Presentation On

Graph Minors and the Robertson-Seymour Theorem

Parameterized Algorithms, Springer

1705066 - Ataf Fazledin Ahamed

Bangladesh University of University of Engineering and Technology

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Outline

Introduction

We discuss the minor relation in graphs and the algorithmic usage of the Robertson-Seymour Theorem

Robertson-Seymour Theorem

1. by Neil Robertson and Paul D. Seymour

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Robertson-Seymour Theorem

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2. proved in a series of 20 papers
3. spanning over 500 pages
4. from 1983 to 2004

No, we are not going to prove it today.

Graph Minors

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Equivalently, graph H is a minor of G, if H can be obtained from G itself by a series of edge deletions, edge contractions, and vertex deletions.

Graph Minors

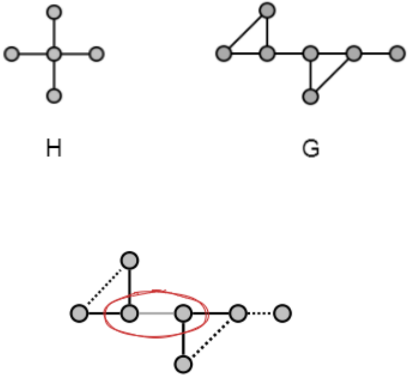

Figure: Graph H is a minor of graph G

Graph Minors

The theory of graph minors began with Wagner's Theorem - that a graph is planar if and only if its minor include neither the complete graph K_5 nor the complete bipartite graph $K_{3,3}$

Graph Minors

Forbidden Graph Minors

A forbidden graph minor is a method of specifying a family of graph by specifying minors that are forbidden from existing within any graph in the family.

Graph Minors

Graph Minors Properties

A number of interesting properties of graphs are preserved under deletions and edge contractions can be characterized by forbidden minors.
Example: Planarity, Coloring, etc.

Few observations:
A graph is a forest if and only if it has no minor isomorphic to the triangle K_3. (Proof is trivial.)

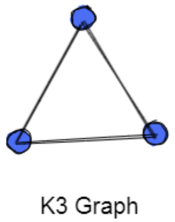

Figure: Graph K_3

Graph Minors

A graph is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$. This is Wagner's theorem.

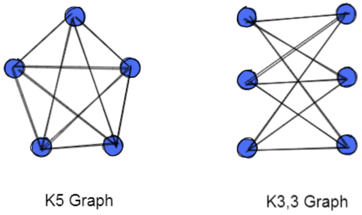

Figure: Graph K_5 and $K_{3,3}$

A graph is outerplanar if and only if it has no minor isomorphic to K_4 or $K_{2,3}$.

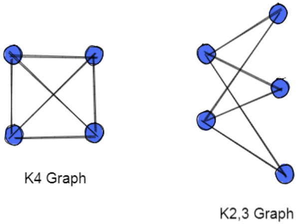

Figure: Graph K_4 and $K_{2,3}$

Few Observations

Proposition 6.8

A minor of a planar graph is also planar.

Removing edges and vertices cannot make a planar graph non-planar. It is the other way round.

Few Observations

1. It is known that a graph can be drawn on a plane if and only if it can be drawn on a sphere.
2. However, there are graphs that can only be drawn on a torus, but not on an sphere. For example- the K_5 graph.
3. Similar cases exist for weird surfaces, such as- Klein Bottle, The Projective Plane, etc.

Few Observations

Statement

If a graph G is embeddable in a surface Σ, and H is a minor of G, then, H is also embeddable on Σ as well.

Such observations and findings have applications in Topological Graph Theory.

Robertson-Seymour Theorem: Few Terminologies

Quasi-ordering or Pre-ordering

A pre-order or quasi-order of a set is a binary relation among its elements that is both reflexive and transitive.

Quasi-ordering or Pre-ordering

Consider a relation $\leq$ on a given set P, so that $\leq$ is a subset of PxP and notation $a \leq b$ is used in place of $(a, b) \in \leq$. Then $\leq$ is called a pre-order or quasi-order if it is both reflexive and transitive.

Reflexivity: $a \leq a$ for all $a \in P$ and
Transitivity: $a \leq b$ and $b \leq c$ then $a \leq c$ for all $a, b, c \in P$

Robertson-Seymour Theorem: Few Terminologies

Well-Quasi-Ordering

A well-quasi-ordering or wqo is a quasi-ordering such that any infinite sequence of elements $x_0, x_1, x_2, ...$ from X contains an increasing pair $x_i \leq x_j$ with $i < j$

Robertson-Seymour Theorem

Theorem 6.10 (Robertson and Seymour)

The class of all graphs is **well-quasi-ordered** by the minor relation. That is, in any infinite family of graphs, there are two graphs such that one is a minor of the other.

Application

$Forb(\mathcal{G}) \leftarrow$ Family of forbidden minors for G

Application

Corollary 6.11

For every minor-closed graph class G, there exists a finite set $Forb(\mathcal{G})$ of graphs with the following property: for every graph G, graph $G \in$ G if and only if there does not exist a minor of G isomorphic to a member of $Forb(\mathcal{G})$.

Application

Corollary 6.11 and Theorem 6.10 gives us the following:

Theorem 6.12 (Robertson and Seymour)

There exists a computable function f and an algorithm that, for every given graphs H and G, checks in time $f(H)|V(G)|^3$ whether $H \leq_m G$

Application

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Theorem 6.12 (Robertson and Seymour)

There exists a computable function f and an algorithm that, for every given graphs H and G, checks in time $f(H)|V(G)|^3$ whether $H \leq_m G$

Which implies that every minor-closed class can be recognized in polynomial time.

Thank You