Pursuing the perfect parallelepiped

Article *in* JP Journal of Algebra, Number Theory and Applications · January 2006

Clifford A. Reiter
Lafayette College

Jordan Olliver Tirrell
Mount Holyoke College

PURSUING THE PERFECT PARALLELEPIPED

CLIFFORD A. REITER
Department of Mathematics, Lafayette College, Easton, PA 18042, USA
e-mail: reiterc@lafayette.edu

JORDAN O. TIRRELL
Farinon Center, Lafayette College, Easton, PA 18042, USA
e-mail: tirrellj@lafayette.edu

Abstract

Whether there exists a parallelepiped with edges, face diagonals, and main diagonals all of integer length is an open question. This is equivalent to thirteen linked quadratic Diophantine equations. We look at the basic Diophantine equation: the structure of integer length vectors in dimensions two, three, and four and give matrix generators for producing all the 3-dimensional integer length integer vectors. Parametric families of parallelepipeds that have good properties and the results of computer searches for perfect parallelepipeds are also described.

1. Introduction

The question of whether there is a rectanglular 3-dimensional box with integer edges, face diagonals, and body diagonal is an open problem often referred to as the perfect cuboid problem [1]. Thus, this question is whether there exist three positive integers x, y, and z such that x^2+y^2, x^2+z^2, y^2+z^2, and $x^2+y^2+z^2$ are all squares. Along with the three conditions that each of the edges is integer length, this results in 7 conditions. It is known that if any one of those seven lengths is allowed to be noninteger, then infinitely many solutions exist [1].

Another variation on the perfect cuboid problem is to relax the condition that the edges should be perpendicular. More precisely, is there a nondegenerate parallelepiped in three dimensions such that its edges, face diagonals, and main diagonals are all of integer length? This question is also open and is known as the perfect parallelepiped problem [1]. A negative answer would imply that no perfect cuboids exist. We investigate mathematics related to the perfect parallelepiped problem.

Notice that a nondegenerate parallelepiped is generated by three independent vectors, say $\vec{u}$, $\vec{v}$, $\vec{w}$. Notice that each face has two diagonals and there are four body diagonals. Thus, a perfect parallelepiped would satisy the 13 conditions that $\|\vec{u}\|$, $\|\vec{v}\|$, $\|\vec{w}\|$, $\|\vec{u}+\vec{v}\|$,$\|\vec{u}-\vec{v}\|$, $\|\vec{u}+\vec{w}\|$, $\|\vec{u}-\vec{w}\|$, $\|\vec{v}+\vec{w}\|$, $\|\vec{v}-\vec{w}\|$, $\|\vec{u}+\vec{v}+\vec{w}\|$, $\|-\vec{u}+\vec{v}+\vec{w}\|$, $\|\vec{u}-\vec{v}+\vec{w}\|$, and $\|\vec{u}+\vec{v}-\vec{w}\|$ are all integers. While there is no requirement that a perfect parallelepiped have integer coordinates, we will be focusing on this stronger version of the perfect parallelepiped problem.

AMS Classification: 11D09, 20H99, Keywords: Perfect Parallelepiped, Perfect Cuboid

Our main tool will be the parametrization of all integer length integer vectors in two, three and four dimensions. While matrix generators are known for the 2-dimensional case (yielding the Pythagorean triples), we give the corresponding, more general, result in the 3-dimensional case. We will conclude by looking at families of vectors having some of the perfect parallelepiped properties, and describe some searches for perfect parallelepipeds using the tools we develop.

2. Integer Length Integer Vectors

Integer length integer vectors correspond to solutions of diophantine equations requiring a sum of squares to be a square. For example, such vectors in Z^2 correspond to integer solutions to $x^2 + y^2 = t^2$. Positive integer solutions are Pythagorean triples corresponding to right triangles with integer sides. For example, the classic $3-4-5$ triangle corresponds to the integer length vector $\vec{u} = \langle 3, 4\rangle$. It will often be convenient to append the length of a vector to itself. We will call these extended coordinates and denote them with a superscript plus. Thus, the above classic triangle has extended coordinates $\vec{u}^{\,+} = \langle 3, 4, 5\rangle$. While that vector is formally in Z^3, we will feel free to regard it as an integer length vector in 2-dimensions. The 3 and 4 are then called the ordinary coordinates of $\vec{u}^{\,+}$. Also, anytime we discuss the length of an extended coordinate vector we will mean the length of its ordinary coordinates. So $\|\vec{u}^{\,+}\| = 5$ for the example above.

The following theorem gives us techniques for parametrizing integer length vectors with integer coordinates in 2, 3, and 4-dimensional space. The 3-dimensional version is similar to the parametrization given in Sierpinski [4]; the 4-dimensional version contains a case similar to the 3-dimensional parameterization, as well as a totally new case. The 2-dimensional version is given for completion and contrast.

Theorem 1. *Up to rearranging the ordinary coordinates, the nonzero integer length integer vectors in two, three, and four dimensions may be parametrized in extended coordinates by the following forms:*
(a) The 2-dimensional form is:

$$\vec{u}^{\,+} = \left\langle \frac{p^2 - n^2}{n}, 2p, \frac{p^2 + n^2}{|n|} \right\rangle$$

where p is any integer and n is a divisor of p^2.
(b) The 3-dimensional form is:

$$\vec{u}^{\,+} = \left\langle \frac{p^2 + q^2 - n^2}{n}, 2p, 2q, \frac{p^2 + q^2 + n^2}{|n|} \right\rangle$$

where p and q are any integers and n is any divisor of $p^2 + q^2$. Moreover, if the first coordinate is positive, and not both p and q are zero, then n may be chosen to be so that $0 < n < \sqrt{p^2 + q^2}$.
(c) The 4-dimensional forms are:

$$\vec{u}^{\,+} = \left\langle \frac{p^2 + q^2 + r^2 - n^2}{n}, 2p, 2q, 2r, \frac{p^2 + q^2 + r^2 + n^2}{|n|} \right\rangle \tag{1}$$

and

$$\vec{u}^{+} = \Big\langle \frac{(2p+1)^2+(2q+1)^2+(2r+1)^2-n^2}{2n}, 2p+1, 2q+1, 2r+1,$$
$$\frac{(2p+1)^2+(2q+1)^2+(2r+1)^2+n^2}{2|n|} \Big\rangle \tag{2}$$

where p, q, and r are any integers and n is any divisor of $p^2+q^2+r^2$ for equation (1) or any divisor of $(2p+1)^2+(2q+1)^2+(2r+1)^2$ for (2).

Proof. First we note that in each case, direct computation verifies that the extended coordinate is the length of the ordinary coordinates and it is easy to check that when the divisibility conditions hold, then all the terms are integer. A *Mathematica* script verifying the length computations may be found at [3].

Now we need to show that any nonzero integer length integer vector in the specified dimensions has one of the given forms. We begin by proving part (b). A nonzero integer length vector in Z^3 corresponds to integer solutions to:

$$x^2+y^2+z^2=t^2 \tag{3}$$

where $t>0$. We note that squares of integers are congruent to 0 or 1 modulo 4, depending on whether the integer is even or odd. Thus, the only possibilities for (3) to hold modulo 4 are if all four of the squares are congruent to 0 or if exactly one of x^2, y^2, z^2 along with t^2 is congruent to 1. In either case, at least two of x, y, and z must be even. Up to rearranging ordinary coordinates, that means that we may assume $y=2p$ and $z=2q$. Substituting those and $t=x+e$ into (3) and simplifying yields:

$$4p^2+4q^2=e(2x+e). \tag{4}$$

Thus $0\equiv e(2x+e)\equiv e^2 \mod 2$ and hence e must be even. Say $e=2n$. Using that in (4) and solving for x (provided $n\neq 0$) we see $x=\frac{p^2+q^2-n^2}{n}$ as desired. If $n<0$, then we need to use $|n|$ in the denominator of the extended coordinate to maintain our convention that the extended coordinate is nonnegative since it is a length. The case $n=0$ arises only when $p=q=0$ and in that case we see $x=\pm t$ and hence

$$\langle x,y,z,t\rangle = \langle -n,0,0,|n|\rangle = \Big\langle \frac{-n^2}{n},0,0,\frac{n^2}{|n|}\Big\rangle$$

with $n\neq 0$ parameterizes the remaining solutions. Note that n divides 0 when $n\neq 0$. Thus, both the form and divisibility condition remain valid for the case when $p=q=0$.

Note that if $x>0$ and y and z are not both zero, then since $p^2+q^2=n(x+n)$, we may select the divisor n to be positive and hence the numerator $p^2+q^2-n^2>0$ whence $n<\sqrt{p^2+q^2}$, which gives the "moreover" portion of (b).

Part (a) follows from the specialization of (b) to the case when $z=0$.

Lastly, we consider part (c). First note that modulo 8, squares are congruent to 0, 1, or 4. Thus, the sum of squares can result in a square, $x^2+y^2+z^2+w^2=t^2$, only if combinations of 0 and 4 on the left sum to 0 or 4, or a single 1 and three 0's add to 1, or four 1's add to 4. In the first two cases, there are at least three even variables. Up to order, we may suppose that $y = 2p$, $z = 2q$, and $w = 2r$. The proof is analogous to the proof for part (b), except that there are sums of three squares where there had been two, and this results in (1). In the case when four 1's add to 4 modulo 8, we take $y = 2p+1$, $z = 2q+1$, $w = 2r+1$ since we know they are odd. Also $y^2 + z^2 + w^2 = t^2 - x^2 = (t+x)(t-x)$. Letting the $m = t + x$ and $n = t - x$, we see $mn = y^2 + z^2 + w^2$ and $x = \frac{m-n}{2} = \frac{mn-n^2}{2n}$. These facts together yield (2) as desired. $\square$

This theorem allows us to easily generate integer length integer vectors. For example, in part (b) if we take $p = 5$ and $q = 7$, then $p^2 + q^2 = 74$, so we can take $n = 2$. This yields $\vec{u}^+ = \langle 35, 10, 14, 39\rangle$.

Several other remarks are in order. The bounds on n given in part (b) will play an important technical role in our main theorem on matrix generators for integer length vectors in Z^3.

Discussions of Pythagorean triples usually focus upon positive integer solutions that are primitive. The triple is primitive if the entries in the triple share no common factor. Since we are interested in using these integer length vectors in the context of the perfect parallelepiped problem, we expect to need to use negative entries (e.g., we consider the length of both $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$) and when we consider collections of such vectors, it may be helpful if some of them are not primitive. Thus, we want to consider negative entries and nonprimitive solutions. However, we do consider a collection of vectors degenerate when the vectors are linearly dependent. Three dependent vectors would form a degenerate parallelpiped with no volume. Thus, we require a perfect parallelepiped to have independent edges.

Notice that if we substitute $p = ab$ and $n = b^2$ into the parametriztion (a), we get $\vec{u}^+ = \langle a^2 - b^2, 2ab, a^2 + b^2\rangle$ which is a more traditional form for the parametrization of primitive Pythagorean triples.

The parametrizations given in Theorem 1 will make it easy for us to generate many integer length integer vectors as part of searches that we will discuss in Section 4. In the next section, we will also see that these play a key role in allowing us to describe matrix generators for all the integer length triples in Z^3.

3. Algebraic Structure of Integer Length Vectors in Z^3

In this section we show that there is one matrix that, together with ordinary coordinate interchanges and sign changes, generates all of the nonzero integer length integer vectors in Z^3. In particular, let

$$J = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

Consider the following illustrations of the multiplication of an extended coordinate integer vector by J; in each case the result is another extended coordinate integer vector.

$$J\begin{pmatrix}1\\2\\2\\3\end{pmatrix}=\begin{pmatrix}7\\6\\6\\11\end{pmatrix},\quad J\begin{pmatrix}1\\2\\-2\\3\end{pmatrix}=\begin{pmatrix}3\\2\\6\\7\end{pmatrix},\quad J\begin{pmatrix}2\\-1\\2\\3\end{pmatrix}=\begin{pmatrix}4\\7\\4\\9\end{pmatrix}$$

The following theorem shows that this is a general property.

Theorem 2. *Multiplying the extended coordinate form for any nonzero integer length integer vector in Z^3 by J produces a nonzero extended coordinate integer length integer vector. Moreover, the multiplication preserves the gcd of the coordinates of the extended vector; in particular, if the original vector was primitive, the resulting vector will be primitive.*

Proof. First notice that

$$J\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=\begin{pmatrix}y+z+t\\x+z+t\\x+y+t\\x+y+z+2t\end{pmatrix};$$

therefore, interchanging ordinary coordinates (e.g. x interchanged with z) does not change the length of the resulting vector. Thus, as we saw in Theorem 1(b), we may assume our vector may be parametrized by:

$$\vec{u}^{+}=\langle\frac{p^2+q^2-n^2}{n},2p,2q,\frac{p^2+q^2+n^2}{|n|}\rangle$$

where n divides p^2+q^2. Direct, but tedious, computation [3] verifies that when $n>0$, the vector $J\vec{u}^{+}$ has length given by:

$$\|J\vec{u}^{+}\|^2=\frac{\left(n^2+2pn+2qn+3p^2+3q^2\right)^2}{n^2}$$

where, as is our convention, the length means the length of the ordinary coordinates. The length $\|J\vec{u}^{+}\|$ must be integer since n divides p^2+q^2. When $n<0$ we get

$$\|J\vec{u}^{+}\|^2=\frac{\left(3n^2-2pn-2qn+p^2+q^2\right)^2}{n^2}.$$

Again, the length is integer.

Let d be a common divisor of the coordinates of $\vec{u}^{+}$, then d is also a divisor of the coordinates of the resulting vector $J\vec{u}^{+}$, since the coordinates are integer linear combinations of the original coordinates. Since the determinant of J is 1, the inverse matrix has integer coefficients, so the converse is also true. Thus, $\vec{u}^{+}$ and $J\vec{u}^{+}$ have the same *gcd*.

Additionally, if the original vector was nonzero, the resulting vector will be nonzero. This follows since J is invertible, so $J\vec{u}^{+}=0$ implies $J^{-1}J\vec{u}^{+}=0$ and hence $\vec{u}^{+}=0$. $\square$

We will adopt a subscript notation to signify columns of J which have had their sign changed; for example:

$$J_{12} = \begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 2 \end{pmatrix}$$

is the same as J except the first and second columns have been negated.

Proposition 3. *The following matrix identities hold:*

(a)

$$J^{-1} = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}$$

(b)

$$J_{123}^2 = J_{12}^4 = J_{13}^4 = J_{23}^4 = I$$

(c)

$$J_{23}J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 2 & 3 \end{pmatrix}$$

(d)

$$J^3 - 3J^2 - 3J + I = 0$$

Proof. These can be checked by direct computation; see [3]. □

A number of authors have described the generation of Pythagorean triples using matrices and resulting in what is called the Barning tree [2]. The matrices they multiply Pythagorean triples by include

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

and variants associated with sign changes of ordinary coordinates. We saw this matrix appeared as a subblock of $J_{23}J$ in Proposition 3(c). In Theorem 4 we will show that J and its variants generate all the nonzero integer length integer vectors in Z^3. Thus, the matrix generators for the Pythagorean triples arise as a special case of Theorem 4.

While we do not use most of the properties in Proposition 3, part (b) will play a crucial role in the proof of Theorem 4. In particular, notice that $J_{123} = JS$ where S is a diagonal matrix with $\langle -1, -1, -1, 1\rangle$ on the diagonal. So $J_{123}^2 = I$ from Proposition 3 implies that $J^{-1} = SJS$. Thus, we see the following.

Remark. *When we have a collection of transformations including multiplication by J and sign changes, we automatically have the inverse transformations as well.*

The proof of Theorem 4 utilizes this reversiblity. We show that any integer length integer vector can be reduced to the trivial case $\langle s, 0, 0, s\rangle$ and hence could have been produced from the trivial case by the reverse process.

Theorem 4. *The extended coordinate form for any nonzero integer length integer vector in Z^3 with the gcd of the coordinates equal to s may be produced from $\langle s, 0, 0, s\rangle$ by a sequence of ordinary coordinate interchanges, ordinary coordinate sign changes, and multiplication by J.*

Proof. Ordinary coordinate interchanges, ordinary coordinate sign changes, and multiplication by J all preserve the *gcd* of the coordinates, so it suffices to consider any primitive vector $\vec{u}^+$. Since the vector is primitive, at least one ordinary coordinate is odd, so by applying coordinate interchanges and sign changes if neccesary, we may assume the first coordinate is odd and positive. If $\vec{u}^+ = \langle 1, 0, 0, 1\rangle$, then we are done. Otherwise, we know by Theorem 1(b) that $\vec{u}^+$ is of the form

$$\vec{u}^+ = \langle \frac{p^2 + q^2 - n^2}{n}, 2p, 2q, \frac{p^2 + q^2 + n^2}{n} \rangle$$

and n may be chosen to be so that $0 < n < \sqrt{p^2 + q^2}$. We claim that at least one of $J_1\vec{u}^+$, $J_{12}\vec{u}^+$, $J_{13}\vec{u}^+$, and $J_{123}\vec{u}^+$ is shorter than $\vec{u}^+$. Once we verify this claim, the proof will follow inductively by repeating the argument until, within a finite number of steps (since the lengths are strictly decreasing sequence of positive integers), it must happen that we reach the alternate case where $\vec{u}^+ = \langle 1, 0, 0, 1\rangle$. As per the remark before the theorem, the reversibility of the process means we could have generated the original primitive vector from $\vec{u}^+ = \langle 1, 0, 0, 1\rangle$.

It remains to show that at least one of $J_1\vec{u}^+$, $J_{12}\vec{u}^+$, $J_{13}\vec{u}^+$, and $J_{123}\vec{u}^+$ is shorter than $\vec{u}^+$. First we compute

$$\|J_1\vec{u}^+\|^2 - \|\vec{u}^+\|^2 = \frac{4}{n}(n + p + q)(2n^2 + pn + qn + p^2 + q^2). \tag{5}$$

The computation is direct but tedious and can be found at [3]. We know n is positive, and by completing the square we can check the last factor is positive regardless of the signs of p and q. Thus,

$$\|J_1\vec{u}^+\| < \|\vec{u}^+\| \text{ exactly when } (n + p + q) < 0.$$

Doing the same computation in equation (5) with J_1 replaced by J_{12}, J_{13}, and J_{123} gives equivalent results up to sign changes. Once again the last factor is positive and so:

$$\|J_{12}\vec{u}^+\| < \|\vec{u}^+\| \text{ exactly when } (n - p + q) < 0,$$
$$\|J_{13}\vec{u}^+\| < \|\vec{u}^+\| \text{ exactly when } (n + p - q) < 0,$$
$$\|J_{123}\vec{u}^+\| < \|\vec{u}^+\| \text{ exactly when } (n - p - q) < 0.$$

The four inequalities on the right of those equivalences have the form $n < \pm p \pm q$ and we claim at least one of these inequalities must hold. Suppose not, then $n \geq \pm p \pm q$ for all choices of $\pm$. That implies $n \geq |p| + |q|$; however, $n < \sqrt{p^2 + q^2} \leq |p| + |q|$ as noted above and using the triangle inequality. This is a contradiction, and hence one of the inequalities must hold and thus one of the four vectors is shorter than $\vec{u}^+$. That completes the induction step and the proof. □

The above theorems show that the J matrix plays a key role in the structure of integer length integer triples. We have remarked that the integer length integer pairs are closely related. We do not know of a generalization to integer length integer vectors in Z^4.

While complications arising from the two types of parameterizations would be expected, we suspect that the difficulties are deeper and that matrix generators do not exist for the nonzero integer length integer vectors in Z^4.

Furthermore, despite the rich structure and beautiful identities described in Proposition 3, we have not found this structure useful for generating families of examples with partial perfect parallelepiped structure. For example, nontrivial identities of the form $f(J) + g(J) = I$ where f and g are any products of J and its variants would mean that for integer length integer vectors $\vec{u}^+$, the three vectors $\vec{f}(J)u^+$, $\vec{g}(J)u^+$, and (their sum) $\vec{I}u^+$ would all be integer length integer vectors; three such integer length vectors would form the sides and positive diagonal of a parallelogram. We know of no such identities.

4. Almost Perfect Parallelepipeds

While we have found no perfect parallelepiped, we have special families and examples to discuss. However, first we note that any examples that we find short of complete perfect parallelepideds are unimpressive in the following way. Consider the almost perfect cuboid from [1] that has only one irrational edge. In vector notation:

$$\begin{aligned} \vec{u} = &\ \langle 7800, && 0, && 0\rangle \\ \vec{v} = &\ \langle 0, && 18720, && 0\rangle \\ \vec{w} = &\ \langle 0, && 0, && \sqrt{211773121}\rangle \end{aligned}$$

Of the thirteen conditions required for a perfect parallelepiped, the only one that fails is that $\|\vec{w}\|$ is not integer (of course, a noninteger entry also appears in $\vec{w}$). The other twelve conditions hold. Thus, we don't expect be able to get nearer to a perfect parallelpiped by counting the number of the thirteen conditions that hold.

We are able to give families of vectors with some of the perfect parallelepiped properties.

Proposition 5. *Consider the parametric family of vectors such that:*

$$\vec{u} = \langle 4q^2(p^2 + q^2 - 1), 8pq^2, 8q^3\rangle$$
$$\vec{v} = \langle 4(r^2 - 1)^2q^2 + (2pr + 1)^2), 8q^2r, -4q(2pr + 1)\rangle$$

For all choices of integers p, q and r, the length of $\vec{u}$, $\vec{v}$ and $\vec{u} + \vec{v}$ are integer.

Proof. Direct computation verifies these three vectors have length $4q^2(p^2 + q^2 + 1)$, $4r^2q^2 + 4q^2 + 4p^2r^2 + 4pr + 1$, and $4q^2 + 4p^2q^2 + 4r^2q^2 + 4p^2r^2 + 4pr + 1$, respectively [3]. $\square$

While the proof of Proposition 5 can be verified as remarked above, we briefly describe how the parametrization was found. The special case of Theorem 1(b) when $n = 1$ is an especially simple integer vector. Creating two such vectors, $\vec{u}$ and $\vec{v}$ formally and completing the square of $\|\vec{u} + \vec{v}\|^2$ yields a linear remainder. Solving for one parameter and clearing the denominator gives the form in Proposition 5.

Notice in Proposition 5 that $\vec{u}$ depends only upon p and q, not r. Thus, we can repeat the construction using the same p and q but different choices for r. This gives three integer length integer vectors with two of the positive diagonals integer. It is not obvious, but three such vectors must be collinear and hence only produce degenerate examples [3].

Proposition 6. *Consider the parametric family of vectors such that:*

$$\begin{array}{rlll} \vec{u} = & \langle 25p^2 - 25q^2, & 50pq, & 0\,\rangle \\ \vec{v} = & \langle 25p^2 - 25q^2, & -50pq, & 0\,\rangle \\ \vec{w} = & \langle \quad 0, & -28pq, & 96pq\rangle \end{array}$$

For all choices of integers p and q the length of $\|\vec{u}\|$, $\|\vec{v}\|$, $\|\vec{w}\|$, $\|\vec{u}+\vec{v}\|$,$\|\vec{u}-\vec{v}\|$, $\|\vec{u}+\vec{v}+\vec{w}\|$, $\|-\vec{u}+\vec{v}+\vec{w}\|$, $\|\vec{u}-\vec{v}+\vec{w}\|$, and $\|\vec{u}+\vec{v}-\vec{w}\|$ are integer. Moreover, of the remaining four conditions, $\|\vec{u}+\vec{w}\| = \|\vec{v}-\vec{w}\|$, and $\|\vec{u}-\vec{w}\| = \|\vec{v}+\vec{w}\|$; therefore, only two additional conditions are required to obtain a perfect parallelepiped.

Proof. Proof by direct computation can be found at [3]. □

The parameterization in Theorem 1 or the structure in Theorem 4 may be used to create large sets of integer length integer vectors in various dimension. We can utilize those in various focused searches for perfect parallelepipeds.

Search Type I. *(i) Create lists of integer length integer vectors in 3 dimensions. (ii) Determine which pairs give rise to parallelograms in 3-dimensions with integer length edges and diagonals. (ii) look for edges duplicated in more than one such paralellogram. (iii) use such overlaps to obtain three promising vectors; then restrict to those that satisfy additional conditions.*

We ran such computations for all choices of parameters in Theorem 1(b) with magnitude less than or equal 45. Only a few examples where found. For example, the following has 11 of the 13 conditions hold:

$$\begin{array}{rlll} \vec{u} = & \langle 21, & 72, & 0\rangle \\ \vec{v} = & \langle 21, & -72, & 0\rangle \\ \vec{w} = & \langle 0, & 0, & 40\rangle. \end{array}$$

In the above example, the two condtions that fail are equivalent: $\|\vec{u}-\vec{v}+\vec{w}\| = \|\vec{-}u+\vec{v}+\vec{w}\|$ is not an integer; hence this is an example where, like the best cuboid examples, we are just one condition away from having a perfect parallelepiped, yet we managed that using integer coordinates. This example also suggests that examples of the form

$$\begin{array}{rlll} \vec{u} = & \langle a, & b, & 0\rangle \\ \vec{v} = & \langle a, & -b, & 0\rangle \\ \vec{w} = & \langle 0, & 0, & c\rangle \end{array}$$

may be a rich hunting ground for perfect parallelepipeds. Indeed, other examples similar to the above, but with larger entries, may be found by searches of vectors with the above pattern.

Not all of the examples found by Type I searches fit the above pattern. For example,

$$\begin{array}{rlll} \vec{u} = & \langle 24, & 27, & 36\rangle \\ \vec{v} = & \langle -24, & 27, & 36\rangle \\ \vec{w} = & \langle 0, & 112, & -84\rangle \end{array}$$

Search Type II. *Here we consider parallelepipeds with a rectangular base and one vector completely free. That is, examples of the form:*

$$\begin{array}{rlll} \vec{u} = & \langle a, & 0, & 0\rangle \\ \vec{v} = & \langle 0, & b, & 0\rangle \\ \vec{w} = & \langle c, & d, & e\rangle \end{array}$$

Here we can efficiently generate integer length pairs $\langle a, b\rangle$ and triples $\langle c, d, e\rangle$ using Theorem 1. We tested all parameters with magnitude below 70 for the pairs and 60 for the triples. The best examples we found, such as the following, satisified 10 of the 13 conditions.

$$\begin{array}{rrrr} \vec{u} = & \langle 42, & 0, & 0\rangle \\ \vec{v} = & \langle 0, & 40, & 0\rangle \\ \vec{w} = & \langle 21, & 56, & 12\rangle \end{array}$$

Search Type III. *Here we consider* 3*-dimensional parallelepipeds embedded in* 4*-dimensions. We use the same strategy as in Type I searches, except we used vectors in* 4*-dimensions.*

We checked all vectors with parameters up to 6 in magnitude in Theorem 1(c). Again we found examples, such as the following, with 11 of the 13 conditions holding.

$$\begin{array}{rrrrr} \vec{u} = & \langle 4, & 6, & 6, & 9\rangle \\ \vec{v} = & \langle -4, & 10, & 2, & 7\rangle \\ \vec{w} = & \langle 12, & -8, & -4, & 10\rangle \end{array}$$

However, in this example the two lengths failing to be integer $\|\vec{u} - \vec{v} + \vec{w}\| = \sqrt{668}$, and $\|\vec{-}u + \vec{v} + \vec{w}\| = \sqrt{160}$ are not equivalent.

Search Type IV. *We directly searched through parallelepipeds of the form given in Proposition 6.*

We checked parameters up to one million without finding any perfect parallelepipeds.

Acknowledgements

This work was supported in part by a Lafayette College EXCEL grant.

References

[1] R. Guy, Unsolved Problems in Number Theory, 2nd ed., Springer-Verlag New York Inc., 1994.

[2] D. McCullough, Height and Excess of Pythagorean Triples, Mathematics Magazine, 78 (2005), 26-44.

[3] C. Reiter and J. Tirrell, Pursuing the Perfect Parallelepiped Auxiliary Materials, http://www.lafayette.edu/~reiterc/nt/ppllpd/index.html

[4] W. Sierpinski, Elementary Theory of Numbers, trans. by A. Hulanicki, Panstwowe Wydawnictwo Naukowe, Warszawa, 1964.