

# Stochastic epidemic models with inference

## Exercise Session 1

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## Exercise 1.1

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# Final Size of Outbreak

## Final Size Equation

$$1 - \tau = e^{-R_0\tau}.$$

Note: This equation only gives the final fraction infected among the **initially susceptible** individuals.

- There is always a solution  $\tau = 0$ .
- If  $R_0 > 1$ , there exists a second solution  $\tau^* > 0$ .
- Final size  $\tau$  shall be the largest solution on  $[0, 1]$ .

# Plot of final size as a function of $R_0$

## Procedure in R:

- Set a function of  $R_0$  solving for  $\tau$  numerically, return  $\tau = \tau(R_0)$ .
- Create a vector of 10000  $R_0$  values in  $[0, 5]$ .
- Create a vector of corresponding values of  $\tau$ .
- Plot  $\tau$  against  $R_0$ .

## Now assuming initial fraction of immune

If there is a fraction  $r$  of **initially immune**, then there is fraction  $(1 - r)$  of initially susceptibles. Then the final size among **initially susceptibles**  $\tau^*$  solves

$$1 - \tau = e^{-R_0(1-r)\tau}.$$

Then the **overall** fraction infected shall be  $\tau^*(1 - r)$ .

## Exercise 1.2

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# Markovian SIR Epidemic Model

- Consider a closed and homogeneous mixing population with fixed size  $N$ .
- At any time point, each individual is susceptible, infectious or recovered.
- The rate of infectious contacts is  $\beta$ , so the rate at which one infectious has contact with a specific other individual is  $\beta/N$ .
- Once infected, one remains to be infectious for a period  $I \sim \text{Exp}(\gamma)$ , after which one becomes recovered and immune.

# Markovian SIR Epidemic Model

Let  $S(t), I(t), R(t)$  be the number of individuals in different states  $S, I, R$  at time  $t$  respectively.

## Two types of events:

- $S \rightarrow I$  : a susceptible gets infected.
- $I \rightarrow R$  : a infectious individual recovers.

## The corresponding rates:

- $\frac{\beta}{N} S(t) I(t)$
- $\gamma I(t)$

**Note:** at any time  $t$ ,  $S(t) + I(t) + R(t) = N$ .



# Deterministic SIR Epidemic Model

SIR differential equation system

$$\begin{aligned}\frac{dS(t)}{dt} &= -\frac{\beta}{N}S(t)I(t), \\ \frac{dI(t)}{dt} &= \frac{\beta}{N}S(t)I(t) - \gamma I(t).\end{aligned}$$

Initial Conditions

$$S(0) = N - 1, I(0) = 1.$$

# Numerical Solution of the SIR ODE

- Define the function to compute derivatives ( $dS(t)/dt, dI(t)/dt$ ) for the SIR ODE.

```
1 gamma <- 0.25
2 beta <- 0.75
3 deter$_sir <- function(t,y, parms) {
4   beta <- parms[1]
5   gamma <- parms[2]
6   N <- parms[3]
7   S <- y[1]
8   I <- y[2]
9   return(list(c(S=...,
10               I=...,
11               )))
12 }
```

# Numerical Solution of the SIR ODE

- Solve the SIR differential equation system with initial conditions (Use `deSolve::lsoda`):

```
1 lsoda(y= ..., \#initial conditions
2 times= ....., \#times at which explicit estimates
   for y are desired
3 func= ..., \#an R-function that computes the
   values of derivatives in the ODE
4 parms= ... \#vector or list of parameters used
   in func
5 )
```

# Stochastic SIR Epidemic Model

Described as a continuous-time Markov process:

Events	Transition	Rates
Infection	$(S(t), I(t)) \rightarrow (S(t) - 1, I(t) + 1)$	$\frac{\beta}{N} S(t) I(t)$
Recovery	$(S(t), I(t)) \rightarrow (S(t), I(t) - 1)$	$\gamma I(t)$

Once  $I(t) = 0$ , the epidemic stops.

**Algorithm to decide which event occurs first:**

- From those two rates, we draw two exponential random numbers for each possible event. `rexp(...)`
- Determine the event with the smaller random number. `which.min(...)`
- Record the event time and update the number of S and I according to the event type.

## Exercise 1.3

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# Markovian SEIR Epidemic Model

- Consider a closed and homogeneous mixing population with fixed size  $N$ .
- The rate of infectious contacts is  $\beta$ , so the rate at which one infectious has contact with a specific other individual is  $\beta/N$ .
- Individuals that get infected are first latent(exposed) for a exponentially distributed period with mean  $1/\rho$ , then they become infectious for a random duration  $I \sim \text{Exp}(\gamma)$ , after which they become recovered and immune.

# Markovian SEIR Epidemic Model

Let  $S(t), E(t), I(t), R(t)$  be the number of individuals in four states  $S, E, I, R$  at time  $t$  respectively. Assume that  $S(0) = N - 1, E(0) = 0, I(0) = 1, R(0) = 1$ .

## Three types of events:

- $S \rightarrow E$  : a susceptible gets infected.
- $E \rightarrow I$  : an exposed individual become infectious.
- $I \rightarrow R$  : a infectious individual recovers.

## The corresponding rates:

# Markovian SEIR Epidemic Model

Let  $S(t), E(t), I(t), R(t)$  be the number of individuals in four states  $S, E, I, R$  at time  $t$  respectively. Assume that  $S(0) = N - 1, E(0) = 0, I(0) = 1, R(0) = 1$ .

## Three types of events:

- $S \rightarrow E$  : a susceptible gets infected.
- $E \rightarrow I$  : an exposed individual become infectious.
- $I \rightarrow R$  : a infectious individual recovers.

## The corresponding rates:

- $\frac{\beta}{N}S(t)I(t)$
- $\rho E(t)$
- $\gamma I(t)$



# Deterministic SEIR Epidemic Model

## SEIR differential equation system

$$\begin{aligned}\frac{\partial S(t)}{\partial t} &= -\frac{\beta}{N}S(t)I(t), \\ \frac{\partial E(t)}{\partial t} &= \frac{\beta}{N}S(t)I(t) - \rho E(t), \\ \frac{\partial I(t)}{\partial t} &= \rho E(t) - \gamma I(t).\end{aligned}$$

## Initial Conditions

$$S(0) = N - 1, E(0) = 0, I(0) = 1, R(0) = 0.$$