On the Integral Cuboid

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Proof. This follows immediately from the proof of the theorem, with $f(n) = n$.

I have not found a similarly elementary and effective solution to the general congruence (1), except in the trivial case when $(u, v) > 1$. (If $p \mid (u, v)$, then (1) implies that $p^n \mid b$. Let $p^N \| b$. Then $n \leqq N$.) In particular, for $n > 1$, solutions of

$$5^n \equiv 2 \pmod{3^n} \tag{4}$$

are unknown. I conjecture that there are none.

References

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2. K. Mahler, On the fractional parts of the powers of a rational number, Acta Arith., 3 (1938) 89–93.

ON THE INTEGRAL CUBOID

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A long-standing problem is whether cuboids (rectangular parallelepipeds) exist for which the edges, face diagonals, and inner diagonals are all integers (Dickson [1, p. 502] and Sierpinski [4, p. 62]). It appears not to have been noted that for the well-known family of solutions yielding integral edges and face diagonals, the inner diagonals cannot be integers.

The problem can be expressed as one of finding solutions in positive integers to the following four equations in seven unknowns:

$$x^2 + y^2 = t^2, \qquad x^2 + z^2 = u^2, \qquad y^2 + z^2 = v^2, \tag{1}$$

$$x^2 + y^2 + z^2 = w^2. \tag{2}$$

Part of the difficulty stems from the fact that the general solution of the system (1) is not known. However, a family of solutions going back to the 18th century (Dickson [1 , p. 497]) is given by

$$x = a(4b^2 - c^2), \qquad y = b(4a^2 - c^2), \qquad z = 4abc \tag{3}$$

for positive integers a, b, c satisfying

$$a^2 + b^2 = c^2. \tag{4}$$

Care must be taken, since x and y may be negative, requiring a sign change. Incidentally, Sierpinski [4, p. 61] slips on this point, saying that solutions of (4) in natural numbers yield solutions of (3) in natural numbers, yet $(a, b, c) = (5, 12, 13)$ gives $(x, y, z) = (2035, -828, 3120)$. This leads to a second slip when he assumes that z has the greatest magnitude, yet $(a, b, c) = (11, 60, 61)$ gives $(x, y, z) = (117469, -194220, 161040)$ and $(a, b, c) = (143, 24, 145)$ gives $(x, y, z) = (-2677103, 1458504, 1990560)$.

One sees that solutions of (3) automatically satisfy the second and third equations in (1), where

$$u = a(4b^2 + c^2), \qquad v = b(4a^2 + c^2), \tag{5}$$

while (4) is needed to establish the first equation in (1).

THEOREM 1. *For x, y, z satisfying* (3) *and* (4), *equation* (2) *is impossible.*

Proof. We have

$$x^2 + y^2 + z^2 = c^2(a^4 + 18a^2b^2 + b^4). \tag{6}$$

The left member however, cannot be a square for positive integers x, y, z since the expression in parentheses is not a square for $ab \neq 0$ (Pocklington [**3**, p. 116]) completing the proof.

The simplest solution of (3) and (4) is given by $(a, b, c) = (3, 4, 5)$, $(x, y, z) = (117, 44, 240)$. There are solutions of (1) of the form (3) not satisfying (4), for example, $(x, y, z) = (-855, 2640, 832)$ for $(a, b, c) = (1, 16, 13)$ and solutions of (1) not of the form (3), for example, $(x, y, z) = (240, 252, 275)$. However, the following theorem demonstrates that (3) in some sense represents all solutions of (1).

THEOREM 2. *Formula* (3) *with* $(a, b, c) = 1$ *represents some integral multiple of every primitive solution of* (1).

Proof. A primitive solution of (1) is one in which x, y, z, t, u, v are positive integers with no common factor. Any solution can be reduced to a primitive solution. In a primitive solution $(x, y, z) = 1$.

For the given x, y, z one solves (3) to get positive real solutions

$$a = \frac{B}{2(AB)^{1/3}}, \qquad b = \frac{A}{2(AB)^{1/3}}, \qquad c = \frac{z}{(AB)^{1/3}}, \tag{7}$$

where

$$A = x + u, \qquad B = y + v. \tag{8}$$

If a, b, and c are integers, the solution itself is represented. If a, b, and c are not all integers, multiply the given solution by the integer $8AB$ to get the primed solution

$$x' = 8ABx, \qquad y' = 8ABy, \qquad z' = 8ABz, \tag{9}$$

where a', b', c' are integers given by

$$a' = y + v, \qquad b' = x + u, \qquad c' = 2z. \tag{10}$$

One readily sees from (3) that the cube of any common factor of a', b', c' is a factor of x', y', z'. Hence a', b', c' can be reduced to a'', b'', c'', where $(a'', b'', c'') = 1$ and the corresponding x'', y'', z'', t'', u'', v'' is a multiple of the original primitive solution of (1), to end the proof.

Normally, there are 6 such representations from permutations of x, y, z, though only 3 significant ones, because an interchange of x and y produces an interchange of a and b. This shows that there is no loss in assuming $2a>2b>c>0$. If in addition x and y are allowed to be negative, there are 24 representations. Thus a way to check if a particular solution of (1) is represented by (3) would be to examine the 24 sets of solutions for a, b, c given in (7) to see if any set is integral.

Lal and Blundon [2] used the formula

$$x = 2mnpq, \qquad y = mn(p^2 - q^2), \qquad z = pq(m^2 - n^2) \tag{11}$$

to generate solutions of (1), requiring that y^2+z^2 be a square. This also represents multiples, $(x, y, z)=(44, 117, 240)$ not being representable. This fact necessitates that solutions be reduced, and furthermore, makes it difficult to give the range of their table. Though formula (3) also has this defect, it may be more effective, involving one less parameter. A still more preferred form may be

$$x = a(b^2 - c^2), \qquad y = b(a^2 - c^2), \qquad z = 2abc, \tag{12}$$

with $a>b>c>0$, $(a, b, c)=1$, and x^2+y^2 a square.

References

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3. H. C. Pocklington, Some Diophantine impossibilities, Proc. Cambridge Phil. Soc., 17 (1914) 110–118.

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REFLECTIONS HAVE REVERSED VECTORS

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1. Introduction. In this note we prove the following elementary theorem, which gives some geometric insight into the notion of a reflection of a metric vector space:

Theorem A. *Every reflection has a reversed vector.*

We also show that the preceding theorem is almost immediately equivalent to the following one:

Theorem B. *Every rotation of a space of odd dimension and every reflection of a space of even dimension has a fixed vector.*

In spite of the elementary nature of these results, we have not been able to locate Theorem A in the literature except in [1] where both theorems are given, but under restrictive hypotheses, namely for real, anisotropic vector spaces. The proof given there rests on properties of the reals, and does not generalize. Theorem B can be found in the literature (see [2], page 131 or [3], Proposition