Reduce measures

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L^1 theory

The problem

$$\begin{cases} -\Delta u + g(u) = f & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases}$$

- $\Omega \subset \mathbb{R}^N$ bounded, open, smooth boundary
- $g : \mathbb{R} \to \mathbb{R}$ continuous, non-decreasing, $g(0) = 0$
- $f \in L^1(\Omega)$

L^1 theory

The problem

$$\begin{cases} -\Delta u + g(u) = f & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

Weak solution

$$\begin{cases} u \in L^1(\Omega), \quad g(u) \in L^1(\Omega), \\ -\int_\Omega u\Delta\varphi + \int_\Omega g(u)\varphi = \int_\Omega f\varphi, \quad \forall\varphi \in C_0^2(\overline{\Omega}). \end{cases}$$

L^1 theory

Theorem (Brezis-Strauss, 1973)

For $f \in L^1(\Omega)$, there is a unique solution to the problem

$$\begin{cases} -\Delta u + g(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Transitions to measure

The problem

$$\begin{cases} -\Delta u + g(u) = \mu & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases}$$

Weak solution

$$\begin{cases} u \in L^1(\Omega), \quad g(u) \in L^1(\Omega), \\ -\int_\Omega u\Delta\varphi + \int_\Omega g(u)\varphi = \int_\Omega \varphi\, d\mu, \quad \forall\varphi \in C_0^2(\overline{\Omega}). \end{cases}$$

Transition to measure

Counterexample

Assume $N \geq 3$. If $p \geq \frac{N}{N-2}$, then, for any $a \in \Omega$, the problem

$$\begin{cases} -\Delta u + |u|^{p-1}u = \delta_a & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has no solution $u \in L^p(\Omega)$.

$$g(t) = |t|^{p-1}t$$

Transition to measure

Theorem

Assume $N \geq 2$ and there exists $p \in [1, \frac{N}{N-2})$ such that

$$|g(t)| \leq C(|t|^p + 1), \quad \forall t \in \mathbb{R}.$$

Then, for every $\mu \in \mathcal{M}(\Omega)$, there is a unique solution to the problem

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$$g(t) = |t|^{p-1}t \quad \Rightarrow \quad |g(t)| \leq C(|t|^p + 1)$$

$$g \text{ bounded} \quad \Rightarrow \quad |g(t)| \leq C(|t|^p + 1)$$

The problem

$$\begin{cases} -\Delta u + g(u) = \mu & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases}$$

- $N \geq 2$
- $g(t) = 0, \ \forall t \leq 0$

$$g_n(t) = \min\{g(t), n\} \leq n, \quad \forall t$$

$$\begin{cases} -\Delta u_n + g_n(u_n) = \mu & in\ \Omega, \\ u_n = 0 & on\ \partial\Omega, \end{cases}$$

Reduce measures

$$\begin{cases} -\Delta u_n + g_n(u_n) = \mu & in\ \Omega, \\ u_n = 0 & on\ \partial\Omega, \end{cases}$$

- $u_n \to u^* \quad in\ L^1(\Omega)$
- $g(u_n) \to g(u^*) \quad a.e\ in\ \Omega$
- $\Rightarrow \quad Fatou$

Weak solution

$$\begin{cases} u \in L^1(\Omega), \quad g(u) \in L^1(\Omega), \\ -\int_\Omega u\Delta\varphi + \int_\Omega g(u)\varphi = \int_\Omega \varphi\, d\mu, \quad \forall \varphi \in C_0^2(\overline{\Omega}). \end{cases}$$

Subsolution

$$\begin{cases} u \in L^1(\Omega), \quad g(u) \in L^1(\Omega), \\ -\int_\Omega u\Delta\varphi + \int_\Omega g(u)\varphi \leq \int_\Omega \varphi\, d\mu, \quad \forall \varphi \in C_0^2(\overline{\Omega}), \varphi \geq 0. \end{cases}$$

Reduce measures

Theorem (1)

We have u^* is the largest subsolution to our problem.

$$-\int_\Omega u^* \Delta\varphi + \int_\Omega g(u^*)\varphi = \int_\Omega \varphi \, d\mu^*, \quad \forall \varphi \in C_0^2(\overline{\Omega})$$

Theorem (2)

We have μ^* is the largest good measure $\leq \mu$.

Thank you for your attention