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Perfect Cuboids and Perfect Square Triangles

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Among the many well-known unsolved diophantine problems is the following:

THE PERFECT CUBOID PROBLEM (PCP): *Is there a rectangular box with all edges, face diagonals, and main diagonals integers?*

An extensive list of references on this problem appears in [1]. In this note, we show that the existence of a solution for the PCP is equivalent to the existence of a solution for an apparently different problem:

THE PERFECT SQUARE TRIANGLE PROBLEM (PSTP): *Is there a triangle whose sides are perfect squares and whose angle bisectors are integers?*

Let us first observe that the word "integers" can be replaced by the word "rationals" in the statement of the PSTP. In order to show that the existence of a solution to the PCP is indeed equivalent to the existence of a solution to the PSTP, assume first that the PCP has a solution. Let x, y, and z be the edges of a perfect cuboid and set

$$a = y^2 + z^2, \quad b = x^2 + z^2, \quad c = x^2 + y^2. \tag{1}$$

Clearly, a, b, and c are the sides of a triangle and are perfect squares. Let $p = \frac{a+b+c}{2}$ be the semiperimeter of this triangle. Since

$$p = x^2 + y^2 + z^2, \quad p - a = x^2, \quad p - b = y^2, \quad p - c = z^2, \tag{2}$$

we conclude that all four numbers p, $p - a$, $p - b$, and $p - c$ are perfect squares. Let l_a, l_b, and l_c be the lengths of the angle bisectors drawn from the angles opposite to the sides a, b, and c, respectively. It is well known that the lengths of these angle bisectors are given in terms of a, b, and c by

$$l_a = 2 \cdot \frac{\sqrt{bcp(p-a)}}{b+c}, \quad l_b = 2 \cdot \frac{\sqrt{acp(p-b)}}{a+c}, \quad l_c = 2 \cdot \frac{\sqrt{abp(p-c)}}{a+b}. \tag{3}$$

Since all the numbers listed in (1) and (2) are perfect squares, it follows, by formula (3), that the triangle with sides a, b, and c is a solution of the PSTP (once "integers" has been replaced by "rationals" in the statement of the problem).

Conversely, assume now that the PSTP has a solution. Let a, b, and c be the sides of a triangle that solves this problem. We may assume that $\gcd(a, b, c) = 1$. Indeed, otherwise, let $d = \gcd(a, b, c)$. Since a, b, and c are all perfect squares, so is d. Then the triangle with sides a/d, b/d, and c/d still solves the PSTP, and $\gcd(a/d, b/d, c/d) = 1$.

By formula (3) and the fact that a, b, and c are perfect squares, it follows that all three integers

$$4p(p-a)=(b+c)^2-a^2,\quad 4p(p-b)=(a+c)^2-b^2,$$
$$4p(p-c)=(a+b)^2-c^2 \tag{4}$$

are perfect squares. Since $\gcd(a,b,c)=1$, it follows that not all of a, b, and c can be even. Reducing modulo 4 the integers listed in (4), one concludes that exactly one of the three numbers a, b, and c is even, and the other two are odd. It now follows that p is an integer, and formula (4) implies that all three integers

$$p(p-a),\quad p(p-b),\quad p(p-c) \tag{5}$$

are perfect squares. We now show that $\gcd(p-a,p-b,p-c)=1$. Indeed, let $e=\gcd(p-a,p-b,p-c)$. Clearly, $e\|(p-b)+(p-c)=a$. By a similar argument, one concludes that $e\mid b$ and $e\mid c$. Since $\gcd(a,b,c)=1$, it follows that $e=1$. Since all three numbers listed in (5) are perfect squares, so is their greatest common divisor. Hence,

$$\gcd(p(p-a),p(p-b),p(p-c))=pe=p$$

is a perfect square. It now follows (again from the fact that the three numbers in (5) are perfect squares) that all four numbers p, $p-a$, $p-b$, $p-c$ are perfect squares. If we now set

$$x=\sqrt{p-a},\quad y=\sqrt{p-b},\quad z=\sqrt{p-c},$$

then one concludes easily that x, y, and z are the edges of a perfect cuboid.

REFERENCE

1. R. K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, New York, NY, 1994, 173–181.