MPhil Econometrics – Limited Dependent Variables and Selection

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Housekeeping

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References

- **Wooldridge (2010) – *Econometric Analysis of Cross Section & Panel Data***
- Cameron & Trivedi (2005) – *Microeconometrics: Methods and Applications*
- Train (2009) – *Discrete Choice Methods with Simulation*

Lecture #1 – Maximum Likelihood Estimation Under Mis-specification

Review: the Poisson Distribution

The Kullback-Leibler Divergence

Example: Consistency of Poisson MLE

Asymptotic Theory for MLE Under Mis-specification

The Information Matrix Equality

Example: Asymptotic Variance Calculations for Poisson MLE

"All models are wrong; some are useful."

Question

What happens if we carry out maximum likelihood estimation, but our model is *wrong*?

This Lecture

Examine a simple example in excruciating detail; present the general theory.

Next Lecture

Apply what we've learned to study Poisson Regression, a model for count data.

Suppose that $y \sim \text{Poisson}(\theta)$

Support Set: $\{0, 1, 2, \ldots\}$

A Poisson Random Variable is a *count*.

$$\sum_{y=0}^{\infty} \frac{e^{-\theta}\theta^y}{y!} = e^{-\theta} \sum_{y=0}^{\infty} \frac{\theta^y}{y!} = e^{-\theta}\left(e^{\theta}\right) = 1$$

Probability Mass Function

$$f(y|\theta) = \frac{e^{-\theta}\theta^y}{y!}$$

Expected Value: $\mathbb{E}(y) = \theta$

Poisson parameter θ equals the mean of y.

$$\begin{aligned}
\mathbb{E}(y) &= \sum_{y=0}^{\infty} y \frac{e^{-\theta}\theta^y}{y!} = \sum_{y=1}^{\infty} y \frac{e^{-\theta}\theta^y}{y!} \\
&= \theta \sum_{y=1}^{\infty} \frac{e^{-\theta}\theta^{y-1}}{(y-1)!} = \theta \sum_{y=0}^{\infty} \frac{e^{-\theta}\theta^y}{y!} = \theta
\end{aligned}$$

Variance: $\text{Var}(y) = \theta$

You will show this on the problem set.

MLE for θ where $y_1, y_2, \ldots, y_N \sim$ iid Poisson(θ).

The Likelihood (iid data)

$L_N(\theta) \equiv \prod_{i=1}^{N} \frac{e^{-\theta}\theta^{y_i}}{y_i!}$

The Log-Likelihood

$\ell_N(\theta) = \sum_{i=1}^{N} [y_i \log(\theta) - \theta - \log(y_i!)]$

Maximum Likelihood Estimator

$\widehat{\theta} \equiv \underset{\theta \in \Theta}{\arg\max}\ \ell_N(\theta) = \bar{y}$

$$\frac{d}{d\theta}\ell_N(\theta) = \sum_{i=1}^{N}\left[\frac{y_i}{\theta} - 1\right]$$

$$\frac{d}{d\theta}\ell_N(\widehat{\theta}) = 0$$

$$\sum_{i=1}^{N}\left[y_i/\widehat{\theta} - 1\right] = 0$$

$$\left(\sum_{i=1}^{N} y_i\right)/\widehat{\theta} = N$$

$$\frac{1}{N}\sum_{i=1}^{N} y_i = \bar{y} = \widehat{\theta}$$

The Kullback-Leibler (KL) Divergence

Motivation

How well does a parametric model $f(\mathbf{y}|\boldsymbol{\theta})$ approximate a *true* density/pmf $p_o(\mathbf{y})$?

Definition

$$KL(p_o; f_{\boldsymbol{\theta}}) \equiv \mathbb{E}\left[\log\left\{\frac{p_o(\mathbf{y})}{f(\mathbf{y}|\boldsymbol{\theta})}\right\}\right]$$

KL Properties

1. *Asymmetric*: $KL(p_o; f_{\boldsymbol{\theta}}) \neq KL(f_{\boldsymbol{\theta}}; p_o)$
2. $KL(p_o; f_{\boldsymbol{\theta}}) \geq 0$; zero iff $p_o = f_{\boldsymbol{\theta}}$
3. Min KL iff max expected log-likelihood

Alternative Expression

$$\boxed{\mathbb{E}\left[\log\left\{\frac{p_o(\mathbf{y})}{f(\mathbf{y}|\boldsymbol{\theta})}\right\}\right] = \underbrace{\mathbb{E}\left[\log p_o(\mathbf{y})\right]}_{\text{Constant wrt } \boldsymbol{\theta}} - \underbrace{\mathbb{E}\left[\log f(\mathbf{y}|\boldsymbol{\theta})\right]}_{\text{Expected Log-like.}}}$$

All expectations are wrt p_o

$p_o(\mathbf{y})$ and $f(\mathbf{y}|\boldsymbol{\theta})$ are merely *functions* of the RV $\mathbf{y}$

$$\mathbb{E}[\log p_o(\mathbf{y})] = \int \log p_o(\mathbf{y}) p_o(\mathbf{y})\, d\mathbf{y}$$

$$\mathbb{E}[\log f(\mathbf{y}|\boldsymbol{\theta})] = \int \log f(\mathbf{y}|\boldsymbol{\theta}) p_o(\mathbf{y})\, d\mathbf{y}$$

Watch Out!

$KL = \infty$ if $\exists \mathbf{y}$ with $f(\mathbf{y}|\boldsymbol{\theta}) = 0$ & $p_o(\mathbf{y}) \neq 0$

$\mathrm{KL}(p_o; f) \geq 0$ with equality iff $p_o = f$

Jensen's Inequality

If φ is convex then $\varphi(\mathbb{E}[y]) \leq \mathbb{E}[\varphi(y)]$, with equality iff φ is linear or y is constant.

log is concave so $(-\log)$ is convex

$$
\begin{aligned}
\mathbb{E}\left[\log\left\{\frac{p_o(y)}{f(y)}\right\}\right] = \mathbb{E}\left[-\log\left\{\frac{f(y)}{p_o(y)}\right\}\right] &\geq -\log\left\{\mathbb{E}\left[\frac{f(y)}{p_o(y)}\right]\right\} \\
&= -\log\left\{\int_{-\infty}^{\infty} \frac{f(y)}{p_o(y)} \cdot p_o(y)\, dy\right\} \\
&= -\log\left\{\int_{-\infty}^{\infty} f(y)\, dy\right\} \\
&= -\log(1) = 0
\end{aligned}
$$

A Simple Example: Calculating the KL Divergence

Remember: all expectations are calculated using p_o.

True Distribution p_o

$y_1, \ldots, y_N \sim$ iid p_o where:

$p_o(0) = \frac{2}{5}, p_o(1) = \frac{1}{5}, p_o(2) = \frac{2}{5}$.

Mis-specified Model f_θ

$y_1, \ldots, y_N \sim$ iid Poisson(θ)

KL Divergence

$KL(p_o; f_\theta) = \theta - \log\theta + (\text{Constant})$

$$\boxed{KL(p_o; f_\theta) = \mathbb{E}[\log p_o(y)] - \mathbb{E}[\log f(y|\theta)]}$$

$$\begin{aligned}\mathbb{E}[\log p_o(y)] &= \sum_{\text{all } y} \log\left[p_o(y)\right] p_o(y)\\ &= \log\left(\frac{2}{5}\right) \times \frac{2}{5} + \log\left(\frac{1}{5}\right) \times \frac{1}{5} + \log\left(\frac{2}{5}\right) \times \frac{2}{5}\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\log f(y|\theta)] &= \sum_{\text{all } y} \log\left[\frac{e^{-\theta}\theta^y}{y!}\right] p_o(y)\\ &= \log\left(e^{-\theta}\right) \times \frac{2}{5} + \log\left(e^{-\theta}\theta\right) \times \frac{1}{5} + \log\left(\frac{e^{-\theta}\theta^2}{2}\right) \times \frac{2}{5}\\ &= -\left[\theta - \log(\theta) + \log(2) \times \frac{2}{5}\right]\end{aligned}$$

A Simple Example Continued: Minimizing the KL Divergence

Model $=$ Poisson(θ); True Dist. $p_o(0) = p_o(2) = \frac{2}{5}$ and $p_o(1) = \frac{1}{5}$

Best Approximation

What parameter value θ_o makes the Poisson(θ) model *as close as possible* to the true distribution p_o, where we measure "closeness" using the KL-divergence?

Using the previous slide

$$KL(p_o; f_\theta) = \theta - \log\theta + (\text{Const.})$$

$$\text{FOC:} \quad 0 = 1 - \frac{1}{\theta} \implies \boxed{\theta = 1}$$

A more direct approach

Min KL $\iff$ Max Expected Log-like.

$$\frac{d}{d\theta}\mathbb{E}[\log f(y|\theta)] = \mathbb{E}\left[\frac{d}{d\theta}\{-\theta + y\log(\theta) - \log(y!)\}\right]$$

$$= \mathbb{E}[-1 + y/\theta] = \mathbb{E}[y]/\theta - 1 = 0$$

$$\implies \boxed{\theta = \mathbb{E}[y]}$$

A Simple Example Continued: Minimizing the KL Divergence

Model = Poisson(θ); True Dist. $p_o(0) = p_o(2) = \frac{2}{5}$ and $p_o(1) = \frac{1}{5}$

Best Approximation

What parameter value θ_o makes the Poisson(θ) model *as close as possible* to the true distribution p_o, where we measure "closeness" using the KL-divergence?

Using the previous slide: $\theta_o = 1$

A more direct approach: $\theta_o = \mathbb{E}[y]$

Both Methods Agree

- For the specified p_o we have: $\mathbb{E}[y] = 0 \times \frac{1}{5} + 1 \times \frac{2}{5} + 2 \times \frac{2}{5} = 1.$
- The "Direct approach" is general: works for *any* p_o (under regularity conditions)

Is this just a coincidence?

We have shown that:

1. Under an iid Poisson(θ) model for $y_1, \ldots, y_N$, the MLE for θ is $\widehat{\theta} = \bar{y}$
2. For *any* (reasonable) p_o, setting $\theta_o = \mathbb{E}[y_i]$ minimizes $KL(p_o; f_\theta)$.

By the (weak) law of large numbers:

If $y_1, \ldots, y_N \sim$ iid, then $\bar{y}$ is a consistent estimator of $\mathbb{E}[y_i]$ as N approaches infinity.

So at least in this example...

The maximum likelihood estimator $\widehat{\theta}$ is a consistent estimator of θ_o, the minimizer the KL divergence from the true distribution p_o to the Poisson(θ) model $f(y|\theta)$.

Maximum Likelihood Estimation Under Mis-specification

Note: expectations and variances are calculated using p_o

Theorem

Suppose that $\mathbf{y}_1, \ldots, \mathbf{y}_N \sim$ iid p_o and let $\widehat{\boldsymbol{\theta}}$ denote the MLE for $\boldsymbol{\theta}$ under the possibly mis-specified model $f(\mathbf{y}|\boldsymbol{\theta})$. Then, under mild regularity conditions:

(i) $\widehat{\boldsymbol{\theta}}$ is consistent for the pseudo-true parameter value $\boldsymbol{\theta}_o$, defined as the minimizer of $KL(p_o, f_{\boldsymbol{\theta}})$ over the parameter space Θ.

(ii) $\sqrt{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \to_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1})$

where we define $\mathbf{J} \equiv -\mathbb{E}\left[\dfrac{\partial^2 \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right]$ and $\mathbf{K} \equiv \text{Var}\left[\dfrac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right]$.

Why is this result such a big deal?

1. Provides an interpretation of MLE when we acknowledge that our models are only an *approximation* or reality: MLE recovers the pseudo-true parameter $\boldsymbol{\theta}_o$.
2. Yields a formula for standard errors that is robust to mis-specification of our model: compare to Heteroskedasticity consistent SEs for regression.
3. If the model is correctly specified, we recover the "classical" MLE result.

A Consistent Asymptotic Variance Matrix Estimator: $\widehat{\mathbf{J}}^{-1}\widehat{\mathbf{K}}\widehat{\mathbf{J}}^{-1}$

$\widehat{\boldsymbol{\theta}} \to_p \boldsymbol{\theta}_o$ plus Uniform Weak Law of Large Numbers: $\boxed{\text{Newey \& McFadden (1994)}}$

$$\boldsymbol{\theta}_o \equiv \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}}{\arg\max}\, \mathbb{E}\left[\log f(\mathbf{y}_i|\boldsymbol{\theta})\right] \qquad \widehat{\theta} \equiv \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}}{\arg\max}\, \frac{1}{N}\sum_{i=1}^{N} \log f(\mathbf{y}|\boldsymbol{\theta})$$

$$\sqrt{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \to_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1}) \qquad \widehat{\boldsymbol{\theta}} \approx \mathcal{N}(\boldsymbol{\theta}_o, \widehat{\mathbf{J}}^{-1}\widehat{\mathbf{K}}\widehat{\mathbf{J}}^{-1}/N)$$

$$\mathbf{J} \equiv -\mathbb{E}\left[\frac{\partial^2 \log f(\mathbf{y}_i|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] \qquad \widehat{\mathbf{J}} \equiv -\frac{1}{N}\sum_{i=1}^{N}\frac{\partial^2 \log f(\mathbf{y}_i|\widehat{\boldsymbol{\theta}})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}$$

$$\mathbf{K} \equiv \mathrm{Var}\left[\frac{\partial \log f(\mathbf{y}_i|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right] \qquad \widehat{\mathbf{K}} \equiv \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial \log f(\mathbf{y}_i|\widehat{\boldsymbol{\theta}})}{\partial\boldsymbol{\theta}}\right]\left[\frac{\partial \log f(\mathbf{y}_i|\widehat{\boldsymbol{\theta}})}{\partial\boldsymbol{\theta}}\right]'$$

Some Notes on the Preceding Slide

What happened to the KL divergence?

$\mathbb{E}[\log p_o(\mathbf{y})]$ does not involve $\boldsymbol{\theta}$. Hence, $\arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \mathbb{E}\left[\log f(\mathbf{y}_i|\boldsymbol{\theta})\right] = \arg\min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} KL(p_o, f_{\boldsymbol{\theta}})$.

Isn't $\widehat{\mathbf{K}}$ missing a term?

The sample variance of $\mathbf{x}$ is given by $\left(\frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i\mathbf{x}_i'\right) - (\bar{\mathbf{x}}\bar{\mathbf{x}}')$ where $\bar{\mathbf{x}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i$. In our formula for $\widehat{\mathbf{K}}$, the "$\bar{\mathbf{x}}\bar{\mathbf{x}}'$" term appears to be missing, but it is in fact equal to zero, since $\widehat{\boldsymbol{\theta}}$ is the solution to the MLE first-order condition.

Some Terminology

I will call $\widehat{\mathbf{J}}^{-1}\widehat{\mathbf{K}}\widehat{\mathbf{J}}^{-1}$ the robust asymptotic variance matrix estimator, since it is correct regardless of whether the model is correctly specified.

Maximum Likelihood Estimation Under Correct Specification

"Classical" large-sample theory for MLE

Theorem

Suppose that $\mathbf{y}_1, \ldots, \mathbf{y}_N \sim$ iid $f(\mathbf{y}|\boldsymbol{\theta}_o)$. Then, under mild regularity conditions:

(i) $\widehat{\boldsymbol{\theta}}$ is consistent for $\boldsymbol{\theta}_o$.

(ii) $\sqrt{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \to_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1})$ where $\mathbf{J} \equiv -\mathbb{E}\left[\dfrac{\partial^2 \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right]$.

Why? If $p_o(\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}_o)$, then:

1. $KL(p_o; f_{\boldsymbol{\theta}})$ equals *zero* at $\boldsymbol{\theta} = \boldsymbol{\theta}_o$.
2. The *information matrix equality* gives $\mathbf{K} = \mathbf{J}$ which implies $\mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1} = \mathbf{J}^{-1}$.

The Information Matrix Equality: if $p_o(\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}_o)$, then $\mathbf{K} = \mathbf{J}$.

$$\boxed{\mathbf{J} \equiv -\mathbb{E}\left[\frac{\partial^2 \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right], \quad \mathbf{K} \equiv \text{Var}\left[\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right]}$$

Step 1: Alternative Expression for $\mathbf{K}$

$$\text{Var}\left[\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right] = \mathbb{E}\left[\left\{\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right\}\left\{\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right\}'\right] - \mathbb{E}\left[\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right]\mathbb{E}\left[\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right]'$$

but since $\boldsymbol{\theta}_o$ minimizes $\mathbb{E}\left[\log f(\mathbf{y}|\boldsymbol{\theta})\right]$,

$$\mathbb{E}\left[\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right] = \frac{\partial}{\partial\boldsymbol{\theta}}\mathbb{E}\left[\log f(\mathbf{y}|\boldsymbol{\theta}_o)\right] = \mathbf{0}$$

so it suffices to show that

$$-\mathbb{E}\left[\frac{\partial^2 \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] = \mathbb{E}\left[\left\{\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right\}\left\{\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right\}'\right]$$

The Information Matrix Equality: if $p_o(\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}_o)$, then $\mathbf{K} = \mathbf{J}$.

$$\boxed{\text{suffices to show} \quad -\mathbb{E}\left[\frac{\partial^2 \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] = \mathbb{E}\left[\left\{\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right\}\left\{\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right\}'\right]}$$

Step 2: Chain Rule & Product Rule

$$\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log f(\mathbf{y}|\boldsymbol{\theta}) = \frac{\partial}{\partial\theta_i}\left[\frac{\partial}{\partial\theta_j}\log f(\mathbf{y}|\boldsymbol{\theta})\right] = \frac{\partial}{\partial\theta_i}\left[\frac{1}{f(\mathbf{y}|\boldsymbol{\theta})}\cdot\frac{\partial}{\partial\theta_j}f(\mathbf{y}|\boldsymbol{\theta})\right]$$

$$= \left[-\frac{1}{f^2(\mathbf{y}|\boldsymbol{\theta})}\cdot\frac{\partial}{\partial\theta_i}f(\mathbf{y}|\boldsymbol{\theta})\right]\left[\frac{\partial}{\partial\theta_j}f(\mathbf{y}|\boldsymbol{\theta})\right] + \frac{1}{f(\mathbf{y}|\boldsymbol{\theta})}\cdot\frac{\partial^2}{\partial\theta_i\partial\theta_j}f(\mathbf{y}|\boldsymbol{\theta})$$

$$= -\left[\frac{1}{f(\mathbf{y}|\boldsymbol{\theta})}\cdot\frac{\partial}{\partial\theta_i}f(\mathbf{y}|\boldsymbol{\theta})\right]\left[\frac{1}{f(\mathbf{y}|\boldsymbol{\theta})}\cdot\frac{\partial}{\partial\theta_j}f(\mathbf{y}|\boldsymbol{\theta})\right] + \frac{1}{f(\mathbf{y}|\boldsymbol{\theta})}\cdot\frac{\partial^2}{\partial\theta_i\partial\theta_j}f(\mathbf{y}|\boldsymbol{\theta})$$

$$= -\frac{\partial}{\partial\theta_i}\log f(\mathbf{y}|\boldsymbol{\theta})\frac{\partial}{\partial\theta_j}\log f(\mathbf{y}|\boldsymbol{\theta}) + \frac{1}{f(\mathbf{y}|\boldsymbol{\theta})}\cdot\frac{\partial^2}{\partial\theta_i\partial\theta_j}f(\mathbf{y}|\boldsymbol{\theta})$$

The Information Matrix Equality: if $p_o(\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}_o)$, then $\mathbf{K} = \mathbf{J}$.

$$\boxed{\text{suffices to show } \; -\mathbb{E}\left[\frac{\partial^2 \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] = \mathbb{E}\left[\left\{\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right\}\left\{\frac{\partial \log f(\mathbf{y}|\boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right\}'\right]}$$

Step 3: Multiply by -1, Evaluate at $\boldsymbol{\theta}_o$, and Take Expectations

$$\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log f(\mathbf{y}|\boldsymbol{\theta}) = -\frac{\partial}{\partial\theta_i}\log f(\mathbf{y}|\boldsymbol{\theta})\frac{\partial}{\partial\theta_j}\log f(\mathbf{y}|\boldsymbol{\theta}) + \frac{1}{f(\mathbf{y}|\boldsymbol{\theta})}\cdot\frac{\partial^2}{\partial\theta_i\partial\theta_j}f(\mathbf{y}|\boldsymbol{\theta})$$

$$-\mathbb{E}\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log f(\mathbf{y}|\boldsymbol{\theta}_o)\right] = \mathbb{E}\left[\frac{\partial}{\partial\theta_i}\log f(\mathbf{y}|\boldsymbol{\theta}_o)\frac{\partial}{\partial\theta_j}\log f(\mathbf{y}|\boldsymbol{\theta}_o)\right] - \underbrace{\mathbb{E}\left[\frac{1}{f(\mathbf{y}|\boldsymbol{\theta}_o)}\cdot\frac{\partial^2}{\partial\theta_i\partial\theta_j}f(\mathbf{y}|\boldsymbol{\theta}_o)\right]}_{\text{suffices to show this is zero!}}$$

The Information Matrix Equality: if $p_o(\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}_o)$, then $\mathbf{K} = \mathbf{J}$.

$$\boxed{\text{suffices to show } \mathbb{E}\left[\frac{1}{f(\mathbf{y}|\boldsymbol{\theta}_o)} \cdot \frac{\partial^2}{\partial\theta_i\partial\theta_j} f(\mathbf{y}|\boldsymbol{\theta}_o)\right] = 0}$$

Step 4: Use $p_o(\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}_o)$

$$\mathbb{E}\left[\frac{1}{f(\mathbf{y}|\boldsymbol{\theta}_o)} \cdot \frac{\partial^2}{\partial\theta_i\partial\theta_j} f(\mathbf{y}|\boldsymbol{\theta}_o)\right] \equiv \int \left[\frac{1}{f(\mathbf{y}|\boldsymbol{\theta}_o)} \cdot \frac{\partial^2}{\partial\theta_i\partial\theta_j} f(\mathbf{y}|\boldsymbol{\theta}_o)\right] p_o(\mathbf{y})\, d\mathbf{y}$$

$$= \int \left[\frac{1}{\color{red}{f(\mathbf{y}|\boldsymbol{\theta}_o)}} \cdot \frac{\partial^2}{\partial\theta_i\partial\theta_j} f(\mathbf{y}|\boldsymbol{\theta}_o)\right] \color{red}{f(\mathbf{y}|\boldsymbol{\theta}_o)}\, d\mathbf{y} = \int \frac{\partial^2}{\partial\theta_i\partial\theta_j} f(\mathbf{y}|\boldsymbol{\theta}_o)\, d\mathbf{y}$$

$$= \frac{\partial^2}{\partial\theta_i\partial\theta_j} \int f(\mathbf{y}|\boldsymbol{\theta}_o)\, d\mathbf{y} = \frac{\partial^2}{\partial\theta_i\partial\theta_j} (1) = 0$$

A Simple Example Continued Again: Asymptotic Variance Calculations

Poisson(θ) model, possibly mis-specified.

Ingredients

$$\boxed{\begin{aligned} \log f(y|\theta) &= -\theta + y\log(\theta) - \log(y!) \\ \frac{d}{d\theta}\log f(y|\theta) &= -1 + y/\theta \\ \frac{d^2}{d\theta^2}\log f(y|\theta) &= -y/\theta^2 \\ \theta_o = \mathbb{E}[y], &\quad \widehat{\theta} = \bar{y} \end{aligned}}$$

$$J = -\mathbb{E}\left[\frac{d^2}{d\theta^2}\log f(y|\theta_o)\right] = 1/\mathbb{E}[y]$$

$$\widehat{J} = -\frac{1}{N}\sum_{i=1}^{N}\frac{d^2}{d\theta^2}\log f(y_i|\widehat{\theta}) = 1/\bar{y}$$

$$K = \text{Var}\left[\frac{d}{d\theta}\log f(y|\theta_o)\right] = \text{Var}(y)/\mathbb{E}[y]^2$$

$$\widehat{K} = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{d}{d\theta}\log f(y_i|\widehat{\theta})\right]^2 = s_y^2/(\bar{y})^2$$

where $s_y^2 \equiv \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{y})^2$ and $\bar{y} \equiv \frac{1}{N}\sum_{i=1}^{n} y_i$

A Simple Example Continued Again: Asymptotic Variance Calculations

From Previous Slide

$$\theta_0 = \mathbb{E}[y], \quad J = 1/\mathbb{E}[y], \quad \widehat{J} = 1/\bar{y}, \quad K = \mathrm{Var}(y)/\mathbb{E}[y]^2, \quad \widehat{K} = s_y^2/(\bar{y})^2$$

Correct Specification

$$\boxed{y_1, \ldots, y_N \sim \text{ iid Poisson}(\theta_o)} \implies \boxed{J = K = 1/\theta_o} \implies \boxed{J^{-1}KJ^{-1} = \theta_o = \mathbb{E}[y]}$$

Potential Mis-specification

$$\boxed{y_1, \ldots, y_N \sim \text{ iid}} \implies \boxed{J = 1/\mathbb{E}[y], \quad K = \mathrm{Var}(y)/\mathbb{E}[y]^2} \implies \boxed{J^{-1}KJ^{-1} = \mathrm{Var}(y)}$$

A Simple Example Continued Again: Asymptotic Variance Calculations

Comparison of Asymptotic Distributions

$$\boxed{y_1, \ldots, y_N \sim \text{ iid Poisson}(\theta_o)} \implies \sqrt{N}(\widehat{\theta} - \theta_o) = \sqrt{N}(\bar{y} - \mathbb{E}[y]) \to_d \mathcal{N}(0, \mathbb{E}[y])$$

$$\boxed{y_1, \ldots, y_N \sim \text{ iid}} \implies \sqrt{N}(\widehat{\theta} - \theta_o) = \sqrt{N}(\bar{y} - \mathbb{E}[y]) \to_d \mathcal{N}(0, \text{Var}[y])$$

Comparison of Asymptotic 95% CIs

$$\boxed{y_1, \ldots, y_N \sim \text{ iid Poisson}(\theta_o)} \implies \bar{y} \pm 1.96 \times \sqrt{\bar{y}/N}$$

$$\boxed{y_1, \ldots, y_N \sim \text{ iid}} \implies \bar{y} \pm 1.96 \times s_y/\sqrt{N}$$

Punch Line

Unless $\text{Var}(y) = \mathbb{E}[y]$, CIs/tests that assume the Poisson model is true are wrong!

Lecture #2 – Poisson Regression

Review: Minimum MSE Predictor / Minimum MSE Linear Predictor

Why not just use OLS?

Conditional Maximum Likelihood Estimation

Poisson Regression: A Robust Model for Count Data

Asymptotic Variance Calculations for Poisson Regression

How to predict a count variable?

Example

Suppose we want to predict y using $\mathbf{x}$, where:

- $y \equiv \#$ of children a woman has: a count variable, i.e. $y \in \{0, 1, 2, \ldots\}$
- $\mathbf{x} \equiv$\{years of schooling, age, married, etc.\}

Minimum MSE Predictor

$\mu(\mathbf{x}) \equiv \mathbb{E}(y|\mathbf{x})$ minimizes $\mathbb{E}\left[\{y - \varphi(\mathbf{x})\}^2\right]$ over all possible predictors $\varphi(\cdot)$.

Minimum MSE Linear Predictor

$\boldsymbol{\beta} \equiv \mathbb{E}\left[\mathbf{x}\mathbf{x}'\right]^{-1}\mathbb{E}[\mathbf{x}y]$ minimizes $\mathbb{E}\left[(y - \mathbf{x}'\boldsymbol{\theta})^2\right]$ over all linear predictors $\mathbf{x}'\boldsymbol{\theta}$.

Proof: $\mathbb{E}(y|\mathbf{x})$ is the minimum MSE predictor

Step 1: add and subtract $\mu(\mathbf{x}) \equiv \mathbb{E}(y|\mathbf{x})$

$$\mathbb{E}\left[\{y-\varphi(\mathbf{x})\}^2\right] = \mathbb{E}\left[\big\{\big(y-\mu(\mathbf{x})\big)-\big(\varphi(\mathbf{x})-\mu(\mathbf{x})\big)\big\}^2\right]$$
$$= \mathbb{E}\left[\{y-\mu(\mathbf{x})\}^2\right] - 2\mathbb{E}\left[\{y-\mu(\mathbf{x})\}\{\varphi(\mathbf{x})-\mu(\mathbf{x})\}\right] + \mathbb{E}\left[\{\varphi(\mathbf{x})-\mu(\mathbf{x})\}^2\right]$$

Step 2: iterated expectations

$$\mathbb{E}\left[\{y-\mu(\mathbf{x})\}\{\varphi(\mathbf{x})-\mu(\mathbf{x})\}\right] = \mathbb{E}\Big(\mathbb{E}\left[\{y-\mu(\mathbf{x})\}\{\varphi(\mathbf{x})-\mu(\mathbf{x})\}|\mathbf{x}\right]\Big)$$
$$= \mathbb{E}\Big(\left[\varphi(\mathbf{x})-\mu(\mathbf{x})\right]\left[\mathbb{E}(y|\mathbf{x})-\mu(\mathbf{x})\right]\Big) = 0$$

Step 3: combine steps 1 & 2

$$\mathbb{E}\left[\{y-\varphi(\mathbf{x})\}^2\right] = \underbrace{\mathbb{E}\left[\{y-\mu(\mathbf{x})\}^2\right]}_{\text{constant wrt } \varphi} + \underbrace{\mathbb{E}\left[\{\varphi(\mathbf{x})-\mu(\mathbf{x})\}^2\right]}_{\text{cannot be negative; zero if } \varphi=\mu}$$

Proof: OLS is the Minimum MSE Linear Predictor

Objective Function

$$\mathbb{E}\left[\left(y - \mathbf{x}'\boldsymbol{\theta}\right)^2\right] = \mathbb{E}[y^2] - 2\mathbb{E}[y\mathbf{x}']\boldsymbol{\theta} + \boldsymbol{\theta}'\mathbb{E}\left[\mathbf{x}\mathbf{x}'\right]\boldsymbol{\theta}$$

Recall: Matrix Differentiation

$$\frac{\partial(\mathbf{a}'\mathbf{z})}{\partial\mathbf{z}} = \mathbf{a}, \quad \frac{\partial(\mathbf{z}'\mathbf{A}\mathbf{z})}{\partial\mathbf{z}} = (\mathbf{A} + \mathbf{A}')\mathbf{z}$$

First-Order Condition

$$-2\mathbb{E}\left[\mathbf{x}y\right] + 2\mathbb{E}[\mathbf{x}\mathbf{x}']\boldsymbol{\beta} = 0 \implies \boldsymbol{\beta} = \mathbb{E}\left[\mathbf{x}\mathbf{x}'\right]^{-1}\mathbb{E}\left[\mathbf{x}y\right]$$

Problems with linear-in-parameters models for count data

Best predictor is $\mathbb{E}(y|\mathbf{x})$ but how can we estimate this?

Plain-vanilla OLS?

- If $\mathbb{E}(y|\mathbf{x}) \approx \mathbf{x}'\boldsymbol{\beta}$, OLS is a reasonable approach.
- Problem: y is a count so it *can't* be negative, but OLS prediction $\mathbf{x}'\boldsymbol{\beta}$ could be.

OLS for $\log(y)$?

- Log-linear model $\log(y) = \mathbf{x}'\beta + \varepsilon$
- Solves the problem of negative predictions: $\log(y)$ *can* be negative.
- Problem: if y is a count it could equal zero but $\log(0) = -\infty$!

A realistic model for count data *must* be nonlinear in parameters.

General Approach

- Assume that $\mathbb{E}(y|\mathbf{x}) = m(\mathbf{x};\boldsymbol{\beta})$ where m is a known parametric function.
- Choose m so that it is always positive, regardless of $\mathbf{x}$ and $\boldsymbol{\beta}$.
- This means m *cannot* be linear.

This Lecture: $m(\mathbf{x};\boldsymbol{\beta}) = \exp(\mathbf{x}'\boldsymbol{\beta})$

- Always strictly positive
- Common choice in practice
- Everything I'll discuss works with other choices of m, making appropriate changes.

How to estimate β_o?

Assumption: $\mathbb{E}(y|\mathbf{x}) = \exp\left(\mathbf{x}'\boldsymbol{\beta}_o\right)$

Using our argument from above, $\boldsymbol{\beta}_o$ minimizes $\mathbb{E}\left[\{y_i - \exp(\mathbf{x}_i'\boldsymbol{\beta})\}^2\right]$ over all $\boldsymbol{\beta}$.

Nonlinear Least Squares (NLLS)

$\widehat{\beta}_{NLLS}$ is the minimizer of $\sum_{i=1}^{N}\left\{y_i - \exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\right\}^2$

Poisson Regression (MLE)

$\widehat{\beta}_{MLE}$ is the MLE for β_o under the model $y_i|\mathbf{x}_i \sim$ indep. $\text{Poisson}\big(\exp(\mathbf{x}_i'\boldsymbol{\beta}_o)\big)$

Conditional versus Unconditional MLE

Last Lecture: Unconditional MLE

Model *unconditional* dist. of a random vector $\mathbf{y}$: $f(\mathbf{y}|\boldsymbol{\theta})$.

This Lecture: Conditional MLE

Model *conditional* dist. of a random variable y *given* a random vector $\mathbf{x}$: $f(\mathbf{y}|\mathbf{x},\boldsymbol{\theta})$.

Why Conditional MLE?

- Unconditional MLE requires joint distribution: $f(y,\mathbf{x}|\boldsymbol{\theta}) = f(y|\mathbf{x},\boldsymbol{\theta})f(\mathbf{x}|\boldsymbol{\theta})$
- $\mathbb{E}(y|\mathbf{x})$ only depends on $f(y|\mathbf{x},\boldsymbol{\theta})$ not $f(\mathbf{x}|\boldsymbol{\theta})$.
- Not interested in $f(\mathbf{x}|\boldsymbol{\theta})$; coming up with a good model for it is challenging.
- Caveat: unconditional MLE is more efficient provided the model for $\mathbf{x}$ is correct.

The Conditional Maximum Likelihood Estimator

Assuming iid data.

Sample

$$\widehat{\boldsymbol{\theta}} \equiv \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \frac{1}{N}\sum_{i=1}^{N} \log f(y_i|\mathbf{x}_i, \boldsymbol{\theta})$$

Population

$$\boldsymbol{\theta}_o \equiv \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \mathbb{E}\left[\log f(y_i|\mathbf{x}_i, \boldsymbol{\theta})\right]$$

Important

- We only model the conditional distribution $y|\mathbf{x}$, but...
- ...the expectation $\mathbb{E}[\log f(y_i|\mathbf{x}_i, \boldsymbol{\theta})]$ is taken over the *joint distribution* of $(y, \mathbf{x})$.
- $f(y_i|\mathbf{x}_i, \boldsymbol{\theta})$ is merely a *function* of the RVs $(y_i, \mathbf{x}_i)$.

Poisson Regression as a Conditional MLE

Model: $y_i|\mathbf{x}_i \sim \text{Poisson}\left(\exp\{\mathbf{x}_i'\boldsymbol{\beta}\} \right)$

$$\ell_i(\boldsymbol{\beta}) \equiv \log f(y_i|\mathbf{x}_i, \boldsymbol{\beta}) = y_i \mathbf{x}_i'\boldsymbol{\beta} - \exp(\mathbf{x}_i'\boldsymbol{\beta}) - \log(y_i!)$$

$$\underbrace{\mathbf{s}_i(\boldsymbol{\beta})}_{\text{score vector}} \equiv \frac{\partial \ell_i(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \mathbf{x}_i\left[y_i - \exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\right]$$

$$\widehat{\boldsymbol{\beta}} \text{ solves } \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i \underbrace{\left[y_i - \exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\right]}_{\text{residual: } u_i} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i u_i(\boldsymbol{\beta}) = \mathbf{0}$$

Average Partial Effects

Partial Effects

For continuous x_j, we call $\frac{\partial}{\partial x_j}\mathbb{E}(y|\mathbf{x})$ the partial effect of x_j. For discrete x_j the partial effect is the difference of $\mathbb{E}(y|\mathbf{x})$ at two different values of x_j

Average Partial Effects (APE)

In nonlinear models, partial effects typically vary with $\mathbf{x}$. The average partial effect is the expectation of the partial effect over the distribution of $\mathbf{x}$.

Average Partial Effects for Poisson Regression

Partial Effect

$$\frac{\partial}{\partial x_j}\mathbb{E}(y|\mathbf{x}) = \frac{\partial}{\partial x_j}\exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right) = \exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\beta_j$$

Average Partial Effect

$$\mathbb{E}\left[\frac{\partial}{\partial x_j}\exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\right] = \mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\right]\beta_j$$

Estimated Partial Effect

$$\exp\left(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}\right)\widehat{\beta}_j$$

Estimated Average Partial Effect

$$\left[\frac{1}{N}\sum_{i=1}^{N}\exp\left(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}\right)\right]\widehat{\beta}_j$$

Relative Effects

The *ratio* of partial effects does not depend on $\mathbf{x}$: relative effects are constant.

Problem Set

Poisson regression: APE$=\bar{y}\widehat{\beta}_j$. Multiply by $\bar{y}$ to put coefficients on the scale of OLS.

Conditional MLE Under Mis-specification

Basically identical to the unconditional version.

Theorem

Suppose that $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^N \sim$ iid p_o and let $\widehat{\boldsymbol{\theta}}$ denote the Conditional MLE for $\boldsymbol{\theta}$ under the possibly mis-specified model $f(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta})$. Then, under mild regularity conditions:

(i) $\widehat{\boldsymbol{\theta}}$ is consistent for the pseudo-true parameter value $\boldsymbol{\theta}_o$, defined as the *maximizer* of the expected log likelihood $\mathbb{E}\left[\log f(y|\mathbf{x}, \boldsymbol{\theta})\right]$ over the parameter space Θ.

(ii) $\sqrt{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \to_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1})$

where we define $\mathbf{J} \equiv -\mathbb{E}\left[\dfrac{\partial^2 \log f(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right]$ and $\mathbf{K} \equiv \text{Var}\left[\dfrac{\partial \log f(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}}\right]$.

Conditional MLE Under Correct Specification

Basically identical to the unconditional version.

Theorem

Suppose that $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^N \sim$ iid where the conditional distribution of $y_i|\mathbf{x}_i$ is given by $f(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}_o)$. Then, under mild regularity conditions,

(i) $\widehat{\boldsymbol{\theta}}$ is consistent for $\boldsymbol{\theta}_o$

(ii) $\sqrt{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1})$ where $\mathbf{J} \equiv -\mathbb{E}\left[\dfrac{\partial^2 \log f(\mathbf{y}|\mathbf{x}, \boldsymbol{\theta}_o)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right]$

What value of $\boldsymbol{\beta}$ maximizes $\mathbb{E}\left[\ell_i(\boldsymbol{\beta})\right]$?

Iterated Expectations

$$\mathbb{E}[\ell_i(\boldsymbol{\beta})] = \mathbb{E}\left\{\mathbb{E}\left[\ell_i(\boldsymbol{\beta})|\mathbf{x}_i\right]\right\} = \mathbb{E}\left\{\mathbb{E}\left[y_i \mathbf{x}_i'\boldsymbol{\beta} - \exp(\mathbf{x}_i'\boldsymbol{\beta}) - \log\left(y_i!\right)|\mathbf{x}_i\right]\right\}$$

Simplify Inner Expectation

$$\mathbb{E}\left[\ell_i(\boldsymbol{\beta})|\mathbf{x}_i\right] = \mathbf{x}_i'\boldsymbol{\beta}\mathbb{E}\left[y_i|\mathbf{x}_i\right] - \exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right) - \underbrace{\mathbb{E}\left[\log\left(y_i!\right)|\mathbf{x}_i\right]}_{\text{constant wrt } \mathbf{x}_i}$$

FOC for Inner Expectation

$$\frac{\partial}{\partial\boldsymbol{\beta}}\mathbb{E}\left[\ell_i(\boldsymbol{\beta})|\mathbf{x}_i\right] = \left\{\mathbb{E}\left[y_i|\mathbf{x}_i\right] - \exp\left(\mathbf{x}_i'\boldsymbol{\beta}\right)\right\}\mathbf{x}_i = \mathbf{0}$$

What value of β maximizes $\mathbb{E}\left[\ell_i(\boldsymbol{\beta})\right]$?

$$\boxed{\frac{\partial}{\partial \boldsymbol{\beta}} \mathbb{E}\left[\ell_i(\boldsymbol{\beta}) | \mathbf{x}_i\right] = \left\{\mathbb{E}\left[y_i | \mathbf{x}_i\right] - \exp\left(\mathbf{x}_i' \boldsymbol{\beta}\right)\right\} \mathbf{x}_i = \mathbf{0}}$$

What does this mean?

Since $\mathbb{E}\left[y_i|\mathbf{x}_i\right] = \exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)$, setting $\boldsymbol{\beta} = \boldsymbol{\beta}_o$ solves the FOC for the inner expectation!

In other words:

For any realization of $\mathbf{x}_i$ and any $\boldsymbol{\beta}$,

$$\mathbb{E}[\ell_i(\boldsymbol{\beta})|\mathbf{x}_i] \leq \mathbb{E}[\ell_i(\boldsymbol{\beta}_o)|\mathbf{x}_i]$$

so taking expectations of both sides:

$$\mathbb{E}\left[\ell_i(\boldsymbol{\beta})\right] = \mathbb{E}\left\{\mathbb{E}[\ell_i(\boldsymbol{\beta})|\mathbf{x}_i]\right\} \leq \mathbb{E}\left\{\mathbb{E}[\ell_i(\boldsymbol{\beta}_o)|\mathbf{x}_i]\right\} = \mathbb{E}\left[\ell_i(\boldsymbol{\beta}_o)\right]$$

Poisson Regression is consistent if $\mathbb{E}(y|\mathbf{x})$ is correctly specified.

We showed this for a particular choice of $m(\mathbf{x};\boldsymbol{\beta})$ but the result is general.

Result

Provided that we have correctly specified $\mathbb{E}(y_i|\mathbf{x}_i)$, it *doesn't matter* if $y_i|\mathbf{x}_i$ actually follows a Poisson distribution: Poisson regression is *still consistent* for $\boldsymbol{\beta}_o$.

Compare

This is very similar to our result for the Poisson(θ) model from last lecture.

Caveat

Strictly speaking we need to show that $\boldsymbol{\beta}_o$ is the *unique* maximizer of the expected log likelihood. *Multiple solutions* if $\mathbf{x}_i$ perfectly co-linear (compare to OLS regression).

Asymptotic Variance Calculations for Poisson Regression

$$\underbrace{\mathbf{s}_i(\boldsymbol{\beta})}_{\text{score vector}} \equiv \frac{\partial \ell_i(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \mathbf{x}_i \left[y_i - \exp\left(\mathbf{x}_i' \boldsymbol{\beta} \right) \right] = \mathbf{x}_i u_i(\boldsymbol{\beta})$$

$$\underbrace{\mathbf{H}_i(\boldsymbol{\beta})}_{\text{Hessian matrix}} \equiv \frac{\partial \mathbf{s}_i(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}'} = -\exp\left(\mathbf{x}_i' \boldsymbol{\beta} \right) \mathbf{x}_i \mathbf{x}_i'$$

$$\mathbf{J} \equiv -\mathbb{E}\left[\mathbf{H}_i(\boldsymbol{\beta}_o) \right] = \mathbb{E}\left[\exp\left(\mathbf{x}_i' \boldsymbol{\beta}_o \right) \mathbf{x}_i \mathbf{x}_i' \right]$$

$$\mathbf{K} \equiv \mathrm{Var}\left[\mathbf{s}_i(\boldsymbol{\beta}_o) \right] = \mathbb{E}\left[\mathbf{s}_i(\boldsymbol{\beta}_o) \mathbf{s}_i(\boldsymbol{\beta}_o)' \right] = \mathbb{E}\left[u_i^2(\boldsymbol{\beta}_o) \mathbf{x}_i \mathbf{x}_i' \right]$$

Asymptotic Variance Calculations for Poisson Regression

$$\boxed{\mathbf{J} = \mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)\mathbf{x}_i\mathbf{x}_i'\right], \quad \mathbf{K} = \mathbb{E}\left[u_i^2(\boldsymbol{\beta}_o)\mathbf{x}_i\mathbf{x}_i'\right]}$$

Notice

$\mathbf{J}$ does not depend on y but $\mathbf{K}$ does:

$$\mathbf{K} = \mathbb{E}\left[u_i^2(\boldsymbol{\beta}_o)\mathbf{x}_i\mathbf{x}_i'\right] = \mathbb{E}\left\{\mathbb{E}\left[u_i^2(\boldsymbol{\beta}_o)|\mathbf{x}_i\right]\mathbf{x}_i\mathbf{x}_i'\right\} = \mathbb{E}\left(\mathbb{E}\left[\{y_i - \mathbb{E}(y_i|\mathbf{x}_i)\}^2|\mathbf{x}_i\right]\mathbf{x}_i\mathbf{x}_i'\right)$$
$$= \mathbb{E}\left[\mathrm{Var}(y_i|\mathbf{x}_i)\mathbf{x}_i\mathbf{x}_i'\right]$$

Assumptions about $\mathrm{Var}(y|\mathbf{x})$ affect the asymptotic variance through $\mathbf{K}$.

Possible Assumptions for $\text{Var}(y|\mathbf{x})$: Strongest to Weakest

1. Poisson Assumption: $\text{Var}(y|\mathbf{x}) = \mathbb{E}(y|\mathbf{x})$
   - holds if Poisson model is correct.
2. Quasi-Poisson Assumption: $\text{Var}(y|\mathbf{x}) = \sigma^2\mathbb{E}(y|\mathbf{x})$
   - Allows for possibility that $y|\mathbf{x}$ is *not* Poisson
   - Overdispersion: $\sigma^2 > 1 \implies \text{Var}(y|\mathbf{x}) > \mathbb{E}(y|\mathbf{x})$
   - Underdispersion $\sigma^2 < 1 \implies \text{Var}(y|\mathbf{x}) < \mathbb{E}(y|\mathbf{x})$
   - If $\sigma^2 = 1$ we're back to the Poisson Assumption.
3. No Assumption: $\text{Var}(y|\mathbf{x})$ unspecified

Asymptotic Variance Under Poisson Assumption

$$\boxed{\mathbf{J} = \mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)\mathbf{x}_i\mathbf{x}_i'\right], \quad \mathbf{K} = \mathbb{E}\left[\mathrm{Var}(y_i|\mathbf{x}_i)\mathbf{x}_i\mathbf{x}_i'\right]}$$

Assumption: $\mathrm{Var}(y_i|\mathbf{x}_i) = \mathbb{E}(y_i|\mathbf{x}_i) = \exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)$

- Implies $\mathbf{K} = \mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)\mathbf{x}_i\mathbf{x}_i'\right]$
- Hence $\mathbf{K} = \mathbf{J}$ (Information Matrix Equality)
- Therefore: $\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \to_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1})$
- Consistent Estimator: $\widehat{\mathbf{J}}^{-1} = \left[\frac{1}{N}\sum_{i=1}^{N}\exp\left(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}\right)\mathbf{x}_i\mathbf{x}_i'\right]^{-1}$

Asymptotic Variance Under Quasi-Poisson Assumption

$$\boxed{\mathbf{J} = \mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)\mathbf{x}_i\mathbf{x}_i'\right], \quad \mathbf{K} = \mathbb{E}\left[\text{Var}(y_i|\mathbf{x}_i)\mathbf{x}_i\mathbf{x}_i'\right]}$$

Assumption: $\text{Var}(y_i|\mathbf{x}_i) = \sigma^2\mathbb{E}(y_i|\mathbf{x}_i) = \sigma^2\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)$

- Implies $\mathbf{K} = \sigma^2\mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)\mathbf{x}_i\mathbf{x}_i'\right] = \sigma^2\mathbf{J}$
- Hence $\mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1} = \sigma^2\mathbf{J}^{-1}$
- Therefore: $\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \to_d \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{J}^{-1})$
- Consistent estimator of $\mathbf{J}^{-1}$ on prev. slide but how can we estimate σ^2?

How to estimate σ^2 under the Quasi-Poisson Assumption?

$$\begin{aligned}
\text{Var}(y|\mathbf{x}) &= \sigma^2 \mathbb{E}(y|\mathbf{x}) \\
\sigma^2 &= \text{Var}(y|\mathbf{x}) / \mathbb{E}(y|\mathbf{x}) \\
\sigma^2 &= \mathbb{E}\left[\{y - \mathbb{E}(y|\mathbf{x})\}^2 \,|\mathbf{x} \right] / \mathbb{E}(y|\mathbf{x}) \\
\sigma^2 &= \mathbb{E}\left[\left. \frac{\{y - \mathbb{E}(y|\mathbf{x})\}^2}{\mathbb{E}(y|\mathbf{x})} \right| \mathbf{x} \right] \\
\sigma^2 &= \mathbb{E}\left[\left. \frac{\{y - \exp(\mathbf{x}'\boldsymbol{\beta}_o)\}^2}{\exp(\mathbf{x}'\boldsymbol{\beta})} \right| \mathbf{x} \right] \\
\mathbb{E}[\sigma^2] &= \mathbb{E}\left(\mathbb{E}\left[\left. \frac{\{y - \exp(\mathbf{x}'\boldsymbol{\beta}_o)\}^2}{\exp(\mathbf{x}'\boldsymbol{\beta})} \right| \mathbf{x} \right] \right) \\
\sigma^2 &= \mathbb{E}\left[\frac{\{y - \exp(\mathbf{x}'\boldsymbol{\beta}_o)\}^2}{\exp(\mathbf{x}'\boldsymbol{\beta})} \right] \\
\sigma^2 &= \mathbb{E}\left[u^2(\boldsymbol{\beta}_o) / \exp(\mathbf{x}'\boldsymbol{\beta}_o) \right]
\end{aligned}$$

Consistent Estimator of σ^2

$$\widehat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{N} \frac{[y_i - \exp(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})]^2}{\exp(\mathbf{x}_i\widehat{\boldsymbol{\beta}})} = \frac{1}{N} \sum_{i=1}^{N} \frac{\widehat{u}_i^2}{\exp(\mathbf{x}_i\widehat{\boldsymbol{\beta}})}$$

Robust Asymptotic Variance Matrix

$$\boxed{\mathbf{J} = \mathbb{E}\left[\exp\left(\mathbf{x}_i'\boldsymbol{\beta}_o\right)\mathbf{x}_i\mathbf{x}_i'\right], \quad \mathbf{K} = \mathbb{E}\left[u_i^2(\boldsymbol{\beta}_o)\mathbf{x}_i\mathbf{x}_i'\right]}$$

No Assumption on $\text{Var}(y_i|\mathbf{x}_i)$

- $\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \to_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1})$
- Consistent Estimator: $\widehat{\mathbf{J}}^{-1} = \left[\frac{1}{N}\sum_{i=1}^{N}\exp\left(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}\right)\mathbf{x}_i\mathbf{x}_i'\right]^{-1}$
- Consistent Estimator: $\widehat{\mathbf{K}} = \frac{1}{N}\sum_{i=1}^{N}\left[y_i - \exp(\mathbf{x}_i\widehat{\boldsymbol{\beta}})\right]^2\mathbf{x}_i\mathbf{x}_i' = \frac{1}{N}\sum_{i=1}^{N}\widehat{u}_i^2\mathbf{x}_i\mathbf{x}_i'$

Why Poisson Regression rather than NLLS?

Assume that $\mathbb{E}(y|\mathbf{x}) = \exp(\mathbf{x}'\beta_o)$

Both Poisson Reg. & NLLS are consistent if the conditional mean is correctly specified.

Count data are typically heteroskedastic.

If $\text{Var}(y|\mathbf{x})$ varies with $\mathbf{x}$, NLLS will be relatively inefficient.

Efficiency of Poisson Regression

- Correct model $\implies$ lowest variance among all estimators that leave the distribution of $\mathbf{x}$ unspecified.
- $\text{Var}(y|\mathbf{x}) = \sigma^2\mathbb{E}(y|\mathbf{x}) \implies$ Poisson regression is more efficient than NLLS and various other count data models.

Lecture #3 – Models for Binary Outcomes

Properties of Binary Outcome Models

Linear Probability Model

Index Models (e.g. Logit & Probit)

Partial Effects

Conditional MLE for Index Models

Pseudo R-squared

Models for Binary Outcomes

Example

- Outcome: $y = 1$ if employed, 0 otherwise
- Predictors/Regressors: $\mathbf{x} = \{\text{age, sex, education, experience, ...}\}$

Question

How does x_j affect our prediction of y holding the other regressors constant?

We'll consider three models:

1. Linear Probability Model (LPM)
2. Logistic Regression (Logit)
3. Probit Regression (Probit)

Properties of Binary Outcome Models: $y \in \{0, 1\}$

Notation

$p(\mathbf{x}) \equiv \mathbb{P}(y = 1|\mathbf{x})$

Conditional Mean

$\mathbb{E}(y|\mathbf{x}) = p(\mathbf{x})$

Conditional Variance

$\text{Var}(y|\mathbf{x}) = p(\mathbf{x})\,[1 - p(\mathbf{x})]$

$$
\begin{aligned}
\mathbb{E}(y|\mathbf{x}) &= 0 \times \mathbb{P}(y = 0|\mathbf{x}) + 1 \times \mathbb{P}(y = 1|\mathbf{x}) \\
&= \mathbb{P}(y = 1|\mathbf{x}) \equiv p(\mathbf{x})
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(y^2|\mathbf{x}) &= \left\{0^2 \times [1 - p(\mathbf{x})] + 1^2 \times p(\mathbf{x})\right\} \\
&= p(\mathbf{x})
\end{aligned}
$$

$$
\begin{aligned}
\text{Var}(y|\mathbf{x}) &= \mathbb{E}(y^2|\mathbf{x}) - \mathbb{E}(y|\mathbf{x})^2 \\
&= \left\{0^2 \times [1 - p(\mathbf{x})] + 1^2 \times p(\mathbf{x})\right\} - p(\mathbf{x})^2 \\
&= p(\mathbf{x})\,[1 - p(\mathbf{x})]
\end{aligned}
$$

The Linear Probability Model: Assume $p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}$

Conditional Mean & Variance

- $\mathbb{E}(y|\mathbf{x}) = p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}$
- $\text{Var}(y|\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}\,(1 - \mathbf{x}'\boldsymbol{\beta})$

This is just Linear Regression!

$$y = \mathbf{x}'\boldsymbol{\beta} + u, \quad \mathbb{E}(u|\mathbf{x}) = 0$$

But u is Heteroskedastic

$$\text{Var}(u|\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}(1 - \mathbf{x}'\boldsymbol{\beta})$$

$$\begin{aligned}
\mathbb{E}(u|\mathbf{x}) &= \mathbb{E}(y - \mathbf{x}'\boldsymbol{\beta}|\mathbf{x}) = \mathbb{E}(y|\mathbf{x}) - \mathbf{x}'\boldsymbol{\beta} \\
&= \mathbf{x}'\boldsymbol{\beta} - \mathbf{x}'\boldsymbol{\beta} = 0
\end{aligned}$$

$$\begin{aligned}
\text{Var}(u|\mathbf{x}) &= \mathbb{E}\left[\{u - \mathbb{E}(u|\mathbf{x})\}^2 |\mathbf{x}\right] = \mathbb{E}\left[u^2|\mathbf{x}\right] \\
&= \mathbb{E}\left[(y - \mathbf{x}'\boldsymbol{\beta})^2 |\mathbf{x}\right] \\
&= \mathbb{E}\left(y^2|\mathbf{x}\right) - 2\mathbb{E}\left(y|\mathbf{x}\right)\mathbf{x}'\boldsymbol{\beta} + \left(\mathbf{x}'\boldsymbol{\beta}\right)^2 \\
&= p(\mathbf{x}) - 2p(\mathbf{x})p(\mathbf{x}) + p(\mathbf{x})^2 \\
&= p(\mathbf{x})\left[1 - p(\mathbf{x})\right]
\end{aligned}$$

The Linear Probability Model: Assume $p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}$

Estimation

Since $\mathbb{E}(u|\mathbf{x}) = 0$ OLS estimation of $y = \mathbf{x}'\boldsymbol{\beta} + u$ is unbiased and consistent.

Inference

Since u is heteroskedastic, tests and CIs should use robust standard errors.

Is the LPM actually reasonable?

- Assumes $p(\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta} \implies$ changing x_j by Δ changes $p(\mathbf{x})$ by $\beta_j\Delta$
- If $\mathbf{x}$ contains regressors without upper/lower bounds, $p(\mathbf{x})$ could be > 1 or < 0!
- LPM could be a reasonable approximation but cannot be *literally* true.

Index Models: Constrain $p(\mathbf{x})$ to lie in $[0, 1]$

Index Model: $p(\mathbf{x}) = G(\mathbf{x}'\boldsymbol{\beta})$

Assume $\mathbf{x}$ includes a constant, $0 \leq G(\cdot) \leq 1$, G is differentiable and strictly increasing, $\lim_{z\to\infty} G(z) = 1$, and $\lim_{z\to-\infty} G(z) = 0$.

Terminology

We call $\mathbf{x}'\boldsymbol{\beta}$ the linear index and G the index function.

Partial Effects

Let $g(z) \equiv \frac{d}{dz}G(z)$. Then $\frac{\partial}{\partial x_j}p(\mathbf{x}) = g(\mathbf{x}'\boldsymbol{\beta})\beta_j$. Hence:

- The partial effect of x_j depends on the value of $\mathbf{x}$ at which we evaluate g.
- G strictly increasing $\implies g(\cdot) > 0 \implies$ sign of partial effect determined by β_j.

Possible Choices of Index Function

CDFs as Index Functions

G satisfies the index model assumptions (prev. slide) iff it is a continuous CDF.

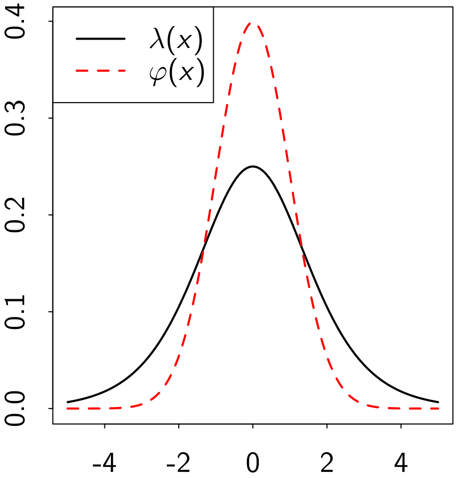
We focus on two examples:

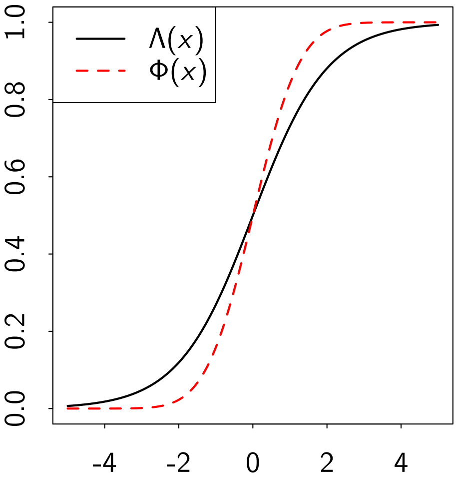
1. Logit: $G(z) = \Lambda(z) \equiv \exp(z)/\left[1 + \exp(z)\right]$
2. Probit: $G(z) = \Phi(z) \equiv \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-t^2/2\right)\, dt$

Notation:

- Λ is the CDF of a "standard logistic" RV and Φ of a standard normal RV.
- λ is the density of a "standard logistic" RV and φ of a standard normal
- To treat Logit and Probit simultaneously, we'll write G as a placeholder.

Standard $\boxed{\text{Logistic}}$ and $\boxed{\text{Normal}}$ Densities and CDFs





Partial Effects: $\partial p(\mathbf{x})/\partial x_j$

LPM

$$\frac{\partial}{\partial x_j}\mathbf{x}'\boldsymbol{\beta} = \beta_j$$

Logit

$$\frac{\partial}{\partial x_j}\Lambda(\mathbf{x}'\boldsymbol{\beta}) = \frac{\beta_j \exp(\mathbf{x}'\boldsymbol{\beta})}{[1+\exp(\mathbf{x}'\boldsymbol{\beta})]^2}$$

Probit

$$\frac{\partial}{\partial x_j}\Phi(\mathbf{x}'\beta) = \frac{\beta_j \exp\{-(\mathbf{x}'\boldsymbol{\beta})^2/2\}}{\sqrt{2\pi}}$$

$$\frac{\partial}{\partial x_j}G(\mathbf{x}'\boldsymbol{\beta}) = g(\mathbf{x}'\boldsymbol{\beta})\beta_j$$

$$\begin{aligned}\frac{d}{dz}\Lambda(z) \equiv \lambda(z) = \frac{d}{dz}\left(\frac{e^z}{1+e^z}\right) &= \frac{e^z(1+e^z)-e^z e^z}{(1+e^z)^2}\\ &= \frac{e^z}{(1+e^z)^2}\end{aligned}$$

$$\frac{d}{dz}\Phi(z) = \varphi(z) = \frac{\exp\left\{-z^2/2\right\}}{\sqrt{2\pi}}$$

Comparing Logit, Probit, and LPM Partial Effects

$$\boxed{\frac{\partial}{\partial x_j} G(\mathbf{x}'\boldsymbol{\beta}) = g(\mathbf{x}'\boldsymbol{\beta})\beta_j, \quad \frac{d}{dz}\Lambda(z) \equiv \lambda(z) = \frac{e^z}{(1+e^z)^2}, \quad \frac{d}{dz}\Phi(z) \equiv \varphi(z) = \frac{\exp\left\{-z^2/2\right\}}{\sqrt{2\pi}}}$$

Maximum Partial Effects

- λ and φ are unimodal with mode at 0

  Logit $\lambda(0) = 0.25$

  Probit $\varphi(0) = (2\pi)^{-1/2} \approx 0.4$

- *Maximum* partial effect when $\mathbf{x}'\boldsymbol{\beta} = 0$

  Logit $\beta_j \lambda(0) = 0.25\beta_j$

  Probit $\beta_j \varphi(0) \approx 0.4\beta_j$

- LPM has constant partial effects β_j

Relative Effects

$$\frac{\frac{\partial}{\partial x_j} p(\mathbf{x})}{\frac{\partial}{\partial x_h} p(\mathbf{x})} = \frac{\beta_j g(\mathbf{x}'\boldsymbol{\beta})}{\beta_h g(\mathbf{x}'\boldsymbol{\beta})} = \frac{\beta_j}{\beta_h}$$

Relative effects do not depend on **x**.

Average Partial Effects for Index Models

Partial Effect

$$\frac{\partial}{\partial x_j} G(\mathbf{x}_i'\boldsymbol{\beta}) = g(\mathbf{x}_i'\boldsymbol{\beta})\beta_j$$

Average Partial Effect

$$\mathbb{E}\left[\frac{\partial}{\partial x_j} G(\mathbf{x}_i'\boldsymbol{\beta})\right] = \mathbb{E}[g(\mathbf{x}_i'\boldsymbol{\beta})]\beta_j$$

Estimated Partial Effect

$$\frac{\partial}{\partial x_j} G(\mathbf{x}_i'\widehat{\boldsymbol{\beta}}) = g(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})\widehat{\beta}_j$$

Estimated Average Partial Effect

$$\left[\frac{1}{N}\sum_{i=1}^{N} g(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})\right]\widehat{\beta}_j$$

Warning:

APE $\neq$ partial effect evaluated at the average value of $\mathbf{x}$ since $\mathbb{E}[f(Z)] \neq f(\mathbb{E}[Z])$.

Conditional MLE for Index Models: iid Observations

Conditional Likelihood

$$f(y_i|\mathbf{x}_i,\boldsymbol{\beta}) = \begin{cases} 1 - G(\mathbf{x}_i'\boldsymbol{\beta}) & \text{if } y_i = 0 \\ G(\mathbf{x}_i'\boldsymbol{\beta}) & \text{if } y_i = 1 \end{cases} \iff f(y_i|\mathbf{x}_i,\boldsymbol{\beta}) = G(\mathbf{x}_i'\boldsymbol{\beta})^{y_i}\left[1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right]^{1-y_i}$$

Conditional Log-Likelihood

$$\ell_i(\boldsymbol{\beta}) \equiv \log f(y_i|\mathbf{x}_i,\boldsymbol{\beta}) = y_i \log\left[G(\mathbf{x}_i'\boldsymbol{\beta})\right] + (1 - y_i)\log\left[1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right]$$

Sample

$$\widehat{\boldsymbol{\beta}} \equiv \underset{\boldsymbol{\beta}\in\boldsymbol{\Theta}}{\arg\max} \frac{1}{N}\sum_{i=1}^{N} \ell_i(\boldsymbol{\beta})$$

Population

$$\boldsymbol{\beta}_o \equiv \underset{\boldsymbol{\beta}\in\boldsymbol{\Theta}}{\arg\max}\, \mathbb{E}\left[\ell(\boldsymbol{\beta})\right]$$

Correct specification: $\mathbb{E}(y|\mathbf{x}) = p(\mathbf{x}) = G(\mathbf{x}'\boldsymbol{\beta}_o)$. Otherwise $\boldsymbol{\beta}_o$ = KL-minimizer.

Asymptotic Variance Calculations for Index Models

Recall from last lecture.

Possibly Mis-specified Model

$\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1})$ where $\mathbf{J} = -\mathbb{E}\left[\mathbf{H}_i(\boldsymbol{\beta}_o)\right]$ and $\mathbf{K} = \mathbb{E}\left[\mathbf{s}_i(\boldsymbol{\beta}_o)\mathbf{s}_i(\boldsymbol{\beta}_o)'\right]$

Correct Specification

$\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \rightarrow_d \mathcal{N}(\mathbf{0}, \mathbf{J}^{-1})$ where $\mathbf{J} = -\mathbb{E}\left[\mathbf{H}_i(\boldsymbol{\beta}_o)\right]$

Asymptotic variance calculations for index models are complicated, but there's a clever trick for computing **J** under correct specification.

Asymptotic Variance Calculation for Correctly Specified Index Models

$$\boxed{\ell_i(\boldsymbol{\beta}) = y_i \log\left\{G(\mathbf{x}_i'\boldsymbol{\beta})\right\} + (1 - y_i)\log\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right\}}$$

Step 1: Calculate The Score Vector

$$\mathbf{s}_i \equiv \frac{\partial}{\partial \boldsymbol{\beta}}\ell_i(\boldsymbol{\beta}) = \frac{y_i g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i}{G(\mathbf{x}_i'\boldsymbol{\beta})} - \frac{(1 - y_i)g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i}{1 - G(\mathbf{x}_i'\boldsymbol{\beta})}$$

$$= \frac{g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i}{G(\mathbf{x}_i'\boldsymbol{\beta})\left[1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right]}\left\{\left[1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right]y_i - G(\mathbf{x}_i'\boldsymbol{\beta})(1 - y_i)\right\}$$

$$= \frac{g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i\left[y_i - G(\mathbf{x}_i'\boldsymbol{\beta})\right]}{G(\mathbf{x}_i'\boldsymbol{\beta})\left[1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right]}$$

Asymptotic Variance Calculation for Correctly Specified Index Models

$$\boxed{\mathbf{s}_i = \frac{g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i \left\{y_i - G(\mathbf{x}_i'\boldsymbol{\beta})\right\}}{G(\mathbf{x}_i'\boldsymbol{\beta})\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right\}}}$$

Step 2: Start Calculating the Hessian but give up because it's a nightmare.

$$\mathbf{H}_i(\boldsymbol{\beta}) \equiv \frac{\partial \mathbf{s}_i}{\partial \boldsymbol{\beta}'} = \frac{\partial}{\partial \boldsymbol{\beta}'}\left([y_i - G(\mathbf{x}_i'\boldsymbol{\beta})] \left[\frac{g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i}{G(\mathbf{x}_i'\boldsymbol{\beta})\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right\}} \right] \right)$$

$$= \frac{-g(\mathbf{x}_i'\boldsymbol{\beta})^2 \mathbf{x}_i\mathbf{x}_i'}{G(\mathbf{x}_i'\boldsymbol{\beta})\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right\}} + [y_i - G(\mathbf{x}_i'\boldsymbol{\beta})] \underbrace{\frac{\partial}{\partial \boldsymbol{\beta}'} \left\{ \frac{g(\mathbf{x}_i'\boldsymbol{\beta})\mathbf{x}_i}{G(\mathbf{x}_i'\boldsymbol{\beta})\left[1 - G(\mathbf{x}_i'\boldsymbol{\beta})\right]} \right\}}_{\text{a nasty awful mess: call it } \mathbf{M}(\mathbf{x}_i, \boldsymbol{\beta})}$$

Asymptotic Variance Calculation for Correctly Specified Index Models

$$\boxed{\mathbf{H}_i(\beta) = \frac{-g(\mathbf{x}_i'\beta)^2\mathbf{x}_i\mathbf{x}_i'}{G(\mathbf{x}_i'\beta)\left\{1 - G(\mathbf{x}_i'\beta)\right\}} + \left[y_i - G(\mathbf{x}_i'\beta)\right]\mathbf{M}(\mathbf{x}_i, \beta)}$$

Step 3: Calculate the *Conditional Expectation* instead...

$$\begin{aligned}
-\mathbb{E}\left[\mathbf{H}_i(\beta)|\mathbf{x}_i\right] &= \frac{g(\mathbf{x}_i'\beta)^2\mathbf{x}_i\mathbf{x}_i'}{G(\mathbf{x}_i'\beta)\left\{1 - G(\mathbf{x}_i'\beta)\right\}} + \underbrace{\mathbb{E}\left[y_i - G(\mathbf{x}_i'\beta)|\mathbf{x}_i\right]}_{\text{equals zero under correct spec.}} \mathbf{M}(\mathbf{x}_i, \beta) \\
&= \frac{g(\mathbf{x}_i'\beta)^2\mathbf{x}_i\mathbf{x}_i'}{G(\mathbf{x}_i'\beta)\left\{1 - G(\mathbf{x}_i'\beta)\right\}}
\end{aligned}$$

Step 4: Iterated Expectations

$$\mathbf{J} = -\mathbb{E}\left[\mathbf{H}_i(\beta_o)\right] = \mathbb{E}\left\{\mathbb{E}\left[\mathbf{H}_i(\beta_o)|\mathbf{x}_i\right]\right\} = \mathbb{E}\left\{\frac{g(\mathbf{x}_i'\boldsymbol{\beta_o})^2\mathbf{x}_i\mathbf{x}_i'}{G(\mathbf{x}_i'\boldsymbol{\beta_o})\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta_o})\right\}}\right\}$$

Asymptotic Variance Calculation for Correctly Specified Index Models

Asymptotic Distribution

$$\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_o) \to_d \mathcal{N}\left(\mathbf{0}, \mathbf{J}^{-1}\right), \quad \mathbf{J}^{-1} = \mathbb{E}\left\{\frac{g(\mathbf{x}_i'\boldsymbol{\beta}_o)^2 \mathbf{x}_i \mathbf{x}_i'}{G(\mathbf{x}_i'\boldsymbol{\beta}_o)\left\{1 - G(\mathbf{x}_i'\boldsymbol{\beta}_o)\right\}}\right\}^{-1}$$

Consistent Estimator

$$\widehat{\mathbf{J}}^{-1} \equiv \left\{\frac{1}{N}\sum_{i=1}^{N} \frac{g(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})^2 \mathbf{x}_i \mathbf{x}_i'}{G(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})\left[1 - G(\mathbf{x}_i'\widehat{\boldsymbol{\beta}})\right]}\right\}^{-1}$$

Notes

- Assumes correct specification, i.e. $p(\mathbf{x}) = \mathbb{E}(y|\mathbf{x}) = G(\mathbf{x}'\boldsymbol{\beta}_o)$
- In contrast, *robust* variance matrix $\mathbf{J}^{-1}\mathbf{K}\mathbf{J}^{-1}$ is complicated, but R can do it.

McFadden (1974) – "Pseudo R-squared"

Model with Intercept Only

$L(\bar{y}) \equiv$ maximized sample Likelihood

$\ell(\bar{y}) \equiv$ maximized sample log-likelihood

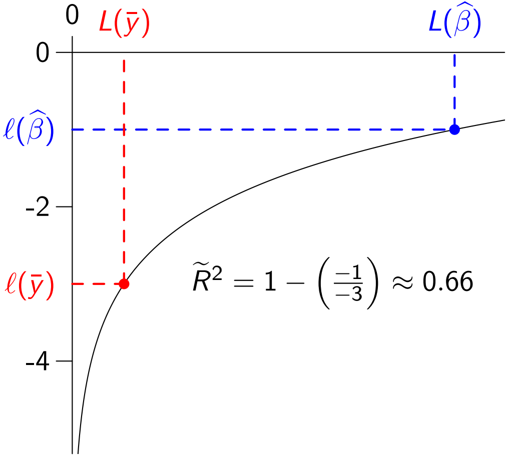
Full Model

$L(\widehat{\beta}) \equiv$ maximized sample Likelihood

$\ell(\widehat{\beta}) \equiv$ maximized sample log-likelihood

Pseudo R-squared

$\widetilde{R}^2 \equiv 1 - \ell(\widehat{\beta})/\ell(\bar{y})$



McFadden (1974) – "Pseudo R-squared"

Pseudo R-squared

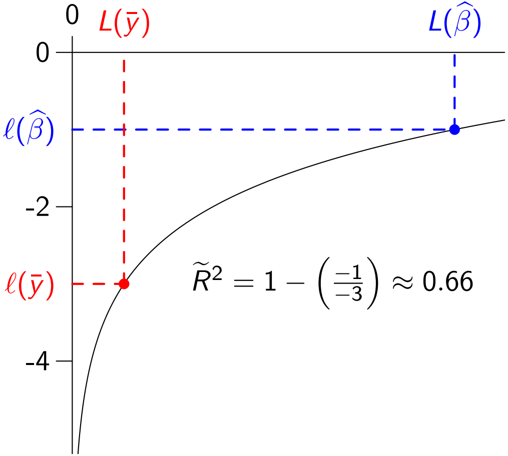
$\widetilde{R}^2 \equiv 1 - \ell(\widehat{\beta})/\ell(\bar{y})$

Always between 0 and 1

Show this on the problem set!

Health Warning

I don't recommend using pseudo-R^2: it's arbitrary and can be misleading. Other people use it so I'm telling you what it is.



Lecture #4 – Random Utility Models

Overview of Random Utility Models

Identification of Choice Models

Index Models as Special Cases (e.g. Logit & Probit)

Multinomial and Conditional Logit

Discrete Choice – Basic Terminology

Decision-maker

Household, person, firm, etc.

Alternatives

Products, courses of action, etc.

Choice Set

The collection of all alternatives available to the decision-maker.

Restrictions on the Choice Set

We assume that:

1. Choices are mutually exclusive: choose only *one* alternative.
2. Choice set is *exhaustive*: contains every alternative (always choose something)
3. The number of alternatives is finite.

We can always redefine the choice set to satisfy 1 and 2

$$\underbrace{\{\text{Beer}, \text{Pizza}\}}_{\text{not mutually exclusive}} \quad \rightarrow \quad \underbrace{\{\text{Beer only}, \text{Pizza only}, \text{Beer and Pizza}\}}_{\text{mutually exclusive}}$$

$$\underbrace{\{\text{Beer only}, \text{Pizza only}, \text{Beer and Pizza}\}}_{\text{not exhaustive}} \quad \rightarrow \quad \underbrace{\{\text{Beer only}, \text{Pizza only}, \text{Beer and Pizza}, \text{Something Else}\}}_{\text{exhaustive}}$$

Random Utility Models (RUMs)

Following Train (2009), use n to index individuals!

Notation

- N decision-makers $n = 1, \ldots, N$
- J alternatives $j = 1, \ldots, J$.

Utility and Decision Rule

- Decision-maker n obtains utility U_{nj} from choosing alternative j
- Maximize utility: decision-maker n chooses alternative i iff $U_{ni} > U_{nj}$ for any $j \neq i$

Random Utility Models

Researcher Observes

- Attributes x_{nj} of each alternative (e.g. product characteristics)
- Attributes s_n of the decision-maker (e.g. demographics)
- Choices but not utilities

Representative Utility V_{nj}

Assume researcher can specify a function $V_{nj}(x_{nj}, s_n)$ relating attributes x_{nj} of each alternative j and attributes s_n of each decision-maker n to her utilities U_{nj}.

Error Terms ε_{nj}

$\varepsilon_{nj} \equiv U_{nj} - V_{nj}$ is the difference between *true* utility U_{nj} and representative utility V_{nj}

Random Utility Models (RUMs)

What are the error terms?

ε_{nj} for $j = 1, \ldots, J$ represent unobserved factors that affect choices but are not captured by representative utilities (i.e. our model)

Treat Errors as Random

Let $\boldsymbol{\varepsilon}' \equiv [\ \varepsilon_{n1} \quad \ldots \quad \varepsilon_{nJ}\ ]$ have density function $f(\boldsymbol{\varepsilon}_n)$. Characterizes unobserved heterogeneity across decision-makers.

Choice Probabilities

$$P_{ni} \equiv \mathbb{P}(U_{ni} > U_{nj} \quad \forall j \neq i) = \int_{\mathbb{R}^J} \mathbb{1}\left\{\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj} \quad \forall j \neq i\right\} f(\boldsymbol{\varepsilon}_n)\, d\boldsymbol{\varepsilon}_n$$

This all sounds a bit abstract...

Basic Idea

1. Write down a parametric model for $V_{nj}(x_{nj}, s_n)$ with unknown parameters $\boldsymbol{\theta}$.
2. Choose a distribution f for the errors (heterogeneity) $\boldsymbol{\varepsilon}_n$.
3. Back out choice probabilities as a function of parameters $\boldsymbol{\theta}$.
4. Use observed choices and attributes to find the MLE $\widehat{\boldsymbol{\theta}}$.

Looking Back; Looking Ahead

- Logit and Probit are special cases of RUMs: choice between two alternatives.
- RUMs provide a framework to estimate more complicated discrete choice models.

Some Complications

Computation

- Integral linking choice probabilities to parameters $\boldsymbol{\theta}$ rarely has a closed form.
- Logit-type models are a well-known *exception*.
- More generally: use *Monte Carlo Simulation* to approximate the integral.

Identification

- Roughly speaking, we say that a parameter is *identified* if it could be uniquely determined by observing the *whole population*.
- What parameters of RUMs are identified from choices and attributes?

A Very Simple Example

Transport Decision

- Exactly two ways to get to work: by car or by bus.
- Observe two attributes: cost in time T and money M of each mode of transport.

Econometrician's Model: (β, γ) unknown

$$V_{\text{car}} = \beta T_{\text{car}} + \gamma M_{\text{car}} \qquad U_{\text{car}} = V_{\text{car}} + \varepsilon_{\text{car}}$$

$$V_{\text{bus}} = \beta T_{\text{bus}} + \gamma M_{\text{bus}} \qquad U_{\text{bus}} = V_{\text{bus}} + \varepsilon_{\text{bus}}$$

Choice Probabilities

$$P_{\text{car}} = \mathbb{P}(\varepsilon_{\text{bus}} - \varepsilon_{\text{car}} < V_{\text{car}} - V_{\text{bus}})$$

$$P_{\text{bus}} = \mathbb{P}(\varepsilon_{\text{car}} - \varepsilon_{\text{bus}} < V_{\text{bus}} - V_{\text{car}}) = 1 - P_{\text{car}}$$

A Very Simple Example: Who drives to work?

What is common?

Parameters: (β, γ). Our goal is to identify and estimate these.

Observed Heterogeneity

- Alice lives next to the bus stop: her T_{bus} is low.
- Bob is 70 and gets a discount on public transport: his M_{bus} is low.
- Clara and her roommates work at the same office and can carpool: her M_{car} is low.

Unobserved Heterogeneity

James hates to drive ($\varepsilon_{\text{car}} - \varepsilon_{\text{bus}} < 0$) but Steve loves driving ($\varepsilon_{\text{car}} - \varepsilon_{\text{bus}} > 0$).

Identification – What can we learn from data?

Only differences in utility matter

- $\mathbb{P}(U_{ni} > U_{nj} \quad \forall j \neq i) = \mathbb{P}\left(U_{ni} - U_{nj} > 0 \quad \forall j \neq i\right)$
- All that matters is how much better or worse a given alternative is than the others.

Consequences

1. We cannot identify a different intercept for each alternative.
2. We can only identify differences of effects for decision-maker attributes.

If there are J alternatives, we can identify only $(J-1)$ intercepts.

Equivalently: normalize one intercept to zero.

Intercept $\Rightarrow \mathbb{E}\left[\varepsilon_{nj}\right] = 0$

- Suppose $U_{nj} = \mathbf{x}'_{nj}\boldsymbol{\beta} + \varepsilon^*_{nj}$ where $\mathbf{x}_{nj}$ *excludes* a constant and $\mathbb{E}[\varepsilon^*_{nj}] \neq 0$.
- Equivalent model: $U_{nj} = \alpha + \mathbf{x}'_{nj}\boldsymbol{\beta} + \varepsilon_{nj}$ where $\mathbb{E}[\varepsilon_{nj}] = 0$ by construction.

Why not a different intercept for each alternative?

$$U_{\text{car}} = \alpha_{\text{car}} + \beta T_{\text{car}} + \gamma M_{\text{car}} + \varepsilon_{\text{car}}$$

$$U_{\text{bus}} = \alpha_{\text{bus}} + \beta T_{\text{bus}} + \gamma M_{\text{bus}} + \varepsilon_{\text{bus}}$$

$$U_{\text{bus}} - U_{\text{car}} = \left(\alpha_{\text{bus}} - \alpha_{\text{car}}\right) + \beta\left(T_{\text{bus}} - T_{\text{car}}\right) + \gamma\left(M_{\text{bus}} - M_{\text{car}}\right) + \left(\varepsilon_{\text{bus}} - \varepsilon_{\text{car}}\right)$$

Only differences of effects for decision-maker attributes are identified.

Can we identify the effects of income Y separately for Bus and Car?

$$U_{\text{car}} = \theta_{\text{car}} Y + \beta T_{\text{car}} + \gamma M_{\text{car}} + \varepsilon_{\text{car}}$$

$$U_{\text{bus}} = \theta_{\text{bus}} Y + \beta T_{\text{bus}} + \gamma M_{\text{bus}} + \varepsilon_{\text{bus}}$$

$$U_{\text{bus}} - U_{\text{car}} = (\theta_{\text{bus}} - \theta_{\text{car}})\, Y + \beta\,(T_{\text{bus}} - T_{\text{car}}) + \gamma\,(M_{\text{bus}} - M_{\text{car}}) + (\varepsilon_{\text{bus}} - \varepsilon_{\text{car}})$$

Equivalent to normalizing one of the θs to zero.

More on Identification – What can we learn from data?

The scale of utility is irrelevant

- Let λ be an arbitrary positive constant.
- Original Model: $U_{nj} = V_{nj} + \varepsilon_{nj}$, $\text{Var}(\varepsilon_{nj}) = \sigma^2$
- Re-scaled Model: $\lambda U_{nj} = \lambda V_{nj} + \lambda \varepsilon_{nj} \iff \boxed{U^*_{nj} = V^*_{nj} + \varepsilon^*_{nj},\ \text{Var}(\varepsilon^*_{nj}) = \lambda^2\sigma^2}$

$\text{Var}(\varepsilon_{nj})$ determines the scale of $\boldsymbol{\beta}$

- $\boxed{U_{nj} = \mathbf{x}'_{nj}\boldsymbol{\beta} + \varepsilon_{nj},\ \text{Var}(\varepsilon_{nj}) = \sigma^2} \iff \boxed{U^*_{nj} = \mathbf{x}'_{nj}(\boldsymbol{\beta}/\sigma) + \varepsilon^*_{nj},\ \text{Var}(\varepsilon^*_{nj}) = 1}$
- Can't directly compare coefs. across models with different normalizations for ε_{nj}.
- Recall: we had to re-scale Logit and Probit coefs. to compare them.

Only differences in utility matter $\implies$ only differences in errors matter.

Notation

- $\widetilde{\varepsilon}_{njk} \equiv \varepsilon_{nj} - \varepsilon_{nk}$ be the *difference* of errors ε_{nj} and ε_{nk}.
- $\widetilde{\boldsymbol{\varepsilon}}_{ni} \equiv$ vector of all unique differences, taking ε_{ni} as the "base case"
  - E.g. $\boldsymbol{\varepsilon}'_n = (\varepsilon_{n1}, \varepsilon_{n2}, \varepsilon_{n3}) \implies \widetilde{\boldsymbol{\varepsilon}}'_{n1} = (\varepsilon_{n2} - \varepsilon_{n1}, \varepsilon_{n3} - \varepsilon_{n1})$
  - Note: J errors $\Rightarrow (J-1)$ unique *differences*
- Let g be the joint density of $\widetilde{\boldsymbol{\varepsilon}}_{ni}$.

Choice Probabilities

$$P_{ni} \equiv \mathbb{P}\left(U_{ni} > U_{nj} \quad \forall j \neq i\right) = \mathbb{P}(\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj} \quad \forall j \neq i)$$

$$= \mathbb{P}(\widetilde{\varepsilon}_{nji} < V_{ni} - V_{nj} \quad \forall j \neq i) = \int_{\mathbb{R}^{J-1}} \mathbb{1}\left\{\widetilde{\varepsilon}_{nji} < V_{ni} - V_{nj} \quad \forall j \neq i\right\} g(\widetilde{\boldsymbol{\varepsilon}}_{ni})\, d\widetilde{\boldsymbol{\varepsilon}}_{ni}$$

How to obtain the index models from last lecture? (E.g. Probit and Logit)

1. Two alternatives, e.g. Bus or Something Else
2. Let $y_n = 1$ if decision-maker n chooses alternative 1; zero otherwise.
3. $V_{nj} = \mathbf{s}_n' \gamma_j$ (representative utility depends only on attributes of decision-maker)
4. $(\varepsilon_{n2} - \varepsilon_{n1}) \sim G$ independently of $\mathbf{s}_n$.

$$
\begin{aligned}
U_{n1} - U_{n2} &= (\mathbf{s}_n'\gamma_1 - \mathbf{s}_n'\gamma_2) + (\varepsilon_{n1} - \varepsilon_{n2}) = \mathbf{s}_n'(\gamma_1 - \gamma_2) + (\varepsilon_{n1} - \varepsilon_{n2}) \\
&= \mathbf{s}_n'\gamma + (\varepsilon_{n1} - \varepsilon_{n2})
\end{aligned}
$$

$$
\mathbb{P}(y_n = 1|\mathbf{s}_n) = \mathbb{P}(U_{n1} - U_{n2} > 0|\mathbf{s}_n) = \mathbb{P}(\varepsilon_{n2} - \varepsilon_{n1} < \mathbf{s}_n'\gamma|\mathbf{s}_n) = G(\mathbf{s}_n'\gamma)
$$

The Logit Family of Choice Models

Theorem

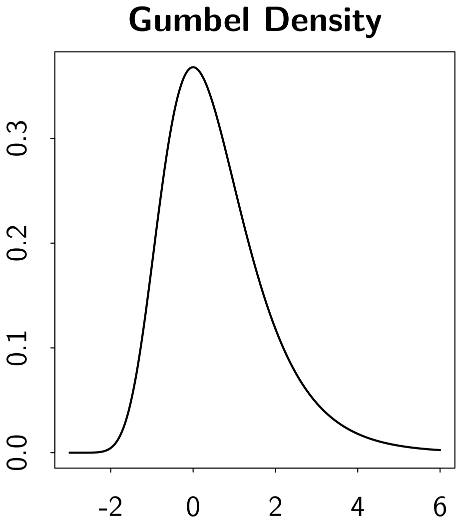
Suppose that $\varepsilon_{n1}, \ldots \varepsilon_{nJ} \sim$ iid F where $F(z) = \exp\{-\exp(-z)\}$. Then,

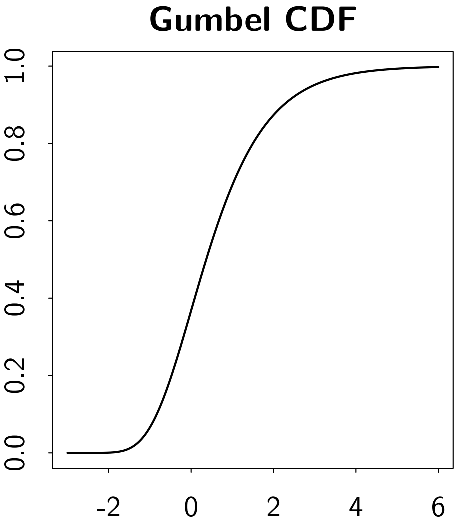
$$P_{ni} = \mathbb{P}(\varepsilon_{nj} - \varepsilon_{ni} < V_{ni} - V_{nj} \quad \forall j \neq i) = \frac{\exp(V_{ni})}{\sum_{j=1}^{J} \exp(V_{nj})}$$

Notes

- This is a special case where the choice probabilities have a closed-form solution!
- $F(z) = \exp\{-\exp(-z)\}$ is the Gumbel aka Type I Extreme Value CDF
- Corollary: the *difference* of independent Gumbel RVs is a standard Logistic RV

The Gumbel Distribution (aka Type I Extreme Value)





Different specifications of V_{nj} yield different models.

Multinomial Logit

- $V_{nj} = \mathbf{s}_n' \gamma_j \quad \leftarrow$ only attributes that are fixed across alternatives (e.g. n's income)
- Can only identify differences $(\gamma_j - \gamma_i)$. Typical to normalize $\boldsymbol{\gamma}_1 = \mathbf{0}$.

Conditional Logit

- $V_{nj} = \mathbf{x}_{nj}' \boldsymbol{\beta} \quad \leftarrow$ only attributes that vary across alternatives (e.g. price)
- Note that $\boldsymbol{\beta}$ is fixed across alternatives.

Mixed Logit

- $V_{nj} = \mathbf{s}_n' \gamma_j + \mathbf{x}_{nj}' \boldsymbol{\beta} \quad \leftarrow$ a combination of the two

The Likelihood for Random Utility Models

Notation

- $y_n \in \{1, \ldots, J\} \equiv n$'s choice.
- $\mathbf{z}_n$ vector of all regressors for n
- $\boldsymbol{\theta}$ vector of all unknown parameters
- Choice Probs. $P_{ni} \equiv \mathbb{P}(y_n = i | \mathbf{z}_n, \boldsymbol{\theta})$

Note

Likelihood is easy, but choice probabilities are usually hard (logit is an exception).

Likelihood

$$f(y_n | \mathbf{z}_n, \boldsymbol{\theta}) = \prod_{j=1}^{J} P_{nj}^{\mathbb{1}\{y_n = j\}}$$

Log Likelihood

$$\ell_N(\boldsymbol{\theta}) = \sum_{n=1}^{N} \sum_{j=1}^{J} \mathbb{1}\{y_n = j\} \log P_{nj}$$

Logit Choice Probabilities

$$P_{ni} = \exp(V_{ni}) / \sum_{j=1}^{J} \exp(V_{nj})$$

Interpreting Multinomial Logit Coefficients

- Partial effects tricky to derive and interpret.
- Better approach: examine log-odds ratios
- Normalizing $\gamma_1 = \mathbf{0}$, we have $\exp(\mathbf{s}_n\gamma_1) = \exp(0) = 1$. Hence,

$$\frac{P_{ni}}{P_{n1}} = \frac{\exp\left(\mathbf{s}_n\gamma_i\right)}{\sum_{j=1}^{J}\exp\left(\mathbf{s}_n\gamma_j\right)} \times \frac{\sum_{j=1}^{J}\exp\left(\mathbf{s}_n\gamma_j\right)}{\exp\left(\mathbf{s}_n\gamma_1\right)} = \frac{\exp(\mathbf{s}_n\gamma_i)}{\exp(\mathbf{s}_n\gamma_1)} = \exp(\mathbf{s}_n\gamma_i)$$

- Taking logs: $\log\left(P_{ni}/P_{n1}\right) = \log\left[\exp(\mathbf{s}_n\gamma_i)\right] = \mathbf{s}_n'\gamma_i$.

Punchline

$\gamma_i^{(k)}$ is the marginal effect of $s_n^{(k)}$ on the relative probability that $y = i$ compared to $y = 1$ measured on the log scale – e.g. taking the bus relative to driving.

Interpreting Conditional Logit Coefficients

You'll derive these on the problem set!

Partial Effects

- The attributes $\mathbf{x}_{nj}$ are *specific* to a particular alternative j.
- Hence: partial effects are much simpler for conditional logit than multinomial.

Own Attribute

$$\frac{\partial P_{nj}}{\partial \mathbf{x}_{nj}} = P_{nj}(1 - P_{nj})\boldsymbol{\beta}$$

Cross-Attribute ($j \neq i$)

$$\frac{\partial P_{nj}}{\partial \mathbf{x}_{ni}} = -P_{nj}P_{ni}\boldsymbol{\beta}$$

If increasing $\mathbf{x}_{nj}^{(k)}$ makes $y = j$ *more likely*, it must make $y = i$ *less likely*

The Independence of Irrelevant Alternatives (IIA)

Or why people don't like logit models...

Logit Choice Probabilities

$$P_{ni} = \frac{\exp(V_{ni})}{\sum_{j=1}^{J} \exp(V_{nj})} \implies \frac{P_{ni}}{P_{nj}} = \exp(V_{ni} - V_{nj})$$

In Words

The relative probability of choosing i versus j only depends on the representative utilties for i and j. This is called the independence of irrelevant alternatives (IIA).

Why is this a problem

IIA arises in logit models because $\varepsilon_{n1}, \ldots, \varepsilon_{nJ}$ are *independent*. In reality "some alternatives are more similar than others," i.e. errors may be correlated.

An Example where IIA is Unreasonable – Choosing Presidential Candidates

Model

- $V_{nj} = (\text{Demographics}_n)'\gamma_j + (\text{Ideology}_{nj})'\beta$
- (Ideology_{nj}) = similarity between voter n's ideology and candidate j's.
- Candidates $= \{\text{Trump}, \text{Sanders}, \text{Warren}\}$

Consider a group of voters who all have the *same* demographics and ideology

E.g. white, centrist, female, mid-westerners between the age of 45 and 50 with an average household income between \$50 and \$55 thousand USD.

Same regressors $\Rightarrow$ same V_{nj}

V_{nj} doesn't vary over n within the group: $\{V_{\text{Trump}}, V_{\text{Sanders}}, V_{\text{Warren}}\}$

An Example where IIA is Unreasonable – Choosing Presidential Candidates

Two-way Race

Suppose 2/3 of this group of voters chooses Sanders over Trump: $P_{\text{Sanders}}/P_{\text{Trump}} = 2$

Assumption

Sanders and Warren are ideologically similar $\implies$ $V_{\text{Warren}} \approx V_{\text{Sanders}}$

Implications of Logit

- Relative choice probabilities are the *same* in a two-way race or a three-way race.
- $P_{\text{Warren}}/P_{\text{Sanders}} = \exp\left(V_{\text{Warren}} - V_{\text{Sanders}}\right) \approx 1$

An Example where IIA is Unreasonable – Choosing Presidential Candidates

Logit Implication for Three-way Race

$$P_{\text{Sanders}} = 2P_{\text{Trump}}, \quad P_{\text{Sanders}} \approx P_{\text{Warren}}, \quad P_{\text{Trump}} + P_{\text{Sanders}} + P_{\text{Warren}} = 1$$

$$\begin{aligned} \implies P_{\text{Trump}} + 2P_{\text{Trump}} + 2P_{\text{Trump}} &= 1 \\ P_{\text{Trump}} &= 1/5 \\ P_{\text{Warren}} = P_{\text{Sanders}} &= 2/5 \end{aligned}$$

What we'd actually expect in a Three-way Race

1/3 Trump, 1/3 Sanders and 1/3 Warren – i.e. Warren "splits" the Sanders vote.

What's going wrong?

Logit assumes $\varepsilon_{\text{Warren}}$ and $\varepsilon_{\text{Sanders}}$ are independent but in reality they're not.

Lecture #5 – Sample Selection

Examples of Sample Selection

The Heckman Selection Model

What is sample selection?

Question

Thus far we have always assumed that $(y_1, \mathbf{x}_1), \ldots, (y_N, \mathbf{x}_N)$ are a random sample from the population of interest. What if they aren't?

Example 1: MPhil Admissions

- Suppose we want to improve admissions decisions at Oxford.
- $y \equiv$ overall marks in 1st year of Oxford Economics MPhil
- $\mathbf{x} \equiv \{\text{undergrad grades}, \text{letters of reference}, \ldots\}$
- What we observe: $\mathbf{x}$ for all applicants; y for applicants who were admitted.
- What we want: $\mathbb{E}(y|\mathbf{x})$ for all applicants.

Example 2: A Model of Wage Offers

Gronau (1974; JPE)

Question

How do wage offers offers w_i^o vary with $\mathbf{x}_i$ for all people in the population.

Problem

Only observe w_i^o for people who *accept* their offer, i.e. those who are employed.

Mathematically

$$\mathbb{E}(w_i^o|\mathbf{x}_i) \neq \mathbb{E}(w_i^o|\mathbf{x}_i, \text{Employed})$$

The Heckman Selection Model (Heckit) — Is β_1 identified?

Outcome Equation

$y_1 = \mathbf{x}_1'\beta_1 + u_1$

Participation Equation

$y_2 = \mathbb{1}\left\{\mathbf{x}'\delta_2 + v_2 > 0\right\}$

Assumptions

(a) Observe $y_2, \mathbf{x}' = (\mathbf{x}_1', \mathbf{x}_2')$; only observe y_1 if $y_2 = 1$.

(b) (u_1, v_2) are mean zero and jointly independent of $\mathbf{x}$.

(c) $v_2 \sim \text{Normal}(0, 1)$

(d) $\mathbb{E}(u_1|v_2) = \gamma_1 v_2$ where γ_1 is an unknown constant.

Notes

- $\mathbb{E}(u_1) = \mathbb{E}(v_2) = 0$ is not restrictive: just include intercepts in both equations.
- Assumption (d) would be *implied* by assuming that (u_1, v_2) are jointly normal.
- These assumptions are strong. They can be weakened a bit, but not too much.

Step 1: Show that u_1 and $\mathbf{x}$ are conditionally independent given v_2.

Assumption (b)

(u_1, v_2) are jointly independent of $\mathbf{x}$.

Equivalently

$$f_{1,2|\mathbf{x}}(u_1, v_2|\mathbf{x}) = f_{1,2}(u_1, v_2), \quad \text{and} \quad f_{1|\mathbf{x}}(u_1|\mathbf{x}) = f_1(u_1), \quad \text{and} \quad f_{2|\mathbf{x}}(v_2|\mathbf{x}) = f_2(v_2)$$

Therefore

$$f_{1|2,\mathbf{x}}(u_1|v_2, \mathbf{x}) = \frac{f_{1,2|\mathbf{x}}(u_1, v_2|\mathbf{x})}{f_{2|\mathbf{x}}(v_2|\mathbf{x})} = \frac{f_{1,2}(u_1, v_2)}{f_2(v_2)} = f_{1|2}(u_1|v_2)$$

In Words

Conditioning on $(v_2, \mathbf{x})$ gives the same information about u_1 as conditioning on v_2 only.

Step 2: Calculate $\mathbb{E}(y_1|\mathbf{x}, v_2)$; show that if v_2 were observed we'd be done.

$$
\begin{aligned}
\mathbb{E}(y_1|\mathbf{x}, v_2) &= \mathbb{E}(\mathbf{x}_1'\boldsymbol{\beta}_1 + u_1|\mathbf{x}, v_2) && \text{(Substitute Outcome Eq.)} \\
&= \mathbf{x}_1'\boldsymbol{\beta}_1 + \mathbb{E}(u_1|\mathbf{x}, v_2) && (\mathbf{x}_1 \text{ is a subset of } \mathbf{x}) \\
&= \mathbf{x}_1'\boldsymbol{\beta}_1 + \mathbb{E}(u_1|v_2) && \text{(apply result of Step 1)} \\
&= \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 v_2 && \text{(apply Assumption (d))}
\end{aligned}
$$

This slide has been cut, but I'm leaving a blank slide in its place so that the slide numbering remains unchanged from lecture.

Step 3: Relate v_2 (unobserved) to $\mathbf{x}$ and y_2 (both observed).

$$
\begin{aligned}
\mathbb{E}(y_1|\mathbf{x}, y_2) &= \mathbb{E}_{v_2|(\mathbf{x},y_2)}\left[\mathbb{E}\left(y_1|\mathbf{x}, y_2, v_2\right)\right] && \text{(Law of Iterated Expectations)} \\
&= \mathbb{E}_{v_2|(\mathbf{x},y_2)}\left[\mathbb{E}\left(y_1|\mathbf{x}, v_2\right)\right] && \text{(Participation Eq: } y_2 = g(\mathbf{x}, v_2)\text{)} \\
&= \mathbb{E}\left[\mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 v_2|\mathbf{x}, y_2\right] && \text{(apply result of Step 2)} \\
&= \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1\mathbb{E}\left(v_2|\mathbf{x}, y_2\right) && \text{(}\mathbf{x}_1 \text{ is a subset of } \mathbf{x}\text{)}
\end{aligned}
$$

Therefore

$$\mathbb{E}\left(y_1|\mathbf{x}, y_2 = 1\right) = \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1\mathbb{E}(v_2|\mathbf{x}, y_2 = 1).$$

What is the significance of Step 3?

- Define $h(\mathbf{x}) \equiv \mathbb{E}(v_2|\mathbf{x}, y_2 = 1)$. Then: $\mathbb{E}(y_1|\mathbf{x}, y_2 = 1) = \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 h(\mathbf{x})$
- Note that $h(\mathbf{x})$ is a random variable: a function of $\mathbf{x}$.
- Step 3 shows that a linear regression of y_1 on $\mathbf{x}_1$ and $h(\mathbf{x})$ for the *selected* sample, those with $y_2 = 1$, identifies $\boldsymbol{\beta}_1$ and γ_1!
- All that remains is to figure out what function h is...

Note: Selection Bias Enters Through γ_1

Assumption (d)

$\mathbb{E}(u_1|v_2) = \gamma_1 v_2$ allows *dependence* between errors in participation and outcome eqs.

Step 3

$\mathbb{E}(y_1|\mathbf{x}, y_2 = 1) = \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 \mathbb{E}(v_2|\mathbf{x}, y_2 = 1)$

Therefore

If $\gamma_1 = 0$ there is no selection bias: in this case $\mathbb{E}(y_1|\mathbf{x}, y_2 = 1) = \mathbf{x}_1'\boldsymbol{\beta}$ so regressing y_1 on $\mathbf{x}_1$ for the subset of individuals with $y_2 = 1$ identifies $\boldsymbol{\beta}_1$.

Step 4: Determine the distribution of v_2 given $(\mathbf{x}, y_2 = 1)$.

$$\mathbb{P}(v_2 \leq t|\mathbf{x}, y_2 = 1) = \mathbb{P}(v_2 \leq t|\mathbf{x}, v_2 > -\mathbf{x}'\boldsymbol{\delta}_2) \qquad \text{(participation eq.)}$$

$$= \frac{\mathbb{P}\left(\{v_2 \leq t\} \cap \{v_2 > -\mathbf{x}'\boldsymbol{\delta}_2\} |\mathbf{x}\right)}{\mathbb{P}(v_2 > -\mathbf{x}'\boldsymbol{\delta}_2|\mathbf{x})} \qquad \text{(defn. of cond. prob.)}$$

$$= \frac{\mathbb{P}\left\{v_2 \in (-\mathbf{x}'\boldsymbol{\delta}_2, t]\right\}}{\mathbb{P}(v_2 > -\mathbf{x}'\boldsymbol{\delta}_2)} \qquad \text{(v_2 and $\mathbf{x}$ are indep.)}$$

$$= \frac{\mathbb{P}\left\{z \in (c, t]\right\}}{\mathbb{P}(z > c)} \qquad \text{(v_2 is standard normal)}$$

where $z \sim \text{Normal}(0, 1)$ and we define the shorthand $c \equiv -\mathbf{x}'\delta_2$.

Step 5: Calculate the Expectation of a Truncated Normal

Recall: $z \sim \text{Normal}(0,1)$ and $c \equiv -\mathbf{x}'\boldsymbol{\delta}_2$

CDF

$$\mathbb{P}(z \leq t | z > c) = \frac{\mathbb{P}\left\{z \in (c, t]\right\}}{\mathbb{P}(z > c)} = \mathbb{1}\left\{c \leq t\right\}\left[\frac{\Phi(t) - \Phi(c)}{1 - \Phi(c)}\right]$$

Density

$$f(t|z > c) = \frac{d}{dt}\mathbb{P}(z \leq t | z > c) = \begin{cases} 0, & t \leq c \\ \varphi(t)/\left[1 - \Phi(c)\right], & t > c \end{cases}$$

Step 5: Calculate the Expectation of a Truncated Normal

Recall: $z \sim \text{Normal}(0,1)$ and $c \equiv -\mathbf{x}'\boldsymbol{\delta}_2$

$$\mathbb{E}(z|z>c) = \int_{-\infty}^{\infty} tf(t|z>c)\,dt = \frac{1}{1-\Phi(c)}\int_c^{\infty} t\varphi(t)\,dt$$

$$= \left[\frac{1}{1-\Phi(c)}\right]\left(\frac{1}{\sqrt{2\pi}}\right)\int_c^{\infty} t\exp\left\{-t^2/2\right\}\,dt$$

$$= \left[\frac{1}{1-\Phi(c)}\right]\left(\frac{1}{\sqrt{2\pi}}\right)\left[-\exp\left\{-t^2/2\right\}\right]_c^{\infty}$$

$$= \left[\frac{1}{1-\Phi(c)}\right]\left(\frac{\exp\left\{-c^2/2\right\}}{\sqrt{2\pi}}\right) = \frac{\varphi(c)}{1-\Phi(c)}$$

Step 6: Put everything together.

Recall: Step 3

$$y_1 = \mathbf{x}_1'\boldsymbol{\beta}_1 + \gamma_1 h(\mathbf{x}) + \eta, \quad \mathbb{E}\left[\eta|\mathbf{x}_1, h(\mathbf{x})\right] = 0, \quad h(\mathbf{x}) \equiv \mathbb{E}(v_2|\mathbf{x}, y_2 = 1)$$

Using Steps 4–5

$$h(\mathbf{x}) = \frac{\varphi(-\mathbf{x}'\boldsymbol{\delta}_2)}{1 - \Phi(-\mathbf{x}'\boldsymbol{\delta}_2)} = \frac{\varphi(\mathbf{x}'\boldsymbol{\delta}_2)}{\Phi(\mathbf{x}'\boldsymbol{\delta}_2)} \quad \text{since } \varphi(-c) = \varphi(c) \text{ and } 1 - \Phi(c) = \Phi(-c).$$

Inverse Mills Ratio

$\varphi(c)/\Phi(c)$ is the inverse Mills Ratio, traditionally denoted by $\lambda \implies h(\mathbf{x}) = \lambda(\mathbf{x}'\boldsymbol{\delta}_2)$.

Careful!

In an earlier lecture λ denoted the standard logistic density. Here it's something else!

The Heckman Two-step Estimator aka "Heckit"

Observables

Observe $(y_{2i}, \mathbf{x}_i)$ for a random sample of size N; only observe y_{1i} for those with $y_{2i} = 1$.

First Step – Estimate δ_2 from Full Sample

- Run Probit on the Participation Eq. $\mathbb{P}(y_{2i} = 1|\mathbf{x}_i) = \Phi(\mathbf{x}_i'\boldsymbol{\delta}_2)$ for the full sample.
- Define $\widehat{\lambda}_i \equiv \lambda(\mathbf{x}_i'\widehat{\boldsymbol{\delta}}_2)$ where $\widehat{\boldsymbol{\delta}}_2$ is the MLE for $\boldsymbol{\delta}_2$.

Second Step – Estimate (β_1, γ_1) from Selected Sample

Using the observations for which y_{i1} is observed, regress y_{i1} on $(\mathbf{x}_{1i}, \widehat{\lambda}_i)$ by OLS to obtain estimates $(\widehat{\boldsymbol{\beta}}_1, \widehat{\gamma}_1)$.

The Heckman Two-step Estimator aka "Heckit"

Theorem

Under the assumptions from above, the 2-step "Heckit" estimators satisfy

$$\begin{bmatrix} \widehat{\boldsymbol{\delta}}_2 \\ \widehat{\boldsymbol{\beta}}_1 \\ \widehat{\gamma}_1 \end{bmatrix} \rightarrow_p \begin{bmatrix} \boldsymbol{\delta}_2 \\ \boldsymbol{\beta}_1 \\ \gamma_1 \end{bmatrix} \quad \text{and} \quad \sqrt{N} \begin{bmatrix} \widehat{\boldsymbol{\delta}}_2 - \boldsymbol{\delta}_2 \\ \widehat{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_1 \\ \widehat{\gamma}_1 - \gamma_1 \end{bmatrix} \rightarrow_d \text{Normal}(\mathbf{0}, \boldsymbol{\Omega}) \quad \text{as } N \rightarrow \infty.$$

Standard Errors

The asymptotic variance matrix $\boldsymbol{\Omega}$ is complicated: the usual OLS standard errors from step two are incorrect as they do not account for the estimation of $\boldsymbol{\delta}_2$ in step one.

The Big Picture: How does Heckit solve the selection problem?

- If we regress y_{1i} on $\mathbf{x}_{1i}$ for the selected sample, there is an omitted variable.
- Under the Heckit assumptions, the omitted variable is precisely $\lambda(\mathbf{x}_i'\boldsymbol{\delta}_2)$.
- Hence: a regression of y_{1i} on $\mathbf{x}_{1i}$ and $\lambda(\mathbf{x}_i'\boldsymbol{\delta}_2)$ is correctly specified.

Why is the second step regression identified?

- If $\mathbf{x}_i$ contains some variables that are *not* in $\mathbf{x}_{1i}$, we have an exclusion restriction: i.e. there are variables that affect participation but not outcomes.
- Even if there are no exclusion restrictions, λ is nonlinear so $\lambda(\mathbf{x}_{1i}'\boldsymbol{\delta}_2)$ will not be perfectly co-linear with $\mathbf{x}_{1i}$.
- Without exclusion restrictions identification comes *solely* from nonlinearity in λ.
- Depending on the values where it is evaluated, λ can be *close* to linear, leading to very imprecise estimates unless you have an exclusion restriction.