Two and Three factor models for Spread Options Pricing

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Outline

- Spot Price Dynamics
- Forward Prices
- Spreads on Forwards
- Model Calibration
- Conclusions

Spot Price Dynamics

Spot Price Dynamics

- Energy and energy commodities markets are unique
- Well known peculiarities with energy commodities:
  - Markets are **illiquid**
  - **Storage costs** (or impossibility of storage) translate into peculiar price behavior
  - Structural issues lead to **high volatility** levels
  - Prices exhibit strong **mean-reversion** tendencies
  - Electricity in particular contains **fast mean-reverting jumps**

Spot Price Dynamics

- Geometric Brownian motion fails to capture price fluctuations
- **Mean-reversion** is an essential feature

$$d \ln S(t) = \kappa \left(\theta - \ln S(t)\right) dt + \sigma \, dX(t)$$

where $X(t)$ is a $\mathbb{P}$ Wiener process

- One-factor models :
  - Fail to capture the term structure of forward rates
  - Fail to treat the long-run mean reversion as dynamic

Spot Price Dynamics

- **Pilipovic** (1997) first proposed the two-factor mean-reverting model

$$
\begin{aligned}
dS_t &= \beta(\theta_{\mathbf{t}} - S_t)\,dt + \sigma_S\,S_t\,dW_t^{(1)} \\
d\theta_{\mathbf{t}} &= \alpha\,\theta_t dt + \sigma_\theta\,\theta_t\,dW_t^{(2)} \\
d[W^{(1)}, W^{(2)}]_t &= \rho\,dt
\end{aligned}
$$

- Long-run mean θ blows up
- No-invariant distribution in long-run
- Does not lead to closed form option prices

Spot Price Dynamics

- A **stationary two-factor mean-reverting** model :

$$
\begin{aligned}
S_t &= \exp\{g_t + \mathbf{X}_t\}\ , \\
d\mathbf{X}_t &= \beta\,(\mathbf{Y_t} - \mathbf{X}_t)\,dt + \sigma_X\,dW_t \\
d\mathbf{Y_t} &= \alpha\,(\phi - \mathbf{Y_t})\,dt + \sigma_Y\,dZ_t \\
d[W,Z]_t &= \rho\,dt
\end{aligned}
$$

- Seasonality is modeled through $\mathbf{g_t}$
- X_t mean-reverts to Y_t
- Y_t mean-reverts to ϕ

Spot Price Dynamics

- Since both X and Y are Gaussian Ornstein-Uhlenbeck processes one finds

$$\mathbf{Y}_t \;=\; \phi + (\mathbf{Y}_s - \phi)\, e^{-\alpha\,(t-s)} + \sigma_Y \int_s^t e^{-\alpha\,(t-u)}\, dZ_u$$

$$\begin{aligned}\mathbf{X}_t \;=\;& G_{s,t} + e^{-\beta(t-s)}\mathbf{X}_s + M_{s,t}\,\mathbf{Y}_s \\ & + \sigma_X \int_s^t e^{-\beta\,(t-u)}\, dW_u + \sigma_Y \int_s^t M_{u,t}\, dZ_u\end{aligned}$$

- Here, $G_{s,t}$ and $M_{s,t}$ are deterministic functions of the model parameters and time

Stochastic Volatility Spot Models

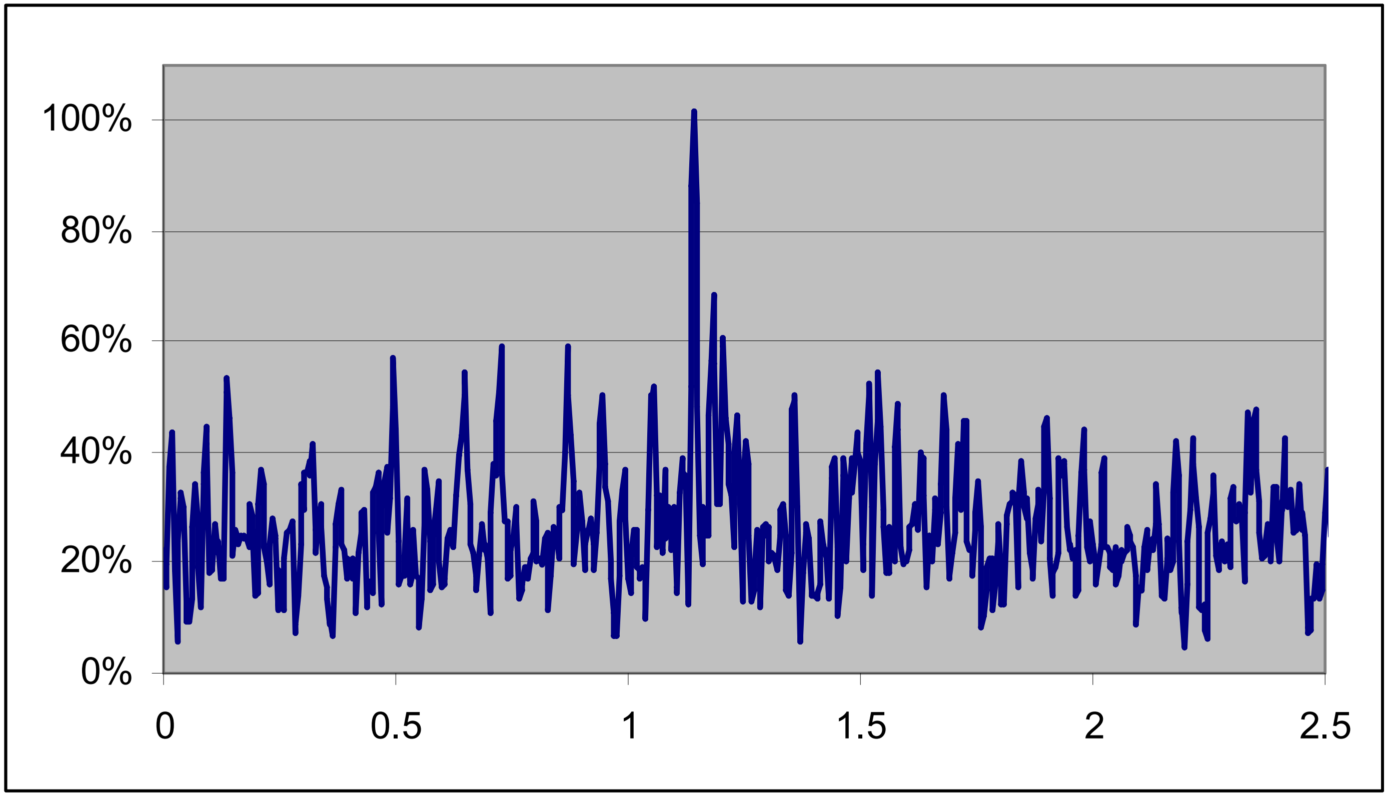
- A **three-factor model**: stochastic long-run mean and stochastic volatility

$$
\begin{aligned}
S_t &= \exp\{g_t + \mathbf{X}_t\}\ , \\
d\mathbf{X}_t &= \beta\,(\mathbf{Y}_\mathbf{t} - \mathbf{X}_t)\,dt + \sigma_{\mathbf{X}}(\mathbf{Z}_\mathbf{t})\,dW_t^{(1)} \\
d\mathbf{Y}_\mathbf{t} &= \alpha\,(\phi - \mathbf{Y}_\mathbf{t})\,dt + \sigma_Y\,dW_t^{(2)} \\
d\mathbf{Z}_\mathbf{t} &= \eta\,(m - \mathbf{Z}_\mathbf{t})\,dt + \sigma_Z\,dW_t^{(3)}
\end{aligned}
$$

$$
\begin{aligned}
d[W^{(1)}, W^{(2)}]_t &= \rho_{xy}\,dt \\
d[W^{(1)}, W^{(3)}]_t &= \rho_{xz}\,dt \\
d[W^{(2)}, W^{(3)}]_t &= 0
\end{aligned}
$$

Stochastic Volatility Spot Models

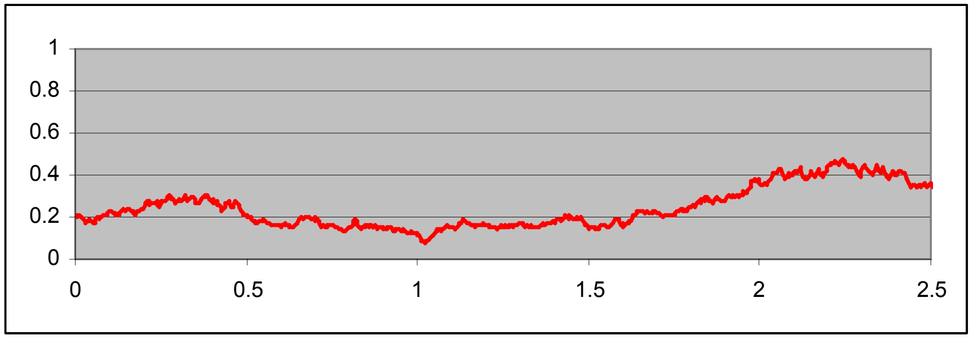
- **Realized volatility** : NYMEX light sweet crude oil



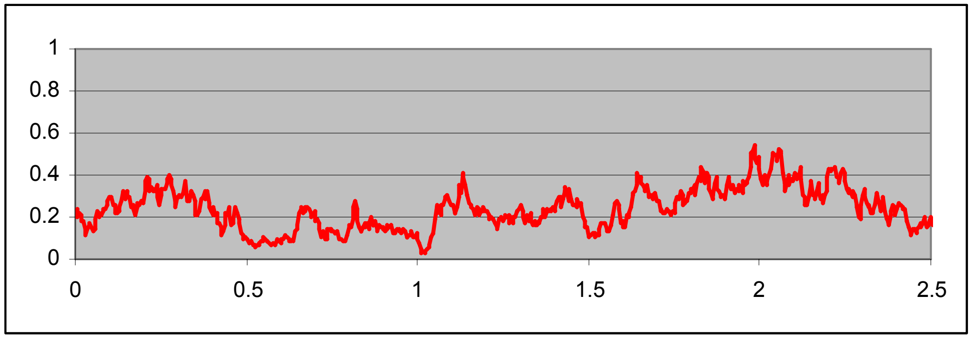
Stochastic Volatility Spot Models

- **Simulated volatility**

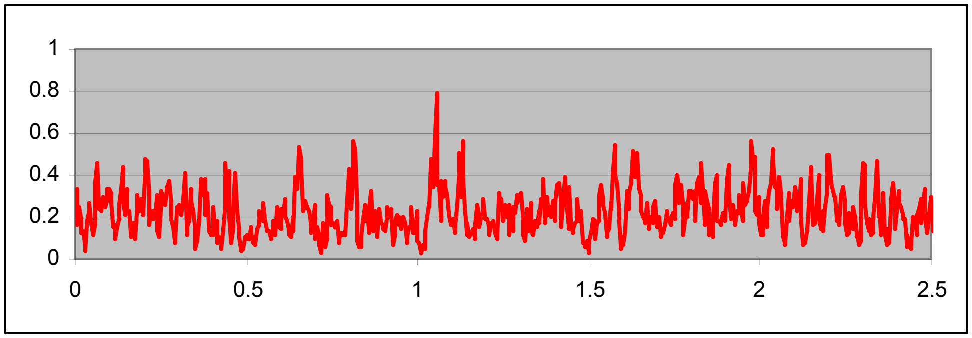
$$\eta = 1, \qquad \frac{\sigma_Z}{\sqrt{2\eta}} = 0.1$$



$$\eta = 10, \qquad \frac{\sigma_Z}{\sqrt{2\eta}} = 0.1$$



$$\eta = 100, \qquad \frac{\sigma_Z}{\sqrt{2\eta}} = 0.1$$



Electricity Spot Price Modeling

- For Electricity, typical modeling assumptions assume

$$d\ln(\mathbf{S_{t-}}) = \alpha(\theta - \ln(\mathbf{S_{t\_}}))\,dt + \sigma\,dW_t + d\mathbf{Q_t}\ .$$

- $\mathbf{Q_t}$ is a Compound Poisson process (or possibly Lévy)

$$Q_t \quad = \quad \int_0^t \int_{-\infty}^{\infty} y\,\mu(dy, dt) = \sum_{n=1}^{N(t)} q_i$$

$$\text{with } q_i \quad \overset{i.i.d}{\sim} \quad F_q(u)$$

- When calibrated to real data one finds:
  - **Mean-reversion is very high** to draw down jumps
  - This pushes diffusive **volatility artificially high**

Electricity Spot Price Modeling

- We propose a natural extension of the two-factor and three-factor diffusion model:

$$S_t := \exp\{g_t + \mathbf{X_t} + \mathbf{J_t}\} \ ,$$

where the new jump component $\mathbf{J_t}$ is defined via

$$d\mathbf{J_t} = -\kappa\, \mathbf{J_{t\_}}\ dt + d\mathbf{Q_t} \ ,$$

- **Jump** and **diffusion** reversion rates are **decoupled**
- No artificially high diffusive volatilities

Forward Prices

Forward Prices

- The **forward price** for **two-factor model** is affine :

$$F^{(i)}(t,T) \;\equiv\; \mathbb{E}_t^{\mathbb{Q}}\left[S_T^{(i)}\right]$$

$$=\; \exp\left(g_T^{(i)} + G_{t,T}^{(i)} + e^{-\beta_i(T-t)}\,\mathbf{X_t^{(i)}} + M_{t,T}^{(i)}\,\mathbf{Y_t^{(i)}}\right)$$

- For the **three-factor model**, we carry out a **singular perturbation expansion** by assuming :

$$\overline{\sigma_Z}^2 := \frac{\sigma_Z^2}{2\eta} < +\infty \qquad \text{while} \quad \eta \to +\infty$$

Forward Prices

- The T-maturity forward price satisfies:

$$\begin{aligned}\left(\epsilon^{-1}\mathcal{A}_{\mathbf{0}}+\epsilon^{-\frac{1}{2}}\mathcal{A}_{\mathbf{1}}+\mathcal{A}_{\mathbf{2}}\right)F^{\epsilon}(t,x,z) &= 0\,,\\ F_T(x,y,z) &= e^{g_T+x}\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}_{\mathbf{0}} &:= (m-z)\,\partial_z+\overline{\sigma_Z}^2\,\partial_{zz}\,,\\ \mathcal{A}_{\mathbf{1}} &:= \sqrt{2}\rho_{xz}\,\overline{\sigma_Z}\,\sigma_X(z)\,\partial_{xz}\,,\\ \mathcal{A}_{\mathbf{2}} &:= \partial_t+\beta(y-x)\,\partial_x+\alpha(\phi-y)\,\partial_y\\ &\quad+\frac{1}{2}\sigma_X^2(z)\,\partial_{xx}+\frac{1}{2}\sigma_y^2\,\partial_{yy}+\rho_{xy}\sigma_X(z)\sigma_Y\,\partial_{xy}\end{aligned}$$

Forward Prices

- We solve this PDE, and prove the error bound, to order $\varepsilon^{1/2}$ using **singular perturbation techniques**

  [á la P. Cotton, J-P Fouque, G. Papanicolaou & R. Sircar (2004) in the context of interest rates]

Corollary 0.1 *For any fixed $(T, x, y, z) \in \mathbb{R}^+ \times \mathbb{R}^3$ and all $t \in [0, T]$, we have*

$$\boxed{\begin{aligned} F^{\epsilon}_{t,T} \quad = \quad & \Big(1 - V_1 h(t,T;3\beta) \\ & \qquad - V_2 \frac{\beta}{\alpha_Y - \beta} \left[h(t,T;3\beta) - h(t,T;\alpha_Y + 2\beta)\right]\Big) F^{(0)}_{t,T} + O(\epsilon)\,, \end{aligned}}$$

where $F^{(0)}_{t,T}$ is the forward price in the two-factor model with

$$\sigma_X = < \sigma_X(Z) >$$

Electricity Forward Prices

- The two-factor jump-diffusion model is also affine:

$$
\begin{aligned}
F_{t,T}^{(1)} &:= \mathbb{E}_t^{\mathbb{Q}}\left[S_T^{(1)}\right] \\
&= \exp\left\{A_{t,T} + B_{t,T}\,X_t^{(1)} + C_{t,T}\,Y_t^{(1)} + D_{t,T}\,J_t\right\}
\end{aligned}
$$

- The infinitesimal generator $\mathcal{A}$ of the joint processes acts on the forward price process rendering it zero

$$
\mathcal{A}\,F_{t,T}^{(1)} = 0
$$

- The **pricing PDE** then reduces to a system of **coupled Riccati ODEs**

$$
\begin{aligned}
B_t - \overline{\beta}_1 B &= 0\,, \\
C_t + \overline{\beta}_1 B - \overline{\alpha}_1 C &= 0\,, \\
D_t - \kappa D &= 0\,,
\end{aligned}
$$

$$
\begin{aligned}
A_t + \overline{\alpha}_1 \overline{\phi}_1 C + \frac{(\sigma_X^{(1)})^2}{2} B^2 & \\
+ \frac{(\sigma_Y^{(1)})^2}{2} C^2 + \rho_1 \sigma_X^{(1)} \sigma_Y^{(1)} BC &= -\int_{-\infty}^{\infty} \lambda(u) \left(e^{D \cdot u} - 1\right) dF_l(u).
\end{aligned}
$$

Spread Options

Exchange Option Pricing

- The **spread on two forward rates** is a very popular product (different assets):

$$\text{pay-off} = \max\left(F^{(1)}_{T,T_1} - \alpha\, F^{(2)}_{T,T_2} - \mathbf{K},\ 0\right)$$

- Exact solution difficult (impossible? – at least so far!) for K ≠ 0 - Put K = 0 – **Margrabe** option

- Risk-neutral Pricing implies :

$$\text{price} = P(t,T)\,\mathbb{E}^{\mathbb{Q}}\left[\max\left(F^{(1)}_{T;T_1} - \alpha\, F^{(2)}_{T;T_2},\ 0\right)\right]$$

- Use an asset as a **numeraire** to reduce the stochastic dimensionality?

Exchange Option Pricing

- Introduce the **measure-$\mathbb{Q}^*$** induced by the following Radon-Nikodym derivative process:

$$\left(\frac{d\mathbb{Q}^*}{d\mathbb{Q}}\right)_t := \frac{\mathbb{E}_t^{\mathbb{Q}}\left[S_T^{(i)}\right]}{\mathbb{E}_0^{\mathbb{Q}}\left[S_T^{(i)}\right]} = \frac{F_{t,T_2}^{(2)}}{F_{0,T_2}^{(2)}}$$

- The **ratio** of two forward prices is a **$\mathbb{Q}^*$-martingale!**

$$F_{t;T_1,T_2} := \frac{F_{t,T_1}^{(1)}}{F_{t,T_2}^{(2)}}$$

Exchange Option Pricing

- In particular, for the **two-factor model** we show

$$F_{T;T_1,T_2} = F_{t;T_1,T_2}\,\exp\{N\} \qquad \textit{with} \quad N \overset{\mathbb{Q}^*}{\sim} \mathcal{N}\left(-\frac{1}{2}(\sigma^*)^2\,;\,(\sigma^*)^2\right)$$

where $(\sigma^*)^2$ is a deterministic function of the model parameters and times.

Proposition 0.1 *The risk-neutral value of the T-maturity forward spread option is*

$$\boxed{\Pi^F_{t,T} = P(t,T)\left[F^{(1)}_{t,T_1}\,\Phi(d^* + \sigma^*_{t,T}) - \alpha\,F^{(2)}_{t,T_2}\,\Phi(d^*)\right]}$$

and

$$d^* := \frac{\ln\left(F_{t;T_1}/\alpha\,F_{t;T_2}\right) - \frac{1}{2}(\sigma^*_{t,T})^2}{\sigma^*_{t,T}}\,.$$

Exchange Option Pricing

- For the **three-factor model** we show, using singular perturbation techniques once again that

Proposition 0.1 *The risk-neutral value of the T-maturity forward spread option is*

$$\boxed{\Pi^F_{t,T} = \Pi^{F(0)}_{t,T} + \Pi^{F(1,1)}_{t,T} + \Pi^{F(1,2)}_{t,T} + O(\epsilon)}$$

where $\Pi^{F(0)}_{t,T}$ is the price in the two-factor model, and the corrections depend on the delta's and the delta-gamma's of the two-factor model.

Electricity Exchange Option Valuation

- The **ratio** of forward prices $F_{T;\,T1,\,T2}$ is still a **martingale** under this new $\mathbb{Q}^*$-measure
- However, it is no longer Gaussian
- Use **Fourier transform** methods to price
- Introduce the m.g.f. process of $Z_T = \ln F_{T;\,T1,\,T2}$

$$\Psi_t^{Z_T}(u) := \mathbb{E}_t^{\mathbb{Q}^*}\left[e^{u\,Z_T}\right]$$

- This too is a martingale under the new measure

Crack Spread Valuation

- Rewrite the price as follows:

make use of a **Fourier transform** result to get:

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}^*}\left[(e^{Z_T-\bar{\alpha}}-1)_+\right] &:= \mathbb{E}_t^{\mathbb{Q}^*}\left[\eta(Z_T-\bar{\alpha})\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\eta}(-p)\,\tilde{f}_{Z_T-\bar{\alpha}}(p)\,dp\end{aligned}$$

The **Fourier transform** of η is standard:

$$\tilde{\eta}(p) := \int_{-\infty}^{\infty} e^{ipx}\eta(x)\,dx = \frac{1}{p(i-p)}$$

whenever $\Im(p) > 1$.

Crack Spread Valuation

- Rewrite the price as follows:

Proposition 0.1 *The price of the calander spark spread option is*

$$\boxed{\Pi_{t,T} = P(t,T)\, e^{\overline{\alpha}}\, F^{(2)}_{t,T_2} \int_{-\infty}^{\infty} \frac{e^{-i\overline{\alpha}p}\;\Psi_t^{Z_T}(ip)}{-p(p+i)}\,\frac{dp}{2\pi}\,,}$$

Spot Price Dynamics

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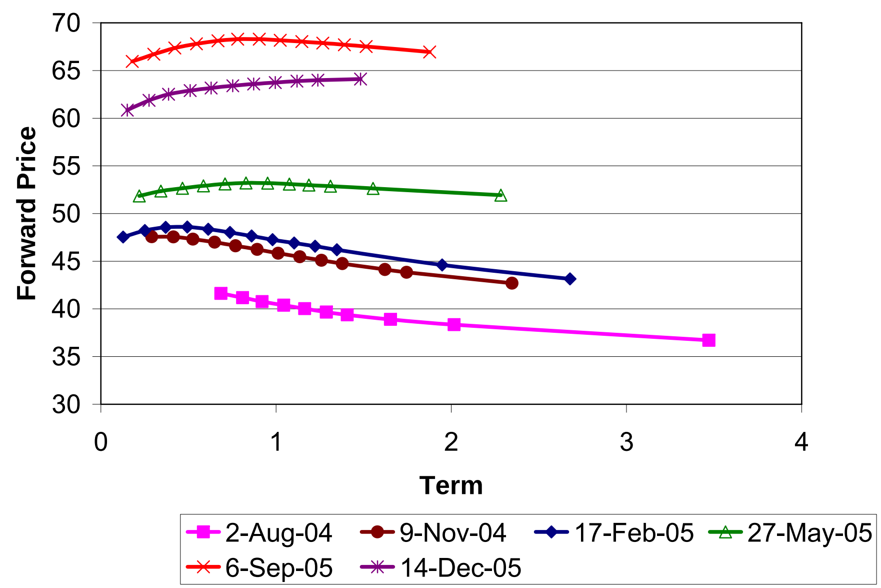
Calibration Results

Calibration

- The data consists of spot prices and forward curves on corresponding days for crude oil



Calibration

- We calibrate the model to both data sets **simultaneously**
- Computing the **sum of squared errors** of the **forward curve** given the spot prices

$$Sum(t_p, \overline{\Theta}) := \sum_{q=1}^{n_p} \left[\log F^{(i)}_{t_p, T^p_q} - \log F^{(i)*}_{t_p, T^p_q} \right]^2 .$$

Minimizing first w.r.t. to the **hidden process Y**

$$Y^{\#(i)}_{t_p}(\overline{\Theta}) = \frac{\sum_{q=1}^{n_p} \left[\overline{M}^{(i)}_{t_p, T^p_q} \left(\log F^{(i)*}_{t_p, T^p_q} - \overline{U}^{(i)}_{t_p, T^p_q} \right) \right]}{\sum_{q=1}^{n_p} \left[\overline{M}^{(i)}_{t_p, T^p_q} \right]^2} .$$

Calibration

- Use this estimate in the sum of squared errors for each day separately
- **Minimize** over the **remaining parameters**

$$\overline{\Theta}^* := ArgMin_{\overline{\Theta}\in\overline{\Omega}} \sum_{p=1}^{m} \sum_{q=1}^{n_m} \left[\overline{U}^{(i)}_{t_p,T_q^p} + \overline{M}^{(i)}_{t_p,T_q^p} \cdot Y^{\#(i)}_{t_p}(\overline{\Theta}) - \log F^{(i)*}_{t_p,T_q^p} \right]^2 ,$$

- This procedure yields the daily estimates for the hidden process Y and the risk-neutral model parameters
- The **time-series** of X and Y are used to **estimate the real-world** parameters via **regression**

Calibration

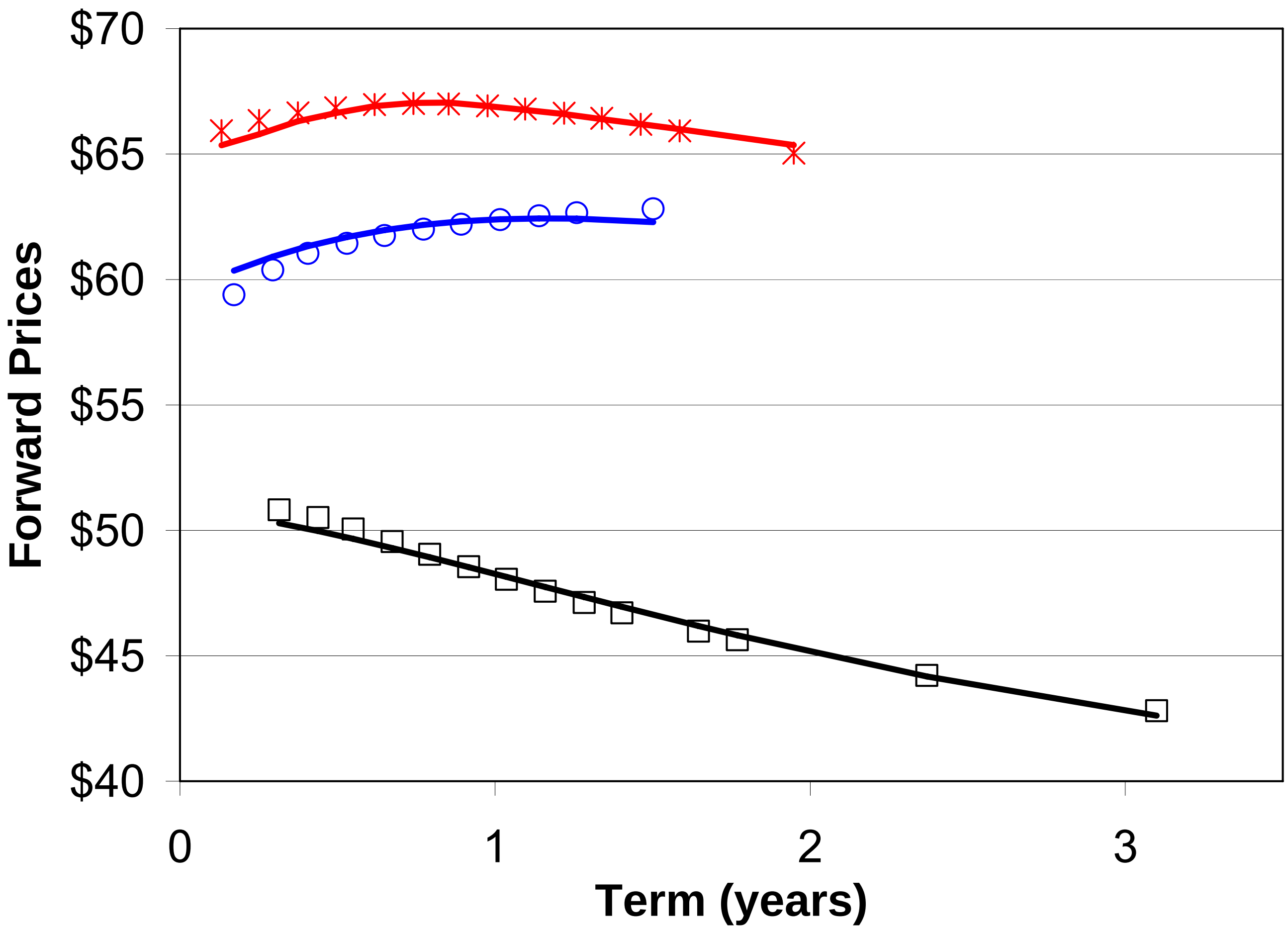
- The calibrated **risk-neutral** model parameters are

β	α	ϕ	σ_X	σ_Y	ρ
0.31	0.15	3.3	33%	63%	-0.96

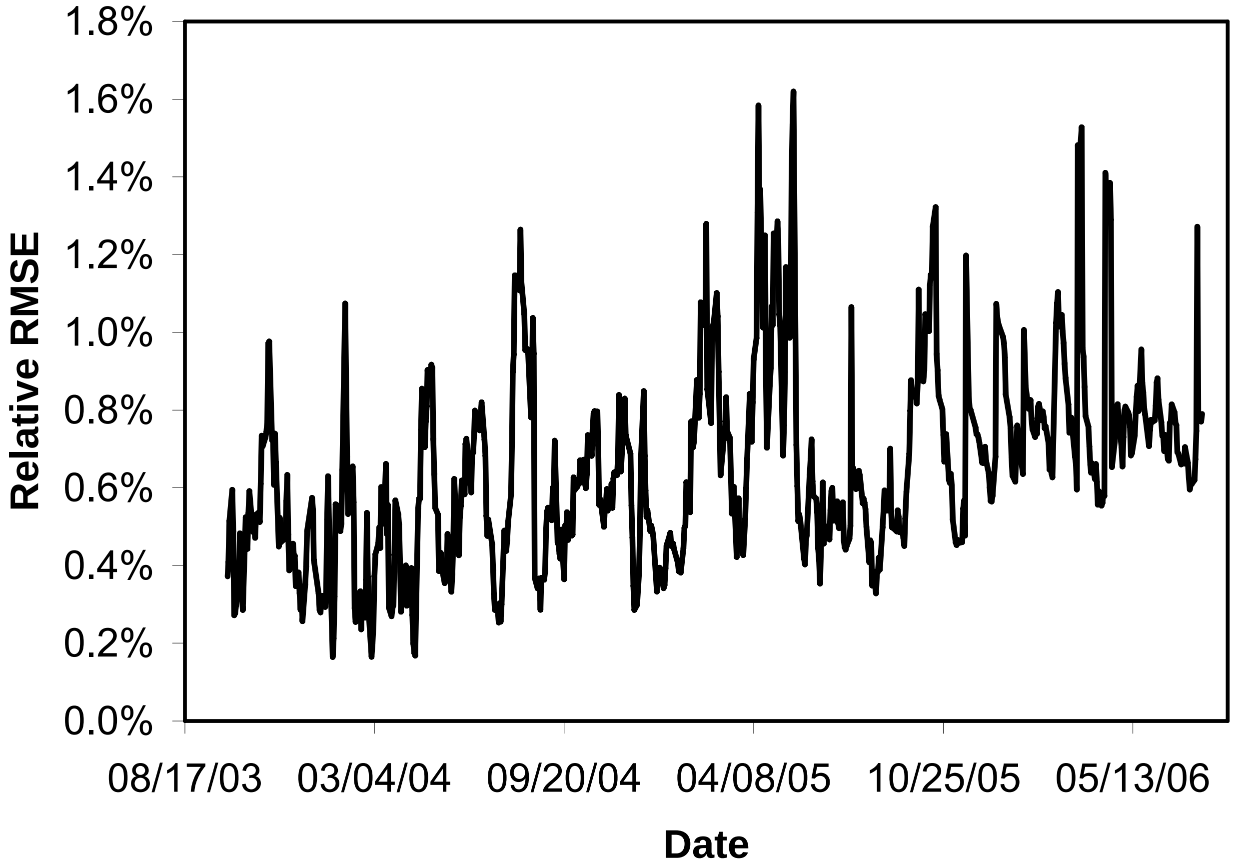
- The calibrated **real-world** model parameters are

β	α	ϕ	σ_X	σ_Y	ρ
1.06	0.73	4.2	33%	63%	-0.96

Calibration



Calibration



Calibration

- We also check the stability of the model parameters through time

# Fwd-Curves	β	α	ϕ	σ_X	σ_Y	ρ
88	0.38	0.26	3.34	33%	19%	-0.97
176	0.52	0.21	3.06	33%	54%	-0.79
264	0.62	0.10	2.36	33%	56%	-0.73
352	0.61	0.08	1.97	35%	60%	-0.64
440	0.52	0.09	2.33	35%	58%	-0.71
528	0.43	0.10	2.98	34%	52%	-0.95
616	0.34	0.13	3.24	34%	58%	-0.96
704	0.31	0.15	3.27	33%	63%	-0.96

Concluding Remarks

- We introduced a **two-factor model** containing mean-reversion to a long run mean-reverting level
  - With and without jumps
- We introduced a third fast mean-reverting **stochastic volatility** into the model
- Obtained forward prices & **spreads on forwards** in closed form
- Future work
  - More thorough model calibration – including calibrating to electricity data using particle filter approaches
  - Adding a slow mean-reverting stochastic volatility

Concluding Remarks

Thank You!