CSC 2240H : Graphs, Matrices, and Optimization Date: 23 Dec 2022

Higher Order Cheeger's Inequalities

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1 Introduction

Recall that given a weighted undirected graph $G = (V, E, w)$ and a subset of vertices $S \subset V$, the conductance of S is defined as

$$\phi(S) \coloneqq \frac{\sum_{u\in S, v\in \overline{S}} w(u,v)}{\min(\mathsf{vol}(S), \mathsf{vol}(\overline{S}))} \tag{1}$$

where $\mathsf{vol}(S) \coloneqq \sum_{v\in S} w(v)$. The conductance of the graph is defined as the minimum conductance over all cuts

$$\phi(G) \coloneqq \min_{S\subset G} \phi(S). \tag{2}$$

The Laplacian is defined as $L \coloneqq D - A$ and the Normalized Laplacian is defined as $\mathcal{L} \coloneqq I - D^{-1/2}AD^{-1/2}$. Finally, let $0 = \lambda_1 \le \lambda_2 \le ... \le \lambda_n \le 2$ represent the eigenvalues of $\mathcal{L}$.

The original Cheeger's inequality states that

$$\frac{\lambda_2}{2} \le \phi(G) \le \sqrt{2\lambda_2}. \tag{3}$$

In this lecture, we look at improving the upper bound by utilizing higher order eigenvalues. Formally, we will prove the following theorem.

Theorem 1.1. *For every undirected graph G and every $k \ge 2$,*

$$\phi(G) = O(k)\frac{\lambda_2}{\sqrt{\lambda_k}}. \tag{4}$$

This inequality is sharp in general as it can be shown that on the circle graph $\phi(G) = \Omega(k\lambda_2/\sqrt{\lambda_k})$.

In the following, we will begin by introducing several definitions and lemmas that will be central to the argument. Then, we will present the proof of Theorem 1.1, which will be divided into three steps.

2 Preliminaries

Given a function $f : V \to \mathbb{R}$, we will introduce several useful definitions. First, we establish a notion of cuts associated to f with different threshold values. Given a threshold $t \in \mathbb{R}$, we define the threshold set to be $V_f(t) \coloneqq \{v \in V : f(v) \ge t\}$. The conductance of f is then defined to be the minimum of the conductance of its threshold sets, i.e. $\phi(f) \coloneqq \min_t \phi(V_f(t))$.

For clarity of presentation, we let $\ell^2(V)$ and $\ell^2(V,w)$ be Hilbert spaces of functions $f : V \to \mathbb{R}$ with the usual dot product (denoted by $\langle .,. \rangle$) and $\langle f, g \rangle_w = \sum_v w(v)f(v)g(v)$ respectively. We also define the w-norm as $\|f\|_w = \sqrt{\langle f, f \rangle_w}$. An object that helps us relate properties of f to the eigenspectrum of $\mathcal{L}$ is the Rayleigh quotient. In this note, we define the Rayleigh quotient of f to be

$$\mathcal{R}(f) \coloneqq \frac{\langle f, Lf \rangle}{\langle f, Df \rangle} = \frac{\left\langle D^{\frac{1}{2}} f, \mathcal{L} D^{\frac{1}{2}} f \right\rangle}{\|f\|_w^2}. \tag{5}$$

The precise relationship between the eigenspectrum of $\mathcal{L}$ and $\mathcal{R}(f)$ is characterized by the following min-max theorem.

Lemma 2.1. *Let $f_1, ..., f_k$ be orthogonal in $\ell^2(V,w)$, then*

$$\lambda_k = \min_{f_1,\ldots,f_k \in \ell^2(V,w)} \max_{f \in \operatorname{span}(f_1,\ldots f_k)} \mathcal{R}(f). \tag{6}$$

Using the above principle, we will prove the following lemma which will be useful for introducing λ_k in our analysis.

Lemma 2.2. *For any k disjointly supported $f_1, ..., f_k$,*

$$\lambda_k \le 2 \max_{1 \le i \le k} \mathcal{R}(f_i).$$

Proof. Notice that since $f_1, \ldots, f_k$ are disjointly supported, they are orthogonal in $\ell^2(V,w)$. Hence from Lemma 2.1, it suffices to show for any $h \in \operatorname{span}(f_1, \ldots, f_k)$ we have $\mathcal{R}(h) \le 2 \max_i \mathcal{R}(f_i)$. Let $h = \sum_{i=1}^k c_i f_i$. Since $f_1, \ldots, f_k$ are disjointly supported, for any $u, v \in V$,

$$\left|h(u) - h(v)\right|^2 \le 2 \sum_{i=1}^k c_i^2 (f_i(u) - f_i(v))^2,$$

where the inequality holds since at most two of the terms in the difference are non-zero. Moreover,

$$\begin{aligned}
\mathcal{R}(h) = \frac{\sum_{\{u,v\} \in E} w(u,v) \left|h(u) - h(v)\right|^2}{\|h\|_w^2} &\le \frac{\sum_{\{u,v\} \in E} \sum_{i=1}^k 2c_i^2 (f_i(u) - f_i(v))^2}{\|h\|_w^2} \\
&= \frac{2 \sum_{i=1}^k c_i^2 \sum_{\{u,v\} \in E} w(u,v)(f_i(u) - f_i(v))^2}{\sum_{i=1}^k c_i^2 \|f_i\|_w^2} \\
&= \frac{2 \sum_{i=1}^k c_i^2 \|f_i\|_w^2 \mathcal{R}(f_i)}{\sum_{i=1}^k c_i^2 \|f_i\|_w^2} \le 2 \max_{1 \le i \le k} \mathcal{R}(f_i).
\end{aligned}$$

□

Finally, we introduce the following lemma which allows us to upper bound the conductance of a function when we can compute certain estimates of that function.

Lemma 2.3. *For every non-negative $h : V \to \mathbb{R}$ with $\mathsf{vol}(\operatorname{supp}(h)) \le \mathsf{vol}(V)/2$,*

$$\phi(h) \le \frac{\sum_{\{u,v\} \in E} w(u,v) \left|h(u) - h(v)\right|}{\sum_v w(v) h(v)} \tag{7}$$

Proof. We begin by normalizing h such that $\max_v h(v) \leq 1$. Let t be a uniform random variable drawn from $[0,1]$. A simple computation shows that $\mathbb{E}[\mathbf{1}(v \in V_h(t))] = h(v)$ and assuming $h(v) \geq h(u)$, $\mathbb{E}[\mathbf{1}(v \in V_h(t))\mathbf{1}(u \in \overline{V_h(t)})] = h(v) - h(u)$. Furthermore, recall from the Cheeger's inequality lecture notes that for any random variable t and resulting random variables X_t and $Y_t > 0$ we have

$$\min_t \frac{X_t}{Y_t} \leq \frac{\mathbb{E}[X_t]}{\mathbb{E}[Y_t]}.$$

Hence,

$$\min_t \frac{\sum_{v \in V_h(t), u \in \overline{V_h(t)}} w(u,v)}{\sum_{v \in V_h(t)} w(v)} \leq \frac{\mathbb{E}\left[\sum_{v \in V_h(t), u \in \overline{V_h(t)}} w(u,v)\right]}{\mathbb{E}\left[\sum_{v \in V_h(t)} w(v)\right]} = \frac{\sum_{\{u,v\} \in E} w(u,v)\left|h(u) - h(v)\right|}{\sum_v w(v)h(v)},$$

Since $\mathsf{vol}(\mathrm{supp}(h)) \leq \mathsf{vol}(V)/2$, the LHS of the first inequality is equal to $\phi(h)$, which completes the proof. $\square$

3 Proof

The proof of the theorem is constructive. We will begin by seeking a "good" function $f : V \to \mathbb{R}$. Next, we will explore the approximation of f using step functions and examine their connection to the higher order eigenvalues of the normalized Laplacian. Finally, we will show how to upper-bound the conductance of a function f using its step-wise approximation. Combining all these steps together, we will conclude that the conductance of f satisfies our theorem. Throughout the proof, we will use the notation $\psi_{t_1,\ldots,t_l}(x) = \arg\min_{t_i} |x - t_i|$.

3.1 Constructing a Function f

We start by considering the second eigenvector of the normalized Laplacian. Let $g \in \ell^2(V)$ be the second eigenvector of $\mathcal{L}$. Furthermore, let $g_+(v) \coloneqq \max(g(v), 0)$ and $g_-(v) \coloneqq \min(g(v), 0)$ be the positive and negative components of g.

Proposition 3.1. *Let $f_+ = D^{-\frac{1}{2}} g_+$ and $f_- = -D^{-\frac{1}{2}} g_-$. Then we have the following inequalities:*

$$\mathcal{R}(f_+) \leq \lambda_2$$
$$\mathcal{R}(f_-) \leq \lambda_2$$

Proof. By definition, we have

$$\mathcal{R}(f_+) = \frac{\left\langle D^{\frac{1}{2}} f_+, \mathcal{L} D^{\frac{1}{2}} f_+ \right\rangle}{\left\langle D^{\frac{1}{2}} f_+, D^{\frac{1}{2}} f_+ \right\rangle} = \frac{\langle g_+, \mathcal{L} g_+ \rangle}{\langle g_+, g_+ \rangle} = \frac{\langle g_+, \mathcal{L} g_+ \rangle}{\|g_+\|^2}.$$

Assuming that $u \in \mathrm{supp}(g_+)$ then

$$(\mathcal{L} g_+)(u) = g_+(u) - \sum_{v:(u,v) \in E} \frac{w(u,v) g_+(v)}{\sqrt{w(u)w(v)}} = g(u) - \sum_{v:(u,v) \in E} \frac{w(u,v) g_+(v)}{\sqrt{w(u)w(v)}}$$

$$
\begin{aligned}
&\le g(u) - \sum_{v:(u,v)\in E} \frac{w(u,v)g(v)}{\sqrt{w(u)w(v)}} \\
&= (\mathcal{L}g)(u) = \lambda_2 \cdot g(u).
\end{aligned}
$$

Therefore

$$
\mathcal{R}(f_+) = \frac{\langle g_+, \mathcal{L}g_+\rangle}{\|g_+\|^2} = \frac{\sum_{u\in\mathrm{supp}(g_+)} g_+(u)\cdot(\mathcal{L}g_+)(u)}{\sum_{u\in\mathrm{supp}(g_+)} g_+(u)^2} \le \frac{\sum_{u\in\mathrm{supp}(g_+)} \lambda_2 \cdot g_+(u)^2}{\sum_{u\in\mathrm{supp}(g_+)} g_+(u)^2} = \lambda_2.
$$

Similarly, we can show that $\mathcal{R}(f_-) \le \lambda_2$. □

Therefore f_+, f_- are two disjointly supported non-negative functions such that $\mathcal{R}(f_+), \mathcal{R}(f_-) \le \lambda_2$. Choosing the one with the smallest support volume and normalizing, we obtain the following corollary.

Corollary 3.2. *There exists $f \in \ell^2(V,w)$ such that*

1. $f \ge 0$
2. $\mathcal{R}(f) \le \lambda_2$
3. $\mathsf{vol}(\mathrm{supp}(f)) \le \frac{\mathsf{vol}(V)}{2}$
4. $\|f\|_w = 1$.

3.2 Approximation with Step Functions and Higher Order Eigenvalues

Definition 3.3. *Given a function $f \in \ell^2(V,w)$, we say $g \in \ell^2(V,w)$ is an l-step approximation of f if there exist $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_{l-1}$ such that*

$$
g(v) = \psi_{t_0,t_1,t_2,\ldots,t_{l-1}}(f(v)).
$$

Next, we show the quantity $\frac{\mathcal{R}(f)}{\lambda_k}$ controls the approximation error of f with at most $O(k)$ steps. Formally, we prove the following lemma:

Lemma 3.4. *Suppose $f \in \ell^2(V,w)$ with $\|f\|_w = 1$. There exists a $2k+1$-step approximation of f, denoted by g, such that*

$$
\|f-g\|_w^2 \le \frac{4\mathcal{R}(f)}{\lambda_k}. \tag{8}
$$

Proof. Let $C := \frac{2\mathcal{R}(f)}{k\lambda_k}$ and $M := \max_v f(v)$. If $0 = t_0, \ldots, t_{2k} \le M$ can be found such that $\sum_{v:f(v)\in[t_{i-1},t_i]} \|f-g\|_w^2 \le C$ for $1 \le i \le 2k$, then

$$
\|f-g\|_w^2 = \sum_{i=1}^{2k} \sum_{v:f(v)\in[t_{i-1},t_i]} w(v) \times |f(v)-g(v)|^2 \le 2kC = \frac{4\mathcal{R}(f)}{\lambda_k}. \tag{9}
$$

Thus, we start from $t_0 = 0$ and recursively construct $(t_i)_{0\le i\le 2k}$. Assuming we have chosen $t_0, \ldots, t_i$, we choose t_{i+1} as the smallest number such that

$$
\sum_{v\,:\,t_i\le f(v)\le t_{i+1}} w(v) \times |f(v) - \psi_{t_i,t_{i+1}}(f(v))|^2 = C. \tag{10}
$$

If there isn't such a smallest number, we let $t_{i+1} = M$. If $t_{2k} = M$ then (9) finishes the proof. Therefore, we suppose on the contrary that $t_{2k} < M$. Then, we can construct $2k$ disjointly supported functions $f_1, \ldots, f_{2k}$ such that at least k of them have Rayleigh quotients less than $\lambda_k/2$, which is a contradiction due to Lemma 2.2.

For $1 \le i \le 2k$, let f_i be the following function:

$$f_i(v) \coloneqq \begin{cases} |f(v) - \psi_{t_{i-1},t_i}(f(v))| & \text{if } t_{i-1} \le f(v) \le t_i \\ 0 & \text{otherwise.} \end{cases}$$

Notice that since $t_{2k} < M$, the identity (10) must hold for all i, hence $\|f_i\|_w^2 = C$. As a result,

$$\sum_{i=1}^{2k} \mathcal{R}(f_i) = \sum_{i=1}^{2k} \frac{\sum_{(u,v)\in E} w(u,v)|f_i(u) - f_i(v)|^2}{\|f_i\|_w^2} = \frac{1}{C} \sum_{(u,v)\in E} w(u,v) \left[\sum_{i=1}^{2k} |f_i(u) - f_i(v)|^2 \right]. \quad (11)$$

We can upper bound the LHS of the equation above by upper bounding $\sum_{i=1}^{2k} |f_i(u) - f_i(v)|^2$. To do so, we will prove the following claim:

Lemma 3.5. *Assuming f_i's are defined as above, for any $u, v \in V$,*

$$\sum_{i=1}^{2k} |f_i(u) - f_i(v)|^2 \le |f(u) - f(v)|^2. \quad (12)$$

We prove this claim by conditioning on whether u and v lie in the support of the same function or 2 different functions. If $u, v \in \mathrm{supp}(f_i)$, the lemma follows by 1-Lipschitzness of $f_i(x)$. If $u \in \mathrm{supp}(f_i), v \in \mathrm{supp}(f_j)$ where $(i < j)$, then we know that $t_{i-1} \le f(u) \le t_i \le t_{j-1} \le f(v) \le t_j$. Given this, we have

$$\begin{aligned} |f_i(u) - f_i(v)|^2 + |f_j(u) - f_j(v)|^2 &= |f_i(u)|^2 + |f_j(v)|^2 \\ &= |f(u) - g(u)|^2 + |f(v) - g(v)|^2 \\ &\le |f(u) - t_i|^2 + |f(v) - t_{j-1}|^2 \\ &\le |f(u) - t_i|^2 + |f(v) - t_i|^2 \\ &\le (|f(u) - t_i| + |f(v) - t_i|)^2 = |f(u) - f(v)|^2 \end{aligned}$$

Plugging this upper bound in (11) yields

$$\sum_{i=1}^{2k} \mathcal{R}(f_i) \le \frac{1}{C} \sum_{(u,v)\in E} w(u,v)(f(u) - f(v))^2 = \frac{\mathcal{R}(f)\, \|f\|_w^2}{\frac{2\mathcal{R}(f)}{k\lambda_k}} = \frac{k\lambda_k}{2}.$$

Hence, at least k of the $2k$ functions have a Rayleigh quotient less than $\frac{\lambda_k}{2}$, which contradicts Lemma 2.2 and completes the proof. □

3.3 Upper Bounding Conductance through Step-wise Approximation

Next, we show that we can use any $2k + 1$-step function g to upper bound $\phi(f)$.

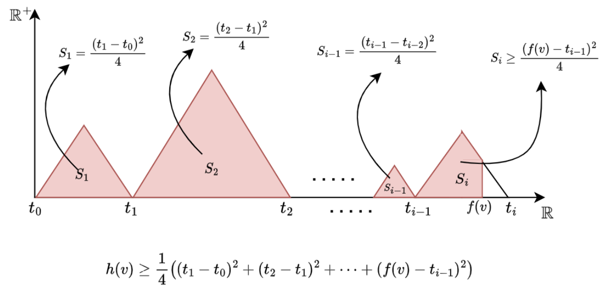

Figure 1: Illustration of $\mu(x)$ and $h(x)$.

Proposition 3.6. *Suppose $f \in \ell^2(V, w)$ with $\|f\|_w = 1$, and let g be any $2k+1$-step approximation of f. Then,*

$$\phi(f) \leq 4k\mathcal{R}(f) + 4k\sqrt{2\mathcal{R}(f)}\,\|f - g\|_w\,. \tag{13}$$

Proof. Using g, we will construct another function $h \in \ell^2(V, w)$ which has the same threshold sets as f, thus $\phi(f) = \phi(h)$, and which we can invoke Lemma 2.3 to upper bound its conductance.

Let $0 = t_0 \leq t_1 \leq \ldots \leq t_{2k}$ be the values of g, i.e. $g(v) = \psi_{t_0,\ldots,t_{2k}}(f(v))$. Define $\mu : \mathbb{R} \to \mathbb{R}$ as

$$\mu(x) = \left|x - \psi_{t_0,\ldots,t_{2k}}(x)\right|.$$

We can think of μ as a distribution over threshold sets. Define h as

$$h(v) = \int_0^{f(v)} \mu(x)dx.$$

Notice that $h(v) \geq h(u)$ iff $f(v) \geq f(u)$, therefore the two functions have the same threshold sets, and $\phi(h) = \phi(f)$. In order to invoke Lemma 2.3, we need to upper bound the numerator and lower bound the denominator that appear in the RHS of (7), which we perform in the next two steps.

Step 1: *Lower Bounding the Denominator.* We begin by establishing the lower bound

$$h(v) \geq \frac{1}{8k} f(v)^2. \tag{14}$$

Suppose $f(v) \in [t_i, t_{i+1}]$. Then,

$$\begin{aligned} h(v) &= \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \mu(x)dx + \int_{t_i}^{f(v)} \mu(x)dx \\ &= \sum_{j=0}^{i-1} \frac{1}{4}(t_{j-1} - t_j)^2 + \int_{t_j}^{f(v)} \mu(x)dx \\ &\geq \sum_{j=0}^{i-1} \frac{1}{4}(t_{j-1} - t_j)^2 + \frac{1}{4}(f(v) - t_i)^2. \end{aligned}$$

Next, we relate the above lower bound to $f(v)^2$ via the following computation

$$f^2(v) = \left(\sum_{j=0}^{i-1}(t_{j+1} - t_j) + f(v) - t_i\right)^2 \leq 2k\left(\sum_{j=0}^{i-1}(t_{j+1} - t_j)^2 + (f(v) - t_i)^2\right),$$

where the inequality follows from the Cauchy-Schwartz inequality. Thus we derived our desired result.

Step 2: *Upper Bounding the Numerator.* We begin by investigating the smoothness of h. WLOG assume $f(u) \leq f(v)$. By definition, for $x \in [f(u), f(v)]$, $\mu(x) \leq \min\{\left|x - g(u)\right|, \left|x - g(v)\right|\}$. Hence we can write

$$\begin{aligned}
\mu(x) &\leq \frac{\left|x - g(u)\right| + \left|x - g(v)\right|}{2} \\
&\overset{(a)}{\leq} \frac{1}{2}\left(\left|x - f(u)\right| + \left|f(u) - g(u)\right| + \left|x - f(v)\right| + \left|f(v) - g(v)\right|\right) \\
&= \frac{1}{2}\left(\left|f(u) - f(v)\right| + \left|f(u) - g(u)\right| + \left|f(v) - g(v)\right|\right),
\end{aligned}$$

where (a) follows from the triangle inequality. Consequently,

$$h(v) - h(u) = \int_{f(u)}^{f(v)} \mu(x)dx \leq \frac{1}{2}\left|f(u) - f(v)\right|\left(\left|f(u) - f(v)\right| + \left|f(u) - g(u)\right| + \left|f(v) - g(v)\right|\right).$$

Next, we upper bound the numerator,

$$\begin{aligned}
\sum_{\{u,v\}\in E} w(u,v)\left|h(u) - h(v)\right| &\leq \sum_{\{u,v\}\in E} \frac{1}{2} w(u,v)\left|f(u) - f(v)\right|\left(\left|f(u) - f(v)\right| + \left|f(u) - g(u)\right| + \left|f(v) - g(v)\right|\right) \\
&\overset{(a)}{\leq} \sum_{\{u,v\}\in E} \frac{1}{2} w(u,v)\left|f(u) - f(v)\right|^2 \\
&\quad + \frac{1}{2}\sqrt{\sum_{\{u,v\}\in E} w(u,v)\left|f(u) - f(v)\right|^2}\sqrt{\sum_{\{u,v\}\in E} w(u,v)\left(\left|f(u) - g(u)\right|^2 + \left|f(v) - g(v)\right|^2\right)} \\
&\overset{(b)}{=} \frac{1}{2}\mathcal{R}(f) + \frac{1}{2}\sqrt{2\mathcal{R}(f)}\left\|f - g\right\|_w,
\end{aligned}$$

where (a) follows from the Cauchy-Schwartz inequality, and (b) holds since $\left\|f\right\|_w = 1$.

Combining steps 1 and 2, we get

$$\frac{\sum_{\{u,v\}\in E} w(u,v)\left|h(u) - h(v)\right|}{\sum_v w(v)h(v)} \leq 4k\mathcal{R}(f) + 4k\sqrt{2\mathcal{R}(f)}\left\|f - g\right\|_w. \tag{15}$$

Invoking Lemma 2.3 and using $\phi(h) = \phi(f)$ completes the proof. □

With Proposition 3.6 in hand, we are ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1: Choose f from Corollary 3.2, and let g be the $2k+1$-step approximation from Lemma 3.4 such that $\left\|f - g\right\|_w \leq 2\sqrt{\frac{\mathcal{R}(f)}{\lambda_k}}$. Applying Proposition 3.6 and using $\mathcal{R}(f) \leq \lambda_2$ we have

$$\phi(f) \leq 4k\lambda_2 + 8\sqrt{2}k\frac{\lambda_2}{\sqrt{\lambda_k}} \leq 12\sqrt{2}k\frac{\lambda_2}{\sqrt{\lambda_k}},$$

where the second inequality holds since $\lambda_k \leq 2$. Finally, by definition $\phi(G) \leq \phi(f)$, which completes the proof of the theorem. □