are zero.

The code verifying the formula (D.32) in Appendix D of [4] looks very similar to the previous code. In this case we have

```
Expr_1:=omega_plus(alpha)*u3+omega_minus(alpha)*u4-2*u1:
Expr_2:=-omega_minus(alpha)*u3+omega_plus(alpha)*u4-2*u2:
```

```
Expr_1:=normal(Expr_1):
Expr_2:=normal(Expr_2):
Expr_1,Expr_2;
```

with the same output `0,0` confirming that `Expr_1` and `Expr_2` both are zero.

The next are the formulas (D.33) and (D.34). They are verified by the code

```
Expr_1:=(u1*u3+u2*u4)/(u1^2+u2^2)-omega_plus(alpha):
Expr_2:=(u1*u4-u2*u3)/(u1^2+u2^2)-omega_minus(alpha):
Expr_1:=normal(Expr_1):
Expr_2:=normal(Expr_2):
Expr_1,Expr_2;
```

The formulas (D.35) and (D.36) in Appendix D of [4] are similar to (D.31) and (D.32). They are verified by the following two fragments of code:

```
Expr_1:=omega_minus(beta)*u1+omega_plus(beta)*u2-u3:
Expr_2:=omega_plus(beta)*u1-omega_minus(beta)*u2-u4:
Expr_1:=normal(Expr_1):
Expr_2:=normal(Expr_2):
Expr_1,Expr_2

Expr_1:=omega_minus(beta)*u3+omega_plus(beta)*u4-2*u1:
Expr_2:=omega_plus(beta)*u3-omega_minus(beta)*u4-2*u2:
Expr_1:=normal(Expr_1):
Expr_2:=normal(Expr_2):
Expr_1,Expr_2;
```

The formulas (D.37) and (D.38) in Appendix D of [4] are similar to (D.33) and (D.34). We use the following code to verify them:

```
Expr_1:=(u1*u4+u2*u3)/(u1^2+u2^2)-omega_plus(beta):
Expr_2:=(u1*u3-u2*u4)/(u1^2+u2^2)-omega_minus(beta):
Expr_1:=normal(Expr_1):
Expr_2:=normal(Expr_2):
Expr_1,Expr_2;
```

The formulas (D.41), (D.42), and (D.43) in Appendix D of [4] are immediate from (D.31) and (D.35). Therefore we omit their verification and proceed to (D.44), (D.45), (D.46). These formulas are verified by the following code:

```
sigma:=(alpha+beta)/2:
delta:=(alpha-beta)/2:
Expr_1:=u1*sin(sigma)-u2*cos(sigma):
Expr_2:=(u1*cos(sigma)+u2*sin(sigma))*omega_plus(delta)-u3:
Expr_3:=(u1*cos(sigma)+u2*sin(sigma))*omega_minus(delta)-u4:
Expr_1:=normal(expand(Expr_1)):
Expr_2:=normal(expand(Expr_2)):
Expr_3:=normal(expand(Expr_3)):
Expr_2:=subs(cos(1/2*alpha)^2=(cos(alpha)+1)/2,
        cos(1/2*beta)^2=(cos(beta)+1)/2,Expr_2):
Expr_3:=subs(cos(1/2*alpha)^2=(cos(alpha)+1)/2,
        cos(1/2*beta)^2=(cos(beta)+1)/2,Expr_3):
```

```
Expr_2:=normal(Expr_2):
Expr_3:=normal(Expr_3):
Expr_1,Expr_2,Expr_3;
```

The formulas (D.47), (D.48), (D.49), (D.50), and (D.51) in Appendix D of [4] are immediate from (D.44), (D.45), (D.46). Therefore we omit their verification.

For the reader's convenience all of the above Maple code is placed ibto the ancillary file `section_04.txt` attached to this submission.

5. Slanted cuboid formulas.

The parallelogram equation (3.5) is written for the parallelogram $ABFE$ in Fig. 1.1. Apart from $ABFE$ there are two other parallelograms associated with the slanted cuboid $ABCDEFGH$. They are $AEGC$ and $EFCD$. The sides and diagonals of the parallelogram $AEGC$ are

$$|AE| = u_1, \qquad |EG| = v_2, \qquad |AG| = v_3, \qquad |EC| = v_4. \tag{5.1}$$

Due to (5.1) the parallelogram equation for the parallelogram $AEGC$ looks like

$$2\,u_1^2 + 2\,v_2^2 = v_3^2 + v_4^2. \tag{5.2}$$

The sides and diagonals of the parallelogram $EFCD$ are

$$|EF| = u_2, \qquad |FC| = v_1, \qquad |FD| = v_3, \qquad |EC| = v_4. \tag{5.3}$$

Due to (5.3) the parallelogram equation for the parallelogram $EFCD$ looks like

$$2\,u_2^2 + 2\,v_1^2 = v_3^2 + v_4^2. \tag{5.4}$$

Combining (5.2) and (5.4) with (3.5), we get a system of three equations:

$$\begin{aligned} &2\,u_1^2 + 2\,u_2^2 = u_3^2 + u_4^2,\\ &2\,u_1^2 + 2\,v_2^2 = v_3^2 + v_4^2,\\ &2\,u_2^2 + 2\,v_1^2 = v_3^2 + v_4^2. \end{aligned} \tag{5.5}$$

The equations (5.5) coincide with (16), (17), and (18) in [4].

Remark. The equations (5.5) follow from the slanted cuboid equations (3.4) and (3.5), but they are not equivalent to (3.4) and (3.5).

The equations (5.5) are parallelogram equations. Applying the formula (4.1) to them, Walter Wyss defines three rational numbers m, m_1, m_2:

$$m = \frac{2\,u_2 + u_3 - u_4}{2\,u_1 + u_3 + u_4}, \tag{5.6}$$

$$m_1 = \frac{2\,v_2 + v_3 - v_4}{2\,u_1 + v_3 + v_4}, \tag{5.7}$$

$$m_2 = \frac{2\,u_2 + v_3 - v_4}{2\,v_1 + v_3 + v_4}. \tag{5.8}$$

These numbers obey the inequalities $0 < m < 1$, $0 < m_1 < 1$, $0 < m_2 < 1$. Therefore they generate three Heron angles α, α_1, α_2 such that

$$0 < \alpha < \frac{\pi}{2}, \qquad 0 < \alpha_1 < \frac{\pi}{2}, \qquad 0 < \alpha_2 < \frac{\pi}{2}. \tag{5.9}$$

Now we proceed to the formulas (3.6). They are coded as follows:

```
restart:

v1:=(s1+1/s1)/2:  v2:=(s2+1/s2)/2:
v3:=(s3+1/s3)/2:  v4:=(s4+1/s4)/2:
u1:=(1/s1-s1)/2:  u2:=(1/s2-s2)/2:
u3:=(1/s3-s3)/2:  u4:=(1/s4-s4)/2:

v1:=normal(v1):  v2:=normal(v2):
v3:=normal(v3):  v4:=normal(v4):
u1:=normal(u1):  u2:=normal(u2):
u3:=normal(u3):  u4:=normal(u4):
```

The cuboid equations (3.4) are verified by substitution:

```
Eq_1:=1+u1^2-v1^2:  Eq_2:=1+u2^2-v2^2:
Eq_3:=1+u3^2-v3^2:  Eq_4:=1+u4^2-v4^2:

normal(Eq_1), normal(Eq_2), normal(Eq_3), normal(Eq_4);
```

The expected output is `0,0,0,0`. It indicates that the equations (3.4) are verified by substituting (3.6) into them.

The next step is to derive the equation (3.9). This is done by the following code:

```
Eq_5:=2*u1^2+2*u2^2-u3^2-u4^2:
Eq_5:=numer(normal(Eq_5));
```

The formulas (5.6), (5.7), and (5.8) are coded as follows:

```
m:=normal((2*u2+u3-u4)/(2*u1+u3+u4));
m1:=normal((2*v2+v3-v4)/(2*u1+v3+v4));
m2:=normal((2*u2+v3-v4)/(2*v1+v3+v4));
```

The rational numbers m, m_1 and m_2 are used as generators for three angles α, α_1, and α_2. This fact is coded as follows:

```
unprotect(sin,cos,tan):
sin:=subsop(3=NULL,eval(sin)):
cos:=subsop(3=NULL,eval(cos)):
tan:=subsop(3=NULL,eval(tan)):
tan(alpha/2):=m:
sin(alpha/2):=m*cos(alpha/2):
sin(alpha):=normal(2*m/(1+m^2)):
cos(alpha):=normal((1-m^2)/(1+m^2)):

tan(alpha1/2):=m1:
sin(alpha1/2):=m1*cos(alpha1/2):
```

```
sin(alpha1):=normal(2*m1/(1+m1^2)):
cos(alpha1):=normal((1-m1^2)/(1+m1^2)):

tan(alpha2/2):=m2:
sin(alpha2/2):=m2*cos(alpha2/2):
sin(alpha2):=normal(2*m2/(1+m2^2)):
cos(alpha2):=normal((1-m2^2)/(1+m2^2)):
```

Now the special functions ω_+ and ω_- and their special values are to be programmed. This is done by the following code:

```
omega_plus:=proc(x) sin(x)+cos(x) end proc:
omega_minus:=proc(x) cos(x)-sin(x) end proc:
```

The number Q is expressed through s_3 and s_4 by means of the formula

$$Q = s_3\,s_4 \tag{5.10}$$

(see (28) in [4]). It is coded by the following line:

```
Q:=s3*s4:
```

The functions $H(x)$, $K(x)$, $M(x)$, $N(x)$ are just notations. They are defined in Appendix E of [4]. Here is the code for them:

```
H:=proc(x) global omega_plus,omega_minus,Q:
       omega_minus(x)-Q*omega_plus(x) end proc:
K:=proc(x) global omega_plus,omega_minus,Q:
       omega_minus(x)+Q*omega_plus(x) end proc:
M:=proc(x) global omega_plus,omega_minus,Q:
       omega_plus(x)-Q*omega_minus(x) end proc:
N:=proc(x) global omega_plus,omega_minus,Q:
       omega_plus(x)+Q*omega_minus(x) end proc:
```

Now we are able to verify the formulas from section 4 in Walter Wyss's paper [4]. Let's begin with the formulas (19) and (20):

```
Eq_19:=2*u1-u3*omega_plus(alpha)-u4*omega_minus(alpha):
Eq_19:=numer(normal(Eq_19)):
Eq_19:=rem(Eq_19,Eq_5,s1):

Eq_20:=2*u2+u3*omega_minus(alpha)-u4*omega_plus(alpha):
Eq_20:=numer(normal(Eq_20)):
Eq_20:=rem(Eq_20,Eq_5,s1):

Eq_19,Eq_20;
```

The expected output of this code is `0,0`. The `rem` operator used in this code means that the formulas (19) and (20) hold modulo the equation (3.9). The same is true for all other formulas in section 4 of the paper [4].

The code below verifies the formulas (21), (22), (23), (24):

```
Eq_21:=2*u1-v3*omega_plus(alpha1)-v4*omega_minus(alpha1):
Eq_21:=numer(normal(Eq_21)):
```

```
Eq_21:=rem(Eq_21,Eq_5,s1):

Eq_22:=2*v2+v3*omega_minus(alpha1)-v4*omega_plus(alpha1):
Eq_22:=numer(normal(Eq_22)):
Eq_22:=rem(Eq_22,Eq_5,s1):

Eq_23:=2*v1-v3*omega_plus(alpha2)-v4*omega_minus(alpha2):
Eq_23:=numer(normal(Eq_23)):
Eq_23:=rem(Eq_23,Eq_5,s1):

Eq_24:=2*u2+v3*omega_minus(alpha2)-v4*omega_plus(alpha2):
Eq_24:=numer(normal(Eq_24)):
Eq_24:=rem(Eq_24,Eq_5,s1):

Eq_21,Eq_22,Eq_23,Eq_24;
```

The next are the formulas (29) through (34). They are verified as follows:

```
Eq_29:=4*Q*u1-s4*M(alpha)-s3*H(alpha):
Eq_29:=numer(normal(Eq_29)):
Eq_29:=rem(Eq_29,Eq_5,s1):

Eq_30:=4*Q*u2+s4*K(alpha)-s3*N(alpha):
Eq_30:=numer(normal(Eq_30)):
Eq_30:=rem(Eq_30,Eq_5,s1):

Eq_31:=4*Q*u1-s4*N(alpha1)-s3*K(alpha1):
Eq_31:=numer(normal(Eq_31)):
Eq_31:=rem(Eq_31,Eq_5,s1):

Eq_32:=4*Q*v2+s4*H(alpha1)-s3*M(alpha1):
Eq_32:=numer(normal(Eq_32)):
Eq_32:=rem(Eq_32,Eq_5,s1):

Eq_33:=4*Q*v1-s4*N(alpha2)-s3*K(alpha2):
Eq_33:=numer(normal(Eq_33)):
Eq_33:=rem(Eq_33,Eq_5,s1):

Eq_34:=4*Q*u2+s4*H(alpha2)-s3*M(alpha2):
Eq_34:=numer(normal(Eq_34)):
Eq_34:=rem(Eq_34,Eq_5,s1):

Eq_29,Eq_30,Eq_31,Eq_32,Eq_33,Eq_34;
```

Though the equations (35), (36), (37), (38) are derived from the previous ones, they can be verified in a straightforward manner:

```
Eq_35:=s4*(M(alpha)-N(alpha1))+s3*(H(alpha)-K(alpha1)):
Eq_35:=numer(normal(Eq_35)):
Eq_35:=rem(Eq_35,Eq_5,s1):
```

```
Eq_36:=-8*Q*u1+s4*(M(alpha)+N(alpha1))+s3*(H(alpha)+K(alpha1)):
Eq_36:=numer(normal(Eq_36)):
Eq_36:=rem(Eq_36,Eq_5,s1):

Eq_37:=-4*Q*s2+s4*(K(alpha)-H(alpha1))-s3*(N(alpha)-M(alpha1)):
Eq_37:=numer(normal(Eq_37)):
Eq_37:=rem(Eq_37,Eq_5,s1):

Eq_38:=-4*Q/s2-s4*(K(alpha)+H(alpha1))+s3*(N(alpha)+M(alpha1)):
Eq_38:=numer(normal(Eq_38)):
Eq_38:=rem(Eq_38,Eq_5,s1):

Eq_35,Eq_36,Eq_37,Eq_38;
```

Let's recall the formulas (4.10). They can be rewritten as follows:

$$\omega_+(x)=\sqrt{2}\,\cos\Bigl(\frac{\pi}{4}-x\Bigr),\qquad\qquad \omega_-(x)=\sqrt{2}\,\sin\Bigl(\frac{\pi}{4}-x\Bigr).\tag{5.11}$$

The formulas (5.11) are verified by means of the following code:

```
omega_plus(x)-expand(sqrt(2)*cos(Pi/4-x)),
omega_minus(x)-expand(sqrt(2)*sin(Pi/4-x));
```

On page 4 of his paper [4] Walter Wyss presents the formulas

$$\omega_+(\sigma_1)=\sqrt{2}\,\cos\psi,\qquad\qquad \omega_-(\sigma_1)=\sqrt{2}\,\sin\psi,\tag{5.12}$$

where $2\,\sigma_1=\alpha+\alpha_1$. Comparing (5.12) with (5.11), we conclude

$$\psi=\frac{\pi}{4}-\sigma_1=\frac{\pi}{4}-\frac{\alpha+\alpha_1}{2}.\tag{5.13}$$

From (5.13) one easily derives

$$\alpha+\psi=\frac{\pi}{4}+\frac{\alpha-\alpha_1}{2}=\frac{\pi}{4}+\delta_1,\tag{5.14}$$

where $2\,\delta_1=\alpha-\alpha_1$. Substituting $x=\alpha+\psi$ into (5.11) and using (5.14), we get

$$\omega_+(\alpha+\psi)=\sqrt{2}\,\cos\delta_1,\qquad\qquad \omega_-(\alpha+\psi)=-\sqrt{2}\,\sin\delta_1.\tag{5.15}$$

The formulas (5.15) coincide with the formulas given by Walter Wyss on page 4 of his paper [4]. So, the formula (5.13) is a key point for understanding what is ψ. This formula is programmed by the following code:

```
psi:=Pi/4-alpha/2-alpha1/2:
```

Note that α, α_1, α_2 are Heron angles (5.9) generated by rational numbers (5.6), (5.7), (5.8). Therefore we have the following formulas

$$\cos^2\Bigl(\frac{\alpha}{2}\Bigr)=\frac{1}{1+m^2},\qquad \cos^2\Bigl(\frac{\alpha_1}{2}\Bigr)=\frac{1}{1+m_1^2},\qquad \cos^2\Bigl(\frac{\alpha_2}{2}\Bigr)=\frac{1}{1+m_2^2}.\tag{5.16}$$

Relying on the formulas (5.16) we introduce a simplification procedure. It is called psi_phi_simplify. We define it with the following code:

```
psi_phi_simplify:=proc(A) local AA: global m,m1,m2:
  AA:=subs(cos(alpha/2)^2=1/(1+m^2),A):
  AA:=subs(cos(alpha1/2)^2=1/(1+m1^2),AA):
  AA:=subs(cos(alpha2/2)^2=1/(1+m2^2),AA):
  return AA:
end proc:
```

Using this procedure, we can proceed to verifying further formulas from Walter Wyss's paper. For the formulas (39), (40), (41), (42) we apply the following code:

```
Eq_39:=s3*cos(psi)*H(alpha+psi)-s4*sin(psi)*K(alpha+psi):
Eq_39:=expand(Eq_39):
Eq_39:=psi_phi_simplify(Eq_39):
Eq_39:=numer(normal(Eq_39)):
Eq_39:=rem(Eq_39,Eq_5,s1):

Eq_40:=-4*Q*u1+s4*cos(psi)*M(alpha+psi)+s3*sin(psi)*N(alpha+psi):
Eq_40:=expand(Eq_40):
Eq_40:=psi_phi_simplify(Eq_40):
Eq_40:=numer(normal(Eq_40)):
Eq_40:=rem(Eq_40,Eq_5,s1):

Eq_41:=-2*Q*s2+s4*cos(psi)*K(alpha+psi)+s3*sin(psi)*H(alpha+psi):
Eq_41:=expand(Eq_41):
Eq_41:=psi_phi_simplify(Eq_41):
Eq_41:=numer(normal(Eq_41)):
Eq_41:=rem(Eq_41,Eq_5,s1):

Eq_42:=-2*Q/s2-s4*sin(psi)*M(alpha+psi)+s3*cos(psi)*N(alpha+psi):
Eq_42:=expand(Eq_42):
Eq_42:=psi_phi_simplify(Eq_42):
Eq_42:=numer(normal(Eq_42)):
Eq_42:=rem(Eq_42,Eq_5,s1):

Eq_39,Eq_40,Eq_41,Eq_42;
```

The formulas (43) and (44) are matrix presentations of the formulas (39), (40), (41), (42). The formulas (45) and (46) are inverse to (43) and (44). We do not verify them. However, we do verify the formulas (47), (48), (49), (50) derived from (45) and (46). This is done by the following code:

```
Eq_47:=-K(alpha+psi)+2*s2*s3*cos(psi):
Eq_47:=expand(Eq_47):
Eq_47:=numer(normal(Eq_47)):
Eq_47:=rem(Eq_47,Eq_5,s1):

Eq_48:=-H(alpha+psi)+2*s2*s4*sin(psi):
Eq_48:=expand(Eq_48):
```

```
Eq_48:=numer(normal(Eq_48)):
Eq_48:=rem(Eq_48,Eq_5,s1):

Eq_49:=-M(alpha+psi)+4*u1*s3*cos(psi)-2*s3/s2*sin(psi):
Eq_49:=expand(Eq_49):
Eq_49:=numer(normal(Eq_49)):
Eq_49:=rem(Eq_49,Eq_5,s1):

Eq_50:=-N(alpha+psi)+4*u1*s4*sin(psi)+2*s4/s2*cos(psi):
Eq_50:=expand(Eq_50):
Eq_50:=numer(normal(Eq_50)):
Eq_50:=rem(Eq_50,Eq_5,s1):

Eq_47,Eq_48,Eq_49,Eq_50;
```

The formulas (51), (52), (53), (54) in [4] are similar to the formulas (35), (36), (37), (38). They are verified by means of the following code:

```
Eq_51:=s3*(N(alpha)-M(alpha2))-s4*(K(alpha)-H(alpha2)):
Eq_51:=numer(normal(Eq_51)):
Eq_51:=rem(Eq_51,Eq_5,s1):

Eq_52:=-8*Q*u2+s3*(N(alpha)+M(alpha2))-s4*(K(alpha)+H(alpha2)):
Eq_52:=numer(normal(Eq_52)):
Eq_52:=rem(Eq_52,Eq_5,s1):

Eq_53:=-4*Q*s1+s4*(N(alpha2)-M(alpha))+s3*(K(alpha2)-H(alpha)):
Eq_53:=numer(normal(Eq_53)):
Eq_53:=rem(Eq_53,Eq_5,s1):

Eq_54:=-4*Q/s1+s4*(N(alpha2)+M(alpha))+s3*(K(alpha2)+H(alpha)):
Eq_54:=numer(normal(Eq_54)):
Eq_54:=rem(Eq_54,Eq_5,s1):

Eq_51,Eq_52,Eq_53,Eq_54;
```

In the next fragment of Walter Wyss's paper [4] the angle ϕ is defined:

$$\phi=\frac{\pi}{4}-\sigma_2=\frac{\pi}{4}-\frac{\alpha+\alpha_2}{2}. \tag{5.17}$$

Here $2\,\sigma_2=\alpha+\alpha_2$. Though the formula (5.17) is not written explicitly, the formulas

$$\omega_+(\sigma_2)=\sqrt{2}\,\cos\phi, \qquad\qquad \omega_-(\sigma_2)=\sqrt{2}\,\sin\phi, \tag{5.18}$$

compared with (5.11) lead to (5.17). Then the following formula with $2\,\delta_2=\alpha-\alpha_2$ is derived from (5.17):

$$\alpha+\phi=\frac{\pi}{4}+\frac{\alpha-\alpha_2}{2}=\frac{\pi}{4}+\delta_2, \tag{5.19}$$

Substituting $x=\alpha+\phi$ into (5.11) and using (5.19), we get

$$\omega_+(\alpha+\phi)=\sqrt{2}\,\cos\delta_2, \qquad\qquad \omega_-(\alpha+\phi)=-\sqrt{2}\,\sin\delta_2. \tag{5.20}$$

The formulas (5.20) are analogous to (5.15), the formula (5.19) is analogous to (5.14), the formulas (5.18) are analogous to (5.12), and the formula (5.17) is analogous to (5.13). The formula (5.17) is programmed by the following code:

```
phi:=Pi/4-alpha/2-alpha2/2:
```

The angle ϕ is used by Walter Wyss in his formulas (55), (56), (57), (58). These formulas are verified as follows:

```
Eq_55:=-s4*cos(phi)*K(alpha+phi)-s3*sin(phi)*H(alpha+phi):
Eq_55:=expand(Eq_55):
Eq_55:=psi_phi_simplify(Eq_55):
Eq_55:=numer(normal(Eq_55)):
Eq_55:=rem(Eq_55,Eq_5,s1):

Eq_56:=-4*Q*u2-s4*sin(phi)*M(alpha+phi)+s3*cos(phi)*N(alpha+phi):
Eq_56:=expand(Eq_56):
Eq_56:=psi_phi_simplify(Eq_56):
Eq_56:=numer(normal(Eq_56)):
Eq_56:=rem(Eq_56,Eq_5,s1):

Eq_57:=-2*Q*s1+s4*sin(phi)*K(alpha+phi)-s3*cos(phi)*H(alpha+phi):
Eq_57:=expand(Eq_57):
Eq_57:=psi_phi_simplify(Eq_57):
Eq_57:=numer(normal(Eq_57)):
Eq_57:=rem(Eq_57,Eq_5,s1):

Eq_58:=-2*Q/s1+s4*cos(phi)*M(alpha+phi)+s3*sin(phi)*N(alpha+phi):
Eq_58:=expand(Eq_58):
Eq_58:=psi_phi_simplify(Eq_58):
Eq_58:=numer(normal(Eq_58)):
Eq_58:=rem(Eq_58,Eq_5,s1):

Eq_55,Eq_56,Eq_57,Eq_58;
```

We omit the formulas (59), (60), (61), (62) just like the formulas (43), (44), (45), (46) above and proceed to (63), (64), (65), (66):

```
Eq_63:=-K(alpha+phi)+2*s1*s3*sin(phi):
Eq_63:=expand(Eq_63):
Eq_63:=numer(normal(Eq_63)):
Eq_63:=rem(Eq_63,Eq_5,s1):

Eq_64:=-H(alpha+phi)-2*s1*s4*cos(phi):
Eq_64:=expand(Eq_64):
Eq_64:=numer(normal(Eq_64)):
Eq_64:=rem(Eq_64,Eq_5,s1):

Eq_65:=-M(alpha+phi)-4*u2*s3*sin(phi)+2*s3/s1*cos(phi):
Eq_65:=expand(Eq_65):
Eq_65:=numer(normal(Eq_65)):
```

```
Eq_65:=rem(Eq_65,Eq_5,s1):

Eq_66:=-N(alpha+phi)+4*u2*s4*cos(phi)+2*s4/s1*sin(phi):
Eq_66:=expand(Eq_66):
Eq_66:=numer(normal(Eq_66)):
Eq_66:=rem(Eq_66,Eq_5,s1):

Eq_63,Eq_64,Eq_65,Eq_66;
```

The next are the formulas (67), (68), (69), (70) in Walter Wyss's paper [4]. They are verified by means of the following code:

```
Eq_67:=-4*Q*u2-s4*K(alpha)+s3*N(alpha):
Eq_67:=numer(normal(Eq_67)):
Eq_67:=rem(Eq_67,Eq_5,s1):

Eq_68:=-4*Q*v2-s4*H(alpha1)+s3*M(alpha1):
Eq_68:=numer(normal(Eq_68)):
Eq_68:=rem(Eq_68,Eq_5,s1):

Eq_69:=-4*Q*u1+s4*M(alpha)+s3*H(alpha):
Eq_69:=numer(normal(Eq_69)):
Eq_69:=rem(Eq_69,Eq_5,s1):

Eq_70:=-4*Q*v1+s4*N(alpha2)+s3*K(alpha2):
Eq_70:=numer(normal(Eq_70)):
Eq_70:=rem(Eq_70,Eq_5,s1):

Eq_67,Eq_68,Eq_69,Eq_70;
```

There are two formulas on page 8 of the paper [4]. They are not numbered:

$$-s_4\,K(\alpha)+s_3\,N(\alpha)=-s_4\,H(\alpha_2)+s_3\,M(\alpha_2), \tag{5.21}$$
$$s_4\,M(\alpha)+s_3\,H(\alpha)=s_4\,N(\alpha_1)+s_3\,K(\alpha_1). \tag{5.22}$$

Giving them the numbers (5.21) and (5.22), we can verify them as follows:

```
Eq_5_21:=-s4*K(alpha)+s3*N(alpha)+s4*H(alpha2)-s3*M(alpha2):
Eq_5_21:=numer(normal(Eq_5_21)):
Eq_5_21:=rem(Eq_5_21,Eq_5,s1):

Eq_5_22:=s4*M(alpha)+s3*H(alpha)-s4*N(alpha1)-s3*K(alpha1):
Eq_5_22:=numer(normal(Eq_5_22)):
Eq_5_22:=rem(Eq_5_22,Eq_5,s1):

Eq_5_21,Eq_5_22;
```

We do not need to follow the proof of the equations (50) and (66) on page 8 of [4]. These equations are verified programmatically above. Similarly, we do not need to follow the proof of the formula (71) on page 9 of this paper. This formula is also verified programmatically by means of the following code:

```
Eq_71:=s1*s2-tan(phi-psi):
Eq_71:=expand(Eq_71):
Eq_71:=numer(normal(Eq_71)):
Eq_71:=rem(Eq_71,Eq_5,s1);
```

Applying the formulas (5.13) and (5.17), we can write the formula (71) as follows:

$$\tan\Bigl(\frac{\alpha_1-\alpha_2}{2}\Bigr)=s_1\,s_2. \tag{5.23}$$

The formula (5.23) can also be verified programmatically:

```
Eq_5_23:=s1*s2-tan(alpha1/2-alpha2/2):
Eq_5_23:=expand(Eq_5_23):
Eq_5_23:=numer(normal(Eq_5_23)):
Eq_5_23:=rem(Eq_5_23,Eq_5,s1);
```

Section 5 of Walter Wyss's paper [4] is slightly different from section 4. Nevertheless, now we proceed to this section and verify some prerequisite formulas therein. The formulas (75), (76), (77), (78) are verified as follows:

```
Eq_75:=-K(alpha)+2*s3*(s2*cos(psi)^2
       +sin(psi)*(2*u1*cos(psi)-1/s2*sin(psi))):
Eq_75:=expand(Eq_75):
Eq_75:=psi_phi_simplify(Eq_75):
Eq_75:=numer(normal(Eq_75)):
Eq_75:=rem(Eq_75,Eq_5,s1):

Eq_76:=-N(alpha)+2*s4*(-s2*sin(psi)^2
       +cos(psi)*(2*u1*sin(psi)+1/s2*cos(psi))):
Eq_76:=expand(Eq_76):
Eq_76:=psi_phi_simplify(Eq_76):
Eq_76:=numer(normal(Eq_76)):
Eq_76:=rem(Eq_76,Eq_5,s1):

Eq_77:=-H(alpha)+2*s4*(s2*sin(psi)*cos(psi)
       +sin(psi)*(2*u1*sin(psi)+1/s2*cos(psi))):
Eq_77:=expand(Eq_77):
Eq_77:=psi_phi_simplify(Eq_77):
Eq_77:=numer(normal(Eq_77)):
Eq_77:=rem(Eq_77,Eq_5,s1):

Eq_78:=-M(alpha)+2*s3*(-s2*sin(psi)*cos(psi)
       +cos(psi)*(2*u1*cos(psi)-1/s2*sin(psi))):
Eq_78:=expand(Eq_78):
Eq_78:=psi_phi_simplify(Eq_78):
Eq_78:=numer(normal(Eq_78)):
Eq_78:=rem(Eq_78,Eq_5,s1):

Eq_75,Eq_76,Eq_77,Eq_78;
```

The formula (79) coincides with (29), the formula (80) coincides with (30). The formula (81) is just a notation. Taking into account this notation, the formulas (82), (83), (84) are verified as follows:

```
lambda:=tan(psi):

unprotect(cot):
cot:=subsop(3=NULL,eval(cot)):
cot(alpha/2):=1/m:
cot(alpha1/2):=1/m1:
cot(alpha2/2):=1/m2:

Eq_82:=-omega_minus(alpha)+lambda*omega_plus(alpha)
       +s2*s3+lambda*s2*s4:
Eq_82:=expand(Eq_82):
Eq_82:=numer(normal(Eq_82)):
Eq_82:=rem(Eq_82,Eq_5,s1):

Eq_83:=-Q*(omega_plus(alpha)+lambda*omega_minus(alpha))
       +s2*s3-lambda*s2*s4:
Eq_83:=expand(Eq_83):
Eq_83:=numer(normal(Eq_83)):
Eq_83:=rem(Eq_83,Eq_5,s1):

Eq_84:=-2*u1*(omega_minus(alpha)-lambda*omega_plus(alpha))
       -s4+lambda*s3+s2*(omega_plus(alpha)
       +lambda*omega_minus(alpha)):
Eq_84:=expand(Eq_84):
Eq_84:=numer(normal(Eq_84)):
Eq_84:=rem(Eq_84,Eq_5,s1):

Eq_82,Eq_83,Eq_84;
```

Concluding the above computations, we can confirm that the formulas (19)–(84) in Walter Wyss's paper are valid.

6. A special solution of the slanted cuboid equation.

Theorem 3.1 provides an exhaustive description of rational slanted cuboids. They constitute rational points within an open subvariety of a three-dimensional real algebraic variety in $\mathbb{R}^4$. This real algebraic variety Γ_3 is defined by the equation (3.9). Its open subvariety $\Gamma_{3++} \subset \Gamma_3$ is outlined by the inequalities (3.7) and (3.10).

Let's consider the equality (85) in Walter Wyss's paper [4]. It is different from all of the previous formulas in this paper. The equality (85) does not hold identically on Γ_3. It makes an auxiliary restriction thus defining a two-dimensional subvariety $\Gamma_2^1 \subset \Gamma_3$. The lower index 2 in Γ_2^1 indicates the dimension of the subvariety. The upper index 1 in Γ_2^1 says that Γ_2^1 is not the only two-dimensional subvariety of Γ_3 that will be considered in what follows. Thus, an auxiliary restriction is set:

$$\lambda = \tan\psi = 0. \tag{6.1}$$

Under the restriction (6.1) the formulas (82), (83), (84) in [4] reduce to the formulas (86), (87), (88) therein. Here are these formulas

$$s_2\,s_3=\omega_-(\alpha),\tag{6.2}$$
$$s_2\,s_3=Q\,\omega_+(\alpha),\tag{6.3}$$
$$s_2\,\omega_+(\alpha)=2\,u_1\,\omega_-(\alpha)+s_4.\tag{6.4}$$

Following Walter Wyss in [4], we multiply both sides of (6.4) by $\omega_+(\alpha)$:

$$s_2\,\omega_+^2(\alpha)=2\,u_1\,\omega_-(\alpha)\,\omega_+(\alpha)+s_4\,\omega_+(\alpha).\tag{6.5}$$

Then we recall the formula (5.10) for Q. Applying (5.10) to (6.3), we get

$$s_2=s_4\,\omega_+(\alpha).\tag{6.6}$$

Due to (6.6) we can replace the last term in (6.5) with s_2:

$$s_2\,\omega_+^2(\alpha)=2\,u_1\,\omega_-(\alpha)\,\omega_+(\alpha)+s_2.\tag{6.7}$$

The formula (6.7) can be transformed as

$$s_2\,(\omega_+^2(\alpha)-1)=2\,u_1\,\omega_-(\alpha)\,\omega_+(\alpha).\tag{6.8}$$

Now we recall the formulas (4.10). From these formulas we derive

$$\omega_+^2(\alpha)-1=\sin(2\,\alpha),\tag{6.9}$$
$$\omega_-(\alpha)\,\omega_+(\alpha)=\cos(2\,\alpha).\tag{6.10}$$

Applying (6.9) and (6.10) to (6.8), we obtain

$$s_2\,\sin(2\,\alpha)=2\,u_1\,\cos(2\,\alpha).\tag{6.11}$$

The formula (6.11) is equivalent to the formula (91) in [4].

Now let's recall that the angle α in (5.9) was introduced through its generator (5.6) (see Definition 4.1), i. e. we have the formula

$$\tan(\alpha/2)=m.\tag{6.12}$$

From (6.12) we derive

$$\cos(\alpha)=\frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)}=\frac{1-m^2}{1+m^2},\tag{6.13}$$

$$\sin(\alpha)=\frac{2\,\tan(\alpha/2)}{1+\tan^2(\alpha/2)}=\frac{2\,m}{1+m^2}.\tag{6.14}$$

Then from (6.13) and (6.14) we derive

$$\sin(2\,\alpha)=2\,\sin\alpha\,\cos\alpha=\frac{4\,m\,\bigl(1-m^2\bigr)}{\bigl(1+m^2\bigr)^2}.\tag{6.15}$$

Again from (6.13) and (6.14) we derive

$$\cos(2\,\alpha)=\cos^2\!\alpha-\sin^2\!\alpha=\frac{\bigl(1-m^2\bigr)^2-4\,m^2}{\bigl(1+m^2\bigr)^2}. \tag{6.16}$$

Now we apply (6.15) and (6.16) to (6.11). As a result we get

$$s_2=u_1\,\frac{\bigl(1-m^2\bigr)^2-4\,m^2}{2\,m\,\bigl(1-m^2\bigr)} \tag{6.17}$$

Let's denote $s_1=s$. Then the first formula (3.6) for $k=1$ is written as

$$u_1=\frac{1-s^2}{2\,s}, \tag{6.18}$$

Substituting (6.18) into (6.17), we derive the formula

$$s_2=\frac{\bigl(1-s^2\bigr)\,\bigl(\bigl(1-m^2\bigr)^2-4\,m^2\bigr)}{4\,m\,s\,\bigl(1-m^2\bigr)}. \tag{6.19}$$

Applying (4.10), (6.13), and (6.14) once more, we obtain the formulas

$$\omega_-(\alpha)=\cos(\alpha)-\sin(\alpha)=\frac{1-m^2-2\,m}{1+m^2}, \tag{6.20}$$

$$\omega_+(\alpha)=\cos(\alpha)+\sin(\alpha)=\frac{1-m^2+2\,m}{1+m^2}. \tag{6.21}$$

Now we substitute (6.19), and (6.20) into (6.2) and we get

$$s_3=\frac{4\,m\,s\,\bigl(1-m^2\bigr)}{\bigl(1-s^2\bigr)\,\bigl(1+m^2\bigr)\,\bigl(1-m^2+2\,m\bigr)}, \tag{6.22}$$

Then we substitute (6.19), and (6.21) into (6.6). As a result we get

$$s_4=\frac{\bigl(1-s^2\bigr)\,\bigl(1+m^2\bigr)\,\bigl(1-m^2-2\,m\bigr)}{4\,m\,s\,\bigl(1-m^2\bigr)}. \tag{6.23}$$

Let's denote through $\theta(s,m)$, $\eta(s,m)$, $\zeta(s,m)$ the right hand sides of the formulas (6.19), (6.22) and (6.23) respectively. The symbol m is linked with the angle α in (5.9). In order unlink the argument m of the functions $\theta(s,m)$, $\eta(s,m)$, $\zeta(s,m)$ from the angle α we replace it with μ. As a result we have

$$\theta(s,\mu)=\frac{\bigl(1-s^2\bigr)\,\bigl(\bigl(1-\mu^2\bigr)^2-4\,\mu^2\bigr)}{4\,\mu\,s\,\bigl(1-\mu^2\bigr)}, \tag{6.24}$$

$$\eta(s,\mu)=\frac{4\,\mu\,s\,\bigl(1-\mu^2\bigr)}{\bigl(1-s^2\bigr)\,\bigl(1+\mu^2\bigr)\,\bigl(1-\mu^2+2\,\mu\bigr)}, \tag{6.25}$$

$$\zeta(s,\mu)=\frac{\bigl(1-s^2\bigr)\,\bigl(1+\mu^2\bigr)\,\bigl(1-\mu^2-2\,\mu\bigr)}{4\,\mu\,s\,\bigl(1-\mu^2\bigr)}. \tag{6.26}$$

Using the functions (6.24), (6.25), (6.26), we define a mapping:

$$\begin{cases} s_1 = s, \\ s_2 = \theta(s,\mu), \\ s_3 = \eta(s,\mu), \\ s_4 = \zeta(s,\mu). \end{cases} \tag{6.27}$$

Theorem 6.1. *The functions (6.27), where $\theta(s,\mu)$, $\eta(s,\mu)$, and $\zeta(s,\mu)$ are given by the formulas (6.24), (6.25) and (6.26) respectively, provide a two-parametric solution of the slanted cuboid equation (3.9).*

The proof of this theorem is pure computations. These computations are performed by means of the following code:

```
restart:
u1:=(1/s1-s1)/2:  u2:=(1/s2-s2)/2:
u3:=(1/s3-s3)/2:  u4:=(1/s4-s4)/2:
u1:=normal(u1):   u2:=normal(u2):
u3:=normal(u3):   u4:=normal(u4):

Eq_5:=2*u1^2+2*u2^2-u3^2-u4^2:
Eq_5:=numer(normal(Eq_5));

theta:=(1-s^2)*((1-mu^2)^2-4*mu^2)/4/mu/s/(1-mu^2);
eta:=4*mu*s*(1-mu^2)/(1-s^2)/(1+mu^2)/(1-mu^2+2*mu);
zeta:=(1-s^2)*(1+mu^2)*(1-mu^2-2*mu)/4/mu/s/(1-mu^2);

Eq_5:=subs(s1=s,s2=theta,s3=eta,s4=zeta,Eq_5):
Eq_5:=numer(normal(Eq_5));
```

Due to (3.7) the parameter s in (6.27) is restricted by the inequalities $0 < s < 1$. The parameter m in (4.3) is restricted by the inequalities $0 < m < 1$. But due to the factor $1-\mu^2-2\,\mu$ in (6.26) and $s_4 > 0$ in (3.7) we have an auxiliary restriction:

$$1-\mu^2-2\,\mu > 0. \tag{6.28}$$

Resolving (6.28) with respect to μ, we get $\mu < \sqrt{2}-1$. Therefore the functions $\theta(s,\mu)$, $\eta(s,\mu)$, and $\zeta(s,\mu)$ are well-defined for

$$0 < s < 1, \qquad\qquad 0 < \mu < \sqrt{2}-1. \tag{6.29}$$

But due to the inequalities (3.7) and (3.10) the actual domain D of the mapping (6.27) could be even smaller than (6.29).

The image of the domain D under the mapping (6.24) is a two-dimensional real algebraic subvariety within Γ_{3++}. Above we have denoted it through Γ^1_2. Note that the slanted cuboid equation (3.9) and the slanted cuboid inequalities (3.7) and (3.10) admit the following two discrete symmetry transformations:

$$s_1 \longleftrightarrow s_2, \qquad\qquad s_3 \longleftrightarrow s_4 \tag{6.30}$$

Applying (6.30) to (6.27), we derive three more mappings:

$$\begin{cases} s_1=\theta(s,\mu),\\ s_2=s,\\ s_3=\eta(s,\mu),\\ s_4=\zeta(s,\mu),\end{cases}\qquad \begin{cases} s_1=s,\\ s_2=\theta(s,\mu),\\ s_3=\zeta(s,\mu),\\ s_4=\eta(s,\mu),\end{cases}\qquad \begin{cases} s_1=\theta(s,\mu),\\ s_2=s,\\ s_3=\zeta(s,\mu),\\ s_4=\eta(s,\mu).\end{cases}\tag{6.31}$$

The images of the domain D under the mappings (6.31) constitute three more two-dimensional real algebraic subvarieties within Γ_{3++}. We denote them Γ_2^2, Γ_2^3, and Γ_2^4 respectively.

In Walter Wyss's paper [4] we find two examples of rational slanted cuboids. The first example on page 14 is produced by choosing

$$s=\frac{1}{2},\qquad\qquad \mu=\frac{1}{3}.\tag{6.32}$$

Substituting (6.32) into (6.27) and taking into account (6.24), (6.25), (6.26), we get

$$s_1=\frac{1}{2},\qquad s_2=\frac{7}{16},\qquad s_3=\frac{16}{35},\qquad s_4=\frac{5}{16}.\tag{6.33}$$

The values (6.33) are produced by the following code:

```
s1=subs(s=1/2,mu=1/3,s);
s2=subs(s=1/2,mu=1/3,theta);
s3=subs(s=1/2,mu=1/3,eta);
s4=subs(s=1/2,mu=1/3,zeta);
```

They do coincide with Walter Wiss's data on page 14. The second example on page 15 of [4] is produced by choosing

$$s=\frac{12}{25},\qquad\qquad \mu=\frac{1}{3}.\tag{6.34}$$

Substituting (6.34) into (6.27) and taking into account (6.24), (6.25), (6.26), we get

$$s_1=\frac{12}{25},\qquad s_2=\frac{3367}{7200},\qquad s_3=\frac{1440}{3367},\qquad s_4=\frac{481}{1440}.\tag{6.35}$$

The values (6.35) again coincide with Walter Wiss's data on page 15 of his paper.

7. Further verifications.

In sections 6 and 7 of his paper [4] Walter Wyss changes some notations. Nevertheless we can continue verifying his formulas relying on Theorem 3.1 and referring them to the basic equation (3.9) of the slanted cuboids. The basic equation (3.9) is programmed by means of the following code:

```
restart:
v1:=(s1+1/s1)/2:  v2:=(s2+1/s2)/2:
v3:=(s3+1/s3)/2:  v4:=(s4+1/s4)/2:
```

```
u1:=(1/s1-s1)/2:  u2:=(1/s2-s2)/2:
u3:=(1/s3-s3)/2:  u4:=(1/s4-s4)/2:
v1:=normal(v1):  v2:=normal(v2):
v3:=normal(v3):  v4:=normal(v4):
u1:=normal(u1):  u2:=normal(u2):
u3:=normal(u3):  u4:=normal(u4):

Eq_5:=2*u1^2+2*u2^2-u3^2-u4^2:
Eq_5:=numer(normal(Eq_5));
```

In the beginning of section 7 of his paper [4] on page 18 Walter Wyss writes 5 equations which are not numbered. Two of them coincide with the equations (16) and (17) on page 3. Other three equations coincide with the equations (9), (10), (11) on page 2. The equation (8) from page 2 is not written on page 18. Like in the case of the equations (16), (18), and (19) on page 3, we have a subset of the slanted cuboid equations (3.4) and (3.5). They are fulfilled once the basic equation (3.9) is fulfilled and the formulas (3.6) are taken for expressing u_1, u_2, u_3, u_4 and v_1, v_2, v_3, v_4 through the generators s_1, s_2, s_3, s_4. The numbers (5.6) and (5.7) are expressed on page 16 of Walter Wyss's paper [4] in a functional form as values of some function $m(x)$ (see formulas (95)). We define this function as

```
m:=proc(x) option remember:  end proc:
```

Its values $m(\alpha)$ and $m(\alpha_1)$ are coded as follows:

```
m(alpha):=normal((2*u2+u3-u4)/(2*u1+u3+u4)):
m(alpha1):=normal((2*v2+v3-v4)/(2*u1+v3+v4)):
```

The formulas (96) from Walter Wyss's paper [4] are coded similarly:

```
m(beta):=normal((2*u2-u3+u4)/(2*u1+u3+u4)):
m(beta1):=normal((2*v2-v3+v4)/(2*u1+v3+v4)):
```

After the formulas (96) on page 16 we see some formulas which are not numbered Some of them coincide with the not numbered formulas on page 4. They lead to (5.12) and (5.13), where σ_1 is defined by means of the formula

$$\sigma_1 := \frac{\alpha + \alpha_1}{2} \tag{7.1}$$

Trigonometric functions of the angle α and its multiples are coded as follows:

```
unprotect(sin,cos,tan,cot):
sin:=subsop(3=NULL,eval(sin)):
cos:=subsop(3=NULL,eval(cos)):
tan:=subsop(3=NULL,eval(tan)):
cot:=subsop(3=NULL,eval(cot)):

tan(alpha/2):=m(alpha):
cot(alpha/2):=1/m(alpha):
sin(alpha/2):=m(alpha)*cos(alpha/2):
sin(alpha):=normal(2*m(alpha)/(1+m(alpha)^2)):
cos(alpha):=normal((1-m(alpha)^2)/(1+m(alpha)^2)):
```

Trigonometric functions of the angle α_1 and its multiples are coded similarly:

```
tan(alpha1/2):=m(alpha1):
cot(alpha1/2):=1/m(alpha1):
sin(alpha1/2):=m(alpha1)*cos(alpha1/2):
sin(alpha1):=normal(2*m(alpha1)/(1+m(alpha1)^2)):
cos(alpha1):=normal((1-m(alpha1)^2)/(1+m(alpha1)^2)):
```

The formulas (5.13) and (7.1) are coded as follows:

```
psi:=Pi/4-alpha/2-alpha1/2:
sigma1:=alpha/2+alpha1/2:
```

The functions $\omega_+(x)$ and $\omega_-(x)$ are defined according to (4.10):

```
omega_plus:=proc(x) sin(x)+cos(x) end proc:
omega_minus:=proc(x) cos(x)-sin(x) end proc:
```

Now we are able to verify the formulas (5.12) which are repeated on page 16 of Walter Wyss's paper [4]. This is done by means of the following code:

```
Eq_5_12_1:=omega_plus(sigma1)-sqrt(2)*cos(psi):
Eq_5_12_1:=expand(Eq_5_12_1):
Eq_5_12_2:=omega_minus(sigma1)-sqrt(2)*sin(psi):
Eq_5_12_2:=expand(Eq_5_12_2):

Eq_5_12_1,Eq_5_12_2;
```

The following formula on page 16 of the paper [4] is immediate from (5.12):

$$\lambda = \tan\psi = \frac{\omega_-(\sigma_1)}{\omega_+(\sigma_1)}. \tag{7.2}$$

The equation (7.2) can be verified directly by means of the following code:

```
Eq_7_2:=tan(psi)-omega_minus(sigma1)/omega_plus(sigma1):
Eq_7_2:=expand(Eq_7_2):
Eq_7_2:=numer(normal(Eq_7_2)):
Eq_7_2:=rem(Eq_7_2,Eq_5,s1);
```

The formula (97) in paper [4] is just a notation. It is coded as follows:

```
k:=(m(alpha)+m(alpha1))/(1-m(alpha)*m(alpha1)):
k:=normal(k):
```

The formulas (98) on page 17 of the paper [4] are different. They should be verified. We verify them by means of the following code:

```
Eq_98_1:=sin(2*sigma1)-2*k/(1+k^2):
Eq_98_1:=expand(Eq_98_1):
Eq_98_1:=numer(normal(Eq_98_1)):
Eq_98_1:=rem(Eq_98_1,Eq_5,s1):

Eq_98_2:=cos(2*sigma1)-(1-k^2)/(1+k^2):
Eq_98_2:=expand(Eq_98_2):
Eq_98_2:=numer(normal(Eq_98_2)):
```

```
Eq_98_2:=rem(Eq_98_2,Eq_5,s1):

Eq_98_1,Eq_98_2;
```

The formula (99) is immediate from the formulas (6.9) and (6.10), which are the identities with respect to the argument α.

The formula (100) in [4] follows from (98) and (99). But, nevertheless, we verify this formula directly by means of the following code:

```
Eq_100:=tan(psi)-(1-k)/(1+k):
Eq_100:=expand(Eq_100):
Eq_100:=numer(normal(Eq_100)):
Eq_100:=rem(Eq_100,Eq_5,s1);
```

The formulas (101) and (102) in [4] are written by analogy to the formulas (97) and (100). These formulas are coded as follows:

```
bar_k:=(m(beta)+m(beta1))/(1-m(beta)*m(beta1)):
bar_lambda:=(1-bar_k)/(1+bar_k):
```

On page 17 of Walter Wyss's paper [4] we see the phrase: “Therefore the parameters u_1, β, $\bar{\lambda}$ also satisfy the general equations, however with the interchange of s_3 with s_4”. We do not know which general equations does he mean. But we suspect that he means the equations (82), (83), (84) on page 11 of his paper. Their analogs for the variables u_1, β, $\bar{\lambda}$ look like

$$\omega_-(\beta) - \bar{\lambda}\,\omega_+(\beta) = s_2\,s_4 + \bar{\lambda}\,s_2\,s_3, \tag{7.3}$$

$$Q\,(\omega_+(\beta) + \bar{\lambda}\,\omega_-(\beta)) = s_2\,s_4 - \bar{\lambda}\,s_2\,s_3, \tag{7.4}$$

$$2\,u_1\,(\omega_-(\beta) - \bar{\lambda}\,\omega_+(\beta)) + s_3 - \bar{\lambda}\,s_4 = s_2\,(\omega_+(\beta) + \bar{\lambda}\,\omega_-(\beta)). \tag{7.5}$$

Here Q is given by the formula (5.10) and $\bar{\lambda}$ is given by the formula (102) in [4]. The formulas (7.3), (7.4), (7.5) are verified as follows:

```
tan(beta/2):=m(beta):
cot(beta/2):=1/m(beta):
sin(beta/2):=m(beta)*cos(beta/2):
sin(beta):=normal(2*m(beta)/(1+m(beta)^2)):
cos(beta):=normal((1-m(beta)^2)/(1+m(beta)^2)):

tan(beta1/2):=m(beta1):
cot(beta1/2):=1/m(beta1):
sin(beta1/2):=m(beta1)*cos(beta1/2):
sin(beta1):=normal(2*m(beta1)/(1+m(beta1)^2)):
cos(beta1):=normal((1-m(beta1)^2)/(1+m(beta1)^2)):

Eq_7_3:=-omega_minus(beta)+bar_lambda*omega_plus(beta)
        +s2*s4+bar_lambda*s2*s3:
Eq_7_3:=expand(Eq_7_3):
Eq_7_3:=numer(normal(Eq_7_3)):
Eq_7_3:=rem(Eq_7_3,Eq_5,s1):
```

```
Q:=s3*s4:
Eq_7_4:=-Q*(omega_plus(beta)+bar_lambda*omega_minus(beta))
        +s2*s4-bar_lambda*s2*s3:
Eq_7_4:=expand(Eq_7_4):
Eq_7_4:=numer(normal(Eq_7_4)):
Eq_7_4:=rem(Eq_7_4,Eq_5,s1):

Eq_7_5:=-2*u1*(omega_minus(beta)-bar_lambda*omega_plus(beta))
        -s3+bar_lambda*s4+s2*(omega_plus(beta)
        +bar_lambda*omega_minus(beta)):
Eq_7_5:=expand(Eq_7_5):
Eq_7_5:=numer(normal(Eq_7_5)):
Eq_7_5:=rem(Eq_7_5,Eq_5,s1):

Eq_7_3,Eq_7_4,Eq_7_5;
```

In the beginning of section 7 of his paper [4] Walter Wyss introduces some new notations replacing previous ones (see (103), (104) (105), (106) on page 18):

$$\sigma=\frac{\alpha+\beta}{2}, \qquad \delta=\frac{\alpha-\beta}{2}. \tag{7.6}$$

$$\sigma_1=\frac{\alpha_1+\beta_1}{2}, \qquad \delta_1=\frac{\alpha_1-\beta_1}{2}. \tag{7.7}$$

The notations (7.7) replace the notations introduced on page 4 of the paper [4]. The second notation (7.6) replaces the notation used on page 9 of the paper [4]. The notations (7.6) do coincide with the notations (4.9). As a whole, the notations (7.6) and (7.7) are coded as follows:

```
sigma:=alpha/2+beta/2:
delta:=alpha/2-beta/2:
sigma1:=alpha1/2+beta1/2:
delta1:=alpha1/2-beta1/2:
```

The next are the formulas (107), (108). The formulas (107) are verified as follows:

```
Eq_107_1:=u2-u1*tan(sigma):
Eq_107_1:=expand(Eq_107_1):
Eq_107_1:=numer(normal(Eq_107_1)):
Eq_107_1:=rem(Eq_107_1,Eq_5,s1):

Eq_107_2:=u3-u1*omega_plus(delta)/cos(sigma):
Eq_107_2:=expand(Eq_107_2):
Eq_107_2:=numer(normal(Eq_107_2)):
Eq_107_2:=rem(Eq_107_2,Eq_5,s1):

Eq_107_3:=u4-u1*omega_minus(delta)/cos(sigma):
Eq_107_3:=expand(Eq_107_3):
Eq_107_3:=numer(normal(Eq_107_3)):
Eq_107_3:=rem(Eq_107_3,Eq_5,s1):

Eq_107_1,Eq_107_2,Eq_107_3;
```

The formulas (108) in [4] are verified similarly:

```
Eq_108_1:=v2-u1*tan(sigma1):
Eq_108_1:=expand(Eq_108_1):
Eq_108_1:=numer(normal(Eq_108_1)):
Eq_108_1:=rem(Eq_108_1,Eq_5,s1):

Eq_108_2:=v3-u1*omega_plus(delta1)/cos(sigma1):
Eq_108_2:=expand(Eq_108_2):
Eq_108_2:=numer(normal(Eq_108_2)):
Eq_108_2:=rem(Eq_108_2,Eq_5,s1):

Eq_108_3:=v4-u1*omega_minus(delta1)/cos(sigma1):
Eq_108_3:=expand(Eq_108_3):
Eq_108_3:=numer(normal(Eq_108_3)):
Eq_108_3:=rem(Eq_108_3,Eq_5,s1):

Eq_108_1,Eq_108_2,Eq_108_3;
```

The formulas (109) and (110) are converse to (107) and (108). Nevertheless, we verify them directly with the use of the following code:

```
Eq_109_1:=tan(sigma)-u2/u1:
Eq_109_1:=expand(Eq_109_1):
Eq_109_1:=numer(normal(Eq_109_1)):
Eq_109_1:=rem(Eq_109_1,Eq_5,s1):

Eq_109_2:=tan(delta)-(u3-u4)/(u3+u4):
Eq_109_2:=expand(Eq_109_2):
Eq_109_2:=numer(normal(Eq_109_2)):
Eq_109_2:=rem(Eq_109_2,Eq_5,s1):

Eq_110_1:=tan(sigma1)-v2/u1:
Eq_110_1:=expand(Eq_110_1):
Eq_110_1:=numer(normal(Eq_110_1)):
Eq_110_1:=rem(Eq_110_1,Eq_5,s1):

Eq_110_2:=tan(delta1)-(v3-v4)/(v3+v4):
Eq_110_2:=expand(Eq_110_2):
Eq_110_2:=numer(normal(Eq_110_2)):
Eq_110_2:=rem(Eq_110_2,Eq_5,s1):

Eq_109_1,Eq_109_2,Eq_110_1,Eq_110_2;
```

The next are the formulas (111), (112), and (113). In the case of the first formula (111) we use the following code in order to verify it:

```
Eq_111_1:=u1^2*(tan(sigma1)^2-tan(sigma)^2)-1:
Eq_111_1:=expand(Eq_111_1):
Eq_111_1:=numer(normal(Eq_111_1)):
Eq_111_1:=rem(Eq_111_1,Eq_5,s1);
```

The second formula (111) is more complicated for computer handling:

```
psi_phi_simplify:=proc(A) local AA: global m:
  AA:=subs(cos(alpha/2)^2=1/(1+m(alpha)^2),A):
  AA:=subs(cos(alpha1/2)^2=1/(1+m(alpha1)^2),AA):
  AA:=subs(cos(beta/2)^2=1/(1+m(beta)^2),AA):
  AA:=subs(cos(beta1/2)^2=1/(1+m(beta1)^2),AA):
  return AA:
end proc:

uuu:=psi_phi_simplify(expand(cos(sigma)^2)):
uuu1:=psi_phi_simplify(expand(cos(sigma1)^2)):
Eq_111_2:=u1^2*(1/uuu1-1/uuu)-1:
Eq_111_2:=expand(Eq_111_2):
Eq_111_2:=numer(normal(Eq_111_2)):
Eq_111_2:=rem(Eq_111_2,Eq_5,s1);
```

Now let's proceed to the formulas (112). For them we use the following code:

```
vvv:=psi_phi_simplify(expand(omega_plus(delta)^2)):
vvv1:=psi_phi_simplify(expand(omega_plus(delta1)^2)):
Eq_112_1:=u1^2*(vvv1/uuu1-vvv/uuu)-1:
Eq_112_1:=numer(normal(Eq_112_1)):
Eq_112_1:=rem(Eq_112_1,Eq_5,s1);

vvv:=expand(1+sin(2*delta)):
vvv1:=expand(1+sin(2*delta1)):
Eq_112_2:=u1^2*(vvv1/uuu1-vvv/uuu)-1:
Eq_112_2:=numer(normal(Eq_112_2)):
Eq_112_2:=rem(Eq_112_2,Eq_5,s1);
```

The formulas (113) are similar to (112). For them we use the following code:

```
vvv:=psi_phi_simplify(expand(omega_minus(delta)^2)):
vvv1:=psi_phi_simplify(expand(omega_minus(delta1)^2)):
Eq_113_1:=u1^2*(vvv1/uuu1-vvv/uuu)-1:
Eq_113_1:=numer(normal(Eq_113_1)):
Eq_113_1:=rem(Eq_113_1,Eq_5,s1);

vvv:=expand(1-sin(2*delta)):
vvv1:=expand(1-sin(2*delta1)):
Eq_113_2:=u1^2*(vvv1/uuu1-vvv/uuu)-1:
Eq_113_2:=numer(normal(Eq_113_2)):
Eq_113_2:=rem(Eq_113_2,Eq_5,s1);
```

The next are the formulas (114) and (115). They are written as follows:

$$\tan^2\sigma_1 - \tan^2\sigma = \tan^2\psi, \tag{7.8}$$

$$\frac{sin(2\,\delta)}{\cos^2\sigma} = \frac{sin(2\,\delta_1)}{\cos^2\sigma_1}. \tag{7.9}$$

Attention! At the bottom of page 18 in his paper [4] Walter Wyss writes: "We rename $\psi_1 = \psi$". This means that in (7.8) ψ does not coincide with (5.13). It coincides with ψ_1 in (3.8). From (3.8) we derive

$$\tan\psi = \tan\psi_1 = \frac{1}{u_1}. \tag{7.10}$$

We use (7.10) in writing code for verifying the formula (7.8):

```
Eq_7_8:=tan(sigma1)^2-tan(sigma)^2-1/u1^2:
Eq_7_8:=expand(Eq_7_8):
Eq_7_8:=numer(normal(Eq_7_8)):
Eq_7_8:=rem(Eq_7_8,Eq_5,s1);
```

In the case of the formula (7.9) we use the following code:

```
vvv:=expand(sin(2*delta)):
vvv1:=expand(sin(2*delta1)):
Eq_7_9:=vvv1/uuu1-vvv/uuu:
Eq_7_9:=numer(normal(Eq_7_9)):
Eq_7_9:=rem(Eq_7_9,Eq_5,s1);
```

The formulas (116), (117), (118), and (119) in Walter Wyss's paper [4] are just notations. They are coded as follows:

```
M:=normal(expand(tan(sigma))):
M1:=normal(expand(tan(sigma1))):
N:=normal(expand(tan(delta))):
N1:=normal(expand(tan(delta1))):
```

The formulas (120) and (121) are trivial. Nevertheless, we can verify them:

```
Eq_120:=sin(alpha)/cos(alpha)-(M+N)/(1-M*N):
Eq_120:=numer(normal(Eq_120)):
Eq_120:=rem(Eq_120,Eq_5,s1):

Eq_121:=sin(alpha1)/cos(alpha1)-(M1+N1)/(1-M1*N1):
Eq_121:=numer(normal(Eq_121)):
Eq_121:=rem(Eq_121,Eq_5,s1):

Eq_120, Eq_121;
```

The formulas (122) and (123) follow from (120) and (121). But we verify them too:

```
Eq_122:=N-(sin(alpha)/cos(alpha)-M)
       /(1+M*sin(alpha)/cos(alpha)):
Eq_122:=numer(normal(Eq_122)):
Eq_122:=rem(Eq_122,Eq_5,s1):
Eq_123:=N1-(sin(alpha1)/cos(alpha1)-M1)
       /(1+M1*sin(alpha1)/cos(alpha1)):
Eq_123:=numer(normal(Eq_123)):
Eq_123:=rem(Eq_123,Eq_5,s1):

Eq_122, Eq_123;
```

Taking into account (7.10), we can verify the formula (124) as follows:

```
Eq_124:=M1^2-M^2-1/u1^2:
Eq_124:=numer(normal(Eq_124)):
Eq_124:=rem(Eq_124,Eq_5,s1);
```

Then we verify the formula (125) by means of the following code:

```
Eq_125:=2*N*(1+M^2)/(1+N^2)-2*N1*(1+M1^2)/(1+N1^2):
Eq_125:=numer(normal(Eq_125)):
Eq_125:=rem(Eq_125,Eq_5,s1);
```

The formula (126) is verified similarly. We use the following code for it:

```
vvv:=normal(expand(sin(2*alpha))):
vvv1:=normal(expand(sin(2*alpha1))):
uuu:=normal(expand(cos(2*alpha))):
uuu1:=normal(expand(cos(2*alpha1))):
Eq_126:=(1-M^2)*vvv-2*M*uuu-(1-M1^2)*vvv1+2*M1*uuu1:
Eq_126:=numer(normal(Eq_126)):
Eq_126:=rem(Eq_126,Eq_5,s1);
```

The formulas (127), (128), (129) are elementary. They follow from the notations (7.6) and (7.7) complemented with the notation (116). So we proceed to the formulas (130) and (131). The formula (130) is verified with the use of the code

```
vvv:=factor(expand(omega_plus(delta))):
uuu:=factor(expand(cos(sigma))):
www:=omega_plus(alpha)+omega_minus(beta):
Eq_130:=normal(vvv/uuu)-1/2*(1+M^2)*www:
Eq_130:=numer(normal(Eq_130)):
Eq_130:=rem(Eq_130,Eq_5,s1);
```

The formula (131) is similar. In this case we use the following code:

```
vvv:=factor(expand(omega_minus(delta))):
uuu:=factor(expand(cos(sigma))):
www:=omega_minus(alpha)+omega_plus(beta):
Eq_131:=normal(vvv/uuu)-1/2*(1+M^2)*www:
Eq_131:=numer(normal(Eq_131)):
Eq_131:=rem(Eq_131,Eq_5,s1);
```

The formulas (132) and (134) are elementary. Therefore we proceed to the formulas (133) and (135). They are verified with the following code:

```
Eq_133:=cos(beta)-(1-M^2)/(1+M^2)*cos(alpha)
       -2*M/(1+M^2)*sin(alpha):
Eq_133:=numer(normal(Eq_133)):
Eq_133:=rem(Eq_133,Eq_5,s1);

Eq_135:=sin(beta)-2*M/(1+M^2)*cos(alpha)
       +(1-M^2)/(1+M^2)*sin(alpha):
Eq_135:=numer(normal(Eq_135)):
Eq_135:=rem(Eq_135,Eq_5,s1);
```

The next are the formulas (136) and (137). The first of these two formulas is verified by means of the following code:

```
Eq_136:=omega_plus(beta)-(1-M^2)/(1+M^2)*omega_minus(alpha)
       -2*M/(1+M^2)*omega_plus(alpha):
Eq_136:=numer(normal(Eq_136)):
Eq_136:=rem(Eq_136,Eq_5,s1);
```

The second one is similar. It is verified with the use of the code

```
Eq_137:=omega_minus(beta)-(1-M^2)/(1+M^2)*omega_plus(alpha)
       +2*M/(1+M^2)*omega_minus(alpha):
Eq_137:=numer(normal(Eq_137)):
Eq_137:=rem(Eq_137,Eq_5,s1);
```

The formulas (138) and (139) are similar to the previous two formulas (136) and (137). They are verified as follows:

```
Eq_138:=omega_plus(alpha)+omega_minus(beta)
       -2/(1+M^2)*(omega_plus(alpha)-M*omega_minus(alpha)):
Eq_138:=numer(normal(Eq_138)):
Eq_138:=rem(Eq_138,Eq_5,s1);

Eq_139:=omega_minus(alpha)+omega_plus(beta)
       -2/(1+M^2)*(omega_minus(alpha)+M*omega_plus(alpha)):
Eq_139:=numer(normal(Eq_139)):
Eq_139:=rem(Eq_139,Eq_5,s1);
```

The next are the formulas (140), (141), (142). They look more simple than the previous ones. We verify these formulas as follows:

```
Eq_140:=u2-u1*M:
Eq_140:=numer(normal(Eq_140)):
Eq_140:=rem(Eq_140,Eq_5,s1);

Eq_141:=u3-u1*(omega_plus(alpha)-M*omega_minus(alpha)):
Eq_141:=numer(normal(Eq_141)):
Eq_141:=rem(Eq_141,Eq_5,s1);

Eq_142:=u4-u1*(omega_minus(alpha)+M*omega_plus(alpha)):
Eq_142:=numer(normal(Eq_142)):
Eq_142:=rem(Eq_142,Eq_5,s1);
```

The formulas (143), (144), (145) are similar to the previous formulas (140), (141), (142). They are verified as follows:

```
Eq_143:=v2-u1*M1:
Eq_143:=numer(normal(Eq_143)):
Eq_143:=rem(Eq_143,Eq_5,s1);

Eq_144:=v3-u1*(omega_plus(alpha1)-M1*omega_minus(alpha1)):
Eq_144:=numer(normal(Eq_144)):
Eq_144:=rem(Eq_144,Eq_5,s1);
```

```
Eq_145:=v4-u1*(omega_minus(alpha1)+M1*omega_plus(alpha1)):
Eq_145:=numer(normal(Eq_145)):
Eq_145:=rem(Eq_145,Eq_5,s1);
```

Exchanging u_3 and u_4, i. e. applying the second transformation (6.30), Walter Wyss derived two more formulas (146) and (147). They are verified as follows:

```
Eq_146:=u3-u1*(omega_minus(beta)+M*omega_plus(beta)):
Eq_146:=numer(normal(Eq_146)):
Eq_146:=rem(Eq_146,Eq_5,s1);

Eq_147:=u4-u1*(omega_plus(beta)-M*omega_minus(beta)):
Eq_147:=numer(normal(Eq_147)):
Eq_147:=rem(Eq_147,Eq_5,s1);
```

The **cuboid limit** or, being more precise, the **rectangular cuboid limit** is the case where the parallelogram ABFE in Fig. 1.1 turns to a rectangle. In this case two its diagonals become equal to each other:

$$|AF| = |EB|. \tag{7.11}$$

Comparing (7.11) with our notations (3.3), we find

$$u_3 = u_4. \tag{7.12}$$

Applying (3.6) to (7.12), we derive the equation

$$\frac{1-s_3^2}{2\,s_3} = \frac{1-s_4^2}{2\,s_4}. \tag{7.13}$$

The equation (7.13) has two solutions

$$s_3 = s_4, \qquad\qquad s_3 = \frac{1}{s_4}.$$

But due to the inequalities (3.7) only the first solution is suitable for us:

$$s_3 = s_4. \tag{7.14}$$

Thus the **rectangular cuboid limit** is the case where either of the two equivalent equalities (7.12) or (7.14) is fulfilled.

On page 21 of his paper [4] Walter Wyss writes that in the rectangular cuboid limit the following equalities are fulfilled:

$$N = 0, \qquad\qquad N_1 = 0. \tag{7.15}$$

The equalities (7.15) are easily verified by means of the following code:

```
subs(s4=s3,N),
subs(s4=s3,N1);
```

On page 23 of his paper [4] Walter Wyss writes that the cuboid limit is given by

$$\alpha = \beta, \qquad\qquad \alpha_1 = \beta_1. \tag{7.16}$$

One can easily verify that the equality (7.14) implies both equalities (7.16). This is done with the use of the following code:

```
normal(subs(s4=s3,cos(alpha)-cos(beta))),
normal(subs(s4=s3,sin(alpha)-sin(beta))),
normal(subs(s4=s3,cos(alpha1)-cos(beta1))),
normal(subs(s4=s3,sin(alpha1)-sin(beta1)));
```

Apart from (7.15) and (7.16) there are two more equalities:

$$M = \tan(\alpha), \qquad\qquad M_1 = \tan(\alpha_1). \tag{7.17}$$

One can verify that the equality (7.14) implies both equalities (7.17). In this case we do it with the use of the following code:

```
subs(s4=s3,normal(M-sin(alpha)/cos(alpha))),
subs(s4=s3,normal(M1-sin(alpha1)/cos(alpha1)));
```

Note that the rectangular cuboid limit is not a singular case though some Walter Wiss's formulas are not applicable to it.

8. A special example.

On page 21 of his paper [4] Walter Wyss considers a special case in the form of an example. This special case is defined by the equality

$$\alpha + \alpha_1 = \frac{\pi}{2}. \tag{8.1}$$

On pages 3 and 4 of his paper Walter Wyss writes that α, α_1, and α_2 are Heron angles in the first quadrant, i. e. they obey the inequalities (5.9). Their generators m, m_1, and m_2 are given by the formulas (25), (26), and (27) on page 3 of the paper [4]. Comparing the formulas (25) and (26) with the formulas (95) on page 16, we conclude that the angles α and α_1 in (8.1) are the same angles which are used in sections 4 and 5 of Walter Wyss's paper [4].

Dividing the equality (8.1) by 2, we derive

$$\frac{\alpha + \alpha_1}{2} = \frac{\pi}{4}. \tag{8.2}$$

Substituting (8.2) into (5.13), we find that

$$\psi = \frac{\pi}{4} - \frac{\alpha + \alpha_1}{2} = 0. \tag{8.3}$$

The equality (8.3) implies the equality

$$\tan\psi = 0. \tag{8.4}$$

Conversely, applying the inequalities (5.9) to (5.13), we derive the inequality

$$-\frac{\pi}{2} < \psi < \frac{\pi}{2}. \tag{8.5}$$

The tangent function is a monotonic increasing function within the interval (8.5). It vanishes exactly once at the point $\psi = 0$. This means that the equality (8.4) implies backward the equality (8.3) and then (8.2) and (8.1), i. e. the equalities (8.1) and (8.4) are equivalent.

Now let's recall that the equality (8.4) in the form of $\lambda = \tan\psi$ and $\lambda = 0$ was used by Walter Wyss in order to construct a special solution of the slanted cuboid equations (see (81) on page 10 and (89) on page 11 of [4]). Thus the conclusion.

Theorem 8.1. *The special solution given by the condition* (6.1) *and the special example defined by the condition* (8.1) *do coincide.*

Attention! Due to renaming variables $\psi_1 = \psi$ at the bottom of page 18 of Walter Wyss's paper [4] the variable ψ in (8.3), (8.4), (8.5) does not coincide with this variable on pages 19, 20, 21 and so on.

On page 21 of the paper [4] we see three formulas, which are not numbered there:

$$u_1 = \cot\psi, \tag{8.6}$$

$$M = \frac{1}{4}\,\bigl(\tan^2\psi\,\tan(2\,\alpha) - 4\,\cot(2\,\alpha)\bigr), \tag{8.7}$$

The formula (8.7) is derived from the formulas (124) and (1.26) in [4] upon expressing α_1 through α by means of (8.1). Indeed, we can writhe the code

```
restart:
Eq_124:=M1^2-M^2-tan(psi)^2:
Eq_126:=(1-M^2)*sin(2*alpha)-2*M*cos(2*alpha)
       -(1-M1^2)*sin(2*alpha1)+2*M1*cos(2*alpha1):
Eq_126:=subs(alpha1=Pi/2-alpha,Eq_126):
sss:=solve({Eq_124,Eq_126},{M,M1}):
assign(sss):
'M'=M;
```

Denoting $s_1 = s$ and applying the first formula (3.6) with $k = 1$, we get

$$u_1 = \frac{1 - s^2}{2\,s}. \tag{8.8}$$

This formula (8.8) coincides with the formula (6.18). The formulas (3.6) with $k = 2$, $k = 3$, and $k = 4$ are written as follows:

$$u_2 = \frac{1 - s_2^2}{2\,s_2}, \qquad u_3 = \frac{1 - s_3^2}{2\,s_3}, \qquad u_4 = \frac{1 - s_4^2}{2\,s_4}. \tag{8.9}$$

Now let's recall that α is a Heron angle with the generator m (see Definition 4.1 and the formula (25) on page 3 of the paper [4]). Definition 4.1 means that the formula (6.12) holds for the angle α. From (6.12) we derive the formulas (6.13), (6.14), (6.15), and (6.16). From (6.15) and (6.16) we derive

$$\tan(2\,\alpha) = \frac{4\,m\,\bigl(1 - m^2\bigr)}{\bigl(1 - m^2\bigr)^2 - 4\,m^2}, \qquad \cot(2\,\alpha) = \frac{\bigl(1 - m^2\bigr)^2 - 4\,m^2}{4\,m\,\bigl(1 - m^2\bigr)}. \tag{8.10}$$

From (6.13) and (6.14), applying the formulas (4.10), we derive

$$\omega_+(\alpha) = \frac{1 - m^2 + 2\,m}{1 + m^2}, \qquad \omega_-(\alpha) = \frac{1 - m^2 - 2\,m}{1 + m^2}. \tag{8.11}$$

Finally, substituting (8.8) into the formula (8.6), we derive

$$\tan\psi = \frac{2\,s}{1-s^2}. \tag{8.12}$$

Substituting (8.12) and (8.10) into (8.7) we derive some definite formula expressing M through s and m. This action is performed by the following code:

```
u1:=(1/s-s)/2:
M:=subs(tan(psi)=1/u1,M):
M:=subs(tan(2*alpha)=4*m*(1-m^2)/((1-m^2)^2-4*m^2),M):
M:=subs(cot(2*alpha)=((1-m^2)^2-4*m^2)/4/m/(1-m^2),M):
```

Then we use the formulas (140), (141), (142) from [4]. Applying the formulas (8.11) to them, we derive some definite formulas expressing u_2, u_3, and u_4 through s and m. This action is performed by means of the following code:

```
u2:=u1*M:
u3:=u1*(omega_plus(alpha)-M*omega_minus(alpha)):
u4:=u1*(omega_minus(alpha)+M*omega_plus(alpha)):

u3:=subs(omega_plus(alpha)=(1-m^2+2*m)/(1+m^2),u3):
u3:=subs(omega_minus(alpha)=(1-m^2-2*m)/(1+m^2),u3):
u4:=subs(omega_plus(alpha)=(1-m^2+2*m)/(1+m^2),u4):
u4:=subs(omega_minus(alpha)=(1-m^2-2*m)/(1+m^2),u4):
```

It turns out that the same formulas expressing u_2, u_3, and u_4 through s and m can be obtained by substituting (6.19), (6.22), and (6.23) into the formulas (8.9). This fact confirms once more that the above observation formulated in Theorem 8.1 is valid. We prove this fact by means of the following code:

```
theta:=(1-s^2)*((1-m^2)^2-4*m^2)/4/m/s/(1-m^2):
eta:=4*m*s*(1-m^2)/(1-s^2)/(1+m^2)/(1-m^2+2*m):
zeta:=(1-s^2)*(1+m^2)*(1-m^2-2*m)/4/m/s/(1-m^2):

Eq_u2:=u2-subs(s2=theta,(1-s2^2)/2/s2):
Eq_u2:=numer(normal(Eq_u2)):

Eq_u3:=u3-subs(s3=eta,(1-s3^2)/2/s3):
Eq_u3:=numer(normal(Eq_u3)):

Eq_u4:=u4-subs(s4=zeta,(1-s4^2)/2/s4):
Eq_u4:=numer(normal(Eq_u4)):

Eq_u2,Eq_u3,Eq_u4:
```

Using (7.15), on pages 21 and 22 of his paper [4] Walter Wyss proves that there are no rectangular rational cuboids within his special example defined by the condition (8.1). Due to Theorem 8.1 we see that the same result is proved in the form of Theorem 2 on pages 12 and 13 of his paper [4].

Walter Wyss's Theorem 2 is valid. It means that there are no rectangular perfect cuboids within two-dimensional subvarieties Γ^1_2, Γ^2_2, Γ^3_2, Γ^4_2 given by the formulas

(6.27) and (6.31). Neither one of the two-dimensional subvarieties nor their union covers the three-dimensional algebraic variety Γ_{3++} given by Theorem 3.1. Therefore rectangular perfect cuboids are still possible.

9. Back to the general case.

On page 22 of his paper [4] and in Appendix F of this paper Walter Wyss studies the equation (126). This equation is written as follows:

$$\begin{aligned}(1-M^2)\,\sin(2\,\alpha)-2\,M\,\cos(2\,\alpha)=\\ =(1-M_1^2)\,\sin(2\,\alpha_1)-2\,M_1\,\cos(2\,\alpha_1).\end{aligned}\tag{9.1}$$

Denoting through $-4\,D$ the value of each side of the equation (9.1), Walter Wyss splits it into two separate equations:

$$(M^2-1)\,\sin(2\,\alpha)+2\,M\,\cos(2\,\alpha)=4\,D,\tag{9.2}$$

$$(M_1^2-1)\,\sin(2\,\alpha_1)+2\,M_1\,\cos(2\,\alpha_1)=4\,D.\tag{9.3}$$

The equation (9.2) is a quadratic equation with respect to M. Walter Wyss denotes through Δ^2 the quoter of its discriminant:

$$\Delta^2=\cos^2(2\,\alpha)+\sin^2(2\,\alpha)+4\,D\,\sin(2\,\alpha).\tag{9.4}$$

The equation (9.4) can be derived by means of the following code:

```
restart:
Eq_9_2:=(M^2-1)*sin(2*alpha)+2*M*cos(2*alpha)-4*D:
Eq_9_4:=Delta^2-discrim(Eq_9_2,M)/4;
```

The equation (9.4) is simplified with the use of the well-known trigonometric identity $\cos^2(2\,\alpha)+\sin^2(2\,\alpha)=1$. As a result we get

$$\Delta^2=1+4\,D\,\sin(2\,\alpha).\tag{9.5}$$

In terms of the machine codes this transformation is performed as follows:

```
Eq_9_5:=subs(cos(2*alpha)^2=1-sin(2*alpha)^2,Eq_9_4);
```

The solution of the equation (9.2) for M is written as

$$M=\frac{\pm\,\Delta-\cos(2\,\alpha)}{\sin(2\,\alpha)}.\tag{9.6}$$

The formula (9.6) is obtained by means of the following code:

```
Eq_9_2:=subs(D=solve(Eq_9_5,D),Eq_9_2):
sss:=solve(Eq_9_2,M):
M_plus:=simplify(subs(cos(2*alpha)^2=1-sin(2*alpha)^2,sss[1]))
         assuming Delta::positive;
M_minus:=simplify(subs(cos(2*alpha)^2=1-sin(2*alpha)^2,sss[2]))
         assuming Delta::positive;
```

Then Walter Wyss considers the equation (9.5) and writes it as follows:

$$(\Delta - 1)(\Delta + 1) = 4\,D\,\sin(2\,\alpha). \tag{9.7}$$

Due to (5.9) we know that $\sin(2\,\alpha) \neq 0$. Assume additionally that

$$D \neq 0. \tag{9.8}$$

Under the assumption (9.8) we have

$$\Delta - 1 \neq 0, \qquad\qquad \Delta + 1 \neq 0. \tag{9.9}$$

Applying (9.8) and (9.9) to (9.7), we can write it as follows:

$$\frac{\Delta - 1}{4\,D} = \frac{\sin(2\,\alpha)}{\Delta + 1} \tag{9.10}$$

The quotients in both sides of (9.10) are nonzero. Let's denote their values through $-r^{-1}/2$, where $r \neq 0$. As a result we split (9.10) into two equations:

$$\frac{\Delta - 1}{4\,D} = -\frac{1}{2\,r}, \qquad\qquad \frac{\sin(2\,\alpha)}{\Delta + 1} = -\frac{1}{2\,r}. \tag{9.11}$$

The equations (9.11) can be written as linear equations with respect to D and Δ:

$$\Delta - 1 = -\frac{2\,D}{r}, \qquad\qquad \sin(2\,\alpha) = -\frac{\Delta + 1}{2\,r}. \tag{9.12}$$

Resolving the equations (9.12), we get

$$D = \sin(2\,\alpha)\,r^2 + r, \qquad\qquad \Delta = -2\,\sin(2\,\alpha)\,r - 1. \tag{9.13}$$

The first formula (9.13) coincides with the second formula (150) on page 22 of Walter Wyss's paper [4]. These two formulas are derived using the code

```
Eq_9_11_1:=(Delta-1)/4/D=-1/2/r;
Eq_9_11_2:=sin(2*alpha)/(Delta+1)=-1/2/r;
sss:=solve({Eq_9_11_1,Eq_9_11_2},{D,Delta});
```

Substituting the second formula (9.13) into (9.6), we derive two solutions for M:

$$\begin{aligned} M_{+} &= -2\,r - \cot(\alpha), \\ M_{-} &= 2\,r + \tan(\alpha). \end{aligned} \tag{9.14}$$

The formulas (9.14) are derived by means of the following code:

```
unprotect(D):
assign(sss):
M_plus:=expand(M_plus);
M_minus:=normal(expand(M_minus)):
```

```
M_minus:=subs(cos(alpha)^2=1-sin(alpha)^2,M_minus):
M_minus:=expand(M_minus);
```

The formulas (9.14) can be verified by substituting them back to the equation (9.2) along with the first formula (9.13). This is done by means of the code

```
simplify(expand(subs(M=M_plus,Eq_9_2)),trig),
simplify(expand(subs(M=M_minus,Eq_9_2)),trig);
```

Walter Wyss presents only the second formula (9.14) for M on page 22 of his paper [4] and actually he does not exploit it.

The equations (9.2) and (9.3) are similar to each other. Using this analogy, we can write the following formulas similar to (9.13):

$$D = \sin(2\,\alpha_1)\,r_1^2 + r_1, \qquad\qquad \Delta_1 = -2\,\sin(2\,\alpha_1)\,r_1 - 1. \tag{9.15}$$

Though being different, the equations (9.2) and (9.3) share the same value of D. Therefore from (9.13) and (9.15) we derive the equation

$$\sin(2\,\alpha)\,r^2 + r = \sin(2\,\alpha_1)\,r_1^2 + r_1. \tag{9.16}$$

Factoring both sides of (9.16), we get the equation

$$r\,(r\,\sin(2\,\alpha) + 1) = r_1\,(r_1\,\sin(2\,\alpha_1) + 1). \tag{9.17}$$

From the inequalities (5.9) we conclude that $\sin(2\,\alpha) \neq 0$ and $\sin(2\,\alpha_1) \neq 0$. Moreover, from the formula (9.11) we derive $r \neq 0$. Similarly $r_1 \neq 0$. Both sides of (9.17) are equal to D, where $D \neq 0$ (see (9.8)). Hence

$$r\,\sin(2\,\alpha) + 1 \neq 0, \qquad\qquad r_1\,\sin(2\,\alpha_1) + 1 \neq 0. \tag{9.18}$$

Due to (9.18) and the inequalities preceding it, the equation (9.17) is written as

$$\frac{r\,\sin(2\,\alpha) + 1}{r_1} = \frac{r_1\,\sin(2\,\alpha_1) + 1}{r} \tag{9.19}$$

Both sides of (9.19) are nonzero. Denoting their value through $1/f$, we split the equation (9.19) into two separate equations:

$$\frac{r_1\,\sin(2\,\alpha_1) + 1}{r} = \frac{1}{f}, \qquad\qquad \frac{r\,\sin(2\,\alpha) + 1}{r_1} = \frac{1}{f}. \tag{9.20}$$

The equations (9.20) can be written as two linear equations for r and r_1:

$$\frac{r}{f} - \sin(2\,\alpha_1)\,r_1 = 1, \qquad\qquad \sin(2\,\alpha)\,r - \frac{r_1}{f} = -1. \tag{9.21}$$

There are two cases for the equations (9.21) — the regular case and the singular case. In the regular case we have the inequality

$$\sin(2\,\alpha)\,\sin(2\,\alpha_1) \neq \frac{1}{f^2}. \tag{9.22}$$

In this case the equations (9.21) are uniquely solvable. Their solution is given by

$$\begin{aligned} r &= \frac{f\,(f\,\sin(2\,\alpha_1)+1)}{1-f^2\,\sin(2\,\alpha_1)\,\sin(2\,\alpha)},\\ r_1 &= \frac{f\,(f\,\sin(2\,\alpha)+1)}{1-f^2\,\sin(2\,\alpha_1)\,\sin(2\,\alpha)}. \end{aligned} \tag{9.23}$$

The formulas (9.23) can be derived by means of the following code:

```
Eq_9_20_1:=(r1*sin(2*alpha1)+1)/r-1/f;
Eq_9_20_2:=(r*sin(2*alpha)+1)/r1-1/f;
sss:=solve({Eq_9_20_1,Eq_9_20_2},{r,r1});
```

The formulas (9.23) are written on page 22 of Walter Wyss's paper [4] (see (153) and (154)). They are consistent. Their denominators are nonzero due to (9.22). Substituting them back to the equations (9.13), we derive

$$D = \frac{f\,(f\,\sin(2\,\alpha_1)+1)\,(f\,\sin(2\,\alpha)+1)}{(f^2\,\sin(2\,\alpha_1)\,\sin(2\,\alpha)-1)^2}, \tag{9.24}$$

$$\Delta = \frac{f^2\,\sin(2\,\alpha_1)\,\sin(2\,\alpha)+2\,f\,\sin(2\,\alpha)+1}{f^2\,\sin(2\,\alpha_1)\,\sin(2\,\alpha)-1}. \tag{9.25}$$

Substituting (9.23) into the equations (9.15), we derive the same formula (9.24) for the parameter D and the following formula for Δ_1:

$$\Delta_1 = \frac{f^2\,\sin(2\,\alpha_1)\,\sin(2\,\alpha)+2\,f\,\sin(2\,\alpha_1)+1}{f^2\,\sin(2\,\alpha_1)\,\sin(2\,\alpha)-1}. \tag{9.26}$$

The formulas (9.24), (9.25), (9.26) are computed by means of the following code:

```
assign(sss):
D:=normal(D);
Delta:=normal(Delta);

D-normal(sin(2*alpha1)*r1^2+r1);
Delta1:=normal(-2*sin(2*alpha1)*r1-1);
```

Due to (9.14) we have two options for M. Similarly, we have two options for M_1:

$$\begin{aligned} M_{1+} &= -2\,r_1-\cot(\alpha_1),\\ M_{1-} &= 2\,r_1+\tan(\alpha_1). \end{aligned} \tag{9.27}$$

Substituting (9.23) into (9.14), we get the following two formulas:

$$M_+ = -\frac{2\,f\,(f\,\sin(2\,\alpha_1)+1)}{1-f^2\,\sin(2\,\alpha_1)\,\sin(2\,\alpha)}-\frac{\cos(\alpha)}{\sin(\alpha)}, \tag{9.28}$$

$$M_- = \frac{2\,f\,(f\,\sin(2\,\alpha_1)+1)}{1-f^2\,\sin(2\,\alpha_1)\,\sin(2\,\alpha)}+\frac{\sin(\alpha)}{\cos(\alpha)}. \tag{9.29}$$

Similarly, substituting (9.23) into (9.27), we get the other two formulas

$$M_{1+}=-\frac{2\,f\,(f\,\sin(2\,\alpha)+1)}{1-f^2\,\sin(2\,\alpha_1)\,\sin(2\,\alpha)}-\frac{\cos(\alpha_1)}{\sin(\alpha_1)},\tag{9.30}$$

$$M_{1-}=\frac{2\,f\,(f\,\sin(2\,\alpha)+1)}{1-f^2\,\sin(2\,\alpha_1)\,\sin(2\,\alpha)}+\frac{\sin(\alpha_1)}{\cos(\alpha_1)}.\tag{9.31}$$

The formulas (9.28), (9.29), (9.30), (9.31) are derived by means of the code

```
M_plus:=M_plus;
M_minus:=M_minus;
M1_plus:=-cos(alpha1)/sin(alpha1)-2*r1;
M1_minus:=sin(alpha1)/cos(alpha1)+2*r1;
```

The formulas (9.28), (9.29), (9.30), (9.31) provides four options for choosing the values of M and M_1 in the equation (9.1):

1). $M=M_+$ and $M_1=M_{1+}$, 2). $M=M_+$ and $M_1=M_{1-}$,
3). $M=M_-$ and $M_1=M_{1+}$, 4). $M=M_-$ and $M_1=M_{1-}$.

Choosing any one of these four options, we get a three-parametric solution of the equation (9.1). **Warning**: a rational solution of the equation (9.1) does not necessarily produce a rational solution for the cuboid equations (3.4) and (3.5).

Walter Wyss does not study the singular case for the equations (9.21) where the inequality (9.22) turns to the equality. He chooses the option 4. Then on page 22 of his paper [4], using the formulas (9.23), he considers two cases for $f\neq 0$:

(i) $r_1\neq r$, where he derives the formula

$$f=\frac{r_1-r}{r\,\sin(2\,\alpha)-r_1\,\sin(2\,\alpha_1)},\tag{9.32}$$

(ii) $r_1=r$, where he derives the formulas

$$f=\frac{r}{1+r\,\sin(2\,\alpha)},\tag{9.33}$$

$$\sin(2\,\alpha)=\sin(2\,\alpha_1).\tag{9.34}$$

The formula (9.32) is verified by means of the following code:

```
Eq_9_32:=f-(r1-r)/(r*sin(2*alpha)-r1*sin(2*alpha1)):
Eq_9_32:=normal(Eq_9_32);
```

The formula (9.34) is derived from (9.23) through the following equation:

$$r_1-r=\frac{f^2\,(\sin(2\,\alpha)-\sin(2\,\alpha_1))}{1-f^2\,\sin(2\,\alpha_1)\,\sin(2\,\alpha)}=0.$$

Then the formula (9.33) is verified by means of the following code:

```
r:=normal(subs(sin(2*alpha1)=sin(2*alpha),r)):
Eq_9_33:=f-r/(1+r*sin(2*alpha)):
Eq_9_33:=normal(Eq_9_33);
```

The formula (9.34) produces two options:

$$\alpha_1 = \alpha, \qquad\qquad \alpha_1 + \alpha = \frac{\pi}{2}. \tag{9.35}$$

The second option (9.35) coincides with the condition (8.1).

10. The rectangular cuboid limit.

The rectangular cuboid limit is the case where the parallelogram ABFE in Fig. 1.1 turns to a rectangle. This case is characterized by the equalities

$$u_3 = u_3, \qquad\qquad s_3 = s_4. \tag{10.1}$$

The equalities (10.1) imply (7.17). Substituting (7.17) into (9.2), we get

$$D = 0. \tag{10.2}$$

Note that the formulas (9.23) were derived under the assumption (9.8). Comparing (9.8) with (10.2), we see that the formulas (9.23) are not applicable to the rectangular cuboid case directly. For this reasom in his paper [4] Walter Wyss applies the formulas (9.23) through the limit procedure

$$D \neq 0, \qquad\qquad D \to 0. \tag{10.3}$$

Then, using a not well-detailed reasoning, from $D \to 0$ in (10.3) he derives

$$r \to 0, \qquad\qquad r_1 \to 0, \tag{10.4}$$

$$f \to 0, \qquad\qquad \frac{r}{r_1} \to 1. \tag{10.5}$$

Looking at (10.4) and (10.5), he says that r and r_1 are "infinitesimally equal" and concludes that the rectangular cuboid limit falls under the case (ii) in (9.33) and (9.34). This is the crucial mistake in his arguments — **infinitesimally equal** does not mean **equal**. There are a lot of rectangular cuboids, no matter rational or irrational, that do not fall under the case (ii) and (9.34), i. e. such that

$$\sin(2\,\alpha) \neq \sin(2\,\alpha_1). \tag{10.6}$$

The equality $D = 0$ for such cuboids can be reached through the limit procedure as $f \to 0$ in (9.23). Indeed, the formulas (9.23) simplify to

$$r = f + f^2\,\sin(2\,\alpha_1) + o(f^2), \qquad\qquad r_1 = f + f^2\,\sin(2\,\alpha) + o(f^2) \tag{10.7}$$

as $f \to 0$. Due to (10.6) r and r_1 in (10.7) tend to zero never being equal for sufficiently small $f \neq 0$. Rational rectangular cuboids are not yet found.

11. Conclusions.

Almost all formulas in Walter Wyss's paper [4] have been verified. They are correct. For the reader's convenience all of the code used for verifying formulas is collected in ancillary files in the section-by-section form according to the sections of the present paper.

Walter Wyss's paper [4] comprises a valuable result for the theory of slanted cuboids. This result is expressed by the explicit formulas (6.24), (6.25), and (6.26) that produce four explicit two-parametric solutions of the basic slanted cuboid equation (3.9) through the formulas (6.27) and (6.31).

As for the main goal of the paper [4], it is not reached. The paper does not contain a correct proof for the no perfect cuboid claim in its title.

References

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