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SYMMETRY-BASED APPROACH TO THE PROBLEM OF A PERFECT CUBOID

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Abstract. A perfect cuboid is a rectangular parallelepiped in which the lengths of all edges, the lengths of all face diagonals, and also the lengths of spatial diagonals are integers. No such cuboid has yet been found, but their nonexistence has also not been proved. The problem of a perfect cuboid is among unsolved mathematical problems. The problem has a natural S_3-symmetry connected to permutations of edges of the cuboid and the corresponding permutations of face diagonals. In this paper, we give a survey of author's results and results of J. R. Ramsden on using the S_3 symmetry for the reduction and analysis of the Diophantine equations for a perfect cuboid.

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1. A brief historical survey. The problem of a perfect cuboid was first mentioned about 300 years ago in the book by Paul Halcke [14], which is a collection of recreational mathematical problems and puzzles. It is reduced to the following system of Diophantine equations:

$$\begin{aligned} &(x_1)^2+(x_2)^2-(d_3)^2=0, \qquad &&(x_2)^2+(x_3)^2-(d_1)^2=0,\\ &(x_3)^2+(x_1)^2-(d_2)^2=0, \qquad &&(x_1)^2+(x_2)^2+(x_3)^2=L^2. \end{aligned} \tag{1.1}$$

Here x_1, x_2, and x_3 are the lengths of the edges of the cuboid, d_1, d_2, and d_3 are the lengths of the diagonals on the faces, and L is the length of the spatial diagonal. In a perfect cuboid, all these values must be integers. If only one of them is not an integer, then such a cuboid is said to be almost perfect. Almost perfect cuboids are known. A big list of almost perfect cuboids found by computer calculations is contained in the appendix to [43]. The first examples of almost perfect cubiods where the length of the spatial diagonal L is not an integer were found by Saunderson in [49]. They are parametrized by the Pythagoras triples of integers $u^2+v^2=w^2$ and are given by the formulas

$$x_1=u\bigl|4v^2-w^2\bigr|, \quad x_2=v\bigl|4u^2-w^2\bigr|, \quad x_3=4uvw. \tag{1.2}$$

L. Euler was interested in this problem after Saunderson. In [37], he found a parametric family of almost perfect cuboids expressed by the formulas

$$\begin{aligned} x_1&=\Bigl|2mn\bigl(3m^2-n^2\bigr)\bigl(3n^2-m^2\bigr)\Bigr|,\\ x_2&=\Bigl|\bigl(m^2-n^2\bigr)\bigl(m^2-4mn+n^2\bigr)\bigl(m^2+4mn+n^2\bigr)\Bigr|,\\ x_3&=\Bigl|8mn\bigl(m^4-n^4\bigr)\Bigr|. \end{aligned} \tag{1.3}$$

The Euler formulas (1.3) are embedded into the Saunderson formulas (1.2) by the substitution

$$u=2mn, \quad v=\bigl|m^2-n^2\bigr|, \quad w=m^2+n^2.$$

Despite this circumstance and contrary to the historical justice, almost perfect cuboids with one noninteger parameter L are called Euler cubes or Euler bricks.

After Euler, the problem on perfect cuboids was forgotten for a long time. Interest in it revived only in the 20th century. Pocklington [37] and Spohn [71] proved that the Saunderson series of almost

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perfect cuboids (1.2) does not contain perfect cuboids. From the Saunderson series of almost perfect cuboids (1.2), one can obtain the following series of almost perfect cuboids:

$$\tilde{x}_1 = x_2\,x_3, \quad \tilde{x}_2 = x_3\,x_1, \quad \tilde{x}_3 = x_1\,x_2. \tag{1.4}$$

Chein (see [5]), Lagrange[1] (see [24]), and Leech [26] proved that the derived series of almost perfect cuboids (1.4) does not contain perfect cuboids as well.

In addition to almost perfect cuboids with noninteger length of the spatial diagonal, almost perfect cuboids were considered in which one of the lengths of face diagonals or one of the lengths of edges is not integer. Such almost perfect cuboids were studied by Colman in [6, 7].

Returning to perfect cuboids, we also mention papers in which certain restrictions on their parameters are proved under the assumption that a perfect cuboid will someday be found. In [13], R. Guy (referring to J. Leech and H. Bergmann) indicated that the product of the lengths of face diagonals and the lengths of edges of a perfect cuboid must be a multiple of the number

$$2^8 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 29 \cdot 37.$$

Some other results on the divisibility of certain parameters of a perfect cuboid were found in [46] by T. S. Roberts.

Another area of research is the construction of numerical examples of almost perfect cuboids, the search for a perfect cuboid by a direct calculation, and obtaining lower estimates for the dimensions of a perfect cuboid. In [21–23], Kraitchik showed that an odd edge of a perfect cuboid cannot be less than 10^6. Later, Lal and Blundon (see [25]) and Korec (see [17, 18]) established by computer calculations that the least edge of a perfect cuboid, independently of its evenness, cannot be less than 10^6. Some estimates were also obtained for the length of the spatial diagonal of a perfect cuboid. Korec (see [19]) found a lower bound $8 \cdot 10^9$ for L. Among recent results, we can also indicate the above-mentioned work [43] of Randall Rathbun (the lower bound $1.55 \cdot 10^{11}$ for the minimal edge) and the work [28] of Matson, where the lower estimate $5 \cdot 10^{11}$ was obtained for the minimal edges and the lower estimate $2.5 \cdot 10^{13}$ for the odd edge.

In 2005, a schoolboy from Tbilisi, Lasha Margishvili, wrote a work in which he claimed that he had proved the nonexistence of perfect cuboids. There are a lot of works with similar statements in the net. But Lasha Margishvili managed to get the Mu Alpha Theta Award in the amount of $1000 at the Science Fair competition in the US. That is why his work resonated with mathematicians and was discussed on the Google groups website:

`https://groups.google.com/forum/#!msg/sci.math/icQFW63yf1E/GiBY1dVdA_gJ`

`https://groups.google.com/forum/#!msg/sci.math/icQFW63yf1E/dvtRtjbGKIgJ`

These references allow one to find the work of Margishvili, which, as it turned out, contains an error and is not currently recognized as a solution of the problem. In later works of his teacher Mamuka Meskhishvili (see [32–34]), the problem on the perfect cuboid is considered as an unsolved mathematical problem.

In a series of papers (see [1, 11, 15, 74, 76]), methods of modern algebraic geometry were applied to the study of the equations of a perfect cuboid (1.1). The ultimate goal of these works is a description of the set of rational points of an algebraic variety determined by Eqs. (1.1). But the matter is complicated by the fact that this variety is not quite typical: typical varieties have no rational points at all, while the variety (1.1) possesses rational points. But all of its known rational points correspond to degenerate cuboids, where at least one of the variables x_1, x_2, x_3 vanishes.

In 2011, using a special change of variables, the author succeeded in transforming the system of equations (1.1) into a single equation of the 12th degree with integer parameters (see [53]). The analysis of this equation led to nontrivial cases where it is reduced to equations of the 10th or 8th

[1] Jean Lagrange is a person bearing the same family name as the famous Joseph-Louis Lagrange, a legend of mechanics and mathematics.

degree. In trivial cases, the degree of the equation is reduced even more, but all these cases correspond to degenerate cuboids. In nontrivial cases, it was not possible to further decrease in the degree of the equation. Hence in [51] three hypotheses on the irreducibility of polynomials in equations of degrees 8, 10, and 12 (three cuboid conjectures) were formulated. Further studies were reflected in [27, 51, 52, 54–59]; however, in this paper we will not discuss this line of research.

Almost perfect cuboids also include integer parallelepipeds, in which the lengths of edges and all diagonals are integers, but the rectangularity condition is violated. Such parallelepipeds exist; they were found by Sawyer and Reiter (see [50]). A more systematic study of nonrectangular perfect cuboids was performed by Sokolowsky, VanHooft, Volkert, and Reiter in [70], and Walter Wyss in [77].

The paper of W. Wyss [77] should be considered separately since it contains a claim to the solution of the problem of the perfect cuboid in its classical formulation for the rectangular case. This is an electronic publication in the archives of Cornell University, widely known as arXiv. At the time of writing the present text (August 11, 2017), the publication of W. Wyss had five versions. The first version was published in 2015 under the title "No perfect cuboids." Each next version corrects and supplements the previous one. The second version is called "On perfect cuboids."

In [77], W. Wyss derived and studied the equations of almost perfect cuboids, in which deviations from rectangularity are allowed at exactly one of the three angles between the edges at each vertex. The class of such almost perfect cuboids includes the class of rectangular perfect cuboids as a special case. Similarly to Eq. (1.1), the equations obtained by Wyss are homogeneous second-degree equations. But the number of variables and the number of equations are greater:

$$\begin{aligned} &(x_1)^2 + (x_2)^2 - (d_3)^2 = 0, && (x_1)^2 + (d_{1a})^2 - (L_a)^2 = 0, \\ &(x_1)^2 + (x_3)^2 - (d_2)^2 = 0, && (x_1)^2 + (d_{1b})^2 - (L_b)^2 = 0, \\ &2\,(x_2)^2 + 2\,(x_3)^2 = (d_{1a})^2 + (d_{1b})^2. \end{aligned} \tag{1.5}$$

Here x_1, x_2, and x_3 are the lengths of the edges, d_2 and d_3 are the lengths of the face diagonals, d_{1a} and d_{1b} are the lengths of the diagonals of the nonrectangular face, and L_a and L_b are the lengths of the spatial diagonals. Equations (1.1) are obtained from (1.5) for $d_{1a} = d_{1b} = d_1$, which implies $L_a = L_b = L$. Using the homogeneity of Eqs. (1.5) and passing from integer solutions to rational, Wyss normalized these solutions by the condition $x_1 = 1$. As a result, he obtained a system of five equations with eight variables. The resulting system of equations defines a three-dimensional real algebraic variety in $\mathbb{R}^8$, and almost perfect cuboids correspond to rational points of this manifold (points with rational coordinates in $\mathbb{R}^8$). Further, Wyss found a two-parameter solution of these equations, which for rational values of the parameters gives a two-parameter family of almost perfect cuboids. Wyss did not compare this family of almost perfect cuboids with the families from [70], but he proved that in the family found, there are no rational points with $d_{1a} = d_{1b} = d_1$ corresponding to classical rectangular perfect cuboids. In the first version of his article, Wyss erroneously concluded from this circumstance that the rectangular perfect cuboids do not exist at all. However, he overlooked the fact that the two-parameter (that is, two-dimensional) family of solutions does not cover the whole three-dimensional algebraic variety in $\mathbb{R}^8$, and there remains the possibility of the existence of rectangular perfect cuboids outside the family constructed by him.

In 2015, the author informed Wyss about the error in his work [77], and after that he published the second version of his work entitled "On perfect cuboids." About a year later, in 2016, the third and the fourth versions of [77] appeared; these two versions are almost identical to each other and have the same name "No perfect cuboid." They contain a new proof of the nonexistence of rectangular perfect cuboids. An analysis of this new proof required a certain amount of time and a computer verification of all Wyss' calculations. In the new proof, I also found an error. I published a detailed report on the results of the analysis of the work [77] and the error found in it in 2017 in [60]. R. Rathbun in his message of March 19 2017 shared my opinion about the presence of a mistake in the new proof of Wyss in the section Number Theory Listserver on the website of the University of North Dakota, USA. Later, he published the paper [44], where he found four new two-parameter families of solutions for Wyss' equations (1.5).

The fifth version of Wyss' paper [77] appeared in 2017 after my publication [60]. It insignificantly differs from the fourth version and contains the same error. Of the two later short notes by Wyss, [78] and [79], the second has nothing to do with the work [77], and in the first, a certain result is proved about rational points of an elliptic curve of a special type under the assumption that the nonexistence of a perfect cuboid is proved. This means that Wyss did not understand or did not accept the arguments from my work [60], although he knew about its existence from 6 April 2017. Further clarification of all the circumstances surrounding the official recognition or nonrecognition of the work [77] of Wyss requires participation of a wider circle of specialists. At the time of writing the present text (August 11, 2017), the classical problem of a rectangular perfect cuboid is officially unresolved.

The further content of this work is based on the earlier results of 2012–2013, obtained by me and J. Ramsden in electronic publications [38–42, 61–68] and provides an overview of these results.

2. Equations of perfect cuboid and the multisymmetric plynomials. We say that a polynomial $p(x_1, \ldots, x_n)$ is symmetric if it is not changed under any permutation of variables σ from the permutation group S_n. The theory of such polynomials is exposed in most textbooks on algebra (see [20]). The concept of a multisymmetric polynomial arises when there is a double, triple, or generally k-fold family of variables:

$$\{x_{ij},\ \text{where } i = 1, \ldots, k,\ j = 1, \ldots, n\}. \tag{2.1}$$

The action of a permutation $\sigma \in S_n$ on a family of variables (2.1) is defined by the formula

$$\sigma(x_{ij}) = x_{i\sigma(j)}. \tag{2.2}$$

A polynomial of a family of variables (2.1) is said to be multisymmetric if it is invariant under all permutations $\sigma \in S_n$ acting by the rule (2.2).

Multisymmetric polynomials are known under many names: they are called vector-symmetric polynomials, diagonally symmetric polynomials, McMahon polynomials, and so on. They were studied in [2–4, 9, 10, 12, 16, 29–31, 35, 36, 45, 47, 48, 69, 72, 73, 75].

In the equations of a perfect cuboid (1.1) we have a two-fold family of variables x_1, x_2, x_3 and d_1, d_2, d_3, and the permutations from the group S_3 act on it. These permutations do not act on the variable L; it can be considered as a parameter, but since it is involved in the equations polynomially, it is also included in the list of variables. In [62], the variables x_1, x_2, x_3 and d_1, d_2, d_3 were included into the matrix

$$M = \begin{Vmatrix} x_1 & x_2 & x_3 \\ d_1 & d_2 & d_3 \end{Vmatrix}. \tag{2.3}$$

This matrix was used for simplifying the notation of the ring of polynomials and the subring of multisymmetric polynomials of the variables x_1, x_2, x_3, d_1, d_2, d_3, and L:

$$\mathbb{Q}\big[x_1, x_2, x_3, d_1, d_2, d_3, L\big] = \mathbb{Q}[M, L], \quad \operatorname{Sym}\mathbb{Q}\,\big[x_1, x_2, x_3, d_1, d_2, d_3, L\big] = \operatorname{Sym}\mathbb{Q}[M, L].$$

Coefficients of such plynomials are chosen in the field of rational numbers $\mathbb{Q}$.

The system of equations of a perfect cuboid (1.1) can be transformed to the form

$$\begin{aligned} &(x_1)^2 + (x_2)^2 - (d_3)^2 = 0, &\quad &(d_3)^2 + (x_3)^2 - L^2 = 0, \\ &(x_2)^2 + (x_3)^2 - (d_1)^2 = 0, &\quad &(d_1)^2 + (x_1)^2 - L^2 = 0, \\ &(x_3)^2 + (x_1)^2 - (d_2)^2 = 0, &\quad &(d_2)^2 + (x_2)^2 - L^2 = 0. \end{aligned} \tag{2.4}$$

It becomes more symmetric, albeit redundant, but remains equivalent to the original system of equations (1.1). Denote by p_1, p_2, p_3, p_4, p_5, and p_6 the polynomials on the left-hand sides of the transformed equations of the perfect cuboid (2.4):

$$\begin{aligned} p_1 &= (x_1)^2 + (x_2)^2 - (d_3)^2, & p_4 &= (d_3)^2 + (x_3)^2 - L^2,\\ p_2 &= (x_2)^2 + (x_3)^2 - (d_1)^2, & p_5 &= (d_1)^2 + (x_1)^2 - L^2,\\ p_3 &= (x_3)^2 + (x_1)^2 - (d_2)^2, & p_6 &= (d_2)^2 + (x_2)^2 - L^2. \end{aligned} \tag{2.5}$$

The polynomials p_1, p_2, p_3, p_4, p_5, and p_6 are not multisymmetric, i.e., they are not invariant under permutations of the variables from S_3. But the system of polynomials (2.5) as a whole is S_3-invariant.

The polynomials p_1, p_2, p_3, p_4, p_5, p_6 from (2.5) generate an ideal in the ring $\mathbb{Q}[M, L]$:

$$I = \big\langle p_1, p_2, p_3, p_4, p_5, p_6 \big\rangle \subset \mathbb{Q}[M, L]. \tag{2.6}$$

Equating to zero any polynomial in the ideal (2.6), we obtain an equation that is a consequence of the original equations (2.4).

The intersection of the ideal (2.6) with the ring of the multisymmetric polynomials $\operatorname{Sym}\mathbb{Q}[M, L]$ yields an ideal in this ring:

$$I_{\text{sym}} = I \cap \operatorname{Sym}\mathbb{Q}[M, L]. \tag{2.7}$$

Definition 1. Each polynomial equation $p(x_1, x_2, x_2, d_1, d_2, d_3, L) = 0$ with a polynomial p from the ideal (2.7) is called a *quotient equation* of the equations of the cuboid (1.1) under the S_3-symmetry.

The passage from Eqs. (1.1) to the quotient equations is considered as a technical simplifying procedure, ultimately directed towards solving the system of perfect cuboid (1.1).

3. Elementary multisymmetric polynomials and the substitution homomorphism. As in the case of symmetric polynomials, there are elementary multisymmetric polynomials in the rings of multisymmetric polynomials, and every multisymmetric polynomial is expressed as a polynomial of the elementary multisymmetric polynomials. In the case of the ring $\operatorname{Sym}\mathbb{Q}[M, L]$ the number of elementary multisymmetric polynomials is 9:

$$\begin{cases} e_{[1,0]} = x_1 + x_2 + x_3,\\ e_{[2,0]} = x_1\,x_2 + x_2\,x_3 + x_3\,x_1,\\ e_{[3,0]} = x_1\,x_2\,x_3, \end{cases} \tag{3.1}$$

$$\begin{cases} e_{[0,1]} = d_1 + d_2 + d_3,\\ e_{[0,2]} = d_1\,d_2 + d_2\,d_3 + d_3\,d_1,\\ e_{[0,3]} = d_1\,d_2\,d_3, \end{cases} \tag{3.2}$$

$$\begin{cases} e_{[2,1]} = x_1\,x_2\,d_3 + x_2\,x_3\,d_1 + x_3\,x_1\,d_2,\\ e_{[1,1]} = x_1\,d_2 + d_1\,x_2 + x_2\,d_3 + d_2\,x_3 + x_3\,d_1 + d_3\,x_1,\\ e_{[1,2]} = x_1\,d_2\,d_3 + x_2\,d_3\,d_1 + x_3\,d_1\,d_2. \end{cases} \tag{3.3}$$

Let us assign to each elementary multisymmetric polynomial in (3.1), (3.2) and (3.3) some new independent variables E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, and E_{12}. Consider the ring of polynomials of these variables and of the variable L. For brevity, we denote this ring by $\mathbb{Q}[E, L]$:

$$\mathbb{Q}[E, L] = \mathbb{Q}\big[E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12}, L\big]. \tag{3.4}$$

The substitution of the elementary multisymmetric polynomials (3.1), (3.2), and (3.3) into the variables E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, and E_{12} transforms the polynomial from the ring (3.4) into a multisymmetric polynomial from the ring $\operatorname{Sym}\mathbb{Q}[M, L]$. The substitution homomorphism

$$\varphi : \mathbb{Q}[E, L] \longrightarrow \operatorname{Sym}\mathbb{Q}[M, L] \tag{3.5}$$

appears.

In the case of the standard classical symmetric polynomials, the substitution homomorphism is bijective, i.e., is an isomorphism. An important difference of multisymmetric polynomials is that the elementary multisymmetric polynomials among them are algebraically dependent. This implies that the substitution homomorphism (3.5) is surjective, but not injective. It has a nontrivial kernel

$$\operatorname{Ker}\varphi = K \neq \{0\}, \tag{3.6}$$

which is an ideal in the ring $\mathbb{Q}[E, L]$. Each ideal in the ring of polynomials $\mathbb{Q}[E, L]$ is finitely generated. It is generated by a finite number of elements:

$$K = \langle q_1, \ldots, q_s \rangle. \tag{3.7}$$

The polynomials $q_1, \ldots, q_s$ that generate the ideal in (3.7) form its basis. An important difference of bases in ideals from bases in linear vector spaces is that the number of the elements in bases of a same ideal can be different. In [62], a basis of the kernel of the substitution homomorphism (3.5) consisting of seven polynomials was found:

$$\begin{aligned} q_1 = {} & 4\,E_{01}\,E_{02}\,E_{20} - E_{02}\,E_{10}^2\,E_{01} - E_{01}^3\,E_{20} + E_{10}\,E_{11}\,E_{01}^2 - E_{11}^2\,E_{01} - 2\,E_{10}\,E_{01}\,E_{12} \\ & + 3\,E_{03}\,E_{10}^2 - 9\,E_{03}\,E_{20} - 3\,E_{21}\,E_{02} + E_{21}\,E_{01}^2 + 3\,E_{11}\,E_{12}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} q_2 = {} & 4\,E_{10}\,E_{20}\,E_{02} - E_{20}\,E_{01}^2\,E_{10} - E_{10}^3\,E_{02} + E_{01}\,E_{11}\,E_{10}^2 - E_{11}^2\,E_{10} - 2\,E_{01}\,E_{10}\,E_{21} \\ & + 3\,E_{30}\,E_{01}^2 - 9\,E_{30}\,E_{02} - 3\,E_{12}\,E_{20} + E_{12}\,E_{10}^2 + 3\,E_{11}\,E_{21}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} q_3 = {} & 9\,E_{21}\,E_{12} - E_{01}^2\,E_{10}\,E_{21} - 6\,E_{10}\,E_{11}\,E_{12} - 6\,E_{01}\,E_{12}\,E_{20} + 5\,E_{01}\,E_{10}^2\,E_{12} \\ & - 3\,E_{11}^3 + 7\,E_{10}\,E_{11}^2\,E_{01} + 12\,E_{11}\,E_{20}\,E_{02} - 3\,E_{01}^2\,E_{11}\,E_{20} - 3\,E_{02}\,E_{10}^2\,E_{11} \\ & - 4\,E_{01}^2\,E_{10}^2\,E_{11} - 81\,E_{03}\,E_{30} + 18\,E_{01}\,E_{02}\,E_{30} - 3\,E_{01}^3\,E_{30} + 36\,E_{20}\,E_{10}\,E_{03} \\ & - 9\,E_{03}\,E_{10}^3 - 16\,E_{01}\,E_{02}\,E_{20}\,E_{10} + 4\,E_{01}^3\,E_{10}\,E_{20} + 4\,E_{01}\,E_{10}^3\,E_{02}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} q_4 = {} & 3\,E_{01}\,E_{21}^2 - 2\,E_{01}^2\,E_{21}\,E_{20} - 9\,E_{01}\,E_{12}\,E_{30} + E_{10}\,E_{12}\,E_{01}\,E_{20} - E_{11}^2\,E_{20}\,E_{01} \\ & + 3\,E_{01}^2\,E_{30}\,E_{11} + E_{11}\,E_{20}\,E_{01}^2\,E_{10} - 3\,E_{01}\,E_{30}\,E_{02}\,E_{10} + 4\,E_{01}\,E_{20}^2\,E_{02} \\ & - E_{01}^3\,E_{20}^2 - E_{01}\,E_{20}\,E_{10}^2\,E_{02}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} q_5 = {} & -27\,E_{10}\,E_{21}\,E_{03} + E_{10}\,E_{01}^3\,E_{21} + 9\,E_{10}\,E_{12}^2 - E_{11}^2\,E_{10}\,E_{01}^2 - 6\,E_{02}\,E_{12}\,E_{10}^2 \\ & - 2\,E_{01}^2\,E_{12}\,E_{10}^2 - 3\,E_{02}\,E_{11}^2\,E_{10} - E_{01}^2\,E_{10}^3\,E_{02} + 9\,E_{11}\,E_{03}\,E_{10}^2 + 3\,E_{01}\,E_{02}\,E_{10}^2\,E_{11} \\ & + E_{01}^3\,E_{11}\,E_{10}^2 - 3\,E_{10}^3\,E_{02}^2 + 3\,E_{10}^3\,E_{01}\,E_{03} + 12\,E_{10}\,E_{20}\,E_{02}^2 + E_{02}\,E_{20}\,E_{01}^2\,E_{10} \\ & - E_{01}^4\,E_{20}\,E_{10} - 18\,E_{10}\,E_{01}\,E_{03}\,E_{20} + 3\,E_{11}\,E_{01}\,E_{10}\,E_{12}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} q_6 = {} & -27\,E_{03}\,E_{21} + E_{21}\,E_{01}^3 + 9\,E_{12}^2 + 3\,E_{12}\,E_{01}\,E_{11} - 2\,E_{01}^2\,E_{10}\,E_{12} - 3\,E_{02}\,E_{11}^2 \\ & - E_{01}^2\,E_{11}^2 + 9\,E_{03}\,E_{11}\,E_{10} - 3\,E_{10}^2\,E_{02}^2 + 3\,E_{01}\,E_{02}\,E_{11}\,E_{10} + E_{01}^3\,E_{11}\,E_{10} \\ & - 18\,E_{20}\,E_{01}\,E_{03} + 3\,E_{03}\,E_{01}\,E_{10}^2 - 6\,E_{02}\,E_{10}\,E_{12} - E_{01}^4\,E_{20} + 12\,E_{02}^2\,E_{20} \\ & + E_{01}^2\,E_{02}\,E_{20} - E_{01}^2\,E_{10}^2\,E_{02}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} q_7 = {} & 3\,E_{21}^2 - 2\,E_{20}\,E_{01}\,E_{21} - 9\,E_{30}\,E_{12} + E_{10}\,E_{12}\,E_{20} - E_{20}\,E_{11}^2 + 3\,E_{30}\,E_{11}\,E_{01} \\ & + {+E_{10}}\,E_{20}\,E_{11}\,E_{01} - 3\,E_{02}\,E_{10}\,E_{30} + 4\,E_{20}^2\,E_{02} - E_{01}^2\,E_{20}^2 - E_{10}^2\,E_{20}\,E_{02}. \end{aligned} \tag{3.14}$$

To verify that the polynomials (3.8)–(3.14) form a basis, we used in [62] some computer programs based on the Gröbner bases and the Buchberger algorithm (see [8]). In addition to the basis given above, we also found a Gröbner basis consisting of 14 polynomials $\tilde{q}_1, \ldots, \tilde{q}_{14}$ (the explicit formulas for these polynomials are quite cumbersome; they are given in the appendix to [64]).

4. Quotient equations of a perfect cuboid and their equivalence to the initial equations. According to Definition 1, the quotient equations of a perfect cuboid are defined by the polynomials of the finitely generated ideal I_{sym} defined in (2.7). The basis of this ideal was computed in [62]. It consists of eight polynomials:

$$I_{\text{sym}} = \langle \tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5, \tilde{p}_6, \tilde{p}_7, \tilde{p}_8 \rangle. \tag{4.1}$$

Let us present the explicit formulas for the basic polynomials from (4.1):

$$\tilde{p}_1 = (x_1)^2 + (x_2)^2 + (x_3)^2 - L^2, \tag{4.2}$$

$$\tilde{p}_2 = (x_2^2 + x_3^2 - d_1^2) + (x_3^2 + x_1^2 - d_2^2) + (x_1^2 + x_2^2 - d_3^2), \tag{4.3}$$

$$\tilde{p}_3 = d_1\,(x_2^2 + x_3^2 - d_1^2) + d_2\,(x_3^2 + x_1^2 - d_2^2) + d_3\,(x_1^2 + x_2^2 - d_3^2), \tag{4.4}$$

$$\tilde{p}_4 = x_1\,(x_2^2 + x_3^2 - d_1^2) + x_2\,(x_3^2 + x_1^2 - d_2^2) + x_3\,(x_1^2 + x_2^2 - d_3^2), \tag{4.5}$$

$$\tilde{p}_5 = x_1\,d_1\,(x_2^2 + x_3^2 - d_1^2) + x_2\,d_2\,(x_3^2 + x_1^2 - d_2^2) + x_3\,d_3\,(x_1^2 + x_2^2 - d_3^2), \tag{4.6}$$

$$\tilde{p}_6 = x_1^2\,(x_2^2 + x_3^2 - d_1^2) + x_2^2\,(x_3^2 + x_1^2 - d_2^2) + x_3^2\,(x_1^2 + x_2^2 - d_3^2), \tag{4.7}$$

$$\tilde{p}_7 = d_1^2\,(x_2^2 + x_3^2 - d_1^2) + d_2^2\,(7x_3^2 + x_1^2 - d_2^2) + d_3^2\,(x_1^2 + x_2^2 - d_3^2), \tag{4.8}$$

$$\tilde{p}_8 = x_1^2\,d_1^2\,(x_2^2 + x_3^2 - d_1^2) + x_2^2\,d_2^2\,(x_3^2 + x_1^2 - d_2^2) \;+\; x_3^2\,d_3^2\,(x_1^2 + x_2^2 - d_3^2). \tag{4.9}$$

The quotient equations of a perfect cuboid with respect to their S_3-symmetry are obtained by equating to zero the polynomials (4.2)–(4.9):

$$\tilde{p}_1 = 0, \quad \tilde{p}_2 = 0, \quad \tilde{p}_3 = 0, \quad \tilde{p}_4 = 0, \quad \tilde{p}_5 = 0, \quad \tilde{p}_6 = 0, \quad \tilde{p}_7 = 0, \quad \tilde{p}_8 = 0. \tag{4.10}$$

Equations (4.10) obtained from (4.2)–(4.9) look more complicated than the initial equations of a perfect cuboid (1.1). However, they follow from (1.1). The reverse, generally speaking, is not valid, i.e., (1.1) do not follow from (4.10). However, a perfect cuboid is a geometric object, and its sizes are expressed by positive integers. This imply the inequalities

$$x_1 > 0, \quad x_2 > 0, \quad x_3 > 0, \quad d_1 > 0, \quad d_2 > 0, \quad d_3 > 0. \tag{4.11}$$

Taking into account Eqs. (4.11), the author proved in [63] the following theorem.

Theorem 1. *Each integer or rational solution of equations* (4.10) *satisfying the inequalities* (4.11) *is an integer or rational solution of Eqs.* (1.1).

Remark. In the problem of a perfect cuboid, only integer and rational solutions are important. However, Theorem 1 also remains valid for real solutions.

Theorem 1 establishes an equivalence of the quotient equations (4.10) to the original equations (1.1) of a perfect cuboid. But Eqs. (4.10) may contain extra solutions, which correspond to degenerate configurations or configurations that cannot be realizable geometrically, which violate at least one of the inequalities (4.11).

5. E-Representations for the quotient equations of a perfect cuboid. Consider the substitution homomorphism (3.5). As we noted above, it is surjective since each polynomial $p \in \operatorname{Sym}\mathbb{Q}[M, L]$ has a preimage q in the ring $\mathbb{Q}[E, L]$. In [38–42, 61–68], the following terminology was suggested.

Definition 2. For each polynomial $p \in \operatorname{Sym}\mathbb{Q}[M, L]$, a polynomial $q \in \mathbb{Q}[E, L]$ possessing the property $p = \varphi(q)$ is called an *E-form* or an *E-representation* of the polynomial p. If a polynomial p determines the polynomial equation $p = 0$, then the polynomial equation $q = 0$ is called an *E-form* or an *E-representation* of the equation $p = 0$.

Since the kernel (3.6) of the substitution homomorphism φ is nontrivial, E-representations of the polynomials and the corresponding equations exist, but they are not unique. For the quotient equations of a prefect cuboid (4.10), their E-representations were computed in [64]. They have the following form:

$$E_{10}^2 - 2\,E_{20} - L^2 = 0, \tag{5.1}$$

$$2\,E_{02} - 4\,E_{20} - E_{01}^2 + 2\,E_{10}^2 = 0, \tag{5.2}$$

$$E_{10}\,E_{11} - 3\,E_{03} - E_{21} + 3\,E_{01}\,E_{02} - E_{20}\,E_{01} - E_{01}^3 = 0, \tag{5.3}$$

$$E_{01}\,E_{11} - E_{12} - 3\,E_{30} + E_{10}\,E_{02} + E_{20}\,E_{10} - E_{01}^2\,E_{10} = 0, \tag{5.4}$$

$$\begin{aligned} &- E_{10}\,E_{21} - E_{01}\,E_{12} - E_{01}\,E_{30} - E_{01}^3\,E_{10} + E_{01}^2\,E_{11} \\ &\qquad - E_{02}\,E_{11} + E_{11}\,E_{20} - E_{10}\,E_{03} + 2\,E_{10}\,E_{01}\,E_{02} = 0, \end{aligned} \tag{5.5}$$

$$\begin{aligned} &4\,E_{01}\,E_{10}\,E_{11} - 3\,E_{01}^2\,E_{10}^2 + 2\,E_{10}^2\,E_{02} + 2\,E_{20}\,E_{01}^2 - 2\,E_{10}\,E_{12} \\ &\qquad - 2\,E_{02}\,E_{20} - 2\,E_{01}\,E_{21} - E_{11}^2 - 12\,E_{10}\,E_{30} + 6\,E_{20}^2 = 0, \end{aligned} \tag{5.6}$$

$$\begin{aligned} &4\,E_{01}\,E_{10}\,E_{11} - 4\,E_{10}^2\,E_{02} - 4\,E_{20}\,E_{01}^2 - 2\,E_{10}\,E_{12} + 10\,E_{02}\,E_{20} \\ &\qquad - 2\,E_{01}\,E_{21} - E_{11}^2 - 12\,E_{01}\,E_{03} - 3\,E_{01}^4 - 6\,E_{02}^2 + 12\,E_{01}^2\,E_{02} = 0, \end{aligned} \tag{5.7}$$

$$\begin{aligned} &9\,E_{01}\,E_{03}\,E_{20} - 7\,E_{01}^2\,E_{02}\,E_{20} + 2\,E_{02}\,E_{10}\,E_{12} - 2\,E_{01}^2\,E_{10}\,E_{12} + 3\,E_{03}\,E_{10}\,E_{11} + 4\,E_{01}^3\,E_{10}\,E_{11} \\ &\quad - 7\,E_{01}\,E_{02}\,E_{10}\,E_{11} - 6\,E_{01}\,E_{03}\,E_{10}^2 + 8\,E_{01}^2\,E_{02}\,E_{10}^2 + 3\,E_{01}\,E_{11}\,E_{30} - 2\,E_{01}\,E_{20}\,E_{21} \\ &\quad + E_{10}\,E_{12}\,E_{20} - E_{02}\,E_{10}^2\,E_{20} + E_{01}\,E_{10}\,E_{11}\,E_{20} + 9\,E_{02}\,E_{10}\,E_{30} - 2\,E_{02}\,E_{20}^2 \\ &\quad + 2\,E_{01}^2\,E_{20}^2 - E_{11}^2\,E_{20} - 3\,E_{12}\,E_{30} + E_{02}\,E_{11}^2 - E_{01}^2\,E_{11}^2 - 2\,E_{02}^2\,E_{10}^2 + 2\,E_{01}^4\,E_{20} \\ &\quad + 2\,E_{02}^2\,E_{20} - 3\,E_{03}\,E_{21} - 2\,E_{01}^3\,E_{21} + 5\,E_{01}\,E_{02}\,E_{21} - 6\,E_{01}^2\,E_{10}\,E_{30} - 3\,E_{01}^4\,E_{10}^2 = 0. \end{aligned} \tag{5.8}$$

Since the E-representations are not unique, Eqs. (5.1)–(5.8) must be replaced by the equations corresponding to the kernel of the substitution homomorphism (3.5), which have the form

$$q_1 = 0, \quad q_2 = 0, \quad q_3 = 0, \quad q_4 = 0, \quad q_5 = 0, \quad q_6 = 0, \quad q_7 = 0, \tag{5.9}$$

where the polynomials q_j, $j = 1, \dots, 7$, are taken from (3.8)–(3.14). Instead of them, we can use in (5.9) any other polynomials that form a basis in the kernel of the substitution homomorphism (3.5), and, as we already noted, their number is not necessarily equal to 7.

The further actions in [64] consist of simplification of the equations in E-form. The variables E_{20} and E_{02} can be expressed through E_{10} and E_{01}:

$$E_{20} = \frac{1}{2}\,E_{10}^2 - \frac{1}{2}\,L^2, \tag{5.10}$$

$$E_{02} = \frac{1}{2}\,E_{01}^2 - L^2. \tag{5.11}$$

The variables E_{03} and E_{30} are expressed through E_{12}, E_{21}, E_{11}, E_{10}, and E_{01}:

$$E_{03} = -\frac{1}{3}\,E_{21} - \frac{1}{6}\,E_{01}\,E_{10}^2 - \frac{5}{6}\,E_{01}\,L^2 + \frac{1}{6}\,E_{01}^3 + \frac{1}{3}\,E_{10}\,E_{11}, \tag{5.12}$$

$$E_{30} = -\frac{1}{3}\,E_{12} - \frac{1}{6}\,E_{10}\,E_{01}^2 - \frac{1}{2}\,E_{10}\,L^2 + \frac{1}{6}\,E_{10}^3 + \frac{1}{3}\,E_{01}\,E_{11}. \tag{5.13}$$

The variables E_{21} and E_{12} can be expressed through E_{11}, E_{10}, and E_{01}:

$$E_{21}=\frac{2\,E_{10}^3\,E_{11}+2\,E_{01}^2\,E_{10}\,E_{11}-E_{01}\,E_{10}^4+E_{01}^5}{8\,(E_{01}^2+E_{10}^2)}+\frac{6\,E_{10}\,E_{11}\,L^2-2\,E_{01}\,E_{10}^2\,L^2-8\,E_{01}^3\,L^2+3\,E_{01}\,L^4}{8\,(E_{01}^2+E_{10}^2)},\qquad(5.14)$$

$$E_{12}=\frac{E_{01}^4\,E_{10}-2\,E_{01}^3\,E_{11}-2\,E_{01}\,E_{10}^2\,E_{11}-E_{10}^5}{8\,(E_{01}^2+E_{10}^2)}+\frac{6\,E_{10}^3\,L^2-6\,E_{01}\,E_{11}\,L^2+3\,E_{10}\,L^4}{8\,(E_{01}^2+E_{10}^2)}.\qquad(5.15)$$

The variable E_{11} satisfies the quadratic equation

$$4\,E_{11}^2+E_{10}^4+E_{01}^4-2\,E_{10}^2\,E_{01}^2-2\,L^2\,E_{10}^2-6\,E_{01}^2\,L^2+L^4=0.\qquad(5.16)$$

The formulas (5.10)–(5.15) allow one to exclude the variables E_{03}, E_{30}, E_{12}, E_{21}, E_{02}, and E_{20} from the remaining equations of the system (5.1)–(5.9). Equation (5.16) allows one to reduce the degree of the variable E_{11} to 1. As it turns out, after this, all equations remaining in the system happily turn out to be identities.

The total number of variables E_{03}, E_{30}, E_{12}, E_{21}, E_{02}, E_{20}, E_{11}, E_{01}, E_{10}, and L is equal to 10. They can be understood as the coordinates of a point in $\mathbb{R}^{10}$. Taking this into account, the result can be formulated as follows.

Theorem 2. *On the set of points of the space $\mathbb{R}^{10}$, where $E_{10}\neq 0$ or $E_{01}\neq 0$, the system of the quotient equations of a perfect cuboid transformed into an E-representation is equivalent to the Diophantine equation* (5.16).

6. Biquadratic Diophantine equation and the Ramsden solution for it. Having obtained equation (5.16) and peering into it, I was going to postpone further actions on it. Indeed, I managed to reduce it to the form

$$(2\,E_{11})^2+(E_{01}^2+L^2-E_{10}^2)^2-8\,E_{01}^2\,L^2=0\qquad(6.1)$$

and found a similarity with an equation arising in the problem of Heron of Alexandria about triangles with integer sides and an integer area. However, I did not know what to do with it further. Then, unexpectedly, I received a letter by e-mail from John Ramsden, whom I had never met before. The letter contained a solution of Eq. (6.1). At the moment, this solution is published by J. Ramsden in [38]. John Ramsden's solution contains several simple cases that are calculated completely and do not lead to perfect cuboids, and one main case, which will be used below.

Let us note that the initial equations of the perfect cuboid (1.1) are homogeneous. Passing from integer solutions of these equations to rational, we can normalize them by the condition

$$L=1.\qquad(6.2)$$

After the substitution of (6.2) into (6.1) we have the equation

$$(2\,E_{11})^2+(E_{01}^2+1-E_{10}^2)^2=8\,E_{01}^2.\qquad(6.3)$$

Ramsden's solution in the main case for (6.3) is expressed by the formulas

$$E_{11}=-\frac{b\,(c^2+2-4\,c)}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b},\qquad(6.4)$$

$$E_{01}=-\frac{b(c^2+2-2\,c)}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b},\qquad(6.5)$$

$$E_{10}=-\frac{b^2\,c^2+2\,b^2-3\,b^2\,c-c}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b}.\qquad(6.6)$$

Here b and c are two arbitrary rational numbers such that the denominator in (6.4)–(6.6) is nonzero.

The substitution of (6.5) and (6.6) into (5.10) and (5.11) yields the following formulas for E_{20} and E_{02}:

$$E_{20} = \frac{b}{2}\,(b\,c^2 - 2\,c - 2\,b)\,(2\,b\,c^2 - c^2 - 6\,b\,c + 2 + 4\,b)\,(b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}, \tag{6.7}$$

$$\begin{aligned} E_{02} = \frac{1}{2}\Big(&28\,b^2\,c^2 - 16\,b^2\,c - 2\,c^2 - 4\,b^2 - b^2\,c^4 + 4\,b^3\,c^4 \\ &- 12\,b^3\,c^3 + 4\,b\,c^3 + 24\,b^3\,c - 8\,b\,c - 2\,b^4\,c^4 + 12\,b^4\,c^3 - 26\,b^4\,c^2 - 8\,b^2\,c^3 \\ &+ 24\,b^4\,c - 16\,b^3 - 8\,b^4\Big)\,\big(b\,c - 1 - b\big)^{-2}\,\big(b\,c - c - 2\,b\big)^{-2}. \end{aligned} \tag{6.8}$$

Then we perform the substitution of (6.4), (6.5), and (6.6) into (5.14) and (5.15) and obtain

$$\begin{aligned} E_{21} = \frac{b}{2}\Big(&5\,c^6\,b - 2\,c^6\,b^2 + 52\,c^5\,b^2 - 16\,c^5\,b - 2\,c^7\,b^2 + 2\,b^4\,c^8 + 142\,b^4\,c^6 - 26\,b^4\,c^7 \\ &- 426\,b^4\,c^5 - 61\,b^3\,c^6 + 100\,b^3\,c^5 + 14\,c^7\,b^3 - c^8\,b^3 - 20\,b\,c^2 - 8\,b^2\,c^2 - 16\,b^2\,c \\ &- 128\,b^2\,c^4 - 200\,b^3\,c^3 + 244\,b^3\,c^2 + 32\,b\,c^3 - 112\,b^3\,c + 768\,b^4\,c^4 - 852\,b^4\,c^3 \\ &+ 568\,b^4\,c^2 + 104\,b^2\,c^3 - 208\,b^4\,c + 8\,c^4 - 4\,c^3 + 16\,b^3 + 32\,b^4 - 2\,c^5\Big) \\ &\times \Big(b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2\Big)^{-1} \\ &\times \big(b\,c - 1 - b\big)^{-2}\,\big(b\,c - c - 2\,b\big)^{-2}, \end{aligned} \tag{6.9}$$

$$\begin{aligned} E_{12} = \Big(&16\,b^6 + 32\,b^5 - 6\,c^5\,b^2 + 2\,c^5\,b - 62\,b^5\,c^6 + 62\,b^6\,c^6 - 180\,b^6\,c^5 + 18\,b^5\,c^7 \\ &- 12\,b^6\,c^7 - 2\,b^5\,c^8 + b^6\,c^8 + 248\,b^5\,c^2 + 248\,b^6\,c^2 - 96\,b^6\,c + 321\,b^6\,c^4 - 180\,b^5\,c^3 \\ &- 144\,b^5\,c - 360\,b^6\,c^3 + b^4\,c^8 + 8\,b^4\,c^6 - 6\,b^4\,c^7 + 18\,b^4\,c^5 + 7\,b^3\,c^6 + 90\,b^5\,c^5 \\ &- 14\,b^3\,c^5 - c^7\,b^3 + 17\,b^2\,c^4 + 28\,b^3\,c^3 - 28\,b^3\,c^2 - 4\,b\,c^3 + 8\,b^3\,c - 57\,b^4\,c^4 \\ &+ 36\,b^4\,c^3 + 32\,b^4\,c^2 - 12\,b^2\,c^3 - 48\,b^4\,c - c^4 + 16\,b^4\Big) \\ &\times \Big(b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2\Big)^{-1} \\ &\times \big(b\,c - 1 - b\big)^{-2}\,\big(b\,c - c - 2\,b\big)^{-2}. \end{aligned} \tag{6.10}$$

Further, substituting the expressions obtained into (5.12) and (5.13), we obtain

$$\begin{aligned} E_{30} = {}& c\,b^2\,(1 - c)\,(c - 2)\,\Big(b\,c^2 - 4\,b\,c + 2 + 4\,b\Big)\,\Big(2\,b\,c^2 - c^2 - 4\,b\,c + 2\,b\Big) \\ &\times \Big(b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2\Big)^{-1}\,\big(b\,c - 1 - b\big)^{-2}\,\big(-c + b\,c - 2\,b\big)^{-2}, \end{aligned} \tag{6.11}$$

$$\begin{aligned} E_{03} = {}& \frac{b}{2}\,\Big(b^2\,c^4 - 5\,b^2\,c^3 + 10\,b^2\,c^2 - 10\,b^2\,c + 4\,b^2 + 2\,b\,c + 2\,c^2 - b\,c^3\Big) \\ &\times \Big(2\,b^2\,c^4 - 12\,b^2\,c^3 + 26\,b^2\,c^2 - 24\,b^2\,c + 8\,b^2 - c^4\,b + 3\,b\,c^3 - 6\,b\,c + 4\,b + c^3 - 2\,c^2 + 2\,c\Big) \\ &\times \Big(b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2\Big)^{-1} \\ &\times (b\,c - 1 - b)^{-2}\,(-c + b\,c - 2\,b)^{-2}. \end{aligned} \tag{6.12}$$

The formulas (6.9)–(6.12) have supplementary denominators in comparison with (6.4)–(6.6). The structure of singularities caused by the denominators in these formulas was studied in [40]. The whole result of these formulas is expressed in the following theorem.

Theorem 3. *The quotient equations of a perfect cuboid* (4.10) *after passage into the E-representation are explicitly solvable. Their solution normalized by the condition $L = 1$ in the general case contains two arbitrary rational parameters b and c and is expressed by the formulas* (6.4)–(6.12).

7. Inverse problems connected with the perfect cuboid. After solving quotient equations (4.10) in E-representation, we met the inverse problem of passage to the initial variables x_1, x_2, x_3, d_1, d_2, d_3, and L with the normalization $L = 1$ introduced in (6.2). For this, we use the formulas of elementary symmetric polynomials (3.1)–(3.3). Note that the formulas (3.1) coincide with the formulas for the classical symmetric polynomials for the three variables x_1, x_2, and x_3. If we consider x_1, x_2, and x_3 as the roots of a cubic equation, then its coefficients are defined by the values of the polynomials (3.1):

$$x^3 - E_{10}\,x^2 + E_{20}\,x - E_{30} = 0. \tag{7.1}$$

The variables d_1, d_2, and d_3 can be considered as roots of another cubic equation:

$$d^3 - E_{01}\,d^2 + E_{02}\,d - E_{03} = 0. \tag{7.2}$$

The coefficients of the cubic equation (7.2) are values of the symmetric polynomials (3.2). The remaining three multisymmetric polynomials (3.3) lead to the following relations for the roots of the cubic equations (7.1) and (7.2):

$$\begin{aligned} &x_1\,x_2\,d_3 + x_2\,x_3\,d_1 + x_3\,x_1\,d_2 = E_{21},\\ &x_1\,d_2 + d_1\,x_2 + x_2\,d_3 + d_2\,x_3 + x_3\,d_1 + d_3\,x_1 = E_{11},\\ &x_1\,d_2\,d_3 + x_2\,d_3\,d_1 + x_3\,d_1\,d_2 = E_{12}. \end{aligned} \tag{7.3}$$

The actual values of the coefficients of Eqs. (7.1) and (7.2), as well as the right-hand sides of the formulas (7.3), are determined by the relations (6.4)–(6.12). Based on (7.1)–(7.3), we formulated in [39] the following inverse problems.

Problem 1. Find all the pairs of rational numbers b and c for which the cubic equations (7.1) and (7.2) with the coefficients defined in (6.6), (6.7), (6.11), (6.5), (6.8), and (6.12) have positive rational roots satisfying the supplementary equations (7.3) with the right-hand sides determined by (6.9), (6.10), and (6.4).

Problem 2. Find at least one pair of rational numbers b and c, for which the cubic equations (7.1) and (7.2) with the coefficients defined in (6.6), (6.7), (6.11), (6.5), (6.8), and (6.12) have positive rational roots satisfying the supplementary equations (7.3) with the right-hand sides determined by (6.9), (6.10), and (6.4).

8. Solvability of inverse problems and the Ramsden lemma. At the moment, there are no effective criteria that allow us, using the (rational) coefficients of the cubic equations, to specify a case where all roots of this equation are rational. The Ramsden lemma presented below can be considered as the first step towards developing such criteria.

Lemma 1 (J. Ramsden)**.** *The reduced cubic equation $y^3 + y^2 + D = 0$ has three rational roots if and only if the equation of sixth degree*

$$D\,(w^2 + 3)^3 + 4\,(w - 1)^2\,(1 + w)^2 = 0 \tag{8.1}$$

has a rational root w. In this case, the roots of the equation $y^3 + y^2 + D = 0$ are expressed through the rational root of Eq. (8.1) *by the formulas*

$$y_1 = -\frac{2\,(w + 1)}{w^2 + 3}, \quad y_2 = \frac{2\,(w - 1)}{w^2 + 3}, \quad y_3 = \frac{1 - w^2}{w^2 + 3}.$$

The proof of the lemma is given in [41].

Under certain restrictions, the cubic equation of the general form

$$A_3\,x^3 + A_2\,x^2 + A_1\,x + A_0 = 0 \tag{8.2}$$

can be reduced to the form $y^3 + y^2 + D = 0$. After this, we can apply Lemma 1. This leads to the following assertion.

Lemma 2. *Assume that the rational numbers A_0, A_1, A_2, and A_3 satisfy the inequalities*

$$A_3 \neq 0, \quad \frac{A_1}{A_3} - \frac{A_2^2}{3\,A_3^2} \neq 0, \quad \frac{A_0}{A_3} - \frac{A_1\,A_2}{3\,A_3^2} + \frac{2\,A_2^3}{27\,A_3^3} \neq 0.$$

Then the cubic equation (8.2) with the coefficients A_0, A_1, A_2, and A_3 has three rational roots if and only if there exists a rational number w, which is a root of the sixth-degree equation (8.1), where the parameter D is given by the formula

$$D = -\frac{(9\,A_1\,A_2\,A_3 - 27\,A_0\,A_3^2 - 2\,A_2^3)^2}{27\,(A_2^2 - 3\,A_1\,A_3)^3}. \tag{8.3}$$

Under these assumptions, the roots of equation (8.2) are expressed through the number w by the formulas

$$x_1 = \frac{1}{18}\Big(\big(2\,A_2^3 - 9\,A_1\,A_2\,A_3 + 27\,A_0\,A_3^2\big)\,w^2 + \big(18\,A_2\,A_1\,A_3 - 6\,A_2^3\big)\,w \\ - 9\,A_1\,A_2\,A_3 + 81\,A_0\,A_3^2\Big)\,A_3^{-1}\,\big(A_2^2 - 3\,A_1\,A_3\big)^{-1}\,(1+w)^{-1}, \tag{8.4}$$

$$x_2 = \frac{1}{18}\Big(\big(2\,A_2^3 - 9\,A_1\,A_2\,A_3 + 27\,A_0\,A_3^2\big)\,w^2 - \big(18\,A_2\,A_1\,A_3 - 6\,A_2^3\big)\,w \\ - 9\,A_1\,A_2\,A_3 + 81\,A_0\,A_3^2\Big)\,A_3^{-1}\,\big(A_2^2 - 3\,A_1\,A_3\big)^{-1}\,(1-w)^{-1}, \tag{8.5}$$

$$x_3 = \frac{1}{9}\Big(\big(A_2^3 - 27\,A_0\,A_3^2\big)\,w^2 + 36\,A_1\,A_2\,A_3 - 81\,A_0\,A_3^2 - 9\,A_2^3\Big) \\ \times\, A_3^{-1}\,\big(A_2^2 - 3\,A_1\,A_3\big)^{-1}\,(1-w)^{-1}\,(1+w)^{-1}. \tag{8.6}$$

The proof of Lemma 2 is given in [41]. This lemma is used in [41] as a criterion of solvability of the inverse problems 1 and 2 formulated above.

9. Algebraic parametrization of the quotient equations of a perfect cuboid. Two algebraic parametrizations of the quotient equations of the perfect cuboid (4.10) are connected to the cubic equations (7.1) and (7.2), i.e., there are two ways of reducing these equations to an algebraic equation of the sixth degree. They are obtained by applying Lemma 2 to these cubic equations. Applying the formula (8.3) to the first cubic equation (7.1) leads to the following expression for D in the corresponding sixth-degree equation (8.1):

$$D_1 = -\frac{2}{27}\Big(7812\,b^4\,c^4 - 216\,b^2\,c^4 - 52\,b^2\,c^3 + 1764\,b^3\,c^4 - 1200\,b^4\,c^3 - 1848\,b^4\,c^2 + 720\,b^4\,c \\ - 36\,c^4\,b - 1512\,b^3\,c^3 - 36\,c^8\,b^3 + 288\,b^3\,c^2 - 108\,c^6\,b^2 + 380\,c^5\,b^2 + 378\,c^7\,b^3 \\ - 231\,c^8\,b^4 - 300\,c^7\,b^4 + 3906\,c^6\,b^4 - 13\,c^7\,b^2 - 8904\,c^5\,b^4 - 882\,c^6\,b^3 + 18\,c^6\,b \\ - 1319\,b^6\,c^8 + 20952\,b^5\,c^3 - 11952\,b^5\,c^2 + 2592\,b^5\,c - 48372\,b^6\,c^4 + 31620\,b^6\,c^3 \\ - 10552\,b^6\,c^2 + 816\,b^6\,c + 1494\,b^5\,c^8 - 5238\,b^5\,c^7 - 4\,c^5 + 7905\,b^6\,c^7 - 24186\,b^6\,c^6 \\ + 288\,b^6 + 43740\,b^6\,c^5 + 7686\,b^5\,c^6 + 576\,b^7 + 128\,b^8 - 15372\,b^5\,c^4 - 1080\,b^7\,c^8$$

$$
\begin{aligned}
&- 3546\,b^7\,c^6 + 51\,c^9\,b^6 + 400\,b^8\,c^8 - 162\,c^9\,b^5 + 8640\,b^7\,c^2 - 3456\,b^7\,c + 2808\,b^7\,c^7 \\
&- 1560\,b^8\,c^7 + 3940\,b^8\,c^6 + 216\,c^9\,b^7 - 960\,b^8\,c - 6240\,b^8\,c^3 + 9\,c^{10}\,b^6 + 7880\,b^8\,c^4 + 4\,c^{10}\,b^8 \\
&- 6732\,b^8\,c^5 + 45\,c^9\,b^4 + 3200\,b^8\,c^2 - 11232\,b^7\,c^3 + 7092\,b^7\,c^4 - 18\,c^{10}\,b^7 - 60\,c^9\,b^8\Big)^2 \\
&\times \Big(2\,c^2 + 2\,b^4\,c^4 - 12\,b^4\,c^3 + 26\,b^4\,c^2 - 24\,b^4\,c + 8\,b^4 - 6\,b^3\,c^4 + 18\,b^3\,c^3 - 36\,b^3\,c \\
&+ 24\,b^3 + 3\,b^2\,c^4 + 8\,b^2\,c^3 - 36\,b^2\,c^2 + 16\,b^2\,c + 12\,b^2 - 6\,b\,c^3 + 12\,b\,c\Big)^{-3} \\
&\times \Big(b^2\,c^4 - 6\,b^2\,c^{-3} + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2\Big)^{-2}. \qquad (9.1)
\end{aligned}
$$

The formulas (8.4), (8.5), and (8.6) applied to Eq. (7.1) give the expressions for x_1, x_2, and x_3 through the root w of Eq. (8.1) with the parameter $D = D_1$ from (9.1):

$$
x_1 = x_1(b, c, w), \quad x_2 = x_2(b, c, w), \quad x_3 = x_3(b, c, w). \qquad (9.2)
$$

In the explicit form, the formulas (9.2) are very cumbersome. They are given in the appendix of [41] in the machine readable form.

The functions (9.2) satisfy the identity $x_1^2 + x_2^2 + x_3^2 = 1$, which follows from the normalization condition (6.1). After finding x_1, x_2, and x_3 from (9.2), the values d_1, d_2, and d_3 can be found from the equations

$$
\begin{aligned}
&x_1\,x_2\,d_3 + x_2\,x_3\,d_1 + x_3\,x_1\,d_2 = E_{21}, \\
&x_1\,d_2 + d_1\,x_2 + x_2\,d_3 + d_2\,x_3 + x_3\,d_1 + d_3\,x_1 = E_{11}, \\
&d_1 + d_2 + d_3 = E_{01}.
\end{aligned} \qquad (9.3)
$$

Equations (9.3) are linear with respect to d_1, d_2, d_3. The first two of them are deduced from the first two formulas in (3.3). The third equation is deduced from the first formula in (3.2). The solution of Eqs. (9.3) yields

$$
d_1 = d_1(b, c, w), \quad d_2 = d_2(b, c, w), \quad d_3 = d_3(b, c, w). \qquad (9.4)
$$

The functions (9.4) are roots of the second cubic equation (7.2). Together with the functions (9.3), they give a solution of the quotient equations of a perfect cuboid (4.10). The explicit formulas for the functions (9.4) are even more cumbersome than the formulas for the functions (9.2).

The only condition for constructing the solutions of the quotient equations of a perfect cuboid (4.10) by the scheme described above is the existence of a pair of rational numbers b and c for which the sixth-degree equation (8.1) with the parameter $D = D_1$ from (9.1) has a rational root.

The question on the validity of the inequalities (4.11) for the functions (9.2) and (9.4) remains open. Hence we cannot assert a priori that the above scheme leads to solutions of the original equations (1.1) of a perfect cuboid.

Applying Lemma 2 to the second cubic equation (7.2), we arrive at mirror results. In this case, the parameter $D = D_2$ in the sixth-degree equation (8.1) is given by the formula

$$
\begin{aligned}
D_2 = -\frac{2\,b^2}{27}\Big(&832\,b^2\,c^2 - 1440\,b^2\,c^4 - 840\,b^2\,c^3 + 4788\,b^3\,c^4 + 396\,b\,c^3 + 720\,b^3\,c + 808\,b^4\,c^4 \\
&+ 3032\,b^4\,c^3 - 2576\,b^4\,c^2 - 96\,b^4\,c + 448\,b^4 - 504\,c^4\,b - 4176\,b^3\,c^3 - 9\,c^8\,b^3 + 72\,b^3\,c^2 - 720\,c^6\,b^2 \\
&+ 2288\,c^5\,b^2 + 1044\,c^7\,b^3 - 322\,c^8\,b^4 + 758\,c^7\,b^4 + 404\,c^6\,b^4 - 210\,c^7\,b^2 - 2464\,c^5\,b^4 - 2394\,c^6\,b^3 \\
&+ 72\,c^4 + 252\,c^6\,b + 3168\,b^6\,c^8 + 441\,c^9\,b^5 - 7056\,b^5\,c + 57960\,b^6\,c^4 - 47232\,b^6\,c^3 + 25344\,b^6\,c^2 \\
&- 8064\,b^6\,c - 1809\,b^5\,c^8 + 14472\,b^5\,c^2 + +3951\,b^5\,c^7 - 72\,c^5 + 36\,c^6 - 11808\,b^6\,c^7 + 1440\,b^5 \\
&+ 28980\,b^6\,c^6 - 49032\,b^6\,c^5 - 4410\,b^5\,c^6 + 8820\,b^5\,c^4 - 15804\,b^5\,c^3 + 1152\,b^6 - 504\,c^9\,b^6 \\
&- 45\,c^9\,b^3 - 6\,c^9\,b^4 + 104\,c^8\,b^2 + 36\,c^{10}\,b^6 + 14\,c^{10}\,b^4 - 45\,c^{10}\,b^5 - 99\,c^7\,b\Big)^2
\end{aligned}
$$

$$\times \Big(6\,b^4\,c^4 - 36\,b^4\,c^3 + 78\,b^4\,c^2 - 72\,b^4\,c + 24\,b^4 - 12\,b^3\,c^4 + 36\,b^3\,c^3 - 72\,b^3\,c + 48\,b^3 + 5\,b^2\,c^4 + 16\,b^2\,c^3 - 68\,b^2\,c^2 + 32\,b^2\,c + 20\,b^2 - 12\,b\,c^3 + 24\,b\,c + 6\,c^2\Big)^{-3} \times \Big(b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2\Big)^{-2}. \tag{9.5}$$

Then the formulas (8.4)–(8.6) imply the formulas for d_1, d_2, and d_3:

$$d_1 = d_1(b, c, w), \quad d_2 = d_2(b, c, w), \quad d_3 = d_3(b, c, w). \tag{9.6}$$

The explicit expressions for the functions (9.6) are given in the appendix to the electronic publication [41].

The functions (9.6) allow one to write the linear equations for x_1, x_2, and x_3:

$$\begin{aligned} &x_1 + x_2 + x_3 = E_{10},\\ &x_1\,d_2 + d_1\,x_2 + x_2\,d_3 + d_2\,x_3 + x_3\,d_1 + d_3\,x_1 = E_{11},\\ &x_1\,d_2\,d_3 + x_2\,d_3\,d_1 + x_3\,d_1\,d_2 = E_{12}. \end{aligned} \tag{9.7}$$

A solution of Eqs. (9.7) yields the following formulas for x_1, x_2, and x_3:

$$x_1 = x_1(b, c, w), \quad x_2 = x_2(b, c, w), \quad x_3 = x_3(b, c, w). \tag{9.8}$$

The functions (9.6) and (9.8) yield a solution of quotient equations of a perfect cuboid (4.10). Generally speaking, these functions differ from the corresponding functions (9.4) and (9.2), i.e., we have two different algebraic parametrizations of the quotient equations (4.10), which reduce them to two different sixth-degree equations of the form (8.1) with the parameters (9.1) and (9.5), respectively.

10. Conclusions. The main result of the symmetry approach to the problem of a perfect cuboid at the moment consists in the derivation of the factor equations (4.10) and in their reduction to a single algebraic sixth-degree equation of the form (8.1). However, at the same time, a number of auxiliary and indirect results have been obtained in [38–42, 61–68], and significance of these results is still difficult to estimate. The studies on the problem of a perfect cuboid continue, and time will tell which of the results will be decisive.

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