SISMID Stochastic Epidemic Models with Inference – Exercise Session 1

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Exercise 1.1 (Final Size of Outbreak)

(a)

Solve the final size equation $1-\tau = \exp(-R_0\tau)$ numerically (with the largest solution for $\tau \in [0,1]$) as a function of R_0 and plot the function for $R_0 \in [0,5]$.

(b)

Now suppose there is a fraction r of initially immune, then the final fraction infected among the initially susceptible, solves $1-\tau = \exp(-R_0(1-r)\tau)$. As in part (a), plot the **overall** fraction infected, as the function of R_0 in $[0,5]$ for $r = 30\%, 50\%, 70\%$.

Exercise 1.2 (Deterministic and Stochastic SIR Model)

(a)

Consider a continuous-time deterministic **SIR** model in a closed and homogeneous mixing population of size $N = 10\ 000$, with rate of contact $\beta = 0.75$ and rate of recovery $\gamma = 0.25$. Let $S(t)$, $I(t)$ and $R(t)$ be the number of susceptibles, infectives and recovered respectively, so we have $S(t) + I(t) + R(t) = N$ at all times. In this model, individuals can only make two moves: from S to I and from I to R. If there is a new infection, S is reduced by one and I is increased by one. I is decreased by one, when there is a recovery. The epidemic stops when there is no infectives. Since we have

$S(t) + I(t) + R(t) = N$ all the times, it is actually sufficient to keep track of $S(t)$ and $I(t)$. Then we have the following SIR differential equation system:

$$\begin{cases} \frac{dS(t)}{dt} = -\frac{\beta}{N} * S(t) * I(t), \\ \\ \frac{dI(t)}{dt} = \frac{\beta}{N} * S(t) * I(t) - \gamma * I(t). \end{cases}$$

with initial conditions: $S(0) = N - 1, I(0) = 1$. Solve the above ODE using R command:

```
library(deSolve)
lsoda(y= ..., #initial conditions
times= ...., #times at which explicit estimates for y are desired
func= ..., #an R-function that computes the values of derivatives
in the ODE
parms= ...  #vector or list of parameters used in func)
```

Moreover, plot the curves of $S(t), I(t)$ and $R(t)$ over time $t \in [0, 100]$.

(b)

Now fix $\gamma = 0.25$, but choose different values of $\beta = 1, 0.75$ and 0.25. In each case, solve the SIR differential equation system with initial conditions: $S(0) = N - 1, I(0) = 1$. Plot the curves of $I(t)$ over time $t \in [0, 100]$. and compare them.

(c)

Take $\beta = 0.75$ and $\gamma = 0.25$ (implying that $R_0 = 3$). There is one way to reduce R_0, which is reducing the number of contacts made by individuals, i.e. reducing β. We pursue a very simple strategy, where the rate β depends on time when different measures take place. Within some time between t_1 and t_2, there are large reduction of contacts, and then the control measures(e.g. social distancing interventions) are slightly relaxed. To be more precise, we have

$$\beta(t) = \begin{cases} \beta_0 \text{ ,if } t \leq t_1, \\ \beta_1 \text{ ,if } t_1 < t \leq t_2, \\ \beta_2 \text{ ,if } t > t_2, \end{cases}$$

with β_0 the ordinary contact rate, $\beta_1 < \beta_2 < \beta_0$. Here we use $\beta_1 = r_1\beta_0$ and $\beta_2 = r_2\beta_0$ with $r_1 \leq r_2$. Take $r_1 = 0.65, r_2 = 0.75, t_1 = 14(\text{days}), t_2 =$

28(days). Assuming that size $N = 10\ 000$ and there is one initial infective, plot the deterministic curve of $I(t)$ over time $t \in [0, 100]$. Compare it with the one in standard deterministic SIR model with $\beta = 0.75$ and $\gamma = 0.25$.

(d)

Here we turn our focus to simulate a Markovian stochastic SIR model with population size N. Assume that there are fraction $c = 10\%$ of initial infectives. There are two possible events: one is from S to I, which occurs at rate $\frac{\beta}{N} * S(t) * I(t)$ and another is from I to R, which occurs at rate $\gamma * I(t)$. The algorithm to decide which event occurs first is as follows. From those two rates, we draw two exponential random numbers for each possible event. Then determine the event with the smaller random number. Finally, record the event time and update the number of S and I according to the event type. Take $\beta = 0.75$ and $\gamma = 0.25$(implying that $R_0 = 3$). Plot $I(t)$ for one (typical) simulated stochastic epidemic and deterministic limit over $t \in [0, 50]$, for different size of population $N = 100, 1000$ and $10\ 000$.

(e)

Let $\beta = 0.375$ and $\gamma = 0.25$ (implying that $R_0 = 1.5$), do 5000 stochastic simulations in three cases when the size of population $N = 500, 1000$ and 5000 with one initial infective. In each simulation, keep track of the final size, i.e. the number of individuals who have been infected at the end of epidemic. Make a histogram of the final size in each case. Give your comments.

Exercise 1.3 (SEIR Model with fixed and time-varying transmission rate)

(a)

In this exercise, we consider the **SEIR** (Susceptible→Exposed→Infectious →Recovered) model in a closed population with size $N = 100$, rate of contact $\beta = 0.4$, rate of recovery $\gamma = 1/7$ and the rate for the E $\rightarrow$ I transition is $\rho = 1/5$, implying a latency period with mean 5 days. Write up the ordinary differential equation system for the above SEIR model. Assume that there is one initial infective, i.e. $I(0) = 1$ and find a numerical approximation for

$I(t), S(t)$ and $E(t)$ using R command `lsoda`. Show a plot of $S(t), E(t)$ and $I(t)$ for time $t \in [0, 100]$.

(b)

Now we modify the SEIR model such that $\beta(t)$ becomes a time-dependent function, which is β_0 until time $t_1 - w$, is β_1 after time $t_1 + w$, and changes linearly from β_0 to β_1 between $t_1 - w$ and $t_1 + w$. The we have the full $\beta(t)$ as follows.

$$\beta(t) = \begin{cases} \beta_0 \text{ ,if } t \leq t_1 - w, \\ \beta_0 + \frac{\beta_1 - \beta_0}{2w} t - \frac{\beta_1 - \beta_0}{2w}(t_1 - w) \text{ ,if } t_1 - w < t \leq t_1 + w, \\ \beta_1 \text{ ,if } t > t_1 + w. \end{cases}$$

Take $t_1 = 30, w = 5, \beta_0 = 0.4$ and $\beta_1 = 0.12$, solve the ODE system for the SEIR model with time-varying transmission rate in part (b) numerically using R command `lsoda` and plot $S(t), E(t)$ and $I(t)$ for $t \in [0, 100]$.

(c)

Now for $N = 100, 1000$ and 10 000, do one simulation of the stochastic SEIR epidemic starting from fraction infected with exponentially distributed incubation period with mean 5 days and the above time-changing rate $\beta(t)$. Overlay it on the plot of the deterministic curve for $I(t)$ for $t \in [0, 100]$ as done in Exercise 1.2.