Perfect Cuboid, Heronian Triangles and Heronian Tetrahedron

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Abstract

The Perfect cuboid problem is equivalent to the problem of a Heronian trirectangular tetrahedron. In other words, if there exists a Perfect cuboid, then a Heronian trirectangular tetrahedron also exists, and vice versa, if there exists a Heronian trirectangular tetrahedron then a Perfect cuboid also exists.

Introduction

Definition 1. *An Euler cuboid (also called an Euler brick) named after Leonhard Euler, is a rectangular cuboid whose edges and face diagonals all have integer lengths.*

Definition 2. *A Perfect cuboid (also called a perfect Euler brick, a perfect box) is an Euler brick whose body diagonal also has integer length.*

The existence of a Perfect cuboid is one of unsolved problems in mathematics. The definition of a Perfect cuboid in geometric terms is equivalent to a solution of the following system of Diophantine equations:

$$\begin{cases} a^2 + b^2 = d^2 \\ a^2 + c^2 = e^2 \\ b^2 + c^2 = f^2 \\ a^2 + b^2 + c^2 = g^2 \end{cases} \tag{1}$$

where a, b, c are three edges, d, e, f three face diagonals and g the body diagonal.

Definition 3. *We call a Perfect cuboid primitive if its edge lengths are relatively prime:* $\gcd(a, b, c) = 1$.

Definition 4. *In geometry, a trirectangular tetrahedron is a tetrahedron where all three face angles at one vertex are right angles.*

Definition 5. *A Heronian tetrahedron is a tetrahedron whose edge lengths, face areas and volume are all integers.*

The existence of a Heronian trirectangular tetrahedron is also one of unsolved mathematical problems.

Definition 6. *A Heronian triangle is a triangle that has side lengths and area that are all integers.*

Heronian triangles

Known Fact 1. *The area of a scaled object will be equal to the scale factor squared.*

Lemma 1. *If (a, b, c) are the side lengths of a Heronian $\triangle ABC$, then a triangle with side lengths (ka, kb, kc) for some $k \in \mathbb{N}$ is also a Heronian triangle.*

Proof. Let (ka, kb, kc) are the side lengths of $\triangle XYZ$.

According to Heron's formula, the areas of triangles $\triangle ABC$ and $\triangle XYZ$ are

$$\begin{aligned} S_{\triangle ABC} &= \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \\ S_{\triangle XYZ} &= \frac{1}{4}\sqrt{(ka+kb+kc)(-ka+kb+kc)(ka-kb+kc)(ka+kb-kc)} \\ &= \frac{k^2}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \\ &= k^2 \cdot S_{\triangle ABC} \in \mathbb{N} \end{aligned}$$

Therefore $\triangle XYZ$ is also a Heronian triangle. □

Lemma 2. *If the side lengths (a, b, c) of a Heronian $\triangle ABC$ has a common divisor k, then the triangle with sides reduced by k times is also Heronian.*

Proof. Suppose k is a common divisor of a, b, c, so there exist three integer numbers $x, y, z \in \mathbb{N}$ as the sides of some $\triangle XYZ$ such that

$$a = kx, b = ky, c = kz$$

Since $\triangle ABC$ is a Heronian, then the area of the triangle $S_{\triangle ABC} \in \mathbb{N}$.

According to Heron's formula the areas of triangles $\triangle ABC$ and $\triangle XYZ$ are

$$\begin{aligned} S_{\triangle ABC} &= \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \\ S_{\triangle XYZ} &= \frac{1}{4}\sqrt{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)} \end{aligned}$$

Assume that

$$s = (x+y+z)(-x+y+z)(x-y+z)(x+y-z) \in \mathbb{N}$$

is not a perfect square. It means that the factorization of s into prime numbers contains an odd power of some prime number p. Multiplying s by k^4 adds to the factorization of s quadruple powers of all primes from the factorization of k. Thus, even if p is present in the factorization of k, the product $s \cdot k^4$ still contains an odd power of p. Thus, $s \cdot k^4$ is not a perfect square too.

Multiplying the precending equality by k^4 we obtain

$$\begin{aligned}k^4 \cdot s &= (kx + ky + kz)(-kx + ky + kz)(kx - ky + kz)(kx + ky - kz)\\ &= (a + b + c)(-a + b + c)(a - b + c)(a + b - c) = 16S^2_{\triangle ABC}\end{aligned}$$

which is definitely a perfect square. It contradicts to the assumption that s is not a perfect square, then one concludes that $\sqrt{s} \in \mathbb{N}$.

Taking a square root from both sides of the preceding equality, we get

$$k^2\sqrt{s} = 4 \cdot S_{\triangle ABC}$$

It follows that $4 \mid k^2\sqrt{s}$.

Suppose that $16 \nmid s$. Due to arbitrariness of selection of k, let set k equal to some coprime to 2 number, for instance $k = 3$. Since $16 \nmid k^4$ and $16 \nmid s$, we obtain that $16 \nmid k^4 s$. But if $4 \mid k^2\sqrt{s}$, then $16 \mid k^4 s$. Contradiction. Hence the area of the triangle $S_{\triangle XYZ} \in \mathbb{N}$ and $\triangle XYZ$ is a Heronian triangle. $\square$

In 2010, Chinese student Ze'en Huo[2] proved the following four theorems and one lemma which establish a connection between a Heronian triangle whose three angle bisectors are rational and four Heronian triangles, one acute and three obtuse, inducing the given. We have significantly redesigned the content and proofs of the theorems, nevertheless expressing immense respect for their author.

Known Fact 2. *The values of sine, cosine, tangent and tangent of half-angle of any angle of a Heronian triangle are rational numbers; the values of radii of incircle and circumcircle are rational numbers.*

Lemma 3. *Let ABC be a Heronian triangle, the angle bisector l_a is rational if and only if* $\sin\frac{A}{2}, \cos\frac{A}{2}$ *are rational numbers.*

Proof. By the Law of Sines, we have

$$\frac{l_a}{\sin B} = \frac{c}{\sin{(B + A/2)}}$$

Suppose $\sin\frac{A}{2}, \cos\frac{A}{2}$ are rational numbers, since $\triangle ABC$ is a Heronian triangle, then $\sin B, \cos B \in \mathbb{Q}$, and so $\sin{(B + A/2)} \in \mathbb{Q}$ and $l_a \in \mathbb{Q}$.

Conversely, if the angle bisector l_a is rational, since the triangle ABC is a Heronian triangle, so $\sin B, \cos B, \tan A/2 \in \mathbb{Q}$. By the above formula, we obtain $\sin{(B + A/2)} \in \mathbb{Q}$. Therefore

$$\cos\frac{A}{2}\left(\sin B + \cos B \tan\frac{A}{2}\right) \in \mathbb{Q},$$

consequently $\sin\frac{A}{2}, \cos\frac{A}{2}$ are rational numbers. This completes the proof. $\square$

Theorem 1. *Let (x, y, z) be a Heronian tuple such that the related triangle is acute. Then the triangle with side lengths (m, n, k) given by*

$$\begin{cases} m = x^2(y^2 + z^2 - x^2) \\ n = y^2(x^2 + z^2 - y^2) \\ k = z^2(x^2 + y^2 - z^2) \end{cases} \tag{2}$$

is a Heronian triangle with three rational angle bisectors.

Proof. Let $\triangle XYZ$ be a Heronian triangle whose lengths of three sides are a Heronian tuple (x, y, z), by the assumptions, we have that $\triangle XYZ$ is an acute triangle, so the center O of the circumscribed circle of $\triangle XYZ$ lies in the interior of $\triangle XYZ$. Do the circumscribed circle O of $\triangle XYZ$; make three tangetial lines of the circle through X, Y, Z, we obtain the intersection $\triangle MNK$ of the three tangential lines (as see the Figure 1)

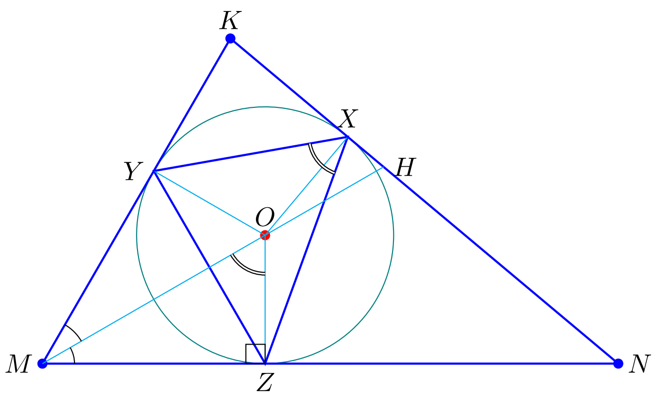

Figure 1

Notice that

$$\angle YOZ = 2\angle YXZ, \angle MOY = \angle MOZ = \frac{1}{2}\angle YOZ = \angle YXZ = \angle X$$

$$\tan X = \tan(\angle YXZ) \in \mathbb{Q}.$$

And let $r = OX = OY = OZ$ be the radius of the circle, then $r \in \mathbb{Q}$. And so

$$MZ = r\tan(\angle MOZ) = r\tan X \in \mathbb{Q}.$$

Similarly, $NZ \in \mathbb{Q}$. So

$$MN = MZ + ZN \in \mathbb{Q}.$$

Similarly, $MK \in \mathbb{Q}, KN \in \mathbb{Q}$. It follows that all sides of $\triangle MNK$ are rational numbers.

Therefore the area of $\triangle MNK$ is

$$S_{\triangle MNK} = \frac{1}{2}(MN + MK + KN)r \in \mathbb{Q}$$

Since

$$\angle NMK = 2\angle OMZ = 2\left(\frac{\pi}{2} - \angle MOZ\right) = \pi - 2\angle YXZ = \pi - 2X$$

, then

$$\sin(\angle NMK) = \sin(2X) = 2\sin X\cos X \in \mathbb{Q}.$$

Similarly,

$$\angle MNK = \pi - 2Y, \angle MKN = \pi - 2Z$$
$$\sin(\angle MNK), \sin(\angle MKN) \in \mathbb{Q}.$$

Now in $\triangle MNK$, we have

$$\cos(M) = \cos(\angle NMK) = -\cos(2X) = \sin^2 X - \cos^2 X \in \mathbb{Q}.$$

Similarly, $\cos N, \cos K \in \mathbb{Q}$.

For the angle bisector MH of the angle $\angle NMK$ in $\triangle MNK$, by the Law of Sines we have

$$\frac{MH}{\sin N} = \frac{MN}{\sin(K + M/2)} = \frac{MN}{\sin(K + (\pi/2 - X))} = \frac{MN}{\cos(K - X)}.$$

Since $\cos(K - X) = \cos K\cos X + \sin K\sin X \in \mathbb{Q}$, thus the angle bisector MH of the angle $\angle NMK$ is a rational number. Similarly, the two other angle bisectors of $\triangle MNK$ are rational numbers.

Summing up, the three sides, area and three angle buisectors of $\triangle MNK$ are rational numbers.

Scaling $\triangle MNK$ we can obtain a primitive Heronian triangle whose lengths of three sides are a primitive Heronian tuple with integer area and rational angle bisectors.

Now we prove that the formulæ 2 hold. Since

$$MC = r\tan X, NC = r\tan Y, \sin X = \frac{a}{2r}, \sin Y = \frac{b}{2r},$$

so

$$\cos X = \frac{y^2 + z^2 - x^2}{2yz}, \cos Y = \frac{x^2 + z^2 - y^2}{2xz}.$$

Hence

$$MN = r(\tan X + \tan Y) = \frac{xyz}{y^2+z^2-x^2} + \frac{xyz}{x^2+z^2-y^2}$$
$$= \frac{2xyz^3}{(y^2+z^2-x^2)(x^2+z^2-y^2)}$$

Similarly,

$$MK = r(\tan X + \tan Z) = \frac{xyz}{y^2+z^2-x^2} + \frac{xyz}{x^2+y^2-z^2}$$
$$= \frac{2xy^3z}{(y^2+z^2-x^2)(x^2+y^2-z^2)}$$
$$KN = r(\tan Z + \tan Y) = \frac{xyz}{x^2+y^2-z^2} + \frac{xyz}{x^2+z^2-y^2}$$
$$= \frac{2x^3yz}{(x^2+y^2-z^2)(x^2+z^2-y^2)}$$

The circumradius r of the circumcircle of $\triangle XYZ$ can be expressed as

$$r = \frac{xyz}{4S_{\triangle XYZ}}$$

Therefore,

$$\begin{aligned} S_{\triangle MNK} &= \frac{1}{2}(MN+MK+KN)r \\ &= \frac{xyz(2x^2y^2+2x^2z^2+2y^2z^2-x^4-y^4-z^4)}{(y^2+z^2-x^2)(x^2+z^2-y^2)(x^2+y^2-z^2)}r \\ &= \frac{xyz\cdot 4S^2_{\triangle XYZ}}{(y^2+z^2-x^2)(x^2+z^2-y^2)(x^2+y^2-z^2)}\cdot\frac{xyz}{4S_{\triangle XYZ}} \\ &= \frac{x^2y^2z^2}{(y^2+z^2-x^2)(x^2+z^2-y^2)(x^2+y^2-z^2)}\cdot S_{\triangle XYZ} \end{aligned}$$

Multiplying by a common rational divisor of MN, MK, KN

$$\mu = \frac{(y^2+z^2-x^2)(x^2+z^2-y^2)(x^2+y^2-z^2)}{xyz} \in \mathbb{Q},$$

we obtain a tuple $(\hat{m}, \hat{n}, \hat{k})$

$$\begin{cases} \hat{m} = 2x^2(y^2+z^2-x^2) \in \mathbb{N} \\ \hat{n} = 2y^2(x^2+z^2-y^2) \in \mathbb{N} \\ \hat{k} = 2z^2(x^2+y^2-z^2) \in \mathbb{N} \end{cases} \tag{3}$$

such that the corresponding $\triangle\hat{M}\hat{N}\hat{K}$ definitely is an integer triangle and is similar to $\triangle MNK$ with a ratio of similarity $\mu \in \mathbb{Q}$.

The area of the triangle $\triangle\hat{M}\hat{N}\hat{K}$ obtains

$$\begin{aligned} S_{\triangle\hat{M}\hat{N}\hat{K}} &= \mu^2 \cdot S_{\triangle MNK} = \\ &= (y^2+z^2-x^2)(x^2+z^2-y^2)(x^2+y^2-z^2)\cdot S_{\triangle XYZ} \in \mathbb{N} \end{aligned}$$

Thus, $\triangle\hat{M}\hat{N}\hat{K}$ is a Heronian triangle too.

Since each of the elements of the $(\hat{m}, \hat{n}, \hat{k})$ has a common integer divisor 2, by Lemma 2 the triangle with side lengths (m, n, k) given by

$$\begin{cases} m = x^2(y^2 + z^2 - x^2) \\ n = y^2(x^2 + z^2 - y^2) \\ k = z^2(x^2 + y^2 - z^2) \end{cases}$$

is also a Heronian triangle.

After scaling of the triangle $\triangle MNK$ by a rational factor, three rational angle bisectors remain rationals. $\square$

Corollary 1.1. *Let* $\lambda = \gcd(m, n, k)$. *By Lemma 2 we can reduce the sizes of the Heronian triangle by a common integer divisor* λ *to obtain a primitive Heronian triangle with side lengths* $(\overline{m}, \overline{n}, \overline{k})$ *given by*

$$\begin{cases} \overline{m} = \dfrac{1}{\lambda}x^2(y^2 + z^2 - x^2) \in \mathbb{N} \\ \overline{n} = \dfrac{1}{\lambda}y^2(x^2 + z^2 - y^2) \in \mathbb{N} \\ \overline{k} = \dfrac{1}{\lambda}z^2(x^2 + y^2 - z^2) \in \mathbb{N} \end{cases} \tag{4}$$

whose three rational angle bisectors are rational.

Example 1.

For a given primitive Heronian tuple $(13, 14, 15)$, find the corresponding primitive Heronian tuple such that the lengths of three angle bisectors of the corresponding Heronian triangle are rational numbers.

Solving: it is easy to see, given by the tuple the corresponding Heronian triangle is acute. By Theorem 1, substituting $x = 13, y = 14, z = 15$ into the formulæ 2 to obtain

$$\begin{cases} m = 13^2(14^2 + 15^2 - 13^2) = 2^3 \times 3^2 \times 7^2 \times 11 = 38808 \\ n = 14^2(13^2 + 15^2 - 14^2) = 2^2 \times 3^2 \times 7 \times 13^2 = 42588 \\ k = 15^2(13^2 + 14^2 - 15^2) = 2^2 \times 3^2 \times 5^3 \times 7 = 31500 \end{cases}$$

Dividing out by the greatest common divisor $\gcd(m, n, k) = 2^2 \times 3^2 \times 7 = 252$, we obtain the corresponding primitive Heronian tuple

$$\begin{cases} \overline{m} = m : 252 = 2 \times 7 \times 11 = 154 \\ \overline{n} = n : 252 = 13^2 = 169 \\ \overline{k} = k : 252 = 5^3 = 125 \end{cases}$$

It is easy to check that the semi-perimeter $s = 224$, the area

$$S = \sqrt{224 \times 70 \times 55 \times 99} = 9240$$

By the formula of angle bisector, the lengths of three angle bisectors are

$$\begin{cases} d_{\overline{m}} = \dfrac{2}{\overline{n} + \overline{k}} \cdot \sqrt{p(p - \overline{m})\overline{n}\overline{k}} = \dfrac{2}{169 + 125}\sqrt{224 \times 70 \times 169 \times 125} = \dfrac{2600}{21} \\ d_{\overline{n}} = \dfrac{2}{\overline{m} + \overline{k}} \cdot \sqrt{p(p - \overline{n})\overline{m}\overline{k}} = \dfrac{2}{154 + 125}\sqrt{224 \times 55 \times 154 \times 125} = \dfrac{30800}{279} \\ d_{\overline{k}} = \dfrac{2}{\overline{n} + \overline{m}} \cdot \sqrt{p(p - \overline{k})\overline{m}\overline{n}} = \dfrac{2}{169 + 154}\sqrt{224 \times 99 \times 169 \times 154} = \dfrac{48048}{323} \end{cases}$$

Therefore, the lengths of three angle bisectors of Heronian triangle with side lengths $(154, 169, 125)$ are rational numbers.

Theorem 2. *If the tuple (m, n, k) is such that the corresponding $\triangle MNK$ is a primitive Heronian triangle whose three angle bisectors are rationals, then the tuple (m, n, k) can be represented as*

$$\begin{cases} m = \dfrac{1}{\lambda} x^2(y^2 + z^2 - x^2) \\ n = \dfrac{1}{\lambda} y^2(x^2 + z^2 - y^2) \\ k = \dfrac{1}{\lambda} z^2(x^2 + y^2 - z^2) \end{cases} \tag{5}$$

where $\lambda = \gcd\left(x^2(y^2 + z^2 - x^2), y^2(x^2 + z^2 - y^2), z^2(x^2 + y^2 - z^2)\right)$ *and* (x, y, z) *is a tuple such that the corresponding acute primitive Heronian triangle is similar to the triangle with side lengths* (a, b, c) *given as:*

$$\begin{cases} a = \dfrac{-m + n + k}{2} \cdot \sqrt{\dfrac{(m - n + k)(m + n - k)}{nk}} \\ b = \dfrac{m - n + k}{2} \cdot \sqrt{\dfrac{(-m + n + k)(m + n - k)}{mk}} \\ c = \dfrac{m + n - k}{2} \cdot \sqrt{\dfrac{(-m + n + k)(m - n + k)}{mn}} \end{cases} \tag{6}$$

Proof. Suppose the tuple (m, n, k) is a Heronian tuple such that the corresponding Heronian $\triangle MNK$ has three rational angle bisectors. By Figure 2 we can construct an incircle O. Let the three tangential points of O with $\triangle MNK$ be A, B, C.

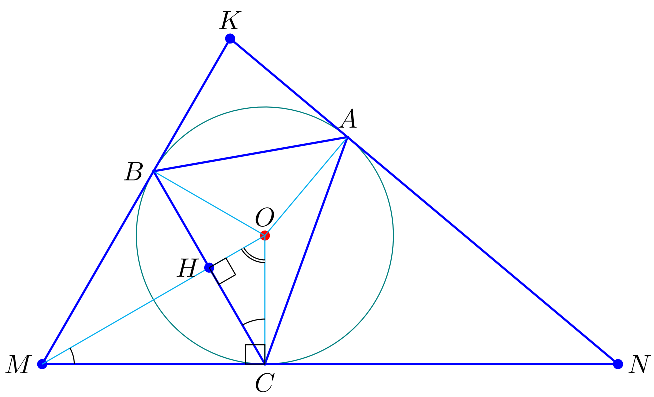

Figure 2

From the proof of Theorem 1, we have

$$\angle NMK = \pi - 2\angle BAC = \pi - 2A$$
$$\angle MNK = \pi - 2\angle ABC = \pi - 2B$$
$$\angle MKN = \pi - 2\angle ACB = \pi - 2C$$

Therefore $\triangle ABC$ is an acute triangle. By the assumptions and Lemma 3, we have $\sin A, \sin B, \sin C \in \mathbb{Q}$, so $BC, AC, AB \in \mathbb{Q}$ and $S_{\triangle ABC} \in \mathbb{Q}$. Let denote the tuple of the sides of $\triangle ABC$ as (a, b, c), where $a = BC, b = AC, c = AB$.

The distances from the vertex M to the two nearest touchpoints B and C are equal and can be obtained as

$$MC = MB = \frac{-m+n+k}{2}$$

The radius r of the incircle O in terms of the sides of $\triangle MNK$ is:

$$r = OA = OB = OC = \frac{S_{\triangle MNK}}{s} = \frac{\sqrt{s(s-m)(s-n)(s-k)}}{s} =$$
$$= \sqrt{\frac{(-m+n+k)(m-n+k)(m+n-k)}{4(m+n+k)}}$$

From the right-angled triangles $\triangle OCM$ the distance from the vertex M to the center of the incircle O obtains

$$MO = \sqrt{OC^2 + MC^2} =$$
$$= \sqrt{\frac{(-m+n+k)(m-n+k)(m+n-k)}{4(m+n+k)} + \frac{(-m+n+k)^2}{4}} =$$
$$= \sqrt{\frac{nk(-m+n+k)}{(m+n+k)}}$$

From the similarity of triangles $\triangle OCM$ and $\triangle OHC$ it implies

$$\frac{CH}{OC} = \frac{MC}{MO}$$

$$CH = \frac{MC \cdot OC}{MO} =$$
$$= \frac{\frac{-m+n+k}{2} \cdot \sqrt{\frac{(-m+n+k)(m-n+k)(m+n-k)}{4(m+n+k)}}}{\sqrt{\frac{nk(-m+n+k)}{(m+n+k)}}} =$$
$$= \frac{-m+n+k}{4} \cdot \sqrt{\frac{(m-n+k)(m+n-k)}{nk}}$$

$$a = BC = 2 \cdot CH = \frac{-m+n+k}{2} \cdot \sqrt{\frac{(m-n+k)(m+n-k)}{nk}}$$

Similarly,

$$b = AC = \frac{m-n+k}{2} \cdot \sqrt{\frac{(-m+n+k)(m+n-k)}{mk}}$$
$$c = AB = \frac{m+n-k}{2} \cdot \sqrt{\frac{(-m+n+k)(m-n+k)}{mn}}$$

From the proof of Theorem 1 the side lengths (m, n, k) of $\triangle MNK$ can be expressed as

$$\begin{cases} m = \dfrac{2a^3bc}{(a^2+b^2-c^2)(a^2+c^2-b^2)} \\ n = \dfrac{2ab^3c}{(b^2+c^2-a^2)(a^2+b^2-c^2)} \\ k = \dfrac{2abc^3}{(b^2+c^2-a^2)(a^2+c^2-b^2)} \end{cases}$$

By scaling the sides of $\triangle ABC$, we can obtain an acute primitive Heronian $\triangle XYZ$, denote the lengths of its three sides as x, y, z. Since $\triangle ABC \sim \triangle XYZ$ and the sides of both triangles are rational numbers, thus there exists some $\lambda \in \mathbb{Q}$, such that

$$a = \frac{x}{\lambda}, \qquad b = \frac{y}{\lambda}, \qquad c = \frac{z}{\lambda}$$

Therefore

$$\begin{cases} m = \dfrac{2a^3bc}{(a^2+b^2-c^2)(a^2+c^2-b^2)} = \dfrac{1}{\lambda} \cdot \dfrac{2x^3yz}{(x^2+y^2-z^2)(x^2+z^2-y^2)} \\ n = \dfrac{2ab^3c}{(b^2+c^2-a^2)(a^2+b^2-c^2)} = \dfrac{1}{\lambda} \cdot \dfrac{2xy^3z}{(y^2+z^2-x^2)(x^2+y^2-z^2)} \\ k = \dfrac{2abc^3}{(b^2+c^2-a^2)(a^2+c^2-b^2)} = \dfrac{1}{\lambda} \cdot \dfrac{2xyz^3}{(y^2+z^2-x^2)(x^2+z^2-y^2)} \end{cases}$$

Multiplying by

$$\frac{(y^2+z^2-x^2)(x^2+z^2-y^2)(x^2+y^2-z^2)}{2xyz}$$

we obtain a tuple (m, n, k)

$$\begin{cases} m = \dfrac{1}{\lambda} \cdot x^2(y^2+z^2-x^2) \\ n = \dfrac{1}{\lambda} \cdot y^2(x^2+z^2-y^2) \\ k = \dfrac{1}{\lambda} \cdot z^2(x^2+y^2-z^2) \end{cases}$$

Since $\triangle MNK$ is a primitive Heronian triangle, so $\gcd(m,n,k)=1$, whereas $m,n,k,x^2(y^2+z^2-x^2),y^2(x^2+z^2-y^2),z^2(x^2+y^2-z^2)\in\mathbb{N}$, so

$$\begin{cases} m=\dfrac{x^2(y^2+z^2-x^2)}{\gcd\left(x^2(y^2+z^2-x^2),y^2(x^2+z^2-y^2),z^2(x^2+y^2-z^2)\right)} \\ n=\dfrac{y^2(x^2+z^2-y^2)}{\gcd\left(x^2(y^2+z^2-x^2),y^2(x^2+z^2-y^2),z^2(x^2+y^2-z^2)\right)} \\ k=\dfrac{z^2(x^2+y^2-z^2)}{\gcd\left(x^2(y^2+z^2-x^2),y^2(x^2+z^2-y^2),z^2(x^2+y^2-z^2)\right)} \end{cases}$$

Therefore

$$\lambda=\gcd\big(x^2(y^2+z^2-x^2),y^2(x^2+z^2-y^2),z^2(x^2+y^2-z^2)\big)$$

This completes the proof. $\square$

Example 2.

For a given Heronian tuple $(84, 125, 169)$, such that the lengths of three angle bisectors of the corresponding Heronian triangle are rational numbers, find the corresponding primitive acute Heronian tuple producing the given.

Solving: By the Theorem 2, substituting $m = 84, n = 125, k = 169$ into the formulæ 6 to obtain

$$\begin{cases} a = \dfrac{-84+125+169}{2} \cdot \sqrt{\dfrac{(84-125+169)(84+125-169)}{125 \cdot 169}} = \dfrac{672}{13} \\ b = \dfrac{84-125+169}{2} \cdot \sqrt{\dfrac{(-84+125+169)(84+125-169)}{84 \cdot 169}} = \dfrac{640}{13} \\ c = \dfrac{84+125-169}{2} \cdot \sqrt{\dfrac{(-84+125+169)(84-125+169)}{84 \cdot 125}} = \dfrac{32}{1} \end{cases}$$

Multiplying by the greatest rational divisor $\lambda = \dfrac{\operatorname{lcm}(13, 13, 1)}{\gcd(672, 640, 32)} = \dfrac{13}{32}$, we obtain the corresponding primitive acute Heronian tuple

$$\begin{cases} x = \lambda \cdot a = \dfrac{13}{32} \cdot \dfrac{672}{13} = 21 \\ y = \lambda \cdot b = \dfrac{13}{32} \cdot \dfrac{640}{13} = 20 \\ z = \lambda \cdot c = \dfrac{13}{32} \cdot \dfrac{32}{1} = 13 \end{cases}$$

It is easy to check that the semi-perimeter $s = 27$, the area

$$S = \sqrt{27 \times 6 \times 7 \times 14} = 126$$

Check the triangle is acute:

$$21^2 = 441 < 569 = 20^2 + 13^2$$

Check the tuple $(13, 20, 21)$ by the formulæ 5 produces the given:

$$\begin{cases} \hat{m} = 13^2(20^2 + 21^2 - 13^2) = 113568 \\ \hat{n} = 20^2(13^2 + 21^2 - 20^2) = 84000 \\ \hat{k} = 21^2(13^2 + 20^2 - 21^2) = 56448 \end{cases}$$

Dividing out by the greatest common divisor $\gcd(\hat{m}, \hat{n}, \hat{k}) = 672$, we obtain the given Heronian tuple $(84, 125, 169)$.

Therefore, the tuple $(13, 20, 21)$ is such that the corresponding primitive acute Heronian triangle producing the given.

Theorem 3. *Let (x, y, z) be a Heronian tuple such that the related triangle is obtuse triangle with $z^2 > x^2 + y^2$. Then the triangle with side lengths (m, n, k) given by*

$$\begin{cases} m = x^2(y^2 + z^2 - x^2) \\ n = y^2(x^2 + z^2 - y^2) \\ k = z^2(z^2 - x^2 - y^2) \end{cases} \tag{7}$$

is a Heronian triangle with three rational angle bisectors.

Proof. Let $\triangle XYZ$ be a Heronian triangle whose lengths of three sides are a Heronian tuple (x, y, z), by the assumptions, we obtain that $\triangle XYZ$ is an obtuse triangle, hence the center O of the circumcircle of $\triangle XYZ$ lies in the exterior of $\triangle XYZ$. Do the circumscribed circle O of $\triangle XYZ$; make three tangetial lines of the circle through X, Y, Z, we obtain the intersection $\triangle MNK$ of the three tangential lines (as see the Figure 3)

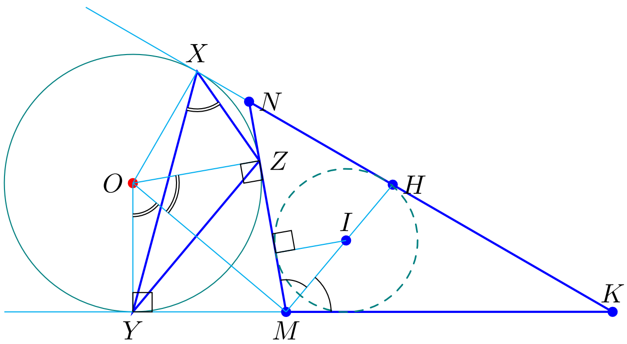

Figure 3

Notice that

$$\angle YOZ = 2\angle YXZ, \angle MOY = \angle MOZ = \frac{1}{2}\angle YOZ = \angle YXZ = \angle X$$

$$\tan X = \tan (YXZ) \in \mathbb{Q}.$$

And let $r = OX = OY = OZ$ be the radius of the circle, then $r \in \mathbb{Q}$. Thus

$$MY = MZ = r \tan (MOZ) = r \tan X \in \mathbb{Q}.$$

Similarly,

$$NX = NZ = r \tan Y \in \mathbb{Q}.$$

And hence

$$MN = MZ + NZ = r \tan X + r \tan Y \in \mathbb{Q}.$$

Since $\angle XOY = 2\pi - 2Z$, so

$$KX = KY = r\tan(\pi - Z) = -r\tan Z \in \mathbb{Q}.$$

Hence

$$MK = KY - MY = -r\tan Z - r\tan X \in \mathbb{Q}$$
$$KN = KY - NX = -r\tan Z - r\tan Y \in \mathbb{Q}$$

For the $\triangle MNK$ r is the exradii opposite K. So the inradius r_i of the incircle I in the $\triangle MNK$ in terms of exradii and semi-perimeter is given by

$$r_i = \frac{s - MN}{s} r \in \mathbb{Q}$$

where $s = \frac{1}{2}(MN + MK + KN)$.

Therefore the area of $\triangle MNK$ is

$$S_{\triangle MNK} = sr_i = (s - MN)r = \frac{1}{2}(MK + KN - MN)r \in \mathbb{Q}$$

So the lengths of three sides of the $\triangle MNK$ are rational numbers.

Now

$$\angle NMK = \pi - 2\angle OMZ = 2\angle YXZ = 2X$$

, then

$$\sin(NMK) = \sin 2X = 2\sin X\cos X \in \mathbb{Q}.$$

Similarly,

$$\angle MNK = 2Y, \angle MKN = 2Z - \pi$$
$$\sin(MNK), \sin(MKN) \in \mathbb{Q}.$$

In $\triangle MNK$, we have

$$\cos M = \cos(NMK) = \cos(2X) = \cos^2 X - \sin^2 X \in \mathbb{Q}.$$

Similarly, $\cos N, \cos K \in \mathbb{Q}$.

For the angle bisector MH of the angle $\angle NMK$ in $\triangle MNK$, by the Law of Sines we have

$$\frac{MH}{\sin N} = \frac{MN}{\sin(MHN)} = \frac{MN}{\sin(K + (\pi/2 - X))} = \frac{MN}{\cos(K - X)}.$$

Since $\cos(K - X) = \cos K\cos X + \sin K\sin X \in \mathbb{Q}$, thus the angle bisector MH of the angle $\angle NMK$ is a rational number. Similarly, the two other angle bisectors of $\triangle MNK$ are rational numbers.

Summing up, the three sides, area and three angle buisectors of $\triangle MNK$ are rational numbers.

Scaling $\triangle MNK$ we can obtain a Heronian triangle whose lengths of three sides are a primitive Heronian tuple with integer area and rational angle bisectors.

Now we prove that the formulæ 7 hold. Since

$$MZ = r\tan X, NZ = r\tan Y, \sin X = \frac{x}{2r}, \sin Y = \frac{y}{2r},$$

so

$$\cos X = \frac{y^2+z^2-x^2}{2yz}, \cos Y = \frac{x^2+z^2-y^2}{2xz}.$$

Hence

$$\begin{aligned} MN = r(\tan X + \tan Y) &= \frac{xyz}{y^2+z^2-x^2} + \frac{xyz}{x^2+z^2-y^2} \\ &= \frac{2xyz^3}{(y^2+z^2-x^2)(x^2+z^2-y^2)}. \end{aligned}$$

Similarly,

$$\begin{aligned} MK = -r(\tan X + \tan Z) &= -\frac{xyz}{y^2+z^2-x^2} - \frac{xyz}{x^2+y^2-z^2} \\ &= \frac{2xy^3z}{(y^2+z^2-x^2)(z^2-x^2-y^2)}, \\ KN = -r(\tan Z + \tan Y) &= -\frac{xyz}{x^2+y^2-z^2} - \frac{xyz}{x^2+z^2-y^2} \\ &= \frac{2x^3yz}{(x^2+y^2-z^2)(z^2-x^2-y^2)}. \end{aligned}$$

The circumradius $r \in \mathbb{Q}$ can be expressed as

$$r = \frac{p}{q}$$

where $p, q \in \mathbb{N}$

Therefore,

$$\begin{aligned} S_{\triangle MNK} &= \frac{1}{2}(MK + KN - MN)r \\ &= \frac{xyz(2x^2y^2 - x^4 - y^4 + z^4)}{(y^2+z^2-x^2)(x^2+z^2-y^2)(z^2-x^2-y^2)} \cdot \frac{p}{q} \end{aligned}$$

Scaling the $\triangle MNK$ in

$$\mu = (y^2+z^2-x^2)(x^2+z^2-y^2)(z^2-x^2-y^2) \cdot q \in \mathbb{N}$$

times, we obtain a tuple $(\hat{m}, \hat{n}, \hat{k})$

$$\begin{cases} \hat{m} = 2xyzq \cdot x^2(y^2 + z^2 - x^2) \in \mathbb{N} \\ \hat{n} = 2xyzq \cdot y^2(x^2 + z^2 - y^2) \in \mathbb{N} \\ \hat{k} = 2xyzq \cdot z^2(z^2 - x^2 - y^2) \in \mathbb{N} \end{cases}$$

such that the corresponding $\triangle\hat{M}\hat{N}\hat{K}$ definitely is an integer triangle and is similar to $\triangle MNK$ with a ratio of similarity $\mu \in \mathbb{N}$.

The area of the triangle $\triangle\hat{M}\hat{N}\hat{K}$ obtains

$$\begin{aligned} S_{\triangle\hat{M}\hat{N}\hat{K}} &= \mu^2 \cdot S_{\triangle MNK} \\ &= (y^2 + z^2 - x^2)(x^2 + z^2 - y^2)(x^2 + y^2 - z^2) \times \\ &\times xyz(2x^2y^2 - x^4 - y^4 + z^4) \cdot pq \in \mathbb{N} \end{aligned}$$

Thus, $\triangle\hat{M}\hat{N}\hat{K}$ is a Heronian triangle.

Since each of the elements of the $(\hat{m}, \hat{n}, \hat{k})$ has a common integer divisor $2xyz \cdot q$, by Lemma 2 the triangle with side lengths (m, n, k) given by

$$\begin{cases} m = x^2(y^2 + z^2 - x^2) \\ n = y^2(x^2 + z^2 - y^2) \\ k = z^2(z^2 - x^2 - y^2) \end{cases}$$

is also a Heronian triangle.

After scaling of the triangle $\triangle MNK$ by a rational factor, three rational angle bisectors remain rationals. $\square$

Corollary 3.1. *Let $\lambda = \gcd(m, n, k)$. By Lemma 2 we can reduce the sizes of the Heronian triangle by a common integer divisor λ to obtain a primitive Heronian triangle with side lengths $(\overline{m}, \overline{n}, \overline{k})$ given by*

$$\begin{cases} \overline{m} = \dfrac{1}{\lambda} x^2(y^2 + z^2 - x^2) \in \mathbb{N} \\ \overline{n} = \dfrac{1}{\lambda} y^2(x^2 + z^2 - y^2) \in \mathbb{N} \\ \overline{k} = \dfrac{1}{\lambda} z^2(z^2 - x^2 - y^2) \in \mathbb{N} \end{cases} \tag{8}$$

whose three rational angle bisectors are rational.

Example 3.

For a given primitive Heronian tuple $(4, 13, 15)$, find the corresponding primitive Heronian tuple such that the lengths of three angle bisectors of the corresponding Heronian triangle are rational numbers.

Solving: it is easy to see, given by the tuple the corresponding Heronian triangle is obtuse. By Theorem 3, substituting $x = 4, y = 13, z = 15$ into the formulæ 7 to obtain

$$\begin{cases} m = 4^2(13^2 + 15^2 - 4^2) = 2^5 \times 3^3 \times 7 = 6048 \\ n = 13^2(4^2 + 15^2 - 13^2) = 2^3 \times 3^2 \times 13^2 = 12168 \\ k = 15^2(15^2 - 4^2 - 13^2) = 2^3 \times 3^2 \times 5^3 = 9000 \end{cases}$$

Dividing by the greatest common divisor $\gcd(m, n, k) = 2^3 \times 3^2 = 72$, we obtain the corresponding primitive Heronian tuple

$$\begin{cases} \overline{m} = m : 72 = 2^2 \times 3 \times 7 = 84 \\ \overline{n} = n : 72 = 13^2 = 169 \\ \overline{k} = k : 72 = 5^3 = 125 \end{cases}$$

It is easy to check that the semi-perimeter $s = 189$, the area

$$S = \sqrt{189 \times 105 \times 20 \times 64} = 5040$$

By the formula of angle bisector, the lengths of three angle bisectors are

$$\begin{cases} d_{\overline{m}} = \dfrac{2}{\overline{n} + \overline{k}} \cdot \sqrt{p(p - \overline{m})\overline{n}\overline{k}} = \dfrac{2}{169 + 125}\sqrt{189 \times 105 \times 169 \times 125} = \dfrac{975}{7} \\ d_{\overline{n}} = \dfrac{2}{\overline{m} + \overline{k}} \cdot \sqrt{p(p - \overline{n})\overline{m}\overline{k}} = \dfrac{2}{84 + 125}\sqrt{189 \times 20 \times 84 \times 125} = \dfrac{12600}{209} \\ d_{\overline{k}} = \dfrac{2}{\overline{n} + \overline{m}} \cdot \sqrt{p(p - \overline{k})\overline{m}\overline{n}} = \dfrac{2}{169 + 84}\sqrt{189 \times 64 \times 84 \times 169} = \dfrac{26208}{253} \end{cases}$$

Therefore, the lengths of three angle bisectors of Heronian triangle with side lengths $(84, 169, 125)$ are rational numbers.

Theorem 4. *If the tuple* (m, n, k) *is such that the corresponding* $\triangle MNK$ *is a primitive Heronian triangle whose three angle bisectors are rationals, then the tuple* (m, n, k) *can be represented as*

$$\begin{cases} m = \dfrac{1}{\lambda} x^2(y^2 + z^2 - x^2) \\ n = \dfrac{1}{\lambda} y^2(x^2 + z^2 - y^2) \\ k = \dfrac{1}{\lambda} z^2(z^2 - x^2 - y^2) \end{cases} \tag{9}$$

where $\lambda = \gcd\left(x^2(y^2 + z^2 - x^2), y^2(x^2 + z^2 - y^2), z^2(z^2 - x^2 - y^2)\right)$ *and* (x, y, z) *is a tuple such that the corresponding obtuse primitive Heronian triangle is similar to the triangle with side lengths* (a, b, c) *given as:*

$$\begin{cases} a = \dfrac{m - n + k}{2} \cdot \sqrt{\dfrac{(m + n + k)(-m + n + k)}{nk}} \\ b = \dfrac{-m + n + k}{2} \cdot \sqrt{\dfrac{(m + n + k)(m - n + k)}{mk}} \\ c = \dfrac{m + n + k}{2} \sqrt{\dfrac{(-m + n + k)(m - n + k)}{mn}} \end{cases} \tag{10}$$

Proof. Suppose the tuple (m, n, k) is a Heronian tuple such that the corresponding Heronian $\triangle MNK$ has three rational angle bisectors. By Figure 4 we can construct an excircle O opposite the vertex K. Let the three, tangential to the sides of the $\triangle MNK$ or their extensions, points of O be A, B, C.

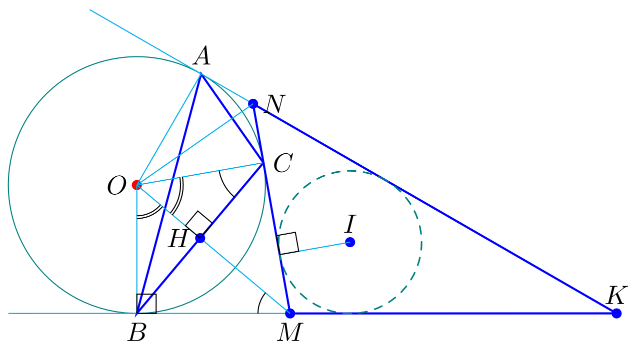

Figure 4

From the proof of Theorem 3, we have

$$\angle NMK = 2\angle BAC = 2A$$
$$\angle MNK = 2\angle ABC = 2B$$
$$\angle MKN = 2\angle ACB - \pi = 2C - \pi$$

It follows that $\angle C$ is obtuse and the triangle $\triangle ABC$ is obtuse. By the assumptions and Lemma 3, we have $\sin A, \sin B, \sin C \in \mathbb{Q}$, so $BC, AC, AB \in \mathbb{Q}$ and $S_{\triangle ABC} \in \mathbb{Q}$. Let denote the tuple of the sides of $\triangle ABC$ as (a, b, c), where $a = BC, b = AC, c = AB$.

The distances from the vertex M to the extouchpoint C can be obtained as

$$MC = MB = \frac{m - n + k}{2}$$

The radius r of the incircle I in terms of the sides of $\triangle MNK$ is:

$$r = \frac{S_{\triangle MNK}}{s} = \frac{\sqrt{s(s-m)(s-n)(s-k)}}{s} = \\ = \sqrt{\frac{(-m+n+k)(m-n+k)(m+n-k)}{4(m+n+k)}}$$

For the $\triangle MNK$ r_K is the exradii opposite K in terms of inradius r and semi-perimeter is given by

$$r_K = OA = OB = OC = \frac{s}{s-k} r = \frac{\sqrt{s(s-m)(s-n)(s-k)}}{s-k} \\ = \sqrt{\frac{(m+n+k)(-m+n+k)(m-n+k)}{4(m+n-k)}}$$

From the right-angled triangles $\triangle OBM$ the distance from the vertex M to the center of the incircle O obtains

$$MO = \sqrt{OB^2 + MB^2} = \\ = \sqrt{\frac{(m+n+k)(-m+n+k)(m-n+k)}{4(m+n-k)} + \frac{(m-n+k)^2}{4}} = \\ = \sqrt{\frac{nk(m-n+k)}{(m+n-k)}}$$

From the similarity of triangles $\triangle OBM$ and $\triangle OHC$ it implies

$$\frac{CH}{OC} = \frac{MB}{MO}$$

$$CH = \frac{MB \cdot OC}{MO} = \\ = \frac{\frac{m-n+k}{2} \cdot \sqrt{\frac{(m+n+k)(-m+n+k)(m-n+k)}{4(m+n-k)}}}{\sqrt{\frac{nk(m-n+k)}{(m+n-k)}}} = \\ = \frac{m-n+k}{4} \cdot \sqrt{\frac{(m+n+k)(-m+n+k)}{nk}}$$

$$a = BC = 2 \cdot CH = \frac{m-n+k}{2} \cdot \sqrt{\frac{(m+n+k)(-m+n+k)}{nk}}$$

Similarly,

$$b = AC = \frac{-m+n+k}{2} \cdot \sqrt{\frac{(m+n+k)(m-n+k)}{mk}}$$

By the Law of Cosine we have

$$\cos \angle MKN = \frac{m^2+n^2-k^2}{2mn}$$

Tangents from the point K to the excircle are equal, therefore $KA = KB = s = \frac{1}{2}(m+n+k)$. By the Law of Cosine we obtain

$$c = AB = \sqrt{2s^2 - 2s^2 \cos \angle MKN} = \frac{m+n+k}{2} \sqrt{\frac{(-m+n+k)(m-n+k)}{mn}}$$

From the proof of Theorem 3 the side lengths (m, n, k) of $\triangle MNK$ can be expressed as

$$\begin{cases} m = \dfrac{2a^3bc}{(a^2+b^2-c^2)(c^2-a^2-b^2)} \\ n = \dfrac{2ab^3c}{(b^2+c^2-a^2)(c^2-a^2-b^2)} \\ k = \dfrac{2abc^3}{(b^2+c^2-a^2)(a^2+c^2-b^2)} \end{cases}$$

By scaling the sides of $\triangle ABC$, we can obtain an acute primitive Heronian $\triangle XYZ$, denote the lengths of its three sides by x, y, z. Since $\triangle ABC \sim \triangle XYZ$ and the sides of both triangles are rational numbers, thus there exists some $\lambda \in \mathbb{Q}$, such that

$$a = \frac{x}{\lambda}, \qquad b = \frac{y}{\lambda}, \qquad c = \frac{z}{\lambda}$$

Therefore

$$\begin{cases} m = \dfrac{2a^3bc}{(a^2+b^2-c^2)(c^2-a^2-b^2)} = \dfrac{1}{\lambda} \cdot \dfrac{2x^3yz}{(x^2+y^2-z^2)(z^2-x^2-y^2)} \\ n = \dfrac{2ab^3c}{(b^2+c^2-a^2)(c^2-a^2-b^2)} = \dfrac{1}{\lambda} \cdot \dfrac{2xy^3z}{(y^2+z^2-x^2)(z^2-x^2-y^2)} \\ k = \dfrac{2abc^3}{(b^2+c^2-a^2)(a^2+c^2-b^2)} = \dfrac{1}{\lambda} \cdot \dfrac{2xyz^3}{(y^2+z^2-x^2)(x^2+z^2-y^2)} \end{cases}$$

Multiplying by

$$\frac{(y^2+z^2-x^2)(x^2+z^2-y^2)(z^2-x^2-y^2)}{2xyz}$$

we obtain a tuple (m, n, k)

$$\begin{cases} m = \dfrac{1}{\lambda} \cdot x^2(y^2 + z^2 - x^2) \\ n = \dfrac{1}{\lambda} \cdot y^2(x^2 + z^2 - y^2) \\ k = \dfrac{1}{\lambda} \cdot z^2(z^2 - x^2 - y^2) \end{cases}$$

Since $\triangle MNK$ is a primitive Heronian triangle, so $\gcd(m, n, k) = 1$, whereas $m, n, k, x^2(y^2 + z^2 - x^2), y^2(x^2 + z^2 - y^2), z^2(z^2 - x^2 - y^2) \in \mathbb{N}$, so

$$\begin{cases} m = \dfrac{x^2(y^2 + z^2 - x^2)}{\gcd(x^2(y^2 + z^2 - x^2), y^2(x^2 + z^2 - y^2), z^2(z^2 - x^2 - y^2))} \\ n = \dfrac{y^2(x^2 + z^2 - y^2)}{\gcd(x^2(y^2 + z^2 - x^2), y^2(x^2 + z^2 - y^2), z^2(z^2 - x^2 - y^2))} \\ k = \dfrac{z^2(x^2 + y^2 - z^2)}{\gcd(x^2(y^2 + z^2 - x^2), y^2(x^2 + z^2 - y^2), z^2(z^2 - x^2 - y^2))} \end{cases}$$

Therefore

$$\lambda = \gcd\big(x^2(y^2 + z^2 - x^2), y^2(x^2 + z^2 - y^2), z^2(z^2 - x^2 - y^2)\big)$$

This completes the proof. $\square$

Example 4.

For a given Heronian tuple $(125, 154, 169)$, such that the lengths of three angle bisectors of the corresponding Heronian triangle are rational numbers, find the three corresponding primitive obtuse Heronian tuples producing the given.

Solving: By the Theorem 4, substituting $m = 125, n = 154, k = 169$ into the formulæ 10 to obtain

$$\begin{cases} a = \dfrac{125 - 154 + 169}{2} \cdot \sqrt{\dfrac{(125 + 154 + 169)(-125 + 154 + 169)}{154 \cdot 169}} = \dfrac{1680}{13} \\ b = \dfrac{-125 + 154 + 169}{2} \cdot \sqrt{\dfrac{(125 + 154 + 169)(125 - 154 + 169)}{125 \cdot 169}} = \dfrac{11088}{65} \\ c = \dfrac{125 + 154 + 169}{2} \sqrt{\dfrac{(-125 + 154 + 169)(125 - 154 + 169)}{125 \cdot 154}} = \dfrac{1344}{5} \end{cases}$$

Multiplying by the greatest rational divisor $\lambda = \dfrac{\operatorname{lcm}(13, 65, 5)}{\gcd(1680, 11088, 1344)} = \dfrac{65}{336}$, we obtain the corresponding primitive acute Heronian tuple

$$\begin{cases} x = \lambda \cdot a = \dfrac{65}{336} \cdot \dfrac{1680}{13} = 25 \\ y = \lambda \cdot b = \dfrac{65}{336} \cdot \dfrac{11088}{65} = 33 \\ z = \lambda \cdot c = \dfrac{65}{336} \cdot \dfrac{1344}{5} = 52 \end{cases}$$

Check that the semi-perimeter $s = 55$, the area

$$S = \sqrt{55 \times 30 \times 22 \times 3} = 330$$

Check the triangle is obtuse:

$$52^2 = 2704 > 1714 = 25^2 + 33^2$$

Check the tuple $(25, 33, 52)$ by the formulæ 9 produces the given:

$$\begin{cases} \hat{m} = 25^2(33^2 + 52^2 - 25^2) = 1980000 \\ \hat{n} = 33^2(25^2 + 52^2 - 33^2) = 2439360 \\ \hat{k} = 52^2(52^2 - 25^2 - 33^2) = 2676960 \end{cases}$$

Dividing out by the greatest common divisor $\gcd(\hat{m}, \hat{n}, \hat{k}) = 15840$, we obtain the given Heronian tuple $(125, 154, 169)$.

Since the formulæ 10 are asymmetric against k, applying them in other orders we obtain two others obtuse primitive Heronian triangles.

Substituting $m = 125, n = 169, k = 154$ into the formulæ 10 to obtain

$$\begin{cases} a = \dfrac{125 - 169 + 154}{2} \cdot \sqrt{\dfrac{(125 + 169 + 154)(-125 + 169 + 154)}{169 \cdot 154}} = \dfrac{1320}{13} \\ b = \dfrac{-125 + 169 + 154}{2} \cdot \sqrt{\dfrac{(125 + 169 + 154)(125 - 169 + 154)}{125 \cdot 154}} = \dfrac{792}{5} \\ c = \dfrac{125 + 169 + 154}{2} \sqrt{\dfrac{(-125 + 169 + 154)(125 - 169 + 154)}{125 \cdot 169}} = \dfrac{14784}{65} \end{cases}$$

Multiplying by the greatest rational divisor $\lambda = \dfrac{\operatorname{lcm}(13, 5, 65)}{\gcd(1320, 792, 14784)} = \dfrac{65}{264}$, we obtain the corresponding primitive acute Heronian tuple

$$\begin{cases} x = \lambda \cdot a = \dfrac{65}{264} \cdot \dfrac{1320}{13} = 25 \\ y = \lambda \cdot b = \dfrac{65}{264} \cdot \dfrac{792}{5} = 39 \\ z = \lambda \cdot c = \dfrac{65}{264} \cdot \dfrac{14784}{65} = 56 \end{cases}$$

Check that the semi-perimeter $s = 60$, the area

$$S = \sqrt{60 \times 35 \times 21 \times 4} = 420$$

Check the triangle is obtuse:

$$56^2 = 3136 > 2146 = 25^2 + 39^2$$

Check the tuple $(25, 39, 56)$ by the formulæ 9 produces the given:

$$\begin{cases} \hat{m} = 25^2(39^2 + 56^2 - 25^2) = 2520000 \\ \hat{n} = 39^2(25^2 + 56^2 - 39^2) = 3407040 \\ \hat{k} = 56^2(56^2 - 25^2 - 39^2) = 3104640 \end{cases}$$

Dividing out by the greatest common divisor $\gcd(\hat{m}, \hat{n}, \hat{k}) = 20160$, we obtain the given Heronian tuple $(125, 169, 154)$.

Similarly, substituting $m = 154, n = 169, k = 125$ into the formulæ 10 to obtain

$$\begin{cases} a = \dfrac{154 - 169 + 125}{2} \cdot \sqrt{\dfrac{(154 + 169 + 125)(-154 + 169 + 125)}{169 \cdot 125}} = \dfrac{1232}{13} \\ b = \dfrac{-154 + 169 + 125}{2} \cdot \sqrt{\dfrac{(154 + 169 + 125)(154 - 169 + 125)}{154 \cdot 125}} = \dfrac{112}{1} \\ c = \dfrac{154 + 169 + 125}{2} \sqrt{\dfrac{(-154 + 169 + 125)(154 - 169 + 125)}{154 \cdot 169}} = \dfrac{2240}{13} \end{cases}$$

Multiplying by the greatest rational divisor $\lambda = \dfrac{\operatorname{lcm}(13, 1, 13)}{\gcd(1232, 112, 2240)} = \dfrac{13}{112}$, we obtain the corresponding primitive acute Heronian tuple

$$\begin{cases} x = \lambda \cdot a = \dfrac{13}{112} \cdot \dfrac{1232}{13} = 11 \\ y = \lambda \cdot b = \dfrac{13}{112} \cdot \dfrac{112}{1} = 13 \\ z = \lambda \cdot c = \dfrac{13}{112} \cdot \dfrac{2240}{13} = 20 \end{cases}$$

Check that the semi-perimeter $s = 22$, the area

$$S = \sqrt{22 \times 11 \times 9 \times 2} = 66$$

Check the triangle is obtuse:

$$20^2 = 400 > 290 = 11^2 + 13^2$$

Check the tuple $(11, 13, 20)$ by the formulæ 9 produces the given:

$$\begin{cases} \hat{m} = 11^2(13^2 + 20^2 - 11^2) = 54208 \\ \hat{n} = 13^2(11^2 + 20^2 - 13^2) = 59488 \\ \hat{k} = 20^2(20^2 - 11^2 - 13^2) = 44000 \end{cases}$$

Dividing out by the greatest common divisor $\gcd(\hat{m}, \hat{n}, \hat{k}) = 352$, we obtain the given Heronian tuple $(154, 169, 125)$.

Therefore, the tuples $(25, 33, 52), (25, 39, 56), (11, 13, 20)$ are such that the corresponding primitive obtuse Heronian triangles producing the given.

Theorem 5. *Let (x, y, z) be a tuple such that the related Heronian triangle is not right-angled. Then the triangle with side lengths (m, n, k) given by*

$$\begin{cases} m = x^2 \cdot |y^2 + z^2 - x^2| \\ n = y^2 \cdot |x^2 + z^2 - y^2| \\ k = z^2 \cdot |x^2 + y^2 - z^2| \end{cases} \tag{11}$$

is a Heronian triangle with three rational angle bisectors.

Proof. Let $\triangle XYZ$ be a Heronian triangle whose lengths of three sides are a Heronian tuple (x, y, z).

If the triangle $\triangle XYZ$ is acute, then for all sides of the triangle the inequalities hold:

$$\begin{cases} x^2 + y^2 > z^2 \\ x^2 + z^2 > y^2 \\ y^2 + z^2 > x^2 \end{cases}$$

Therefore the formulæ

$$\begin{cases} m = x^2 \cdot |y^2 + z^2 - x^2| = x^2(y^2 + z^2 - x^2) \\ n = y^2 \cdot |x^2 + z^2 - y^2| = y^2(x^2 + z^2 - y^2) \\ k = z^2 \cdot |x^2 + y^2 - z^2| = z^2(x^2 + y^2 - z^2) \end{cases}$$

are same the Theorem 1 gives.

If the triangle $\triangle XYZ$ is obtuse, then for the largest side, say z, the inequality

$$x^2 + y^2 < z^2$$

hold.

Therefore the formulæ

$$\begin{cases} m = x^2 \cdot |y^2 + z^2 - x^2| = x^2(y^2 + z^2 - x^2) \\ n = y^2 \cdot |x^2 + z^2 - y^2| = y^2(x^2 + z^2 - y^2) \\ k = z^2 \cdot |x^2 + y^2 - z^2| = z^2(z^2 - x^2 - y^2) \end{cases}$$

are same the Theorem 3 gives. ∎

Perfect Cuboid and Perfect Square Triangle

Definition 7 (The Perfect Cuboid Problem (PCP))**.** *Is there a rectangular box with all edges, face diagonals, and main diagonals integers?*

Definition 8 (The Perfect Square Triangle Problem (PSTP))**.** *Is there a triangle whose sides are perfect squares and whose angle bisectors are integer?*

In 2000, Mexican mathematician Luca[3] proved the following interesting theorem which implies that PCP is equivalent to PSTP, the proof of which we present almost unchanged:

Theorem 6. *In this note, we show that the existence of a solution for the PCP is equivalent to the existence of a solution for the PSTP.*

Proof. Let us first observe that the word "integers" can be replaced by the word "rationals" in the statement of the PSTP. In order to show that the existence of a solution to the PCP is indeed equivalent to the existence of a solution to the PSTP, assume first that the PCP has a solution. Let x, y, and z be the edges of a perfect ,cuboid and set

$$a = y^2 + z^2, \qquad b = x^2 + z^2, \qquad c = x^2 + y^2. \tag{12}$$

Clearly, a, b, and c are the sides of a triangle and are perfect squares. Let $p = \frac{a+b+c}{2}$ be the semiperimeter of this triangle. Since

$$p = x^2 + y^2 + z^2, \qquad p - a = x^2, \qquad p - b = y^2, \qquad p - c = z^2, \tag{13}$$

we conclude that all four numbers $p, p-a, p-b$, and $p-c$ are perfect squares. Let l_a, l_b, and l_c be the lengths of the angle bisectors drawn from the angles opposite to the sides a, b, and c, respectively. It is well known that the lengths of these angle bisectors are given in terms of a, b, and c by

$$l_a = 2 \cdot \frac{\sqrt{bcp(p-a)}}{b+c}, \qquad l_b = 2 \cdot \frac{\sqrt{acp(p-b)}}{a+c}, \qquad l_c = 2 \cdot \frac{\sqrt{abp(p-c)}}{a+b}. \tag{14}$$

Since all the numbers listed in 12 and 13 are perfect squares, it follows, by formula 14, that the triangle with sides a, b, and c is a solution of the PSTP (once "integers" has been replaced by "rational" in the statement of the problem).

Conversely, assume now that the PSTP has a solution. Let a, b, and c be the sides of a triangle that solves this problem. We may assume that $\gcd(a, b, c) = 1$. Indeed, otherwise, let $d = \gcd(a, b, c)$. Since a, b, and c are all perfect squares, so is d. Then the triangle with sides $a/d, b/d$, and c/d still solves the PSTP, and $\gcd(a/d, b/d, c/d) = 1$.

By formula 14 and the fact that a, b, and c are perfect squares, it follows that all three integers

$$4p(p-a) = (b+c)^2-a^2, \quad 4p(p-b) = (a+c)^2-b^2, \quad 4p(p-c) = (a+b)^2-c^2 \tag{15}$$

are perfect squares. Since $\gcd(a, b, c) = 1$, it follows that not all of a, b, and c can be even. Reducing modulo 4 the integers listed in 15, one concludes that exactly one of the three numbers a, b, and c is even, and the other two are odd. It now follows that p is an integer, and formula 15 implies that all three integers

$$p(p-a), \qquad p(p-b), \qquad p(p-c) \tag{16}$$

are perfect squares. We now show that $\gcd(p-a, p-b, p-c) = 1$. Indeed, let $e = \gcd(p-a, p-b, p-c)$. Clearly, $e|(p-b)+(p-c) = a$. By a similar argument, one concludes that $e|b$ and $e|c$, Since $\gcd(a, b, c) = 1$, it follows that $e = 1$. Since all three numbers listed in 16 are perfect squares, so is their greatest common divisor. Hence,

$$\gcd(p(p-a), p(p-b), p(p-c)) = pe = p$$

is a perfect square. It now follows (again from the fact that the three numbers in 16 are perfect squares) that all four numbers $p, p-a, p-b, p-c$ are perfect squares.

$$x = \sqrt{p-a}, \qquad y = \sqrt{p-b}, \qquad z = \sqrt{p-c}, \tag{17}$$

then one concludes easily that x, y, and z are the edges of a perfect cuboid. $\square$

Perfect Cuboid and Heronian Triangles

Applying the conclusions of the Theorem 6 to the original system of equations 1, we obtain

Theorem 7. *If a Perfect cuboid exists and a, b, c are its edges, the corresponding face diagonals d, e, f and the body diagonal g, then the triangle with side lengths (d^2, e^2, f^2) is a Heronian triangle an area $abcg$ with three rational angle bisectors.*

Proof.

$$a, b, c, d, e, f, g \in \mathbb{N}$$

In the triangle with side lengths (d^2, e^2, f^2) semi-perimeter s obtains

$$s = \frac{d^2 + e^2 + f^2}{2} = g^2$$
$$s - d^2 = \frac{-d^2 + e^2 + f^2}{2} = c^2$$
$$s - e^2 = \frac{d^2 - e^2 + f^2}{2} = b^2$$
$$s - f^2 = \frac{d^2 + e^2 - f^2}{2} = a^2$$

The area of the triangle obtains

$$S_\triangle = \sqrt{s(s - d^2)(s - e^2)(s - f^2)} = abcg \in \mathbb{N}$$

Well-known formula of angle bisectors in terms of triangle sides gives

$$l_d = 2 \cdot \frac{\sqrt{e^2 f^2 s(s - d^2)}}{e^2 + f^2} = \frac{2cefg}{e^2 + f^2} \in \mathbb{Q}$$
$$l_e = 2 \cdot \frac{\sqrt{d^2 f^2 s(s - e^2)}}{d^2 + f^2} = \frac{2bdfg}{d^2 + f^2} \in \mathbb{Q}$$
$$l_f = 2 \cdot \frac{\sqrt{d^2 e^2 s(s - f^2)}}{d^2 + e^2} = \frac{2adeg}{d^2 + e^2} \in \mathbb{Q}$$

Thus, if a Perfect cuboid exists, the squares of three its face diagonals (d^2, e^2, f^2) should construct a Heronian triangle with rational angle bisectors. □

Theorem 8. *If a Perfect cuboid exists and a, b, c are its edges, the corresponding face diagonals d, e, f and the body diagonal g, then the acute triangle with side lengths (af, be, cd), the obtuse triangles with side lengths $(bf, ae, gd), (ad, cf, ge), (ce, bd, gf)$ are Heronian triangles an equal area $\dfrac{abcg}{2}$.*

Proof.

$$a, b, c, d, e, f, g \in \mathbb{N} \Rightarrow af, be, cd, bf, ae, gd, ad, cf, ge, ce, bd, gf \in \mathbb{N}$$

Applying Heron's formula, the area of a triangle with side lengths (af, be, cd) obtains

$$\begin{aligned}S_{\triangle} &= \frac{1}{4}\sqrt{4a^2f^2b^2e^2 - (a^2f^2 + b^2e^2 - c^2d^2)^2} \\ &= \frac{1}{4}\sqrt{4a^2f^2b^2e^2 - 4a^4b^4} = \frac{ab}{2}\sqrt{e^2f^2 - a^2b^2} \\ &= \frac{ab}{2}\sqrt{c^2g^2} = \frac{abcg}{2} \in \mathbb{N}\end{aligned}$$

since at least two edges of an Euler cuboid are even.

The area of a triangle with side lengths (bf, ae, gd) obtains

$$\begin{aligned}S_{\triangle} &= \frac{1}{4}\sqrt{4b^2f^2a^2e^2 - (b^2f^2 + a^2e^2 - g^2d^2)^2} \\ &= \frac{1}{4}\sqrt{4a^2b^2e^2f^2 - 4a^4b^4} = \frac{ab}{2}\sqrt{e^2f^2 - a^2b^2} \\ &= \frac{ab}{2}\sqrt{c^2g^2} = \frac{abcg}{2} \in \mathbb{N}\end{aligned}$$

The area of a triangle with side lengths (ad, cf, ge) obtains

$$\begin{aligned}S_{\triangle} &= \frac{1}{4}\sqrt{4a^2d^2c^2f^2 - (a^2d^2 + c^2f^2 - g^2e^2)^2} \\ &= \frac{1}{4}\sqrt{4a^2c^2d^2f^2 - 4a^4c^4} = \frac{ac}{2}\sqrt{d^2f^2 - a^2c^2} \\ &= \frac{ac}{2}\sqrt{b^2g^2} = \frac{abcg}{2} \in \mathbb{N}\end{aligned}$$

The area of a triangle with side lengths (ce, bd, gf) obtains

$$\begin{aligned}S_{\triangle} &= \frac{1}{4}\sqrt{4c^2e^2b^2d^2 - (c^2e^2 + b^2d^2 - g^2f^2)^2} \\ &= \frac{1}{4}\sqrt{4b^2c^2d^2e^2 - 4b^4c^4} = \frac{bc}{2}\sqrt{d^2e^2 - b^2c^2} \\ &= \frac{bc}{2}\sqrt{a^2g^2} = \frac{abcg}{2} \in \mathbb{N}\end{aligned}$$

Since

$$\begin{cases} a^2f^2 + b^2e^2 = a^2(b^2 + c^2) + b^2(a^2 + c^2) = c^2d^2 + 2a^2b^2 < c^2d^2 \\ a^2f^2 + c^2d^2 = a^2(b^2 + c^2) + c^2(a^2 + b^2) = b^2e^2 + 2a^2c^2 < b^2e^2 \\ b^2e^2 + c^2d^2 = b^2(a^2 + c^2) + c^2(a^2 + b^2) = a^2f^2 + 2b^2c^2 < a^2f^2 \end{cases}$$

So the triangle with side lengths (af, be, cd) is acute.

$$\begin{aligned} b^2f^2 + a^2e^2 &= b^2(b^2 + c^2) + a^2(a^2 + c^2) = a^4 + b^4 + c^2d^2 \\ &< (a^2 + b^2)^2 + c^2d^2 = d^2(c^2 + d^2) = d^2g^2 \end{aligned}$$

Briefly $b^2f^2+a^2e^2 < d^2g^2$, therefore the triangle with side lengths (bf, ae, gd) is obtuse. Similarly we prove, that the triangles with side lengths (ad, cf, ge), (ce, bd, gf) are obtuse. $\square$

Perfect Cuboid and Heronian Tetrahedron

Theorem 9. *If there exists a Heronian trirectangular tetrahedron with three orthogonal edges a, b, c, then the cuboid with edges (ab, ac, bc) is a Perfect cuboid.*

Proof. The Euler cuboid with edges a, b, c and the face diagonals d, e, f uniquely corresponds to the given trirectangular tetrahedron (the edges that meet at the right angle highlighted in teal in the Figure 5).

Clearly, the face diagonals (d, e, f) of an Euler cuboid constitute an acute triangle (highlighted in blue in the Figure 5).

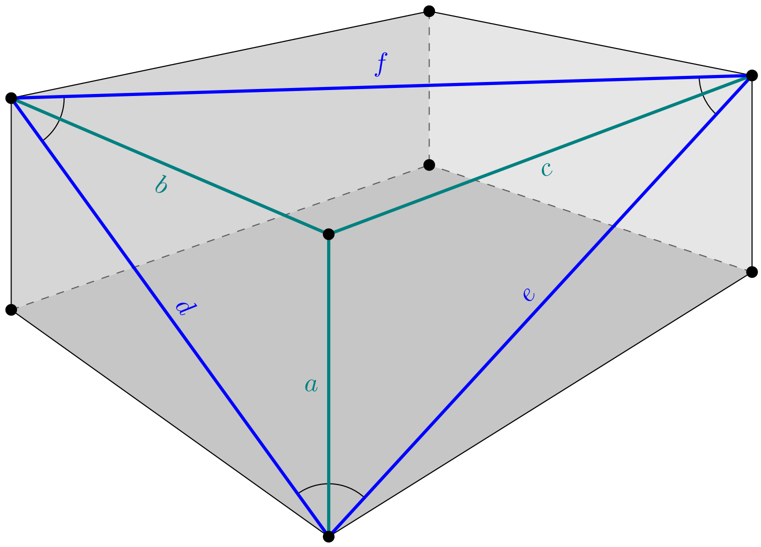

Figure 5

By the Heron's formula the area of the triangle $\triangle def$ obtains

$$S_{\triangle def} = \frac{1}{4}\sqrt{4d^2e^2 - (d^2+e^2-f^2)^2} = \frac{1}{4}\sqrt{4(a^2+b^2)^2(a^2+c^2)^2 - 4a^4}$$
$$= \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}$$

Since the triangle is Heronian, so firstly $S_{\triangle def} \in \mathbb{N}$.

Secondly, by the Theorem 1 the triangle with side lengths (m, n, k) given by

$$\begin{cases} m = d^2(e^2+f^2-d^2) = 2c^2d^2 \\ n = e^2(d^2+f^2-e^2) = 2b^2e^2 \\ k = f^2(d^2+e^2-f^2) = 2a^2f^2 \end{cases}$$

is a Heronian triangle with three rational angle bisectors.

Due to the Lemma 2 the sides of the triangle can be reduced by 2 times without losing the integrality of the sides and area.

Therefore the triangle with sides (a^2f^2, b^2e^2, c^2d^2) is a Heronian triangle with rational angle bisectors. The semi-perimeter of this triangle obtains

$$p = \frac{a^2f^2 + b^2e^2 + c^2d^2}{2}$$

Applying the conclusions of the Theorem 6 to the original system of equations 1, from the formulæ 17 we obtain that the cuboid with sides

$$\begin{cases} A = \sqrt{p - a^2f^2} = \sqrt{\dfrac{-a^2f^2 + b^2e^2 + c^2d^2}{2}} = \sqrt{b^2c^2} = bc \in \mathbb{N} \\ B = \sqrt{p - b^2e^2} = \sqrt{\dfrac{a^2f^2 - b^2e^2 + c^2d^2}{2}} = \sqrt{a^2c^2} = ac \in \mathbb{N} \\ C = \sqrt{p - c^2d^2} = \sqrt{\dfrac{a^2f^2 + b^2e^2 - c^2d^2}{2}} = \sqrt{a^2b^2} = ab \in \mathbb{N} \end{cases}$$

is a Perfect cuboid.

In fact, let D, E, F be the face diagonals of this cuboid given as

$$\begin{cases} D = \sqrt{A^2 + B^2} = \sqrt{b^2c^2 + a^2c^2} = \sqrt{c^2d^2} = cd \in \mathbb{N} \\ E = \sqrt{A^2 + C^2} = \sqrt{b^2c^2 + a^2b^2} = \sqrt{b^2e^2} = be \in \mathbb{N} \\ F = \sqrt{B^2 + C^2} = \sqrt{a^2c^2 + a^2b^2} = \sqrt{a^2f^2} = af \in \mathbb{N} \end{cases}$$

The body diagonal G can be expressed as

$$G = \sqrt{A^2 + B^2 + C^2} = \sqrt{a^2b^2 + a^2c^2 + b^2c^2} = 2 \cdot S_{\triangle def} \in \mathbb{N}$$

□

Theorem 10. *If there exists a Perfect cuboid with edges a, b, c, then the trirectangular tetrahedron with three orthogonal legs $2ab, 2ac, 2bc$ is Heronian.*

Proof. If there exist a Perfect cuboid with edges (a, b, c), the face diagonals (d, e, f) and the body diagonal g, then according to the Theorem 8, the acute triangle with side lengths (af, be, cd) is Heronian.

Clearly, the triangle $\triangle XYZ$ with side lengths $(2af, 2be, 2cd)$ is also Heronian.

In the cuboid with three integer edges $(2ab, 2ac, 2bc)$, the face diagonals obtain

$$\begin{cases} \sqrt{4a^2b^2 + 4a^2c^2} = 2\sqrt{a^2f^2} = 2af \in \mathbb{N} \\ \sqrt{4a^2b^2 + 4b^2c^2} = 2\sqrt{b^2e^2} = 2be \in \mathbb{N} \\ \sqrt{4a^2c^2 + 4b^2c^2} = 2\sqrt{c^2d^2} = 2cd \in \mathbb{N} \end{cases}$$

In the presented cuboid three orthogonal edges constitute a trirectangular tetrahedron. In the tetrahedron all edges are integer, as well as the areas of faces and the volume. Indeed, above we have already showed that the triangle with side lengths $(2af, 2be, 2cd)$ is Heronian. Clearly, the areas of three other faces of the tetrahedron are integer, $2a^2b^2, 2a^2c^2$ and $2b^2c^2$ respectively. The volume of the tetrahedron by the well-know formula obtains $a^2b^2c^2$. Therefore the presented trirectangular tetrahedron is Heronian. $\square$

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