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On the rational cuboids with a given face[☆]

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Abstract

A rational cuboid is a rectangular parallelepiped whose edges and face diagonals all have rational lengths. In this paper, we consider the problem: are there rational cuboids with a given face? In a sense, we reduce the problem to a finite calculation.

MSC: 11D09

Keywords: Rational cuboids; Pythagorean triplets; Diophantine equations

1. Introduction

A rational cuboid is a rectangular parallelepiped whose edges and face diagonals all have rational lengths. This is equivalent to the problem of solving the system of Diophantine equations $x^2 + y^2 = l^2$, $x^2 + z^2 = m^2$ and $y^2 + z^2 = n^2$. The problem has attracted much historical interest (see [3]). In 1719, Paul Halcke (see [3]) found that $44^2 + 240^2$, $44^2 + 117^2$ and $240^2 + 117^2$ are all squares. In 1772, Euler (see [3]) found

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that for

$$x = 8f(f^4 - 1), \quad y = (1 - f^2)(f^4 - 14f^2 + 1) \quad \text{and} \quad z = 2f(3f^4 - 10f^2 + 3),$$

$x^2 + y^2$, $x^2 + z^2$ and $y^2 + z^2$ are all squares. For the other related research, one may refer to [1] and [2,4–6].

In this paper, we pose the following problem:

Problem. For given positive integers a, b with $a^2 + b^2$ being a square and $(a, b) = 1$, are there positive integers c, d such that both $c^2 + a^2d^2$ and $c^2 + b^2d^2$ are squares?

The problem is not trivial even in the simple cases: $(a, b) = (4, 3), (12, 5), (24, 7)$, etc. It is well known that if $a^2 + b^2$ is a square with $2|a$ and $(a, b) = 1$, then $4|a$. In this paper we develop a general theory to deal with the problem.

Theorem 1. *For given positive integers a, b with $4|a$ and $(a, b) = 1$, if there are positive integers c, d such that both $c^2 + a^2d^2$ and $c^2 + b^2d^2$ are squares, then there exist positive integers a_1, a_2, b_1, b_2, M_1 and M_2 with $a = a_1a_2$, $b = b_1b_2$, $M_1|a^2 - b^2$, $M_2|a^2 - b^2$ and either $2 \nmid a_1$ or $2|a_1$ and $8|a_2$ such that for $i = 1, 2$ and any odd prime p,*

(i) *if $p|a$, then* $\left(\frac{b_iM_i}{p}\right) = 1$;
(ii) *if $p|b$, then* $\left(\frac{a_iM_i}{p}\right) = 1$;
(iii) *if $p|M_i$, then* $\left(\frac{-a_ib_i}{p}\right) = 1$;
(iv) *if* $p \,\Big|\, \frac{M}{M_i}$, *then* $\left(\frac{a_ib_i}{p}\right) = 1$;
(v) $b_1 \equiv M_1 \pmod 8$ *and* $b_2 + a_2 \equiv M_2 \pmod 8$;
(vi) *if a_1, b_1, M_1 are all squares, then $d \leqslant M_1/\sqrt{a_2b_2}$; if a_2, b_2, M_2 are all squares, then $d \leqslant \max\{1, M_2/\sqrt{a_1b_1}\}$, where d is the least positive integer with the property.*

Theorem 1 gives a sufficient condition for the problem being negative. For given a, b, there are only finitely many possibilities for a_1, a_2, b_1, b_2, M_1 and M_2. If none of cases satisfies Theorem 1(i)–(vi), then the problem is negative for a, b. Theorem 1 is powerful for giving the restrictions for both $i = 1, 2$. We conjecture that if a_1, a_2, b_1, b_2, M_1 and M_2 satisfy Theorem 1(i)–(v), neither a_1, b_1, M_1 nor a_2, b_2, M_2 are all squares, then the problem is affirmative. The conjecture is true for $a < 100$ and $b < 100$. That is,

Theorem 2. *The problem is negative for* $(a, b) = (4, 3)$, $(8, 15)$, $(12, 5)$, $(12, 35)$, $(16, 63)$, $(28, 45)$, $(36, 77)$, $(40, 9)$, $(56, 33)$, $(72, 65)$;

The problem is affirmative for $(a, b) = (20, 21)$, $(20, 99)$, $(24, 7)$, $(48, 55)$, $(60, 11)$, $(60, 91)$, $(80, 39)$, $(84, 13)$.

2. Proof of Theorem 1

Suppose that there exist positive integers c, d such that

$$c^2 + a^2d^2 = m^2, \qquad c^2 + b^2d^2 = n^2.$$

Further, we assume that d is the least positive integer with the property. Then $(c, d) = 1$. Let

$$(c, a) = u_1, \quad (c, b) = v_1, \quad M = a^2 - b^2,$$

$$a = u_1u_2, \quad b = v_1v_2, \quad c = u_1c_1 = v_1c_2.$$

Then

$$c_1^2 + u_2^2d^2 = \left(\frac{m}{u_1}\right)^2, \quad c_2^2 + v_2^2d^2 = \left(\frac{n}{v_1}\right)^2. \tag{1}$$

Lemma. *Let u, v, s, t, e, f, r and w be positive integers with $(s, t) = (e, f) = 1$, $a = uw$, $b = vr$, $e \neq f$ and*

$$ust = vef, \qquad r(s^2 - t^2) = w(e^2 - f^2).$$

Then there exists a positive integer X with $X|M$ such that for any odd prime p,

(i) *if $p|a$, then* $\left(\frac{rX}{p}\right) = 1$;
(ii) *if $p|b$, then* $\left(\frac{wX}{p}\right) = 1$;
(iii) *if $p|X$, then* $\left(\frac{-rw}{p}\right) = 1$;
(iv) *if $p \mid \frac{M}{X}$, then* $\left(\frac{rw}{p}\right) = 1$;
(v) *if $2 \nmid u$, then $r + w \equiv X \pmod 8$;*
if $2|u$, then $r \equiv X \pmod 8$;
(vi) *if r, w and X are all squares, then*

$$\frac{ef}{u} = \frac{st}{v} \geqslant d^2, \qquad \frac{s^2 - t^2}{w} = \frac{e^2 - f^2}{r} \geqslant \frac{2\sqrt{uv}d^2}{X}.$$

Remark. The parameters e, f (and also s, t) are used for Pythagorean triangles

$$(2ef)^2 + (e^2 - f^2)^2 = (e^2 + f^2)^2$$

in (1) and also in the three cases of the proof of Theorem 1.

Proof. Let

$$s_2 = (s, e), \quad s_3 = (s, f), \quad t_2 = (t, e), \quad t_3 = (t, f),$$

$$s_1 = \frac{s}{s_2 s_3}, \quad t_1 = \frac{t}{t_2 t_3}, \quad e_1 = \frac{e}{s_2 t_2}, \quad f_1 = \frac{f}{s_3 t_3}.$$

Since $(s, t) = (e, f) = 1$, we have $s_2 s_3 | s$, $t_2 t_3 | t$, $s_2 t_2 | e$ and $s_3 t_3 | f$. Hence s_1, t_1, e_1, f_1 are positive integers and

$$u s_1 t_1 = v e_1 f_1. \tag{2}$$

By

$$\left(\frac{s}{s_2}, \frac{e}{s_2}\right) = 1, \quad s_1 \Big| \frac{s}{s_2}, \quad e_1 \Big| \frac{e}{s_2},$$

we have $(s_1, e_1) = 1$. Similarly, we have $(s_1, f_1) = 1$, $(t_1, e_1) = 1$ and $(t_1, f_1) = 1$. By (2) we have

$$e_1 f_1 | u, \quad s_1 t_1 | v.$$

By $(u, v) = 1$ and (2), we have

$$u | e_1 f_1, \qquad v | s_1 t_1.$$

Hence $u = e_1 f_1$, $v = s_1 t_1$. By $r(s^2 - t^2) = w(e^2 - f^2)$, we have

$$r(s_1^2 s_2^2 s_3^2 - t_1^2 t_2^2 t_3^2) = w(e_1^2 s_2^2 t_2^2 - f_1^2 s_3^2 t_3^2).$$

Thus

$$(r s_1^2 s_2^2 + w f_1^2 t_3^2) s_3^2 = (r t_1^2 t_3^2 + w e_1^2 s_2^2) t_2^2.$$

By $(s_3, t_2) = 1$, there exists a positive integer X such that

$$r s_1^2 s_2^2 + w f_1^2 t_3^2 = X t_2^2, \tag{3}$$

$$r t_1^2 t_3^2 + w e_1^2 s_2^2 = X s_3^2. \tag{4}$$

By (3) $\times\, rt_1^2 -$ (4) $\times\, wf_1^2$ and (3) $\times\, we_1^2 -$ (4) $\times\, rs_1^2$, we have

$$-Ms_2^2 = X(rt_1^2t_2^2 - wf_1^2s_3^2), \tag{5}$$

$$Mt_3^2 = X(we_1^2t_2^2 - rs_1^2s_3^2). \tag{6}$$

Since $(s_2, t_3) = 1$, we have $X|M$.

By $(s, t) = (e, f) = (u, v) = 1$, we have

$$(s_1s_2, f_1t_3) = (s_1s_2, t_2) = (f_1t_3, t_2) = 1, \tag{7}$$

$$(t_1t_3, e_1s_2) = (t_1t_3, s_3) = (e_1s_2, s_3) = 1, \tag{8}$$

$$(t_1t_2, f_1s_3) = (t_1t_2, s_2) = (f_1s_3, s_2) = 1, \tag{9}$$

$$(e_1t_2, s_1s_3) = (e_1t_2, t_3) = (s_1s_3, t_3) = 1. \tag{10}$$

By $X\frac{M}{X} = a^2 - b^2 = u^2w^2 - v^2r^2$ and $(uw, vr) = 1$, we have

$$(r, w) = (r, X) = (w, X) = 1. \tag{11}$$

By (3)–(11) we have

$$(rs_1^2s_2^2, wf_1^2t_3^2) = (rs_1^2s_2^2, Xt_2^2) = (wf_1^2t_3^2, Xt_2^2) = 1,$$

$$(rt_1^2t_3^2, we_1^2s_2^2) = (rt_1^2t_3^2, Xs_3^2) = (we_1^2s_2^2, Xs_3^2) = 1,$$

$$(rt_1^2t_2^2, wf_1^2s_3^2) = \left(rt_1^2t_2^2, \frac{M}{X}\, s_2^2\right) = \left(wf_1^2s_3^2, \frac{M}{X}\, s_2^2\right) = 1,$$

$$(rs_1^2s_3^2, we_1^2t_2^2) = \left(rs_1^2s_3^2, \frac{M}{X}t_3^2\right) = \left(we_1^2t_2^2, \frac{M}{X}t_3^2\right) = 1.$$

If $p|a$, then, either $p|wf_1^2$ or $p|we_1^2$. By (3) and (4) we have

$$\left(\frac{rX}{p}\right) = 1.$$

That is, (i). Similarly, by (3)–(6), we have (ii)–(iv). If $2\nmid u$, then, by $2|a$ and $a = uw$, we have $2|w$. Thus $2\nmid s_1s_2$, $2\nmid t_2$ and $2\nmid f_1t_3$. By (3) we have

$$r + w \equiv X \pmod 8.$$

If $2|u$ and $4\nmid u$, then by $4|a$ and $a = uw$ we have $2|w$. Thus $2\nmid s_1s_2$, $2\nmid t_2$ and $2\nmid t_1t_3$, $2\nmid s_3$ and either $8|wf_1^2$ or $8|we_1^2$. By (3) and (4), we have $r \equiv X \pmod 8$. If $4|u$, then, by $(e_1, f_1) = 1$ and $u = e_1f_1$, we have either $4|e_1$ or $4|f_1$. Similarly, by (3) and (4) we have $r \equiv X \pmod 8$. Thus we have (v). Now we prove (vi).

By (3) and (4) we have

$$r^2s_1^2t_1^2s_2^2 + rwt_1^2f_1^2t_3^2 = rXt_1^2t_2^2,$$

$$rwf_1^2t_1^2t_3^2 + w^2f_1^2e_1^2s_2^2 = wXf_1^2s_3^2.$$

That is,

$$a^2s_2^2 + rw(f_1t_1t_3)^2 = wX(f_1s_3)^2,$$

$$b^2s_2^2 + rw(f_1t_1t_3)^2 = rX(t_1t_2)^2.$$

Similarly, we have

$$a^2t_3^2 + rw(e_1s_1s_2)^2 = wX(e_1t_2)^2,$$

$$b^2t_3^2 + rw(e_1s_1s_2)^2 = rX(s_1s_3)^2.$$

Since rw, wX and rX are all squares, by the assumption for d, we have $s_2 \geqslant d$ and $t_3 \geqslant d$. Thus

$$\frac{ef}{u} = s_2s_3t_2t_3 \geqslant d^2,$$

$$\begin{aligned}
\frac{e^2 - f^2}{r} &= \frac{e_1^2s_2^2t_2^2X - f_1^2s_3^2t_3^2X}{rX} \\
&= \frac{1}{rX}(e_1^2s_2^2(rs_1^2s_2^2 + wf_1^2t_3^2) - f_1^2t_3^2(rt_1^2t_3^2 + we_1^2s_2^2)) \\
&= \frac{1}{X}(e_1^2s_1^2s_2^4 - f_1^2t_1^2t_3^4) \\
&\geqslant \frac{1}{X}(e_1s_1s_2^2 + f_1t_1t_3^2) \\
&\geqslant \frac{2}{X}\sqrt{uv}s_2t_3 \geqslant \frac{2}{X}\sqrt{uv}d^2.
\end{aligned}$$

This completes the proof of the lemma. $\square$

Proof of Theorem 1. *Case* 1: $2\nmid u_1$. Then $2\nmid c$. Hence $2\nmid c_1$ and $2\nmid c_2$. By (1) there exist integers s, t, e, f such that

$$u_2 d = 2ef, \quad c_1 = e^2 - f^2, \quad (e, f) = 1, \quad 2|ef, \quad e > f \geqslant 1,$$

$$v_2 d = 2st, \quad c_2 = s^2 - t^2, \quad (s, t) = 1, \quad 2|st, \quad s > t \geqslant 1.$$

Thus

$$u_2 st = v_2 ef, \quad v_1(s^2 - t^2) = u_1(e^2 - f^2), \quad d = \frac{2ef}{u_2}.$$

By the lemma there exists the corresponding X_1. Let $s' = e + f$, $t' = e - f$, $e' = s + t$, $f' = s - t$. Then $(s', t') = 1$, $(e', f') = 1$ and

$$u_1 s' t' = v_1 e' f', \quad v_2(s'^2 - t'^2) = u_2(e'^2 - f'^2), \quad 2d = \frac{4ef}{u_2} = \frac{s'^2 - t'^2}{u_2}.$$

By the lemma there exists the corresponding X_2. Let $a_i = u_i$, $b_i = v_i$ and $M_i = X_i$ $(i = 1, 2)$. Now Theorem 1 follows from the lemma.

Case 2: $2\nmid u_2$. Then $2|u_1$ and $2|c$. By $(c, d) = 1$ we have $2\nmid d$. By (1) there exist integers s, t, e, f such that

$$c_1 = 2st, \quad u_2 d = s^2 - t^2, \quad (s, t) = 1, \quad 2|st, \quad s > t \geqslant 1,$$

$$c_2 = 2ef, \quad v_2 d = e^2 - f^2, \quad (e, f) = 1, \quad 2|ef, \quad e > f \geqslant 1.$$

Thus

$$u_1 st = v_1 ef, \quad v_2(s^2 - t^2) = u_2(e^2 - f^2), \quad d = \frac{s^2 - t^2}{u_2}.$$

By the lemma there exists the corresponding X_2. Let $s' = e + f$, $t' = e - f$, $e' = s + t$, $f' = s - t$. Then $(s', t') = 1$, $(e', f') = 1$ and

$$u_2 s' t' = v_2 e' f', \quad v_1(s'^2 - t'^2) = u_1(e'^2 - f'^2), \quad d = \frac{s^2 - t^2}{u_2} = \frac{e' f'}{u_2}.$$

By the lemma there exists the corresponding X_1. Let $a_1 = u_2$, $a_2 = u_1$, $b_1 = v_2$, $b_2 = v_1$, $M_1 = X_2$ and $M_2 = X_1$. Now Theorem 1 follows from the lemma.

Case 3: $2|u_1$ and $2|u_2$. By $2|u_1$ we have $2|c$. By $(c, d) = 1$ we have $2\nmid d$. Since $2\nmid b$, we have $2|c_2$. By (1) we have $4|c_2$ and $4|u_2$ (note that if R, S, T are integers with $2|R$, $(R,S)=1$ and $R^2+S^2=T^2$, then $4|R$). Thus $4|c$ and then $4|u_1$. Hence, the case $2|u_1$ and $2|u_2$ may happen only if $16|a$. In this case, we have $4|u_1$ and $4|u_2$. By (1) there exist integers s,t,e,f such that

$$c_2=2st,\quad v_2d=s^2-t^2,\quad (s,t)=1,\quad 2|st,\quad s>t\geqslant 1,$$

$$c_1=e^2-f^2,\quad u_2d=2ef,\quad (e,f)=1,\quad 2|ef,\quad e>f\geqslant 1.$$

Thus

$$u_2(s^2-t^2)=2v_2ef,\quad 2v_1st=u_1(e^2-f^2).$$

Let $s'=s+t$, and $t'=s-t$. Then $(s',t')=1$ and

$$\frac{u_2}{2}s't'=v_2ef,\quad v_1(s'^2-t'^2)=2u_1(e^2-f^2),\quad d=\frac{ef}{u_2/2}.$$

By the lemma there exists the corresponding X_1. Let $s''=e+f$, $t''=e-f$, $e''=s$ and $f''=t$. Then $(s'',t'')=1$, $(e'',f'')=1$ and

$$\frac{u_1}{2}s''t''=v_1e''f'',\quad v_2(s''^2-t''^2)=2u_2(e''^2-f''^2),\quad d=\frac{s^2-t^2}{v_2}=\frac{e''^2-f''^2}{v_2}.$$

By the lemma there exists the corresponding X_2. Let $a_1=u_2/2$, $a_2=2u_1$, $b_1=v_2$, $b_2=v_1$, $M_1=X_2$ and $M_2=X_1$. Note that

$$\left(\frac{2u_2X_2}{p}\right)=\left(\frac{(u_2/2)X_2}{p}\right)=\left(\frac{a_1M_1}{p}\right)$$

and

$$\left(\frac{2u_2v_2}{p}\right)=\left(\frac{(u_2/2)v_2}{p}\right)=\left(\frac{a_1b_1}{p}\right).$$

Now Theorem 1 follows from the lemma.

This completes the proof of Theorem 1. $\square$

3. Proof of Theorem 2

1. $a=4$, $b=3$, $M=a^2-b^2=7$: By $2\nmid a_1$ we have $a_1=1$. By $b_1\equiv M_1 \pmod 8$ we have $b_1=1$ and $M_1=1$. Thus $a_1=b_1=M_1=1$, a contradiction with Theorem 1(vi), $d\leqslant\frac{M_1}{\sqrt{a_2b_2}}=\frac{1}{2\sqrt{3}}<1$.

2. $a = 8$, $b = 15$, $M = a^2 - b^2 = -7 \times 23$: By $2 \nmid a_1$ we have $a_1 = 1$. By $\left(\frac{a_1 M_1}{3}\right) = 1$ and $\left(\frac{a_1 M_1}{5}\right) = 1$, we have $M_1 = 1$. By $b_1 \equiv M_1 \pmod 8$ we have $b_1 = 1$. Thus $a_1 = b_1 = M_1 = 1$, a contradiction with Theorem 1(vi) $d \leqslant \frac{M_1}{\sqrt{a_2 b_2}} < 1$.
3. $a = 12$, $b = 5$, $M = a^2 - b^2 = 7 \times 17$: By $2 \nmid a_1$ we have $a_1 = 1, 3$. By $b_1 \equiv M_1 \equiv \pm 1 \pmod 8$, we have $b_1 = 1$. By $M_1 \equiv b_1 \pmod 8$ we have $M_1 = 1, 17$. By $\left(\frac{b_1 M_1}{3}\right) = 1$ we have $M_1 = 1$. By $\left(\frac{a_1 M_1}{5}\right) = 1$ we have $a_1 = 1$. Thus $a_1 = b_1 = M_1 = 1$, a contradiction with Theorem 1(vi) $d \leqslant \frac{M_1}{\sqrt{a_2 b_2}} < 1$.
4. $a = 12$, $b = 35$, $M = a^2 - b^2 = -23 \times 47$: By $2 \nmid a_1$ we have $a_1 = 1, 3$. By $b_1 \equiv M_1 \equiv \pm 1 \pmod 8$, we have $b_1 = 1, 7$. By $\left(\frac{b_1 M_1}{3}\right) = 1$ we have $M_1 = 1, 23 \times 47$. By $b_1 \equiv M_1 \equiv 1 \pmod 8$ we have $b_1 = 1$. By $\left(\frac{a_1 M_1}{5}\right) = 1$ we have $a_1 = 1$. By $\left(\frac{a_1 M_1}{7}\right) = 1$ we have $M_1 = 1$. Thus $a_1 = b_1 = M_1 = 1$, a contradiction with Theorem 1(vi) $d \leqslant \frac{M_1}{\sqrt{a_2 b_2}} < 1$.
5. $a = 16$, $b = 63$, $M = a^2 - b^2 = -79 \times 47$: By $a = 16$ we have $a_1 = 1, 2$. By $b_1 \equiv M_1 \equiv \pm 1 \pmod 8$, we have $b_1 = 1, 7, 9, 63$. By $\left(\frac{a_1 M_1}{7}\right) = 1$ we have $M_1 = 1, 79$. If $b_1 = 1, 9$, then, by $M_1 \equiv b_1 \equiv 1 \pmod 8$ we have $M_1 = 1$. By $\left(\frac{a_1 M_1}{3}\right) = 1$ we have $a_1 = 1$. Thus $a_1 = 1, b_1 = 1, 9, M_1 = 1$, a contradiction with Theorem 1(vi). Hence $b_1 = 7, 63$. Then $b_2 = 9, 1$ and $a_2 = 16, 8$. By $\left(\frac{a_2 M_2}{7}\right) = 1$ we have $M_2 = 1, 79$. By $M_2 \equiv b_2 + a_2 \equiv 1 \pmod 8$ we have $M_2 = 1$. By $\left(\frac{a_2 M_2}{3}\right) = 1$ we have $a_2 = 16$. Thus $a_2 = 16, b_2 = 1, 9, M_2 = 1$. By Theorem 1(vi) $d \leqslant \max\{1, \frac{M_2}{\sqrt{a_1 b_1}}\} = 1$. Hence $d = 1$. By directly calculation we know that there are no positive integers c, m, n with $c^2 + 16^2 = m^2$ and $c^2 + 63^2 = n^2$.
6. $a = 28$, $b = 45$, $M = -73 \times 17$: By $2 \nmid a_1$ we have $a_1 = 1, 7$. By $b_1 \equiv M_1 \equiv 1 \pmod 8$, we have $b_1 = 1, 9$. By $\left(\frac{a_1 M_1}{3}\right) = 1$ we have $M_1 = 1, 73$. By $\left(\frac{b_1 M_1}{7}\right) = 1$ we have $M_1 = 1$. By $\left(\frac{a_1 M_1}{5}\right) = 1$ we have $a_1 = 1$. Thus $a_1 = 1$, $b_1 = 1, 9$, $M_1 = 1$, a contradiction with Theorem 1(vi).
7. $a = 36$, $b = 77$, $M = -113 \times 31$: By $2 \nmid a_1$ we have $a_1 = 1, 3, 9$. By $b_1 \equiv M_1 \equiv \pm 1 \pmod 8$, we have $b_1 = 1, 7$. By $\left(\frac{b_1 M_1}{3}\right) = 1$ we have $M_1 = 1, 31$. By Theorem 1(iv) $\left(\frac{a_1 b_1}{113}\right) = 1$ we have $a_1 = 1, 9$. By $\left(\frac{a_1 M_1}{7}\right) = 1$ we have $M_1 = 1$. By $b_1 \equiv M_1 \equiv 1 \pmod 8$, we have $b_1 = 1$. Thus $a_1 = 1, 9$, $b_1 = 1$, $M_1 = 1$, a contradiction with Theorem 1(vi).
8. $a = 40$, $b = 9$, $M = 7^2 \times 31$: By $2 \nmid a_1$ we have $a_1 = 1, 5$. By $b_1 \equiv M_1 \equiv \pm 1 \pmod 8$, we have $b_1 = 1, 9$. By $M_1 \equiv b_1 \equiv 1 \pmod 8$, we have $M_1 = 1, 7^2, 7 \times 31$. By $\left(\frac{a_1 M_1}{3}\right) = 1$ we have $a_1 = 1$. By $\left(\frac{-a_1 b_1}{7}\right) = -1$ we have $M_1 = 1$. Thus $a_1 = 1$, $b_1 = 1, 9$, $M_1 = 1$, a contradiction with Theorem 1(vi).
9. $a = 56$, $b = 33$, $M = 89 \times 23$: By $2 \nmid a_1$ we have $a_1 = 1, 7$. By $b_1 \equiv M_1 \equiv \pm 1 \pmod 8$, we have $b_1 = 1, 33$. By $M_1 \equiv b_1 \equiv 1 \pmod 8$, we have $M_1 = 1, 89$.

By $\left(\frac{a_1M_1}{3}\right)=1$ we have $M_1=1$. By $\left(\frac{b_1M_1}{7}\right)=1$ we have $b_1=1$. By $\left(\frac{a_1M_1}{11}\right)=1$ we have $a_1=1$. Thus $a_1=1,\ b_1=1,\ M_1=1$, a contradiction with Theorem 1(vi).

10. $a=72,\ b=65,\ M=137\times 7$: By $2\nmid a_1$ we have $a_1=1,3,9$. By $b_1\equiv M_1\equiv \pm 1\pmod 8$, we have $b_1=1,65$. By $M_1\equiv b_1\equiv 1\pmod 8$, we have $M_1=1,137$. By $\left(\frac{a_1b_1}{7}\right)=1$ we have $a_1=1,9$. By $\left(\frac{a_1M_1}{5}\right)=1$ we have $M_1=1$. By $\left(\frac{b_1M_1}{3}\right)=1$ we have $b_1=1$. Thus $a_1=1,\ b_1=1,9,\ M_1=1$, a contradiction with Theorem 1(vi).

11. $a=20,\ b=21$:

$$275^2+21^2\times 12^2=373^2,\qquad 275^2+20^2\times 12^2=365^2.$$

12. $a=20,\ b=99$:

$$231^2+20^2\times 8^2=281^2,\qquad 231^2+99^2\times 8^2=825^2.$$

13. $a=24,\ b=7$:

$$693^2+24^2\times 20^2=843^2,\qquad 693^2+7^2\times 20^2=707^2.$$

14. $a=48,\ b=55$:

$$1100^2+48^2\times 21^2=1492^2,\qquad 1100^2+55^2\times 21^2=1595^2.$$

15. $a=60,\ b=11$:

$$85^2+60^2\times 12^2=725^2,\qquad 85^2+11^2\times 12^2=157^2.$$

16. $a=60,\ b=91$:

$$5643^2+60^2\times 236^2=15243^2,\qquad 5643^2+91^2\times 236^2=22205^2.$$

17. $a=80,\ b=39$:

$$44^2+80^2\times 3^2=244^2,\qquad 44^2+39^2\times 3^2=125^2.$$

18. $a=84,\ b=13$:

$$2144115^2+84^2\times 38324^2=3867891^2,$$

$$2144115^2+13^2\times 38324^2=2201237^2.$$

This completes the proof of Theorem 2. $\square$

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