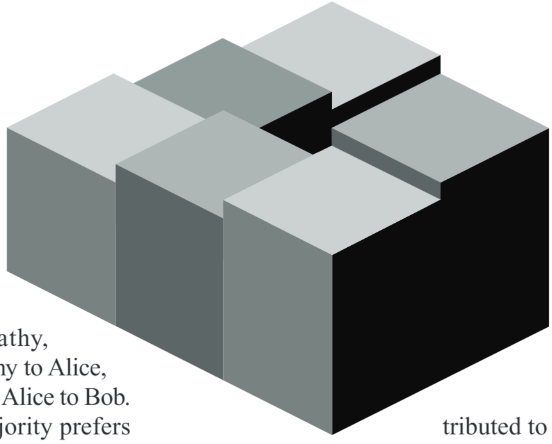
Politics as (Un)usual



Politics, it's said, is the art of the possible. And for decades, mathematical voting theorists have pointed to the possibility that different voting systems—from simple plurality to instant run-off to the rank-ordered Borda count—could produce vastly different results. Donald Saari, director of the Institute for Mathematical Behavioral Sciences at the University of California, Irvine, and a leading voting theorist, likes to boast that if you tell him the result you want (say, for a textbook selection or a new hire for your department) and your voters' actual preferences, he'll find a perfectly fair way of guaranteeing the desired result.

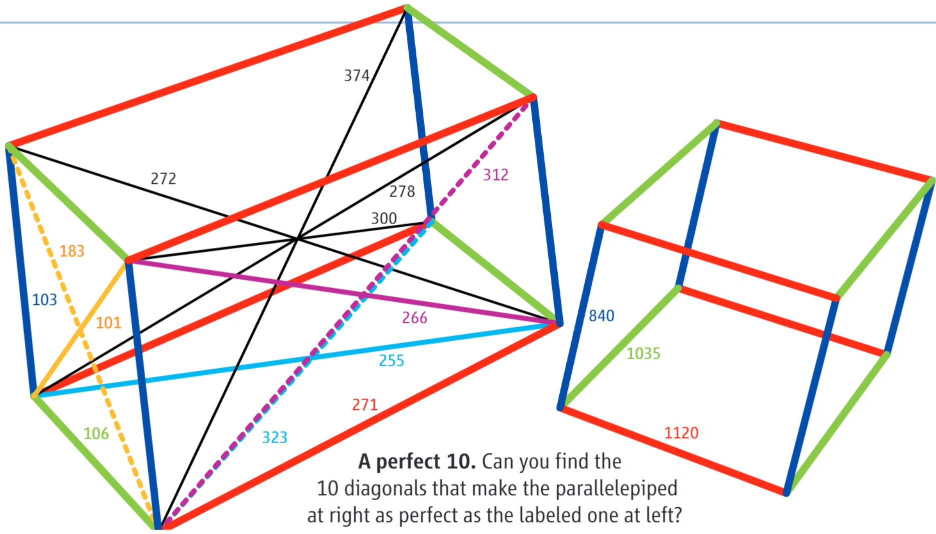
But how common are such paradoxes in practice? Not very, if a recent case study is typical. In a session on voting theory at the joint meeting of the American Mathematical Society and the Mathematical Association of America, Anna Popova, a graduate student in psychology at the University of Illinois, Urbana-Champaign, described an analysis of ranked ballots from 8 years of voting for the presidency of the American Psychological Association (APA). She and her adviser, Michel Regenwetter, found that different voting methods gave the same result much more often than not.

One classic voting paradox is known as a Condorcet cycle. An extreme example occurs when a third of the voters prefer Alice to Bob to Cathy, another third prefer Bob to Cathy to Alice, and a final third prefer Cathy to Alice to Bob. In such a case, an outright majority prefers Alice to Bob, a different outright majority prefers Bob to Cathy, and yet another outright majority prefers Cathy to Alice. Subtler examples, with the numbers not exactly equal, produce cases in which a reasonable argument can be made—and a voting system adopted—for electing any of the candidates.

Popova and Regenwetter obtained data sets from APA elections from 1998 to 2005, each with upward of 20,000 voters giving full or partial rankings to five candidates for president. (APA uses a form of instant-runoff voting, in which candidates with the fewest first-place votes are successively eliminated and their supporters' ballots are redistributed to those voters' second-place selections.) The research, which compared seven different voting methods, turned up no examples of Condorcet cycles and found only one case in which one method (plurality vote) produced a different winner from the others.

Steven Brams, a voting theory expert at New York University in New York City, questions the generality of the psychologists' findings, noting that balloting in professional societies is often heavily influenced by the process of selecting nominees. "I don't allege that its nominating committees pick some preferred candidate and then fill in the rest of the ballot with weak opponents," he says.

Perfection in a Box



A perfect 10. Can you find the 10 diagonals that make the parallelepiped at right as perfect as the labeled one at left?

Here's a good way to keep mathematicians busy for centuries: Make up an object, call it "perfect," and then try to find out if it exists. Take so-called perfect numbers: whole numbers such as 6 or 28, which equal the sum of their divisors ($6 = 1 + 2 + 3$; $28 = 1 + 2 + 4 + 7 + 14$). It's hard enough to find ones that are even, but no one has ever spotted a single one that's odd.

Things are even worse with "perfect cuboids": rectangular bricks whose three different edges and four different diagonals (three on faces of the brick and one angling through its "body") all have lengths that are exact integers. Nobody knows whether they exist at all. But something similar has just turned up: At the meeting, researchers reported the first sightings of "perfect parallelepipeds."

A parallelepiped is basically a brick whose sides are allowed to be parallelograms rather than strict 90º rectangles. Like a cuboid, a parallelepiped has three different edges, but it has 10 different diagonals: two on each of its three different sides, and four crisscrossing the body. Perfection is achieved if all 13 of these numbers are exact integers.

Now Clifford Reiter of Lafayette College in Easton, Pennsylvania, and Jorge Sawyer, an undergraduate at Lafayette, have found the first examples of perfect parallelepipeds. Their search depended on a simple property of all parallelograms, established by simple algebra: The sum of the squares of a parallelogram's two diagonals is twice the sum of the squares of its two edges. With that formula and a bit of number theory to guide them, Reiter and Sawyer could easily run a systematic search for all parallelograms with edges of "short" integer length, say, into the thousands, whose diagonals are also integers. They then looked for combinations

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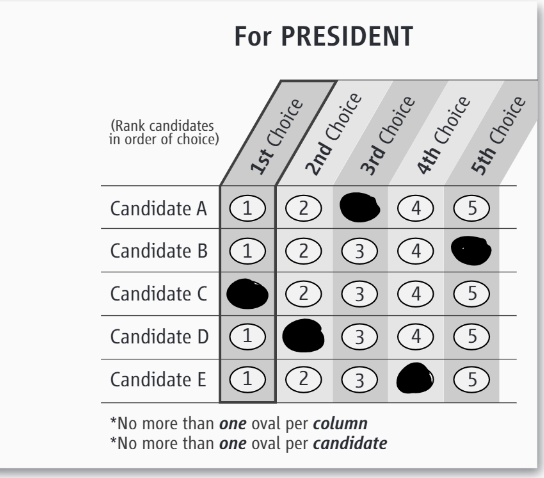
Published by AAAS



Who's on first? Optical illusion (*left*) mimics a paradoxical "everybody wins" outcome that can actually occur in ranked-choice voting.

"But it may be more than coincidence that one candidate almost always handily beats four opponents." Saari is similarly unsure. "The only way such an empirical result can happen is if people have remarkably similar preferences, much more so than we would expect in general society," he says.

Regenwetter agrees there's a lot left to be explained. But it's becoming clear, he says, that "the empirical world is highly different from the picture that has generally emerged out of the mathematical theory of social choice."

of these "perfect parallelograms" to serve as sides of candidate parallelepipeds and devised a computer algorithm to pick out the perfect ones.

"The biggest surprise for me was how quickly we got results," Sawyer says. Their first example has edges of length 271, 106, and 103 (see figure, left). In all, the computer search found 30 perfect parallelepipeds, with edge lengths up to 3920. Some are tantalizingly close to being cuboids, with one or two rectangular sides. Such findings "may spark even more interest in the perfect cuboid problem," Sawyer says.

Ezra Brown, a number theorist at Virginia Polytechnic Institute and State University in Blacksburg, agrees. "The size of Reiter and Sawyer's smallest solution is very surprising," he says. "Apparently, easing the restrictions that the faces be rectangles is more crucial than anyone thought." Nonetheless, he notes, perfect cuboids, if they exist at all, are a long way off: Computers have looked at all possible bricks with edge lengths up to 10 billion without finding a single one that's perfect.

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What Comes Next?

One of mathematicians' most beloved Web sites is getting ready for a makeover. The Online Encyclopedia of Integer Sequences, established by Neil Sloane at AT&T Labs Research in 1996 and run largely as a one-man shop, is poised to go "wiki," with 50 associate editors taking over much of the workload.

The OEIS, or simply "Sloane" as it's known to sequence fanatics, is a database of nearly 200,000 lists of numbers—a mathematical equivalent to the FBI's voluminous fingerprint files. Much as fingerprints give police a quick way to link a new crime to earlier ones, sequences enable researchers to make connections between mathematical problems that might otherwise go unnoticed. The innocuous-seeming sequence 1, 2, 5, 14, 42, 132, …, for example, arises in a huge number of different contexts, from counting the arrangements of nonintersecting chords inside a circle to enumerating secondary structure possibilities of RNA. To sequence fanatics, such "Catalan numbers," as they're known, are even more famous than the ubiquitous Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, … .

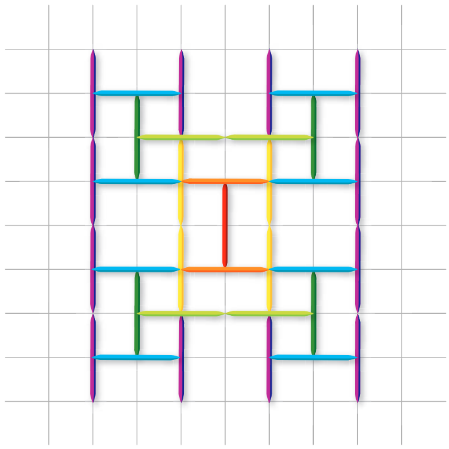
Sloane began compiling sequences in 1965 as a graduate student at Cornell University. By 1973 he had 2372 of them, which he published as *A Handbook of Integer Sequences*. An updated edition, with 5487 sequences, appeared in 1995, with the help of Simon Plouffe of the University of Quebec, Montreal. But by then, Sloane was already moving online. The OEIS made its debut a year later, with a database of 10,000 sequences.

The OEIS "is one of the most useful tools available online for the working mathematician," says Doron Zeilberger, a combinatorialist at Rutgers University, New Brunswick. "It also is a great tool for determining the novelty of a new sequence. If it is not in Sloane, it is most likely to be new!"

The Web site invites users to submit new sequences or comment on existing ones. Such contributions have fueled the database's steady growth. In 2009 alone, the total increased by 18,709 sequences, or more than 50 a day. "Sequences are still pouring in," Sloane says.

Even as he edits the constant stream of new sequences, Sloane takes time to admire some of the contributions. His latest favorite is the "toothpick sequence," added in 2008 by Omar Pol, an OEIS contributor from Buenos Aires. The toothpick sequence registers the increasing size of a geometric arrangement of toothpicks, in which a new batch is added at each stage, centered on and at right angles to the exposed tips of the previous batch (see figure). The picture that emerges displays surprising fractal growth. "It's got beautiful structure," Sloane says. He and his AT&T colleague David Applegate have written a paper on the sequence's mathematical properties, and Applegate has contributed a movie of its geometric growth, linked to its entry in the OEIS.

Sloane set up the OEIS Foundation last year and transferred intellectual-property rights to the nonprofit organization. With Applegate's help, he plans to move the database to a wiki format, giving each sequence its own Web page, with new submissions moderated by a board of editors. The transition has hit a snag, however: Search-engine software in the "wikiverse" can't yet handle sequences of numbers.



Chewy. Database managed by Neil Sloan (*top*) includes the "toothpick" sequence (1, 3, 7, 11, 15, 23, 35,...), shown here in color-coded steps.

Once that technicality is overcome, Sloane expects the wiki format to be an improvement. "It's the correct mechanism for handling the database," he says. As for his anticipated reduced workload, "it'll mostly be a relief. On the other hand, I'll miss all the e-mails."

–BARRY CIPRA