Piecewise Cubic Splines

The fitting of a polynomial curve to a set of data points has applications in CAD (computer-assisted design), CAM (computer-assisted manufacturing), and computer graphics systems. An operator wants to draw a smooth curve through data points that are not subject to error. Traditionally, it was common to use a french curve or an architect's spline and subjectively draw a curve that looks smooth when viewed by the eye. Mathematically, it is possible to construct cubic functions $S_k(x)$ on each interval $[x_k, x_{k+1}]$ so that the resulting piecewise curve $y = S(x)$ and its first and second derivatives are all continuous on the larger interval $[x_0, x_N]$. The continuity of $S'(x)$ means that the graph $y = S(x)$ will not have sharp corners. The continuity of $S''(x)$ means that the *radius of curvature* is defined at each point.

Definition 5.1. Suppose that $\{(x_k, y_k)\}_{k=0}^{N}$ are $N + 1$ points, where $a = x_0 < x_1 < \cdots < x_N = b$. The function $S(x)$ is called a ***cubic spline*** if there exist N cubic polynomials $S_k(x)$ with coefficients $s_{k,0}$, $s_{k,1}$, $s_{k,2}$, and $s_{k,3}$ that satisfy the following properties:

I. $S(x) = S_k(x) = s_{k,0} + s_{k,1}(x - x_k) + s_{k,2}(x - x_k)^2 + s_{k,3}(x - x_k)^3$ for $x \in [x_k, x_{k+1}]$ and $k = 0, 1, \ldots, N - 1$.

II. $S(x_k) = y_k$ for $k = 0, 1, \ldots, N$.

III. $S_k(x_{k+1}) = S_{k+1}(x_{k+1})$ for $k = 0, 1, \ldots, N - 2$.

IV. $S'_k(x_{k+1}) = S'_{k+1}(x_{k+1})$ for $k = 0, 1, \ldots, N - 2$.

V. $S''_k(x_{k+1}) = S''_{k+1}(x_{k+1})$ for $k = 0, 1, \ldots, N - 2$. ▲

Property I states that $S(x)$ consists of piecewise cubics. Property II states that the piecewise cubics interpolate the given set of data points. Properties III and IV require that the piecewise cubics represent a smooth continuous function. Property V states that the second derivative of the resulting function is also continuous.

Existence of Cubic Splines

Let us try to determine if it is possible to construct a cubic spline that satisfies properties I through V. Each cubic polynomial $S_k(x)$ has four unknown constants ($s_{k,0}$, $s_{k,1}$, $s_{k,2}$, and $s_{k,3}$); hence there are $4N$ coefficients to be determined. Loosely speaking, we have $4N$ degrees of freedom or conditions that must be specified. The data points supply $N+1$ conditions, and properties III, IV, and V each supply $N-1$ conditions. Hence, $N+1+3(N-1)=4N-2$ conditions are specified. This leaves us two additional degrees of freedom. We will call them ***endpoint constraints***: they will involve either $S'(x)$ or $S''(x)$ at x_0 and x_N and will be discussed later. We now proceed with the construction.

Since $S(x)$ is piecewise cubic, its second derivative $S''(x)$ is piecewise linear on $[x_0, x_N]$. The linear Lagrange interpolation formula gives the following representation for $S''(x) = S_k''(x)$:

$$S_k''(x) = S''(x_k)\frac{x-x_{k+1}}{x_k-x_{k+1}} + S''(x_{k+1})\frac{x-x_k}{x_{k+1}-x_k}. \tag{4}$$

Use $m_k = S''(x_k)$, $m_{k+1} = S''(x_{k+1})$, and $h_k = x_{k+1} - x_k$ in (4) to get

$$S_k''(x) = \frac{m_k}{h_k}(x_{k+1}-x) + \frac{m_{k+1}}{h_k}(x-x_k) \tag{5}$$

for $x_k \le x \le x_{k+1}$ and $k = 0, 1, \ldots, N-1$. Integrating (5) twice will introduce two constants of integration, and the result can be manipulated so that it has the form

$$S_k(x) = \frac{m_k}{6h_k}(x_{k+1}-x)^3 + \frac{m_{k+1}}{6h_k}(x-x_k)^3 \quad + p_k(x_{k+1}-x) + q_k(x-x_k). \tag{6}$$

Substituting x_k and x_{k+1} into equation (6) and using the values $y_k = S_k(x_k)$ and $y_{k+1} = S_k(x_{k+1})$ yields the following equations that involve p_k and q_k, respectively:

$$y_k = \frac{m_k}{6}h_k^2 + p_k h_k \qquad \text{and} \qquad y_{k+1} = \frac{m_{k+1}}{6}h_k^2 + q_k h_k. \tag{7}$$

These two equations are easily solved for p_k and q_k, and when these values are substituted into equation (6), the result is the following expression for the cubic function $S_k(x)$:

$$\begin{aligned} S_k(x) = {} & -\frac{m_k}{6h_k}(x_{k+1}-x)^3 + \frac{m_{k+1}}{6h_k}(x-x_k)^3 \\ & + \left(\frac{y_k}{h_k} - \frac{m_k h_k}{6}\right)(x_{k+1}-x) + \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1}h_k}{6}\right)(x-x_k). \end{aligned} \tag{8}$$

Notice that the representation (8) has been reduced to a form that involves only the unknown coefficients $\{m_k\}$. To find these values, we must use the derivative of (8), which is

$$S'_k(x) = -\frac{m_k}{2h_k}(x_{k+1} - x)^2 + \frac{m_{k+1}}{2h_k}(x - x_k)^2 - \left(\frac{y_k}{h_k} - \frac{m_k h_k}{6}\right) + \frac{y_{k+1}}{h_k} - \frac{m_{k+1}h_k}{h_k}. \tag{9}$$

Evaluating (9) at x_k and simplifying the result yield

$$S'_k(x_k) = -\frac{m_k}{3}h_k - \frac{m_{k+1}}{6}h_k + d_k, \qquad \text{where } d_k = \frac{y_{k+1} - y_k}{h_k}. \tag{10}$$

Similarly, we can replace k by $k-1$ in (9) to get the expression for $S'_{k-1}(x)$ and evaluate it at x_k to obtain

$$S'_{k-1}(x_k) = \frac{m_k}{3}h_{k-1} + \frac{m_{k-1}}{6}h_{k-1} + d_{k-1}. \tag{11}$$

Now use property IV and equations (10) and (11) to obtain an important relation involving m_{k-1}, m_k, and m_{k+1}:

$$h_{k-1}m_{k-1} + 2(h_{k-1} + h_k)m_k + h_k m_{k+1} = u_k, \tag{12}$$

where $u_k = 6\,(d_k - d_{k-1})$ for $k = 1, 2, \ldots, N-1$.

Construction of Cubic Splines

Observe that the unknowns in (12) are the desired values $\{m_k\}$, and the other terms are constants obtained by performing simple arithmetic with the data points $\{(x_k, y_k)\}$. Therefore, in reality, system (12) is an underdetermined system of $N-1$ linear equations involving $N+1$ unknowns. Hence two additional equations must be supplied. They are used to eliminate m_0 from the first equation and m_N from the $(N-1)$st equation in system (12). The standard strategies for the endpoint constraints are summarized in Table 5.8.

Consider strategy (v) in Table 5.8. If m_0 is given, then $h_0 m_0$ can be computed, and the first equation (when $k = 1$) of (12) is

$$2(h_0 + h_1)m_1 + h_1 m_2 = u_1 - h_0 m_0. \tag{13}$$

Similarly, if m_N is given, then $h_{N-1}m_N$ can be computed, and the last equation (when $k = N-1$) of (12) is

$$h_{N-2}m_{N-2} + 2(h_{N-2} + h_{N-1})m_{N-1} = u_{N-1} - h_{N-1}m_N. \tag{14}$$

Table 5.8 Endpoint Constraints for a Cubic Spline

Description of the strategy		Equations involving m_0 and m_N
(i)	*Clamped cubic spline:* specify $S'(x_0)$, $S'(x_n)$ (the "best choice" if the derivatives are known)	$m_0 = \frac{3}{h_0}(d_0 - S'(x_0)) - \frac{m_1}{2}$ $m_N = \frac{3}{h_{N-1}}(S'(x_N) - d_{N-1}) - \frac{m_{N-1}}{2}$
(ii)	*Natural cubic spline* (a "relaxed curve")	$m_0 = 0, m_N = 0$
(iii)	Extrapolate $S''(x)$ to the endpoints	$m_0 = m_1 - \frac{h_0(m_2 - m_1)}{h_1}$, $m_N = m_{N-1} + \frac{h_{N-1}(m_{N-1} - m_{N-2})}{h_{N-2}}$
(iv)	$S''(x)$ is constant near the endpoints	$m_0 = m_1, m_N = m_{N-1}$
(v)	Specify $S''(X)$ at each endpoint	$m_0 = S''(x_0), m_N = S''(x_N)$

Equations (13) and (14) with (12) used for $k = 2, 3, \ldots, N-2$ form $N-1$ linear equations involving the coefficients $m_1, m_2, \ldots, m_{N-1}$.

Regardless of the particular strategy chosen in Table 5.8, we can rewrite equations 1 and $N-1$ in (12) and obtain a tridiagonal linear system of the form $\boldsymbol{HM} = \boldsymbol{V}$, which involves $m_1, m_2, \ldots, m_{N-1}$:

$$
(15) \qquad \begin{bmatrix} b_1 & c_1 & & & & \\ a_1 & b_2 & c_2 & & & \\ & & \ddots & & & \\ & & & a_{N-3} & b_{N-2} & c_{N-2} \\ & & & & a_{N-2} & b_{N-1} \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{N-2} \\ m_{N-1} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-2} \\ v_{N-1} \end{bmatrix}.
$$

The linear system in (15) is strictly diagonally dominant and has a unique solution (see Chapter 3 for details). After the coefficients $\{m_k\}$ are determined, the spline coefficients $\{s_{k,j}\}$ for $S_k(x)$ are computed using the formulas

$$s_{k,0} = y_k, \qquad s_{k,1} = d_k - \frac{h_k(2m_k + m_{k+1})}{6}, \tag{16}$$
$$s_{k,2} = \frac{m_k}{2}, \qquad s_{k,3} = \frac{m_{k+1} - m_k}{6h_k}.$$

Each cubic polynomial $S_k(x)$ can be written in nested multiplication form for efficient computation:

$$S_k(x) = ((s_{k,3}w + s_{k,2})w + s_{k,1})w + y_k, \qquad \text{where } w = x - x_k \tag{17}$$

and $S_k(x)$ is used on the interval $x_k \le x \le x_{k+1}$.

Equations (12) together with a strategy from Table 5.8 can be used to construct a cubic spline with distinctive properties at the endpoints. Specifically, the values for m_0 and m_N in Table 5.8 are used to customize the first and last equations in (12) and form the system of $N - 1$ equations given in (15). Then the tridiagonal system is solved for the remaining coefficients $m_1, m_2, \ldots, m_{N-1}$. Finally, the formulas in (16) are used to determine the spline coefficients. For reference, we now state how the equations must be prepared for each different type of spline.

Endpoint Constraints

The following five lemmas show the form of the tridiagonal linear system that must be solved for each of the different endpoint constraints in Table 5.8.

Lemma 5.1 (Clamped Spline). There exists a unique cubic spline with the first derivative boundary conditions $S'(a) = d_0$ and $S'(b) = d_N$.

Proof. Solve the linear system

$$\left(\frac{3}{2}h_0 + 2h_1\right)m_1 + h_1 m_2 = u_1 - 3(d_0 - S'(x_0))$$
$$h_{k-1}m_{k-1} + 2(h_{k-1} + h_k)m_k + h_k m_{k+1} = u_k \qquad \text{for } k = 2, 3, \ldots, N-2$$
$$h_{N-2}m_{N-2} + \left(2h_{N-2} + \frac{3}{2}h_{N-1}\right)m_{N-1} = u_{N-1} - 3(S'(x_N) - d_{N-1}). \qquad \bullet$$

Remark. The clamped spline involves slope at the ends. This spline can be visualized as the curve obtained when a flexible elastic rod is forced to pass through the data points, and the rod is clamped at each end with a fixed slope. This spline would be useful to a draftsman for drawing a smooth curve through several points.

Lemma 5.2 (Natural Spline). There exists a unique cubic spline with the free boundary conditions $S''(a) = 0$ and $S''(b) = 0$.

Proof. Solve the linear system

$$
\begin{aligned}
2(h_0+h_1)m_1 + h_1 m_2 &= u_1 \\
h_{k-1}m_{k-1} + 2(h_{k-1}+h_k)m_k + h_k m_{k+1} &= u_k \qquad \text{for } k = 2,\ 3,\ \ldots,\ N-2. \\
h_{N-2}m_{N-2} + 2(h_{N-2}+h_{N-1})m_{N-1} &= u_{N-1}.
\end{aligned}
$$
•

Remark. The natural spline is the curve obtained by forcing a flexible elastic rod through the data points but letting the slope at the ends be free to equilibrate to the position that minimizes the oscillatory behavior of the curve. It is useful for fitting a curve to experimental data that are significant to several significant digits.

Lemma 5.3 (Extrapolated Spline). There exists a unique cubic spline that uses extrapolation from the interior nodes at x_1 and x_2 to determine $S''(a)$ and extrapolation from the nodes at x_{N-1} and x_{N-2} to determine $S''(b)$.

Proof. Solve the linear system

$$
\begin{aligned}
&\left(3h_0 + 2h_1 + \frac{h_0^2}{h_1}\right)m_1 + \left(h_1 - \frac{h_0^2}{h_1}\right)m_2 = u_1 \\
&h_{k-1}m_{k-1} + 2(h_{k-1}+h_k)m_k + h_k m_{k+1} = u_k \qquad \text{for } k = 2,\ 3,\ \ldots,\ N-2 \\
&\left(h_{N-2} - \frac{h_{N-1}^2}{h_{N-2}}\right)m_{N-2} + \left(2h_{N-2} + 3h_{N-1} + \frac{h_{N-1}^2}{h_{N-2}}\right)m_{N-1} = u_{N-1}.
\end{aligned}
$$
•

Remark. The extrapolated spline is equivalent to assuming that the end cubic is an extension of the adjacent cubic; that is, the spline forms a single cubic curve over the interval $[x_0, x_2]$ and another single cubic over the interval $[x_{N-2}, x_N]$.

Lemma 5.4 (Parabolically Terminated Spline). There exists a unique cubic spline that uses $S'''(x) \equiv 0$ on the interval $[x_0, x_1]$ and $S'''(x) \equiv 0$ on $[x_{N-1}, x_N]$.

Proof. Solve the linear system

$$
\begin{aligned}
(3h_0+2h_1)m_1 + h_1 m_2 &= u_1 \\
h_{k-1}m_{k-1} + 2(h_{k-1}+h_k)m_k + h_k m_{k+1} &= u_k \qquad \text{for } k = 2,\ 3,\ \ldots,\ N-2 \\
h_{N-2}m_{N-2} + (2h_{N-2}+3h_{N-1})m_{N-1} &= u_{N-1}.
\end{aligned}
$$
•

Remark. The assumption that $S''(x) \equiv 0$ on the interval $[x_0, x_1]$ forces the cubic to degenerate to a quadratic over $[x_0, x_1]$, and a similar situation occurs over $[x_{N-1}, x_N]$.

Lemma 5.5 (Endpoint Curvature-Adjusted Spline). There exists a unique cubic spline with the second derivative boundary conditions $S''(a)$ and $S''(b)$ specified.

Proof. Solve the linear system

$$\begin{aligned}
2(h_0 + h_1)m_1 + h_1 m_2 &= u_1 - h_0 S''(x_0)\\
h_{k-1}m_{k-1} + 2(h_{k-1} + h_k)m_k + h_k m_{k+1} &= u_k \qquad \text{for } k = 2,\ 3,\ \ldots,\ N-2\\
h_{N-2}m_{N-2} + 2(h_{N-2} + h_{N-1})m_{N-1} &= u_{N-1} - h_{N-1}S''(x_N).
\end{aligned}$$

•

Remark. Imposing values for $S''(a)$ and $S''(b)$ permits the practitioner to adjust the curvature at each endpoint.

The next five examples illustrate the behavior of the various splines. It is possible to mix the end conditions to obtain an even wider variety of possibilities, but we leave these variations to the reader to investigate.

Example 5.7. Find the clamped cubic spline that passes through (0, 0), (1, 0.5), (2, 2.0), and (3, 1.5) with the first derivative boundary conditions $S'(0) = 0.2$ and $S'(3) = -1$.

First, compute the quantities

$$\begin{aligned}
&h_0 = h_1 = h_2 = 1\\
&d_0 = (y_1 - y_0)/h_0 = (0.5 - 0.0)/1 = 0.5\\
&d_1 = (y_2 - y_1)/h_1 = (2.0 - 0.5)/1 = 1.5\\
&d_2 = (y_3 - y_2)/h_2 = (1.5 - 2.0)/1 = -0.5\\
&u_1 = 6(d_1 - d_0) = 6(1.5 - 0.5) = 6.0\\
&u_2 = 6(d_2 - d_1) = 6(-0.5 - 1.5) = -12.0.
\end{aligned}$$

Then use Lemma 5.1 and obtain the equations

$$\begin{aligned}
\left(\frac{3}{2} + 2\right)m_1 + m_2 &= 6.0 - 3(0.5 - 0.2) = 5.1,\\
m_1 + \left(2 + \frac{3}{2}\right)m_2 &= -12.0 - 3(-1.0 - (-0.5)) = -10.5.
\end{aligned}$$

When these equations are simplified and put in matrix notation, we have

$$\begin{bmatrix} 3.5 & 1.0\\ 1.0 & 3.5 \end{bmatrix} \begin{bmatrix} m_1\\ m_2 \end{bmatrix} = \begin{bmatrix} 5.1\\ -10.5 \end{bmatrix}.$$

It is a straightforward task to compute the solution $m_1 = 2.25$ and $m_2 = -3.72$. Now apply the equations in (i) of Table 5.8 to determine the coefficients m_0 and m_3:

$$\begin{aligned}
m_0 &= 3(0.5 - 0.2) - \frac{2.52}{2} = -0.36,\\
m_3 &= 3(-1.0 + 0.5) - \frac{-3.72}{2} = 0.36.
\end{aligned}$$

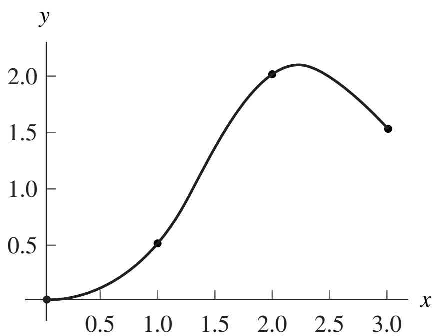

Figure 5.12 The clamped cubic spline with derivative boundary conditions: $S'(0) = 0.2$ and $S'(3) = -1$.

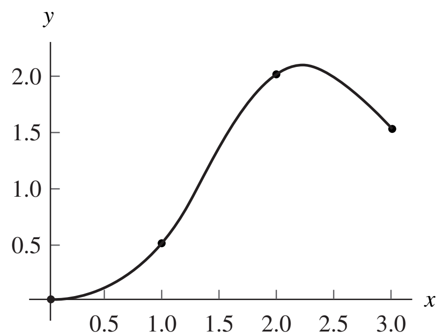

Figure 5.13 The natural cubic spline with $S''(0) = 0$ and $S''(3) = 0$.

Next, the values $m_0 = -0.36$, $m_1 = 2.25$, $m_2 = -3.72$, and $m_3 = 0.36$ are substituted into equations (16) to find the spline coefficients. The solution is

$$
\begin{aligned}
S_0(x) &= 0.48x^3 - 0.18x^2 + 0.2x && \text{for } 0 \le x \le 1,\\
S_1(x) &= -1.04(x-1)^3 + 1.26(x-1)^2 &&\\
&\qquad + 1.28(x-1) + 0.5 && \text{for } 1 \le x \le 2,\\
S_2(x) &= 0.68(x-2)^3 - 1.86(x-2)^2 &&\\
&\qquad + 0.68(x-2) + 2.0 && \text{for } 2 \le x \le 3.
\end{aligned}
\tag{18}
$$

This clamped cubic spline is shown in Figure 5.12. ■

Example 5.8. Find the natural cubic spline that passes through (0, 0.0), (1, 0.5), (2, 2.0), and (3, 1.5) with the free boundary conditions $S''(x) = 0$ and $S''(3) = 0$.

Use the same values $\{h_k\}$, $\{d_k\}$, and $\{u_k\}$ that were computed in Example 5.7. Then use Lemma 5.2 and obtain the equations

$$
\begin{aligned}
2(1+1)m_1 + m_2 &= 6.0,\\
m_1 + 2(1+1)m_2 &= -12.0.
\end{aligned}
$$

The matrix form of this linear system is

$$
\begin{bmatrix} 4.0 & 1.0 \\ 1.0 & 4.0 \end{bmatrix}
\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}
=
\begin{bmatrix} 6.0 \\ -12.0 \end{bmatrix}.
$$

It is easy to find the solution $m_1 = 2.4$ and $m_2 = -3.6$. Since $m_0 = S''(0) = 0$ and

$m_3 = S''(3) = 0$, when equations (16) are used to find the spline coefficients, the result is

$$
\begin{aligned}
S_0(x) &= 0.4x^3 + 0.1x && \text{for } 0 \le x \le 1,\\
S_1(x) &= -(x-1)^3 + 1.2(x-1)^2 &&\\
&\quad + 1.3(x-1) + 0.5 && \text{for } 1 \le x \le 2,\\
S_2(x) &= 0.6(x-2)^3 - 1.8(x-2)^2 &&\\
&\quad + 0.7(x-2) + 2.0 && \text{for } 2 \le x \le 3.
\end{aligned} \tag{19}
$$

This natural cubic spline is shown in Figure 5.13. ■

Example 5.9. Find the extrapolated cubic spline through (0, 0.0), (1, 0.5), (2, 2.0), and (3, 1.5).

Use the values $\{h_k\}$, $\{d_k\}$, and $\{u_k\}$ from Example 5.7 with Lemma 5.3 and obtain the linear system

$$
\begin{aligned}
(3+2+1)m_1 + (1-1)m_2 &= 6.0,\\
(1-1)m_1 + (2+3+1)m_2 &= -12.0.
\end{aligned}
$$

The matrix form is

$$
\begin{bmatrix} 6.0 & 0.0 \\ 0.0 & 6.0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 6.0 \\ -12.0 \end{bmatrix},
$$

and it is trivial to obtain $m_1 = 1.0$ and $m_2 = -2.0$. Now apply the equations in (iii) of Table 5.8 to compute m_0 and m_3:

$$
\begin{aligned}
m_0 &= 1.0 - (-2.0 - 1.0) = 4.0,\\
m_3 &= -2.0 + (-2.0 - 1.0) = -5.0.
\end{aligned}
$$

Finally, the values for $\{m_k\}$ are substituted in equations (16) to find the spline coefficients. The solution is

$$
\begin{aligned}
S_0(x) &= -0.5x^3 + 2.0x^2 - x && \text{for } 0 \le x \le 1,\\
S_1(x) &= -0.5(x-1)^3 + 0.5(x-1)^2 &&\\
&\quad + 1.5(x-1) + 0.5 && \text{for } 1 \le x \le 2,\\
S_2(x) &= -0.5(x-2)^3 - (x-2)^2 &&\\
&\quad + (x-2) + 2.0 && \text{for } 2 \le x \le 3.
\end{aligned} \tag{20}
$$

The extrapolated cubic spline is shown in Figure 5.14. ■

Example 5.10. Find the parabolically terminated cubic spline through (0, 0.0), (1, 0.5), (2, 2.0), and (3, 1.5).

Use $\{h_k\}$, $\{d_k\}$, and $\{u_k\}$ from Example 5.7 and then apply Lemma 5.4 to obtain

$$
\begin{aligned}
(3+2)m_1 + m_2 &= 6.0,\\
m_1 + (2+3)m_2 &= -12.0.
\end{aligned}
$$

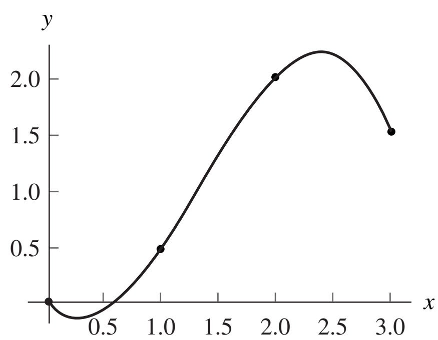

Figure 5.14 The extrapolated cubic spline.

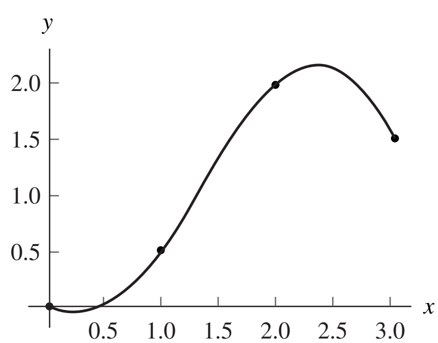

Figure 5.15 The parabolically terminated cubic spline.

The matrix form is

$$\begin{bmatrix} 5.0 & 1.0 \\ 1.0 & 5.0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 6.0 \\ -12.0 \end{bmatrix},$$

and the solution is $m_1 = 1.75$ and $m_2 = -2.75$. Since $S''(x) \equiv 0$ on the subinterval at each end, formulas (iv) in Table 5.8 imply that we have $m_0 = m_1 = 1.75$ and $m_3 = m_2 = -2.75$. Then the values for $\{m_k\}$ are substituted in equations (16) to get the solution

$$\begin{aligned} S_0(x) &= 0.875x^2 - 0.375x && \text{for } 0 \le x \le 1, \\ S_1(x) &= -0.75(x-1)^3 + 0.875(x-1)^2 \\ &\quad + 1.375(x-1) + 0.5 && \text{for } 1 \le x \le 2, \\ S_2(x) &= -1.375(x-2)^2 + 0.875(x-2) + 2.0 && \text{for } 2 \le x \le 3. \end{aligned} \tag{21}$$

This parabolically terminated cubic spline is shown in Figure 5.15. ■

Example 5.11. Find the curvature-adjusted cubic spline through $(0, 0.0)$, $(1, 0.5)$, $(2, 2.0)$, and $(3, 1.5)$ with the second derivative boundary conditions $S''(0) = -0.3$ and $S''(3) = 3.3$.

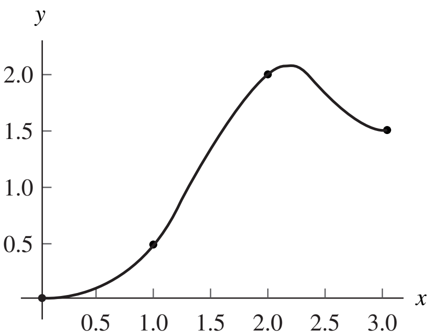

Figure 5.16 The curvature adjusted cubic spline with $S''(0) = -0.3$ and $S''(3) = 3.3$.

Use $\{h_k\}$, $\{d_k\}$, and $\{u_k\}$ from Example 5.7 and then apply Lemma 5.5 to obtain

$$\begin{aligned} 2(1+1)m_1 + m_2 &= 6.0 - (-0.3) = 6.3, \\ m_1 + 2(1+1)m_2 &= -12.0 - (3.3) = -15.3. \end{aligned}$$

The matrix form is

$$\begin{bmatrix} 4.0 & 1.0 \\ 1.0 & 4.0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 6.3 \\ -15.3 \end{bmatrix},$$

and the solution is $m_1 = 2.7$ and $m_2 = -4.5$. The given boundary conditions are used to determine $m_0 = S''(0) = -0.3$ and $m_3 = S''(3) = 3.3$. Substitution of $\{m_k\}$ in equations (16) produces the solution

$$\begin{aligned} S_0(x) &= 0.5x^3 - 0.15x^2 + 0.15x && \text{for } 0 \le x \le 1, \\ S_1(x) &= -1.2(x-1)^3 + 1.35(x-1)^2 \\ &\quad + 1.35(x-1) + 0.5 && \text{for } 1 \le x \le 2, \\ S_2(x) &= 1.3(x-2)^3 - 2.25(x-2)^2 \\ &\quad + 0.45(x-2) + 2.0 && \text{for } 2 \le x \le 3. \end{aligned} \tag{22}$$

This curvature-adjusted cubic spline is shown in Figure 5.16. ∎

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