No Perfect Cuboid

Walter Wyss

Abstract

A rectangular parallelepiped is called a cuboid (standing box). It is called perfect if its edges, face diagonals and body diagonal all have integer length. Euler gave an example where only the body diagonal failed to be an integer (Euler brick). Are there perfect cuboids? We prove that there is no perfect cuboid.

1 Introduction

Cuboids have been studied extensively. It suffices to look at rational cuboids [1, 2]. Rational cuboids are characterized by seven positive rational numbers (three different edges, three different face diagonals and the body diagonal). Examples are known where all but one of the seven quantities are rational. Our approach uses the concept of a rational leaning box. This is a parallelepiped with two different rectangular faces and a face that is a parallelogram. Rational leaning boxes are characterized by nine positive rational numbers (three different edges, two different face diagonals belonging to the rectangular faces, two different face diagonals belonging to the face parallelogram and two different body diagonals). If the parallelogram face becomes a rectangle, then we have a standing box. Computer aided discoveries have shown the existence of perfect leaning boxes [3, 4, 5]. We found a two-parameter family of solutions for rational leaning boxes analytically. The two diagonals of the face parallelogram can never be equal. Thus there is no standing rational box in this family. Finally we use an equivalent description of leaning boxes to show that in general there is no perfect cuboid.
In the appendices we use generic symbols which do not necessarily coincide with the ones used in the main text.

2 The equations for the perfect leaning box

All the following nine quantities are positive integers.
x,y,z, denote the three different edges. The face rectangle (x,y) has diagonal a, the face rectangle (x,z) has diagonal b and the face parallelogram (y,z) has diagonals c_1, c_2. The two different body diagonals are denoted by d_1, d_2.

These quantities satisfy the equations

$$x^2 + y^2 = a^2 \tag{1}$$
$$x^2 + z^2 = b^2 \tag{2}$$
$$x^2 + c_1^2 = d_1^2 \tag{3}$$
$$x^2 + c_2^2 = d_2^2 \tag{4}$$
$$2y^2 + 2z^2 = c_1^2 + c_2^2 \tag{5}$$

The last equation represents a perfect parallelogram [7, 8]

3 Parameterization for the rational leaning box

We look for solutions of the equations (1), (2), (3), (4), (5) in rational positive numbers.

We now scale these equations as follows

$$u_1 = \frac{y}{x}, \qquad u_2 = \frac{z}{x}, \qquad u_3 = \frac{c_1}{x}, \qquad u_4 = \frac{c_2}{x} \tag{6}$$
$$v_1 = \frac{a}{x}, \qquad v_2 = \frac{b}{x}, \qquad v_3 = \frac{d_1}{x}, \qquad v_4 = \frac{d_2}{x} \tag{7}$$

Then u_k and v_k, k=1,2,3,4, are positive rational numbers.

The scaled equations are

$$1 + u_1^2 = v_1^2 \tag{8}$$
$$1 + u_2^2 = v_2^2 \tag{9}$$
$$1 + u_3^2 = v_3^2 \tag{10}$$
$$1 + u_4^2 = v_4^2 \tag{11}$$
$$2u_1^2 + 2u_2^2 = u_3^2 + u_4^2 \tag{12}$$

The last equation represents a rational parallelogram (Appendix C)

The scaled equations can be parameterized by the four Heron angles ψ_k and their generators s_k, k=1,2,3,4 (Appendix A) as follows

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$$u_k = \cot \psi_k = \frac{1 - s_k^2}{2s_k} = \frac{1}{2}(\frac{1}{s_k} - s_k) \tag{13}$$

$$v_k = \frac{1}{\sin \psi_k} = \frac{1 + s_k^2}{2s_k} = \frac{1}{2}(\frac{1}{s_k} + s_k) \tag{14}$$

$$v_k - u_k = s_k, v_k + u_k = \frac{1}{s_k}, 0 < s_k < 1 \tag{15}$$

4 The three parallelograms

Besides the face parallelogram there are two interior parallelograms

I.
$$2u_1^2 + 2u_2^2 = u_3^2 + u_4^2 \tag{16}$$

II.
$$2u_1^2 + 2v_2^2 = v_3^2 + v_4^2 \tag{17}$$

III.
$$2v_1^2 + 2u_2^2 = v_3^2 + v_4^2 \tag{18}$$

From Appendix (D.32) we have the representation

I.
$$2u_1 = u_3\omega_+(\alpha) + u_4\omega_-(\alpha) \tag{19}$$
$$2u_2 = -u_3\omega_-(\alpha) + u_4\omega_+(\alpha) \tag{20}$$

II.
$$2u_1 = v_3\omega_+(\alpha_1) + v_4\omega_-(\alpha_1) \tag{21}$$
$$2v_2 = -v_3\omega_-(\alpha_1) + v_4\omega_+(\alpha_1) \tag{22}$$

III.
$$2v_1 = v_3\omega_+(\alpha_2) + v_4\omega_-(\alpha_2) \tag{23}$$
$$2u_2 = -v_3\omega_-(\alpha_2) + v_4\omega_+(\alpha_2) \tag{24}$$

with m as the generator of α

$$m = \frac{2u_2 + u_3 - u_4}{2u_1 + u_3 + u_4} \tag{25}$$

and with m_1 as the generator of α_1

$$m_1 = \frac{2v_2 + v_3 - v_4}{2u_1 + v_3 + v_4} \tag{26}$$

and with m_2 as the generator of α_2

$$m_2 = \frac{2u_2 + v_3 - v_4}{2v_1 + v_3 + v_4} \tag{27}$$

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$\alpha, \alpha_1, \alpha_2$ are Heron angles in the first quadrant.

In terms of generators s_3, s_4, with

$$Q = s_3 s_4 \tag{28}$$

we get from Appendix E the representation

I.
$$4Qu_1 = s_4 M(\alpha) + s_3 H(\alpha) \tag{29}$$
$$4Qu_2 = -s_4 K(\alpha) + s_3 N(\alpha) \tag{30}$$
II.
$$4Qu_1 = s_4 N(\alpha_1) + s_3 K(\alpha_1) \tag{31}$$
$$4Qv_2 = -s_4 H(\alpha_1) + s_3 M(\alpha_1) \tag{32}$$
III.
$$4Qv_1 = s_4 N(\alpha_2) + s_3 K(\alpha_2) \tag{33}$$
$$4Qu_2 = -s_4 H(\alpha_2) + s_3 M(\alpha_2) \tag{34}$$

Comparing Equations (29-32) and using (15) we find

$$0 = s_4[M(\alpha) - N(\alpha_1)] + s_3[H(\alpha) - K(\alpha_1)] \tag{35}$$
$$8Qu_1 = s_4[M(\alpha) + N(\alpha_1)] + s_3[H(\alpha) + K(\alpha_1)] \tag{36}$$
$$4Qs_2 = s_4[K(\alpha) - H(\alpha_1)] - s_3[N(\alpha) - M(\alpha_1)] \tag{37}$$
$$4Q\frac{1}{s_2} = -s_4[K(\alpha) + H(\alpha_1)] + s_3[N(\alpha) + M(\alpha_1)] \tag{38}$$

With

$$\alpha + \alpha_1 = 2\sigma_1, \qquad \alpha - \alpha_1 = 2\delta_1$$
$$\omega_+(\sigma_1) = \sqrt{2}\cos\psi, \qquad \omega_-(\sigma_1) = \sqrt{2}\sin\psi$$
$$\sqrt{2}\cos\delta_1 = \omega_+(\alpha + \psi), \qquad \sqrt{2}\sin\delta_1 = -\omega_-(\alpha + \psi)$$

and Appendix E (Lemma 7) Equations (35-38) become

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$$0 = s_3 \cos \psi H(\alpha + \psi) - s_4 \sin \psi K(\alpha + \psi) \tag{39}$$

$$4Qu_1 = s_4 \cos \psi M(\alpha + \psi) + s_3 \sin \psi N(\alpha + \psi) \tag{40}$$

$$2Qs_2 = s_4 \cos \psi K(\alpha + \psi) + s_3 \sin \psi H(\alpha + \psi) \tag{41}$$

$$2Q\frac{1}{s_2} = -s_4 \sin \psi M(\alpha + \psi) + s_3 \cos \psi N(\alpha + \psi) \tag{42}$$

or in matrix form

$$\begin{pmatrix} s_4 \sin \psi & -s_3 \cos \psi \\ s_4 \cos \psi & s_3 \sin \psi \end{pmatrix} \begin{pmatrix} K(\alpha + \psi) \\ H(\alpha + \psi) \end{pmatrix} = 2Q \begin{pmatrix} 0 \\ s_2 \end{pmatrix} \tag{43}$$

$$\begin{pmatrix} s_4 \cos \psi & s_3 \sin \psi \\ -s_4 \sin \psi & s_3 \cos \psi \end{pmatrix} \begin{pmatrix} M(\alpha + \psi) \\ N(\alpha + \psi) \end{pmatrix} = 2Q \begin{pmatrix} 2u_1 \\ \frac{1}{s_2} \end{pmatrix} \tag{44}$$

The inverse equations are

$$\begin{pmatrix} K(\alpha + \psi) \\ H(\alpha + \psi) \end{pmatrix} = \begin{pmatrix} s_3 \sin \psi & s_3 \cos \psi \\ -s_4 \cos \psi & s_4 \sin \psi \end{pmatrix} \begin{pmatrix} 0 \\ 2s_2 \end{pmatrix} \tag{45}$$

$$\begin{pmatrix} M(\alpha + \psi) \\ N(\alpha + \psi) \end{pmatrix} = \begin{pmatrix} s_3 \cos \psi & -s_3 \sin \psi \\ s_4 \sin \psi & s_4 \cos \psi \end{pmatrix} \begin{pmatrix} 4u_1 \\ \frac{2}{s_2} \end{pmatrix} \tag{46}$$

Explicitly

$$K(\alpha + \psi) = 2s_2 s_3 \cos \psi \tag{47}$$

$$H(\alpha + \psi) = 2s_2 s_4 \sin \psi \tag{48}$$

$$M(\alpha + \psi) = 4u_1 s_3 \cos \psi - 2\frac{s_3}{s_2} \sin \psi \tag{49}$$

$$N(\alpha + \psi) = 4u_1 s_4 \sin \psi + 2\frac{s_4}{s_2} \cos \psi \tag{50}$$

Comparing equations 29,30,33,34 and using (15) we find

$$0 = s_3[N(\alpha) - M(\alpha_2)] - s_4[K(\alpha) - H(\alpha_2)] \tag{51}$$

$$8Qu_2 = s_3[N(\alpha) + M(\alpha_2)] - s_4[K(\alpha) + H(\alpha_2)] \tag{52}$$

$$4Qs_1 = s_4[N(\alpha_2) - M(\alpha)] + s_3[K(\alpha_2) - H(\alpha)] \tag{53}$$

$$4Q\frac{1}{s_1} = s_4[N(\alpha_2) + M(\alpha)] + s_3[K(\alpha_2) + H(\alpha)] \tag{54}$$

With

$$\alpha + \alpha_2 = 2\sigma_2, \qquad\qquad \alpha - \alpha_2 = 2\delta_2$$

$$\omega_+(\sigma_2) = \sqrt{2}\cos\phi, \qquad\qquad \omega_-(\sigma_2) = \sqrt{2}\sin\phi$$

$$\sqrt{2}\cos\delta_2 = \omega_+(\alpha + \phi), \qquad\qquad \sqrt{2}\sin\delta_2 = -\omega_-(\alpha + \phi)$$

and Appendix E (Lemma 7), equations (51-54) become

$$0 = -s_4\cos\phi K(\alpha + \phi) - s_3\sin\phi H(\alpha + \phi) \tag{55}$$

$$4Qu_2 = -s_4\sin\phi M(\alpha + \phi) + s_3\cos\phi N(\alpha + \phi) \tag{56}$$

$$2Qs_1 = s_4\sin\phi K(\alpha + \phi) - s_3\cos\phi H(\alpha + \phi) \tag{57}$$

$$2Q\frac{1}{s_1} = s_4\cos\phi M(\alpha + \phi) + s_3\sin\phi N(\alpha + \phi) \tag{58}$$

or in matrix form

$$\begin{pmatrix} -s_4\cos\phi & -s_3\sin\phi \\ s_4\sin\phi & -s_3\cos\phi \end{pmatrix} \begin{pmatrix} K(\alpha + \phi) \\ H(\alpha + \phi) \end{pmatrix} = \begin{pmatrix} 0 \\ 2Qs_1 \end{pmatrix} \tag{59}$$

$$\begin{pmatrix} -s_4\sin\phi & s_3\cos\phi \\ s_4\cos\phi & s_3\sin\phi \end{pmatrix} \begin{pmatrix} M(\alpha + \phi) \\ N(\alpha + \phi) \end{pmatrix} = \begin{pmatrix} 4Qu_2 \\ 2Q\frac{1}{s_1} \end{pmatrix} \tag{60}$$

The inverse equations are

$$\begin{pmatrix} K(\alpha + \phi) \\ H(\alpha + \phi) \end{pmatrix} = \begin{pmatrix} -s_3 \cos\phi & s_3 \sin\phi \\ -s_4 \sin\phi & -s_4 \cos\phi \end{pmatrix} \begin{pmatrix} 0 \\ 2s_1 \end{pmatrix} \tag{61}$$

$$\begin{pmatrix} M(\alpha + \phi) \\ N(\alpha + \phi) \end{pmatrix} = \begin{pmatrix} -s_3 \sin\phi & s_3 \cos\phi \\ s_4 \cos\phi & s_4 \sin\phi \end{pmatrix} \begin{pmatrix} 4u_2 \\ 2\frac{1}{s_1} \end{pmatrix} \tag{62}$$

Explicitly

$$K(\alpha + \phi) = 2s_1 s_3 \sin\phi \tag{63}$$

$$H(\alpha + \phi) = -2s_1 s_4 \cos\phi \tag{64}$$

$$M(\alpha + \phi) = -4u_2 s_3 \sin\phi + 2\frac{s_3}{s_1} \cos\phi \tag{65}$$

$$N(\alpha + \phi) = 4u_2 s_4 \cos\phi + 2\frac{s_4}{s_1} \sin\phi \tag{66}$$

Finally, comparing equations (31-34), we get from equations (37,38)

$$4Qu_2 = -s_4 K(\alpha) + s_3 N(\alpha) \tag{67}$$

$$4Qv_2 = -s_4 H(\alpha_1) + s_3 M(\alpha_1) \tag{68}$$

and from equations (53,54)

$$4Qu_1 = s_4 M(\alpha) + s_3 H(\alpha) \tag{69}$$

$$4Qv_1 = s_4 N(\alpha_2) + s_3 K(\alpha_2) \tag{70}$$

Thus we are left with the equations

$$4Qu_2 = -s_4 K(\alpha) + s_3 N(\alpha)$$
$$4Qu_1 = s_4 M(\alpha) + s_3 H(\alpha)$$
$$4Qu_2 = -s_4 H(\alpha_2) + s_3 M(\alpha_2)$$
$$4Qu_1 = s_4 N(\alpha_1) + s_3 K(\alpha_1)$$

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or

$$-s_4 K(\alpha) + s_3 N(\alpha) = -s_4 H(\alpha_2) + s_3 M(\alpha_2)$$
$$s_4 M(\alpha) + s_3 H(\alpha) = s_4 N(\alpha_1) + s_3 K(\alpha_1)$$

But these are precisely equations (51,35). Thus there are only the equations (47-50) and (63-66).

Remark 1. Equation 50 follows from the equations (47-49) and the identity (E.22)

Proof.

$$
\begin{aligned}
N(\alpha + \psi) &= \frac{4Q + H(\alpha + \psi)M(\alpha + \psi)}{K(\alpha + \psi)} \\
&= \frac{4Q + 4Q[2u_1 s_2 \sin \psi \cos \psi - \sin^2 \psi]}{2 s_2 s_3 \cos \psi} \\
&= \frac{4Q \cos \psi [2u_1 s_2 \sin \psi + \cos \psi]}{2 s_2 s_3 \cos \psi} \\
&= 2u_1 2 s_4 \sin \psi + 2 \frac{s_4}{s_2} \cos \psi
\end{aligned}
$$

$\square$

Remark 2. Equation 66 follows from equations (63-65) and the identity (E.22)

Proof.

$$
\begin{aligned}
N(\alpha + \phi) &= \frac{4Q + H(\alpha + \phi)M(\alpha + \phi)}{K(\alpha + \phi)} \\
&= \frac{4Q - 2 s_1 s_4 \cos \phi [-4 u_2 s_3 \sin \phi + 2 \frac{s_3}{s_1} \cos \phi]}{2 s_1 s_3 \sin \phi} \\
&= 4 u_2 s_4 \cos \phi + \frac{2 s_4}{s_1} \sin \phi
\end{aligned}
$$

$\square$

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Remark 3. Equations (47,48,63,64) result in the relation

$$s_1 s_2 = \tan (\phi - \psi) \tag{71}$$

Proof.

$$K(\alpha + \psi)H(\alpha + \phi) - K(\alpha + \phi)H(\alpha + \psi)$$
$$= - 2s_2 s_3 \cos \psi 2 s_1 s_4 \cos \phi - 2 s_1 s_3 \sin \phi 2 s_2 s_4 \sin \psi$$
$$= - 4Q s_1 s_2 \cos (\phi - \psi)$$

From the identity (E.24) we get

$$K(\alpha + \psi)H(\alpha + \phi) - K(\alpha + \phi)H(\alpha + \psi)$$
$$= 4Q \sin (\psi - \phi)$$

and then

$$s_1 s_2 \cos (\phi - \psi) = \sin (\phi - \psi)$$

□

Remark 4. Equations (47,63) together with equation 71 result in equation 65

Proof.

$$\text{let} \quad \phi - \psi = \delta, \quad \psi = \phi - \delta$$
$$\text{Then} \quad K(\alpha + \psi) = K(\alpha + \phi - \delta) = K(\alpha + \phi) \cos \delta + M(\alpha + \phi) \sin \delta$$
$$\text{and} \quad 2 s_2 s_3 \cos (\phi - \delta) = \cos \delta K(\alpha + \phi) + \sin \delta M(\alpha + \phi)$$
$$2 s_2 s_3 [\cos \phi + \sin \phi \tan \delta] = 2 s_1 s_3 \sin \phi + \tan \delta M(\alpha + \phi)$$
$$2 s_2 s_3 \cos \phi + 2 s_1 s_3 \sin \phi s_2^2 = 2 s_1 s_3 \sin \phi + s_1 s_2 M(\alpha + \phi)$$
$$2 s_2 s_3 \cos \phi - s_1 s_2 M(\alpha + \phi) = 2 s_1 s_3 \sin \phi [1 - s_2^2]$$
$$2 \frac{s_3}{s_1} \cos \phi - M(\alpha + \phi) = 4 s_3 \sin \phi u_2$$

□

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5 The equations for the rational leaning box

The unknowns are

$$s_1, s_2, s_3, s_4; \quad 0 < s_k < 1 \tag{72}$$

$$u_k = \frac{1 - s_k^2}{2s_k}, \quad Q = s_3 s_4 \tag{73}$$

The parameters are α Heron angle with generator m, ψ Euler angle and u_1 (74)

Using Appendix E the equations (47-50) result in the relations

$$K(\alpha) = 2s_3[s_2 \cos^2 \psi + \sin \psi \{2u_1 \cos \psi - \frac{1}{s_2} \sin \psi\}] \tag{75}$$

$$N(\alpha) = 2s_4[-s_2 \sin^2 \psi + \cos \psi \{2u_1 \sin \psi + \frac{1}{s_2} \cos \psi\}] \tag{76}$$

$$H(\alpha) = 2s_4[s_2 \sin \psi \cos \psi + \sin \psi \{2u_1 \sin \psi + \frac{1}{s_2} \cos \psi\}] \tag{77}$$

$$M(\alpha) = 2s_3[-s_2 \sin \psi \cos \psi + \cos \psi \{2u_1 \cos \psi - \frac{1}{s_2} \sin \psi\}] \tag{78}$$

These relations reproduce equations (29-30)

$$s_3 H(\alpha) + s_4 M(\alpha) = 4Q u_1 \tag{79}$$
$$s_3 N(\alpha) - s_4 K(\alpha) = 4Q u_2 \tag{80}$$

With

$$\lambda = \tan \psi \tag{81}$$

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the independent equations for the rational leaning box become

$$\omega_-(\alpha) - \lambda\omega_+(\alpha) = s_2s_3 + \lambda s_2s_4 \tag{82}$$

$$Q[\omega_+(\alpha) + \lambda\omega_-(\alpha)] = s_2s_3 - \lambda s_2s_4 \tag{83}$$

$$2u_1[\omega_-(\alpha) - \lambda\omega_+(\alpha)] + s_4 - \lambda s_3 = s_2[\omega_+(\alpha) + \lambda\omega_-(\alpha)] \tag{84}$$

From equations (82-84) with

$$\lambda = 0 \tag{85}$$

the equations for the rational leaning box thus become

$$s_2s_3 = \omega_-(\alpha) \tag{86}$$

$$s_2s_3 = Q\omega_+(\alpha) \tag{87}$$

$$s_2\omega_+(\alpha) = 2u_1\omega_-(\alpha) + s_4 \tag{88}$$

Theorem 1. For $\lambda = 0$ the solutions of the equations for the rational leaning box are given by the two rational parameters s_1, m, where

$$0 < s_1 < 1, \quad u_1 = \frac{1 - s_1^2}{2s_1}, \tag{89}$$

and α has the generator m,

$$0 < \alpha < \frac{\pi}{4}, \quad 0 < m < \sqrt{2} - 1 \tag{90}$$

as follows

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$$s_2 = 2u_1 \cot(2\alpha) \tag{91}$$

$$s_3 = \frac{\omega_-(\alpha)}{s_2} \tag{92}$$

$$s_4 = \frac{s_2}{\omega_+(\alpha)} \tag{93}$$

We also have to respect the inequality (D.13)

Proof. Equation 92 follows from equation 86.
Equation 87 reads

$$s_2 = s_4\omega_+(\alpha)$$

and gives equation 93.

From equation 88 we get

$$s_2\omega_+^2(\alpha) = 2u_1\omega_-(\alpha)\omega_+(\alpha) + s_4\omega_+(\alpha)$$

or

$$s_2[\omega_+^2(\alpha) - 1] = 2u_1\omega_-(\alpha)\omega_+(\alpha)$$

resulting in

$$s_2 \sin(2\alpha) = 2u_1 \cos(2\alpha)$$

Equation 90 assures that $\omega_-(\alpha) > 0$

$\square$

Theorem 2. For $\lambda = 0$ the cuboid limit $s_4 = s_3$ is impossible. Thus there is no perfect cuboid in this family.

Proof. From equation (92,93) we find

$$s_2^2 s_3 = s_4 \cos(2\alpha) \qquad (94)$$

Since 2α is a Heron angle, $\cos(2\alpha)$ can not be the square of a rational number, (Appendix A, Lemma 1)

Remark 5. For $\psi = 0$, we get from page 4 that

$$\omega_+(\sigma_1) = \sqrt{2}, \qquad\qquad \omega_-(\sigma_1) = 0$$

or

$$\cos\sigma_1 = \frac{1}{\sqrt{2}}, \qquad\qquad \sin\sigma_1 = \frac{1}{\sqrt{2}}$$

$$\text{meaning that} \quad \sigma_1 = \frac{\pi}{4}$$
$$\text{and thus} \quad \alpha + \alpha_1 = 2\sigma_1 = \frac{\pi}{2}$$

$\square$

Example 1.

$$s_1 = \frac{1}{2}, \quad m = \frac{1}{3}$$

$$\cos\alpha = \frac{4}{5}, \quad \sin\alpha = \frac{3}{5}, \quad \cos(2\alpha) = \frac{7}{25}, \quad \sin(2\alpha) = \frac{24}{25}, \quad \omega_+(\alpha) = \frac{7}{5}, \quad \omega_-(\alpha) = \frac{1}{5}$$

Then

$$s_2 = \frac{7}{16}, \quad s_3 = \frac{16}{35}, \quad s_4 = \frac{5}{16}$$

and

$$u_1 = \frac{840}{1120}, \quad u_2 = \frac{1035}{1120}, \quad u_3 = \frac{969}{1120}, \quad u_4 = \frac{1617}{1120}$$
$$v_1 = \frac{1400}{1120}, \quad v_2 = \frac{1525}{1120}, \quad v_3 = \frac{1481}{1120}, \quad v_4 = \frac{1967}{1120}$$

The leaning box is then given by

$$x = 1120, \qquad y = 840, \qquad z = 1035$$
$$a = 1400, \qquad b = 1525$$
$$c_1 = 969, \qquad c_2 = 1617$$
$$d_1 = 1481, \qquad d_2 = 1967$$

Example 2.

$$s_1 = \frac{12}{25}, \quad m = \frac{1}{3}$$

$$\cos(2\alpha) = \frac{7}{25}, \quad \sin(2\alpha) = \frac{24}{25}, \quad \omega_+(\alpha) = \frac{7}{5}, \quad \omega_-(\alpha) = \frac{1}{5}$$

Then

$$s_2 = \frac{3367}{7200}, \quad s_3 = \frac{1440}{3367}, \quad s_4 = \frac{2405}{7200}$$

and

$$u_1 = \frac{38868648}{48484800}, \quad u_2 = \frac{40503311}{48484800}, \quad u_3 = \frac{46315445}{48484800}, \quad u_4 = \frac{64478365}{48484800}$$

$$v_1 = \frac{62141352}{48484800}, \quad v_2 = \frac{63176689}{48484800}, \quad v_3 = \frac{67051445}{48484800}, \quad v_4 = \frac{80673635}{48484800}$$

The leaning box is then given by

$$
\begin{aligned}
x &= 48484800, & y &= 38868648, & z &= 40503311 \\
a &= 62141352, & b &= 63176689 \\
c_1 &= 46315445, & c_2 &= 64478365 \\
d_1 &= 67051445, & d_2 &= 80673635
\end{aligned}
$$

6 Symmetry and the general equations

We have solved the general equations (82-84) for

$$\lambda = \tan\psi = 0$$

For a given s_1, we found s_2, s_3, s_4 in terms of u_1 and α (91-93).
The two parallelograms I, II in (16-17) are described by the two Heron angles α and α_1 and according to the observation in Appendix D are also described by the two Heron angles β and β_1. Their generators are given by

$$m(\alpha) = \frac{2u_2 + u_3 - u_4}{2u_1 + u_3 + u_4}, \quad m(\alpha_1) = \frac{2v_2 + v_3 - v_4}{2u_1 + v_3 + v_4} \tag{95}$$

$$m(\beta) = \frac{2u_2 - u_3 + u_4}{2u_1 + u_3 + u_4}, \quad m(\beta_1) = \frac{2v_2 - v_3 + v_4}{2u_1 + v_3 + v_4} \tag{96}$$

Now the equation (16) is invariant under the interchange of u_3 and u_4, and the equation (17) is invariant under the interchange of v_3 and v_4. According to (13-14) this means that the equations (16-17) are invariant under the interchange of s_3 with s_4.

From (p.4) and (81) we find

$$\alpha + \alpha_1 = 2\sigma_1$$

$$\omega_+(\sigma_1) = \sqrt{2}\cos\psi, \quad \omega_-(\sigma_1) = \sqrt{2}\sin\psi, \quad \lambda = \tan\psi = \frac{\omega_-(\sigma_1)}{\omega_+(\sigma_1)}$$

let k be the generator of $\alpha + \alpha_1$

$$k = \frac{m(\alpha) + m(\alpha_1)}{1 - m(\alpha)m(\alpha_1)} \tag{97}$$

Then

$$\sin(2\sigma_1) = \frac{2k}{1+k^2}, \quad \cos(2\sigma_1) = \frac{1-k^2}{1+k^2} \tag{98}$$

and

$$\lambda = tan\psi = \frac{\omega_-(\sigma_1)}{\omega_+(\sigma_1)} = \frac{\omega_-^2(\sigma_1)}{\omega_+(\sigma_1)\omega_-(\sigma_1)} = \frac{1-\sin(2\sigma_1)}{\cos(2\sigma_1)} \tag{99}$$

resulting in

$$\lambda = \frac{1-k}{1+k} \tag{100}$$

Let $\bar{k}$ be the generator of $\beta + \beta_1$

$$\bar{k} = \frac{m(\beta) + m(\beta_1)}{1 - m(\beta)m(\beta_1)} \tag{101}$$

and thus

$$\bar{\lambda} = \frac{1-\bar{k}}{1+\bar{k}} \tag{102}$$

Therefore the parameters $u_1, \beta, \bar{\lambda}$ also satisfy the general equations, however with the interchange of s_3 with s_4.

7 An equivalent set of equations for the rational leaning box

For the three parallelograms we use

$$\text{I.} \qquad 2u_1^2 + 2u_2^2 = u_3^2 + u_4^2$$
$$\text{II.} \qquad 2u_1^2 + 2v_2^2 = v_3^2 + v_4^2$$
$$\text{III.} \qquad v_2^2 - u_2^2 = 1,\ v_3^2 - u_3^2 = 1,\ v_4^2 - u_4^2 = 1$$

where $u_k = \cot \psi_k$, $v_k = \frac{1}{\sin \psi_k}$, ψ_k Heron angles with generators $s_k \in (0,1)$, k=1,2,3,4.

We give s_1 resulting in giving u_1 and v_1.

According to Appendix D we have the following parameterization

$$\text{I. parameters} \qquad u_1, \alpha, \beta; \alpha, \beta \in (0, \frac{\pi}{2}) \tag{103}$$

$$\alpha + \beta = 2\sigma, \alpha - \beta = 2\delta; \alpha = \sigma + \delta, \beta = \sigma - \delta \tag{104}$$

$$\text{II. parameters} \qquad u_1, \alpha_1, \beta_1; \alpha_1, \beta_1 \in (0, \frac{\pi}{2}) \tag{105}$$

$$\alpha_1 + \beta_1 = 2\sigma_1, \alpha_1 - \beta_1 = 2\delta_1; \alpha_1 = \sigma_1 + \delta_1, \beta_1 = \sigma_1 - \delta_1 \tag{106}$$

and thus the representation

$$u_2 = u_1 \tan \sigma, \qquad u_3 = u_1 \frac{\omega_+(\delta)}{\cos \sigma}, \qquad u_4 = u_1 \frac{\omega_-(\delta)}{\cos \sigma} \tag{107}$$

$$v_2 = u_1 \tan \sigma_1, \qquad v_3 = u_1 \frac{\omega_+(\delta_1)}{\cos \sigma_1}, \qquad v_4 = u_1 \frac{\omega_-(\delta_1)}{\cos \sigma_1} \tag{108}$$

conversely

$$\tan \sigma = \frac{u_2}{u_1}, \qquad\qquad \tan \delta = \frac{u_3 - u_4}{u_3 + u_4} \tag{109}$$

$$\tan \sigma_1 = \frac{v_2}{u_1}, \qquad\qquad \tan \delta_1 = \frac{v_3 - v_4}{v_3 + v_4} \tag{110}$$

Observe that ψ_1 is a Heron angle and $\sigma, \delta, \sigma_1, \delta_1$ are Euler angles. We rename $\psi_1 = \psi$

We now have the conditions

$$u_1^2[\tan^2 \sigma_1 - \tan^2 \sigma] = u_1^2[\frac{1}{\cos^2 \sigma_1} - \frac{1}{\cos^2 \sigma}] = 1 \tag{111}$$

$$u_1^2[\frac{\omega_+{}^2(\delta_1)}{\cos^2 \sigma_1} - \frac{\omega_+{}^2(\delta)}{\cos^2 \sigma}] = u_1^2[\frac{1 + \sin(2\delta_1)}{\cos^2 \sigma_1} - \frac{1 + \sin(2\delta)}{\cos^2 \sigma}] = 1 \tag{112}$$

$$u_1^2[\frac{\omega_-{}^2(\delta_1)}{\cos^2 \sigma_1} - \frac{\omega_-{}^2(\delta)}{\cos^2 \sigma}] = u_1^2[\frac{1 - \sin(2\delta_1)}{\cos^2 \sigma_1} - \frac{1 - \sin(2\delta)}{\cos^2 \sigma}] = 1 \tag{113}$$

or using eq.(111), the conditions reduce to

$$\tan^2 \sigma_1 - \tan^2 \sigma = \tan^2 \psi \tag{114}$$

$$\frac{\sin(2\delta)}{\cos^2 \sigma} = \frac{\sin(2\delta_1)}{\cos^2 \sigma_1} \tag{115}$$

Using the generators, we introduce the abbreviations

$$M = m(\alpha + \beta) = m(2\sigma) = \tan \sigma > 0 \tag{116}$$

$$M_1 = m(\alpha_1 + \beta_1) = m(2\sigma_1) = \tan \sigma_1 > 0 \tag{117}$$

$$N = m(\alpha - \beta) = m(2\delta) = \tan \delta \tag{118}$$

$$N_1 = m(\alpha_1 - \beta_1) = m(2\delta_1) = \tan \delta_1 \tag{119}$$

Then

$$\tan \alpha = \tan(\sigma + \delta) = \frac{\tan \sigma + \tan \delta}{1 - \tan \sigma \tan \delta} = \frac{M + N}{1 - MN} \tag{120}$$

$$\tan \alpha_1 = \tan(\sigma_1 + \delta_1) = \frac{\tan \sigma_1 + \tan \delta_1}{1 - \tan \sigma_1 \tan \delta_1} = \frac{M_1 + N_1}{1 - M_1 N_1} \tag{121}$$

resulting in

$$N = \frac{\tan \alpha - M}{1 + M \tan \alpha} \tag{122}$$

$$N_1 = \frac{\tan \alpha_1 - M_1}{1 + M_1 \tan \alpha_1} \tag{123}$$

The conditions (114) (115) then become

$$M_1^2 - M^2 = \tan^2 \psi \tag{124}$$

$$\frac{2N}{1 + N^2}(1 + M^2) = \frac{2N_1}{1 + N_1^2}(1 + M_1^2) \tag{125}$$

Or, using eqs (122), (123) we find

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$$(1 - M^2)\sin(2\alpha) - 2M\cos(2\alpha) = (1 - M_1^2)\sin(2\alpha_1) - 2M_1\cos(2\alpha_1) \quad (126)$$

Now

$$\cos\delta\cos\sigma = \frac{1}{2}[\cos(\delta - \sigma) + \cos(\delta + \sigma)] = \frac{1}{2}[\cos\beta + \cos\alpha] \quad (127)$$

$$\sin\delta\cos\sigma = \frac{1}{2}[\sin(\delta - \sigma) + \sin(\delta + \sigma)] = \frac{1}{2}[\sin\alpha - \sin\beta] \quad (128)$$

$$\frac{1}{\cos^2\sigma} = 1 + \tan^2\sigma = 1 + M^2 \quad (129)$$

gives the following representation

$$\frac{\omega_+(\delta)}{\cos\sigma} = \frac{\cos\delta + \sin\delta}{\cos^2\sigma}\cos\sigma = \frac{1}{2}(1 + M^2)[\omega_+(\alpha) + \omega_-(\beta)] \quad (130)$$

$$\frac{\omega_-(\delta)}{\cos\sigma} = \frac{\cos\delta - \sin\delta}{\cos^2\sigma}\cos\sigma = \frac{1}{2}(1 + M^2)[\omega_-(\alpha) + \omega_+(\beta)] \quad (131)$$

From

$$\cos\beta = \cos(\alpha + \beta - \alpha) = \cos(\alpha + \beta)\cos\alpha + \sin(\alpha + \beta)\sin\alpha \quad (132)$$

$$\cos\beta = \frac{1 - M^2}{1 + M^2}\cos\alpha + \frac{2M}{1 + M^2}\sin\alpha \quad (133)$$

$$\sin\beta = \sin(\alpha + \beta - \alpha) = \sin(\alpha + \beta)\cos\alpha - \cos(\alpha + \beta)\sin\alpha \quad (134)$$

$$\sin\beta = \frac{2M}{1 + M^2}\cos\alpha - \frac{1 - M^2}{1 + M^2}\sin\alpha \quad (135)$$

we find

$$\omega_+(\beta) = \frac{1 - M^2}{1 + M^2}\omega_-(\alpha) + \frac{2M}{1 + M^2}\omega_+(\alpha) \quad (136)$$

$$\omega_-(\beta) = \frac{1 - M^2}{1 + M^2}\omega_+(\alpha) - \frac{2M}{1 + M^2}\omega_-(\alpha) \quad (137)$$

and then

$$\omega_+(\alpha) + \omega_-(\beta) = \frac{2}{1 + M^2}[\omega_+(\alpha) - M\omega_-(\alpha)] \quad (138)$$

$$\omega_-(\alpha) + \omega_+(\beta) = \frac{2}{1 + M^2}[\omega_-(\alpha) + M\omega_+(\alpha)] \quad (139)$$

Finally, from eq.(107)

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$$u_2 = u_1 M \tag{140}$$
$$u_3 = u_1[\omega_+(\alpha) - M\omega_-(\alpha)] \tag{141}$$
$$u_4 = u_1[\omega_-(\alpha) + M\omega_+(\alpha)] \tag{142}$$

and from eq.(108)

$$v_2 = u_1 M_1 \tag{143}$$
$$v_3 = u_1[\omega_+(\alpha_1) - M_1\omega_-(\alpha_1)] \tag{144}$$
$$v_4 = u_1[\omega_-(\alpha_1) + M_1\omega_+(\alpha_1)] \tag{145}$$

From the symmetry of interchanging α and β, which corresponds to the interchange of u_3 and u_4, we also have the representation

$$u_3 = u_1[\omega_-(\beta) + M\omega_+(\beta)] \tag{146}$$
$$u_4 = u_1[\omega_+(\beta) - M\omega_-(\beta)] \tag{147}$$

Example

For the special case of $\alpha + \alpha_1 = \frac{\pi}{2}$, resulting in

$$\cos(2\alpha_1) = -\cos(2\alpha), \sin(2\alpha_1) = \sin(2\alpha)$$

and given the two Heron angles ψ, α ; $u_1 = \cot\psi$,
the conditions (124) (126) read

$$M_1 - M = 2\cot(2\alpha), M_1 + M = \frac{1}{2}\tan^2\psi\tan(2\alpha)$$

or

$$M = \frac{1}{4}[\tan^2\psi\tan(2\alpha) - 4\cot(2\alpha)]$$
$$M_1 = \frac{1}{4}[4\cot(2\alpha) + \tan^2\psi\tan(2\alpha)]$$

Observe that this two-parameter family of rational leaning boxes has no cuboid limit, because in the cuboid limit N=0, $N_1 = 0$, implying

$$M = \tan\alpha, M_1 = \tan\alpha_1 = \frac{1}{\tan\alpha}$$

Then

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$$M_1^2 - M^2 = \tan^2 \psi$$

reads

$$1 - \tan^4 \alpha = \tan^2 \alpha \tan^2 \psi$$

which has no rational solutions [6].

We now split the eq.(126) into two parts

$$(M^2 - 1)\sin(2\alpha) + 2M\cos(2\alpha) = 4D \tag{148}$$

$$(M_1^2 - 1)\sin(2\alpha_1) + 2M_1\cos(2\alpha_1) = 4D \tag{149}$$

According to the Appendix F, they have the following parameter representations, replacing λ by r, respectively by r_1.

$$M = 2r + \tan\alpha \qquad\qquad D = r[1 + r\sin(2\alpha)] \tag{150}$$

$$M_1 = 2r_1 + \tan\alpha_1 \qquad\qquad D = r_1[1 + r_1\sin(2\alpha_1)] \tag{151}$$

Now, using [8], the equation

$$r[1 + r\sin(2\alpha)] = r_1[1 + r_1\sin(2\alpha_1)] \tag{152}$$

has the parameter representation

$$r = f\frac{1 + f\sin(2\alpha_1)}{1 - f^2\sin(2\alpha)\sin(2\alpha_1)} \tag{153}$$

$$r_1 = f\frac{1 + f\sin(2\alpha)}{1 - f^2\sin(2\alpha)\sin(2\alpha_1)} \tag{154}$$

Conversely, for $f \neq 0$

case (i)

$$r_1 \neq r, f = \frac{r_1 - r}{r\sin(2\alpha) - r_1\sin(2\alpha_1)} \tag{155}$$

case (ii)

$$r_1 = r, f = \frac{r}{1 + r\sin(2\alpha)} \tag{156}$$

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and
$$\sin(2\alpha_1) = \sin(2\alpha)$$
or
$$\cos(\alpha_1 + \alpha)\sin(\alpha_1 - \alpha) = 0$$

For the non-trivial case, $\alpha \neq \alpha_1$ we have

$$\alpha_1 + \alpha = \frac{\pi}{2}, \text{ which is the example above}$$

This parameter representation can be verified directly.

8   Cuboid Limit

The cuboid limit is given by

$$\beta \to \alpha \quad \text{and} \quad \beta_1 \to \alpha_1 \tag{157}$$

From (118, 119) this means

$$N \to 0 \quad \text{and} \quad N_1 \to 0 \tag{158}$$

and from (122, 123) we get

$$M \to \tan\alpha \quad \text{and} \quad M_1 \to \tan\alpha_1 \tag{159}$$

or equivalently from (150,151) that

$$r \to 0 \quad \text{and} \quad r_1 \to 0 \tag{160}$$

Looking at the equation (152) and letting $\frac{r_1}{r} = s$, we have two cases

$$\text{(i)} \quad s = 1 \quad \text{implies} \quad \sin(2\alpha) = \sin(2\alpha_1) \quad \text{or}$$
$$\alpha + \alpha_1 = \frac{\pi}{2} \tag{161}$$

which, according to the example on page 21 has no cuboid limit. This is also the family found in section 5.

$$\text{(ii)} \quad s \neq 1 \ . \ \text{Let}$$
$$F(r, r_1) = \frac{r_1}{r} - \frac{1 + r\sin(2\alpha)}{1 + r_1\sin(2\alpha_1)} \tag{162}$$

For the cuboid limit

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$$F(r, r_1) = s - 1 \neq 0, \tag{163}$$

which contradicts equation (152).

Thus there is also no cuboid limit.

In conclusion, there is no perfect cuboid.

9   Consequence

Corollary

Let α_1, α be Euler angles and ψ be a Heron angle. Then in the equation

$$\tan^2 \alpha_1 - \tan^2 \alpha = \tan^2 \psi \tag{164}$$

not both α and α_1 can be Heron angles

Proof. In the cuboid limit only equation (124) is the surviving condition. If both α and α_1 were Heron angles we would have a perfect cuboid. This is a contradiction. $\qquad\square$

Example

Euler Cuboid (body diagonal not rational)

$$\tan \alpha_1 = \frac{125}{240}, \quad \cos \alpha_1 = \frac{48}{\sqrt{2929}}, \quad \sin \alpha_1 = \frac{25}{\sqrt{2929}}$$

$$\tan \alpha = \frac{44}{240}, \quad \cos \alpha = \frac{60}{61}, \quad \sin \alpha = \frac{11}{61}, \quad m(\alpha) = \frac{1}{11}$$

$$\tan \psi = \frac{117}{240}, \quad \cos \psi = \frac{80}{89}, \quad \sin \psi = \frac{39}{89}, \quad m(\psi) = \frac{3}{13}$$

Example

Face Cuboid (one face diagonal not rational)

$$\tan \alpha_1 = \frac{765}{520}, \quad \cos \alpha_1 = \frac{104}{185}, \quad \sin \alpha_1 = \frac{153}{185}, \quad m(\alpha_1) = \frac{9}{17}$$

$$\tan \alpha = \frac{756}{520}, \quad \cos \alpha = \frac{130}{\sqrt{52621}}, \quad \sin \alpha = \frac{189}{\sqrt{52621}}$$

$$\tan \psi = \frac{117}{520}, \quad \cos \psi = \frac{40}{41}, \quad \sin \psi = \frac{9}{41}, \quad m(\psi) = \frac{1}{9}$$

Appendix A

Generator of an angle
Heron angle, Euler angle

<u>Definition 1</u>

For an arbitrary angle α, its generator is defined by

$$m(\alpha) = \frac{\sin \alpha}{1 + \cos \alpha} = \tan(\frac{\alpha}{2}) \tag{A.1}$$

Consequently

$$\cos \alpha = \frac{1 - m^2(\alpha)}{1 + m^2(\alpha)}, \quad \sin \alpha = \frac{2m(\alpha)}{1 + m^2(\alpha)} \tag{A.2}$$

We have the following properties

$$m(-\alpha) = -m(\alpha) \tag{A.3}$$

$$m^2(\alpha) = \frac{1 - \cos \alpha}{1 + \cos \alpha} \tag{A.4}$$

$$m(\alpha + \beta) = \frac{m(\alpha) + m(\beta)}{1 - m(\alpha)m(\beta)} \tag{A.5}$$

$$\frac{dm(\alpha)}{d\alpha} = \frac{1}{2}\frac{1}{\cos^2(\frac{\alpha}{2})} > 0 \tag{A.6}$$

Then m(α) is an increasing function of α, with

$$m(0) = 0, \quad m(\frac{\pi}{2}) = 1$$

For $0 < \alpha < \frac{\pi}{2}$ the generator satisfies

$$0 < m(\alpha) < 1 \tag{A.7}$$

Definition 2

An angle α is called a Heron angle if both $\sin\alpha$ and $\cos\alpha$ are rational.

The generator of a Heron angle is rational and vise versa. The sum and the difference of two Heron angles are Heron angles.

The complement

$$\bar{\alpha} = \frac{\pi}{2} - \alpha \tag{A.8}$$

of a Heron angle α is a Heron angle.

Definition 3

An angle α is called an Euler angle if $\tan\alpha$ is rational.

If α is an Euler angle then 2α is a Heron angle. A Heron angle is an Euler angle.

Lemma 1

Let α be a Heron angle, $0 \leq \alpha \leq \pi$.
Then the equation

$$\sin\alpha = \lambda^2 \tag{A.9}$$

where λ is a rational number, $0 \leq \lambda \leq 1$, has only the trivial solutions $\alpha = 0, \quad \alpha = \frac{\pi}{2}, \quad \alpha = \pi$

Proof. let $\lambda = \frac{a}{b}, \quad 0 < a < b, \quad$ a and b integers.

Then $\sin\alpha$ is rational and

$$\cos^2\alpha = 1 - \sin^2\alpha = 1 - \lambda^4 = \frac{b^4 - a^4}{b^4} \tag{A.10}$$

But according to Euler [6], $b^4 - a^4$ can not be the square of an integer. Thus $\cos\alpha$ is not rational, i.e. α is not a Heron angle, except for the trivial cases. $\square$

Lemma 2

Let α be a Heron angle, $0 \leq \alpha \leq \frac{\pi}{2}$.

Then the equation

$$\tan \alpha = \lambda^2 \tag{A.11}$$

where λ is a rational number, $\lambda \geq 0$, has only the trivial solution $\alpha = 0$.

Proof. Let $\lambda = \frac{a}{b}$, $a > 0$, $b > 0$, a and b integers.

Then $\tan \alpha$ is rational and

$$\frac{1}{\cos^2 \alpha} = 1 + \tan^2 \alpha = 1 + \lambda^4 = \frac{b^4 + a^4}{b^4} \tag{A.12}$$

But according to Euler [6], $b^4 + a^4$ is not the square of an integer. Thus $\cos \alpha$ is not rational, except for the trivial case.

Thus in any case, for a Heron angle α, $\sin \alpha, \cos \alpha, \tan \alpha, \cot \alpha$ can not be the square of a rational number, except for the trivial cases.

$\square$

Corollary 1

The elliptic curve

$$y^2 = x(1 - x^2) \tag{A.13}$$

has only the trivial rational points (x,y), namely (-1,0),(0,0),(1,0)

Proof. Let α be a Heron angle and

$$\lambda^2 = \sin 2\alpha$$

Then λ can not be rational, except for the trivial cases.
Let x be the generator of α

$$\lambda^2 = 2 \sin \alpha \cos \alpha = 2 \frac{2x}{1+x^2} \frac{1-x^2}{1+x^2}$$

$$\lambda^2 = [\frac{2}{1+x^2}]^2 x(1-x^2)$$

Then

$$y^2 = x(1-x^2) \tag{A.14}$$

has only the trivial rational points.

$\square$

Corollary 2

The elliptic curve

$$y^2 = 2x(1-x^2) \tag{A.15}$$

has only the trivial rational points (x,y), namely (-1,0),(0,0),(1,0)

Proof. Let α be a Heron angle with generator x, and

$$\lambda^2 = \tan \alpha$$

Then λ can not be rational, except for the trivial case.

Now

$$\lambda^2 = \frac{2x}{1-x^2} = [\frac{1}{1-x^2}]^2 2x(1-x^2)$$

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Then

$$y^2 = 2x(1 - x^2)$$

has only the trivial rational points. $\square$

Appendix B

Rotations

A rotation in two dimensions is given by the matrix

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{B.1}$$

These form an Abelian group.

Group multiplication	$R(\alpha)R(\beta) = R(\alpha + \beta)$	(B.2)
Identity	$R(0)$	(B.3)
Inverse	$R(\alpha)^{-1} = R(-\alpha)$	(B.4)

For two two-dimensional vectors, related by a rotation, we have

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = R(\alpha) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \tag{B.5}$$

They have the same length, i.e.

$$x_1^2 + y_1^2 = x_2^2 + y_2^2 \tag{B.6}$$

Conversely we get the rotation angle α through

$$\cos\alpha = \frac{1}{x_2^2 + y_2^2}[x_1 x_2 + y_1 y_2] \tag{B.7}$$

$$\sin\alpha = \frac{1}{x_2^2 + y_2^2}[y_1 x_2 - x_1 y_2] \tag{B.8}$$

Appendix C

The ω-functions

Definition

For an angle α we introduce the ω-functions by

$$\omega_+(\alpha) = \cos\alpha + \sin\alpha, \quad \omega_-(\alpha) = \cos\alpha - \sin\alpha \tag{C.1}$$

We then have the following properties

$$\omega_+(-\alpha) = \omega_-(\alpha) \tag{C.2}$$
$$\omega_-(-\alpha) = \omega_+(\alpha) \tag{C.3}$$
$$\omega_+^2(\alpha) + \omega_-^2(\alpha) = 2 \tag{C.4}$$
$$\omega_+(\alpha)\omega_-(\alpha) = \cos(2\alpha) \tag{C.5}$$
$$\omega_+^2(\alpha) = 1 + \sin(2\alpha) \tag{C.6}$$
$$\omega_-^2(\alpha) = 1 - \sin(2\alpha) \tag{C.7}$$

For two angles α and β we introduce

$$\alpha + \beta = 2\sigma, \quad \alpha - \beta = 2\delta \tag{C.8}$$
$$\alpha = \sigma + \delta, \quad \beta = \sigma - \delta \tag{C.9}$$

We then find

$$\omega_+(\alpha)\omega_+(\beta) = \cos(2\delta) + \sin(2\sigma) \tag{C.10}$$
$$\omega_-(\alpha)\omega_-(\beta) = \cos(2\delta) - \sin(2\sigma) \tag{C.11}$$
$$\omega_+(\alpha)\omega_-(\beta) = \cos(2\sigma) + \sin(2\delta) \tag{C.12}$$

We also have the relation

$$R(\beta)\begin{pmatrix} \omega_+(\alpha+\beta) \\ \omega_-(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \omega_+(\alpha) \\ \omega_-(\alpha) \end{pmatrix} \tag{C.13}$$

or explicitly

$$\omega_+(\alpha+\beta) = \cos\beta\,\omega_+(\alpha) + \sin\beta\,\omega_-(\alpha) \tag{C.14}$$
$$\omega_+(\alpha+\beta) = \cos\alpha\,\omega_+(\beta) + \sin\alpha\,\omega_-(\beta) \tag{C.15}$$
$$\omega_-(\alpha+\beta) = -\sin\beta\,\omega_+(\alpha) + \cos\beta\,\omega_-(\alpha) \tag{C.16}$$
$$\omega_-(\alpha+\beta) = -\sin\alpha\,\omega_+(\beta) + \cos\alpha\,\omega_-(\beta) \tag{C.17}$$

This gives the following relations

$$\omega_+(\alpha) + \omega_+(\beta) = 2\cos\delta\omega_+(\sigma) \qquad (C.18)$$
$$\omega_+(\alpha) - \omega_+(\beta) = 2\sin\delta\omega_-(\sigma) \qquad (C.19)$$
$$\omega_-(\alpha) + \omega_-(\beta) = 2\cos\delta\omega_-(\sigma) \qquad (C.20)$$
$$\omega_-(\alpha) - \omega_-(\beta) = -2\sin\delta\omega_+(\sigma) \qquad (C.21)$$
$$\omega_+(\alpha) + \omega_-(\beta) = 2\cos\sigma\omega_+(\delta) \qquad (C.22)$$
$$\omega_+(\alpha) - \omega_-(\beta) = 2\sin\sigma\omega_-(\delta) \qquad (C.23)$$
$$\omega_-(\alpha) + \omega_+(\beta) = 2\cos\sigma\omega_-(\delta) \qquad (C.24)$$
$$\omega_-(\alpha) - \omega_+(\beta) = -2\sin\sigma\omega_+(\delta) \qquad (C.25)$$

and finally

$$\begin{pmatrix} \omega_-(\alpha) \\ \omega_+(\alpha) \end{pmatrix} = R(2\alpha) \begin{pmatrix} \omega_+(\alpha) \\ \omega_-(\alpha) \end{pmatrix} \qquad (C.26)$$

Appendix D

The Rational Parallelogram

A parallelogram with its sides u_1, u_2 and diagonals u_3, u_4 being positive rational numbers is called a rational parallelogram It is governed by the parallelogram equation

$$2u_1^2 + 2u_2^2 = u_3^2 + u_4^2 \tag{D.1}$$

In [7, 8] we found a bijective parameter representation for all rational parallelograms.
It is given by the rational scaling parameter

$$u > 0 \tag{D.2}$$

and two rational parameters m,n

$$0 < m < 1, \quad 0 < n < 1 \tag{D.3}$$

The representation is given by

$$u_1 = (1 - mn)u \tag{D.4}$$
$$u_2 = (m + n)u \tag{D.5}$$
$$u_3 = (1 + mn - n + m)u \tag{D.6}$$
$$u_4 = (1 + mn + n - m)u \tag{D.7}$$

Conversely

$$4u = 2u_1 + u_3 + u_4 \tag{D.8}$$
$$m = \frac{2u_2 + u_3 - u_4}{4u}, \quad n = \frac{2u_2 - u_3 + u_4}{4u} \tag{D.9}$$

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Special cases:

$$\text{Rectangle:} \qquad u_4 = u_3; \quad n = m \tag{D.10}$$

$$\text{Rhomboid:} \qquad u_2 = u_1; \quad n = \frac{1-m}{1+m} \tag{D.11}$$

From (D.4) and (D.5) we find

$$(m+n)u_1 = (1-mn)u_2 \tag{D.12}$$

and then

$$n = \frac{u_2 - mu_1}{u_1 + mu_2}, \quad 0 < n < 1 \tag{D.13}$$

$$m = \frac{u_2 - nu_1}{u_1 + nu_2}, \quad 0 < m < 1 \tag{D.14}$$

From (D.13) we find

$$1 - mn = \frac{u_1(1+m^2)}{u_1 + mu_2} \tag{D.15}$$

$$1 + mn = \frac{u_1(1-m^2) + 2mu_2}{u_1 + mu_2} \tag{D.16}$$

$$m - n = \frac{2mu_1 - u_2(1-m^2)}{u_1 + mu_2} \tag{D.17}$$

$$u = \frac{u_1 + mu_2}{1 + m^2} \tag{D.18}$$

and then

$$u_3 = (u_2 + u_1)\frac{2m}{1+m^2} - (u_2 - u_1)\frac{1-m^2}{1+m^2} \tag{D.19}$$

$$u_4 = (u_2 + u_1)\frac{1-m^2}{1+m^2} + (u_2 - u_1)\frac{2m}{1+m^2} \tag{D.20}$$

This is a parameterization of a rational parallelogram by

$$u_1 > 0, u_2 > 0, \text{ and } 0 < m < 1$$

From (D.14) we find

$$1 - mn = \frac{u_1(1 + n^2)}{u_1 + nu_2} \tag{D.21}$$

$$1 + mn = \frac{u_1(1 - n^2) + 2nu_2}{u_1 + nu_2} \tag{D.22}$$

$$m - n = \frac{-2nu_1 + u_2(1 - n^2)}{u_1 + nu_2} \tag{D.23}$$

$$u = \frac{u_1 + nu_2}{1 + n^2} \tag{D.24}$$

and then

$$u_3 = (u_2 - u_1)\frac{2n}{1 + n^2} + (u_2 + u_1)\frac{1 - n^2}{1 + n^2} \tag{D.25}$$

$$u_4 = (u_2 + u_1)\frac{2n}{1 + n^2} - (u_2 - u_1)\frac{1 - n^2}{1 + n^2} \tag{D.26}$$

This is a parameterization of a rational parallelogram by

$$u_1 > 0, u_2 > 0, \text{ and } 0 < n < 1$$

Now the two parameters m,n in (D.3) give rise to the Heron angles α, β through

$$\cos\alpha = \frac{1-m^2}{1+m^2}, \quad \sin\alpha = \frac{2m}{1+m^2}, \quad 0 < \alpha < \frac{\pi}{2} \tag{D.27}$$

$$\cos\beta = \frac{1-n^2}{1+n^2}, \quad \sin\beta = \frac{2n}{1+n^2}, \quad 0 < \beta < \frac{\pi}{2} \tag{D.28}$$

And conversely

$$m = \frac{\sin\alpha}{1+\cos\alpha} = \tan(\frac{\alpha}{2}) \tag{D.29}$$

$$n = \frac{\sin\beta}{1+\cos\beta} = \tan(\frac{\beta}{2}) \tag{D.30}$$

Using the ω-functions we get in matrix notation the representation

$$\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} \omega_+(\alpha) & -\omega_-(\alpha) \\ \omega_-(\alpha) & \omega_+(\alpha) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{D.31}$$

Inversely

$$2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \omega_+(\alpha) & \omega_-(\alpha) \\ -\omega_-(\alpha) & \omega_+(\alpha) \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \tag{D.32}$$

and conversely

$$\omega_+(\alpha) = \frac{1}{u_1^2 + u_2^2} [u_1 u_3 + u_2 u_4] \tag{D.33}$$

$$\omega_-(\alpha) = \frac{1}{u_1^2 + u_2^2} [u_1 u_4 - u_2 u_3] \tag{D.34}$$

Similarly we find the representation

$$\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} \omega_-(\beta) & \omega_+(\beta) \\ \omega_+(\beta) & -\omega_-(\beta) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{D.35}$$

Inversely

$$2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \omega_-(\beta) & \omega_+(\beta) \\ \omega_+(\beta) & -\omega_-(\beta) \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \tag{D.36}$$

and conversely

$$\omega_+(\beta) = \frac{1}{u_1^2 + u_2^2}[u_1 u_4 + u_2 u_3] \tag{D.37}$$

$$\omega_-(\beta) = \frac{1}{u_1^2 + u_2^2}[u_1 u_3 - u_2 u_4] \tag{D.38}$$

Finally we introduce the two Euler angles σ, δ through

$$\alpha + \beta = 2\sigma, \quad \alpha - \beta = 2\delta \tag{D.39}$$

$$\alpha = \sigma + \delta, \quad \beta = \sigma - \delta \tag{D.40}$$

From

$$u_1 \omega_+(\alpha) - u_2 \omega_-(\alpha) = u_1 \omega_-(\beta) + u_2 \omega_+(\beta) \tag{D.41}$$

and

$$2u_3 = u_1 \omega_+(\alpha) - u_2 \omega_-(\alpha) + u_1 \omega_-(\beta) + u_2 \omega_+(\beta) \tag{D.42}$$

$$2u_4 = u_1 \omega_-(\alpha) + u_2 \omega_+(\alpha) + u_1 \omega_+(\beta) - u_2 \omega_-(\beta) \tag{D.43}$$

we find, using (C.19), (C.22), (C.24), (C.25) the relations

$$u_1 \sin \sigma = u_2 \cos \sigma \tag{D.44}$$

$$u_3 = [u_1 \cos \sigma + u_2 \sin \sigma]\omega_+(\delta) \tag{D.45}$$

$$u_4 = [u_1 \cos \sigma + u_2 \sin \sigma]\omega_-(\delta) \tag{D.46}$$

resulting in the representation

$$u_2 = u_1 \tan \sigma \tag{D.47}$$

$$u_3 = u_1 \frac{\omega_+(\delta)}{\cos \sigma} \tag{D.48}$$

$$u_4 = u_1 \frac{\omega_-(\delta)}{\cos \sigma} \tag{D.49}$$

Conversely

$$\tan \sigma = \frac{u_2}{u_1} \tag{D.50}$$

$$\tan \delta = \frac{u_3 - u_4}{u_3 + u_4} \tag{D.51}$$

From (D.27), (D.28), (D.39), (D.40) we find the inequalities

$$\tan \sigma > 0 \tag{D.52}$$

$$-1 < \tan \delta < 1 \tag{D.53}$$

(D.47-49) is a parameterization of a rational parallelogram, given one side $u_1 > 0$, by two Euler angles.

Observe, that given u_1 and u_2 there are two representations of u_3 and u_4, (D.31) (D.35). One is through the Heron angle α, with generator $m = m(\alpha)$ and the other through the Heron angle β, with generator $n = m(\beta)$. According to (D.8) and (D.9) they are however related through the interchange of u_3 with u_4.

Appendix E

Auxiliary Functions

For an angle α and a number Q we introduce the functions

$$H(\alpha) = \omega_-(\alpha) - Q\omega_+(\alpha) \qquad (\text{E.1})$$
$$K(\alpha) = \omega_-(\alpha) + Q\omega_+(\alpha) \qquad (\text{E.2})$$
$$M(\alpha) = \omega_+(\alpha) - Q\omega_-(\alpha) \qquad (\text{E.3})$$
$$N(\alpha) = \omega_+(\alpha) + Q\omega_-(\alpha) \qquad (\text{E.4})$$

or in matrix form

$$\begin{pmatrix} H(\alpha) \\ N(\alpha) \end{pmatrix} = \begin{pmatrix} \omega_-(\alpha) & -\omega_+(\alpha) \\ \omega_+(\alpha) & \omega_-(\alpha) \end{pmatrix} \begin{pmatrix} 1 \\ Q \end{pmatrix} = R(\alpha) \begin{pmatrix} 1 - Q \\ 1 + Q \end{pmatrix} \qquad (\text{E.5})$$

$$\begin{pmatrix} K(\alpha) \\ M(\alpha) \end{pmatrix} = \begin{pmatrix} \omega_-(\alpha) & \omega_+(\alpha) \\ \omega_+(\alpha) & -\omega_-(\alpha) \end{pmatrix} \begin{pmatrix} 1 \\ Q \end{pmatrix} = R(\alpha) \begin{pmatrix} 1 + Q \\ 1 - Q \end{pmatrix} \qquad (\text{E.6})$$

We now introduce the T-matrix

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (\text{E.7})$$

and find the following properties

$$T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (\text{E.8})$$

$$TR(\alpha)T = R(-\alpha) \qquad (\text{E.9})$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \qquad (\text{E.10})$$

$$R(\alpha)TR(-\alpha) = \begin{pmatrix} -\sin(2\alpha) & \cos(2\alpha) \\ \cos(2\alpha) & \sin(2\alpha) \end{pmatrix} \qquad \text{(E.11)}$$

Then we have the following statements

Lemma 1

$$2 = H(\alpha)\omega_-(\alpha) + N(\alpha)\omega_+(\alpha) \qquad \text{(E.12)}$$
$$2 = K(\alpha)\omega_-(\alpha) + M(\alpha)\omega_+(\alpha) \qquad \text{(E.13)}$$
$$2Q = N(\alpha)\omega_-(\alpha) - H(\alpha)\omega_+(\alpha) \qquad \text{(E.14)}$$
$$2Q = K(\alpha)\omega_+(\alpha) - M(\alpha)\omega_-(\alpha) \qquad \text{(E.15)}$$

Lemma 2

$$\begin{pmatrix} H(\alpha+\beta) \\ N(\alpha+\beta) \end{pmatrix} = R(\alpha) \begin{pmatrix} H(\beta) \\ N(\beta) \end{pmatrix} = R(\beta) \begin{pmatrix} H(\alpha) \\ N(\alpha) \end{pmatrix} \qquad \text{(E.16)}$$

$$\begin{pmatrix} K(\alpha+\beta) \\ M(\alpha+\beta) \end{pmatrix} = R(\alpha) \begin{pmatrix} K(\beta) \\ M(\beta) \end{pmatrix} = R(\beta) \begin{pmatrix} K(\alpha) \\ M(\alpha) \end{pmatrix} \qquad \text{(E.17)}$$

Lemma 3
With $\bar{\alpha} = \frac{\pi}{2} - \alpha$ we have

$$H(\bar{\alpha}) = -K(\alpha), \quad N(\bar{\alpha}) = M(\alpha) \qquad \text{(E.18)}$$

Lemma 4

$$\begin{pmatrix} H(\alpha) \\ N(\alpha) \end{pmatrix} = R(2\alpha) \begin{pmatrix} M(\alpha) \\ K(\alpha) \end{pmatrix} \qquad \text{(E.19)}$$

$$\begin{pmatrix} K(\alpha) \\ M(\alpha) \end{pmatrix} = R(2\alpha) \begin{pmatrix} N(\alpha) \\ H(\alpha) \end{pmatrix} \qquad \text{(E.20)}$$

Lemma 5

$$1) \quad H^2(\alpha) + N^2(\alpha) = K^2(\alpha) + M^2(\alpha) = 2(1 + Q^2) \qquad \text{(E.21)}$$

$$2) \quad K(\alpha)N(\alpha) - H(\alpha)M(\alpha) = 4Q \qquad \text{(E.22)}$$

$$3) \quad K(\alpha)N(\alpha) + H(\alpha)M(\alpha) = 2(1 + Q^2)\cos(2\alpha) \qquad \text{(E.23)}$$

$$4) \quad K(\alpha + \delta)H(\alpha + \beta) - K(\alpha + \beta)H(\alpha + \delta) = 4Q\sin(\delta - \beta) \qquad \text{(E.24)}$$

$$5) \quad N(\alpha + \beta)M(\alpha + \delta) - N(\alpha + \delta)M(\alpha + \beta) = 4Q\sin(\delta - \beta) \qquad \text{(E.25)}$$

Lemma 6

$$1) \quad K(\alpha) + H(\alpha) = 2\omega_-(\alpha) \qquad \text{(E.26)}$$

$$2) \quad K(\alpha) - H(\alpha) = 2Q\omega_+(\alpha) \qquad \text{(E.27)}$$

$$3) \quad N(\alpha) + M(\alpha) = 2\omega_+(\alpha) \qquad \text{(E.28)}$$

$$4) \quad N(\alpha) - M(\alpha) = 2Q\omega_-(\alpha) \qquad \text{(E.29)}$$

$$5) \quad M(\alpha) + H(\alpha) = 2\cos\alpha[1 - Q] \qquad \text{(E.30)}$$

$$6) \quad M(\alpha) - H(\alpha) = 2\sin\alpha[1 + Q] \qquad \text{(E.31)}$$

$$7) \quad N(\alpha) + K(\alpha) = 2\cos\alpha[1 + Q] \qquad \text{(E.32)}$$

$$8) \quad N(\alpha) - K(\alpha) = 2\sin\alpha[1 - Q] \qquad \text{(E.33)}$$

Now with

$$\alpha + \beta = 2\sigma, \quad \alpha - \beta = 2\delta$$
$$\alpha = \sigma + \delta, \quad \beta = \sigma - \delta$$

we introduce the angle ψ through

$$\omega_+(\sigma) = \sqrt{2}\cos\psi, \quad \omega_-(\sigma) = \sqrt{2}\sin\psi \qquad \text{(E.34)}$$

Then ψ is an Euler angle and for

$$0 < \alpha < \frac{\pi}{2}, \quad 0 < \beta < \frac{\pi}{2}, \quad \text{we find} \quad -\frac{\pi}{4} < \psi < \frac{\pi}{4}$$

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From $\sigma = \alpha - \delta$ and (C.13) we find

$$\begin{pmatrix} \omega_+(\sigma) \\ \omega_-(\sigma) \end{pmatrix} = R(\delta) \begin{pmatrix} \omega_+(\alpha) \\ \omega_-(\alpha) \end{pmatrix} = R(\psi) \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \qquad \text{(E.35)}$$

$$\begin{pmatrix} \omega_+(\alpha + \psi) \\ \omega_-(\alpha + \psi) \end{pmatrix} = R(-\psi) \begin{pmatrix} \omega_+(\alpha) \\ \omega_-(\alpha) \end{pmatrix} = R(-\delta) \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \qquad \text{(E.36)}$$

Explicitly

$$\sqrt{2}\cos\delta = \omega_+(\alpha + \psi), \quad \sqrt{2}\sin\delta = -\omega_-(\alpha + \psi) \qquad \text{(E.37)}$$

Lemma 7

$$
\begin{array}{llr}
1) & M(\alpha) - N(\beta) = -2\sin\psi K(\alpha + \psi) & \text{(E.38)} \\
2) & M(\alpha) + N(\beta) = 2\cos\psi M(\alpha + \psi) & \text{(E.39)} \\
3) & N(\alpha) - M(\beta) = -2\sin\psi H(\alpha + \psi) & \text{(E.40)} \\
4) & N(\alpha) + M(\beta) = 2\cos\psi N(\alpha + \psi) & \text{(E.41)} \\
5) & K(\alpha) + H(\beta) = 2\sin\psi M(\alpha + \psi) & \text{(E.42)} \\
6) & K(\alpha) - H(\beta) = 2\cos\psi K(\alpha + \psi) & \text{(E.43)} \\
7) & H(\alpha) + K(\beta) = 2\sin\psi N(\alpha + \psi) & \text{(E.44)} \\
8) & H(\alpha) - K(\beta) = 2\cos\psi H(\alpha + \psi) & \text{(E.45)}
\end{array}
$$

The easy proofs are left to the reader.

Appendix F

A Convenient Bijective Parameterization

For a given Heron angle α, $0 < \alpha < \frac{\pi}{2}$, we look at the quadratic equation

$$(M^2 - 1)\sin(2\alpha) + 2M\cos(2\alpha) = 4D, M > 0 \tag{F.1}$$

or

$$M^2 \sin(2\alpha) + 2M\cos(2\alpha) - [\sin(2\alpha) + 4D] = 0 \tag{F.2}$$

Then the solutions are given by

$$M_+ = \frac{1}{\sin(2\alpha)}[-\cos(2\alpha) + \Delta] \tag{F.3}$$

$$M_- = \frac{1}{\sin(2\alpha)}[-\cos(2\alpha) - \Delta] \tag{F.4}$$

where

$$\begin{aligned}
\Delta^2 &= \cos^2(2\alpha) + \sin(2\alpha)[\sin(2\alpha) + 4D] \\
\Delta^2 &= 1 + 4\sin(2\alpha)D, \Delta \geq 0
\end{aligned} \tag{F.5}$$

or

$$\Delta^2 + [\sin(2\alpha) - D]^2 = 1 + [\sin(2\alpha) + D]^2 \tag{F.6}$$

According to [8] this relation has the following bijective parameter representation with the parameters a_1, a_2, λ

$$\Delta = a_1 + \lambda a_2 \quad 1 = a_1 - \lambda a_2 \tag{F.7}$$

$$\sin(2\alpha) - D = a_2 - \lambda a_1 \quad \sin(2\alpha) + D = a_2 + \lambda a_1 \tag{F.8}$$

Conversely

$$2a_1 = 1 + \Delta \tag{F.9}$$

$$a_2 = \sin(2\alpha) \tag{F.10}$$

$$\lambda a_1 = D \tag{F.11}$$

From (F.5) and (F.9) we find that

$$a_1 \neq 0, \quad 1 + \lambda a_2 = a_1 \neq 0 \tag{F.12}$$

Then

$$a_1 = 1 + \lambda a_2 = 1 + \lambda \sin(2\alpha) \tag{F.13}$$

$$\Delta = 1 + 2\lambda a_2 = 1 + 2\lambda \sin(2\alpha) \tag{F.14}$$

and from (F.3), (F.4) the equation (F.1) is bijectively parameterized by the parameter λ as

$$M_+ = 2\lambda + \tan\alpha \tag{F.15}$$

$$M_- = -2\lambda - \cot\alpha \tag{F.16}$$

$$D = \lambda[1 + \lambda\sin(2\alpha)] \tag{F.17}$$

Conversely

$$2\lambda = M_+ - \tan\alpha \tag{F.18}$$

Remark 6.

1) From (F.13) we find

$$1 + \lambda\sin(2\alpha) \neq 0 \tag{F.19}$$

2) For the limit

$$\lambda \to 0 \tag{F.20}$$

we find from (F.15-17) that

$$M_+ \to \tan\alpha \tag{F.21}$$

$$M_- \to -\cot\alpha \tag{F.22}$$

$$D \to 0 \tag{F.23}$$

Since the solution of (F.1) has to be positive, the only acceptable solution for the limit (F.20) is

$$M = M_+ \tag{F.24}$$

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Department of Physics, University of Colorado Boulder, Boulder, CO 80309
Walter.Wyss@Colorado.EDU

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