SOME IDEAS ON OPTION PRICING

1. The model

We are going to look at the example given by the Black-Scholes model, something that will allow us to do our simulations and compare them easily. Suppose S_t denotes the price of a share at time t, where t is measured in years, we shall assume it satisfies the stochastic differential equation (SDE)

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{1}$$

where the interest rate r and the volatility σ are supposed to be constant; further B_t denotes Brownian motion at time t, a stochastic process associated with the normal distribution function with standard deviation $\sqrt{t}$, namely

$$P\{B_t < x\} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} e^{-\frac{y^2}{2t}} dy;$$

here we assume $B_0 = 0$.[1] Equation (1) has the well-known solution

$$S_t = S_0 e^{(r-1/2\sigma^2)t+\sigma B_t}, \tag{2}$$

which we'll use throughout.
Now, the payoff of a call option to be exercised at time T is given by $(S_T - K)_+ = \max(S_T, K)$, where K is the strike price. The fundamental theorem of asset pricing then states that the price C_K of the call option is given by

$$C_K = E[e^{-rT}(S_T - K)_+] \tag{3}$$

and it is this quantity that will be the object of our investigations. Naturally, this being the Black-Scholes model, we already know that

$$C_K = xN(d_+(T,x)) - e^{-rT}KN(d_(T,x)), \quad \text{where } x = S_0,$$

$$\begin{aligned} d_-(T,x) &= \frac{1}{\sigma\sqrt{T}} \left[\log\frac{x}{K} + (r - (1/2)\sigma^2)T\right] \\ d_+(T,x) &= d_-(T,x) + \sigma\sqrt{T} = \frac{1}{\sigma\sqrt{T}} \left[\log\frac{x}{K} + (r + (1/2)\sigma^2)T\right] \end{aligned}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

is the normal distribution function.
However, we wish to compare how long it takes to compute (3) for many values of K using Monte Carlo simulation with the time necessary to do the same using Fourier

[1] In reality we should say $P\{B_0 = 0\} = 1$, or $B_0 = 0$ almost surely, but I won't bother any further.

analysis. We make some adjustments; since S_t has a log-normal distribution, we set

$$X_t = \log S_t, \quad h = \log K \Longrightarrow X_t = X_0 + (r - (1/2)\sigma^2)t + \sigma B_t, \quad X_0 = \log S_0$$
$$\Longrightarrow C_K = C_{\exp h} = E[e^{-rT}(S_T - K)_+] = E[e^{-rT}(e^{X_T} - e^h)_+].$$

Since the function $C_{\exp h}$ is not integrable in h, we introduce the modified call price

$$c(h) \equiv \exp(\alpha h) C_{\exp h}, \quad \alpha > 0, \tag{4}$$

which is integrable (the choice of α is something to be discussed later). Now, the Fourier transform with respect to h is given by

$$\widehat{c}(\xi) = \int e^{i\xi h} e^{\alpha h} E[e^{-rT}(e^{X_T} - e^h)_+] dh = \\ e^{-rT} \int e^{i\xi h} e^{\alpha h} \left(\int_h^\infty (e^x - e^h)\phi_T(x) dx \right) dh$$

where ϕ_T indicates the probability density function associated with X_T. Changing the order of integration we get

$$e^{-rT} \int \left(\int_{-\infty}^x e^{i\xi h} e^{\alpha h} (e^x - e^h) dh \right) dx = \\ e^{-rT} \int \left(\frac{e^x e^{(i\xi+\alpha)h}}{i\xi + \alpha} - \frac{e^{(i\xi+1+\alpha)h}}{i\xi + 1 + \alpha} \right) \Bigg|_{-\infty}^x \phi_T(x) dx = \\ e^{-rT} \int \frac{e^{(i\xi+1+\alpha)x}}{(i\xi+\alpha)(i\xi+1+\alpha)} \phi_T(x) dx = \\ \frac{e^{-rT}}{\alpha^2 + \alpha - \xi^2 + i(2\alpha+1)\xi} \int e^{ix(\xi - i(\alpha+1))} \phi_T(x) dx = \frac{e^{-rT} \widehat{\phi_T}(\xi - (\alpha+1)i)}{\alpha^2 + \alpha - \xi^2 + i(2\alpha+1)\xi}.$$

By the Fourier inversion formula and equation (4) we have

$$C_{\exp h} = \frac{\exp(-\alpha h)}{2\pi} \int e^{-i\xi h} \widehat{c}(\xi) d\xi = \frac{\exp(-\alpha h)}{\pi} \int_0^\infty \Re \left[e^{-i\xi h} \widehat{c}(\xi) \right] d\xi. \tag{5}$$

Naturally, to utilise this formula we'll need to know that

$$\widehat{\phi_T}(\eta) = \exp \left[i \left(X_0 + (r - (1/2)\sigma^2)T \right) \eta - \sigma^2 T \eta^2 / 2 \right].$$

At this stage we wish to turn (5) into a sum which can be computed effectively using the Fast Fourier transform algorithm. We are going to write

$$C_{\exp h} \approx \frac{\exp(-\alpha h)}{\pi} \sum_{j=1}^N \Re \left[e^{-ih\Delta(j-1/2)} \widehat{c}(\Delta(j - 1/2)) \right] \times \Delta, \tag{6}$$

where we have taken the Riemann sum with N intervals of width Δ and the integrand has been eveluated at the midpoints. Note that this implies a truncation of the integral at the upper limit $N\Delta$. We shall have the freedom to choose Δ and N when we run our simulations depending on the level of accuracy we want.
Now, remember that the Discrete Fourier transform of a vector $x = (x_1, \ldots, x_N)$ is a new vector $X = (X_1, \ldots, X_N)$ where

$$X_m = \sum_{j=1}^N e^{-i\frac{2\pi}{N}(j-1)(m-1)} x_j, \quad m = 1, \ldots, N. \tag{7}$$

To put the sum in (6) in the form given by (7) we will take a spacing $\lambda = 2\pi/N\Delta$ between consecutive values of h. We are going to compute $C_{\exp h}$ at values $h_m = h_1 + \lambda(m-1)$, $m = 1, \ldots, N$ where we may choose the starting value h_1 as we please; we should nonetheless choose h_1 so that the interval $[h_1, h_1 + (N-1)\lambda]$ contains all the values we are interested in. Now we are going to write

$$C_{\exp h_m} = \frac{\Delta}{\pi \exp(\alpha h_m)} \sum_{j=1}^{N} \Re\left[\widehat{c}(\Delta(j-1/2))e^{-i(j-1/2)(h_1+\lambda(m-1))\Delta}\right] =$$

$$\frac{\Delta}{\pi \exp(\alpha h_m)} \Re\left[e^{-i\lambda(m-1)\Delta/2} \sum_{j=1}^{N} \widehat{c}(\Delta(j-1/2))e^{-i\lambda\Delta(j-1)(m-1)}e^{-i(j-1/2)h_1\Delta}\right] =$$

$$\frac{\Delta}{\pi \exp(\alpha h_m)} \Re\left[e^{-i\pi(m-1)/N} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(m-1)}\widehat{c}(\Delta(j-1/2))e^{-i(j-1/2)h_1\Delta}\right],$$

so in our case the vector x has coordinates

$$x_j = \widehat{c}(\Delta(j-1/2))e^{-i(j-1/2)h_1\Delta}, \quad j = 1, \ldots, N$$

and the vector X has coordinates

$$X_m = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(m-1)}\widehat{c}(\Delta(j-1/2))e^{-i(j-1/2)h_1\Delta}, \quad m = 1, \ldots, N$$

whilst

$$C_{\exp h_m} = \frac{\Delta}{\pi \exp(\alpha h_m)} \Re\left[e^{-i\pi(m-1)/N}X_m\right].$$

2. Some comments

There are several parameters that we have the freedom to choose in the model; in the modified call price given by equation (4) the parameter α can be chosen; we note that $\widehat{c}(0) = \int c(h)dh$, so since $c \geq 0$, the modified call price is integrable if $\widehat{c}(0) < \infty \iff \widehat{\phi}(-(\alpha+1)i) < \infty$. Given the definition of the Fourier transform this gives the condition

$$E[S_T^{\alpha+1}] < \infty.$$

In the model we are using this is always satisfied and we can choose α depending on the speed of computations. Now, to run simulations we can just choose interest rate r and volatility σ as we wish (say $r = 3\% \equiv 0.03$ and $\sigma = 0.2$) and initial cost of the share $S_0 = 1$. Then we wish to compute call prices for a lot of strikes near the share price, so near $K = 1$ and hence $h = 0$. To implement the Fast Fourier transform algorithm we must choose where to truncate the integral on the right-hand side of (5), so we have to choose Δ and N. Depending on the accuracy we wish to have we can choose $\Delta = 0.1$ and pick $N = 8192$ or $N = 16384$ or a higher power of 2 (a power of 2 is required to apply the Cooley-Tukey algorithm). In any case, the choice of the parameters depends on the computational time required to perform the simulations, and a bit of trial-and-error will help. Always keep in mind that $h = \log K$ in the calculations.